CONFORMAL AND HARMONIC MEASURES ON LAMINATIONS ASSOCIATED WITH RATIONAL MAPS

VADIM A. KAIMANOVICH AND MIKHAIL LYUBICH

Dedicated to Dennis Sullivan on the occasion of his 60th birthday

Abstract. In this work we continue the exploration of affine and hyperbolic laminations associated with rational maps, which were introduced in [LM97]. Our main goal is to construct natural geometric measures on these laminations: transverse conformal measures on the affine laminations and harmonic measures on the hyperbolic laminations. The exponent δ of the transverse conformal measure does not exceed 2, and is related to the eigenvalue of the harmonic measure by the formula $\lambda = \delta(\delta - 2)$. In the course of the construction we introduce a number of geometric objects on the laminations: the basic cohomology class of an affine lamination (an obstruction to flatness), leafwise and transverse conformal streams, the backward and forward Poincaré series and the associated critical exponents. We discuss their relations to the Busemann and the Anosov–Sinai cocycles, the curvature form, currents and transverse invariant measures, λ-harmonic functions, Patterson–Sullivan and Margulis measures, etc. We also prove that the dynamical laminations in question are never flat except for several explicit special cases (rational functions with parabolic Thurston orbifold).

Contents

Date: May 30, 2001.

Stony Brook IMS Preprint #2001/5
May 2001

0. Introduction

The field of holomorphic dynamics consists of at least three closely related branches:

- Iteration theory of rational endomorphisms of the Riemann sphere;
- The theory of Kleinian groups;
- The theory of holomorphic foliations.

The construction of Lyubich and Minsky [LM97] brings these three branches together: its input is a rational endomorphism, and the output is a hyperbolic lamination analogous to the hyperbolic manifold of a Kleinian group (or rather to the unit tangent bundle of that manifold).

The modern theory of Kleinian groups is intimately related with the 3-dimensional hyperbolic geometry which provides many deep insights and powerful tools in both ways (see Mostow [Mo68], Thurston [Th91], Minsky [Mi99], etc.). This relation is based on Poincaré's observation that a Kleinian group G can be extended to a discrete group of isometries of the hyperbolic space $\mathbf{H}^3$. The quotient $M = \mathbf{H}^3/G$ is a 3-dimensional hyperbolic manifold (or rather orbifold) whose topology and geometry reflect the combinatorial and geometric properties of G.

Sullivan's dictionary between the first two branches of holomorphic dynamics (see [S85]) made it natural to wonder whether there exists an analogous object associated with a rational endomorphism $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree $d > 1$. Such an object, a hyperbolic 3-dimensional (orbifold) lamination $\mathcal{M}_f$, was constructed in [LM97]. The *hyperbolization* (a functorial passage from dimension 2 to dimension 3) in this construction is based on an idea different from that of the "Poincaré hyperbolization" and consisting in the observation that a natural one-dimensional fiber bundle over an arbitrary affine Riemann surface (whose fibers consist of all conformal metrics on the tangent space to a given point) carries a canonical hyperbolic metric. Thus, one produces first an *affine Riemann surface lamination* $\mathcal{A}_f$ whose leaves are isomorphic to $\mathbb{C}$, then a *hyperbolic 3-lamination* $\mathcal{H}_f$ by applying to $\mathcal{A}_f$ the hyperbolization functor, and then finally one obtains the *quotient* *hyperbolic lamination* $\mathcal{M}_f$ by factorizing $\mathcal{H}_f$ with respect to the action of the automorphism $\widehat{f} : \mathcal{H}_f \hookleftarrow$ (which is a natural lift of f).

Any Riemannian manifold is endowed with the associated volume. It is not the case for a (leafwise) Riemannian lamination: leafwise volumes can be organized into a global measure only in the presence of a holonomy invariant transverse measure. To handle this problem, L. Garnett [Ga83] introduced the notion of a *harmonic measure* on a Riemannian foliation, which can play the role of the Riemannian volume on a manifold. She showed that for foliations of compact manifolds such a measure always exists. Actually, the results of Garnett can be placed into a more general context of the theory of Markov chains. In these terms Garnett's harmonic measures are interpreted as stationary measures of the leafwise Brownian motion, and their existence follows from existence of a stationary measure for an arbitrary Markov chain with a compact state space and weak* continuous transition probabilities.

However, the lamination $\mathcal{M}_f$ associated with a rational map is usually non-compact and it is *a priori* not clear whether there exists a harmonic measure on this lamination. Our goal is to construct such a measure ω assuming that the lamination $\mathcal{M}_f$ is locally compact. [More precisely, the measure which we construct is not harmonic but rather λ-*harmonic*, i.e., is a λ-eigenmeasure of the leafwise Laplacian.] The measure ω is in fact a very special λ-harmonic measure satisfying the property that its Radon-Nikodym cocycle is equal to $\exp[\delta_{\mathrm{cr}}\beta]$, where β is the *Busemann cocycle* on $\mathcal{M}_f$, and δ_{cr} is the *critical exponent* of an appropriately defined *Poincaré series*. The eigenvalue λ and the critical exponent are related by the formula $\lambda = \delta_{\mathrm{cr}}(\delta_{\mathrm{cr}} - 2)$ coinciding with the familiar relation between the bottom eigenvalue of the Laplacian and the critical exponent in the theory of Kleinian groups (see [Pa76], [S87]).

The harmonic measure ω is constructed by integrating the leafwise hyperbolic volume with respect to a special transverse measure. The latter comes as a natural lift of an $\widehat{f}$-invariant *parallel transverse conformal stream* on the affine lamination $\mathcal{A}_f$. We define it as a family of transverse measures parameterized by leafwise conformal metrics and which are transformed in a natural geometric way under the map $\widehat{f}$ and under the holonomies on $\mathcal{A}_f$. The former is controlled by the leafwise differential of f, while the latter are controlled by the *basic cocycle* β_σ on $\mathcal{A}_f$.

This cocycle is a very interesting object on its own right. To define it, one should make a choice of a leafwise conformal Riemannian metric on $\mathcal{A}_f$, i.e., of a section $\sigma : \mathcal{A}_f \to \mathcal{H}_f$ of the bundle $\mathcal{H}_f \to \mathcal{A}_f$. A change of the metric leads to replacing β_σ with a cohomologous cocycle. Thus we obtain a well-defined *basic class* $\mathsf{b} \in H^1(\mathcal{A}_f)$. This class vanishes if and only if $\mathcal{A}_f$ is Euclidean. We prove that it happens if and only if f is a very special function with "parabolic Thurston orbifold" (such a function is equal, up to a Möbius conjugacy, to $z \mapsto z^d$, a Chebyshev polynomial, or a Lattès example).

Remark. The first example of a non-Euclidean affine foliation was given by E. Ghys [Gh97] (see also [Gh99] and §**5.6**). This phenomenon is quite different from the situation with compact 2-dimensional hyperbolic laminations, which can always be uniformized (see Verjovsky [Ve87] and Candel [Ca93]). On the other hand, our result suggests that a "generic" affine lamination is not Euclidean.

Let us now outline the structure of the paper. We start §**1** with a discussion of geometric structures on conformal manifolds: connection between affine and hyperbolic structures, Busemann and basic cocycles, and their relation to the curvature form. Then we carry over this discussion to the level of laminations. It leads us to the hyperbolization functor and to the basic cohomology class of an affine lamination.

In §**2** we discuss measure-theoretic structures on laminations: measures, currents, conformal streams (leafwise and transverse), the Brownian motion and the interplay between them. In particular, we note that paring a transverse and a leafwise streams of the same dimension δ on an affine lamination $\mathcal{A}$ gives us a global "conformal Gibbs measure" on $\mathcal{A}$. This measure can be further lifted to an invariant, with respect to the "vertical flow", measure on the corresponding hyperbolic lamination $\mathcal{H}$ analogous to the Margulis measure for the geodesic flows (compare Sullivan [S79], Kaimanovich [Ka90], Bedford–Fisher–Urbansky [BFU]).

In our situation the Margulis measure is usually singular on the leaves (in particular, this is always the case when $\delta < 2$). In order to produce a leafwise absolutely continuous measure, we first lift the transverse δ-stream of $\mathcal{A}$ to the hyperbolic lamination $\mathcal{H}$. This yields a transverse measure on the latter whose Jacobian under the holonomy ($\equiv$ the Radon–Nikodym cocycle) is equal to $\exp[\delta\beta]$, where β is the Busemann cocycle on $\mathcal{H}$. The result of the integration of this measure with respect to the leafwise hyperbolic volume is a λ-harmonic measure on $\mathcal{H}$ with $\lambda = \delta(\delta - 2)$.

We begin the next section, §**3**, with a discussion of laminations $\mathcal{A}_f$ and $\mathcal{H}_f$ associated with rational maps. First, we recap the construction of [LM97] of these laminations, and prove some useful properties of them. Then we construct a special leafwise Riemannian metric on $\mathcal{A}_f$ which is locally uniformly contracted by the backward iterates of f. This property allows us to express the basic cocycle by a dynamical formula as the "distortion" of the metric along infinite backward orbits. This relates the geometric basic cocycle to the dynamical cocycle considered by Anosov–Sinai [AS67], Ledrappier [Le81] and others.

An important feature of the lamination $\mathcal{A}_f$ is that it is endowed with a special class of transversals (we call them *dual fibers*), which makes it similar to a product lamination. By using the dual fibration we introduce the *dual basic cocycle* which measures the distortion of the Riemannian metric along the forward orbits. We finish §**3** with the above mentioned theorem characterizing the rational maps f for which the affine lamination $\mathcal{A}_f$ is Euclidean.

We begin §**4** with a discussion of the *balanced measure* κ for f constructed in [Br65], [Ly93], and its lift $\widehat{\kappa}$ to $\mathcal{A}_f$. The conditional measures of $\widehat{\kappa}$ give rise to a transverse holonomy invariant measure m on $\mathcal{A}_f$ (compare [Su97]). Under the action of $\widehat{f}$ this measure transforms as $\widehat{f}m = d \cdot m$. We prove (§**4.2**), using the results of [Br65], [Ly93], that the leaves of $\mathcal{A}_f$ are asymptotically equidistributed with respect to this measure (compare with the results of Bedford–Smillie [BS91] and Fornaess–Siboni [FS92] for polynomial automorphisms of $\mathbb{C}^2$). The transverse invariant measure m will play an auxiliary role of the "counting measure" on the transversals.

Then we pass to the main constructions of the paper. In §**4.3** we introduce the *backward Poincaré series* and the associated *critical exponent* $\delta_{\rm cr}$ for $\mathcal{A}_f$. By comparing the

Poincaré series at exponent 2 with the leafwise area on $\mathcal{A}_f$ (using the equidistribution of leaves), we prove that $\delta_{\text{cr}} \le 2$. Then (§**4.4**) we construct an $\widehat{f}$-invariant parallel transverse δ_{cr}-conformal stream μ on $\mathcal{A}_f$. To this end we generalize the Patterson method to the lamination context by replacing points with transversals and replacing the counting measure on the group orbits with the transverse invariant measure. Finally, we lift μ to an $\widehat{f}$-invariant λ-harmonic measure $\omega = \omega^\mu$ on the 3-dimensional hyperbolic lamination $\mathcal{H}_f$, and then push it down onto the quotient $\mathcal{M}_f = \mathcal{H}_f/\widehat{f}$.

This construction can be also realized directly in terms of "global" measures on the hyperbolic lamination $\mathcal{H}_f$. Namely, the balanced transverse measure m gives rise to a measure θ on $\mathcal{H}_f$ whose Radon–Nikodym cocycle (with respect to the leafwise Riemannian volume) is identically one, but $\widehat{f}\theta = d \cdot \theta$. The Patterson construction applied to the measure θ allows us to obtain a measure ω on $\mathcal{H}_f$ which is $\widehat{f}$-invariant, but on the contrary its Radon–Nikodym cocycle is $\exp[\delta\beta]$, where β is the leafwise Busemann cocycle on $\mathcal{H}_f$ (there are no measures on $\mathcal{H}_f$ which would simultaneously have a trivial Radon–Nikodym cocycle and be $\widehat{f}$-invariant).

In §**4.5** we introduce the *forward Poincaré exponent* of $\mathcal{A}_f$ and the corresponding leafwise conformal stream. One way to construct such a stream is to lift the Sullivan conformal measure on the Julia set $J(f)$ [S83] to the lamination $\mathcal{A}_f$, but we can also go the other way round and construct intrinsically the desired objects on the laminations. In the convex co-compact case (when f has only non-recurrent critical points on the Julia set and does not have parabolic points) the backward and the forward critical exponents coincide, and the product of the transverse and leafwise streams yields an invariant "conformal Gibbs measure" on $\mathcal{A}_f$. By pushing this measure down to the Julia set $J(f)$ we obtain, in a new way, the conformal Gibbs measure on $J(f)$, which was first constructed by Denker and Urbansky [DU91b], [U94]. The point of our approach is that upstairs (on the lamination) the critical points disappear and the dynamics become hyperbolic (at some expense though, as orbifold singular points appear on the leaves).

Note that our method of construction of conformal Gibbs measures by paring the transverse and leafwise conformal streams can also be used in the general context of the Gibbs theory (in the absence of conformal structure) for constructing measures with prescribed Radon–Nikodym derivatives. In this way we can construct the classical SRB measures on hyperbolic attractors (see §**2.1.3**), as well as on strong unstable foliations of partially hyperbolic systems. Since our method based on the Patterson averaging procedure does not need Markov partitions and requires only "soft hyperbolicity", it can potentially be applied in a broad range of situations.

At the end of §**4** we give a list of problems motivated by our results which continues the list given in [LM97].

In order to illustrate the tight link of our results and methods to the setting of Kleinian groups, we include an Appendix (§**5**) with a description of laminations associated with Kleinian groups. We first observe that there is a natural $\mathbb{C}$-fibration $\mathring{\mathcal{A}} \cong \overline{\mathbb{C}} \times \overline{\mathbb{C}} \setminus \text{diag}$ over $\overline{\mathbb{C}}$. Then for a Kleinian group G the unit tangent bundle $\mathring{\mathcal{M}}_G \equiv UM$ over the associated quotient 3-manifold $M = \mathbf{H}^3/G$ (i.e., the phase space of the geodesic flow on

M) is obtained by applying the hyperbolization functor to $\mathring{\mathcal{A}}$ and then factorizing it by the action of G.

By restricting the transversal $\overline{\mathbb{C}}$ of $\mathring{\mathcal{A}}$ to the limit set $\Lambda(G)$, we obtain a G-invariant affine lamination $\mathcal{A}_G = \overline{\mathbb{C}} \times \Lambda(G) \setminus \text{diag}$. Let $\mathcal{M}_G \subset \mathring{\mathcal{M}}_G$ be the hyperbolization of $\mathcal{A}_G$ modulo the group action. The Patterson–Sullivan measure on the limit set $\Lambda(G)$ gives rise both to a leafwise and a transverse conformal streams on $\mathcal{A}_G$. The latter then determines a λ-harmonic measure on $\mathcal{M}_G$, where λ is the bottom eigenvalue of the Laplacian on M. This measure is precisely analogous to the λ-harmonic measure ω we have constructed on $\mathcal{M}_f$. The product of the leafwise and transverse conformal streams is a geodesic current. We show equivalence of this and several other constructions of the geodesic current determined by the Patterson–Sullivan measure. We also give a new direct construction of the associated invariant measure of the geodesic flow as a local product of conformal measures on strongly stable and strongly unstable horospheres.

Correspondences between various objects associated with affine laminations determined by Kleinian groups and rational maps are listed in §**5.5**.

Finally, in §**5.6** we prove (generalizing an earlier result of Ghys [Gh97]) that some affine foliations associated with Kleinian groups are non-Euclidean and establish a link between their parallel transverse conformal streams and invariant measures of the horosphere foliation on the homology cover of a compact hyperbolic manifold constructed by Babillot and Ledrappier [BL98].

The main results of this paper were presented at the Texas Geometry and Topology Conference in Rice (November 1997), at the AMS meeting in Davis (April 1998, [KL98]), at the Meeting on Complex Analysis in Dynamical Systems in Rio-de Janeiro (September 1998), and at many other meetings since then.

Acknowledgement. The authors thank Etienne Ghys, John Milnor, Yair Minsky, and Anatoly Vershik for useful comments. Parts of this work were done during the authors' stays at IHES (1996), University of Manchester (1996–2000), UNAM at Cuernavaca (1997), as well as at one of the authors' home Universities. These visits were partially supported by the Rosenbaum Foundation, NSF, and the Clay Institute. We thank all of the above Institutes and Foundations for their hospitality and support.

0.1. **The list of notations.** The end of a proof is denoted by the symbol $\square$. Occasionally we split proofs into separate assertions or steps, in which case the end of the proof of each of them is denoted by the symbol $\triangle$. We also use the symbol $\triangle$ to denote the end of a definition.

- $\mathbb{N} = \{0, 1, 2, \dots\}$ — set of natural numbers,
  $\mathbb{Z}$ — set of integers,
  $\mathbb{R}$ — set of reals,
  $\mathbb{C}$ — complex plane,
  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ — Riemann sphere,
  ς — spherical metric on $\overline{\mathbb{C}}$,
  spher — spherical area form on $\overline{\mathbb{C}}$,
  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ — punctured complex plane,
  $\mathbb{D}(z, r) = \{\zeta \in \mathbb{C} : |z - \zeta| < r\}$, $\mathbb{D}_r \equiv \mathbb{D}_r(0)$, $\mathbb{D} \equiv \mathbb{D}_1$,

$U \Subset V$ — U is *compactly contained in* V, i.e., there exists a compact set K such that $U \subset K \subset V$,
$\langle f, m\rangle$ — integral of function f with respect to measure m.

§1.1.1 $\mathbf{H}^3$ — hyperbolic 3-space,
$\partial\mathbf{H}^3$ — sphere at infinity (visibility sphere),
$q \in \partial\mathbf{H}^3$ — point at infinity,
$(\mathbf{H}^3, q)$ — pointed at infinity hyperbolic space,
dist — hyperbolic metric on $\mathbf{H}^3$,
β_q — Busemann cocycle on $(\mathbf{H}^3, q)$,
$\omega_q^{\uparrow}$ — vertical form,
$v_q^{\uparrow}$ — vertical vector field,
$\xi_q = \{\xi_q^{\tau}\}_{\tau\in\mathbb{R}}$ — vertical flow,
$\mathrm{Hor}_q(h)$ — horosphere centered at $q \in \partial\mathbf{H}^3$ and passing through $h \in \mathbf{H}^3$,
$\mathrm{Hor}(\mathbf{H}^3, q)$ — space of horospheres centered at q,
$\mathrm{Hor}(\mathbf{H}^3)$ — space of all horospheres in $\mathbf{H}^3$,
$\Upsilon_\infty \in \partial\mathbf{H}^3$ — center of a horosphere $\Upsilon \in \mathrm{Hor}(\mathbf{H}^3)$,
$\mathcal{P}_q = \partial\mathbf{H}^3 \setminus \{q\}$ — punctured visibility sphere,
$\xi_q(h)$ — vertical geodesic passing through h,
$\mathfrak{p}_q : \mathbf{H}^3 \mapsto \mathcal{P}_q$ — the map assigning to h the limit point of the geodesic $\xi_q(h)$.

