EARTHQUAKE MEASURE AND CROSS-RATIO DISTORTION

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ABSTRACT. Given an orientation-preserving circle homeomorphism h, let $(E, \mathcal{L})$ denote a Thurston's left or right earthquake representation of h and σ the transversal shearing measure induced by $(E, \mathcal{L})$. We first show that the Thurston norm $||\cdot||_{Th}$ of σ is equivalent to the cross-ratio distortion norm $||\cdot||_{cr}$ of h, i.e., there exists a constant $C > 0$ such that

$$\frac{1}{C}||h||_{cr} \leq ||\sigma||_{Th} \leq C||h||_{cr}$$

for any h. Secondly we introduce two new norms on the cross-ratio distortion of h and show they are equivalent to the Thurston norms of the measures of the left and right earthquakes of h. Together it concludes that the Thurston norms of the measures of the left and right earthquakes of h and the three norms on the cross-ratio distortion of h are all equivalent. Furthermore, we give necessary and sufficient conditions for the measures of the left and right earthquakes to vanish in different orders near the boundary of the hyperbolic plane. Vanishing conditions on either measure imply that the homeomorphism h belongs to certain classes of circle diffeomorphisms classified by Sullivan in [7].

1. Introduction

Given any orientation-preserving homeomorphism h of the unit circle $\mathbb{S}^1$, according to Thurston [4], there exists a left or right earthquake map $(E, \mathcal{L})$ in the hyperbolic plane $\mathbb{H}$ such that its extension to the boundary circle $\mathbb{S}^1$ is exactly equal to h. Any earthquake map $(E, \mathcal{L})$ naturally introduces a transversal shearing measure σ, which quantifies the amount of shearing along the geodesic lines in the lamination $\mathcal{L}$. Let $\mathcal{M}$ denote the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\textit{the diagonal}\}$ factored by the euqivalence relation $(a, b) \sim (b, a)$. Then the measure σ can be viewed as a Borel measure on $\mathcal{M}$ with the support consisting of the pairs of the endpoints of the lines in $\mathcal{L}$. The Thurston norm $||\sigma||_{Th}$ of σ is the supremum, over all hyperbolic geodesic segments β of length one in the hyperbolic plane, of the total amount of shearing along the lines in the support of σ that intersect β.

For any homeomorphism h, we define the cross-ratio distortion norm $||h||_{cr}$ of h as

$$||h||_{cr} = \sup_{Q} |\log cr(h(Q))| \, ,$$

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where the supremum is taken over all quadruples $Q = \{a, b, c, d\}$ of four points arranged in counterclockwise order on the circle with $cr(Q) = 1$, where

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} \text{ and } cr(h(Q)) = \frac{(h(b)-h(a))(h(d)-h(c))}{(h(c)-h(b))(h(d)-h(a))}.$$

It is shown in [1] that there exists a constant $C > 0$ such that

$$||\sigma||_{Th} \leq C||h||_{cr} \tag{1}$$

for any orientation-preserving homeomorphism h. Here we show there exists another constant $C > 0$ such that

$$||h||_{cr} \leq C||\sigma||_{Th} \tag{2}$$

for any h. Therefore we have the first main theorem.

Theorem 1. *The Thurston norm $||\cdot||_{Th}$ is equivalent to the cross-ratio distortion norm $||\cdot||_{cr}$, i.e., there exists a constant $C > 0$ such that for any orientation-preserving homeomorphism h,*

$$\frac{1}{C}||h||_{cr} \leq ||\sigma||_{Th} \leq C||h||_{cr}.$$

Remark. The following weaker version of the inquality (2) was shown in [1]: for any $C_0 > 0$, there exists $C > 0$, depending only on C_0, such that $||h||_{cr} \leq C||\sigma||_{Th}$ for any σ with $||\sigma||_{Th} \leq C_0$. It was proved by a quite different method by studying the tangent vectors to the earthquake curve of σ and then integrating the vector field.

Given an orientation-preserving homeomorphism h, let σ_h^l and σ_h^r denote the transversal shearing measures inducecd by a left earthquake and a right one of h. Since Theorem 1 holds for both σ_h^l and σ_h^r, we have the following corollary, which is not trivial because the lamination for a right earthquake representation of h may be quite different from the lamination for its left earthquake, for example, the lamination for the right earthquake of a homeomorphism h corresponding to a simple left earthquake consists of infinitely many geodesic lines ([4]).

Corollary 1. *For any orientation-preserving homeomorphism h,*

$$\frac{1}{C^2}||\sigma_h^l||_{Th} \leq ||\sigma_h^r||_{Th} \leq C^2||\sigma_h^l||_{Th},$$

where C is the same constant in Theorem 1.

In the course of proving Theorem 1, we find that the Thurston norms of σ_h^l and σ_h^r are related to the upper and lower bounds on the cross-ratio distortions of h on certain quadruples. Therefore we introduce two new norms on the cross-ratio distortion of h.

Definition 1. *The upper and lower cross-ratio distortion norms of h are defined as*

$$||h||_{cr+} = \sup_{\tilde{Q}} \ln cr(h(\tilde{Q}))$$

and

$$||h||_{cr-} = -\inf_{\tilde{Q}} \ln cr(h(\tilde{Q})),$$

where the supremum is taken over all quadruples $\tilde{Q} = \{a, b, c, d\}$ of points arranged in counterclockwise order on the unit circle satisfying $cr(\tilde{Q}) = 1$, and $|b-c|$ and $|c-d|$ are the two smallest lengths among $|a-b|$, $|b-c|$, $|c-d|$ and $|d-a|$.

Let $l = \overline{ac}$ be a geodesic line in the lamination of a left earthquake and $l' = \overline{bd}$ a geodesic line perpendicular to l such that a, b, c, d are arranged on the circle in counterclockwise direction and $|b-c|$ and $|c-d|$ are the two shortest lengths among $|a-b|$, $|b-c|$, $|c-d|$ and $|d-a|$. Then $\{a, b, c, d\}$ forms a quadruple $\tilde{Q}$. Since l is a geodesic line in the lamination of a left earthquake, one can easily check that $cr(h(\tilde{Q})) \geq 1$. Therefore $||h||_{cr+} \geq 0$. Correspondingly, by using a right earthquake of h, one can show $||h||_{cr-} \geq 0$.

By a method similar to the one used to prove Theorem 1, we show

Theorem 2. *For any orientation-preserving homeomorphism h,*

$$\frac{1}{C}||h||_{cr+} \leq ||\sigma_h^l||_{Th} \leq C||h||_{cr+}$$

and

$$\frac{1}{C}||h||_{cr-} \leq ||\sigma_h^r||_{Th} \leq C||h||_{cr-},$$

where C can be taken as the same constant in Theorem 1.

Corollary 2. *The cross-ratio norm, the upper cross-ratio norm and the lower cross-ratio norm are equivalent, that is, for any orientation preserving circle homeomorphism h,*

$$\frac{1}{C^2}||h||_{cr+} \leq ||h||_{cr} \leq C^2||h||_{cr+}$$

and

$$\frac{1}{C^2}||h||_{cr-} \leq ||h||_{cr} \leq C^2||h||_{cr-},$$

where C is the same constant in the previous two theorems.

Suppose we take the open unit disk $\mathbb{D}$ centered at the origin in the complex plane $\mathbb{C}$ as a model for the hyperbolic plane $\mathbb{H}$. Let D be a disk in $\mathbb{D}$ and $\delta(D)$ denote the Euclidean distance from D to the boundary $\mathbb{S}^1$ of $\mathbb{D}$ and $mass_\sigma(D)$ the total amount of shearing along the lines of the lamination of σ that intersect D. Given any $\alpha \geq 0$, we say that a measure σ is *vanishing* (resp. *strongly vanishing*) of *order* α if

$$mass_\sigma(D) \leq O(\delta(D)^\alpha) \;\; (resp. \;\; mass_\sigma(D) \leq o(\delta(D)^\alpha))$$

for all disks D of hyperbolic diameter ≤ 1.

The second main part of this work is to give necessary and sufficient conditions for σ to vanish or strongly vanish in different order α.

Assume that four points a, b, c, d on the unit circle $\mathbb{S}^1$ are labelled in counterclockwise order. Define the *minimum scale* of a quadruple $Q = \{a, b, c, d\}$ to be

$$s_{min}(Q) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

Let $\tilde{Q}$ be a quadruple as the same as in Definition 1. We say that an orientation-preserving homeomorphism h of the unit circle $\mathbb{S}^1$ is *smooth of order α from above* (resp. *strongly smooth of order α from above*) if

$$\ln cr(h(\tilde{Q})) \leq O(s_{min}(\tilde{Q})^\alpha) \;\; (resp. \;\; \ln cr(h(\tilde{Q})) \leq o(s_{min}(\tilde{Q})^\alpha))$$

for all quadruples $\tilde{Q}$. Similarly we say h is *smooth of order* α *from below* (resp. *strongly smooth of order* α *from below*) if

$$\ln cr(h(\tilde{Q})) \geq -O(s_{min}(\tilde{Q})^{\alpha}) \ \ (resp. \ \ \ln cr(h(\tilde{Q})) \geq -o(s_{min}(\tilde{Q})^{\alpha}))$$

for all quadruples $\tilde{Q}$. And finally we say that h is *smooth* (resp. *strongly smooth*) of *order* α if h is smooth (resp. strongly smooth) of order α from above and below.

Theorem 3. *For each $\alpha \geq 0$, any orientation-preserving circle homeomorphism h is smooth (resp. strongly smooth) of order α from above if and only if σ_h^l is vanishing (resp. strongly vanishing) of order α.*

Theorem 4. *For each $\alpha \geq 0$, any orientation-preserving circle homeomorphism h is smooth (resp. strongly smooth) of order α from below if and only if σ_h^r is vanishing (resp. strongly vanishing) of order α.*

Theorem 5. *For each $\alpha \geq 0$, any orientation-preserving circle homeomorphism h is smooth (resp. strongly smooth) of order α if and only if both σ_h^l and σ_h^r are vanishing (resp. strongly vanishing) of order α.*

When $\alpha = 0$, h is smooth of order 0 means h is quasisymmetric. The previous theorem includes a known result that h is quasisymmetric if and only if σ is Thurston bounded, which was proved in [1] and [5] by different methods. The method of [1] is to study the tangent vectors to the earthquake curves of finite approximations of σ and then to integrate the vector fields. The method of [5] is to show that the earthquake curve of σ depends holomorphically on the parameter t and then to apply Slodkowski's theorem for the extension of a holomorphic motion [6]. In this paper, we provides a unified proof for the equivalence of two conditions on h and σ for all orders by improving the method used to show Theorem 1. Also, observe that Theorems 1, 2 and Corollary 1 imply that the quasisymmetry of h is equivalent to the boundness of either σ_h^l or σ_h^r. It also raises a question whether or not σ_h^l and σ_h^r vanish in the same order α for any $\alpha \geq 0$, which may have different answers for different values of α.

