ON THE CONFORMAL DIMENSIONS OF QUASICONVEX POST-CRITICALLY FINITE SELF SIMILAR SETS

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Abstract. The conformal dimension of a metric space is the infimum of the Hausdorff dimensions of all quasisymmetrically equivalent metrics on the space. We show that certain classical self-similar fractal subsets of Euclidean space are not minimal for conformal dimension by constructing explicit metrics in the quasisymmetry class of the Euclidean metric with reduced Hausdorff dimension.

1. Introduction

In most cases, the Hausdorff dimension of a metric space is not a quasisymmetric invariant. We define the *conformal dimension* $\mathcal{C}\dim X$ of a metric space $X = (X, d)$ to be the infimum of the Hausdorff dimensions of all metric spaces quasisymmetrically equivalent to (X, d):

$$\mathcal{C}\dim X = \inf_{d' \in QS(d)} \dim(X, d').$$

Here "dim" denotes Hausdorff dimension and $QS(d)$ denotes the collection of metrics on X quasisymmetrically equivalent to d: $d' \in QS(d)$ if and only if there exists an increasing homeomorphism η of $[0, \infty)$ to itself so that

$$d(x, y) \le t d(x, z) \quad \Rightarrow \quad d'(x, y) \le \eta(t) d'(x, z) \tag{1.1}$$

for all $x, y, z \in X$.

The concept of conformal dimension was introduced by Pansu [13] in connection with his study of quasiconformal and quasisymmetric maps on Carnot groups. The conformal dimension of any space is clearly greater than or equal to the topological dimension; this inequality can be strict. In [16], the author showed that each Ahlfors regular space X with nontrivial conformal modulus has minimal conformal dimension, that is, $\mathcal{C}\dim X = \dim X$. (Such spaces necessarily have $\dim X \ge 1$—it is an open question whether spaces with minimal dimension exist in the case $0 < \dim X < 1$.) As a corollary, it follows that a space X with minimal conformal dimension exists for each value of $\dim X$ in the interval $[1, \infty)$.[1] In fact, a set of this type can be constructed as a subset of the Euclidean space $\mathbb{R}^n$ for any $n \ge \dim X$. These spaces can be chosen, for instance, to be classical self-similar fractal sets. For further results in this subject, we refer the reader to [17], [1], [3], [4] and section 15 in [6].

It is of some interest to calculate or (at the least) estimate the conformal dimension of a given metric space. In this paper, we show that certain classical self-similar fractal subsets of Euclidean spaces are not minimal for conformal dimension, i.e., there exist quasisymmetrically equivalent metrics on the sets with reduced Hausdorff dimension. The sets in question are given as the

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[1]This result was already known to Pansu and can be found in [13].

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invariant sets for finite collections of contractive similarities. They thus admit a type of "strict" self-similarity which interacts naturally with quasisymmetry. A basic theme throughout this paper will be the use of iteration to obtain uniform estimates as in (1.1). In order to construct new metrics on these sets, we will impose certain finite combinatorial assumptions at the first level of the construction; the iterative procedure by which the sets are built will then transfer these assumptions to all levels and guarantee that they hold uniformly throughout the final space.

Our main results (Theorem 5.1 and Theorem 5.2) provide nontrivial upper bounds for the conformal dimension for self-similar sets satisfying two assumptions: post-critical finiteness and accessibility. Lower bounds for the conformal dimension are in general much more difficult to obtain, see [16] and the references previously mentioned. In sections 3 and 6 we present several examples, indicating the explicit bounds for the conformal dimension which follow from Theorems 5.1 and 5.2. For example, we show that the Sierpinski gasket SG (Figure 1) satisfies

$$\mathcal{C}\dim SG \le 1.4160\ldots < 1.5849\ldots = \frac{\log 3}{\log 2} = \dim SG.$$

Similarly, we show that the *hexagasket* HG (Figure 7) satisfies

$$\mathcal{C}\dim HG \le \frac{\log(2+\sqrt{6})}{\log 3} = 1.3588\ldots < 1.6309\ldots = \frac{\log 6}{\log 3} = \dim HG.$$

A few words about our assumptions are in order. Roughly speaking, post-critical finiteness of a self-similar set states that the set is "nearly disconnected"—for example, it admits finite cut sets at all scales and locations. On the other hand, accesibility states that the set is "fairly well-connected"—for example, it implies that the space is quasiconvex, see Proposition 2.10. Thus on an informal level these two assumptions appear to work against each other. It is not clear precisely how many examples of sets satisfying these two conditions exist. Nevertheless, we hope that the ideas and techniques which we develop here in a specific context will be of use in the future for the study of the conformal geometry of more general "quasi-self-similar" sets which arise naturally in complex dynamics and geometric group theory, see Remark 4 in section 7. The self-similar sets which we consider in this paper provide a convenient and manageable framework for a preliminary understanding of the degree to which quasisymmetric maps can distort the dimension of a general metric space.

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2. Preliminary definitions and results

We follow Hutchinson's approach [7], viewing self-similar sets as invariant sets for collections of similarity mappings. Fix $n \ge 2$ and consider a collection of $N \ge 2$ contractive similarities $f_i : \mathbb{R}^n \to \mathbb{R}^n$ with contraction ratios $\lambda_i < 1$ and fixed points $a_i \in \mathbb{R}^n$. Thus $f_i(x) = a_i + \lambda_i \cdot M_i(x - a_i)$ for some orthogonal matrices M_i. Then there exists a unique compact set $K \subset \mathbb{R}^n$ which is invariant for the transformations $f_1, \ldots, f_N$, i.e.

$$K = \bigcup_{i=1}^{N} K_i, \qquad K_i = f_i(K).$$

This follows, for instance, from the completeness of the space of all compact subsets of $\mathbb{R}^n$ with the Hausdorff metric, see [12, 4.13].

Let $F = \{1, \dots, N\}$. For any string $\sigma = (\sigma_0, \dots, \sigma_{m-1}) \in F^m$, $m \in \mathbb{N}$, set $f_\sigma = f_{\sigma_{m-1}} \circ \cdots \circ f_{\sigma_0}$, $\lambda_\sigma = \lambda_{\sigma_{m-1}} \cdot \cdots \cdot \lambda_{\sigma_0}$, and $K_\sigma = f_\sigma(K)$. Then

$$K = \bigcup_{\sigma \in F^m} K_\sigma \tag{2.1}$$

for each $m \in \mathbb{N}$ and

$$\max_{\sigma \in F^m} \operatorname{diam} K_\sigma \le \lambda_{\max}^m \operatorname{diam} K, \tag{2.2}$$

which tends to zero as $m \to \infty$ since $\lambda_{\max} = \max\{\lambda_1, \dots, \lambda_N\} < 1$.

We denote by P the finite-sided convex polyhedron which is the closed convex hull of the fixed points $a_1, \dots, a_N$.

Lemma 2.3. *P is the closed convex hull of K.*

Proof. It suffices to prove that $K \subset P$. If not, then there exists a closed half-space $H \subset \mathbb{R}^n$ so that $a_i \in H$ for each i but $K \setminus H \neq \emptyset$. Let $x_0 \in K \setminus H$. Choose a backward orbit

$$\cdots \longmapsto x_k \overset{f_{j_k}}{\longmapsto} x_{k-1} \longmapsto \cdots \overset{f_{j_2}}{\longmapsto} x_1 \overset{f_{j_1}}{\longmapsto} x_0.$$

Denote by $h(x)$ the coefficient of $x \in \mathbb{R}^n$ in the direction orthogonal to ∂H (thus $h(a_i) \ge 0$ for each i and $h(x_0) < 0$). Then $h(x_k) = \lambda_{j_k}^{-1} h(x_{k-1})$ for each k and so

$$h(x_k) = \lambda_{j_1}^{-1} \lambda_{j_2}^{-1} \cdots \lambda_{j_k}^{-1} h(x_0) \ge \lambda_{\max}^{-k} h(x_0) \to \infty$$

as $k \to \infty$, which contradicts the compactness of K. □

Lemma 2.4. *$\bigcup_{i=1}^N P_i \subset P$ and $K = \bigcap_{m=1}^\infty \bigcup_{\sigma \in F^m} P_\sigma$, where $P_i = f_i(P)$ and $P_\sigma = f_\sigma(P)$.*

Proof. Since the operation of taking the closed convex hull commutes with each contractive similarity, P_i is the closed convex hull of K_i. Then the inclusion $P_i \subset P$ follows since P is a closed convex set containing K_i.

For each m, the inclusion $K \subset \bigcup_{\sigma \in F^m} P_\sigma$ follows from (2.1). The inclusion $K \supset \bigcap_{m=1}^\infty \bigcup_{\sigma \in F^m} P_\sigma$ follows from (2.2) and the compactness of K. □

Let O denote the interior of the convex polyhedron P. Then $O_i = f_i(O) \subset O$ for each $i = 1, \dots, N$. We impose the further assumption that

$$O_i \cap O_j = \emptyset \tag{2.5}$$

for each $i \neq j$. This is the so-called *open set condition* (page 118 in [5]). From (2.5) it follows that the Hausdorff dimension of K is the unique solution s to the equation

$$\sum_{i=1}^N \lambda_i^s = 1. \tag{2.6}$$

See, for example, Theorem 9.3 in [5].

Informally speaking, post-critically finite self-similar sets admit finite cut sets at all scales and locations. In the physics literature, these are often referred to as “finitely ramified” sets.