§1.1.2 $\mathbf{H}^3 = \{(x, y, t) : (x, y) \cong z \in \mathbb{R}^2 \cong \mathbb{C},\ t = e^s \in \mathbb{R}_+\}$ — the upper half-space model,
vol — hyperbolic volume form,
vol — hyperbolic volume measure,
$\Delta_{\mathbf{H}}$ — hyperbolic Laplace–Beltrami operator,
$\mathbf{E}^2 \cong \mathbb{C}$ — Euclidean plane,
eucl — Euclidean area form,
ℓ — Lebesgue measure,
$\Delta_{\mathbf{E}}$ — Euclidean Laplacian,

§1.1.3 ξ_M — vertical flow on a pointed at infinity hyperbolic orbifold M,
$v_M^{\uparrow}$ — vertical vector field,
$\omega_M^{\uparrow}$ — vertical form,
$\mathsf{B} = \mathsf{B}(M) = [\omega_M^{\uparrow}] \in H^1(M)$ — Busemann class of M.

§1.1.4 $\mathfrak{p} : \mathfrak{H}M \to M$ — scaling bundle over a conformal orbifold M,
ε_h — conformal Euclidean metric on $T_{\mathfrak{p}h}M$,
ξ — scaling flow on $\mathfrak{H}M$,
$\beta(h_1, h_2) = \log \frac{\varepsilon_{h_1}}{\varepsilon_{h_2}}$ — Busemann cocycle on fibers of $\mathfrak{p}$,
ℓ_z, $z \in M$ — Lebesgue measures on fibers of $\mathfrak{p}$,
$\widetilde{\lambda}$ — lift of a measure λ from M to $\mathfrak{H}M$,
$\widetilde{\omega}$ — lift of a differential form ω from M to $\mathfrak{H}M$,
ρ_σ — Riemannian metric on M associated with a section $\sigma : M \to \mathfrak{H}M$,
$h_\sigma = \sigma \circ \mathfrak{p}(h)$ — intersection of σ with the fiber of $\mathfrak{p}$ passing through $h \in \mathfrak{H}M$,
$b_\sigma(h) = \beta(h_\sigma, h)$ — relative hyperbolic height of h with respect to section σ.

§1.1.5 $\mathfrak{H}$ — hyperbolization functor

§1.1.6 s_σ — relative hyperbolic height of a section $\sigma : A \to \mathfrak{H}A$ with respect to a (local) parallel section,
$\omega_\sigma^{\uparrow} = ds_\sigma$ — basic 1-form,

$\mathsf{b} = \mathsf{b}(A) = [\omega_\sigma^\uparrow] \in H^1(A)$ — basic class of A,
$\beta_\sigma(z_1, z_2) = \beta\big(\sigma(z_1), \sigma(z_2)\big)$ — basic cocycle on A.

§1.1.7 area_σ — area form determined by a section $\sigma : A \to \mathfrak{H}A$,
$area_\sigma$ — Riemannian area measure determined by σ,
K_σ — Gauss curvature of the metric ρ_σ,
Ω_σ — curvature form of the metric ρ_σ,
$d^c = I^{-1} \circ d \circ I = \frac{1}{i}(\partial - \overline{\partial})$ — "twisted differential" on a Riemann surface,
$\varphi.\,\sigma$ — image of a section $\sigma : A_1 \to \mathfrak{H}A_1$ under an isomorphism $\varphi : A_1 \to A_2$.
$\|D\varphi(z)\|_{\sigma_1,\sigma_2}$ — norm of the differential $D\varphi$ with respect to metrics $\rho_{\sigma_1}, \rho_{\sigma_2}$.

§1.2.1 $\psi : \mathcal{B} \to B \times T$, $B \cong \mathbb{D}^n$ — coordinate chart,
$\mathcal{B} \cong B \times T$ — flow box,
$B_t = \psi^{-1}(B \times \{t\}) \subset \mathcal{B}\,, t \in T$ — local leaves ($\equiv$ plaques),
$T_x = \psi^{-1}(\{x\} \times T) \subset \mathcal{B}\,, x \in B$ — transversals,
$H_{\mathcal{B}}$ — holonomy on $\mathcal{B}$,
$\psi_{ij} = \psi_i \circ \psi_j^{-1}$ — transition maps between coordinate charts,
$\mathfrak{T}(\mathcal{L})$ — set of all transversals of a lamination,
$L(x) = L_{\mathcal{L}}(x)$ — leaf of $\mathcal{L}$ passing through a point x,
$\mathsf{graph}\,\mathcal{L} = \{(x_1, x_2) : L(x_1) = L(x_2)\}$ — graph of $\mathcal{L}$.

§1.2.2 $U\mathcal{L}$ — unit tangent lamination over a Riemannian lamination.

§1.2.3 $\mathfrak{H}\mathcal{L}$ — scaling bundle lamination over a conformal lamination.

§1.3.1 $T\mathcal{L}, T^*\mathcal{L}$ — tangent and cotangent bundles of a lamination,
$\mathbf{\Omega}^p(\mathcal{L})$ — space of leafwise p-forms,
$H^*_{\mathrm{dR}}(\mathcal{L}) \cong H^*(\mathcal{L})$ — leafwise de Rham cohomology,
$[\omega] \in H^p(\mathcal{L})$ — cohomology class of a closed leafwise p-form.

§1.3.2 $\mathcal{H}$ — pointed at infinity hyperbolic 3-lamination
$\mathsf{B} = [\omega^\uparrow] \in H^1(\mathcal{H})$ — Busemann class of $\mathcal{H}$,
$\mathcal{A}$ — affine lamination,
$\mathsf{b} = [\omega_\sigma^\uparrow] \in H^1(\mathcal{A})$ — basic class of $\mathcal{A}$.

§1.3.3 $\mathsf{e} = \frac{1}{2\pi}[\Omega_\rho] \in H^2(\mathcal{L})$ — Euler class of a Riemann surface lamination $\mathcal{L}$.

§2.1.1 $\operatorname{Jac} F(x) = \frac{d(F^{-1}\beta)}{d\alpha}(x)$ — Radon–Nikodym Jacobian of $F : (X, \alpha) \to (Y, \beta)$,
Δ_μ — modulus of a quasi-invariant transverse measure μ,
$\lambda \star \mu$ — "product" of a leafwise measure λ and a transverse measure μ.

§2.1.2 $\theta|_{\mathcal{B}}$ — restriction of a measure θ onto a flow box $\mathcal{B} \cong B \times T$,
$\overline{\theta}_{\mathcal{B}}$ — projection of $\theta|_{\mathcal{B}}$ onto T,
$\theta^t_{\mathcal{B}},\ t \in T$ — conditional measures of $\theta|_{\mathcal{B}}$,
$\Delta_{\theta,\lambda}$ — Radon–Nikodym cocycle (the modulus) of θ with respect to a leafwise measure λ,
$d\Theta(x, y) = d\theta(x)\, d\lambda_{L(x)}(y)$ — "counting measure" on $\mathsf{graph}\,\mathcal{L}$,
$\Sigma : (x, y) \mapsto (y, x)$ — flip transformation on $\mathsf{graph}\,\mathcal{L}$.

§2.1.3 $\mathcal{W}^u$ — unstable foliation,
$\operatorname{Jac}^u f$ — Jacobian of f with respect to the leafwise Riemannian volume on $\mathcal{W}^u$,
Δ_{AS} — Anosov–Sinai cocycle.

§2.1.4 $\mathcal{V}(\mathcal{L})$ — space of leafwise volume forms,

$\mathcal{V}_0(\mathcal{L}) \subset \mathcal{V}(\mathcal{L})$ — subspace of volume forms with compact support,
ω — leafwise volume form,
λ_ω — leafwise measure determined by ω,
$\omega \star \mu \equiv \lambda_\omega \star \mu$ — global measure determined by ω and a transverse measure μ,
$\mathcal{C}(\mathcal{L}) = [\mathcal{V}_0(\mathcal{L})]^*$ — space of currents,
$\langle \omega, c \rangle$ — pairing of a form $\omega \in \mathcal{V}_0(\mathcal{L})$ and a current $c \in \mathcal{C}(\mathcal{L})$,
$[D]$ — integration current over a leafwise domain D,
$[\mu]$ — Ruelle–Sullivan current of a transverse measure μ.

§**2.2.1** $\widetilde{\lambda}$ — lift of a leafwise measure λ from conformal lamination $\mathcal{L}$ to its scaling bundle $\mathfrak{H}\mathcal{L}$,
$\widetilde{\theta}$ — lift of a global measure θ from $\mathcal{L}$ to $\mathfrak{H}\mathcal{L}$,
$\widetilde{\mu}$ — lift of a transverse measure μ from $\mathcal{L}$ to $\mathfrak{H}\mathcal{L}$.

§**2.2.2** $\eta = \{\eta_\rho\}$ — conformal stream on a conformal manifold M,
$vol = \{vol_\rho\}$ — volume stream,
ℓ^δ_ρ — δ-Hausdorff measure of a metric ρ,
$\mathrm{HD}(X)$ — Hausdorff dimension of a set X,
$\overline{\eta} = \exp[-\delta b_\rho]\widetilde{\eta}_\rho$ — measure on $\mathfrak{H}M$.

§**2.2.3** $\lambda = \{\lambda_\rho\}$ — leafwise conformal stream on conformal lamination $\mathcal{L}$,
$\overline{\lambda} = \exp[-\delta b_\rho]\widetilde{\lambda}_\rho$ — leafwise measure on $\mathfrak{H}\mathcal{L}$,
$\mu = \{\mu_\rho\}$ — transverse conformal stream on $\mathcal{L}$,
$\boldsymbol{\upsilon} = \lambda \star \mu$ — "product" of leafwise and transverse conformal streams of the same dimension.

§**2.2.4** μ — parallel transverse conformal stream μ on affine lamination $\mathcal{A}$,
$\overline{\mu} = \exp[\delta b_\rho] \cdot \widetilde{\mu}_\rho$ — transverse measure on $\mathfrak{H}\mathcal{A}$,
$\widetilde{\boldsymbol{\upsilon}} = \overline{\lambda} \star \overline{\mu}$ — ξ-invariant measure on $\mathfrak{H}\mathcal{A}$.

§**2.3.1** $p(t,x,y)$ — heat kernel on a Riemannian manifold,
π^t_x — transition probabilities of the Brownian motion,
$P^t = e^{t\Delta}$ — Markov semigroup of the Brownian motion,
$Q^t = (P^t)^*$ — dual semigroup.

§**2.3.3** $\omega^\mu = vol \star \overline{\mu}$ — $\delta(\delta-2)$-harmonic measure on $\mathfrak{H}\mathcal{A}$.

§**2.3.4** η — conformal stream on $\overline{\mathbb{C}} \cong \partial \mathbf{H}^3$,
Φ^η — $\delta(\delta-2)$-harmonic function on $\mathbf{H}^3$,
Φ^λ — leafwise $\delta(\delta-2)$-harmonic function on $\mathfrak{H}\mathcal{A}$.

§**2.4** $\mathcal{M} = \mathcal{M}_G = \mathfrak{H}\mathcal{A}/G$ — quotient hyperbolic lamination,
$\omega^\mu_{\mathcal{M}}$ — $\delta(\delta-2)$-harmonic measure on $\mathcal{M}$,
$\Phi^\lambda_{\mathcal{M}}$ — $\delta(\delta-2)$-harmonic function on $\mathcal{M}$,
$\widetilde{\boldsymbol{\upsilon}}_{\mathcal{M}} = \overline{\lambda} \star \overline{\mu}/G$ — ξ-invariant measure on $\mathcal{M}$.

§**3.1.1** f — rational endomorphism of $\overline{\mathbb{C}}$,
d — degree of f,
invariant set — $X : fX \subset X$,
completely invariant set — $X : f^{-1}X = X$,
pre-periodic point — preimage of a periodic point under some iterate of f,
critical point of f — $c \in \overline{\mathbb{C}} : Df(c) = 0$,
$C = C(f)$ — set of critical points,
$C_l = \bigcup_{n=1}^{l} f^n C \subset \overline{\mathbb{C}}$, $\quad l > 0$ — l-postcritical set,

C_∞ — postcritical set,
$\widehat{z} = \{\dots, z_{-1}, z_0\}$ — backward trajectory,
$\mathcal{N} = \mathcal{N}_f = \{\widehat{z}\}$ — space of backward trajectories,
$\widehat{f} : \{\dots, z_{-1}, z_0\} \mapsto \{\dots, z_{-1}, z_0, f z_0\}$ — natural extension of f,
$\pi : \widehat{z} = \{\dots, z_{-2}, z_{-1}, z_0\} \mapsto z_0$,
$\pi_n(\widehat{z}) = \pi \circ \widehat{f}^n(\widehat{z}) = z_n$,
$\mathcal{R}$ — regular part of $\mathcal{N}$,
$H_\vartheta : \pi^{-1}(z) \to \pi^{-1}(\zeta)$ — holonomy in $\mathcal{R}$ along a path ϑ with endpoints $z, \zeta \in \overline{\mathbb{C}}$,
$L_0 \subset \mathcal{R}$ — the special leaf,
$\mathcal{A}^n \subset \mathcal{R}$ — union of all parabolic leaves excluding L_0.

§**3.1.2** $\mathcal{U}$ — space of all non-constant meromorphic functions on $\overline{\mathbb{C}}$,
Aff — group of complex affine maps $A : \mathbb{C} \to \mathbb{C}$,
$\mathcal{K} = \bigcap_{n \ge 0} f^n(\mathcal{U})$ — global attractor of f,
$\widehat{\mathcal{K}}$ — natural extension of $\mathcal{K}$,
$\boldsymbol{\mathcal{A}} = \widehat{\mathcal{K}}/\mathbb{C}^*$ — universal orbifold affine lamination,
$\psi : (L(\widehat{z}), \widehat{z}) \to (\mathbb{C}, 0)$ — affine chart on leaf $L(\widehat{z})$,
$\varphi_{-n} = \pi_{-n} \circ \psi^{-1}$ — sequence of meromorphic functions representing a point $\widehat{z} \in \mathcal{A}^n$,
$\iota : \mathcal{A}^n \to \boldsymbol{\mathcal{A}}$,
$\mathcal{A}^l = \iota \mathcal{A}^n \subset \boldsymbol{\mathcal{A}}$,
$\mathcal{A} = \mathrm{cl}(\mathcal{A}^l) \subset \boldsymbol{\mathcal{A}}$ — (orbifold) affine lamination associated with f,
$\wp : \mathcal{A} \to \mathcal{A}^l \cong \mathcal{A}^n$,
$\pi \cong \pi \circ \wp : \mathcal{A} \to \overline{\mathbb{C}},\ \{\varphi_{-n}\} \mapsto \varphi_0(0)$.

§**3.1.3** $J = J(f)$ — Julia set of f,
$\widehat{J} = \pi_{\mathcal{N}}^{-1} J \subset \mathcal{N}$ — natural extension of the Julia set,
$\mathcal{J}^n = \widehat{J} \cap \mathcal{A}^n$ — affine part of the natural extension supplied with the turbulent topology,
$\mathcal{J}^l = \iota(\mathcal{J}^n) \subset \mathcal{A}^l$ — affine part of the natural extension supplied with the laminar topology,
$\mathcal{J} = \mathrm{cl}(\mathcal{J}^l) = \pi^{-1} J \subset \mathcal{A}$ — Julia set in the affine lamination $\mathcal{A}$.

§**3.1.4** $\overline{\mathcal{T}}(\boldsymbol{z}) \equiv \overline{\mathcal{T}}_z = \pi^{-1} z \subset \mathcal{A},\ \boldsymbol{z} \in \mathcal{A}, z = \pi(\boldsymbol{z}) \in \overline{\mathbb{C}}$ — fibers of the dual fibration of $\mathcal{A}$,
$\mathcal{T}(\boldsymbol{z}) \equiv \mathcal{T}_z = \overline{\mathcal{T}}_z \cap \mathcal{A}^l$ — fibers of the dual fibration of $\mathcal{A}^l$,
$\overline{\mathcal{T}}^n(\boldsymbol{z}) \equiv \overline{\mathcal{T}}^n_\zeta = \widehat{f}^n \overline{\mathcal{T}}_\zeta \subset \overline{\mathcal{T}}(\boldsymbol{z}),\ n \in \mathbb{N}, \zeta = z_{-n}$ — rank n cylinders in $\overline{\mathcal{T}}(\boldsymbol{z})$,
$\mathcal{T}^n(\boldsymbol{z}) \equiv \mathcal{T}^n_\zeta$ — rank n cylinders in $\mathcal{T}(\boldsymbol{z})$.

§**3.1.5** $V_{\mathcal{B}}$ — dual holonomy on a standard flow box $\mathcal{B}$,
$\mathcal{O}_m(\widehat{z}, V),\ m \in \mathbb{N}$ — standard univalent flow boxes,
$\mathcal{O}(\widehat{z}, V) \equiv \mathcal{O}_0(\widehat{z}, V)$.

§**3.1.6** $\mathcal{H} = \mathfrak{H}\mathcal{A}$ — orbifold pointed at infinity $\mathbf{H}^3$-lamination associated with f,
$\mathcal{M} = \mathcal{H}/\widehat{f}$ — quotient hyperbolic lamination.

§**3.1.7** germ — uniform structure associated with the germ topology.

§**3.2.1** $\boldsymbol{z}_n = \widehat{f}^n \boldsymbol{z}$ — $\widehat{f}$-orbit of $\boldsymbol{z} \in \mathcal{A}$.

§**3.2.2** $\mathcal{A}'$ — subset of $\mathcal{A}$ consisting of leaves which do not correspond to repelling periodic points of f,
$\mathcal{H}'$ — corresponding subset of $\mathcal{H}$,
$\mathcal{M}' = \mathcal{H}'/\widehat{f}$.

§**3.3.1** $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ — meromorphic function,

$I(\varphi, X) = \int_X \|D\varphi\|$ eucl — spherical area of $\varphi(X)$ taken with multiplicity,
$I(\varphi, z, r) \equiv I(\varphi, \mathbb{D}(z, r))$,
$R(\varphi, z)$ — the radius determined from $I(\varphi, z, R) = 1$,
$I(\varphi, r) \equiv I(\varphi, 0, r)$,
$R(\varphi) \equiv R(\varphi, 0)$.

§**3.3.4** $\mathrm{dist}_n(\varphi, \psi) = \sup_{|z|\le n} \varsigma(\varphi(z), \psi(z))$,
$\mathrm{dist}(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathrm{dist}_n(\varphi, \psi)$ — metric on the space of meromorphic functions,
$\mathcal{U}^0 = \{\varphi \in \mathcal{U} :\ R(\varphi) = 1\}$ — space of normalized meromorphic functions.

§**3.4.2** $\|DV\|_\rho$ — norm of the derivative of the dual holonomy.

§**3.4.3** α_ρ — dual basic cocycle.

§**3.5** O_f — Thurston orbifold.

§**4.1.1** κ — balanced measure on the Julia set $J(f) \subset \overline{\mathbb{C}}$.
$\widehat{\kappa}$ — balanced measure on $\widehat{J}$ (the natural extension of κ),
$\boldsymbol{\kappa} \equiv \iota(\widehat{\kappa})$ — balanced measure on $\mathcal{J}^l \subset \mathcal{J}$.

§**4.1.2** m_z, $z \in J \setminus C_\infty$ — uniform measure on $\pi^{-1}(z)$,
$m = \{m_\tau\}$ — transverse balanced measure.

§**4.1.3** $\theta = vol \star \widetilde{m}$ — global balanced measure.

§**4.2.1** $\eta^n_{\Delta,z}$ — normalized counting measure on $\widehat{f}^n \Delta \cap \overline{\mathcal{T}}_z$,
$\widetilde{\kappa}_L$ — pullback of the balanced measure to a leaf L,
$\widetilde{\kappa} = \{\widetilde{\kappa}_L\}$ — associated leafwise measures on $\mathcal{A}$,
$\mathcal{S}$ — spaces of test functions and test forms on $\mathcal{A}$.