Let us notice that if h is smooth from above (resp. from below) of order $\alpha > 2$ or h is strongly smooth from above (resp from below) of order $\alpha \geq 2$, then h has Schwarzian derivative ≤ 0 (resp. ≥ 0). And therefore if h is smooth of order $\alpha > 2$ or strongly smooth of order $\alpha \geq 2$, then h has Schwarzian derivative 0, and hence h is a Möbius transformation and $\sigma = 0$. Therefore the previous theorem is only interesting when $0 \leq \alpha \leq 2$ in smooth cases and $0 \leq \alpha < 2$ in strongly smooth cases.

Another consequence of the technique developed in this paper is that the vanishing conditions on either σ_h^l or σ_h^r in different orders imply that h belongs to certain classes of circle diffeomorphisms classified by Sullivan in [7]. Define the *maximum scale* of a quadruple $Q = \{a, b, c, d\}$ to be

$$s_{max}(Q) = \max\{|a-b|, |b-c|, |c-d|, |d-a|\}.$$

Theorem 6. *For any $\alpha \geq 0$, if σ_h^l or σ_h^r is vanishing (resp. strongly vanishing) of order α, then*

$$|\ln cr(h(Q))| = O(s_{max}(Q)^{\alpha}) \ \ (\text{resp. } \ |\ln cr(h(Q))| = o(s_{max}(Q)^{\alpha}))$$

for all quadruples $Q = \{a, b, c, d\}$ with $cr(Q) = 1$, where a, b, c, d are arranged on the circle in counterclockwise direction.

Corollary 3. *For any $\alpha \geq 0$, if σ_h^l or σ_h^r is vanishing (resp. strongly vanishing) of order α, then*

$$|\ln 3cr(h(Q))| = O(s_{max}(Q)^\alpha) \text{ (resp. } |\ln 3cr(h(Q))| = o(s_{max}(Q)^\alpha))$$

for all quadruples $Q = \{a, b, c, d\}$ with $cr(Q) = \frac{1}{3}$, where a, b, c, d are arranged on the circle in counterclockwise direction.

Four points $a < b < c < d$ on the real line is called a *standard 4-tuple* if $b - a = c - b = d - c$, denoted by Q'. In [7], Sullivan classified the smoothness of one-dimensional homeomorphisms h according to the comparison between the cross-ratio distortion of h on Q' and the scale of Q'. Let us say that a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is Sullivan smooth (resp. strongly Sullivan smooth) of order α for $\alpha \geq 0$ if the cross-ratio distortion of h on any standard 4-tuple Q' is $O(s(Q')^\alpha)$ (resp. $o(s(Q')^\alpha)$). And then a circle homeomorphism h is Sullivan smooth (resp. strongly Sullivan smooth) of order α if there is a smooth coordinate system for the circle, consisting of finitely many charts, such that h is Sullivan smooth (resp. strongly Sullivan smooth) of order α on each chart. One can also find an outline in [7] to prove the following dichotomy: for $0 < \alpha < 1$, h is Sullivan smooth of order α if and only if h is a diffeomorphism and $\phi = \ln h'$ is α-Hölder continuous; h is Sullivan smooth of order 1 if and only if h is a diffeomorphism and ϕ satisfies Zygmund condition; for $1 < \alpha \leq 2$, h is Sullivan smooth of order α if and only if h is a diffeomorphism and ϕ is $C^{1,\,\alpha-1}$; and h is strongly Sullivan smooth of order 2 if and only if h is a Möbius transformation.

Let h be an orientation-preserving circle homeomorphism, σ_h^l and σ_h^r be the transversal shearing measures induced by a left earthquake and a right one of h respectively. From Corollary 3, it is easy to see that for each $\alpha \geq 0$, if either σ_h^l or σ_h^r is vanishing (resp. strongly vanishing) of order α, then h is Sullivan smooth (resp. strongly Sullivan smooth) of order α. On the other hand, we have

Theorem 7. *For each $0 \leq \alpha < 1$, if h is Sullivan smooth (resp. strongly Sullivan smooth) of order α, then h is smooth of order α.*

Theorems 3 and 4, Corollary 3 and Theorem 7 imply

Theorem 8. *For each $0 \leq \alpha < 1$, the following statements are equivalent:*
(1) h is smooth (resp. strongly smooth) of order α;
(2) h is smooth (resp. strongly smooth) of order α from above;
(3) h is smooth (resp. strongly smooth) of order α from below;
(4) σ_h^l is vanishing (resp. strongly vanishing) of order α;
(5) σ_h^r is vanishing (resp. strongly vanishing) of order α;
(6) h is Sullivan smooth (resp. strongly Sullivan smooth) of order α.

Corollary 4. *The following statements are equivalent:*
(1) h is quasisymmetric;
(2) $||h||_{cr}$ is finite;
(3) $||h||_{cr+}$ is finite;
(4) $||h||_{cr-}$ is finite;
(5) σ_h^l is Thurston bounded;
(6) σ_h^r is Thurston bounded;
(7) h is Sullivan smooth of order 0.

Corollary 5. *The following statements are equivalent:*

(1) h is symmetric;
(2) h is strongly smooth of order 0;
(3) h is strongly smooth of order 0 from above;
(4) h is strongly smooth of order 0 from below;
(5) $\sigma_h^l(D)$ vanishes uniformly as the disk D approaches the boundary;
(6) $\sigma_h^r(D)$ vanishes uniformly as the disk D approaches the boundary;
(7) h is strongly Sullivan smooth of order 0.

Remark. The equivalence of (1) and (4) in Theorem 8 was first proved in [1], however, the proof of the part (4) $\Longrightarrow$ (1) in [1] is quite long because it applies the method of studying the regularities of the tangent vectors to the earthquake curves and then integrating the vector fields to derive the smooth regularities for h. In this paper, we give a direct and relatively shorter proof of that part by using some work of Sullivan in [7] and our corollary 3 and theorem 7.

We conclude the introduction by extending Sullivan's dichotomy to a trichotomy that includes characterizations corresponding to earthquake measures. In the following table, let Q denote a standard 4-tuple in one of the finitely many charts in a smooth coordinate system of the unit circle, D a disk in the hyperbolic plane of hyperbolic diameter 1, and $s(Q)$ the scale of Q and $\delta(D)$ the Euclidean distance from D to the boundary circle of the hyperbolic plane. When h is a diffeomorphism, we denote by $\phi = \ln h'$. Furthermore, $\phi \in C^\alpha$ means that ϕ is α-Hölder continuous; $\phi \in C^Z$ means that ϕ satisfies Zygmund condition; $\phi \in C^{1,\alpha}$, $0 < \alpha < 1$, means that ϕ is C^1 and ϕ' is α-Hölder continuous; and finally $\phi \in C^{1,1}$ means that $\phi \in C^1$ and ϕ' satisfies Lipschitz condition. Also, observe that if ϕ satisfies Zygmund condition then ϕ is α-Hölder continuous for any $0 < \alpha < 1$ ([3]).

Order α	**Smoothness** h		**Distortion** $\lvert \ln cr(h(Q))\rvert$		**Measure** $\sigma(D)$
0	quasisymmetric	$\Longleftrightarrow$	$\lVert h\rVert_{cr} < +\infty$	$\Longleftrightarrow$	$\lVert\sigma\rVert_{Th} < +\infty$
0	symmetric	$\Longleftrightarrow$	$o(1)$	$\Longleftrightarrow$	$o(1)$
$0 < \cdot < 1$	diffeo., $\phi \in C^\alpha$	$\Longleftrightarrow$	$O(s(Q)^\alpha)$	$\Longleftrightarrow$	$O(\delta(D)^\alpha)$
$0 < \cdot < 1$	diffeo., $\phi \in C^\alpha$, Hölder const. $= o(1)$	$\Longleftrightarrow$	$o(s(Q)^\alpha)$	$\Longleftrightarrow$	$o(\delta(D)^\alpha)$
1	diffeo., $\phi \in C^Z$	$\Longleftrightarrow$	$O(s(Q))$	$\Longleftarrow$	$O(\delta(D))$
1	diffeo., $\phi \in C^Z$, Zygmund const. $= o(1)$	$\Longleftrightarrow$	$o(s(Q))$	$\Longleftarrow$	$o(\delta(D))$
$1 < \cdot < 2$	diffeo, $\phi \in C^{1,\,\alpha-1}$	$\Longleftrightarrow$	$O(s(Q)^\alpha)$	$\Longleftarrow$	$O(\delta(D)^\alpha)$
$1 < \cdot < 2$	diffeo, $\phi \in C^{1,\,\alpha-1}$, Hölder const. $= o(1)$	$\Longleftrightarrow$	$o(s(Q)^\alpha)$	$\Longleftarrow$	$o(\delta(D)^\alpha)$
2	diffeo., $\phi \in C^{1,1}$	$\Longleftrightarrow$	$O(s(Q)^2)$	$\Longleftarrow$	$O(\delta(D)^2)$
2	Möbius trans.	$\Longleftrightarrow$	$o(s(Q)^2)$	$\Longleftrightarrow$	$o(\delta(D)^2)$

The paper is arranged as follows. Necessary backgrounds on earthquake maps and transversal shearing measures are given in section 2. Theorems 1 and 2 are proved in section 3 and the proofs of the remaining theorems are included in the last section.

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2. Earthquake map and shearing measure

Assume that we take the unit disk $\mathbb{D}$ centered at the origin of the complex plane $\mathbb{C}$ as a model for the hyperbolic plane $\mathbb{H}$.

A *geodesic lamination* $\mathcal{L}$ in the hyperbolic plane $\mathbb{H}$ is a collection of geodesics which foliate a closed subset L of $\mathbb{H}$. The set L is called *the locus* of $\mathcal{L}$, the geodesics are called the leaves of $\mathcal{L}$, the connected components of $\mathbb{H} \setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathcal{L}$ are called the *strata* of the lamination.