Definition 2.7. The *critical set* for the self-similar fractal K is $C(K) = \bigcup_{i \neq j} K_i \cap K_j$. The *post-critical set* $PC(K)$ is the backward orbit of $C(K)$, i.e.,

$$PC(K) = \bigcup_{m=0}^\infty \bigcup_{\sigma \in F^m} f_\sigma^{-1}(C(K)) \cap K.$$

We say that K is *post-critically finite* (*PCF*) if $PC(K)$ is a finite set.

Post-critically finite self-similar fractals were introduced by Kigami [9], [8] as a natural setting for a non-Euclidean theory of Dirichlet forms and harmonic analysis. For recent advances in this direction, see [2], [10] and [11]. The expository article of Strichartz [14] is an excellent introduction to this subject.

A typical example of a PCF self-similar fractal is the *Sierpinski gasket SG* (Figure 1) which is the invariant set for the collection of three planar contractive similarities

$$f_1(z) = \tfrac{1}{2}z + \tfrac{1}{2}, \quad f_2(z) = \tfrac{1}{2}z + \tfrac{1}{2}e^{2\pi i/3}, \quad f_3(z) = \tfrac{1}{2}z + \tfrac{1}{2}e^{-2\pi i/3}.$$

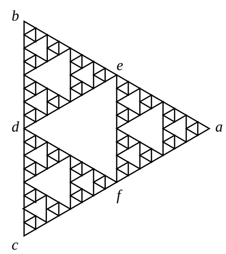

FIGURE 1. The Sierpinski gasket SG

The gasket SG satisfies the open set condition and hence has dimension

$$\frac{\log 3}{\log 2} = 1.5849\ldots$$

by (2.6). The critical and post-critical sets are $C(SG) = \{d,e,f\}$ and $PC(SG) = \{a,b,c,d,e,f\}$, respectively.

Lemma 2.8. *Let $K \subset \mathbb{R}^n$ be a PCF self-similar set. Then K has Hausdorff dimension strictly less than n and topological dimension at most one.*

Proof. Since K is PCF, $K \neq P$ and hence $\bigcup_{i=1}^N P_i \subsetneq P$. Thus

$$|P| > \sum_i |P_i| = \sum_i \lambda_i^n |P|$$

(where $|\cdot|$ denotes the Lebesgue measure) and so $\dim K < n$ by (2.6). On the other hand, every point in K has a countable neighborhood basis $K_{\sigma_1}, K_{\sigma_2}, \ldots$, where $\sigma_m \in F^m$, and each of these sets has boundary (in K) consisting of a finite (possibly zero) number of points by the PCF assumption. Thus K has topological dimension at most one. $\square$

We now introduce a further subclass of PCF self-similar sets (so-called *accessible* sets). In these sets, any two parts of the set can be joined by a "reasonably short" curve; for a precise version of this statement, see Proposition 2.10.

Definition 2.9. We say that a PCF self-similar set K is *accessible* if there exists a connected graph $\Gamma \subset P$ with vertex set $V = PC(K)$ such that the following three conditions are satisfied:

(i) $\bigcup_{i=1}^N \Gamma_i$ is connected;
(ii) $\Gamma \subset \bigcup_{i=1}^N \Gamma_i$;
(iii) for all $i \neq j$, $\Gamma_i \cap \Gamma_j = \partial P_i \cap \partial P_j$.

Here, as before, we denote by Γ_i the graph obtained by contracting Γ by the similarity map f_i.

The Sierpinski gasket SG is clearly accessible as we may choose Γ to be the graph on the vertex set $PC(SG)$ with edges $\{\overline{ae}, \overline{be}, \overline{bd}, \overline{cd}, \overline{cf}, \overline{af}\}$; see Figure 1. In this case Γ coincides with ∂P, where P is the closed unit triangle with vertices at a, b and c. In general, the graph Γ for a accessible set K need not agree with the 1-skeleton of P. A more complicated example of a accessible PCF self-similar set where Γ is not the 1-skeleton of P is the hexagasket, which we discuss in section 6.

We begin with some trivial observations about accessible PCF sets. First, by condition (ii) in Definition 2.9, $\Gamma \subset \bigcup_{\sigma \in F^m} \Gamma_\sigma$ for all m. Since $\Gamma_\sigma \subset P_\sigma$, Lemma 2.4 implies that $\Gamma \subset K$. Thus the topological dimension of K is equal to one and $1 \le \dim K < n$. We denote by

$$\mathcal{V} = \bigcup_{m=1}^{\infty} \bigcup_{\sigma \in F^m} V_\sigma$$

and

$$\mathcal{E} = \bigcup_{m=1}^{\infty} \bigcup_{\sigma \in F^m} \Gamma_\sigma$$

the *total vertex set* and *total edge set* of K. By (2.2), $\mathcal{V}$ is dense in $\mathcal{E}$, which in turn is dense in K.

A rectifiably connected set $A \subset \mathbb{R}^n$ is called *C-quasiconvex* ($C \ge 1$) if every two points $x, y \in A$ can be joined by a curve of length $\le C|x-y|$. The following proposition explains the remark preceding Definition 2.9.

Proposition 2.10. *For any accessible PCF self-similar fractal K, the total edge set $\mathcal{E}$ is C-quasiconvex for some constant $C < \infty$ depending only on the initial data $f_1, \dots, f_N$.*

Specifically, the quasiconvexity constant for $\mathcal{E}$ will depend on the following geometric quantities associated with the initial data. Note that computation of these values is a finite calculation if the initial set of transformations $f_1, \dots, f_N$ is specified.

First, we define the *minimal relative distance between nonadjacent subpolyhedra*:

$$\delta = \min \left\{ \frac{\operatorname{dist}(P_i, P_j)}{\operatorname{diam} P} : i, j = 1, \dots, N, P_i \cap P_j = \emptyset \right\}. \tag{2.11}$$

Next, we define the *minimal angle between adjacent subpolyhedra*:

$$\theta = \min \left\{ \begin{matrix} \text{angle between } \partial P_i \cap H \\ \text{and } \partial P_j \cap H \end{matrix} : \begin{matrix} i, j = 1, \dots, N, \ P_i \cap P_j \ne \emptyset, \\ H \text{ a half-plane containing } \partial P_i \cap P_j \end{matrix} \right\}. \tag{2.12}$$

Finally, we let C_0 be a specific (finite) coefficient of quasiconvexity for the initial edge set $\mathcal{E}_0 = \Gamma$. (Note that such a coefficient exists because Γ is a finite polygonal graph.)

The following lemma is a simple application of the Law of Cosines.

Lemma 2.13. *Let ABC be a triangle with sides of length a, b and c and angle ϕ opposite to the side of length c. Then $a + b \le \csc(\phi/2) c$.*

Proof of Proposition 2.10. We will show that $\mathcal{E}$ is C-quasiconvex with

$$C = \frac{1 + \lambda_{\max}}{1 - \lambda_{\max}} \cdot \frac{1}{\delta} \cdot \csc(\theta/2) \cdot C_0. \tag{2.14}$$

For each m, let $\mathcal{E}_m = \bigcup_{\sigma \in F^m} \Gamma_\sigma$. We will prove the proposition by showing that for each m, $\mathcal{E}_m$ is C_m-quasiconvex with

$$C_m = \max \left\{ 1 + 2 \sum_{l=1}^{m} \lambda_{\max}^l, \left(1 + 2 \sum_{l=1}^{m-1} \lambda_{\max}^l \right) \csc(\theta/2) \right\} \cdot \frac{1}{\delta} \cdot C_0. \tag{2.15}$$

Since $\mathcal{E} = \cup_m \mathcal{E}_m$, the result follows.

The proof is by induction. By assumption $\mathcal{E}_0$ is C_0-quasiconvex. Assume that $\mathcal{E}_{m-1}$ is C_{m-1}-quasiconvex and let $x, y \in \mathcal{E}_m$. If x and y are contained in a common subpolyhedron P_i then (since P_i is similar to P) x and y can be joined by a curve in Γ_i of length $\le C_{m-1}|x-y| \le C_m|x-y|$ by the induction hypothesis.

We may thus assume that $x \in \Gamma_\sigma$ and $y \in \Gamma_\tau$ with $\sigma = (i_1, \dots, i_m)$ and $\tau = (j_1, \dots, j_m)$ in F^m and $i_1 \ne j_1$. There are two cases: (i) $P_{i_1} \cap P_{j_1} = \emptyset$ and (ii) $P_{i_1} \cap P_{j_1} \ne \emptyset$.