§**4.3.1** $\mu^{\delta,n} = \widehat{f}^n(\|D\widehat{f}^n\|_\sigma^{-\delta} m)$,
$\Xi_T(\delta) = \sum_{n\in\mathbb{N}} \|\mu_T^{\delta,n}\|$ — backward Poincaré series,
$\delta_{\mathrm{cr}}(T)$ — critical exponent of the series $\Xi_T(\delta)$,
δ_{cr} — critical exponent of the map f.

§**4.3.3** $area_\sigma = \mathsf{area}_\sigma \star m$ — global measure on $\mathcal{A}$.

§**4.3.4** $\mathcal{H}^-_\sigma = \{\boldsymbol{h} \in \mathcal{H} : b_\sigma(\boldsymbol{h}) < 0\}$ — part of the hyperbolic lamination $\mathcal{H}$ under the graph of σ,
$vol_\sigma = \widetilde{area_\sigma} = \widetilde{\mathsf{area}}_\sigma \star \widetilde{m}$ — lift of the measure $area_\sigma$ to $\mathcal{H}$,
$\mathsf{vol}^\varepsilon_\sigma = \exp[\varepsilon b_\sigma] \cdot \widetilde{\mathsf{area}}_\sigma$,
$vol^\varepsilon_\sigma = \exp[\varepsilon b_\sigma] \cdot vol_\sigma = \mathsf{vol}^\varepsilon_\sigma \star \widetilde{m}$,
$\ell^\varepsilon = \{\ell^\varepsilon_{\boldsymbol{z}}\}$, $\boldsymbol{z} \in \mathcal{A}$ — measures on vertical geodesics $\mathfrak{p}^{-1}(\boldsymbol{z})$ with densities $\exp[\varepsilon b_\sigma]$,
$D(\boldsymbol{z}) = D_\sigma(\boldsymbol{z}) \subset L_{\mathcal{A}}(\boldsymbol{z})$ — leafwise disk with radius 1 with respect to Euclidean structure $\sigma_{\boldsymbol{z}}$,
$I(\boldsymbol{z}) = \{\boldsymbol{h} \in \mathfrak{p}^{-1}(\boldsymbol{z}) : -1 < b_\sigma(\boldsymbol{h}) < 0\}$,
$W(\boldsymbol{z}) = \bigcup_{\boldsymbol{\zeta}\in D(\boldsymbol{z})} I_{\boldsymbol{\zeta}} \subset L_{\mathcal{H}}(\boldsymbol{z})$.

§**4.4.1** μ — $\widehat{f}$-invariant parallel transverse conformal stream of dimension δ_{cr} on $\mathcal{A}$,
$\omega = \omega^\mu$ — $\widehat{f}$-invariant λ-harmonic measure with $\lambda = \delta_{\mathrm{cr}}(\delta_{\mathrm{cr}} - 2)$ corresponding to μ.

§**4.5.1** λ — $\widehat{f}$-invariant parallel transverse conformal stream on $\mathcal{A}$,
$\eta \equiv \eta_\varsigma$ — continuous conformal measure on the Julia set of f,
$\widetilde{\eta}$ — lift of η to a leafwise Radon measure on $\mathcal{A}$,
υ — f-invariant measure on $J(f)$ equivalent to a δ-conformal measure η,

$\widehat{v}$ — lift of v to the natural extension.

§4.5.2 $\lambda^{\gamma,n} = \widehat{f}^{-n}(\|D\widehat{f}^{-n}\|_\rho^\gamma \cdot \widetilde{\kappa})$,
$\Theta_D(\gamma) = \sum_{n\in\mathbb{N}} \lambda^{\gamma,n}(D)$ — forward Poincaré series,
$\gamma_{\rm cr}(D)$ — critical exponent of Θ_D,
$\gamma_{\rm cr}$ — forward critical exponent.

§4.5.3 $\boldsymbol{v} = \lambda \star \mu$,
$\widetilde{\boldsymbol{v}} = \overline{\lambda} \star \overline{\mu} = \widetilde{\lambda \star \mu}$ — lift of $\boldsymbol{v}$ to $\mathcal{H} = \mathfrak{H}\mathcal{A}$,
$\widetilde{\boldsymbol{v}}_{\mathcal{M}}$ — image of $\widetilde{\boldsymbol{v}}$ on the quotient lamination $\mathcal{M} = \mathcal{H}/\widehat{f}$.

§4.5.4 $\mathcal{C}$ — convex core of the lamination $\mathcal{H}$.

§5.1.1 $\mathring{\mathcal{A}}$ — tautological $\mathbb{C}$-foliation of $\partial\mathbf{H}^3 \times \partial\mathbf{H}^3 \setminus \text{diag}$,
$\mathring{\mathcal{H}} = \mathfrak{H}\mathring{\mathcal{A}}$ — tautological pointed at infinity hyperbolic foliation of $\mathbf{H}^3 \times \partial\mathbf{H}^3$.

§5.1.2 $p : U\mathbf{H}^3 \to \mathbf{H}^3$ — unit tangent bundle over $\mathbf{H}^3$,
$\gamma = \{\gamma^\tau\}_{\tau\in\mathbb{R}}$ — geodesic flow on $U\mathbf{H}^3$,
$\gamma(v)$ — geodesic determined by a tangent vector $v \in U\mathbf{H}^3$,
$\gamma^{\pm\infty}(v) \in \partial\mathbf{H}^3$ — endpoints of $\gamma(v)$,
$\text{Hor}(v) = \text{Hor}_{\gamma^\infty(v)}(p(v))$ — horosphere centered at $\gamma^\infty(v)$ and passing through $p(v)$.

§5.1.3 $\mathcal{W}^s$ — weakly stable foliation of the geodesic flow.
$\mathcal{W}^{ss}$ — strongly stable foliation of the geodesic flow.

§5.2.1 G — Kleinian group,
$\Lambda = \Lambda(G) \subset \partial\mathbf{H}^3$ — limit set of G,
$\mathcal{A}_G$ — affine lamination of $\partial\mathbf{H}^3 \times \Lambda$,
$\mathcal{H}_G = \mathfrak{H}\mathcal{A}_G \subset U\mathbf{H}^3$ — hyperbolization of $\mathcal{A}_G$,
$M = \mathbf{H}^3/G$ — quotient hyperbolic manifold,
$\mathcal{M}_G = \mathcal{H}_G/G \subset UM$ — quotient hyperbolic lamination.

§5.2.2 $\Lambda_0 = \Lambda_0(G)$ — set of fixed points of all hyperbolic elements of G,
$\mathcal{H}'_G = \mathbf{H}^3 \times (\Lambda \setminus \Lambda_0)$ — union of leaves from $\mathcal{H}_G$ parameterized by points from $\Lambda \setminus \Lambda_0$,
$\mathcal{M}'_G = \mathcal{H}'_G/G$.

§5.3.1 ς_h — visual metric on $\partial\mathbf{H}^3$ centered at $h \in \mathbf{H}^3$.

§5.3.2 $l_h(q_-, q_+) = \beta_{q_-}(h, o) + \beta_{q_+}(h, o)$ — "cut length",
$\varepsilon_{h,q}$ — Euclidean metric on $\mathcal{P}_q$ induced by the hyperbolic metric on $\text{Hor}_q(h)$.

§5.4.1 ν — Patterson–Sullivan stream of a Kleinian group G,
$\lambda(\nu)$ — G-invariant leafwise conformal stream on $\mathcal{A}_G$,
$\mu(\nu)$ — G-invariant parallel transverse conformal stream on $\mathcal{A}_G$.

§5.4.2 $\boldsymbol{v}$ — geodesic current,
$\widetilde{\boldsymbol{v}}$ — invariant measure of the geodesic flow on $U\mathbf{H}^3$,
$\widetilde{\boldsymbol{v}}_M$ — invariant measure of the geodesic flow on $M = \mathbf{H}^3/G$,
$\boldsymbol{v}(\nu) = \lambda(\nu) \star \mu(\nu)$.

§5.4.3 $|q_- - q_+|$ — the Euclidean chordal distance between points from the unit sphere,
$d\boldsymbol{v}^{[1]}(q_-, q_+) = d\nu_o(q_-)\, d\nu_o(q_+)/|q_- - q_+|^{2\delta}$,
$d\boldsymbol{v}^{[2]}(q_-, q_+) = \exp\big[\delta l_h(q_-, q_+)\big] \cdot d\nu_h(q_-)\, d\nu_h(q_+)$,
ε_v — induced Euclidean metric on the horosphere $\text{Hor}(v)$,
ν^{ss} — leafwise measure on $\mathcal{W}^{ss}$ assigned by the stream ν to the leafwise metrics ε_v,
ν^{su} — analogous measure on $\mathcal{W}^{su}$,

ν_{h,q_+} — the measure on $\partial\mathbf{H}^3$ assigned by the stream ν to the metric ε_{h,q_+},
$\boldsymbol{v}_{q_-,q_+} = \nu_v \otimes \nu_{-v}$,
$d\boldsymbol{v}^{[3]}(q_-,q_+) = d\boldsymbol{v}_{q_-,q_+}(q_-,q_+)$,
$\mathcal{R}(q_1,q_2,q_3,q_4)$ — cross ratio of points $q_1,q_2,q_3,q_4 \in \partial\mathbf{H}^3$.

§**5.4.5** $d\omega^\nu(h,q) = d\,vol(h)d\nu_h(q)$ — G-invariant $\delta(\delta-2)$-harmonic measure on $\mathcal{H}_G$,
$\Phi^\nu(h,q) = \|\nu_h\|$ — G-invariant leafwise $\delta(\delta-2)$-harmonic function,

§**5.6.1** H — compact hyperbolic manifold with $H^1(H,\mathbb{R}) \neq \{0\}$,
$G = \pi_1(H)$ — fundamental group,
$\mathrm{Hom}(G,\mathbb{R}) \cong H^1(H,\mathbb{R})$ — group of additive characters of G,
$\chi \in \mathrm{Hom}(G,\mathbb{R})$ — character of G,
$T^g_\chi v = g \circ \gamma^{\chi(g)}(v)$ — twisted action of G on $U\mathbf{H}^3$ determined by χ.

§**5.6.2** $l(g) = \min\{\mathrm{dist}(h,gh) : h \in \mathbf{H}^3\}$ — length of the closed geodesic representing the conjugacy class of $g \in G$,
$\|\chi\| = \sup_{g\in G} \chi(g)/l(g)$ — stable norm on $\mathrm{Hom}(G,\mathbb{R})$.

§**5.6.3** $\mathcal{B}_\chi = \mathcal{W}^{ss}/T_\chi$ — affine foliation of the quotient manifold $U\mathbf{H}^3/T_\chi$

§**5.6.4** $\Sigma(s) = \sum_{g\in G} \exp\big[-s\,\mathrm{dist}(o,go) - \chi(g)\big]$,
$\delta(\chi)$ — critical exponent of the Poincaré series $\Sigma(s)$.

1. Affine and hyperbolic laminations

1.1. **Affine plane and hyperbolic space.**

1.1.1. *Pointed at infinity hyperbolic spaces.* Let $\mathbf{H}^3$ be the 3-dimensional *hyperbolic space*. The *sphere at infinity* $\partial\mathbf{H}^3$ is the boundary of the *visibility compactification* of $\mathbf{H}^3$: an escaping to infinity sequence of points $h_n \in \mathbf{H}^3$ converges in this compactification iff the directing vectors of the geodesic rays $[o,h_n]$ issued from a certain ($\equiv$ any) reference point $o \in \mathbf{H}^3$ converge.

This and all other facts formulated in this section can be easily checked in terms of the upper half-space model of $\mathbf{H}^3$ described in §**1.1.2**.

Definition 1.1. A hyperbolic space $\mathbf{H}^3$ with a distinguished boundary point $q \in \partial\mathbf{H}^3$ is called *pointed at infinity*. △

Definition 1.2 ([Ka90])**.** Let $(\mathbf{H}^3,q)$ be a pointed at infinity hyperbolic space. The choice of the point $q \in \partial\mathbf{H}^3$ determines the *Busemann cocycle* β_q on $\mathbf{H}^3 \times \mathbf{H}^3$ by the formula

$$\beta_q(h_1,h_2) = \lim_h \Big[\mathrm{dist}(h_1,h) - \mathrm{dist}(h_2,h)\Big]\,, \tag{1.3}$$

where $h \in \mathbf{H}^3$ converges to q in the visibility topology. △

In other words, $\beta_q(h_1,h_2)$ can be considered as a "regularization" of the formal expression $\mathrm{dist}(h_1,q) - \mathrm{dist}(h_2,q)$. The Busemann cocycle obviously satisfies the cocycle *chain rule*:

$$\beta_q(h_1,h_3) = \beta_q(h_1,h_2) + \beta_q(h_2,h_3) \qquad \forall\, h_i \in \mathbf{H}^3\,. \tag{1.4}$$

Remark 1.5. Sometimes the Busemann cocycle is defined with the opposite sign, e.g., see [Ka00a].

Definition 1.6. The *vertical 1-form*

$$\omega_q^\uparrow = d\beta_q(h_1, \cdot)$$

is the differential of the Busemann cocycle β_q with respect to the second argument (due to the chain rule (1.4) it is independent of the choice of the first argument h_1), so that

$$\beta_q(h_1, h_2) = \int_\vartheta \omega_q^\uparrow \ , \tag{1.7}$$

for any smooth path ϑ with endpoints h_1, h_2. The *vertical vector field* $v_q^\uparrow$ is dual to the form $\omega_q^\uparrow$ with respect to the hyperbolic metric and consists of unit length vectors "pointing" at the point q in the sense that the integral curves of the field $v_q^\uparrow$ are the *vertical geodesics* which converge to the point q at $+\infty$ in the visibility topology. The *vertical flow*

$$\xi_q = \{\xi_q^\tau\}_{\tau\in\mathbb{R}}$$

on $\mathbf{H}^3$ is the motion with unit hyperbolic speed along the vertical geodesics to the point q (i.e., it is isomorphic to the restriction of the geodesic flow onto the field $v_q^\uparrow$). △

Definition 1.8. The *horosphere* centered at q and passing through a point $h \in \mathbf{H}^3$ is the level set of the Busemann cocycle:

$$\operatorname{Hor}_q(h) = \left\{ h' \in \mathbf{H}^3 : \beta_q(h, h') = 0 \right\} . \tag{1.9}$$

We denote the *space of horospheres centered at* q by $\operatorname{Hor}(\mathbf{H}^3, q)$, and the *space of all horospheres in* $\mathbf{H}^3$ by

$$\operatorname{Hor}(\mathbf{H}^3) = \bigcup_{q\in\partial\mathbf{H}^3} \operatorname{Hor}(\mathbf{H}^3, q) \ .$$

The center of a horosphere $\Upsilon \in \operatorname{Hor}(\mathbf{H}^3)$ is denoted $\Upsilon_\infty \in \partial\mathbf{H}^3$. △

For a fixed point $q \in \partial\mathbf{H}^3$ the horospheres centered at q are obviously tangent to the plane distribution determined by the form $\omega_q^\uparrow$ and orthogonal to the vertical vector field $v_q^\uparrow$ (this is why we shall often refer to them as *horizontal*). They foliate the hyperbolic space, and the vertical flow ξ_q acts transitively on $\operatorname{Hor}(\mathbf{H}^3, q)$. More precisely,

$$\beta_q(h_1, h_2) = s \qquad \Longleftrightarrow \qquad \xi_q^s \operatorname{Hor}_q(h_1) = \operatorname{Hor}_q(h_2) \ .$$

Therefore, the Busemann cocycle $\beta_q(h_1, h_2)$ is the signed distance between the horospheres $\operatorname{Hor}_q(h_1)$ and $\operatorname{Hor}_q(h_2)$, where the sign is chosen according to the direction of the vertical flow: it is positive if h_2 is "closer" to q than h_1 (see Fig. 1). Any isometry between pointed at infinity hyperbolic spaces (i.e., such that the point at infinity is mapped to the point at infinity) obviously preserves the Busemann cocycle and conjugates the vertical flows.

Denote by

$$\mathcal{P}_q = \partial\mathbf{H}^3 \setminus \{q\} \tag{1.10}$$

the punctured visibility sphere. For a point $h \in \mathbf{H}^3$ let $\xi_q(h)$ be the vertical geodesic passing through h, and let $\mathfrak{p}_q(h) \in \mathcal{P}_q$ be its limit point at $-\infty$ (the limit point at $+\infty$ being q). Then the map

$$\mathfrak{p}_q : \mathbf{H}^3 \mapsto \mathcal{P}_q \tag{1.11}$$

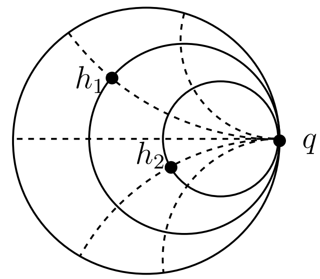

FIGURE 1

is a (trivial) fiber bundle over $\mathcal{P}_q$ whose fiber $\mathfrak{p}_q^{-1}(z)$ at a point $z \in \mathcal{P}_q$ is the vertical geodesic joining the points z and q. The horospheres (1.9) are sections of $\mathfrak{p}_q$ (see Fig. 2).

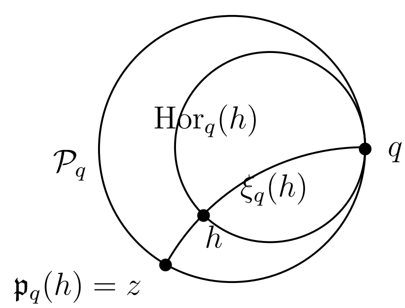

FIGURE 2

Below we shall usually omit the subscript q when the point at infinity q is fixed.

1.1.2. *The upper half-space model.* By using the term "vertical" in Definition 1.6 we were implicitly referring to the *upper half-space model* of the hyperbolic space which we shall now briefly recall. The state space of this model is

$$\mathbf{H}^3 \cong \mathbb{R}^2 \times \mathbb{R}_+ = \Big\{(x, y, t) : x, y \in \mathbb{R},\, t > 0\Big\} . \tag{1.12}$$

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by putting

$$z = x + iy ,$$

and alongside with the Euclidean length t also use the *hyperbolic length* parameterization of $\mathbb{R}_+$

$$s = \log t .$$

The *hyperbolic metric* on $\mathbf{H}^3$ is in this model

$$\frac{|dz|^2 + dt^2}{t^2} = \frac{|dz|^2}{e^{2s}} + ds^2 \tag{1.13}$$

with

$$|dz|^2 = dz \otimes d\overline{z} = dx^2 + dy^2$$

being the standard metric on the Euclidean plane $\mathbf{E}^2 \cong \mathbb{C}$. The *hyperbolic volume form* vol and the *hyperbolic measure vol* on $\mathbf{H}^3$ are

$$\mathsf{vol} = \frac{\mathsf{eucl} \wedge dt}{t^3} = \frac{\mathsf{eucl} \wedge ds}{e^{2s}} , \qquad d\,vol = \frac{d\ell\, dt}{t^3} = \frac{d\ell\, ds}{e^{2s}} , \tag{1.14}$$

where

$$\mathsf{eucl} = dx \wedge dy , \qquad d\ell = dxdy ,$$

are, respectively, the Euclidean area form and the Lebesgue measure on $\mathbf{E}^2$. The Laplace–Beltrami operator of the hyperbolic metric (1.13) is

$$\begin{aligned} \Delta_{\mathbf{H}} &= t^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2} \right) - t\frac{\partial}{\partial t} = t^2 \Delta_{\mathbf{E}} + t^2 \frac{\partial^2}{\partial t^2} - t\frac{\partial}{\partial t} \\ &= e^{2s} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\partial^2}{\partial s^2} - 2\frac{\partial}{\partial s} = e^{2s}\Delta_{\mathbf{E}} + e^{2s}\frac{\partial^2}{\partial s^2} - 2\frac{\partial}{\partial s} , \end{aligned} \tag{1.15}$$

where

$$\Delta_{\mathbf{E}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian of the Euclidean metric on $\mathbf{E}^2$.