Let $\mathcal{L}$ be a geodesic lamination in $\mathbb{H}$. By a $\mathcal{L}$-*left earthquake map* E we mean that E is an injective and surjective (and often discontinuous) map from $\mathbb{H}$ to $\mathbb{H}$ satisfying

(1) the restriction of E on each stratum A of the lamination is the restriction of a Möbius transformation, which maps $\mathbb{H}$ onto $\mathbb{H}$, on A, and

(2) for any two strata A and B, the *comparison map*

$$cmp(A, B) = (E|_A)^{-1} \circ E|_B : \mathbb{H} \to \mathbb{H}$$

is a hyperbolic transformation whose axis weakly separates A and B and which translates to the left as viewed from A. Here $E|_A$ and $E|_B$ denote the Möbius transformations representing E on A and B. A line l *weakly separates* two sets A and B if any path connecting a point $a \in A$ to a point $b \in B$ intersects l.

Thurston [4] showed that each left earthquake map $(E, \mathcal{L})$ extends uniquely to a map $\tilde{E}$ defined on $\mathbb{H} \cup \mathbb{S}^1$. The extension is continous at each point $x \in \mathbb{S}^1$, and the restriction of $\tilde{E}$ to $\mathbb{S}^1$ is a homeomorphism. Conversely, every circle homeomorphism f is realized in this way, in other words, for any circle homeomorphism f there exists a left earthquake map E with lamination $\mathcal{L}$ such that $\tilde{E}|_{\mathbb{S}^1} = f$. We will say that $(E, \mathcal{L})$ is a *left earthquake representation* of f.

Similarly one can define a right earthquake map. And by an earthquake map we mean it is either a left earthquake or a right one. In general, left and right earthquakes have parallel results. In this paper, unless otherwise indicated, we assume earthquakes are left earthquakes.

Each earthquake map naturally introduces a transversal shearing measure which quantifies the amount of shearing along the geodesics in the lamination.

Given a hyperbolic Möbius transformation s from $\mathbb{H}$ onto $\mathbb{H}$, the *translation length* of s can be defined to be the logarithm of the derivative at its expanding fixed point, it is denoted by $\tau(s)$. We say that two hyperbolic Möbius tranformations s and t from $\mathbb{H}$ onto $\mathbb{H}$ with disjoint axes *translate in the same direction* if the region between the axes of s and t induces an orientation on the axes that agrees with the direction of one axis and disagrees with the other.

Let s and t be two hyperbolic Möbius transformations form $\mathbb{H}$ onto $\mathbb{H}$ which have disjoint axes and translate in the same direction. From Figure 2.5 in [4], one can show that the composition $s \circ t$ is a hyperbolic Möbius transformation from $\mathbb{H}$ onto $\mathbb{H}$ whose axis weakly separates the axes of s and t, and the translation length of $s \circ t$ is greater than or equal to the sum of the translation lengths of s and t, i.e.,

$$\tau(s \circ t) \geq \tau(s) + \tau(t). \tag{3}$$

See the details for its proof in [1].

Given an earthquake map $(E, \mathcal{L})$ and two geodesic lines l_* and l^* in $\mathcal{L}$, let β be a closed geodesic segment which is transversal to both l_* and l^* and intersects them at its endpoints. The amount $\nu(\beta)$ of *relative transversal shearing* of the earthquake map $(E, \mathcal{L})$ along β is defined as follows. Let $P = \{I_i\}_{i=1}^n$ be a partition of β into small geodesic segments, and T_i the translation length of the comparison map of the strata containing the endpoints of the segment I_i. Define

$$\nu(P) = \sum_{i=1}^{n} T_i.$$

By the inequality (3), $\nu(P)$ decreases as the partition P is refined. We define the *relative transversal shearing* of $(E, \mathcal{L})$ along β to be

$$\nu(\beta) = \inf_P \nu(P).$$

As pointed out by Thurston in [4], $\nu(\beta)$ can be well approximated by the sum of the translation lengths of comparison maps.

Proposition 1 ([4]). *Let $(E, \mathcal{L})$ be an earthquake map and $\nu(\beta)$ be defined as above. Then for any infinite sequence $\{P_n\}_{n=1}^{\infty}$ of nested partitions of β with $\lim_{n\to\infty} d(P_n) = 0$, $\nu(P_n)$ converges to $\nu(\beta)$ as $n \to \infty$. Furthermore, for any partition P of β, we have*

$$\nu(P) - O(\nu(P)d(P)^2) \leq \nu(\beta) \leq \nu(P), \tag{4}$$

where $d(P)$ denotes the maximal hyperbolic length of the segments in a partition P of β.

Lemma 1. *Let d denote a small positive number. Suppose that s and t are two hyperbolic Möbius transformations from $\mathbb{H}$ onto $\mathbb{H}$ which have disjoint axes and translate in the same direction, and suppose that the distance from the axis of s to the axis of t in the hyperbolic metric is d. Then*

$$|\tau(s) + \tau(t) - \tau(s \circ t)| = O(\min\{\tau(s), \tau(t)\}d^2). \tag{5}$$

This lemma was pointed out in [4], and a complete proof was given in [1] (Lemma 8). We now apply this lemma to prove Proposition 1.

Proof. Given two infinite sequences $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ of nested partitions of β with $\lim_{n\to\infty} d(P_n) = \lim_{n\to\infty} d(Q_n) = 0$, $\{P_n \cup Q_n\}_{n=1}^{\infty}$ is also an infinite sequence of nested partitions of β with $\lim_{n\to\infty} d(P_n \cup Q_n) = 0$. By Lemma 1,

$$|\nu(P_n) - \nu(P_n \cup Q_n)| = O(\nu(P_n)d(P_n)^2),$$

and

$$|\nu(Q_n) - \nu(P_n \cup Q_n)| = O(\nu(Q_n)d(Q_n)^2),$$

and by the inequality (3), $\nu(P_n)$, $\nu(Q_n)$ and $\nu(P_n \cup Q_n)$ are decreasing sequences. Together, $\nu(P_n)$ and $\nu(Q_n)$ converge to the limit of $\nu(P_n \cup Q_n)$ as $n \to \infty$. This implies the first part of the proposition.

To show the other part, we consider a special infinite sequence $\{P_n\}_{n=1}^{\infty}$ of nested partitions of β such that $P_1 = P$ and P_{n+1} is an even bisection of P_n for any $n \in \mathbb{N}$. Applying Lemma 1 again,

$$|\nu(P_{n+1}) - \nu(P_n)| = O(\nu(P_n)d(P_n)^2).$$

By the inequality (3), $\nu(P_{n+1}) \leq \nu(P_n)$ for any $n \in \mathbb{N}$, and therefore,

$$|\nu(P_n) - \nu(P)| = O(\nu(P)d(P)^2 \sum_{i=0}^{n-1} \frac{1}{2^{2i}}).$$

The proof follows by taking the limit as $n \to \infty$. □

Notice that if a geodesic segment β is transversal to one leaf of a lamination $\mathcal{L}$ then it transversally intersects any leaf of $\mathcal{L}$ at most once. The *earthquake measure* $\sigma(\beta)$ of a closed geodesic segment β, induced by the earthquake map $(E, \mathcal{L})$, is defined to be

$$\sigma(\beta) = \inf_{\beta'} \nu(\beta') \tag{6}$$

where β' is a closed geodesic segment containing β in its interior.

It is easy to see that the earthquake measure induced by an earthquake map with finitely many leaves is an atomic measure supported on those leaves. In general, σ extends to a Borel measure on the space $\mathcal{M}$ with the support consisting of all pairs of the endpoints of the leaves in $\mathcal{L}$, where $\mathcal{M}$ denotes the space $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{\textit{the diagonal}\}$ factored by the euqivalence relation $(a, b) \sim (b, a)$. The *Thurston norm* of σ is

$$||\sigma||_{Th} = \sup_{l(\beta)\leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta),$$

where β is a closed geodesic segment transversal to the lamination $\mathcal{L}$ and $l(\beta)$ denotes the hyperbolic length of β. We also define the norm of ν by

$$||\nu|| = \sup_{l(\beta)\leq 1} \nu(\beta) = \sup_{l(\beta)=1} \nu(\beta),$$

where β is a closed geodesic segment transversal to the lamination $\mathcal{L}$. The Thurston norm of σ is equivalent to the norm of ν.

Proposition 2. *For any earthquake map $(E, \mathcal{L})$, the induced ν and σ satisfy*

$$||\nu|| \leq ||\sigma||_{Th} \leq 2||\nu||.$$

Proof. Without loss of generality, let β denote a closed geodesic segment of hyperbolic length 1. Given any closed geodesic segment β' containing β in its interior, $\nu(\beta') \geq \nu(\beta)$, and hence

$$\sigma(\beta) = \inf_{\beta'} \nu(\beta') \geq \nu(\beta).$$

Therefore $||\sigma||_{Th} \geq ||\nu||$.

On the other hand, let β' be a closed geodesic segment of hyperbolic length 2 and containing β in its interior. Suppose that $\beta' = \beta_1 \cup \beta_2$, where both β_1 and β_2 are closed geodesic segments and have hyperbolic length 1. Then

$$\sigma(\beta) \leq \nu(\beta') = \nu(\beta_1) + \nu(\beta_2) \leq 2||\nu||.$$

And therefore $||\sigma||_{Th} \leq 2||\nu||$. □

Notice that the constant 2 in the previous proposition is sharp in the sense that there exists an earthquake map such that the induced σ and ν satisfy $||\sigma||_{Th} = 2||\nu||$. For an example, let $\mathcal{L}$ consist of two geodesic lines l_1 and l_2 which have hyperbolic distance equal to 1. Let G_1, G_2 and G_3 denote the gaps, and G_2 is in the middle, G_1 and G_2 share one side l_1 and G_2 and G_3 share one side l_2. Suppose that $E|_{G_1}$ is the identity, $E|_{l_1} = E|_{G_2}$, $E|_{l_2} = E|_{G_3}$, and the comparison maps of G_2 to G_1 and G_3 to G_1 have the same nonzero translation length. One can easily check that $||\sigma||_{Th} = 2||\nu||$.