In case (i), choose two sequences of points

$$\begin{array}{ll} x_1 \in V_{i_1} \subset \Gamma_{i_1} & y_1 \in V_{j_1} \subset \Gamma_{j_1} \\ x_2 \in V_{(i_1,i_2)} \subset \Gamma_{(i_1,i_2)} & y_2 \in V_{(j_1,j_2)} \subset \Gamma_{(j_1,j_2)} \\ \vdots & \vdots \\ x_m \in V_{(i_1,\dots,i_m)} = V_\sigma \subset \Gamma_\sigma & y_m \in V_{(j_1,\dots,j_m)} = V_\tau \subset \Gamma_\tau \\ x_{m+1} = x \in \Gamma_\sigma & y_{m+1} = y \in \Gamma_\tau, \end{array}$$

where $V_\sigma = f_\sigma(V)$. As before, since each subpolyhedron (at any size) is similar to P, we find that x_l can be joined to x_{l+1} by a curve in $\Gamma_{(i_1,\dots,i_l)}$ of length $\le C_0|x_l - x_{l+1}|$. A similar statement holds for the points y_l. Finally, x_1 can be joined to y_1 by a curve of length $\le C_0|x_1 - y_1|$. Let γ be the curve from x to y formed by the concatenation of all of these curves. Then the length of γ is bounded by

$$\begin{aligned} & C_0|x_1 - y_1| + C_0 \sum_{l=1}^m |x_l - x_{l+1}| + C_0 \sum_{l=1}^m |y_l - y_{l+1}| \\ \le & C_0 \left(\operatorname{diam} P + \sum_{l=1}^m \operatorname{diam} P_{(i_1,\dots,i_l)} + \sum_{l=1}^m \operatorname{diam} P_{(j_1,\dots,j_l)} \right) \\ \le & C_0 \left(1 + 2 \sum_{l=1}^m \lambda_{\max}^l \right) \operatorname{diam} P \\ \le & C_0 \left(1 + 2 \sum_{l=1}^m \lambda_{\max}^l \right) \frac{\operatorname{dist}(P_i, P_j)}{\delta} \le C_m |x-y| \end{aligned}$$

which completes the proof in this case.

In case (ii), choose $z \in \Gamma_i \cap \Gamma_j = \partial P_i \cap \partial P_j$. By the induction hypothesis, we can choose curves from x to z in P_i and from z to y in P_j with lengths $\le C_{m-1}|x-z|$ and $\le C_{m-1}|z-y|$ respectively. Again, the length of the concatenation of these two curves is at most

$$C_{m-1}(|x-z| + |z-y|) \le C_{m-1} \csc(\theta/2)|x-y| \le C_m|x-y|$$

by Lemma 2.13. In all cases, we see that the desired conclusion is verified. Thus the proof of Proposition 2.10 is complete. □

3. An example: the Sierpinski gasket

To aid the reader in understanding the details of our construction, we will begin by briefly summarizing the results of the following two sections in the special case of the Sierpinski gasket SG. We introduce a class of new metrics on the gasket which are in the quasisymmetry class of the standard metric. We then show that SG is not minimal for conformal dimension by estimating the Hausdorff dimension of the gasket in these new metrics.

We denote by T the closed triangle with vertices 1, $e^{2\pi i/3}$ and $e^{-2\pi i/3}$; this corresponds to the polyhedron P of the previous section. Observe that SG is 2-quasiconvex and has topological dimension equal to one.

Let Σ denote the collection of triplets $(\sigma_0, \sigma_1, \sigma_2) \in \{1,2,3\}^3$ for which T_σ is disjoint from ∂T. Explicitly,

$$\Sigma = \{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}.$$

See Figure 2. Fixing $\epsilon > 0$, we define a function $d = d_\epsilon : SG \times SG \to [0,\infty)$ in several steps. First, suppose that x and y are the endpoints of an edge in $\mathcal{E}_{3m}$ for some $m \in \mathbb{N}$. In this case we set

$$d(x,y) = \epsilon^{\mu(\sigma)}|x-y|, \tag{3.1}$$

where $\mu(\sigma) \in \{0,1,\dots,m\}$ is equal to the number of triplets $(\sigma_0,\sigma_1,\sigma_2)$, $(\sigma_3,\sigma_4,\sigma_5)$, $\dots$, $(\sigma_{3m-3},\sigma_{3m-2},\sigma_{3m-1})$ which lie in Σ. We then extend d to $\mathcal{V} \times \mathcal{V}$ as a path metric: for $x,y \in \mathcal{V}$ we set

$$d(x,y) = \inf \left\{ d-\mathrm{length}(\gamma) : \begin{array}{c} \gamma \subset \mathcal{E} \text{ is a finite} \\ \text{polygonal arc joining } x \text{ to } y \end{array} \right\} \tag{3.2}$$

where the d-length of a finite polygonal arc γ connecting vertices $x = x_0, x_1, \dots, x_{r-1}, x_r = y$ is just $\sum_{\nu=1}^r d(x_{\nu-1}, x_\nu)$.

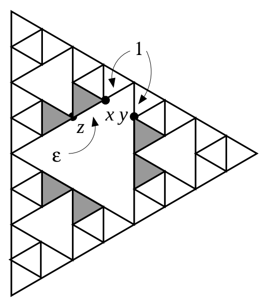

FIGURE 2. The triangles T_σ and an extremal configuration for weak quasiconformality of $\mathrm{id} : \mathcal{V}_3 \to (\mathcal{V}_3, d)$

We claim that d is a metric on $\mathcal{V}$. The proof of this relies on two facts: first, that the value of $d(x,y)$ (as computed in (3.2)) is unchanged from the value given in (3.1) in the case when x and y are the endpoints of a single edge, and second, that the distance between distinct points in $\mathcal{V}$ is positive. See Proposition 4.5 and Corollary 4.6. The identity map on $\mathcal{V}$ from the Euclidean metric to the new metric d is 2-Lipschitz. We can thus extend d to $SG \times SG$ by density: for $x,y \in SG$ we set

$$d(x,y) = \lim_{n\to\infty} d(x_n, y_n) \tag{3.3}$$

for any sequences (x_n) and (y_n) in $\mathcal{V}$ with $x_n \to x$ and $y_n \to y$. Then d continues to be a metric on SG and the identity map from SG to (SG,d) is a 2-Lipschitz homeomorphism. We claim that this map is also quasisymmetric. In the interest of brevity, we give here only a short plausibility argument for this fact. By [18, Theorem 6.6] it suffices to verify that the map is *weakly quasisymmetric*:

$$|x-y| \le |x-z| \qquad \Rightarrow \qquad d(x,y) \le H d(x,z) \tag{3.4}$$

for all $x,y,z \in \mathcal{V}$ and some (absolute) constant $H < \infty$. If $x,y,z \in \mathcal{V}_3$ then (3.4) holds with $H = 2/\epsilon$ (see Figure 2); in general, (3.4) holds for all triples $x,y,z \in \mathcal{V}$ with a slightly larger choice of $H = H(\epsilon) < \infty$.

See Theorem 5.1 for a detailed proof of the quasisymmetry of this map.

To estimate the Hausdorff dimension of the new metric space (SG, d), we first show that

$$\epsilon^{\mu(\sigma)}(\frac{1}{8})^m \le d - \operatorname{diam} T_\sigma \le 2\epsilon^{\mu(\sigma)}(\frac{1}{8})^m \tag{3.5}$$

for every triangle T_σ with $\sigma \in \{1,2,3\}^{3m}$. See Proposition 4.10. By using the coverings of SG with the collection of these triangles for each $m \in \mathbb{N}$ and estimating the d-diameters of these triangles using (3.5), we find that $\dim(SG, d) \le s_0$, where $s_0 = s_0(\epsilon)$ is the unique solution to the equation

$$21 + 6\epsilon^{s_0} = 8^{s_0}.$$

Taking the limit as $\epsilon \to 0$, we find that

$$\mathcal{C} \dim SG \le \frac{\log 21}{\log 8} < \frac{\log 3}{\log 2} = \dim SG. \tag{3.6}$$

In order to sharpen this estimate for $\mathcal{C} \dim SG$ we can repeat the above construction, replacing $\{1,2,3\}^3$ with $\{1,2,3\}^k$ for larger values of k and using a new set of weights on the resulting 3^k triangles. In section 6 we will carry out this procedure to deduce that

$$\mathcal{C} \dim SG \le 1.4160\ldots$$

It is unclear precisely how far the dimension of the gasket may be reduced by using this construction, or whether one can achieve the conformal dimension by constructions of this type. Indeed, we still do not know the exact value of the conformal dimension of the gasket.[2]

4. Constructing new metrics on an accessible PCF fractal

Following the method used in the preceding section, we now introduce a class of new metrics on a general accessible PCF self-similar fractal K. In the following section, we will show that these new metrics are quasisymmetrically equivalent to the standard metric and we will estimate the Hausdorff dimension of K in these new metrics.

Let $K \subset \mathbb{R}^n$ be an accessible PCF self-similar set. Fix an integer $k \ge 1$. We consider weight functions $\boldsymbol{\rho}$ on F^k taking values in the interval $(0,1]$. (Recall that $F = \{1, \ldots, N\}$, where N denotes the number of contractions used in the construction of K.) We denote the value of $\boldsymbol{\rho}$ on $\sigma \in F^k$ by ρ_σ. For m-tuples $\sigma = (\sigma_0, \ldots, \sigma_{m-1}) \in F^{mk}$, $m \in \mathbb{N}$, we write $\rho_\sigma = \rho_{\sigma_{m-1}} \cdot \cdots \cdot \rho_{\sigma_0}$. As in (3.1), we define $d(x,y)$ for vertices x, y which are the endpoints of an edge E in $\mathcal{E}_{km}$ by the formula

$$d(x,y) = \rho_\sigma |x-y|, \qquad E \subset P_\sigma, \sigma \in F^{km}. \tag{4.1}$$

Observe that E is contained in a *unique* polyhedron P_σ by virtue of the PCF assumption.