The geodesics in $\mathbf{H}^3$ are either Euclidean circles orthogonal to the boundary plane or vertical lines. Therefore, the visibility sphere in this model is the union of the distinguished point at infinity $q = \infty$ and the boundary plane

$$\mathcal{P} = \Big\{ (z,t) : z \in \mathbb{C},\, t = 0 \Big\} \cong \partial \mathbf{H}^3 \setminus \{q\} .$$

The Busemann cocycle with respect to the point $q = \infty$ is the logarithm of the ratio of the “Euclidean heights” of points h_2 and h_1, or the difference between their “hyperbolic heights” (so that $\beta(h_1, h_2)$ is positive if h_2 is “higher” than h_1):

$$\beta(h_1, h_2) = \log \frac{t(h_2)}{t(h_1)} = s(h_2) - s(h_1) . \tag{1.16}$$

Thus, the horosphere $\mathrm{Hor}(h)$ is the horizontal coordinate plane passing through h. The vertical flow ξ acts as

$$\xi^\tau(z,t) = (z, e^\tau t) , \qquad \text{or,} \qquad \xi^\tau(z,s) = (z, s+\tau) .$$

The geodesic $\xi(h)$ passing through a point h is, indeed, a vertical line, and the projection $\mathfrak{p}$ (1.11) is just the coordinate projection $h \mapsto z(h)$ (see Fig. 3). The vertical vector field and the vertical 1-form are, respectively,

$$v^\uparrow = \frac{\partial}{\partial s} = t\frac{\partial}{\partial t} , \qquad \omega^\uparrow = ds = \frac{dt}{t} .$$

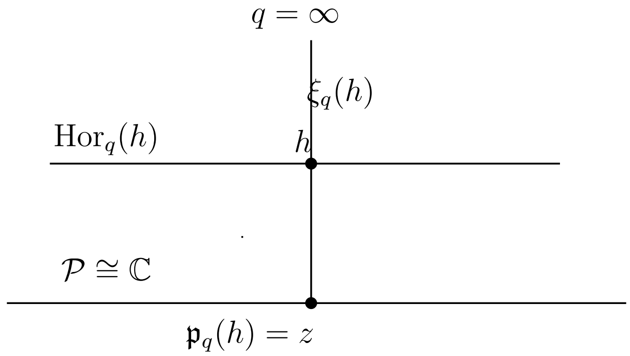

FIGURE 3

1.1.3. *Non-simply connected hyperbolic manifolds and orbifolds.* In what follows all manifolds will be assumed oriented. Recall that a *hyperbolic 3-manifold* is a 3-dimensional manifold endowed with a complete Riemannian metric of constant negative curvature -1 (this metric is called *hyperbolic*). A *Kleinian group* is a discrete subgroup of the group of orientation preserving isometries $\mathrm{Iso}_0(\mathbf{H}^3)$. Any hyperbolic 3-manifold is locally isometric to $\mathbf{H}^3$, and it can be presented as the quotient $M \cong \mathbf{H}^3/\Gamma$ of $\mathbf{H}^3$ by a freely acting Kleinian group Γ. Conversely, any such group determines a hyperbolic 3-manifold.

Using the vertical flow one can extend the notion of a pointed space to the non-simply connected case.

Definition 1.17. A hyperbolic 3-manifold M is *pointed at infinity* if it is endowed with a unit speed *vertical flow* ξ_M whose orbits are asymptotic (as time goes to $+\infty$) geodesics. $\triangle$

Such a manifold is obtained by factorizing a pointed at infinity hyperbolic space $(\mathbf{H}^3, q)$ by a discrete freely acting group Γ of hyperbolic motions which fixes q (the description of such groups is well-known, see Proposition 1.22 below). Then Γ preserves the vertical vector field $v^\uparrow$ and the vertical form $\omega^\uparrow$, so that they descend from $\mathbf{H}^3$ to the *vertical vector field* $v_M^\uparrow$ and the closed *vertical form* $\omega_M^\uparrow$ on the quotient manifold $M = \mathbf{H}^3/\Gamma$. Therefore, the *horosphere foliation* also descends from $\mathbf{H}^3$ to M.

Remark 1.18. Equivalently, a pointed at infinity hyperbolic 3-manifold can be defined as a $(G, \mathbf{H}^3)$-manifold, where $G = \mathrm{Par}_0(\mathbf{H}^3, q)$ is the group of orientation preserving isometries of $\mathbf{H}^3$ which fix the point at infinity q.

Definition 1.19 ([Th91], [Sc83])**.** An *n-dimensional orbifold* is a topological space M covered with a family $\mathcal{U}$ of neighborhoods U_i such that:

(i) The family $\mathcal{U}$ is closed under finite intersections;
(ii) For each U_i there exists a homeomorphism $\psi_i : U_i \to \widetilde{U}_i/G_i$, where $\widetilde{U}_i/G_i$ is the quotient of a domain $\widetilde{U}_i \subset \mathbb{R}^n$ by an action of a finite group G_i of homeomorphisms;
(ii) If $U_i \subset U_j$, then the group G_i embeds into G_j, and there is an equivariant embedding $\psi_{ij} : \widetilde{U}_i \to \widetilde{U}_j$.

The space M is called the *underlying space* of the orbifold. The domains $\widetilde{U}_i$ endowed with the maps

$$\phi_i : \widetilde{U}_i \to \widetilde{U}_i/G_i \xrightarrow{\psi_i^{-1}} U_i$$

are *orbifold local charts*. The *orbifold singular set* $S \subset M$ is the union of sets $\phi_i(\widetilde{S}_i)$, where $\widetilde{S}_i \subset \widetilde{U}_i$ is the set of points whose stabilizers in the group G_i are non-trivial. △

Therefore, an orbifold is locally homeomorphic to quotients of $\mathbb{R}^n$ by actions of finite groups. The orbit space X/G of a properly discontinuous action of a group G on a manifold X has a natural orbifold structure. However, not every orbifold can be globally obtained in this way.

Definition 1.20. By using local charts one can *rig* orbifolds with various geometric structures (smooth, conformal, hyperbolic, etc.) and tensor fields. They have to be defined on the sets $\widetilde{U}_i$ and be invariant with respect to the groups G_i and the transition maps ϕ_{ij}. In this way one can also talk about *morphisms* of rigged orbifolds. △

In particular, a 3-dimensional hyperbolic orbifold M is *pointed at infinity* if it is endowed with a vector field $v_M^{\uparrow}$ (called *vertical*) whose integral curves are asymptotic geodesics. Denote be $\omega_M^{\uparrow}$ the dual *vertical 1-form*. The singular set of such an orbifold is clearly invariant under the corresponding vertical flow.

Definition 1.21. The cohomology class

$$\mathsf{B} = \mathsf{B}(M) = [\omega_M^{\uparrow}] \in H^1(M)$$

of the 1-form $\omega_M^{\uparrow}$ is called the *Busemann class* of a pointed at infinity 3-dimensional hyperbolic orbifold M. △

Any three-dimensional hyperbolic orbifold is isometric to the quotient orbifold $\mathbf{H}^3/\Gamma$ for a certain Kleinian group Γ [Th91, Proposition 5.4.3] (note that singular points of such orbifolds are organized into "axes" of elliptic rotations). On the other hand, any Kleinian group is a finite extension of a freely acting subgroup, and we have

Proposition 1.22. *Let M be a pointed at infinity 3-dimensional hyperbolic orbifold.*

(a) *The orbifold M is isomorphic to $\mathbf{H}^3/\Gamma$, where Γ is an elementary Kleinian group which preserves the point at infinity q. The group Γ is a finite cyclic extension of a freely acting normal subgroup Γ_0, so that $M \cong \widetilde{M}/Z$, where $\widetilde{M} = \mathbf{H}^3/\Gamma_0$ and $Z = \Gamma/\Gamma_0$.*

(b) *The group Γ_0 belongs to one of the following two types:*
  - (i) *Γ_0 is a cyclic group generated by a hyperbolic element which fixes q;*
  - (ii) *Γ_0 is a discrete parabolic group, i.e., it preserves the horospheres centered at q and acts on each horosphere as a discrete group of Euclidean motions. In this case Γ_0 is isomorphic to one of the groups $\mathbb{Z}^0 \cong \{e\}, \mathbb{Z}^1, \mathbb{Z}^2$.*

(c) *If $\Gamma_0 \neq \{e\}$, then the manifold $\widetilde{M}$ is a solid 3-torus in case (i), and either a rank 1 cusp (product of the bi-infinite cylinder $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ by $\mathbb{R}$) or a rank 2 cusp (product of the 2-torus $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ by $\mathbb{R}$) in case (ii). In case (i) the orbits of the vertical flow converge to the meridian geodesic of the torus; in case (ii) they escape through the cusp.*

The underlying space of the orbifold M is a solid 3-torus in case (i), and either an orbifold cusp $\mathbb{C} \times \mathbb{R}$ of rank 1, or an orbifold cusp $\overline{\mathbb{C}} \times \mathbb{R}$ of rank 2 in case (ii). In case (i) the (unique) closed geodesic in M is singular; in case (ii) there are several singular geodesics escaping through the cusp.

(d) *The Busemann class $\mathsf{B}(M)$ vanishes iff the group Γ_0 belongs to type (ii). In this case Γ_0 (as well as its finite extension Γ) preserves the horospheres in $\mathbf{H}^3$ centered at the point q ($\equiv$ the level sets of the Busemann cocycle), so that the Busemann cocycle β descends from $\mathbf{H}^3$ onto the orbifold M and gives the Busemann cocycle β_M on M which is related to the vertical form $\omega_M^{\uparrow}$ by formula* (1.7).

Remark 1.23. According to (d) above, if the group Γ_0 belongs to type (i) from Proposition 1.22, then $\mathsf{B} \neq 0$, i.e., the vertical form $\omega_M^{\uparrow}$ is not exact. However, even in this case *locally* one can still consider the Busemann cocycle β_M as a function of pairs of points defined by formula (1.7).

1.1.4. *Conformal orbifolds.* Two Riemannian metrics ρ, ρ' on the same orbifold are called *conformally equivalent* if $\rho' = \varphi\rho$, where φ is a positive scalar multiplier.

Definition 1.24. A *conformal orbifold* is a connected orbifold M endowed with a class of all pairwise conformally equivalent Riemannian metrics. Metrics from this class (and also corresponding Euclidean metrics on the tangent spaces T_zM, $z \in M$) are called *conformal*. △

Definition 1.25. Let M be a conformal orbifold. The elements of the *scaling bundle*

$$\mathfrak{p} : \mathfrak{H}M \to M \tag{1.26}$$

over M are conformal spheres in the tangent spaces T_zM, $z \in M$. Any sphere $h \in \mathfrak{H}M$ can be considered as the unit sphere of the associated conformal Euclidean metric ε_h on $T_{\mathfrak{p}h}M$. Below we shall often identify h and ε_h, and consider $\mathfrak{H}M$ as the *bundle of conformal Euclidean metrics on tangent spaces T_zM, $z \in M$.* △

In other words, $\mathfrak{H}M$ is the result of first removing zero tangent vectors from the tangent bundle TM and then factorizing with respect to rotations (which are well-defined due to the presence of a conformal structure). The fibers of $\mathfrak{H}M$ are isomorphic to $\mathbb{R}$. By

$$\xi(h) = \mathfrak{p}^{-1} \circ \mathfrak{p}(h)$$

we denote the fiber of $\mathfrak{p}$ passing through a point $h \in \mathfrak{H}M$. The *scaling flow* $\{\xi^\tau\}_{\tau\in\mathbb{R}}$ acts on $\mathfrak{H}M$ by conformal rescalings:

$$\varepsilon_{\xi^\tau h} = e^{-\tau}\varepsilon_h \ , \tag{1.27}$$

and its orbits are the fibers of $\mathfrak{p}$. For any two points $h_1, h_2 \in \mathfrak{H}M$ from the same fiber put

$$\beta(h_1, h_2) = \log \frac{\varepsilon_{h_1}}{\varepsilon_{h_2}} \ , \tag{1.28}$$

i.e.,

$$\beta(h_1, h_2) = \tau \iff \xi^\tau h_1 = h_2 \ .$$

Therefore, the fibers of $\mathfrak{H}M$ are endowed with the metric

$$d(h_1, h_2) = \left|\beta(h_1, h_2)\right| \ , \qquad \mathfrak{p}h_1 = \mathfrak{p}h_2 \ . \tag{1.29}$$

By $\omega^{\uparrow}$ denote the differential of the function (1.28) with respect to the second coordinate (cf. Definition 1.6).

Remark 1.30. We reproduce the notations introduced in §**1.1.1** above; the reason for this will become clear in §**1.1.5**.

Definition 1.31. In view of formula (1.29), the fibers $\mathfrak{p}^{-1}(z) \cong \mathbb{R}$, $z \in M$ of the bundle $\mathfrak{p} : \mathfrak{H}M \to M$ carry canonical Lebesgue measures ℓ_z. Using these measures one can *lift* measures from M to $\mathfrak{H}M$. Namely, given a measure λ on M, its lift is the measure on $\mathfrak{H}M$ which is obtained by integrating the measures ℓ_z against λ:

$$d\widetilde{\lambda}(z,s) = d\lambda(z)\, d\ell_z(s)\ .$$

In the same way, denote by

$$\widetilde{\omega} = \omega \wedge \omega^{\uparrow}$$

the lift of a differential form ω from M to $\mathfrak{H}M$. △

By Definition 1.25 points $h \in \mathfrak{H}M$ correspond to Euclidean conformal metrics ε_h on $T_{\mathfrak{p}h}M$, so that *sections* $\sigma : M \to \mathfrak{H}M$ of the fiber bundle $\mathfrak{p}$ are in one-to-one correspondence with conformal Riemannian metrics ρ_σ on M. Note that by assigning to the metric ρ_σ the associated Riemannian volume form the bundle $\mathfrak{p}$ can be identified with the bundle of positive volume forms on M.

Remark 1.32. Although we call the metric ρ_σ Riemannian, usually we only assume continuity of σ unless otherwise specified.

By (1.28), for any two sections σ, σ'

$$\frac{\rho_{\sigma'}}{\rho_\sigma}(z) = \frac{\varepsilon_{\sigma'(z)}}{\varepsilon_{\sigma(z)}} = \exp\Big[\beta\Big(\sigma'(z), \sigma(z)\Big)\Big]\ . \tag{1.33}$$

We shall use the notations

$$h_\sigma = \sigma \circ \mathfrak{p}(h)\ , \qquad h \in \mathfrak{H}M$$

for the natural projection of a point h to the section σ (i.e, the intersection of the fiber passing through h with σ), and

$$b_\sigma(h) = \beta(h_\sigma, h) \tag{1.34}$$

for the *relative hyperbolic height* of h with respect to the section σ (see Fig. 4). Clearly, for another section σ'

$$b_{\sigma'}(h) - b_\sigma(h) = \beta(h_{\sigma'}, h_\sigma) = \log \frac{\rho_{\sigma'}}{\rho_\sigma}(\mathfrak{p}h)\ . \tag{1.35}$$

1.1.5. *Hyperbolization of affine surfaces.* An oriented 2-dimensional conformal manifold is called a *Riemann surface*. Equivalently, a Riemann surface is a 1-dimensional complex manifold. By endowing a Riemann surface S with an atlas of coordinate charts with transition maps from a given pseudo-group $\mathcal{C}$ (contained in the pseudo-group of all holomorphic maps) one can define finer geometric structures on S (*stiffen* S in the terminology of [Th91]).

Definition 1.36. We shall say that S is

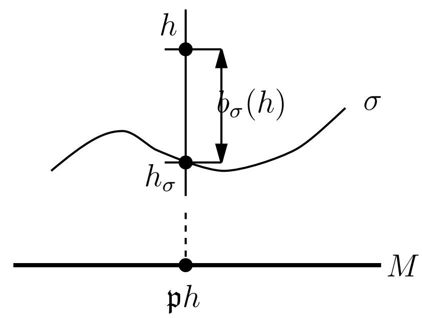

FIGURE 4

(i) an *affine Riemann surface*, if $\mathcal{C}$ is the group of all complex affine maps $z \mapsto az+b,\ a, b \in \mathbb{C}, a \neq 0$;
(ii) a *Euclidean surface*, if $\mathcal{C}$ is the group of all maps $z \mapsto az+b,\ a, b \in \mathbb{C}, |a| = 1$ (so that the transitions are Euclidean motions);
(ii) a *translation surface*, if $\mathcal{C}$ is the group of all maps $z \mapsto z+b,\ b \in \mathbb{C}$.

△

If S is an affine surface, then its tangent and cotangent bundles (and hence all tensor bundles) are endowed with a natural flat connection. Being *parallel* with respect to this connection means to have constant coefficients in any affine coordinate chart (the reader is referred to [Ca88] and [Go] for general notions from the theory of affine manifolds). So, one can talk about parallel vector fields, forms, Riemannian metrics, etc. on S. In these terms a Euclidean surface is an affine surface endowed with a parallel conformal metric, and a translation surface is an affine surface endowed with a non-zero parallel vector field (such a field obviously determines a parallel conformal metric: the vectors from the field have unit length in this metric).

An affine Riemann surface structure is the same as a *complex affine structure*, or, in "real terms", a *projective Euclidean* ($\equiv$ *similarity*) structure. In other words, affine Riemann surfaces are $(G, \mathbf{E}^2)$-manifolds, where G is the group of all similarities of the Euclidean plane $\mathbf{E}^2$ (i.e., compositions of rotations, translations and expansions or contractions by a scalar factor). In particular, an *affine plane* is $\mathbb{R}^2 \cong \mathbb{C}$ endowed with the class of all rescalings of a given Euclidean structure.

All these notions obviously carry over to *Riemann orbifold surfaces* (cf. Definition 1.20).

> *In what follows by a surface we shall mean an orbifold surface endowed with a complex (conformal) structure. An "affine surface" will always mean a complete complex ($\equiv$ Riemann) orbifold affine surface.*

Remark 1.37. A two-dimensional affine orbifold with underlying space $\mathbb{C}$ is isomorphic either to $\mathbb{C}/\mathbb{Z}_d$, where the cyclic group $\mathbb{Z}_d$ acts on $\mathbb{C}$ by rotations, or to $\mathbb{C}/\widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}}$ is the infinite dihedral group (i.e., the $\mathbb{Z}_2$-extension of $\mathbb{Z}$) generated by a translation and a central symmetry. The branched covering map which determines the orbifold is $z \mapsto z^d$ in the first case and $z \mapsto \cos z$ in the second case.