Given a geodesic lamination $\mathcal{L}$ and a closed disk D in $\mathbb{H}$, we say that $\mathcal{L}$ intersects D in *a parallel fashion* if there are two geodesic lines l_* and l^* among those intersecting D such that any other intersecting line separates l_* from l^*. And we say that *an open disk intersects a lamination in a parallel fashion* if its closure does so. There exists a constant $a > 1$ (the maximal value of such constants is $\ln(1 + \frac{\sqrt{3}}{2}))$, independent of the lamination $\mathcal{L}$, such that for any disk D in $\mathbb{H}$ of hyperbolic diameter $< a$, $\mathcal{L}$ intersects D in a parallel fashion. Suppose that D is a disk in $\mathbb{H}$ of hyperbolic diameter ≤ 1. Let l_* and l^* be the two leaves which bound all leaves intersecting D. Let I and J denote the smallest arcs on $\mathbb{S}^1$ bounded by the endpoints of l_* and l^* such that any leaf intersecting D connects a point of I to a point of J, and let r be the closed geodesic segment perpendicular to both l_* and l^*. Then any leaf in $\mathcal{L}$ intersecting r connects a point of I to a point of J. We call $\sigma(r)$ the *mass* of the earthquake measure σ in the disk D, denoted by

$$mass_\sigma(D) = \sigma(r).$$

3. Equivalence of two norms

Let $Q = \{a, b, c, d\}$ be a quadruple consisting of four points a, b, c and d on the unit circle arranged in the counterclockwise direction. Given an orientation-preserving homeomorphism h on the unit circle $\mathbb{S}^1$, the *cross-ratio distortion norm* of h is

$$||h||_{cr} = \sup_{cr(Q)=1} |\ln cr(h(Q))|,$$

where

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} \text{ and } cr(h(Q)) = \frac{(h(b)-h(a))(h(d)-h(c))}{(h(c)-h(b))(h(d)-h(a))}.$$

It is easy to see that the cross-ratio distortion norm of h is invariant under pre- or post-composition by Möbius transformations. It is also true that h is quasisymmetric if and only if $||h||_{cr}$ is finite.

Let $(E, \mathcal{L})$ be an earthquake map representing h and σ be the induced the earthquake measure, and let ν denote the relative transveral shearing. In this section, we show that there exists a constant $C > 0$, independent of h and σ, such that

$$||h||_{cr} \leq C||\nu||.$$

Thus, by Proposition 2, we have

$$||h||_{cr} \leq C||\sigma||_{Th}.$$

Combining this result with Theorem 8 of [1], we have Theorem 1. In the following we first summarize some techniques into several lemmas. We use $\mathbb{D}$ to denote the unit open disk centered at the origin of the complex plane $\mathbb{C}$ and $\mathbb{H}$ to denote the upper half plane.

Lemma 2. *A quadruple Q has $cr(Q) = 1$ if and only if the geodesic $\overline{ac}$ from a to c is perperdicular to the geodesic $\overline{bd}$ from b to d, and if and only if the hyperbolic distance from $\overline{ab}$ to $\overline{cd}$ (or $\overline{bc}$ to $\overline{da}$) is equal to* $\ln(3 + 2\sqrt{2})$.

Proof. Let A be the Möbius transformation mapping $\mathbb{D}$ onto $\mathbb{H}$ such that $A(a) = -1, A(b) = 0$ and $A(c) = 1$. Since $cr(A(Q)) = cr(Q) = 1$, $A(d) = \infty$. Clearly, the geodesic from -1 to 1 is perpendicular to the geodesic from 0 to ∞, and then the geodesic $\overline{ac}$ is perpendicular to $\overline{bd}$.

Let β denote the geodesic in $\mathbb{D}$ which is perpendicular to both the geodesics $\overline{ab}$ to $\overline{cd}$. There exists a Möbius transformation B from $\mathbb{D} \to \mathbb{H}$ which maps β to the imaginary axis, and a to -1 and b to 1. Assume that $B(d) = -x$ and $B(c) = x$ with $x > 1$. Since $cr(B(Q)) = 1$, $x = 3 + 2\sqrt{2}$. And therefore, taking $\frac{|dz|}{y}$ as the hyperbolic metric in $\mathbb{H}$, the distance from $\overline{ab}$ to $\overline{cd}$ is $\ln(3 + 2\sqrt{2})$. By a symmetry argument, the hyperbolic distance from $\overline{bc}$ to $\overline{da}$ is equal to the same value. $\square$

In this paper, we let $C_0 = 2$, which is the smallest positive integer greater than $\ln(3 + 2\sqrt{2}) = 1.7627471\cdots$.

Lemma 3. *Consider in the hyperbolic plane $\mathbb{H}$. Let l_n denote the geodesic connecting $-e^{-n}$ to e^{-n} for each $n \in \{0\}\cup\mathbb{N}$, $\mathcal{L}$ the lamination consisting of l_n's, and G_{n+1} the gap between l_n and l_{n+1} and G_0 the remained gap. Suppose that an earthquake map E is defined as follows: $E|_{G_0}$ is the indentity map, for each $n \in \{0\} \cup \mathbb{N}$, the comparison map $E|_{G_n}^{-1} \circ E|_{G_{n+1}}$ is the hyperbolic Möbius transformation with axis l_n and hyperbolic translation length $\ln \lambda_n$, and $E|_{l_n} = E|_{G_{n+1}}$. Let h denote the extension of E to the boundary of $\mathbb{H}$, and Q the quadruple $\{1, \infty, -1, 0\}$. If there exists $\lambda \geq 1$ such that $\lambda_n \leq \lambda$ for each $n \in \{0\} \cup \mathbb{N}$, then there exists a constant $C_1 > 0$, independent of λ, such that*

$$0 \leq \ln cr(h(Q)) \leq C_1 \ln \lambda.$$

Proof. Denote by $A_0 = E|_{G_0}$, and $A_n = E|_{G_{n-1}}^{-1} \circ E|_{G_n}$ for each $n \in \mathbb{N}$. Clearly $h(Q) = \{1, \infty, -1, h(0)\}$, and

$$h(0) = \lim_{n\to\infty} A_1 \circ A_2 \circ \cdots \circ A_n(0).$$

Let x_n denote the point $-e^{-n}$ and y_n the point e^{-n} on the real axis for each $n \in \{0\} \cup \mathbb{N}$. Since the derivative of A_n decreases from λ_n to $\frac{1}{\lambda_n}$ on the interval $[x_n, y_n]$, by the mean value theorem,

$$y_n - A_n(0) = A_n(y_n) - A_n(0) \geq y_n\lambda_n^{-1} \geq e^{-n}\lambda^{-1}.$$

Then

$$\begin{aligned} y_{n-1} - A_{n-1} \circ A_n(0) &= A_{n-1}(y_{n-1}) - A_{n-1} \circ A_n(0) \geq (y_{n-1} - A_n(0))\lambda_{n-1}^{-1} \\ &\geq (y_{n-1} - y_n + y_n - A_n(0))\lambda^{-1} \geq (e^{-n+1} - e^{-n})\lambda^{-1} + e^{-n}\lambda^{-2}. \end{aligned}$$

Inductively,

$$\begin{aligned} &y_0 - A_0 \circ A_2 \circ \cdots \circ A_n(0) \\ &\geq (1 - e^{-1})\lambda^{-1} + (e^{-1} - e^{-2})\lambda^{-2} + \cdots + (e^{-n+1} - e^{-n})\lambda^{-n} + e^{-n}\lambda^{-n-1} \\ &= (e - 1)[e^{-1}\lambda^{-1} + e^{-2}\lambda^{-2} + \cdots + e^{-n}\lambda^{-n}] + e^{-n}\lambda^{-(n+1)}. \end{aligned}$$

As $n \to \infty$,

$$y_0 - h(0) \geq \frac{e - 1}{e\lambda - 1}.$$

Now we estimate $cr(h(Q))$. Clearly

$$cr(h(Q)) = cr(h(\{1, \infty, -1, 0\})) = cr(h(\{\infty, -1, 0, 1\}))^{-1} = \frac{h(0)+1}{1-h(0)}.$$

Therefore

$$1 \le cr(h(Q)) \le \frac{2 - \frac{e-1}{e\lambda-1}}{\frac{e-1}{e\lambda-1}} = \frac{2e\lambda - e - 1}{e-1}.$$

Let $\sigma = \ln\lambda$, then

$$0 \le \ln cr(h(Q)) \le \ln \frac{2e^{1+\sigma} - e - 1}{e-1}.$$

Let $\phi(\sigma) = \ln \frac{2e^{1+\sigma}-e-1}{e-1}$. Clearly $\phi'(\sigma) = \frac{1}{1-\frac{e+1}{2e^{1+\sigma}}}$. Then $1 \le \phi'(\sigma) \le \frac{2e}{e-1}$ for any $\sigma \in [0, +\infty)$. By the mean value theorem again,

$$\phi(\sigma) - \phi(0) \le \frac{2e}{e-1}\sigma.$$

Since $\phi(0) = 0$,

$$0 \le \ln cr(h(Q)) \le \phi(\sigma) \le C_1 \ln\lambda,$$

where $C_1 = \frac{2e}{e-1}$. □

The following two lemmas are technical tools needed to compare the norms of σ and h. They have been applied to prove several results in [1], for example, Theorem 8 there. See the corollaries 1 and 2 in [1] for their proofs.

Lemma 4 (GHL)**.** *Let $Q = \{a, b, c, d\}$ be a quadruple on the real line with $-\infty \le a < b < c < d$, and $c \le s \le d$ and $d < t$. Suppose that $A_{(s,t)}$ is the hyperbolic Möbius transformation with the repelling fixed point at s and the attracting fixed point at t and its derivative at the repelling fixed point equal to $\lambda > 1$, and $f_{(s,t)} : \mathbb{R} \to \mathbb{R}$ is defined to be equal to $A_{(s,t)}$ on the interval $[s, t]$ and equal to the identity on the complement of $[s, t]$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ considered as a function of two variables $s \in [y, z]$ and $t \in (z, +\infty)$ decreases in s for each fixed t and increases in t for each fixed s.*

Lemma 5 (GHL)**.** *With the same notations as in the previous corollary, suppose $b \le s \le c$ and $d \le t$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ is increasing in s for each fixed t and also increasing in t for each fixed s.*

Let h denote an orientation-preserving circle homeomorphism and $(E, \mathcal{L})$ an earthquake representation of h, and σ the induced earthquake measure by $(E, \mathcal{L})$.

Theorem 9. *There exists a universal constant $C > 0$, independent of h and σ, such that*

$$||h||_{cr} \le C||\sigma||_{Th}.$$

In fact, one can take $C = C_0 + 2C_1C_2$, where C_0 is the smallest positive integer greater than or equal to $\ln(3 + 2\sqrt{2})$, $C_1 = \frac{2e}{e-1}$, and C_2 is the smallest positive integer greater than or equal to $\ln(e + \sqrt{e^2-1})$.