Definition 4.2. We say that $\boldsymbol{\rho} : F^k \to (0,1]$ is *admissible* if the following two conditions hold:

(i) $\rho_\sigma = 1$ if $\sigma \in F^k$ is such that $P_\sigma \cap \mathcal{E}_0 \ne \emptyset$;

(ii) for all pairs of vertices $x, y \in \mathcal{V}_0$ which are the two endpoints of a single edge in $\mathcal{E}_0$ and all finite polygonal arcs γ in $\mathcal{E}_k$ joining x to y,

$$d - \operatorname{length}(\gamma) \ge d(x,y) = |x-y|. \tag{4.3}$$

(Observe that the right hand equality in (4.3) follows from part (i).)

Here as before by a *finite polygonal arc* we mean a topological arc formed by a finite number of edges in $\mathcal{E}$ and by the *d-length* of a finite polygonal arc γ passing through vertices $x = x_0, x_1, \ldots, x_r = y$ we mean the sum $\sum_{\nu=1}^r d(x_{\nu-1}, x_\nu)$.

[2] Tomi Laakso (personal communication) has indicated a variation on this construction which should show that the conformal dimension of the Sierpinski gasket is one.

Essentially, $\boldsymbol{\rho}$ is admissible if it does not distort the sizes of polyhedra which meet the initial edge set and does not reduce the distances between adjacent vertices in the initial vertex set.

The function $\boldsymbol{\rho}$ on the Sierpinski gasket used in section 3, i.e., $\rho_\sigma = 1$ if $\sigma \notin \Sigma(k)$ and $\rho_\sigma = \epsilon$ if $\sigma \in \Sigma(k)$, is admissible. Consider Figure 3.

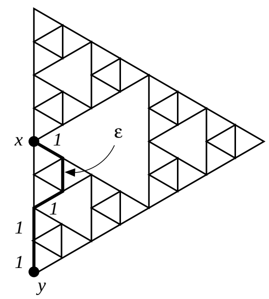

FIGURE 3. Sample curve joining two elements of $\mathcal{V}_0(SG)$ in $\mathcal{E}_3(SG)$

Conditions (i) and (ii) in Definition 4.2 comprise a finite combinatorial assumption which is in principle decidable for a given weight function $\boldsymbol{\rho}$. Indeed, there are only finitely many vertices in $\mathcal{V}_k$ and any two such vertices can be connected by at most finitely many polygonal arcs in $\mathcal{E}_k$. We are at present unable to give a more explicit description of the admissible weights, or to exhibit a sufficiently large collection of examples of these weights for general fractals (see remark 2 in section 7).

For a given $k \in \mathbb{N}$ and admissible $\boldsymbol{\rho} : F^k \to (0, 1]$, we now define $d = d_{k,\rho}$ on $\mathcal{V} \times \mathcal{V}$ by

$$d(x, y) = \inf \left\{ d - \text{length}(\gamma) : \begin{array}{c} \gamma \subset \mathcal{E} \text{ is a finite} \\ \text{polygonal arc joining } x \text{ to } y \end{array} \right\}. \tag{4.4}$$

For the remainder of this section, we assume that $\boldsymbol{\rho}$ is an admissible weight and we investigate properties of the distance function d on the total edge set $\mathcal{E}$ of K. We prove that d is a metric and we establish some elementary results regarding distances between points and the sizes of the subpolyhedra P_σ in this new metric.

Proposition 4.5. *Any two points in $\mathcal{V}$ can be joined by a d-geodesic in $\mathcal{E}$. More specifically, if $x, y \in \mathcal{V}_{km}$ for some $m \in \mathbb{N}$, then there exists a finite polygonal arc $\gamma_0 \subset \mathcal{E}_{k(m+1)}$ joining x to y such that $d(x, y) = d - \text{length}(\gamma_0)$. If x and y are the vertices associated with a single edge E in $\mathcal{E}_{km}$, then the value of $d(x, y)$ as computed in (4.4) is still equal to $\rho_\sigma|x - y|$, where P_σ is the unique polyhedron containing E as one of its sides.*

Corollary 4.6. *$d(x, y) > 0$ if $x, y \in \mathcal{V}$ and $x \neq y$. Thus d is a metric on $\mathcal{V}$.*

Proof of Corollary 4.6. Assume that $x, y \in \mathcal{V}_{km}$. By Proposition 4.5,

$$d(x, y) = d - \text{length}(\gamma_0) \geq \rho_{\min}^{m+1} \text{length}(\gamma_0) \geq \rho_{\min}^{m+1}|x - y| > 0$$

where $\rho_{\min} = \min\{\rho_\sigma : \sigma \in F^k\} > 0$ and γ_0 denotes a d-geodesic joining x to y. Since d as defined in (4.4) is clearly symmetric and satisfies the triangle inequality, it is a metric on $\mathcal{V}$. □

Proof of Proposition 4.5. Let $x, y \in \mathcal{V}_{km}$. Since there are only finitely many polygonal arcs joining x to y in $\mathcal{E}_{k(m+1)}$ (and at least one since Γ is connected), there must exist at least one such are γ_0 of minimal d-length. We will show that any other finite polygonal arc joining x to y in $\mathcal{E}$ must have d-length no less than $d - \text{length}(\gamma_0)$.

Let γ be such an arc. Since γ consists of finitely many segments, $\gamma \subset \mathcal{E}_{km'}$ for some $m' \geq m$. The proof proceeds by replacing γ with a new finite polygonal arc $\gamma_1 \subset \mathcal{E}_{k(m'-1)}$ whose d-length is less than or equal to that of γ. Repeating this process, we produce a sequence of arcs

$$\begin{gathered} \gamma \subset \mathcal{E}_{km'} \\ \gamma_1 \subset \mathcal{E}_{k(m'-1)} \\ \vdots \\ \gamma_{m'-m-1} \subset \mathcal{E}_{k(m+1)}, \end{gathered}$$

all joining x to y, such that the d-length is not increased at any stage. Then

$$d - \text{length}(\gamma) \geq d - \text{length}(\gamma_{m'-m+1}) \geq d - \text{length}(\gamma_0)$$

and the proof is complete.

We now indicate how to carry out the typical step in this replacement procedure. Let γ_i be the curve in $\mathcal{E}_{k(m'-i)}$. By the PCF assumption, there exists a sequence of points $x = x_0, x_1, \ldots, x_r = y$ on γ_i with $x_\nu \in \mathcal{V}_{k(m'-i-1)}$ so that $x_{\nu-1}$ and x_ν are the endpoints of an edge in $\mathcal{E}_{k(m'-i-1)}$. Then for each ν, we can replace the portion of γ_i between $x_{\nu-1}$ and x_ν by the corresponding straight-line edge and not increase the d-length (see assumption (ii) in Definition 4.2). Concatenating these straight-line segments yields a new curve γ_{i+1} with $d - \text{length}(\gamma_{i+1}) \leq d - \text{length}(\gamma_i)$ as desired.

To complete the proof, we observe that if x and y are the endpoints of a single edge, then γ_0 must be equal to that edge (again by 4.2(ii)). □

Proposition 4.7. (a) *The identity map from $\mathcal{V}$ to $(\mathcal{V}, d)$ is Lipschitz.*
(b) *The metric $d = d_{k,\rho}$ extends to a metric on K.*
(c) *The identity map from K to (K, d) is a Lipschitz homeomorphism.*

Proof. For x and y in $\mathcal{V}$, the inequality

$$d(x,y) \leq C_{qcvx}|x-y| \tag{4.8}$$

follows from (4.4), where C_{qcvx} denotes the quasiconvexity constant of K determined in Proposition 2.10. By appealing to Proposition 4.7, we may extend d to a metric on all of K by density:

$$d(x,y) = \lim_{n\to\infty} d(x_n, y_n), \qquad x, y \in K, \tag{4.9}$$

for any sequences (x_n) and (y_n) in $\mathcal{V}$ with $x_n \to x$ and $y_n \to y$. The identity map from K to (K, d) is also Lipschitz since (4.8) extends to all points $x, y \in K$. Finally, this map is a homeomorphism since K is compact. □

Proposition 4.10. *Let K be a accessible PCF self-similar fractal. Then*
(a) *there exist constants $0 < c_1 \leq c_2 < \infty$ so that*

$$c_1\rho_\sigma|x-y| \leq d(x,y) \leq c_2\rho_\sigma|x-y| \tag{4.11}$$

whenever x and y are points in K and P_σ is the smallest polyhedron (with $\sigma \in F^{km}$ for some m) containing x and y;
(b) *there exist constants $0 < c_3 \leq c_4 < \infty$ so that*

$$c_3\rho_\sigma \operatorname{diam} P_\sigma \leq d - \operatorname{diam} P_\sigma \leq c_4\rho_\sigma \operatorname{diam} P_\sigma \tag{4.12}$$

for all σ.

Here and henceforth, we write diam for the diameter in the Euclidean metric and $d - \text{diam}$ for the diameter in the new metric.

Proof. (a) It suffices to verify (4.11) for $x, y \in \mathcal{V}$. Fix $x, y \in \mathcal{V}$ and let P_σ, $\sigma \in F^{km}$, be the smallest polyhedron containing both x and y. Then x can be joined to y by a curve γ of length at most $C_{qcvx}|x-y|$ contained entirely in P_σ; see Proposition 2.10. By (4.4) and the fact that $\rho_\sigma \le 1$ for all σ, it follows that

$$d(x,y) \le d-\mathrm{length}(\gamma) \le \rho_\sigma \,\mathrm{length}(\gamma) \le C_{qcvx}\, \rho_\sigma |x-y|.$$

Thus the right hand inequality holds with $c_2 = C_{qcvx}$.