Proposition 1.38. *The punctured visibility sphere $\mathcal{P} = \partial \mathbf{H}^3 \setminus \{q\}$ of a pointed at infinity hyperbolic space $(\mathbf{H}^3, q)$ is endowed with a natural structure of an affine plane.*

Proof. The projection $\mathfrak{p}$ (1.11) allows one to identify any horosphere $\mathrm{Hor}(h)$ (1.9) with the punctured visibility sphere $\mathcal{P}$. Denote by ε_h the hyperbolic metric on $\mathcal{P}$ obtained by restricting the hyperbolic metric onto $\mathrm{Hor}(h)$ and then projecting it onto $\mathcal{P}$. By formula (1.13)

$$d\varepsilon_h^2 = \frac{|dz|^2}{t^2} = \frac{|dz|^2}{e^{2s}} \tag{1.39}$$

in the upper half-plane model, so that $(\mathcal{P}, \varepsilon_h)$ is a Euclidean plane. Further, by (1.16) the structures ε_h for different h are all proportional, and

$$\frac{\varepsilon_{h_1}}{\varepsilon_{h_2}} = \exp\Big[\beta(h_1, h_2)\Big] . \tag{1.40}$$

$\square$

Conversely, let us show that any affine surface gives rise to a pointed at infinity hyperbolic 3-orbifold.

Proposition 1.41. *If A is an affine surface, then the fiber bundle $\mathfrak{H}A$ is given a natural structure of a pointed at infinity hyperbolic 3-orbifold. In particular, if A is an affine plane, then $\mathfrak{H}A$ is a pointed at infinity hyperbolic space.*

Proof. The fibers of the bundle $\mathfrak{p} : \mathfrak{H}A \to A$ are endowed with a natural metric (1.29), and the points of $\mathfrak{H}A$ are themselves metrics on tangent spaces $T_z A$. In order to combine them and produce a metric on $\mathfrak{H}A$ we need the affine connection over A.

Since the affine connection is flat, any metric ε_h, $h \in \mathfrak{H}A$, can be transported parallelly to give a family of metrics on tangent spaces $T_z A$ for all z from a certain neighbourhood of $\mathfrak{p}h$, which gives the *parallel foliation* of $\mathfrak{H}A$. In other words, any metric ε_h extends from the tangent space $T_{\mathfrak{p}h}A$ to a Euclidean metric (also denoted ε_h) on a neighbourhood of $\mathfrak{p}h$. The parallel foliation together with the fibration $\mathfrak{p}$ provide us with natural *parallel local coordinates* (z, s) on $\mathfrak{H}A$, where z is the "horizontal" complex coordinate, and s is the "vertical" real coordinate.

Combining the local parallel metric ε_h with the metric (1.29) on the fibers gives a Riemannian metric on $\mathfrak{H}A$, which in terms of local coordinates (z, s) takes precisely the form (1.13). Therefore, the constructed metric on $\mathfrak{H}A$ is hyperbolic, the fibers $\xi(h)$ are the asymptotic geodesics of this metric, and the parallel foliation of $\mathfrak{H}A$ becomes the horosphere foliation in the sense of §**1.1.3**. The scaling flow on $\mathfrak{H}A$ coincides with the vertical flow. The Busemann cocycle on $\mathfrak{H}A$ is (locally) given by formula (1.28) which precisely coincides with (1.40). Note that in (1.28) we no longer require that points h_1 and h_2 lie on the same fiber, which is possible due to the presence of the flat affine connection. $\square$

It is clear that any isomorphism $\varphi : A_1 \to A_2$ between two affine surfaces extends to an isometry between the corresponding pointed at infinity hyperbolic orbifolds $\mathfrak{H}A_i$

conjugating the vertical flows, and, conversely, any isometry between pointed at infinity hyperbolic spaces $(\mathbf{H}^3_i, q_i)$ induces an isomorphism between the corresponding boundary affine planes $\mathcal{P}_i$. As a rule, we shall use the same notations both for an affine map $A_1 \to A_2$ and for the associated isometry $\mathfrak{H}A_1 \to \mathfrak{H}A_2$, which should not lead to a confusion.

Definition 1.42. We call the correspondence $A \mapsto \mathfrak{H}A$ between affine orbifold surfaces and pointed at infinity hyperbolic orbifolds the *hyperbolization functor*. △

If the surface A is complete (as a (G, X)-orbifold, see [Th91, Chapter 3] for a definition and discussion), then $\mathfrak{H}A$ is also complete (also as a (G, X)-orbifold). However, the structure group of $\mathfrak{H}A$ has compact stabilizers, so that completeness of $\mathfrak{H}A$ as a (G, X)-orbifold is equivalent to its completeness as a metric space [Th91, Proposition 3.4.15], and we have

Proposition 1.43. *The hyperbolization functor applied to a complete affine surface gives rise to a pointed at infinity hyperbolic orbifold of type (ii) from Proposition 1.22, and, conversely, any such orbifold can be obtained in this way.*

Remark 1.44. The simplest examples of non-complete affine surfaces are provided by the plane $\mathbb{R}^2$ (and its quotients by $\mathbb{Z}$ and $\mathbb{Z}^2$) with the affine structure determined by the Euclidean metric $e^{\varphi}(dx^2 + dy^2)$, where φ is a linear function.

Remark 1.45. We have considered the hyperbolization for affine surfaces only. However, the hyperbolization construction from Proposition 1.41 *verbatim* carries over to higher dimensional manifolds (orbifolds) endowed with a projective Euclidean ($\equiv$ similarity) structure.

1.1.6. *The basic class.* Let A be an affine surface, and $\sigma : A \to \mathfrak{H}A$ be a section of the fiber bundle $\mathfrak{p} : \mathfrak{H}A \to A$. Fix a (local) parallel section $\sigma_0 : A \to \mathfrak{H}A$, denote by

$$s_\sigma(z) = \beta_{\sigma_0}\Big(\sigma(z)\Big) = \beta\Big(\sigma_0(z), \sigma(z)\Big) \tag{1.46}$$

the *relative hyperbolic height* of σ with respect to σ_0 (cf. (1.34)), and put

$$\omega^\uparrow_\sigma = ds_\sigma \ . \tag{1.47}$$

Clearly, $\omega^\uparrow_\sigma$ does not depend on the choice of σ_0, and the forms $\omega^\uparrow_\sigma$ corresponding to different sections σ are all pairwise cohomologous.

Definition 1.48. We shall call $\omega^\uparrow_\sigma$ the *basic 1-form* on an affine surface A associated with the section σ. The cohomology class

$$\mathsf{b} = \mathsf{b}(A) = [\omega^\uparrow_\sigma] \in H^1(A)$$

will be called the *basic class* of A. △

One can easily check

Proposition 1.49. *The following properties are equivalent:*

(i) *The basic class $\mathsf{b}(A)$ of an affine surface A vanishes;*
(ii) *The Busemann class $\mathsf{B}(\mathfrak{H}A)$ of the hyperbolization $\mathfrak{H}A$ vanishes;*
(ii) *The fiber bundle $\mathfrak{p} : \mathfrak{H}A \to A$ admits a parallel section;*
(iv) *The affine structure on A can be refined to a Euclidean structure.*

Corollary 1.50. *As it follows from Proposition 1.22 and Proposition 1.43, the basic class vanishes for any complete affine surface (although this is not necessarily so for non-complete surfaces, cf. Remark 1.44).*

Remark 1.51. We emphasize that the triviality of the basic class of individual leaves of a *lamination* does *not* imply the triviality of the basic class of the whole lamination. The corresponding examples are given below in §**3.5** and §**5**.

Definition 1.52. If the class b vanishes, i.e., if all the basic 1-forms $\omega_\sigma^\uparrow$ are exact, we denote by β_σ the cocycle on A determined by the form $\omega_\sigma^\uparrow$ by formula (1.7) and call it the *basic cocycle on A induced by the section σ*. In other words,

$$\beta_\sigma(z_1, z_2) = \beta\Big(\sigma(z_1), \sigma(z_2)\Big) , \tag{1.53}$$

where β is the Busemann cocycle on $\mathfrak{H}A$ (which in this case is well-defined by Proposition 1.49). △

The basic cocycle on A is connected with the Busemann cocycle on $\mathfrak{H}A$ with the formula

$$\beta(h_1, h_2) = \beta_\sigma(\mathfrak{p}h_1, \mathfrak{p}h_2) + b_\sigma(h_2) - b_\sigma(h_1) , \tag{1.54}$$

where b_σ is the relative hyperbolic height (1.34) with respect to the section σ.

Remark 1.55. Even if $\mathsf{b} \neq 0$, Definition 1.52 and formulas (1.53), (1.54) still make sense locally (cf. Remark 1.23).

1.1.7. *Conformal metrics on affine surfaces.* Recall (see §**1.1.4**) that sections σ of the fiber bundle $\mathfrak{p} : \mathfrak{H}A \to A$ are in one-to-one correspondence with conformal Riemannian metrics ρ_σ on A. A parallel section σ corresponds to a parallel metric ρ_σ, i.e., it refines the affine structure on A to a Euclidean structure (cf. Definition 1.36). Denote by area_σ and $area_\sigma$ the Riemannian area form and the Riemannian measure, respectively, associated with a section σ. Conversely, any conformal metric (resp., any positive area form, or any absolutely continuous measure with continuous density) determines a section σ of the fiber bundle $\mathfrak{p}$.

As it follows from (1.33), in terms of the height function s_σ (1.46) with respect to a (local) parallel section σ_0

$$d\rho_\sigma^2 = \frac{|dz|^2}{e^{2s_\sigma}} , \qquad \mathsf{area}_\sigma = \frac{\mathsf{eucl}}{e^{2s_\sigma}} , \qquad d\, area_\sigma = \frac{d\ell}{e^{2s_\sigma}} , \tag{1.56}$$

where in the right-hand sides are the Euclidean metric $d\rho_{\sigma_0}^2 = |dz|^2$, its area form and the corresponding area Lebesgue measure, respectively.

Remark 1.57. Since for an affine surface there is a one-to-one correspondence between conformal metrics and the associated area forms, the Busemann cocycle can be considered as a particular case of the "volume cocycle" defined for an arbitrary affine manifold (e.g., see [Go]).

Formulas (1.14), (1.16), (1.34) and (1.56) imply the following relation between the lift of the form area_σ (resp., of the measure $area_\sigma$) to $\mathfrak{H}A$ (see Definition 1.31) and the hyperbolic volume form vol (resp., the hyperbolic measure vol) on the hyperbolic orbifold $\mathfrak{H}A$.

Proposition 1.58. *Let A be an affine surface, and let $\sigma : A \to \mathfrak{H}A$ be a section of the fiber bundle $\mathfrak{p} : A \to \mathfrak{H}A$. Then*

$$\frac{\widetilde{\mathsf{area}_\sigma}}{\mathsf{vol}} = \frac{d\,\widetilde{area_\sigma}}{d\,vol} = \exp[2b_\sigma]\ ,$$

where b_σ is the relative hyperbolic height (1.34) *with respect to the section σ.*

The Gauss curvature of the metric ρ_σ (1.56) is

$$K_\sigma = e^{2s_\sigma}\,\Delta_{\mathbf{E}} s_\sigma\ ,$$

and the curvature form is

$$\Omega_\sigma = K_\sigma \cdot \mathsf{area}_\sigma = \Delta_{\mathbf{E}} s_\sigma \cdot \mathsf{eucl}\ , \tag{1.59}$$

where eucl and $\Delta_{\mathbf{E}}$ are the area form and the Laplacian of the Euclidean parallel metric ρ_{σ_0}, respectively (for example, see the computation in [GH78, §0.5]).

In order to connect the curvature form Ω_σ with the basic 1-form $\omega^{\uparrow}_\sigma$ (1.47) let us recall that any Riemann surface S alongside with the usual differential d is endowed with the "twisted differential"

$$d^c = I^{-1} \circ d \circ I = \frac{1}{i}(\partial - \overline{\partial})\ , \tag{1.60}$$

where I is the almost complex structure on S, and $\partial, \overline{\partial}$ are the holomorphic and the antiholomorphic differentials, respectively, (e.g., see [We58, §II.2]).

Both operators d, d^c are real, and

$$\begin{aligned} 2\partial = d + id^c\ , \qquad 2\overline{\partial} = d - id^c\ ,\\ 2i\partial\overline{\partial} = -2i\overline{\partial}\partial = dd^c = -d^c d\ . \end{aligned} \tag{1.61}$$

In terms of a local coordinate $z = x + iy$

$$d^c\varphi = \frac{\partial\varphi}{\partial x}dy - \frac{\partial\varphi}{\partial y}dx\ , \qquad dd^c\varphi = \Delta_{\mathbf{E}}\varphi \cdot \mathsf{eucl}\ .$$

Applying the latter formula to the height function s_σ (1.46) and taking into account formulas (1.47) and (1.59) we obtain

Proposition 1.62. *Let A be an affine surface, and let $\sigma : A \to \mathfrak{H}A$ be a section of the fiber bundle $\mathfrak{p} : A \to \mathfrak{H}A$. Then the curvature form Ω_σ of the metric ρ_σ is connected with the basic 1-form $\omega^{\uparrow}_\sigma$ by the formula*

$$\Omega_\sigma = -d^c\omega^{\uparrow}_\sigma\ . \tag{1.63}$$

Finally, let $\varphi : A_1 \to A_2$ be an isomorphism between two affine surfaces A_i. Let us extend φ to an isometry $\mathfrak{H}A_1 \to \mathfrak{H}A_2$. Then φ maps sections of the fiber bundle $\mathfrak{p}_1 : \mathfrak{H}A_1 \to A_1$ to sections of the fiber bundle $\mathfrak{p}_2 : \mathfrak{H}A_2 \to A_2$ by the formula

$$\sigma \mapsto \varphi.\,\sigma\ , \qquad (\varphi.\,\sigma)\Big(\varphi(z)\Big) = \varphi\Big(\sigma(z)\Big)\ . \tag{1.64}$$

Let σ_i be sections of the fibre bundles $\mathfrak{p}_i : A_i \to \mathfrak{H}A_i$. Denote the norm of the differential $D\varphi$ at a point $z \in A_1$ measured with respect to the Riemannian metrics $\rho_{\sigma_1}, \rho_{\sigma_2}$ by

$$\|D\varphi(z)\|_{\sigma_1,\sigma_2} = \frac{\rho_{\sigma_2}}{\rho_{\varphi.\,\sigma_1}}\Big(\varphi(z)\Big) = \frac{\rho_{\varphi^{-1}.\,\sigma_2}}{\rho_{\sigma_1}}(z)\ .$$

Proposition 1.65. *Let $\varphi : A_1 \to A_2$ be an isomorphism between two complete affine surfaces, and let σ_i be sections of the fibre bundles $\mathfrak{p}_i : A_i \to \mathfrak{H}A_i$. Then for any two points $z, \zeta \in A_1$*

$$\log \frac{\|D\varphi(z)\|_{\sigma_1,\sigma_2}}{\|D\varphi(\zeta)\|_{\sigma_1,\sigma_2}} = \beta_{\sigma_2}\Big(\varphi(z), \varphi(\zeta)\Big) - \beta_{\sigma_1}(z, \zeta) , \tag{1.66}$$

where β_{σ_i} is the basic cocycle on the surface A_i induced by the section σ_i.

Proof. By formula (1.33)

$$\begin{aligned} \log \|D\varphi(z)\|_{\sigma_1,\sigma_2} &= \beta_2\Big(\sigma_2(\varphi(z)), \varphi.\, \sigma_1(\varphi(z))\Big) \\ &= \beta_1\Big(\varphi^{-1}.\, \sigma_2(z), \sigma_1(z)\Big) , \end{aligned} \tag{1.67}$$

where β_i is the Busemann cocycle on $\mathfrak{H}A_i$. Therefore, since φ preserves the Busemann cocycle,

$$\begin{aligned} \log \frac{\|D\varphi(z)\|_{\sigma_1,\sigma_2}}{\|D\varphi(\zeta)\|_{\sigma_1,\sigma_2}} &= \beta_2\Big(\sigma_2(\varphi(z)), \varphi.\, \sigma_1(\varphi(z))\Big) - \beta_2\Big(\sigma_2(\varphi(\zeta)), \varphi.\, \sigma_1(\varphi(\zeta))\Big) \\ &= \beta_2\Big(\sigma_2(\varphi(z)), \sigma_2(\varphi(\zeta))\Big) - \beta_2\Big(\varphi.\, \sigma_1(\varphi(z)), \varphi.\, \sigma_1(\varphi(\zeta))\Big) \\ &= \beta_{\sigma_2}\Big(\varphi(z)), \varphi(\zeta)\Big) - \beta_{\varphi.\,\sigma_1}\Big(\varphi(z), \varphi(\zeta)\Big) \\ &= \beta_{\sigma_2}\Big(\varphi(z)), \varphi(\zeta)\Big) - \beta_{\sigma_1}(z, \zeta) . \end{aligned}$$

□

1.2. **The notion of lamination.**

1.2.1. *General laminations.*

Definition 1.68. A topological space $\mathcal{B}$ is given a structure of dimension n *product lamination* if it is endowed with a homeomorphism

$$\psi : \mathcal{B} \to B \times T ,$$

where $B \cong \mathbb{D}^n$ is the open unit ball in $\mathbb{R}^n$, and T is another topological space. The sets

$$B_t = \psi^{-1}(B \times \{t\}) \subset \mathcal{B} , \qquad t \in T$$

are called *leaves*, and the sets

$$T_x = \psi^{-1}(\{x\} \times T) \subset \mathcal{B} , \qquad x \in B$$

are *transversals*. The arising "sliding" homeomorphisms $H_{\mathcal{B}}$ between the transversals T_{x_1} and T_{x_2} with $x_1, x_2 \in B$ are called *holonomies* (see Fig. 5). △

Definition 1.69. A dimension n *lamination* $\mathcal{L}$ of a topological space $\mathcal{X}$ (called the *total space of* $\mathcal{L}$) is determined by a family of local product laminations $\psi_i : \mathcal{B}_i \to B_i \times T_i$, such that the sets $\mathcal{B}_i$ (called *flow boxes*) form an open covering of $\mathcal{X}$, and the *transition maps*

$$\psi_{ij} = \psi_i \circ \psi_j^{-1} : \psi_j(\mathcal{B}_i \cap \mathcal{B}_j) \to \psi_i(\mathcal{B}_i \cap \mathcal{B}_j)$$

between the *coordinate charts* ψ_i are homeomorphisms taking local leaves to local leaves. The latter requirement on the transition maps implies that the local leaves (sometimes also called *plaques*) piece together to form *global leaves*, which are n-manifolds immersed

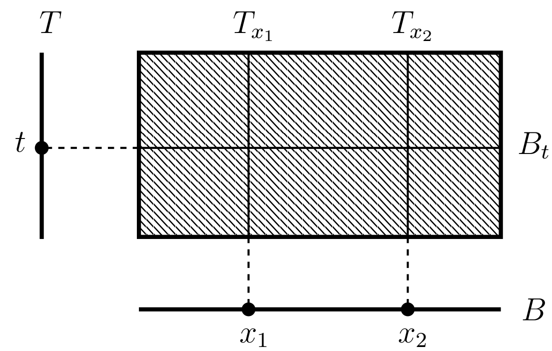

FIGURE 5

injectively in $\mathcal{X}$. Denote by $\mathfrak{T}(\mathcal{L})$ the set of all transversals of $\mathcal{L}$. The holonomies on all flow boxes taken together generate the *holonomy pseudogroup* acting on the disjoint union of all transversals $T \in \mathfrak{T}(\mathcal{L})$. △

The notion of lamination is a natural generalization of that of *foliation*, which corresponds to the case when $\mathcal{X}$ is a manifold and T_i are open unit balls of the complementary dimension $\dim \mathcal{X} - n$.

In what follows we shall always assume that the total space $\mathcal{X}$ is separable and metrizable. In order to simplify the notations we shall often denote the total space $\mathcal{X}$ and the lamination $\mathcal{L}$ by the same letter $\mathcal{L}$, which should not lead to a confusion.