We divide the proof into several cases. Given a quadruple $Q = \{a, b, c, d\}$ with $cr(Q) = 1$, we first assume that three points a, b and c belong to the same stratum A, and estimate $cr(h(Q))$ in this case. By a Möbius change of coordinates, we may assume that $a = 1, b = \infty, c = -1, d = 0$, and by postcomposing E with another

Möbius transformation, we may also assume that the earthquake E is the identity map on the stratum A. We will show there is a constant C such that

$$0 \leq \ln cr(h(Q)) \leq C||\sigma||_{Th}.$$

Let x_n denote the point $-e^{-n}$ and y_n the point e^{-n} on the real axis for each $n \in \{0\} \cup \mathbb{N}$. And let $\mathcal{L}'$ denote the collection of the lines in $\mathcal{L}$ which connect points of the interval $[-1, 0)$ to points of $(0, 1]$. Let $\mathcal{L}_0^-$ denote the lines in $\mathcal{L}'$ which connect points of $[x_0, x_1)$ to points of $(0, y_0]$, and $\mathcal{L}_0^+$ denote the lines in $\mathcal{L}'$ which connect points of $[x_0, 0)$ to points of $(y_1, y_0]$. Let $\mathcal{L}_0 = \mathcal{L}_0^- \cup \mathcal{L}_0^+$. Then any line in $\mathcal{L}' \setminus \mathcal{L}_0$ must connect a point in $[x_1, 0)$ to a point in $(0, y_1]$. Inductively, for each $n \in \mathbb{N}$, let $\mathcal{L}_n^-$ denote the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \mathcal{L}_{n-1})$ which connect points of $[x_n, x_{n+1})$ to points of $(0, y_n]$, and $\mathcal{L}_n^+$ the collection of the lines in $\mathcal{L}' \setminus (\mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \mathcal{L}_{n-1})$ which connect points of $[x_n, 0)$ to points of $(y_{n+1}, y_n]$, and $\mathcal{L}_n = \mathcal{L}_n^- \cup \mathcal{L}_n^+$. We have the following three lemmas.

Lemma 6. *For each $n \in \{0\} \cup \mathbb{N}$, any line in $\mathcal{L}_n$ must connect a point in $[x_n, 0)$ to a point in $(0, y_n]$.*

Proof. It can be easily proved by an induction on n. □

Lemma 7. *There exists a constant $C_2 > 0$, independent of h and σ, such that $\sigma(\mathcal{L}_n) \leq C_2||\sigma||_{Th}$ for any $n \in \{0\} \cup \mathbb{N}$.*

Proof. For each $n \in \{0\} \cup \mathbb{N}$, let l_n denote the geodesic line connecting the point x_n to the point y_n. And for any $n \in \mathbb{N}$, let l_n^- denote the geodesic connecting the point x_n to 0, and l_n^+ the geodesic connecting 0 to the point y_n. The hyperbolic distance from l_n to l_{n+1}^- (or l_{n+1}^+), $n \in \{0\} \cup \mathbb{N}$, is equal to a constant, which is equal to $\ln(e + \sqrt{e^2 - 1})$. Let C_2 denote the smallest positive integer which is greater than or equal to $\ln(e + \sqrt{e^2 - 1})$. Then $\sigma(\mathcal{L}_n) \leq C_2||\sigma||_{Th}$ for each $n \in \{0\} \cup \mathbb{N}$. □

Let $\tilde{E}$ be the same earthquake map as the one defined in Lemma 3 with $\ln \lambda_n = \sigma(\mathcal{L}_n)$.

Lemma 8. *One has the following inequality:*

$$cr(h(Q)) \leq cr(\tilde{E}(Q)).$$

Proof. For each $n \in \{0\} \cup \mathbb{N}$, let l'_n denote the geodesic line in $\mathcal{L}_n$ above all of the other geodesic lines in $\mathcal{L}_n$, let E_n be the earthquake which induces the earthquake measure $(\sigma|_{\mathcal{L}_n}, \mathcal{L}_n)$ and which is the identity on the stratum above the geodesic line l'_n. In the case when $\mathcal{L}_n$ is an empty collection, we let E_n be the identity map. Denote $h_n = E_0 \circ E_1 \circ \cdots \circ E_n$. Clearly

$$h(d) = h(0) = \lim_{n \to \infty} h_n(0).$$

We only need to show that for each $n \in \{0\} \cup \mathbb{N}$,

$$cr(h_n(Q)) \leq cr(A_0 \circ A_1 \circ \cdots \circ A_n(Q)),$$

where A_i's are the maps defined in the proof of Lemma 3. Now we compare $cr(A_n(Q))$ with $cr(E_n(Q))$. Let the geodesic lines l_i's be the same ones defined in Lemma 3. By Lemmas 4 and 6, if we move the weights of the geodesic lines in $\mathcal{L}_n$ to the geodesic line l_n, we only increase the cross ratio of the image of Q, that is, $cr(E_n(0)) \leq cr(A_n(Q))$. Therefore $E_n(0) \leq A_n(0)$. Since A_{n-1} is monotone increasing on the interval $[-e^n, e^n]$,

$$A_{n-1}(E_n(0)) \leq A_{n-1}(A_n(0)).$$

By Lemmas 4 and 6 again, we move the weights of the geodesic lines in $\mathcal{L}_{n-1}$ to the geodesic line l_{n-1}, we only increse the cross ratio of the image of the quadruple $\{1, \infty, -1, E_n(0)\}$, that is,

$$cr(E_{n-1}(\{1, \infty, -1, E_n(0)\})) \leq cr(A_{n-1}(\{1, \infty, -1, E_n(0)\})).$$

Hence

$$E_{n-1} \circ E_n(0) \leq A_{n-1} \circ E_n(0) \leq A_{n-1} \circ A_n(0).$$

Inductively, for each $0 \leq i \leq n$, we see that

$$E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0) \leq A_{i+1} \circ \cdots \circ A_{n-1} \circ A_n(0).$$

By the monotonicity of A_i on the interval $[-e^i, e^i]$, one has

$$A_i(E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0)) \leq A_i(A_{i+1} \circ \cdots \circ A_{n-1} \circ A_n(0)).$$

And by Lemmas 4 and 6 and moving the weights of the geodesic lines in $\mathcal{L}_i$ to the geodesic line l_i, we have

$$\begin{aligned} &cr(E_i(\{1, \infty, -1, E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0)\})) \\ &\leq cr(A_i(\{1, \infty, -1, E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0)\})), \end{aligned}$$

and therefore

$$\begin{aligned} E_i \circ E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0) &\leq A_i \circ E_{i+1} \circ \cdots \circ E_{n-1} \circ E_n(0) \\ &\leq A_i \circ A_{i+1} \circ \cdots \circ A_{n-1} \circ A_n(0). \end{aligned}$$

When $i = 0$, the above inequality implies

$$cr(h_n(Q)) \leq cr(A_0 \circ A_1 \circ \cdots \circ A_n(Q)).$$

By taking the limit as $n \to \infty$, we have

$$cr(h(Q)) \leq cr(\tilde{E}(Q)).$$

□

Lemmas 8, 3 and 7 imply the following proposition.

Proposition 3. *If $cr(Q) = 1$ and a, b, c belong to the same stratum of the earthquake represention $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq C_1 C_2 ||\sigma||_{Th}.$$

Proposition 4. *If $cr(Q) = 1$ and a, c belong to the same stratum of the earthquake represention $(E, \mathcal{L})$ of h, then*

$$0 \leq \ln cr(h(Q)) \leq 2C_1 C_2 ||\sigma||_{Th}.$$

Proof. Recall that we may assume that $a = \infty$, $b = -1$, $c = 0$, $d = 1$, and the earthquake E is the identity map on the stratum A containing the points a and c. By the proof of the previous proposition,

$$0 \leq \ln cr(\{a, b, c, h(d)\} \leq C_1 C_2 ||\sigma||_{Th}$$

and

$$0 \leq \ln cr(\{a, h(b), c, d\} \leq C_1 C_2 ||\sigma||_{Th}.$$

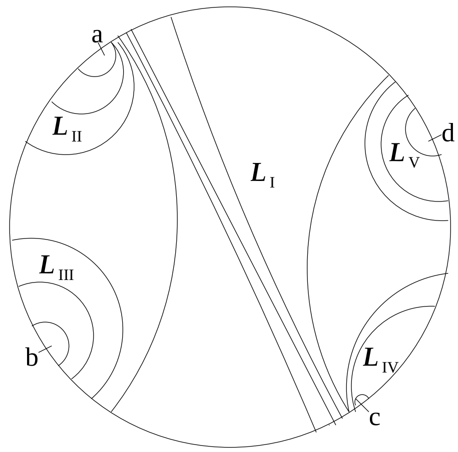

FIGURE 1. Five subcollections of $\mathcal{L}$ in the proof of Proposition 5.

And hence

$$1 \le h(d) \le e^{C_1C_2||\sigma||_{Th}} \text{ and } 1 \ge -h(b) \ge e^{-C_1C_2||\sigma||_{Th}}.$$

Therefore

$$1 \le cr(a, h(b), c, h(d)) = \frac{h(d)}{-h(b)} \le e^{2C_1C_2||\sigma||_{Th}},$$

which implies

$$0 \le \ln cr(h(Q)) \le 2C_1C_2||\sigma||_{Th}.$$

□

Proposition 5. *If $cr(Q) = 1$, and assume that there exists at least one geodesic line in the lamination $\mathcal{L}$ which separates the vertices a, b from the vertices c, d, then*

$$|\ln(cr(Q))| \le (C_0 + 2C_1C_2)||\sigma||_{Th}.$$

Proof. Given two points x and y on the unit circle, we use $[x, y]$ to denote the arc on $\mathbb{S}^1$ from x to y in counterclockwise direction. Let $\mathcal{L}_I$ denote the collection of the geodesic lines in $\mathcal{L}$ that connect points of the arc $[d, a]$ to points of the arc $[b, c]$. By Lemma 2, $\sigma(\mathcal{L}_I) \le C_0||\sigma||_{Th}$. Let $\mathcal{L}_{II}$ denote the collection of the lines in $\mathcal{L}$ that connect points of the arc (d, a) to points of the arc (a, b), $\mathcal{L}_{III}$ the collection of the lines in $\mathcal{L}$ that connect points of the arc (a, b) to points of the arc (b, c), $\mathcal{L}_{IV}$ the collection of the lines in $\mathcal{L}$ that connect points of the arc (b, c) to points of the arc (c, d), and finally $\mathcal{L}_V$ the collection of the lines in $\mathcal{L}$ that connect points of the arc (c, d) to points of the arc (d, a). First notice that the motion of a (resp. c) under the earthquake map E along the lines in $\mathcal{L}_{II}$ (resp. $\mathcal{L}_{IV}$) only decreases the cross ratio of Q. Therefore the cross ratio $cr(h(Q))$ is less than or equal to the cross ratio $cr(E'(Q))$, where E' is the new earthquake by omitting the earthquake motion along the geodesic lines in $\mathcal{L} \setminus (\mathcal{L}_I \cup \mathcal{L}_{III} \cup \mathcal{L}_V)$.