For the left hand inequality, we let P_α and P_β be subpolyhedra of P_σ with $\alpha, \beta \in F^{k(m+1)}$ and $x \in P_\alpha \setminus P_\beta$ and $y \in P_\beta \setminus P_\alpha$. Let γ_0 be a d-geodesic joining x to y.

If P_α and P_β are disjoint, then γ_0 travels through at least one other P_ω, $\omega \in F^{k(m+1)}$, which is a child of P_σ. Denote by $\min.\mathrm{edge}(P_\omega)$ the minimal length of an edge in Γ_ω and let

$$H_P = \frac{\mathrm{diam}\, P}{\min.\mathrm{edge}\, P}. \tag{4.13}$$

Then

$$\begin{aligned} d(x,y) &\ge \rho_\omega \min.\mathrm{edge}(P_\omega) \ge \frac{\rho_{\min}}{H_P}\rho_\sigma\, \mathrm{diam}\, P_\omega \\ &\ge \frac{\rho_{\min}\lambda_{\min}}{H_P}\rho_\sigma\, \mathrm{diam}\, P_\sigma \ge \frac{\rho_{\min}\lambda_{\min}}{H_P}\rho_\sigma |x-y|. \end{aligned}$$

On the other hand, if P_α and P_β are not disjoint, then $P_\alpha \cap P_\beta$ consists of a single point w (by the PCF assumption). Choose further subpolyhedra $P_{\alpha'}, P_{\beta'}$ of maximal size so that $w \in P_{\alpha'}$, $x \notin P_{\alpha'}$, $w \in P_{\beta'}$, and $y \notin P_{\beta'}$. In this case $\rho_{\alpha'} = \rho_\alpha$ and $\rho_{\beta'} = \rho_\beta$ (by assumption (i) in Definition 4.2) and $\mathrm{diam}\, P_{\alpha'} + \mathrm{diam}\, P_{\beta'} \ge \lambda_{\min}|x-y|$ (by the maximality of $P_{\alpha'}$ and $P_{\beta'}$). Then

$$\begin{aligned} d(x,y) &\ge \rho_{\alpha'} \min.\mathrm{edge}(P_{\alpha'}) + \rho_{\beta'} \min.\mathrm{edge}(P_{\beta'}) \\ &\ge \frac{1}{H_P}(\rho_\alpha\, \mathrm{diam}\, P_{\alpha'} + \rho_\beta\, \mathrm{diam}\, P_{\beta'}) \\ &\ge \frac{\rho_{\min}}{H_P}\rho_\sigma(\mathrm{diam}\, P_{\alpha'} + \mathrm{diam}\, P_{\beta'}) \ge \frac{\rho_{\min}\lambda_{\min}}{H_P}\rho_\sigma |x-y|. \end{aligned}$$

In every case, we see that the left hand inequality holds with $c_1 = \rho_{\min}\lambda_{\min}/H_P$.

(b) The right hand inequality in (4.12) follows from the corresponding inequality in (4.11) with $c_4 = c_2 = C_{qcvx}$. To see why the left hand inequality holds true, we let x, y be two vertices in $\mathcal{V}_\sigma = f_\sigma(\mathcal{V}_0)$ which are joined by an edge in $\mathcal{E}$. By Proposition 4.5,

$$d-\mathrm{diam}\, P_\sigma \ge d(x,y) = \rho_\sigma |x-y| \ge \frac{1}{H_P}\rho_\sigma\, \mathrm{diam}\, P_\sigma,$$

where H_P is as in (4.13). Thus the left hand inequality in (4.11) holds with $c_3 = H_P^{-1}$. □

5. Estimating the conformal dimension

The new metrics constructed in the previous section allow us to estimate from above the conformal dimensions of certain accessible PCF self-similar sets. In this section, we will prove the following two theorems.

Theorem 5.1. *Let $K \subset \mathbb{R}^n$ be an accessible PCF self-similar set generated by contraction mappings $f_i : \mathbb{R}^n \to \mathbb{R}^n$. Assume that all of the contraction factors are equal: $\lambda_i \equiv \lambda$. Then $d_{k,\rho}$ is in the quasisymmetry class of the standard metric on K for each $k \in \mathbb{N}$ and each admissible $\boldsymbol{\rho} : F^k \to (0,1]$.*

Theorem 5.2. *Let K be any accessible PCF self-similar set. For k and $\boldsymbol{\rho}$ as in Theorem 5.1, $\dim(K, d_{k,\rho}) \le s_0$, where $s_0 = s_0(k, \boldsymbol{\rho})$ is the unique positive solution to the equation*

$$\sum_{\sigma \in F^k} \rho_\sigma^{s_0} \lambda_\sigma^{s_0} = 1. \tag{5.3}$$

Observe that under the assumptions of Theorem 5.1, equation (5.3) reduces to

$$\sum_{\sigma \in F^k} \rho_\sigma^{s_0} = \left(\frac{1}{\lambda}\right)^{ks_0}. \tag{5.4}$$

Note that in the undeformed case $\rho_\sigma \equiv 1$, (5.4) becomes $N^k = (1/\lambda)^{ks_0}$ and $s_0 = \log N / \log(1/\lambda) = \dim K$.

By combining Theorems 5.1 and 5.2, we derive an upper bound for the conformal dimension of K.

Corollary 5.5. *Under the assumptions of Theorem 5.1,*

$$\mathcal{C} \dim K \le \inf_k \inf_{\boldsymbol{\rho}} s_0(k, \boldsymbol{\rho}),$$

where the second infimum is taken over all admissible weights $\boldsymbol{\rho} : F^k \to (0, 1]$.

Proof of Theorem 5.2. Let $s > s_0(k, \boldsymbol{\rho})$. Then $\sum_{\sigma \in F^k} \rho_\sigma^{s_0} \lambda_\sigma^{s_0} < 1$. For each $m \in \mathbb{N}$, K can be covered by the N^{km} sets P_σ, $\sigma \in F^{km}$. By Proposition 4.10(b), $d - \operatorname{diam} P_\sigma \le c_4 \rho_\sigma \operatorname{diam} P_\sigma$. Denoting by $\mathcal{H}_s^\delta$ the s-dimensional Hausdorff premeasure at scale $\delta > 0$, we have

$$\begin{aligned} \mathcal{H}_s^{c_4 \lambda_{\max}^{km} \operatorname{diam} P}(K, d) &\le \sum_{\sigma \in F^{km}} (d - \operatorname{diam} P_\sigma)^s \\ &\le c_4^s \sum_{\sigma \in F^{km}} \rho_\sigma^s \lambda_\sigma^s \\ &= c_4^s \left(\sum_{\sigma \in F^k} \rho_\sigma^s \lambda_\sigma^s \right)^m \end{aligned}$$

which tends to zero as $m \to \infty$. The conclusion follows. □

Proof of Theorem 5.1. Recall the statement we seek to prove: there exists an increasing homeomorphism $\eta = \eta_{d,\rho} : (0, \infty) \to (0, \infty)$ such that

$$\frac{d(x, y)}{d(x, z)} \le \eta \left(\frac{|x - y|}{|x - z|} \right) \tag{5.6}$$

for all $x, y, z \in K$, $x \ne y$, $x \ne z$, where $d = d_{k,\rho}$. Recall also that we require $\lambda_i \equiv \lambda$ in the hypotheses.

We begin with a few obvious simplifications. By a preliminary scaling, we may assume that $\operatorname{diam} P = 1$. By [15, Theorem 2.25], it suffices to verify (5.6) for $x, y, z \in \mathcal{V}$. Finally, the self-similarity of K implies that it is enough to verify (5.6) in the case when the smallest polyhedron containing all three points x, y, and z is the initial polyhedron P.

Denote by P_σ the smallest polyhedron containing x and y and by P_τ the smallest polyhedron containing x and z. By Proposition 4.10(a), $d(x, y) \approx \rho_\sigma |x - y|$ and $d(x, z) \approx \rho_\tau |x - z|$, where the notation $A \approx B$ means $A/C \le B \le CA$ for some constant C. We consider three cases:

(i) $P_\sigma \subset P_\tau = P$;
(ii) $P_\tau \subset P_\sigma = P$;

(iii) $P_\sigma \not\subset P_\tau = P$ and $P_\tau \not\subset P_\sigma$.

In case (i), $\rho_\tau = 1$ and

$$\frac{d(x,y)}{d(x,z)} \approx \frac{\rho_\sigma |x-y|}{|x-z|} \le \frac{|x-y|}{|x-z|}.$$

In case (iii) we must have $P_\sigma \cap P_\tau = \{x\}$. Thus P_σ and P_τ are adjacent polyhedra (possibly at different levels) and

$$\rho_\sigma \le \frac{1}{\rho_{\min}} \rho_\tau$$

by the admissibility condition 4.2(i) for the weight $\boldsymbol{\rho}$. In this case

$$\frac{d(x,y)}{d(x,z)} \approx \frac{\rho_\sigma |x-y|}{\rho_\tau |x-z|} \le \frac{1}{\rho_{\min}} \cdot \frac{|x-y|}{|x-z|}.$$

It remains to consider case (ii). Here $\rho_\sigma = 1$ and so

$$\frac{d(x,y)}{d(x,z)} \approx \frac{1}{\rho_\tau} \frac{|x-y|}{|x-z|}. \tag{5.7}$$

Assume that $\tau \in F^{km}$, $m \in \mathbb{N}$. We divide this case into two further subcases, according to whether the distance between x and y is comparable to the diameter of P or relatively small compared to $\operatorname{diam} P$.