Remark 1.70. Although the above definition of a lamination refers to open flow boxes only, below we shall also often use closed flow boxes and their (closed) transversals.

Definition 1.71. Denote the global leaf of the lamination $\mathcal{L}$ passing through a point $x \in \mathcal{X}$ by $L(x) = L_{\mathcal{L}}(x)$. The equivalence relation

$$\mathsf{graph}\,\mathcal{L} = \{(x_1, x_2) : L(x_1) = L(x_2)\} \subset \mathcal{X} \times \mathcal{X} \tag{1.72}$$

is called the *graph* of the lamination $\mathcal{L}$. △

1.2.2. *Geometric structures on laminations.* As usual, one can restrict the class of transition maps to preserve finer structures on local leaves. For example, one can endow the lamination $\mathcal{L}$ with a (leafwise) smooth, Riemannian or conformal structure. The latter means that there is an atlas of local charts $\psi : \mathcal{B} \to B \times T$ which are supplied with transversely continuous leafwise Riemannian metrics, such that the transition maps are conformal and transversely continuous.

Definition 1.73. For $n = 2$, if $\mathbb{D}^2 \subset \mathbb{R}^2 \cong \mathbb{C}$ is given the standard complex structure, and ψ_{ij} are leafwise conformal maps, $\mathcal{L}$ is called a *Riemann surface lamination*; then the global leaves have the structure of Riemann surfaces. Further restricting the class of leafwise transition maps (see Definition 1.36) one gets affine laminations, Euclidean laminations and translation laminations. Their leaves are affine, Euclidean and translation surfaces, respectively. If the leaves of an affine lamination are isomorphic to the standard affine plane $\mathbb{C}$, we also call it a $\mathbb{C}$-*lamination*. △

Definition 1.74. For $n = 3$, if the local leaves are given a Riemannian metric of constant negative curvature -1, and ψ_{ij} are local isometries, then we call $\mathcal{L}$ a 3-*dimensional hyperbolic lamination*, or a *hyperbolic* 3-*lamination*. In the case when all leaves of a hyperbolic 3-lamination are simply connected (i.e., isometric to $\mathbf{H}^3$), we call it a $\mathbf{H}^3$-*lamination*. If we continuously assign to each leaf $L \cong \mathbf{H}^3$ of a $\mathbf{H}^3$-lamination a point $q = q(L) \in \partial L$ on the sphere at infinity of L, we say that $\mathcal{L}$ is a *pointed at infinity* $\mathbf{H}^3$-*lamination*. △

To make the last definition more rigorous one has to pass from the $\mathbf{H}^3$-lamination $\mathcal{L}$ to its *unit tangent lamination* $U\mathcal{L}$. The leaves of the latter are S^2-bundles over the leaves of the original lamination. Continuous sections of $U\mathcal{L}$ are unit vector fields on $\mathcal{L}$ (tangent to the leaves). By definition, a pointed lamination is endowed with a unit vector field on $\mathcal{L}$ whose integral curves on every leaf form a family of asymptotic geodesics (with respect to the leaf hyperbolic metric). This vector field and the associated flow on a pointed lamination will be naturally called *vertical*. Then the distribution orthogonal to the vertical flow is integrable and gives leafwise horosphere foliations. In the same way one also defines a general *pointed at infinity hyperbolic* 3-*lamination* — its leaves are pointed at infinity hyperbolic 3-manifolds, cf. §**1.1.3**.

Definition 1.75. By analogy with the notion of orbifold, we can define an *orbifold lamination* (see [LM97]). The space $\mathcal{X}$ of such a lamination contains a subset of *singular points*. Any singular point $x \in \mathcal{X}$ has a neighbourhood $\mathcal{O}$ (an *orbifold box*) which is the quotient $\widetilde{\mathcal{O}}/G$ of a certain local product lamination $\widetilde{\mathcal{O}}$ (a *covering box*) by a finite group G of homeomorphisms fixing x. The group G is required to preserve the leaf structure of the local lamination, *not* its individual leaves. A leaf of an orbifold lamination is *singular* (i.e., is a true orbifold) if it passes through a singular point. As before, if all boxes carry an additional leafwise structure preserved by transition maps, and on covering boxes this structure is invariant with respect to the action of the corresponding finite groups G, then we say that the orbifold lamination is endowed with this additional structure. Thus, we can talk about orbifold affine (Euclidean, hyperbolic, etc.) laminations. △

In what follows by a lamination we shall always mean an orbifold lamination.

1.2.3. *The hyperbolization functor.*

Definition 1.76. Let us call a map between two laminations *laminar* if it carries leaves to leaves. Given any of the above additional leafwise structures (affine, Euclidean, hyperbolic, etc.), we can define the *category* of (orbifold) laminations endowed with this structure. A *morphism* in this category is a continuous laminar map from the space of one lamination to the space of the other lamination respecting the corresponding leafwise structures. For instance, an *affine map* between laminations is a continuous laminar map which is affine on the leaves. △

If G is a properly discontinuous group of automorphisms of an (orbifold) lamination $\mathcal{L}$, then the quotient $\mathcal{M} = \mathcal{L}/G$ also is an orbifold lamination and inherits the geometric structure of $\mathcal{L}$. For example, if $\mathcal{H}$ is a pointed at infinity hyperbolic lamination, then the quotient $\mathcal{M} = \mathcal{H}/G$ inherits the structure of a pointed at infinity hyperbolic lamination (however, even if $\mathcal{H}$ is an $\mathbf{H}^3$-lamination, $\mathcal{M}$ is not, generally speaking, an $\mathbf{H}^3$-lamination: if G has invariant leaves in $\mathcal{H}$, then the corresponding leaves of $\mathcal{M}$ are quotients of $\mathbf{H}^3$).

The hyperbolization functor $\mathfrak{H}$ introduced in Definition 1.42 extends to a covariant functor (also denoted $\mathfrak{H}$) from the category of (orbifold) affine 2-laminations to the category of (orbifold, pointed at infinity) hyperbolic 3-laminations.

Definition 1.77. Given a conformal lamination $\mathcal{L}$ denote by $\mathfrak{H}\mathcal{L}$ the *scaling bundle lamination* over $\mathcal{L}$ whose leaves are the scaling bundles over the leaves of $\mathcal{L}$ (see Definition 1.25). For an affine 2-lamination $\mathcal{A}$ the 3-lamination $\mathfrak{H}\mathcal{A} = \mathcal{H}$ is hyperbolic. Its leaves $L_{\mathcal{H}} = \mathfrak{H}L_{\mathcal{A}}$ are pointed at infinity hyperbolic 3-orbifolds corresponding to the leaves $L_{\mathcal{A}}$ of $\mathcal{A}$ (which are affine surfaces). More precisely, the space of $\mathcal{H}$ is the set of all pairs $\boldsymbol{h} = (\boldsymbol{z}, \varepsilon)$, where $\boldsymbol{z}$ is a point of the lamination $\mathcal{A}$, and ε is a local parallel Euclidean structure on a neighbourhood of the point $\boldsymbol{z}$ on its leaf $L_{\mathcal{A}}(\boldsymbol{z})$. Denote by $q(\boldsymbol{h}) \equiv q(\boldsymbol{z})$ the distinguished point on the ideal boundary of the leaf $L_{\mathcal{H}}(\boldsymbol{h}) \equiv L_{\mathcal{H}}(\boldsymbol{z})$ of $\mathcal{H}$ passing through a point $\boldsymbol{h}$ with $\mathfrak{p}\boldsymbol{h} = \boldsymbol{z}$. △

> *Without further notice we shall use below for various leafwise objects associated with affine 2-laminations (resp., pointed at infinity hyperbolic 3-laminations) the notations introduced in* §**1.1** *for a single affine surface (resp., pointed at infinity hyperbolic orbifold). Sometimes we shall add the subscript L indicating dependence on the leaf L.*

In particular, the leafwise projections $\mathfrak{p}_L$ (1.11) form a fiber bundle

$$\mathfrak{p} : \mathcal{H} \to \mathcal{A}\,, \qquad \boldsymbol{h} = (\boldsymbol{z}, \varepsilon) \mapsto \boldsymbol{z} \tag{1.78}$$

whose fibers $\mathfrak{p}^{-1}(\boldsymbol{z})$ are vertical geodesics on the pointed at infinity hyperbolic leaves; the leafwise vertical flows ξ_L and the leafwise vertical vector fields $v_L^{\uparrow}$ are organized into global continuous *vertical flow* ξ and *vertical vector field* $v^{\uparrow}$ on the lamination $\mathcal{H}$, respectively, etc.

Lemma 1.79. *For any affine lamination $\mathcal{A}$ there exists a continuous section $\sigma : \mathcal{A} \to \mathcal{H}$ of the fiber bundle $\mathfrak{p} : \mathcal{H} = \mathfrak{H}\mathcal{A} \to \mathcal{A}$.*

Proof. (cf. [MS74, Lemma 5.9]). Select a countable open covering U_i of $\mathcal{A}$ such that the bundle $\mathfrak{p} : \mathcal{H} \to \mathcal{A}$ is trivial over U_i. Let $\mathfrak{p}^{-1}U_i \equiv \mathcal{H}_i \cong U_i \times \mathbb{R}$ be its trivialization, and $\sigma_i : U_i \to \mathcal{H}_i$ be a local continuous section. Take a partition of unity $\psi_i : U_i \to \mathbb{R}$ on $\mathcal{A}$, and set $\sigma = \sum \psi_i \sigma_i$. Since the fibers of $\mathfrak{p}$ bear an affine structure (determined by the hyperbolic length parameterization), these convex combinations make sense and define a global continuous section of the bundle. Note that in fact this proof shows that the section σ can be chosen smooth. □

> *Below all sections are assumed to be continuous.*

Remark 1.80. The functor $\mathfrak{H}$ on the category of laminations is *not* invertible. The affine 2-lamination corresponding to a pointed at infinity hyperbolic 3-lamination $\mathcal{H}$ is well-defined if and only if the vertical flow is *proper*, i.e., the set

$$\{t : \xi^t K \cap K \neq \varnothing\}$$

is bounded in $\mathbb{R}$ for any compact $K \subset \mathcal{H}$. It is equivalent to saying that the quotient of $\mathcal{H}$ by the vertical flow is Hausdorff (cf. Proposition 1.43 and Remark 3.14).

Any affine map $\varphi : \mathcal{A}_1 \to \mathcal{A}_2$ between affine laminations extends to a map

$$\varphi_{\mathfrak{H}} : \mathcal{H}_1 = \mathfrak{H}\mathcal{A}_1 \to \mathfrak{H}\mathcal{A}_2 = \mathcal{H}_2$$

between the associated hyperbolic 3-laminations which acts as a local isometry between the leaves $L_{\mathcal{H}_1}(\boldsymbol{h})$ and $L_{\mathcal{H}_2}(\varphi_{\mathfrak{H}}\boldsymbol{h})$ (in the sequel we shall use the same notation φ for both the affine and the hyperbolic maps). If $\mathcal{A}_i$ are in fact $\mathbb{C}$-laminations, then $\varphi_{\mathfrak{H}}$ is a global leafwise isometry. Obviously, $\varphi_{\mathfrak{H}}$ also conjugates the vertical flows on $\mathcal{H}_1$ and $\mathcal{H}_2$.

Remark 1.81. It may well happen that an automorphism group of an affine lamination $\mathcal{A}$ is not properly discontinuous, whereas it becomes properly discontinuous on the hyperbolization $\mathfrak{H}\mathcal{A}$, cf. Remark 3.14 below.

1.3. **Cohomology of an affine lamination.**

1.3.1. *Leafwise cohomology.*

Definition 1.82. Given a C^∞-lamination $\mathcal{L}$, we can naturally define its *leafwise tangent and cotangent bundles*, $T\mathcal{L}$ and $T^*\mathcal{L}$, and the associated tensor bundles. Unless otherwise specified, all sections of these bundles are assumed to be smooth leafwise and continuous transversely. A section ω of the bundle $\bigwedge^p T^*\mathcal{L}$ is called a (leafwise) *p-form* on $\mathcal{L}$. By $\boldsymbol{\Omega}^p(\mathcal{L})$ we denote the linear space of all p-forms on $\mathcal{L}$. The *leafwise de Rham complex*

$$\boldsymbol{\Omega}^0(\mathcal{L}) \xrightarrow{d} \boldsymbol{\Omega}^1(\mathcal{L}) \xrightarrow{d} \dots \xrightarrow{d} \boldsymbol{\Omega}^n(\mathcal{L}) \xrightarrow{d} \dots$$

determined by the *leafwise differential* d gives rise to the *leafwise de Rham cohomology* $H^*_{\mathrm{dR}}(\mathcal{L}) \cong H^*(\mathcal{L})$ (it is also often called *tangential*). By $[\omega] \in H^p(\mathcal{L})$ we denote the *cohomology class* of a closed leafwise p-form ω. △

Remark 1.83. Of course, triviality of the cohomology of individual leaves does *not* imply triviality of the leafwise cohomology of the whole lamination.

Remark 1.84. The cohomology introduced in Definition 1.82 can be called *transversely continuous* (we are using transverse continuity stipulated in the definition of a lamination and require all the considered objects to be transversely continuous). For foliated manifolds (in the presence of a smooth transverse structure) one usually deals with the *transversely smooth* cohomology. On the other hand, replacing transverse continuity with just measurability with respect to the Borel structure one gets the *transversely Borel* cohomology. In the presence of a quasi-invariant transverse measure type (see below §**2**) one may neglect sets of measure 0, which leads to the *transversely measurable* cohomology. See [MS88, Chapter 3] for more details.

1.3.2. *The basic class.*

Definition 1.85. Let $\mathcal{H}$ be a pointed at infinity hyperbolic 3-lamination. The cohomology class $\mathsf{B}(\mathcal{H}) = [\omega^{\uparrow}] \in H^1(\mathcal{H})$ of the leafwise vertical 1-forms is called the *Busemann class* of $\mathcal{H}$.

Let now $\mathcal{A}$ be an affine lamination. We shall continue using the notations introduced in §**1.1.6** for the leafwise objects associated with the leafwise sections σ_L determined by a continuous global section $\sigma : \mathcal{A} \to \mathcal{H} = \mathfrak{H}\mathcal{A}$.

Definition 1.86. The cohomology class $\mathsf{b} \in H^1(\mathcal{A})$ of the basic 1-forms $\omega_\sigma^{\uparrow}$ (1.47) is called the *basic class* of the affine lamination $\mathcal{A}$. △

In complete analogy with Proposition 1.49 we have

Proposition 1.87. *The following properties are equivalent:*

(i) *The basic class $\mathsf{b}(\mathcal{A})$ of an affine lamination $\mathcal{A}$ vanishes;*
(ii) *The Busemann class $\mathsf{B}(\mathfrak{H}\mathcal{A})$ of the hyperbolization $\mathfrak{H}\mathcal{A}$ vanishes;*
(iii) *The fiber bundle $\mathfrak{p} : \mathfrak{H}\mathcal{A} \to \mathcal{A}$ admits a leafwise parallel section;*
(iv) *The leafwise affine structure on $\mathcal{A}$ can be refined to a leafwise Euclidean structure.*

Therefore, *the basic class b is the cohomological obstruction to the existence of a Euclidean structure on an affine lamination.*

Remark 1.88. We remind that the triviality of the basic class of individual leaves of $\mathcal{A}$ does *not* imply the triviality of the basic class of the whole lamination. See §**3.5** and §**5** for examples.

1.3.3. *Relation to the Euler class.* The operators d_L^c (1.60) on the leaves L of a Riemann surface lamination $\mathcal{L}$ piece together a global operator d^c acting on leafwise forms and satisfying formulas (1.61), where d is the leafwise differential. Therefore, we have

Proposition 1.89. *For a Riemann surface lamination $\mathcal{L}$ the operator d^c determines a homomorphism*

$$d^c : H^1(\mathcal{L}) \to H^2(\mathcal{L}) \ . \tag{1.90}$$

Definition 1.91. The *Euler class* $\mathsf{e} \in H^2(\mathcal{L})$ of a Riemann surface lamination $\mathcal{L}$ is

$$\mathsf{e} = \frac{1}{2\pi}[\Omega_\rho] \ ,$$

where $[\Omega_\rho]$ is the cohomology class of the leafwise curvature forms Ω_ρ of leafwise conformal metrics ρ. △

Formula (1.63) then implies

Proposition 1.92. *The basic class $\mathsf{b} \in H^1(\mathcal{A})$ and the Euler class $\mathsf{e} \in H^2(\mathcal{A})$ of an affine lamination $\mathcal{A}$ are related by the formula*

$$\mathsf{e} = -\frac{1}{2\pi} d^c \mathsf{b} \ . \tag{1.93}$$

Remark 1.94. If the leaves of an affine lamination $\mathcal{A}$ are complete affine surfaces, then, as it follows from Proposition 1.87, the basic class $\mathsf{b} \in H^1(\mathcal{A})$ is the cohomological obstruction to the existence of a transversely continuous family of *complete* leafwise flat metrics, whereas the Euler class $\mathsf{e} \in H^2(\mathcal{A})$ is the cohomological obstruction to the existence of a transversely continuous family of flat metrics which are not necessarily leafwise complete. It would be interesting to find out under what general conditions on $\mathcal{A}$ the classes b and e must be zero or non-zero simultaneously.

2. Measures and currents on laminations

2.1. **Measures on general laminations.** First we recall some notions from the general measure theory of equivalence relations and foliated spaces [FM77], [MS88] adapted to our situation. There are two ways of dealing with measures on laminations. One is based on separating variables and consists in considering leafwise (tangential) and transverse measures. The other approach deals with measures on the whole laminated space. Integration of a transverse measure against a leafwise measure gives a global measure. Conversely, the transverse and the leafwise measures can be, roughly speaking, recovered from a global measure as its conditional measures. We shall use both languages switching from one to the other at our convenience.

2.1.1. *Leafwise, transverse and global measures.*

Definition 2.1. Given a measure type preserving invertible transformation F from a *source measure space* (X, α) to a *target measure space* (Y, β), we shall use the term *"Radon–Nikodym Jacobian"* (or, simply *"Jacobian"*) for the Radon–Nikodym derivative

$$\operatorname{Jac} F(x) = \frac{d(F^{-1}\beta)}{d\alpha}(x) = \frac{d\beta}{d(F\alpha)}(Fx) ,$$

so that for any measurable subset $A \subset X$

$$\beta(FA) = \int_A \operatorname{Jac} F(x)\, d\alpha(x) . \tag{2.2}$$

If F is not invertible, but the preimage $F^{-1}(y)$ is finite for β-a.e. $y \in Y$, then we define the Jacobian as

$$\operatorname{Jac} F(x) = \frac{d(F^{-1}\beta)}{d\alpha}(x) = \frac{1}{\mathsf{card}[F^{-1}(F(x))]} \cdot \frac{d\beta}{d(F\alpha)}(Fx) ,$$

where $F^{-1}\beta$ is the integral of the uniform probability measures on the sets $F^{-1}(y)$ with respect to β, so that formula (2.2) still holds for any measurable $A \subset X$ such that the restriction of F onto A is one-to-one. △

In order to indicate dependence on the source measure α and the target measure β we shall sometimes add the corresponding subscripts (or just one subscript if the source and the target spaces coincide). Clearly,

$$\operatorname{Jac}_{\varphi\alpha,\psi\beta} F(x) = \frac{\psi(F(x))}{\varphi(x)} \operatorname{Jac}_{\alpha,\beta} F(x) \tag{2.3}$$

for positive densities φ, ψ on the spaces X and Y, respectively. The Radon–Nikodym Jacobian coincides with the usual Jacobian in the case when X and Y are Riemannian manifolds with Riemannian volumes α and β, respectively, and F is smooth.