By a Möbius change of coordinates, we may assume $a = -\infty$, $b = -1$, $c = 0$, $d = 1$. Then the goedesic lines in $\mathcal{L}_I$ connect the points of $[-1, 0]$ to the points of $[1, +\infty]$. By Lemma 5, if one moves all the lines in $\mathcal{L}_I$ to the geodesic line from 0 to ∞ without changing the total amount of shearing along the lines in $\mathcal{L}_I$ to obtain a new earthquake E'', then the cross ratio of the image quadruple of Q under the earthquake is possibly increased, that is, $cr(E'(Q)) \leq cr(E''(Q))$. As usual, by postcomposing with a Möbius transformation, we may assume that E'' is the identity map on the geodesic line from 0 to ∞.

Let E_{III} (resp. E_V) denote the earthquake obtained by omitting the earthquake motion along all lines in $\mathcal{L} \setminus \mathcal{L}_{III}$ (resp. $\mathcal{L} \setminus \mathcal{L}_V$). Then by Proposition 3 and the same argument in the proof of Proposition 4, we have

$$1 \leq E_V(d) \leq e^{C_1C_2||\sigma||_{Th}} \text{ and } 1 \geq E_{III}(b) \geq e^{-C_1C_2||\sigma||_{Th}}.$$

Then

$$1 \leq E''(d) \leq e^{(C_1C_2+C_0)||\sigma||_{Th}} \text{ and } 1 \geq E''(b) \geq e^{-C_1C_2||\sigma||_{Th}}.$$

Therefore

$$cr(E(Q)) \leq cr(E'(Q)) \leq cr(E''(Q)) \leq e^{(2C_1C_2+C_0)||\sigma||_{Th}}.$$

Now we omit the motion of E along the geodesic lines in $\mathcal{L}_{III}$ and $\mathcal{L}_V$, and move all geodesic lines in $\mathcal{L}_I$ to the goedesic line $\overline{bd}$. By similar reasoning, we obtain

$$cr(E(Q)) \geq e^{-(2C_1C_2+C_0)||\sigma||_{Th}}.$$

This completes the proof of the proposition. □

Now the proof of Theorem 9 can be organized as follows.

Proof. Let $C = C_0 + 2C_1C_2$. We show that for any quadruple Q with $cr(Q) = 1$,

$$|\ln cr(h(Q))| \leq C||\sigma||_{Th}.$$

We divide the proof into three cases.

Case 1: The quadruple Q has three vertices belonging to the same stratum. Then either a, b, c or b, c, d or c, d, a or d, a, b belongs to the same stratum. By Proposition 3, either

$$0 \leq \ln cr(h(Q)) \leq C_1C_2||\sigma||_{Th}$$

or

$$0 \leq \ln cr(h(\{b, c, d, a\})) \leq C_1C_2||\sigma||_{Th}.$$

Clearly $cr(h(\{b, c, d, a\})) = \frac{1}{cr(h(Q))}$, and hence

$$|\ln cr(h(Q))| \leq C_1C_2||\sigma||_{Th} < C||\sigma||_{Th}.$$

Case 2: The quadruple Q has two opposite vertices belonging to the same stratum. Then either a and c or b and d belong to the same stratum. By Proposition 4 and the same reasoning in Case 1, we have

$$|\ln cr(h(Q))| \leq 2C_1C_2||\sigma||_{Th} < C||\sigma||_{Th}.$$

Case 3: The quadruple Q has no opposite vertices belonging to the same stratum. Then either there exists a geodesic line in $\mathcal{L}$ which separates a and b from c and d or there exists a geodesic line in $\mathcal{L}$ which separates b and c from d and a. By Proposition 5 and the same reasoning in Case 1, we have

$$|\ln cr(h(Q))| \leq (C_0 + 2C_1C_2)||\sigma||_{Th} = C||\sigma||_{Th}.$$

This completes the proof. □

Our Theorem 9 and Theorem 8 of [1] imply Theorem 1.

Now we prove Theorem 2.

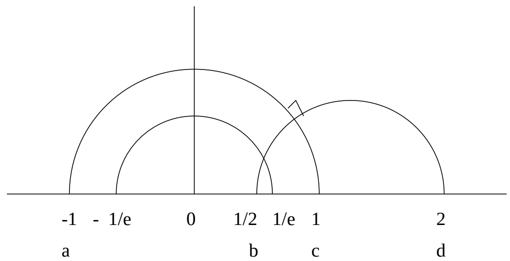

FIGURE 2. Construction 1 of the quadruple $\tilde{Q}$ in the proof of Theorem 2.

Proof. Let σ_h^l be the earthquake measure induced by a left earthquake representation E of h. Theorem 9 implies

$$||h||_{cr+} \leq C||\sigma_h^l||_{Th},$$

where C is the same constant in Theorem 9. It remains to show that there exists another constant $C' > 0$ such that $||\sigma_h^l||_{Th} \leq C'||h||_{cr+}$. The proof for this part is very similar to the proof of Theorem 8 in [1] with some extra work on the selection of four points a, b, c, d in order to have a quadruple $\tilde{Q}$ satisfying the conditions in the definition of $||h||_{cr+}$ (Definition 1). It is divided into two parts: (i) there exists a constant $C_3 > 0$ such that $||\sigma_h^l||_{Th} \leq C_3||h||_{cr+}$ if $||h||_{cr+} < \frac{1}{2}\ln\frac{1}{3}\frac{e+1}{e-1}$; (ii) there exists another constant $C_4 > 0$ such that $||\sigma_h^l||_{Th} \leq ||h||_{cr+} + C_4$. Clearly (ii) implies $||\sigma_h^l||_{Th} \leq (1 + \frac{C_4}{\frac{1}{2}\ln\frac{1}{3}\frac{e+1}{e-1}})||h||_{cr+}$ if $||h||_{cr+} \geq \frac{1}{2}\ln\frac{1}{3}\frac{e+1}{e-1}$. Therefore

$$||\sigma_h^l||_{Th} \leq \max\{C_3, 1 + \frac{C_4}{\frac{1}{2}\ln\frac{1}{3}\frac{e+1}{e-1}}\}||h||_{cr+}.$$

Let D be a closed disk in $\mathbb{D}$ of hyperbolic diameter 1, l_1 and l_2 be the lines in the lamination $\mathcal{L}_l$ of σ_h^l which have the maximal hyperbolic distance among the lines of $\mathcal{L}_l$ that intersect D. Let β denote the geodesic perpendicular to both l_1 and l_2 (in the case that l_1 and l_2 only share one endpoint, we will only require β to be perpendicular to l_1 with the hyperbolic length of the segment on β between l_1 and l_2 is smaller than or equal to $\frac{1}{2}$). Label the endpoints of β by x and y so that the arc from x to y going in the counterclockwise direction is no longer than the arc from y to x. By postcomposition by a Möbius transformation, we may assume that the earthquake map is the identity map on the geodesic line l_1. Let $A : \mathbb{D} \to \mathbb{H}$ be a Möbius transformation mapping x to 0, y to ∞, the arc (xy) to the positive half real line, and the geodesic l_1 to the geodesic connecting -1 to 1 and l_2 to a geodesic line connecting $-s$ to s with $\frac{1}{e} \leq s \leq 1$. Let $E' = A \circ E \circ A^{-1}$. If $a = -1$, $b = \frac{1}{2}$, $c = 1$ and $d = 2$, then $\{a, b, c, d\}$ forms a quadruple $\tilde{Q}$. Moreover, $A^{-1}(\tilde{Q})$ also forms a quadruple $\tilde{Q}$ on $\mathbb{S}^1$ for E. We now show that $cr(E'(\tilde{Q}))$ is bounded from below by a function $\phi(\lambda)$ satisfying $\phi(1) = 1$, $\phi'(1) > 0$ and $\phi(\lambda) \geq 1$ for any $\lambda \geq 1$, where $\lambda = e^{\sigma_h^l(D)}$.

Let E'' be the earthquake by omitting all earthquake shifts of E' along the lines in $A(\mathcal{L}_l)$ except those intersecting the disk $A(D)$. Note that

$$cr(E'(\tilde{Q})) \leq cr(E''(\tilde{Q})).$$

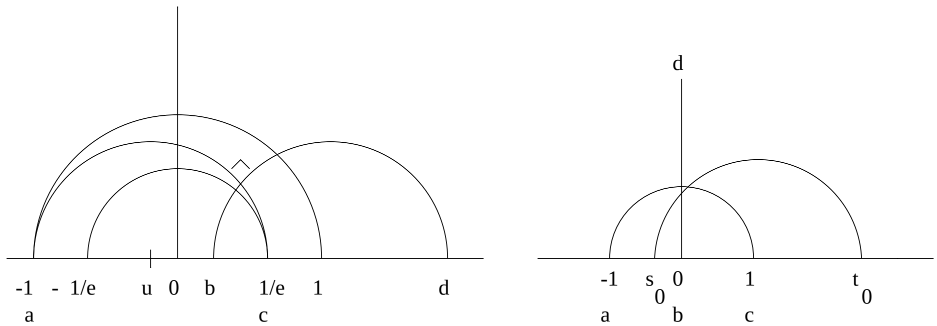

FIGURE 3. Construction 2 of the quadruple $\tilde{Q}$ in the proof of Theorem 2.