Suppose first that $|x-y| \ge \lambda / H_P$ (recall that we assume $\operatorname{diam} P = 1$). Since

$$|x-z| \le \operatorname{diam} P_\tau \le \lambda^{km},$$

we see that

$$\frac{|x-y|}{|x-z|} \ge \frac{\lambda}{H_P} \left(\frac{1}{\lambda}\right)^{km}.$$

Set

$$q = \frac{\log(1/\rho_{\min})}{k \log(1/\lambda)}. \tag{5.8}$$

Then

$$\begin{aligned} \frac{d(x,y)}{d(x,z)} &\approx \frac{1}{\rho_\tau} \frac{|x-y|}{|x-z|} \\ &\le \left(\frac{1}{\rho_{\min}}\right)^m \frac{|x-y|}{|x-z|} \\ &= \left(\frac{1}{\lambda}\right)^{kmq} \frac{|x-y|}{|x-z|} \\ &\le \left(\frac{H_P}{\lambda}\right)^q \left(\frac{|x-y|}{|x-z|}\right)^{1+q}. \end{aligned}$$

We are thus reduced to the study of the case when x and y are relatively close: $|x-y| \le \lambda/H_P$. It follows from this that x and y are contained in adjacent subpolyhedra P_α and P_β with $\alpha, \beta \in F^k$, i.e., $\{w\} = P_\alpha \cap P_\beta \ne \emptyset$. Write

$$\tau = (\alpha, \tau_1, \ldots, \tau_{m-1}) \in F^{km},$$

where $\tau_i \in F^k$ for each i. Define an integer $p = p(w,x) \in \{0,1,\dots,m-1\}$ to be the largest index for which $P_{(\alpha,\tau_1,\dots,\tau_p)}$ contains w. Then

$$|x-w| \geq \frac{\lambda^{k(p+2)}}{H_P} \tag{5.9}$$

since $P_{(\alpha,\tau_1,\dots,\tau_p,\tau_{p+1})} \not\ni w$.

Observe that $\rho_{\tau_i} = 1$ for each $i = 1,2,\dots,p$. Thus

$$\rho_\tau = \rho_{\tilde\tau}$$

where $\tilde\tau = (\alpha,\tau_{p+1},\dots,\tau_{m-1}) \in F^{k(m-p)}$. Returning to (5.7), we estimate

$$\begin{aligned} \frac{d(x,y)}{d(x,z)} &\approx \frac{1}{\rho_\tau}\,\frac{|x-y|}{|x-z|} = \frac{1}{\rho_{\tilde\tau}}\,\frac{|x-y|}{|x-z|} \\ &\leq \left(\frac{1}{\rho_{\min}}\right)^{m-p} \frac{|x-y|}{|x-z|} \\ &= \left(\frac{1}{\lambda}\right)^{k(m-p)q} \frac{|x-y|}{|x-z|}. \end{aligned} \tag{5.10}$$

We next rewrite (5.9) in the form

$$\lambda^{pk} \leq \frac{H_P|x-w|}{\lambda^{2k}}.$$

Using this together with the trivial estimate $|x-z| \leq \lambda^{km}$, we deduce that

$$\left(\frac{1}{\lambda}\right)^{k(m-p)} \leq \frac{H_P|x-w|}{\lambda^{2k}|x-z|}. \tag{5.11}$$

Combining (5.10) and (5.11), we find that

$$\frac{d(x,y)}{d(x,z)} \leq C\frac{|x-w|^q|x-y|}{|x-z|^{1+q}} \leq C\left(\frac{|x-y|}{|x-z|}\right)^{1+q},$$

where the inequality $|x-w| \leq C'|x-y|$ with $C' = \csc(\theta/2)$ (θ as in (2.12)) follows from Lemma 2.13.

Thus in every case we see that (5.6) holds with $\eta(t) = C\max\{t,t^{1+q}\}$, where q is defined in 5.8. This completes the proof of the quasisymmetric equivalence of d with the Euclidean metric on K. □

Remark 5.12. The reason why we must impose the additional assumption that all of the λ_i's are equal is clear from the last case in the proof. If we consider the situation for general values of λ_i, we find that (5.9) must be replaced by

$$|x-w| \geq \frac{\lambda_{\min}^{k(p+2)}}{H_P}.$$

However, the only upper bound for $|x-z|$ which we can give in general is the trivial estimate

$$|x-z| \leq \lambda_{\max}^{km}$$

and it is clear that in this case we cannot uniquely define q as in (5.8) so that the distortion estimate

$$\frac{d(x,y)}{d(x,z)} \leq C\left(\frac{|x-y|}{|x-z|}\right)^{1+q}$$

holds.

6. Examples

1. Further estimates for the conformal dimension of SG. We can apply Corollary 5.5 to other admissible metrics on the Sierpinski gasket to improve the upper bound (3.6) for the conformal dimension. For example, when $k = 4$ we can choose the admissible metric $\boldsymbol{\rho} = \boldsymbol{\rho}_{4,\epsilon} : \{1,2,3\}^4 \to (0,1]$ indicated in Figure 4. Here $\epsilon > 0$ can be taken to be any positive value. Among the 81 total subtriangles T_σ, $\sigma \in \{1,2,3\}^4$, there are 45 triangles with $\rho_\sigma = 1$, 24 triangles with $\rho_\sigma = \frac{1}{2}$, and 12 triangles with $\rho_\sigma = \epsilon$. According to Corollary 5.5,

$$\mathcal{C}\dim SG \leq s_0(4,\epsilon),$$

where $s_0 = s_0(4,\epsilon)$ is the unique positive solution to the equation

$$45 + 24 \cdot 2^{-s_0} + 12\epsilon^{s_0} = 16^{s_0}.$$

Letting $\epsilon \to 0$,

$$\mathcal{C}\dim SG \leq s_0 = 1.43778\ldots,$$

where now s_0 is the unique positive solution to the equation

$$45 + 24 \cdot 2^{-s_0} = 16^{s_0}.$$

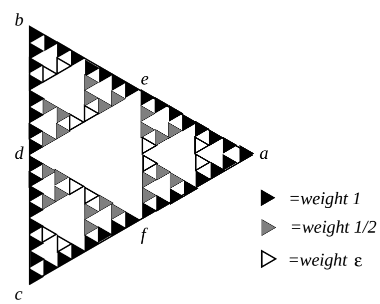

Figure 4. Values of $\boldsymbol{\rho}_{4,\epsilon}$

It is not clear how far this construction can be extended, i.e., how small the upper bound for $\mathcal{C}\dim SG$ can be made by this method. We have found a collection of admissible weight functions $\boldsymbol{\rho}_{6,\epsilon}$ on $\{1,2,3\}^6$. The following table gives the number of triangles T_σ, $\sigma \in \{1,2,3\}^6$, which recieve each of the values taken on by $\boldsymbol{\rho}_{6,\epsilon}$. (In the interest of simplicity, we omit the figure showing where these triangles are located. The reader is invited to reconstruct the missing figure using the table as a hint.)

Value of $\boldsymbol{\rho}_{6,\epsilon}$	1	1/2	1/3	1/5	1/6	1/15	1/30	ϵ
Number of triangles	333	24	36	84	24	36	24	168

As before, using Corollary 5.5 and letting $\epsilon \to 0$ we obtain the bound

$$\mathcal{C}\dim SG \leq s_0 = 1.4160\ldots$$

mentioned in the introduction, where s_0 is obtained as the unique positive solution to the equation

$$333 + 24 \cdot 2^{-s_0} + 36 \cdot 3^{-s_0} + 84 \cdot 5^{-s_0} + 24 \cdot 6^{-s_0} + 36 \cdot 15^{-s_0} + 24 \cdot 30^{-s_0} = 64^{s_0}.$$

2. Higher dimensional "Sierpinski simplices". Let T denote the unit simplex in $\mathbb{R}^{n-1}$, $n \geq 3$. Thus T is the closed convex hull of a set of n points in $\mathbb{R}^{n-1}$, any two of which are at a unit distance apart. For $i = 1, \ldots, n$, let f_i denote the conformal contraction with scale factor $1/2$ whose fixed point is the ith vertex of T. Then the maps $f_1, \ldots, f_n$ give rise to a accessible PCF fractal K of dimension

$$\frac{\log n}{\log 2}.$$

The level n weight function which puts weight one on each boundary simplex and an arbitrarily small weight ϵ on each interior simplex is admissible. It is clear that the number of interior simplices at level n is $n!$ (since each of the n indices must occur). Thus the conformal dimension of K is

$$\leq \frac{\log(n^n - n!)}{\log(2^n)} = \frac{\log n}{\log 2} - \frac{\log(1 - n!/n^n)}{n \log 2}.$$

This bound could of course be further reduced by using more elaborate weights as in the previous subsection.

3. Dendrites. Define five planar contractive similarities $f_j(z) = \frac{1}{3}z + \frac{2}{3}a_j$, where $a_1 = -1 + i$, $a_2 = 1 + i$, $a_3 = 0$, $a_4 = -1 - i$ and $a_5 = 1 - i$, and consider the invariant set X for this collection. See Figure 5(a). It is clear that X satisfies the open set condition and hence has dimension

$$\frac{\log 5}{\log 3} = 1.46497\ldots$$

Moreover, X is PCF, with critical set $C(K) = \{e, f, g, h\}$ and post-critical set $PC(K) = \{a, b, c, d, e, f, g, h\}$ and accessible; the graph Γ (Figure 5(b)) with edges $\{\overline{ae}, \overline{eh}, \overline{hd}, \overline{bf}, \overline{fg}, \overline{gc}\}$ satisfies the conditions in Definition 2.9.