Definition 2.4 ([MS88])**.** Let $\Delta : \mathsf{graph}\,\mathcal{L} \to \mathbb{R}_+$ be a measurable multiplicative cocycle of a lamination $\mathcal{L}$ (see §**1.3.1**). A family $\mu = \{\mu_T\}$ of Radon ($\equiv$ locally finite) measures on transversals T of the lamination $\mathcal{L}$ is *quasi-invariant with modulus* Δ if for any two transversals T, T' connected by a holonomy $H : T \to T'$ the Jacobian of this holonomy with respect to the measure μ is

$$\mathrm{Jac}_\mu H(t) = \Delta(t, H(t)) , \qquad t \in T . \tag{2.5}$$

We call such a family μ a *transverse measure with modulus* $\Delta = \Delta_\mu$. In particular, for the trivial cocycle $\Delta \equiv 1$ we have the usual definition of a *holonomy invariant measure* (e.g., see [Pl75]). △

Remark 2.6. If the lamination $\mathcal{L}$ is minimal, then any quasi-invariant measure μ has full support, i.e., $\operatorname{supp}\mu_{\mathrm{T}} = \mathrm{T}$ for any closed transversal T.

In view of formula (2.3), multiplying a transverse measure μ by a positive density $\varphi : \mathcal{L} \to \mathbb{R}_+$ gives a new transverse measure $\mu' = \varphi\mu$ with cohomologous modulus

$$\Delta_{\mu'}(x, y) = \frac{\varphi(y)}{\varphi(x)} \Delta_\mu(x, y) . \tag{2.7}$$

Definition 2.8. A *leafwise measure* $\lambda = \{\lambda_L\}$ on $\mathcal{L}$ is a family of Borel measures on leaves of $\mathcal{L}$ which is transversely measurable. The latter means that for any flow box $\mathcal{B} \cong B \times T$ the restrictions λ_t of the leafwise measure onto the local leaves B_t depend on the parameter $t \in T$ measurably in the weak topology: for any function $f \in C(B)$ the map $t \mapsto \langle f, \lambda_t \rangle$ is Borel. △

An absolutely continuous leafwise measure can be constructed for any (metrizable) leafwise smooth lamination by using a partition of the unity (cf. Lemma 1.79). Note that the individual leafwise measures λ_L are usually infinite.

Definition 2.9. A leafwise measure λ and a transverse measure μ with modulus Δ_μ determine a global measure $\lambda \star \mu$ on the (total space of the) lamination $\mathcal{L}$ in the following way. Given a flow box $\mathcal{B} \cong B \times T$ take a transversal $T = T_{x_0}$, $x_0 \in B$ and put

$$d(\lambda \star \mu)(x, t) = \Delta_\mu\Big((x_0, t), (x, t)\Big)\, d\lambda_t(x)\, d\mu_T(t) , \tag{2.10}$$

where λ_t is the leafwise measure on the local leaf B_t. One can easily check that for a given flow box $\mathcal{B}$ the measure $\lambda \star \mu$ does not depend on the choice of x_0, and any two intersecting flow boxes determine the same measure on their intersection, so that $\lambda \star \mu$ is a well-defined measure on $\mathcal{L}$. △

Both the leafwise measure λ and the transverse measure μ may be multiplied by a density $\varphi : \mathcal{L} \to \mathbb{R}$, cf. (2.7). Clearly,

$$(\varphi\lambda) \star \mu = \lambda \star (\varphi\mu) = \varphi(\lambda \star \mu) . \tag{2.11}$$

2.1.2. *Conditional measures and the Radon–Nikodym cocycle.* We shall now look at the inverse procedure: to what extent the measures λ and μ can be recovered from the global measure $\lambda \star \mu$.

Definition 2.12. Let θ be a measure on the lamination $\mathcal{L}$. Take a flow box $\mathcal{B} \cong B \times T$ with $\theta(\mathcal{B}) < \infty$. Then the restriction $\theta|_{\mathcal{B}}$ of θ to $\mathcal{B}$ can be uniquely (up to subsets of measure 0) decomposed as

$$d\theta|_{\mathcal{B}}(x,t) = d\overline{\theta}_{\mathcal{B}}(t)\, d\theta^t_{\mathcal{B}}(x) \ ,$$

where $\overline{\theta}_{\mathcal{B}}$ is the projection of $\theta|_{\mathcal{B}}$ onto T, and the measures $\theta^t_{\mathcal{B}},\ t \in T$ (called the *conditional measures* of $\theta|_{\mathcal{B}}$) are probability measures on the local leaves B_t (see [Ro49], [CFS82]). In other words, if f is an integrable function on $\mathcal{B}$, then the integrals of f with respect to the measures $\theta^t_{\mathcal{B}}$ depend on t measurably, and

$$\langle f, \theta|_{\mathcal{B}} \rangle = \int \left\langle f(\cdot, t), \theta^t_{\mathcal{B}} \right\rangle d\overline{\theta}_{\mathcal{B}}(t) \ .$$

The latter formula can be looked at as a generalization of the Fubini theorem. △

Definition 2.13. A global measure θ on $\mathcal{L}$ is *absolutely continuous* with respect to a leafwise measure λ if for any flow box $\mathcal{B} \cong B \times T$ with $\theta(\mathcal{B}) < \infty$ almost every conditional measure $\theta^t_{\mathcal{B}}$ is equivalent to the restriction λ_t of the corresponding leafwise measure onto the local leaf B_t. △

Remark 2.14. Clearly, Definition 2.13 only depends on the type of the measure θ and the leafwise measures λ. If the leafwise measures λ on a leafwise smooth lamination belong to the smooth leafwise measure type, then we shall usually omit the reference to λ and talk just about absolutely continuous measures on $\mathcal{L}$.

Due to the absence of a natural normalization, the conditional measures on global leaves are defined only projectively. Indeed, if we take two intersecting flow boxes $\mathcal{B}, \mathcal{B}'$, then on their intersection the conditional measures $\theta^t_{\mathcal{B}'}$ are proportional to the conditional measures $\theta^t_{\mathcal{B}}$ with leafwise constant densities. Hence almost all global leaves L of the lamination bear "conditional measures" θ_L well-defined up to a multiplicative constant (note that these measures may only be finite for "dissipative leaves", i.e., those which are ergodic components of $\mathsf{graph}\, \mathcal{L}$). It leads to the following definition:

Definition 2.15. If a measure θ on a lamination $\mathcal{L}$ is absolutely continuous with respect to a leafwise measure λ, then the measurable multiplicative cocycle

$$\Delta_{\theta,\lambda} : \mathsf{graph}\, \mathcal{L} \to \mathbb{R}_+ \ , \qquad \Delta_{\theta,\lambda}(x,y) = \frac{d\theta_L/d\lambda_L(y)}{d\theta_L/d\lambda_L(x)}$$

is well-defined. The cocycle $\Delta_{\theta,\lambda}$ is called the *Radon–Nikodym cocycle* (or, the *modulus*) of the measure θ with respect to the leafwise measure λ. △

We shall now explain why we use the term "Radon–Nikodym cocycle". Given a measure θ on $\mathcal{L}$ and a leafwise measure $\lambda = \{\lambda_L\}$, define the measure Θ on $\mathsf{graph}\, \mathcal{L} \subset \mathcal{L} \times \mathcal{L}$ by putting

$$d\Theta(x,y) = d\theta(x)\, d\lambda_{L(x)}(y) \ . \tag{2.16}$$

Then, by considering flow boxes, one arrives at

Proposition 2.17. *A measure θ on a lamination $\mathcal{L}$ is absolutely continuous with respect to a leafwise measure λ iff the associated measure Θ (2.16) is quasi-invariant with respect to the "flip transformation"*

$$\Sigma : (x,y) \mapsto (y,x)$$

on $\mathsf{graph}\,\mathcal{L}$. Then the Radon–Nikodym cocycle $\Delta_{\theta,\lambda}$ coincides with the Jacobian of the transformation Σ with respect to the measure Θ:

$$\Delta_{\theta,\lambda}(x,y) = \mathrm{Jac}_\Theta\,\Sigma(x,y) \ .$$

Clearly, for any density φ

$$\begin{aligned} \Delta_{\theta,\varphi\lambda}(x,y) &= \frac{\varphi(x)}{\varphi(y)}\Delta_{\theta,\lambda}(x,y) \ , \\ \Delta_{\varphi\theta,\lambda}(x,y) &= \frac{\varphi(y)}{\varphi(x)}\Delta_{\theta,\lambda}(x,y) \ , \end{aligned} \tag{2.18}$$

so that the (multiplicative) cohomology class of the Radon–Nikodym cocycle (we call it the *Radon–Nikodym cohomology class*) remains the same when the measures θ or λ are replaced with equivalent ones.

Remark 2.19. Proposition 2.17 is a straightforward generalization of an analogous result for countable equivalence relations proved in [FM77]. The measure Θ (2.16) is an analogue of the (right) counting measure on an equivalence relation: in our setup we need the leafwise measures λ instead of the leafwise counting measures used in [FM77].

Taking stock of previous definitions we now obtain

Proposition 2.20. *If μ is a transverse measure with modulus Δ_μ, and λ is a leafwise measure on a lamination $\mathcal{L}$, then the measure $\lambda\star\mu$ is absolutely continuous with respect to λ, and*

$$\Delta_{\lambda\star\mu,\lambda} = \Delta_\mu \ .$$

Conversely, if a measure θ on $\mathcal{L}$ is absolutely continuous with respect to λ, then the formula

$$\frac{d\overline{\theta}_{\mathcal{B}}}{d\mu_T}(t) = \int \Delta_{\theta,\lambda}\big(T(t),x\big)\,d\lambda_t(x) \ ,$$

defines a transverse measure μ with the modulus $\Delta_\mu = \Delta_{\theta,\lambda}$, and $\theta = \lambda\star\mu$. Here $\mathcal{B} \cong B\times T$ is a flow box, $T = T_{x_0}$, $x_0 \in B$ its transversal, and λ_t denotes the restriction of the leafwise measure $\lambda_{L(T(t))}$ onto the local leaf B_t.

Remark 2.21. The modulus Δ of a transverse measure μ is a function on $\mathsf{graph}\,\mathcal{L}$, i.e., it is cohomologically leafwise trivial. However, the modulus is only defined "mod 0", i.e., up to subsets of $\mathcal{L}$ of zero measure, so that it would be more rigorous to talk about the modulus as about a class of cocycles which differ mod 0. It may well happen that this class contains a cocycle (necessarily unique) which is *locally continuous* in the following sense: there exists a transversely continuous leafwise 1-form ω such that for μ-a.e. leaf L the restriction ω_L of ω onto L is cohomologically trivial, and $\log\Delta(x,y) = \int_\vartheta \omega_L$ for any smooth leafwise path ϑ joining any two points $x,y \in L$.

2.1.3. *SRB measures and affine structures.* As an important illustration to the above discussion let us consider a *hyperbolic attractor* A of a C^2-diffeomorphism f of a Riemannian manifold M (for example, see [Yo89] for the background on hyperbolic dynamics and smooth ergodic theory needed for understanding this section). Let $\mathcal{W}^u$ be the *unstable foliation* of f. As usual, the leaf of $\mathcal{W}^u$ passing through a point z is denoted by $\mathcal{W}^u(z)$. Let $\mathrm{Jac}^u f(z)$ denote the Jacobian of $f : \mathcal{W}^u(z) \to \mathcal{W}^u(fz)$ with respect to the leafwise Riemannian volume λ^u on the unstable leaves. Given a point z, put $z_k = f^k z$.

Definition 2.22. The locally continuous cocycle $\Delta_{\mathrm{AS}} : \mathsf{graph}\, \mathcal{W}^u \to \mathbb{R}_+$ defined as

$$\Delta_{\mathrm{AS}}(z,\zeta) = \prod_{k=1}^{\infty} \frac{\mathrm{Jac}^u f(z_{-k})}{\mathrm{Jac}^u f(\zeta_{-k})} \tag{2.23}$$

is called the *Anosov–Sinai cocycle* of the diffeomorphism f. △

This cocycle was first introduced by Anosov and Sinai in [AS67, p.151] in the context of $\mathcal{Y}$-diffeomorphisms. Note that the leafwise projective measure $\Delta_{\mathrm{AS}}\lambda^u$ is invariant under f. It turns out that this measure can be realized as the conditional measure for a certain global invariant measure called *Sinai–Ruelle–Bowen* or briefly *SRB* :

Theorem 2.24 ([Si72], [BR75])**.** *Any hyperbolic attractor carries a unique invariant measure m which is absolutely continuous on the unstable foliation. The Radon–Nikodym cocycle of this measure with respect to the leafwise Riemannian volume coincides with the Anosov–Sinai cocycle Δ_{AS}.*

Remark 2.25. The above discussion can be translated into the geometric language of connections (compare §**1.3.2**, §**3.2.1**). Consider the line bundle $\mathfrak{p} : \mathfrak{V}\mathcal{W}^u \to M$ of positive leafwise volume elements of the foliation $\mathcal{W}^u$ (if the leaves carry an additional conformal structure, then $\mathfrak{V}\mathcal{W}^u$ coincides with the scaling bundle $\mathfrak{H}\mathcal{W}^u$). The fibers of $\mathfrak{V}\mathcal{W}^u$ are endowed with the "hyperbolic metric" $d(v,w) = |\log(w/v)|$ (1.29). Sections of $\mathfrak{p}$ are leafwise measures with positive densities (or, rather, positive leafwise volume forms if the leaves are oriented), cf. the discussion in §**1.1.4**. One can say that $\mathfrak{p}$ is the "positive part" of the leafwise determinant bundle of $\mathcal{W}^u$.

Flat connections of $\mathfrak{p}$ are leafwise projective measures. In the same way as in §**1.1.6**, the flat connection determined by a section $\sigma : x \mapsto v_x$ gives rise to the "basic cocycle" of $\mathfrak{p}$

$$\beta_\sigma(x,y) = \log \frac{T_{x,y}(v_x)}{v_y} \ ,$$

where $T_{x,y}$ is the parallel translation from $\mathfrak{p}^{-1}(x)$ to $\mathfrak{p}^{-1}(y)$ (it is well-defined since the leaves of $\mathcal{W}^u$ are simply connected). Turning on the dynamics, we can express β_σ by means of the dynamical formula (2.23) (compare with Theorem 3.22) below) and obtain an invariant flat connection of $\mathfrak{p}$. Theorem 2.24 tells us that this connection is associated with a global invariant measure of f.

Remark 2.26. A similar discussion can be carried out for the strongly unstable foliation of a *partially* hyperbolic system as well.

In the case when the unstable foliation is one-dimensional, the above discussion admits a nice geometric refinement. Namely, a measure $d\mu = \rho dx$ on $\mathbb{R}$ with a positive continuous density ρ determines a Euclidean structure on R (compatible with its smooth structure) by means of the "local chart"

$$\psi(x) = \int_0^x \rho(t)\,dt,$$

and vice versa (cf. §**1.1.5**). Hence, a projective measure of that kind determines an affine structure on $\mathbb{R}$, and vice versa. Thus, if the unstable foliation $\mathcal{W}^u$ is one-dimensional, then the leafwise projective measure $\Delta_{\mathrm{AS}}\lambda^u$ determines an invariant affine structure on this foliation. In fact, it is a unique invariant affine structure on $\mathcal{W}^u$ compatible with its smooth structure.

In particular, this construction applies to the geodesic flow on a surface of non-constant negative curvature. In this situation the arising flow invariant affine structure on the leaves of the strongly unstable foliation (i.e., on punctured spheres at infinity) determines a hyperbolic metric on the leaves of the stable foliation (which are copies of the universal covering surface), i.e., uniformizes the universal covering surface, cf. §**4.6** below.

Let us now consider a one-dimensional C^2-map $f : I \to I$ with an absolutely continuous invariant measure m with positive characteristic exponent. Let

$$\widehat{f} : (\widehat{I}, \widehat{m}) \to (\widehat{I}, \widehat{m})$$

be its natural extension. As above, lift a Riemannian metric from I to the leaves of the Pesin unstable "foliation" of $\widehat{m}$ and denote by λ^u the leafwise Riemannian measures. Then

$$\mathrm{Jac}\, f(\widehat{x}) = |f'(x)| \;, \qquad x = \pi(\widehat{x}) \;.$$

Proposition 2.27 (Ledrappier [Le81, Proposition 3.6])**.** *The Radon–Nikodym cocycle of the measure $\widehat{m}$ on the Pesin unstable foliation with respect to the leafwise Riemannian measure λ^u is given by the explicit formula*

$$\Delta_{\widehat{m}}(\widehat{x}, \widehat{y}) = \prod_{k=1}^{\infty} \frac{|f'(x_{-k})|}{|f'(y_{-k})|} \;. \tag{2.28}$$

In other words, this formula gives the densities of the conditional measures of $\widehat{m}$ with respect to the Riemannian measure λ^u on the unstable leaves.

Again, the leafwise projective measure $\Delta_{\widehat{m}}\lambda$ determines an invariant affine structure on the (measurable) unstable foliation. In the case when $f : S^1 \to S^1$ is an expanding circle endomorphism we obtain an affine structure on the *one-dimensional solenoid* (the natural extension of f). This affine structure was used by Sullivan in the one-dimensional Renormalization Theory [MS93], [S92].

2.1.4. *Leafwise forms and currents.* The notion of leafwise measures can be also introduced in terms of leafwise forms (see §**1.3.1**). Denote by $\mathcal{V}(\mathcal{L}) = \mathbf{\Omega}^n(\mathcal{L})$ the space of leafwise volume forms of $\mathcal{L}$ (here n is the leafwise dimension of $\mathcal{L}$), and let $\mathcal{V}_0(\mathcal{L})$ be the subspace of volume forms with compact support. In the 2-dimensional case ($n = 2$), volume forms will be also called *area forms*. Locally, a volume n-form can be presented as

$$\omega = \omega(x,t)\, dx_1 \wedge \cdots \wedge dx_n \;,$$

where $(x,t) = (x_1, \ldots, x_n, t)$ are the local coordinates on a flow box $\mathcal{B} \cong B \times T$, and the function $(x,t) \mapsto \omega(x,t)$ is continuous in both variables. We emphasize that unlike in the usual definition, ω is only required to be continuous (and not necessarily smooth) in the leafwise direction. If the lamination is orientable (in particular, if $\mathcal{L}$ is a 2-dimensional lamination endowed with a leafwise complex structure), then it makes sense to talk about *positive* volume forms, i.e., those for which the function ω is strictly positive. Then, obviously, there is a one-to-one correspondence

$$\lambda_\omega \longleftrightarrow \omega_\lambda \tag{2.29}$$

between leafwise measures with positive continuous densities $\lambda \equiv \lambda_\omega$ and positive volume n-forms $\omega \equiv \omega_\lambda$, and an arbitrary volume form can be uniquely decomposed into a difference of two non-negative volume form (its positive and negative parts). A particular case is when the lamination is orientable and endowed with a transversely continuous leafwise Riemannian structure, in which situation the leafwise Riemannian structure gives rise to the corresponding positive leafwise Riemannian volume forms, and the associated leafwise measures are just the leafwise Riemannian volumes. In view of the identification (2.29) we shall use the notation

$$\omega \star \mu \equiv \lambda_\omega \star \mu \tag{2.30}$$

for the global measure (2.10) on $\mathcal{X}$ determined by a leafwise volume form ω and a transverse measure μ.