Let B be the hyperbolic transformation with axis equal to the geodesic connecting $-\frac{1}{e}$ to $\frac{1}{e}$, translation length equal to $\lambda = e^{\sigma^l_h(D)}$ and repelling fixed point at $-\frac{1}{e}$. Define $f : \mathbb{R} \to \mathbb{R}$ to be equal to B on the interval $[-\frac{1}{e}, \frac{1}{e}]$ and the indentity on the complement. Applying Lemma 4 to $cr(E''(\{c, d, a, b\}))$,

$$cr(E''(\tilde{Q})) = cr(E''(\{c, d, a, b\})) \geq cr(f(\{c, d, a, b\})).$$

Define $\phi(\lambda) = cr(f(\{c, d, a, b\}))$. By elementary calculation,

$$\phi(\lambda) = cr(f(\{c, d, a, b\})) = \frac{\lambda \frac{1-s}{t-1} t + s}{\lambda \frac{1-s}{t-1} + 1},$$

where $s = \frac{1}{3}\frac{e-1}{e+1} < 1$ and $t = \frac{1}{3}\frac{e+1}{e-1} > 1$. It is easy to see that $\phi(\lambda)$ increases from 1 to t as λ increases from 1 to $+\infty$. Let λ_0 be the positive real number (greater than 1) such that $\phi(\lambda_0) = \sqrt{\frac{1}{3}\frac{e+1}{e-1}}$, and C'_3 be the minimal value of the derivative of ϕ on $[1, \lambda_0]$. One can check that $C'_3 > 0$. By the mean value theorem,

$$\phi(\lambda) - 1 \geq C'_3(\lambda - 1)$$

for each $\lambda \in [1, \lambda_0]$. Now we have

$$e^{||h||_{cr+}} \geq cr(E'(\tilde{Q})) \geq cr(E''(\tilde{Q})) \geq cr(f(\tilde{Q})) = \phi(\lambda).$$

Therefore, if $||h||_{cr+} \leq \frac{1}{2} \ln \frac{1}{3}\frac{e+1}{e-1}$ then $e^{||h||_{cr+}} \leq \sqrt{\frac{1}{3}\frac{e+1}{e-1}}$, and hence

$$e^{||h||_{cr+}} \geq \phi(\lambda) \geq 1 + C'_3(\lambda - 1).$$

Then

$$\sigma^l_h(D) = \ln \lambda \leq \lambda - 1 \leq \frac{1}{C'_3}(e^{||h||_{cr+}} - 1).$$

Since $||h||_{cr+} \leq \frac{1}{2} \ln \frac{1}{3}\frac{e+1}{e-1}$, there exists a constant $C_3 > 0$ such that

$$\sigma^l_h(D) \leq C_3 ||h||_{cr+},$$

which implies (i).

To prove (ii), we construct a quadruple $\tilde{Q}$ for $E' = A \circ E \circ A^{-1}$ as follows. Let u be the midpoint between -1 and $\frac{1}{e}$ and b be the midpoint between u and $\frac{1}{e}$. Let $\overline{bd}$ be the geodesic perpendicular to the geodesic connecting -1 to $\frac{1}{e}$, where d is on the real line. Therefore $b = \frac{3-e}{4e}$ and $d = \frac{3+e}{2e}$. Notice that $d > 1$. Let $a = -1$ and

$c = \frac{1}{e}$. Then $\{a, b, c, d\}$ forms a quadruple $\tilde{Q}$ for E', and moreover $A^{-1}(\tilde{Q})$ also forms a quadruple $\tilde{Q}$ on $\mathbb{S}^1$ for E. Now we show that $cr(E'(\tilde{Q}))$ is bounded from below by a function $\xi(\lambda)$, where $\lambda = e^{\sigma_h^l(D)}$. Again let E'' be the earthquake by omitting the shifts along the goedesics in $A(\mathcal{L}_l)$ except those intersecting the disk $A(D)$. Notice that

$$cr(E'(\tilde{Q})) \leq cr(E''(\tilde{Q})).$$

Let F be the hyperbolic Möbius transformation with axis equal to the geodesic connecting $-\frac{1}{e}$ to 1, repelling fixed point at $-\frac{1}{e}$ and translation length λ. Define $g : \mathbb{R} \to \mathbb{R}$ to be equal to F on the interval $[-\frac{1}{e}, 1]$ and the identity on the complement. By moving the weights along the geodesics intersecting $A(D)$ to the geodesic connecting $-\frac{1}{e}$ to 1 and applying Lemma 5, we have

$$cr(E''(\tilde{Q})) \geq cr(g(\tilde{Q})).$$

In order to have the exact expression for $cr(g(\tilde{Q}))$, let G be the Möbius transformation which maps b to 0, d to ∞ and fixes a (and therefore it maps c to 1). Then $W = G \circ F \circ G^{-1}$ is the hyperbolic Möbius transformation with axis equal to the geodesic connecting s_0 to t_0, repelling fixed point at s_0 and translation length λ, where $s_0 = -cr(\{-\frac{1}{e}, b, c, d\})$ and $t_0 = cr(\{d, a, b, 1\})$. Notice that $-1 < s_0 < 0$ and $t_0 > 1$. Clearly

$$\frac{W(z) - s_0}{t_0 - W(z)} = \lambda \frac{z - s_0}{t_0 - z},$$

and hence

$$W(z) = \frac{(\lambda t_0 - s_0)z + (s_0 t_0 - \lambda s_0 t_0)}{(\lambda - 1)z + (t_0 - \lambda s_0)}.$$

Define $\xi(\lambda) = cr(g(\tilde{Q}))$. Therefore

$$\xi(\lambda) = cr(g(\tilde{Q})) = cr((G \circ g \circ G^{-1})(\{-1, 0, 1, \infty\})) = \frac{W(0) + 1}{W(1) - W(0)}.$$

Clearly

$$W(0) + 1 = \frac{s_0 t_0 - \lambda s_0 t_0 - \lambda s_0 + t_0}{t_0 - \lambda s_0}$$

and

$$W(1) - W(0) = \frac{\lambda (t_0 - s_0)^2}{(\lambda - 1 + t_0 - \lambda s_0)(t_0 - \lambda s_0)},$$

hence

$$\xi(\lambda) = \frac{[(1 - s_0)\lambda + (t_0 - 1)][-s_0(t_0 + 1)\lambda + (s_0 + 1)t_0]}{\lambda (t_0 - s_0)^2}.$$

Since the coefficients of λ and the constant terms in the above expression are positive, for any $\lambda \geq 1$,

$$\xi(\lambda) \geq C_4' \lambda,$$

where $C_4' = \frac{(1-s_0)(-s_0(t_0+1))}{(t_0-s_0)^2} > 0$. Therefore

$$e^{||h||_{cr+}} \geq cr(E'(\tilde{Q})) \geq cr(E''(\tilde{Q})) \geq \xi(\lambda) \geq C_4' \lambda,$$

and then

$$\sigma_h^l(D) = \ln \lambda \leq ||h||_{cr+} + C_4,$$

where $C_4 = -\ln C_4'$. This completes the proof of (ii).

The proof for $||h||_{cr-}$ and σ_h^r to be equivalent is similar. $\square$

REMARK. In fact, the proof of (ii) provides an alternative proof of (i), and hence a unified way to prove $||\sigma||_{Th} \leq C||h||_{cr}$ in Theorem 1 and $||\sigma_h^l||_{Th} \leq C||h||_{cr+}$ and $||\sigma_h^r||_{Th} \leq C||h||_{cr-}$ in Theorem 2. Through explicit calculations, one can find $s_0 = -\frac{7-e}{5+e}$ and $t_0 = \frac{5e-3}{3-e}$ in the part (ii) of the previous proof, and hence

$$(s_0+1)(t_0+1) < 2.$$

Therefore

$$\xi'(1) = \frac{2-(s_0+1)(t_0+1)}{t_0-s_0} > 0.$$

Applying the inverse function theorem and mean value theorem to $\xi(\lambda)$ in a small neighborhood of 1, there exist $\delta > 0$ and $C_5' > 0$ such that if $1 \leq \xi(\lambda) < 1+\delta$ then $\xi(\lambda) \geq 1+C_5'(\lambda-1)$. Therefore if $||h||_{cr+} < \ln(1+\delta)$ then $1 \leq \xi(\lambda) \leq e^{||h||_{cr+}} < 1+\delta$, and then

$$e^{||h||_{cr+}} \geq \xi(\lambda) \geq 1 + C_5'(\lambda - 1),$$

which implies that

$$\sigma_h^l(D) = \ln\lambda \leq \lambda - 1 \leq C_5||h||_{cr+}$$

for some constant $C_5 > 0$.

4. Smooth circle homeomorphisms

The purpose of this section is to prove Theorems 3, 6 and 7. The proof of Theorem 4 is similar to the proof of Theorem 3, and Theorem 5 is the consequence of Theorems 3 and 4.

Now we first show Theorem 3.

Proof. Let $(E, \mathcal{L})$ be a left earthquake representation of an orientation-preserving circle homeomorphism h and σ_h^l the earthquake measure induced by $(E, \mathcal{L})$. Clearly, Theorem 2 implies that h is smooth from above of order 0 if and only if σ_h^l is vanishing of order 0. It remains to prove the statement for other cases. We will give details to show that for each $\alpha > 0$, h is smooth from above of order α if and only if σ_h^l is vanishing of the same order. The proof for the equivalence between strongly smooth of h and strongly vanishing of σ_h^l is similar.

Let $\alpha > 0$. We first show that if σ_h^l is vanishing of order α then h is smooth from above of the same order.

Consider $\mathbb{D}$ as the hyperbolic plane. Let $\tilde{Q}$ be a quadruple consisting of four points a, b, c, d arranged on the unit circle $\mathbb{S}^1$ in the counterclockwise direction satisfying that $cr(\tilde{Q}) = 1$ and $|b-c|$ and $|d-c|$ are the two smallest lengths among $|b-a|, |c-b|, |d-c|$ and $|a-d|$.

Let β be the common perpendicular geodesic segment to the geodesics $\overline{ab}$ and $\overline{cd}$ and β' be the common perpendicular geodesic segement to $\overline{bc}$ and $\overline{da}$. Note that the maximal Euclidean distance from a point on β (or β') to the boundary circle is $O(s_{min}(\tilde{Q}))$.

In order to estimate the cross ratio $cr(h(\tilde{Q}))$ of the image quadruple $h(\tilde{Q})$, we need to consider the relative patterns between the lamination $\mathcal{L}$ and the quadruple $\tilde{Q}$. The following two cases are crucial. One case is that there is a geodesic in $\mathcal{L}$ which connects the arc (b, c) to the arc (d, a), and the other case is that there is a geodesic in $\mathcal{L}$ which connects the arc (c, d) to the arc (a, b). There may be no geodesic in $\mathcal{L}$ connecting (b, c) to (d, a) nor any geodesic in $\mathcal{L}$ connecting (c, d) to

(a, b), but this situation can be included in the first or second case. We give the details for the first case. The proof for the second case is similar.

In the first case, let $\mathcal{L}_I, \mathcal{L}_{II}, \mathcal{L}_{III}, \mathcal{L}_{IV}$ and $\mathcal{L}_V$ be the same as they are defined in the proof of Proposition 5. First notice again that the motion of a or b under the earthquake map E along the lines in $\mathcal{L}_{II}$ or $\mathcal{L}_{IV}$ only decreases the cross ratio of $\tilde{Q}$. Therefore the cross ratio $cr(h(\tilde{Q}))$ is less than or equal to the cross ratio $cr(E'(\tilde{Q}))$, where E' is the new earthquake obtained from E by omitting the earthquake motion along the geodesic lines in $\mathcal{L} \setminus (\mathcal{L}_I \cup \mathcal{L}_{III} \cup \mathcal{L}_V)$. Let E_{III} (resp. E_V) denote the earthquake obtained from E' by omitting the earthquake motion along the lines in $\mathcal{L} \setminus \mathcal{L}_{III}$ (resp. $\mathcal{L} \setminus \mathcal{L}_V$) and fixing c, d, a (resp. a, b, c).