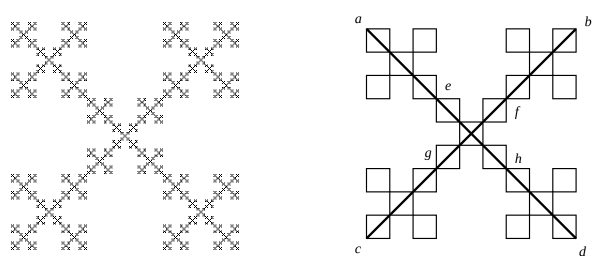

FIGURE 5. (a) A dendrite X; (b) An accessible graph on X

For any $k \geq 2$ and $\epsilon > 0$ let $\boldsymbol{\rho}_{k,\epsilon}$ denote the weight function on $\{1, 2, 3, 4, 5\}^k$ which assigns weight 1 to all subsquares which meet the initial graph $\Gamma_0 = \Gamma$ and weight ϵ to all other subsquares. See Figure 6 for the cases $k = 2, 3$. This weight function is admissible. The number of level k subsquares which recieve weight 1 is $2 \cdot 3^k - 1$.

Let $d_{k,\epsilon}$ denote the corresponding metric on X (as defined in section 4). According to Theorem 5.2, the dimension of $(X, d_{k,\epsilon})$ is at most $s_0 = s_0(k, \epsilon)$, the unique positive solution to the equation

$$3^{ks_0} = 2 \cdot 3^k - 1 + (5^k - 2 \cdot 3^k + 1)\epsilon^{s_0}.$$

Letting $\epsilon \to 0$ and applying Corollary 5.5, we find that

$$\mathcal{C} \dim X \leq \inf_k \frac{\log(2 \cdot 3^k - 1)}{k \log 3} = 1.$$

Consequently, $\mathcal{C} \dim X = 1$.

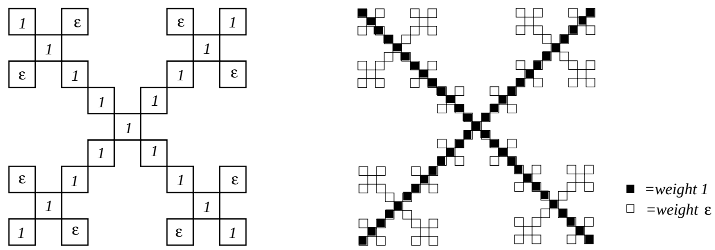

FIGURE 6. Values of $\boldsymbol{\rho}_{k,\epsilon}$ for $k = 2, 3$

The preceding argument clearly applies to a wide variety of dendritic fractals. Furthermore—although it is not clear from the arguments in sections 4 and 5—the conformal dimension of these sets can in fact be attained by considering only the images of X under quasiconformal self-maps of $\mathbb{C}$. In fact, for each k and ϵ, there exists a quasiconformal map $f : \mathbb{C} \to \mathbb{C}$ so that $(X, d_{k,\epsilon})$ is bi-Lipschitz equivalent with $f(X) \subset \mathbb{C}$. (This is an easy modification of the main result of [3].) These examples and the examples of [4] are the only known examples of self-similar fractals for which the conformal dimension can be calculated exactly.

4. The hexagasket. For our final example, we consider the so-called hexagasket. Consider the six planar contractive similarities

$$f_j(z) = \frac{1}{3}z + \frac{2}{3}e^{2\pi i j/6}, \qquad j = 1, \dots, 6.$$

The invariant set for this collection of mappings is called the *hexagasket* and we denote it by HG. See Figure 7(a). It is clear that HG satisfies the open set condition and hence has dimension

$$\frac{\log 6}{\log 3} = 1.6309\dots$$

Furthermore, HG is PCF, with critical set $C(K) = \{g, h, i, j, k, l\}$ and post-critical set $PC(K) = \{a, b, c, d, e, f, g, h, i, j, k, l\}$ and accessible; the "star of David" graph Γ (Figure 7(b)) with edges $\{\overline{ag}, \overline{gh}, \overline{ha}, \overline{bh}, \overline{hi}, \overline{ib}, \dots, \overline{fg}, \overline{gl}, \overline{lf}\}$ satisfies the conditions in Definition 2.9.

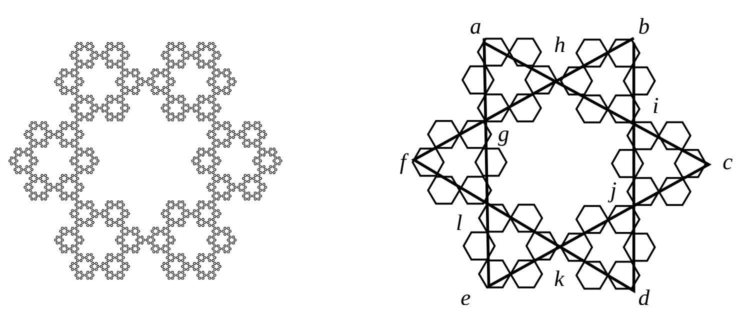

FIGURE 7. (a) The hexagasket HG; (b) An accessible graph on the hexagasket

As in the previous subsection, we define a collection of admissible metrics $d_{k,\epsilon}$ on HG for each integer $k \geq 3$ and $\epsilon > 0$. Each $d_{k,\epsilon}$ will correspond to a weight $\boldsymbol{\rho}_{k,\epsilon} : \{1, \dots, 6\}^k \to (0, 1]$ which will take on only the values $1, \frac{1}{3}, \dots, \frac{1}{3^{k-3}}, \epsilon$. Figure 8 shows a magnification of one of the six primary subhexagons in HG, indicating the values taken on by $\boldsymbol{\rho}_{k,\epsilon}$ for $k = 3, 4$. Observe that $\boldsymbol{\rho}_{k,\epsilon} = 1$ on each hexagon which meets the initial edge set $\mathcal{E}_0$ as required by 4.2(i).

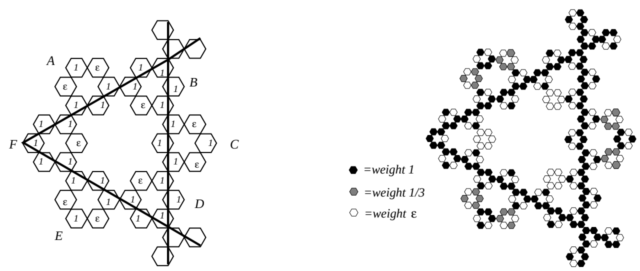

FIGURE 8. Values of $\boldsymbol{\rho}_{k,\epsilon}$ for $k = 3, 4$

We now proceed with a more precise description of these admissible weights. Let us call the second level hexagons marked A, C, E in Figure 8 *medial hexagons* and those marked B, D, F *terminal hexagons*. For $k \geq 3$ and $j = 0, 1, \ldots, k-3$, let

$$m_{k,j} = \begin{array}{c}\text{number of level } k \text{ subhexagons of a medial} \\ \text{hexagon which are given weight } 3^{-j} \text{ by } \boldsymbol{\rho}_{k,\epsilon},\end{array}$$

$$t_{k,j} = \begin{array}{c}\text{number of level } k \text{ subhexagons of a terminal} \\ \text{hexagon which are given weight } 3^{-j} \text{ by } \boldsymbol{\rho}_{k,\epsilon},\end{array}$$

$$m_{k,\epsilon} = \begin{array}{c}\text{number of level } k \text{ subhexagons of a medial} \\ \text{hexagon which are given weight } \epsilon \text{ by } \boldsymbol{\rho}_{k,\epsilon},\end{array}$$

$$t_{k,\epsilon} = \begin{array}{c}\text{number of level } k \text{ subhexagons of a terminal} \\ \text{hexagon which are given weight } \epsilon \text{ by } \boldsymbol{\rho}_{k,\epsilon}.\end{array}$$

Then $m_{3,0} = 4$, $m_{3,\epsilon} = 2$, $t_{3,0} = 5$, $t_{3,\epsilon} = 1$, and we have the recursion relations

$$\begin{aligned} m_{k+1,j} &= 4m_{k,j} + 2m_{k,j-1} \\ m_{k+1,\epsilon} &= 6m_{k,\epsilon} \\ t_{k+1,j} &= t_{k,j} + 4m_{k,j} \\ t_{k+1,\epsilon} &= t_{k,\epsilon} + 4m_{k,\epsilon} + 6^{k-2}. \end{aligned} \tag{6.1}$$

where $m_{k,j} = t_{k,j} = 0$ if $j > k-3$. Note that

$$\sum_{j=1}^{k-3} m_{k,j} + m_{k,\epsilon} = \sum_{j=1}^{k-3} t_{k,j} + t_{k,\epsilon} = 6^{k-2} \tag{6.2}$$

for each $k \geq 3$,[3] which is the number of level k hexagons contained in any second level hexagon (medial or terminal).

[3] Exercise: prove (6.2) by induction using (6.1) and the initial values $m_{3,0}, t_{3,0}, m_{3,\epsilon}, t_{3,\epsilon}$.