By Lemma 2 and the fact that the maximal Euclidean distance from a point on β to the boundary circle is $O(s_{min}(\tilde{Q}))$,

$$\sigma(\mathcal{L}_I) = \sigma(\beta) = C_0 O(s_{min}(\tilde{Q})^\alpha).$$

Let p denote the intersection point between $\overline{ac}$ and $\overline{bd}$. Let $z_0 = p$ and take a sequence $\{z_n\}_{n=0}^\infty$ of the points on $\overline{pd}$ such that the hyperbolic distance between adjacent points z_n and z_{n+1} is 1 for each $n \in \{0\} \cup \mathbb{N}$. For each $n \in \{0\} \cup \mathbb{N}$, let $l_n = \overline{x_n y_n}$ be the geodesic perpendicular to $\overline{pd}$, and assume $x_0 = c$, x_n, $n \in \mathbb{N}$, lie on the arc (c, d) in counterclockwise direction, and $y_0 = a$, y_n, $n \in \mathbb{N}$, lie on the arc (d, a) in clockwise direction. Now divide the geodesic lines of $\mathcal{L}_V$ into the same groups $\mathcal{L}_n$, $n \in \{0\} \cup \mathbb{N}$, as we have done in the proof of Proposition 3. Then $\sigma(\mathcal{L}_n) = C_2 O(s_{min}(\tilde{Q})^\alpha)$ for each $n \in \{0\} \cup \mathbb{N}$. Therefore

$$0 \le \ln cr(\{a, b, c, E_V(d)\}) \le C_1 C_2 O(s_{min}(\tilde{Q})^\alpha).$$

Similarly, one can show

$$0 \le \ln cr(\{a, E_{III}(b), c, d\}) \le C_1 C_2 (s_{min}(\tilde{Q})^\alpha).$$

The same method used to prove Proposition 5 shows

$$\ln cr(h(Q)) \le (C_0 + 2C_1 C_2) O(s_{min}(\tilde{Q})^\alpha).$$

In the second case that there exists a geodesic line in $\mathcal{L}$ connecting the arc (c, d) to the arc (a, b), a similar method shows

$$\ln cr(h(Q)) \le (C_0 + 2C_1 C_2) O(s_{min}(\tilde{Q})^\alpha).$$

Together, we have shown that if σ_h^l is vanishing of order α then h is smooth from above of the same order.

It remains to show that if h is smooth from above of order α then σ_h^l is vanishing of the same order. The proof goes as the same as the proof of part (i) in the proof of Theorem 2 plus showing that $s_{min}(A^{-1}(\tilde{Q})) = O(\delta(D))$, which is indeed true. We omit the details here. $\square$

Now let Q denote a quadruple consisting of four points a, b, c, d arranged on $\mathbb{S}^1$ in counterclockwise direction with $cr(Q) = 1$. Let β (resp. β') be the common perpendicular geodesic segment to $\overline{ab}$ and $\overline{cd}$ (resp. $\overline{bc}$ and $\overline{da}$). Note that the maximal Euclidean distance from a point on β (or β') to the boundary circle is $O(s_{max}(Q))$ and the maximal Euclidean distance from a point on the geodesic line $\overline{ac}$ (or $\overline{bd}$) to the boundary circle is also $O(s_{max}(Q))$. Applying the above method, which is used to prove that for each $\alpha > 0$ if σ_h^l is vanishing of order α then h is smooth from above of the same order, one can also show that for each $\alpha > 0$ if σ_h^l is vanishing of order α then

$$|\ln cr(h(Q))| = O(s_{max}(Q)^\alpha)$$

for all quadruples $Q = \{a, b, c, d\}$ with $cr(Q) = 1$. This is Theorem 6.

Similarly one can prove Corollary 3 by using the quadruple Q with $cr(Q) = \frac{1}{3}$.

In the final part of this section, we show Theorem 7. We first introduce a proposition proved by Sullivan in [7].

Proposition 6 ([7]). *For each $0 \leq \alpha < 1$, a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is Sullivan smooth of order α if and only if h is a diffeomorphism and $\phi = \ln h'$ is α-Hölder continuous. Furthermore, h is strongly Sullivan smooth of order α if and only if h is a diffeomorphism and the α-Hölder constant of $\phi = \ln h'$ is a little o of the scale.*

The following lemma is due to a similar result in [2].

Lemma 9. *Let $0 \leq \alpha < 1$ and suppose that $h : \mathbb{R} \to \mathbb{R}$ is Sullivan smooth of order α. Let $C > 0$ and suppose that a quadruple Q of four points $a < b < c < d$ on the real line satisfies $cr(Q) = 1$, $|d-a| < C$, and $|b-c|$ and $|c-d|$ are the two smallest lengths among $|a-b|$, $|b-c|$ and $|d-c|$. Then*

$$|\ln cr(h(Q))| = O(s_{min}(Q)^\alpha),$$

where the constant O depends on C and the α-Hölder constant of ϕ. Furthermore, if h is strongly Sullivan smooth of order α then

$$|\ln cr(h(Q))| = o(s_{min}(Q)^\alpha).$$

Proof. Let $L = |b-a|, M = |c-b|, R = |d-c|$ and $T = |d-a|$, and $h(L) = |h(b)-h(a)|, h(M) = |h(c)-h(b)|, h(R) = |h(d)-h(c)|$ and $h(T) = |h(d)-h(a)|$. Since $cr(Q) = \frac{LR}{MT} = 1$, $\frac{L}{M} = \frac{T}{R} > 1$ and $\frac{R}{M} = \frac{T}{L} > 1$. Furthermore, $\frac{R}{M} = \frac{T}{L} = \frac{L+M+R}{L} \leq 3$ since M and R are the two smallest ones among L, M and R. Therefore, $1 < \frac{R}{M} \leq 3$.

We will give the details to show the first half of the lemma, the proof for the second half only needs a slight modification. We also assume $\alpha > 0$. When $\alpha = 0$, the proof is in fact quite easy.

Applying Proposition 6 and the mean value theorem,

$$|\ln \frac{h(R)}{h(M)} / \frac{R}{M}| = O(M^\alpha).$$

Now we estimate $\ln cr(h(Q))$. Rewrite $\ln cr(h(Q))$ as

$$\ln cr(h(Q)) = \ln \frac{cr(h(Q)}{cr(Q)} = \ln \frac{\frac{h(L)h(R)}{h(M)h(T)}}{\frac{LR}{MT}} = \ln \frac{\frac{h(L)}{h(T)}}{\frac{L}{T}} + \ln \frac{\frac{h(R)}{h(M)}}{\frac{R}{M}}.$$

It remains to show that $\ln \frac{h(L)}{h(T)} / \frac{L}{T}$ is a big O of M^α. If $\frac{M}{L} \geq \delta$ for a positive constant δ, then L is commensurable to M and R, and hence Proposition 6 and the mean value theorem imply

$$|\ln \frac{h(L)}{h(T)} / \frac{L}{T}| = O(M^\alpha),$$

where the constant O depends on δ and the α-Hölder constant of ϕ. When $\frac{M}{L}$ is sufficiently small, rewrite

$$\ln \frac{\frac{h(L)}{h(T)}}{\frac{L}{T}} = \ln \frac{\frac{T}{L}}{\frac{h(T)}{h(L)}} = \ln \frac{1+\frac{M}{L}+\frac{R}{L}}{1+\frac{h(M)}{h(L)}+\frac{h(R)}{h(L)}}.$$

Clearly $\frac{h(M)}{h(L)} \leq \frac{M}{L} e^{O(L^\alpha)}$ and $L \leq C$. If $\frac{M}{L}$ is sufficiently small, then $\frac{h(M)}{h(L)}$ is also sufficiently small. One can see that $\frac{R}{L}$ and $\frac{h(R)}{h(L)}$ are also sufficiently small. Then

$$|\ln \frac{\frac{h(L)}{h(T)}}{\frac{L}{T}}| = O([\frac{M}{L}-\frac{h(M)}{h(L)}]+[\frac{R}{L}-\frac{h(R)}{h(L)}]) = O(\frac{M}{L}[1-\frac{h(M)}{h(L)}/\frac{M}{L}]+\frac{R}{L}[1-\frac{h(R)}{h(L)}/\frac{R}{L}])$$

$$= O(\frac{M}{L}O(L^\alpha)+\frac{R}{L}O(L^\alpha)) = O((\frac{M}{L})^{1-\alpha}M^\alpha+(\frac{R}{L})^{1-\alpha}R^\alpha) = O((s_{min}(Q))^\alpha).$$

It completes the proof. □

Now we show Theorem 7.

Proof. Let $\mathbb{S}^1 = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$ and $U_1 = \mathbb{S}^1 \setminus \{1\}$, $U_2 = \mathbb{S}^1 \setminus \{i\}$, $U_3 = \mathbb{S}^1 \setminus \{-1\}$ and $U_4 = \mathbb{S}^1 \setminus \{-i\}$. For each $1 \leq n \leq 4$, let $x_n = \mathbb{S}^1 \setminus U_n$ and define

$$\Phi_n : U_n \to \mathbb{R} : z \mapsto \frac{z+x_n}{i(z-x_n)}.$$

Consider $U_n, n = 1,2,3,4$, as a open cover of $\mathbb{S}^1$ and $\Phi_n, n = 1,2,3,4$, as coordinate charts.

Given a quadruple Q of four points a, b, c, d arranged on the circle in counterclockwise direction, without loss of generality, we assume $s_{min}Q$ is small enough and $|b-c|$ and $|c-d|$ are the two smallest lengths among $|b-a|$, $|c-b|$, $|d-c|$ and $|a-d|$. There exists U_k for some $1 \leq k \leq 4$ such that $\Phi_k(Q)$ is contained in an interval centered at the origin with length no longer than a constant C, where C only depends on the coordinate charts. Therefore $s_{max}(\Phi_k(Q)) \leq C$ and $s_{min}(\Phi_k(Q))$ is commensurable with $s_{min}(Q)$.

Applying Lemma 9 to the composition $\tilde{h} = \Phi_j \circ h \circ \Phi_k^{-1}$, we have

$$|\ln cr(h(Q))| = |\ln \tilde{h}(\Phi_k(Q))| = O((s_{min}(\Phi_k(Q)))^\alpha) = O((s_{min}(Q))^\alpha).$$

This completes the proof. □

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