The recursive equations in (6.1) can be solved explicitly and we find that

$$\begin{aligned}
m_{k,j} &= \binom{k-3}{j} 2^{2k-j-4}, \qquad j = 0,\dots,k-3,\\
m_{k,\epsilon} &= 2\cdot 6^{k-3},\\
t_{k,j} &= \begin{cases} \frac{1}{3}(16\cdot 4^{k-3}-1), & j=0,\\ 2^{j+4}\sum_{l=0}^{k-j-4}\binom{l+j}{j}2^{2l}, & j=1,\dots,k-4,\\ 0, & j=k-3,\end{cases}\\
t_{k,\epsilon} &= \frac{6\cdot 4^{k-3}-1}{3}.
\end{aligned}$$

The total hexagasket HG contains 18 medial hexagons and 18 terminal hexagons. Thus the weight $\boldsymbol{\rho}_{k,\epsilon}$ on HG assigns weight 3^{-j} to $18m_{k,j}+18t_{k,j}$ level k hexagons and assigns weight ϵ to $18m_{k,\epsilon}+18t_{k,\epsilon}$ level k hexagons. According to Theorem 5.2, the dimension of $(HG, d_{k,\epsilon})$ is at most $s_0 = s_0(k,\epsilon)$, the unique positive solution to the equation

$$3^{ks_0} = \sum_\sigma \rho_\sigma^{s_0} = \sum_{j=0}^{k-3}(18m_{k,j}+18t_{k,j})3^{-js_0} + (18m_{k,\epsilon}+18t_{k,\epsilon})\epsilon^{s_0}.$$

Letting $\epsilon\to 0$ and applying Corollary 5.5, we find that

$$\mathcal{C}\dim HG \le \inf_k s_0(k),$$

where $s_0 = s_0(k)$ is the unique positive solution to the equation

$$\begin{aligned}
3^{ks_0} &= 18\sum_{j=0}^{k-3}(m_{k,j}+t_{k,j})3^{-js_0}\\
&= 18\sum_{j=0}^{k-3}\binom{k-3}{j}2^{2k-j-4}3^{-js_0} + 96\cdot 4^{k-3} - 6\\
&\quad + 18\sum_{j=1}^{k-4}2^{j+4}\sum_{l=0}^{k-j-4}\binom{l+j}{j}2^{2l}3^{-js_0}\\
&= 18\frac{(28+8\cdot 3^{-s_0})(4+2\cdot 3^{-s_0})^{k-3}+10\cdot 3^{-s_0}-1}{3+2\cdot 3^{-s_0}}.
\end{aligned}$$

Let

$$F_k(x) := 18x^k\frac{(28+8x)(4+2x)^{k-3}+10x-1}{3+2x}$$

and let $x = x_k$ denote the unique positive root of the polynomial equation $F_k(x) = 1$. Then $x_k = 3^{-s_0(k)}$ and so

$$\mathcal{C}\dim HG \le \inf_k \frac{\log(1/x_k)}{\log 3} = \frac{\log(1/\sup_k x_k)}{\log 3}.$$

Writing $F_k(x)$ in the form

$$F_k(x) = \frac{72x^3(7+2x)(4x+2x^2)^{k-3}+18(10x-1)x^k}{3+2x}$$

shows that $F_k(a) < 1$ for sufficiently large values of k if $a < \frac{1}{2}\sqrt{6}-1$. (Observe that $\frac{1}{2}\sqrt{6}-1$ is the unique positive solution to the quadratic equation $4x+2x^2=1$.) Thus $\frac{1}{2}\sqrt{6}-1 \le \sup_k x_k$.

On the other hand, if $a > \frac{1}{2}\sqrt{6} - 1 > 0.1$ then for each k

$$F_k(a) > \frac{72a^3(7+2a)}{3+2a}.$$

The right hand side is an increasing function of a and so

$$F_k(a) > \frac{72(5+\sqrt{6})(\frac{1}{2}\sqrt{6}-1)^3}{1+\sqrt{6}} = 1.765\ldots > 1.$$

Consequently $\sup_k x_k \le \frac{1}{2}\sqrt{6} - 1$.

In conclusion, $\sup_k x_k = \frac{1}{2}\sqrt{6} - 1 = \frac{1}{2+\sqrt{6}}$ and

$$\mathcal{C}\dim HG \le \frac{\log(2+\sqrt{6})}{\log 3} = 1.3588\ldots \tag{6.3}$$

as mentioned in the introduction.

It is unlikely that this estimate is best possible, even if we only consider those bounds that can be obtained from Theorem 2.6. It should be possible to arrange the weights on the level k hexagons more evenly and reduce this upper bound even further. Also, other metrics on HG may turn out to have even smaller Hausdorff dimension. The construction presented here, however, has the advantages that it is reasonably straightforward to compute the new dimensions as well as to show the quasisymmetric equivalence with the standard metric.

7. Further questions

In this final section, we mention some questions related to this paper which would be of interest for further study.

1. What is the connection between accessibility and post-critical finiteness? The PCF assumption says (roughly speaking) that the set is "almost not connected", while accessibility implies quasi-convexity, i.e., any two points in the set can be joined by a relatively short curve. Observe that among the N-sided polygaskets (see Figure 9), only the usual Sierpinski gasket ($N = 3$) and the hexagasket ($N = 6$) are both accessible and PCF, although these sets are always PCF whenever N is not a multiple of 4. All of the planar examples of accessible PCF fractals in this paper arise from lattices. Which lattices give rise to (at least) one accessible PCF fractal?

2. Can we determine more explicitly the class of admissible weights $\boldsymbol{\rho}$ for a given accessible PCF fractal K? In other words, can we solve the finite combinatorial problem embedded in Definition 4.2(ii)? Observe that this is essentially just an optimization problem; we must minimze the (implicitly defined) value of $s_0(k, \boldsymbol{\rho})$ (see Theorem 5.2) subject to the system of linear constraints on $\boldsymbol{\rho}$ implied by the admissibility condition 4.2. The solution to this problem would lead to a more explicit upper bound for the conformal dimension of K in Theorem 2.6. What additional conditions (if any) must we impose to ensure that the conformal dimension of K is one?

3. Is it possible to estimate the conformal dimension if some of the assumptions on K are relaxed? Recall the three assumptions which we impose: accessibility, post-critical finiteness, and the requirement that the subpolyhedra $P_i = f_i(P)$ be of equal size. The last condition was a technical assumption which was needed in the proof of Theorem 5.1; without it we are at present unable to show that the new metrics are quasisymmetrically equivalent with the standard metric. Accessibility was used in an essential way in defining the new metrics $d = d_{k,\rho}$ via the path metric definition

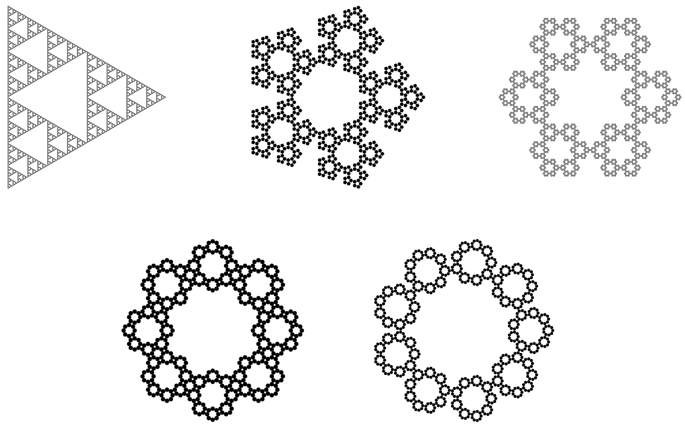

FIGURE 9. N-sided polygaskets for $N = 3, 5, 6, 8, 9$

(4.4). (Note that K must be at least rectifiably connected for this definition to have any chance of yielding a metric.) It may be possible construct other metrics on self-similar fractals without using rectifiable connectedness and thus avoid using accessibility. Finally, the PCF condition was used at several points (Proposition 4.5, Proposition 4.10(i) and Theorem 5.1) to conclude that adjacent subpolyhedra intersect at precisely one point. A typical example of a non-PCF set is the Sierpinski carpet SC, see Figure 10. We still do not know the value of the conformal dimension of SC, in particular, we do not know if its dimension can be reduced by any quasisymmetric map. Note, however, that it is certainly **not** possible for SC to have conformal dimension one by the results of [16], since $SC \supset C \times [0,1]$ (where C denotes the usual $\frac{1}{3}$-Cantor set) and

$$\mathcal{C}\dim(C \times [0,1]) = \dim(C \times [0,1]) = 1 + \dim C > 1.$$

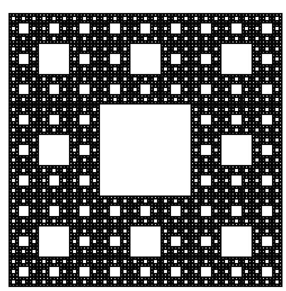

FIGURE 10. The Sierpinski carpet SC

4. What can be done for sets with less self-similarity? For example, what can we say if we assume that the maps f_i are merely bi-Lipschitz (rather than similarities)? Note that estimates for the conformal dimension are of particular interest in complex dynamics (for Julia sets of rational maps) and in hyperbolic geometry (for the limit sets of Kleinian groups), and these sets typically admit a "quasi-self-similar" structure of precisely this type. The PCF assumption is often satisfied in these settings, but the accessibility condition is rarely satisfied as these sets typically admit no rectifiable curves.

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