Denote by $\mathcal{C}(\mathcal{L})$ the dual to the space $\mathcal{V}_0(\mathcal{L})$. Elements of $\mathcal{C}(\mathcal{L})$ are called *currents*. [Note that currents are usually meant to be functionals on the space of C^∞ forms, whereas our space $\mathcal{V}_0(\mathcal{L})$ consists of C^0 forms.] The result of applying a current c to a form ω will be denoted by $\langle \omega, c\rangle$. The space of currents is endowed with the topology of weak convergence on compact sets:

$$c_n \to c \qquad \Longleftrightarrow \qquad \langle \omega, c_n\rangle \to \langle \omega, c\rangle \qquad \forall\, \omega \in \mathcal{V}_0(\mathcal{L}) \ . \tag{2.31}$$

Given a domain D on a leaf of the lamination $\mathcal{L}$ denote by $[D]$ the *integration current* over D:

$$\langle \omega, [D]\rangle = \int_D \omega \ .$$

Definition 2.32. For any given transverse measure μ the pairing $\star$ (2.30) determines a continuous functional on the space $\mathcal{V}_0(\mathcal{L})$ (i.e., a current) by the formula

$$\langle \omega, [\mu]\rangle = \omega \star \mu(\mathcal{X}) \ .$$

The current $[\mu]$ is called the *Ruelle–Sullivan current* associated with the transverse measure μ [MS88]. $\triangle$

Proposition 2.33 ([MS88]). *The Ruelle–Sullivan current $[\mu]$ is closed if and only if the transverse measure μ is invariant.*

2.2. **Measures and streams on affine and hyperbolic laminations.** Let us consider the special case of a conformal lamination $\mathcal{L}$ (in particular, of an affine lamination) and the associated scaling bundle lamination $\mathcal{H} = \mathfrak{H}\mathcal{L}$ (recall that the scaling bundle lamination over an affine lamination is a hyperbolic lamination). We shall look at the relations between the measures on $\mathcal{L}$ and $\mathcal{H}$ which follow from the fact that $\mathcal{H}$ is fibered over $\mathcal{L}$ and the fibers are endowed with canonical Lebesgue measures.

2.2.1. *Lifted measures.*

Definition 2.34. Recall (see §**1.1.4**) that given a conformal manifold M one can associate with any measure λ on M its *lift* $\widetilde{\lambda}$ to the scaling bundle $\mathfrak{H}M$ obtained by integrating the Lebesgue measures ℓ_z on fibers $\mathfrak{p}^{-1}(z)$, $z \in M$ against the measure λ. Given a conformal lamination $\mathcal{L}$, one can apply this construction both to leafwise and to global measures on $\mathcal{L}$. The *lift of a leafwise measure* $\lambda = \{\lambda_L\}$ is a leafwise measure $\widetilde{\lambda} = \{\widetilde{\lambda}_L\}$ on $\mathcal{H}$, and the *lift of a global measure* θ is a global measure $\widetilde{\theta}$ on $\mathcal{H}$. △

We shall now define the lift of transverse measures from $\mathcal{L}$ to $\mathcal{H}$. Recall that the points $\boldsymbol{z}$ of $\mathcal{L}$ can be identified with the fibers $\mathfrak{p}^{-1}(\boldsymbol{z})$ on the leaves of $\mathcal{H}$. Therefore, for any transversal $T_{\mathcal{H}}$ of $\mathcal{H}$ the image $T_{\mathcal{L}} = \mathfrak{p}T_{\mathcal{H}}$ is a transversal of $\mathcal{L}$. On the other hand, in order to pass from a transversal $T_{\mathcal{L}}$ of $\mathcal{L}$ to a transversal $T_{\mathcal{H}}$ of $\mathcal{H}$ we have to take care of the additional dimension of the leaves of $\mathcal{H}$ and to specify the location of the points of $T_{\mathcal{H}}$ on the fibers $\mathfrak{p}^{-1}(\boldsymbol{z})$, $\boldsymbol{z} \in T_{\mathcal{L}}$. In particular, given a section $\sigma : \mathcal{L} \to \mathcal{H}$, the image $\sigma(T_{\mathcal{L}})$ is a transversal of $\mathcal{H}$.

Let now μ be a quasi-invariant transverse measure of the lamination $\mathcal{L}$ with modulus Δ_μ. For any transversal $T \in \mathfrak{T}(\mathcal{H})$ define the measure $\widetilde{\mu}_T$ on T as

$$\widetilde{\mu}_T(X) = \mu_{\mathfrak{p}T}(\mathfrak{p}X) , \qquad X \subset T . \tag{2.35}$$

One can easily check

Proposition 2.36. *The measures $\widetilde{\mu}_T$, $T \in \mathfrak{T}(\mathcal{H})$ determine a quasi-invariant transverse measure $\widetilde{\mu}$ with the modulus*

$$\Delta_{\widetilde{\mu}}(\boldsymbol{h}_1, \boldsymbol{h}_2) = \Delta_\mu(\mathfrak{p}\boldsymbol{h}_1, \mathfrak{p}\boldsymbol{h}_2) , \tag{2.37}$$

in particular, the measure $\widetilde{\mu}$ is invariant with respect to all holonomies consisting in moving along the leafwise fibers of the bundle $\mathfrak{p}$.

Definition 2.38. The transverse measure $\widetilde{\mu}$ (2.35) of the lamination $\mathcal{H}$ is called the *lift* of the transverse measure μ of the lamination $\mathcal{L}$. △

Definition 2.9 then implies

Proposition 2.39. *Let λ and μ be a leafwise and a quasi-invariant transverse measure of a conformal lamination, respectively. Then*

$$\widetilde{\lambda \star \mu} = \widetilde{\lambda} \star \widetilde{\mu} .$$

2.2.2. *Conformal streams.*

Definition 2.40. A *conformal stream* of dimension δ on a conformal manifold M is a correspondence $\eta : \rho \to \eta_\rho$ which assigns to any conformal metric ρ a (non-zero) Radon measure η_ρ on M in such a way that all the measures η_ρ are pairwise equivalent with the Radon–Nikodym derivatives

$$\frac{d\eta_{\rho'}}{d\eta_\rho} = \left(\frac{\rho'}{\rho}\right)^\delta . \tag{2.41}$$

In other words, $d\eta_\rho \otimes d\rho^{-\delta}$ is a *conformally covariant tensor* (cf. the discussion of tensor fields of non-integer degree in [Fu86, Section 1.2.2]). We will also say that the dependence of the measure η_ρ on the metric $\rho \in \mathcal{C}$ is *δ-covariant*. △

Remark 2.42. Clearly, any conformal isomorphism between two conformal manifolds M and M' allows one to transfer conformal streams from M to M'.

The simplest example is provided by the *volume stream vol*, which consists in assigning to a metric ρ the Riemannian volume vol_ρ, and its dimension coincides with the dimension of M. [We slightly abuse the notations, since depending on the context vol may denote both the Riemannian volume $vol = vol_\rho$ associated with a fixed Riemannian metric ρ and the volume stream $vol = \{vol_\rho\}$, which should not lead to a confusion.] More generally, let ℓ^δ_ρ denote the Hausdorff measure on M in dimension $\delta \in (0, \dim M]$ corresponding to a conformal metric ρ. Take some closed subset $X \subset M$ such that $\ell^\delta_\rho|_X$ is a non-zero Radon measure on X (thus, $\mathrm{HD}(X) = \delta$, where HD stands for the Hausdorff dimension). Then the correspondence $\rho \mapsto \ell^\delta_\rho|_X$ is a conformal stream on M of dimension δ (supported on X). Somewhat loosely, it will be called the *Hausdorff stream* on X.

Remark 2.43. A more traditional term would be *conformal measure* or *conformal density*, see Sullivan [S79].

Theorem 2.44. *There is a one-to-one correspondence between conformal streams η of given dimension δ on a conformal manifold M and measures $\overline{\eta}$ on the scaling bundle $\mathfrak{H}M$ such that*

$$\xi^\tau \overline{\eta} = \exp[\delta\tau] \cdot \overline{\eta} \, , \qquad \tau \in \mathbb{R} \, , \tag{2.45}$$

where ξ is the scaling flow on $\mathfrak{H}M$.

Proof. Take the conformal metric $\rho = \rho_\sigma$ corresponding to a section $\sigma : M \to \mathfrak{H}M$, and put

$$\overline{\eta} = \exp[-\delta b_\rho] \cdot \widetilde{\eta}_\rho \, , \tag{2.46}$$

where $\widetilde{\eta}_\rho$ is the lift of the measure η_ρ to $\mathfrak{H}M$ (Definition 1.31), and b_ρ is the relative hyperbolic height (1.34) with respect to the section σ. Then, as it follows from (1.35), the measure $\overline{\eta}$ is the same for all conformal measures ρ and depends on the conformal stream η only. Since the measure $\widetilde{\eta}_\rho$ in (2.46) is invariant with respect to the scaling flow, the measure $\overline{\eta}$ satisfies the relation (2.45).

Conversely, if a measure $\overline{\eta}$ on $\mathfrak{H}M$ satisfies (2.45), then it decomposes into an integral of measures on the fibers of $\mathfrak{p}$ also satisfying the same scaling condition. Therefore, the latter measures must be equivalent to the Lebesgue measure with the density proportional to the exponential function $\tau \mapsto \exp[-\delta\tau]$, and formula (2.46) then allows one to recover the conformal stream from the measure $\overline{\eta}$. □

A conformal stream can be always obtained just by assigning a certain measure η_0 to a fixed metric ρ_0 and then putting

$$\eta_\rho = \left(\frac{\rho}{\rho_0}\right)^\delta \eta_0 \, ,$$

which is not very interesting. However, the point of this notion is in looking for conformal streams with additional invariance properties (cf. §**4.4** and §**5.4** below).

Definition 2.47. If F is a diffeomorphism of M preserving its conformal structure, then F acts on δ-dimensional conformal streams by the formula

$$(F\eta)_\rho = F(\eta_{F^{-1}\rho}) . \tag{2.48}$$

A conformal stream η is called *F-invariant* if $F\eta = \eta$, in other words, if the correspondence $\rho \mapsto \eta_\rho$ is equivariant with respect to the action of F on the spaces of metrics and of measures. △

Clearly, the action (2.48) on conformal streams on M is conjugate to the action of F on associated measures on $\mathfrak{H}M$. In particular, a conformal stream η on M is F-invariant iff the corresponding measure $\overline{\eta}$ on $\mathfrak{H}M$ is F-invariant. A straightforward check also yields

Proposition 2.49. *A conformal stream η is F-invariant iff*

$$\mathrm{Jac}_{\eta_\rho} F = \|DF\|_\rho^\delta$$

for any ($\equiv$ some) conformal metric ρ.

Important examples of non-trivial conformal streams on the Riemann sphere $\overline{\mathbb{C}}$ are provided by the results of Patterson and Sullivan asserting that any Kleinian group or rational map admits an invariant conformal stream of some dimension $\delta \in (0, 2]$ supported on its limit (resp., Julia) set X (see §**5.4**, [S83], [DU91a]). Note that the Hausdorff stream on X is invariant whenever it exists (i.e., when $0 < \ell_\rho^\delta(X) < \infty$ for $\delta = \mathrm{HD}(X)$). So, the Patterson–Sullivan stream on X can be viewed as a dynamical replacement of the Hausdorff stream on X, in the case when the latter does not exist.

2.2.3. *Leafwise and transverse conformal streams.*

Definition 2.50. A *leafwise conformal stream λ of dimension $\delta \geq 0$* on a conformal lamination $\mathcal{L}$ is a family of δ-conformal streams λ_L (see Definition 2.40) on the leaves of $\mathcal{L}$ which is locally continuous in the transverse direction. △

Leafwise application of Theorem 2.44 then immediately yields

Theorem 2.51. *Given a conformal lamination $\mathcal{L}$, there is a one-to-one correspondence between leafwise conformal streams λ of given dimension δ on $\mathcal{L}$ and transversely continuous leafwise measures*

$$\overline{\lambda} = \exp[-\delta b_\rho]\widetilde{\lambda}_\rho \tag{2.52}$$

on the lamination $\mathfrak{H}\mathcal{L}$ such that

$$\xi^\tau \overline{\lambda} = \exp[\delta\tau] \cdot \overline{\lambda} , \qquad \tau \in \mathbb{R} , \tag{2.53}$$

where ξ is the leafwise scaling flow on $\mathfrak{H}\mathcal{L}$.

Alongside with the leafwise conformal streams one can also introduce a dual transverse notion.

Definition 2.54. A *transverse conformal stream of dimension $\delta \geq 0$* on a conformal lamination $\mathcal{L}$ is a family of quasi-invariant transverse measures μ_ρ associated with leafwise conformal metrics ρ and such that

$$\frac{d\mu_{\rho'}}{d\mu_\rho} = \left(\frac{\rho'}{\rho}\right)^{-\delta} . \tag{2.55}$$

△

Remark 2.56. Note that the sign of the exponent in the right-hand side of formula (2.55) is opposite to the sign in formula (2.41) for leafwise conformal streams. This definition has the following motivation. Let θ be a global measure on $\mathcal{L}$ which is leafwise smooth, and let $vol = \{vol_\rho\}$ be the leafwise volume stream. Then disintegrating the measure θ with respect to the leafwise measure vol_ρ associated with a leafwise conformal metric ρ provides us with a transverse measure μ_ρ (see Proposition 2.20). As it follows from formula (2.11), then

$$\frac{d\mu_{\rho'}}{d\mu_\rho} = \left(\frac{d\,vol_{\rho'}}{d\,vol_\rho}\right)^{-1} = \left(\frac{\rho'}{\rho}\right)^{-d} ,$$

where d is the leafwise dimension of $\mathcal{L}$, so that μ is a transverse conformal stream of dimension d.

The next result immediately follows from Definition 2.50 and Definition 2.54.

Theorem 2.57. *If λ and μ are a leafwise and a transverse conformal streams of the same dimension of a lamination $\mathcal{L}$, then the global measure on $\mathcal{L}$*

$$\boldsymbol{v} = \lambda \star \mu = \lambda_\rho \star \mu_\rho$$

is independent of the choice of a leafwise metric ρ.

Let now $\mathcal{A}$ be an affine lamination. Recall that leafwise conformal metrics $\rho = \rho_\sigma$ on $\mathcal{A}$ are in one-to-one correspondence with the sections σ of the fiber bundle $\mathfrak{p} : \mathcal{H} = \mathfrak{H}\mathcal{A} \to \mathcal{A}$, and we shall sometimes indicate dependence of conformal measures on sections σ rather than on the associated metrics ρ_σ. In terms of the Busemann cocycle β on $\mathfrak{H}\mathcal{A}$ the definitions of leafwise and transverse conformal streams take, respectively, the form

$$\begin{aligned} \frac{d\lambda_{\sigma'}}{d\lambda_\sigma} &= \left(\frac{\rho_{\sigma'}}{\rho_\sigma}\right)^{\delta} = \exp\Big[-\delta\beta(\sigma,\sigma')\Big] , \\ \frac{d\mu_{\sigma'}}{d\mu_\sigma} &= \left(\frac{\rho_{\sigma'}}{\rho_\sigma}\right)^{-\delta} = \exp\Big[\delta\beta(\sigma,\sigma')\Big] . \end{aligned} \tag{2.58}$$

Note that by the definition of an affine lamination $\mathcal{A}$ its leaves are endowed with projective Euclidean metrics, so that a leafwise conformal stream determines a leafwise projective measure on $\mathcal{A}$.

2.2.4. *Parallel transverse conformal streams.* For transverse conformal streams there is yet another notion of invariance which is based on the existence of a flat leafwise connection and ensuing possibility to make parallel transport of conformal metrics along the leaves:

Definition 2.59. A transverse conformal stream on an affine lamination $\mathcal{A}$ is called *parallel* if it assigns to any (local) parallel leafwise conformal metric ρ a (local) holonomy invariant transverse measure μ_ρ. △

Remark 2.60. This notion makes sense for leafwise conformal streams as well, but the only parallel leafwise conformal stream is the area stream.

Formula (2.58) readily implies

Proposition 2.61. *A transverse conformal stream μ of dimension δ on an affine lamination $\mathcal{A}$ is parallel iff for any ($\equiv$ some) leafwise conformal metric ρ the Radon–Nikodym derivative of the associated transverse measure μ_ρ is*

$$\Delta_{\mu_\rho} = \exp[\delta\beta_\rho] \,, \tag{2.62}$$

where $\beta_\rho = \beta_\sigma$ is the basic cocycle determined by the section $\sigma : \mathfrak{H}\mathcal{A} \to \mathcal{A}$ associated with the metric $\rho = \rho_\sigma$.

Theorem 2.63. *Let $\mathcal{A}$ be an affine lamination and let $\mathcal{H} = \mathfrak{H}\mathcal{A}$ be its hyperbolization. Then there exists a natural bijective correspondence between*

(i) *Parallel transverse conformal streams μ on $\mathcal{A}$ of dimension $\delta \geq 0$;*
(ii) *Transverse measures $\overline{\mu}$ of the lamination $\mathcal{H}$ with modulus* $\exp[\delta\beta]$*;*
(iii) *Absolutely continuous measures ω^μ on the lamination $\mathcal{H}$ with the Radon–Nikodym cocycle* $\exp[\delta\beta]$ *with respect to the leafwise hyperbolic volume vol.*

Proof. We begin with a parallel transverse conformal stream μ of dimension δ. Fix a leafwise conformal metric $\rho = \rho_\sigma$ corresponding to a section $\sigma : \mathcal{A} \to \mathcal{H}$ and put

$$\overline{\mu} = \exp[\delta b_\rho] \cdot \widetilde{\mu}_\rho \,, \tag{2.64}$$

where $\widetilde{\mu}_\rho$ is the transverse measure of $\mathcal{H}$ obtained by lifting the measure μ_ρ (see Definition 2.38), and b_ρ is the relative hyperbolic height (1.34) with respect to the section σ. For another metric ρ' and the associated section σ'

$$\frac{d\widetilde{\mu}_{\rho'}}{d\widetilde{\mu}_\rho} = \frac{d\mu_{\rho'}}{d\mu_\rho} \circ \mathfrak{p} = \exp\Big[\delta\beta(\sigma,\sigma')\Big] \circ \mathfrak{p} = \exp\Big[\delta(b_{\rho'} - b_\rho)\Big] \,,$$

so that the measure $\overline{\mu}$ is independent of the choice of the metric ρ. As it follows from (2.7) and (1.34), the modulus of the measure $\overline{\mu}$ is

$$\begin{aligned}
\Delta_{\overline{\mu}}(\boldsymbol{h}_1, \boldsymbol{h}_2) &= \exp\Big[\delta b_\rho(\boldsymbol{h}_2) - \delta b_\rho(\boldsymbol{h}_1)\Big] \cdot \Delta_{\widetilde{\mu}_\rho}(\boldsymbol{h}_1, \boldsymbol{h}_2) \\
&= \exp\Big[\delta\beta(\boldsymbol{h}_1, \boldsymbol{h}_2) - \delta\beta_\rho(\mathfrak{p}\boldsymbol{h}_1, \mathfrak{p}\boldsymbol{h}_2)\Big] \cdot \Delta_{\mu_\rho}(\mathfrak{p}\boldsymbol{h}_1, \mathfrak{p}\boldsymbol{h}_2) \\
&= \exp\Big[\delta\beta(\boldsymbol{h}_1, \boldsymbol{h}_2)\Big] \,.
\end{aligned}$$

Now the measure

$$\omega^\mu = vol \star \overline{\mu} \,, \tag{2.65}$$

where *vol* is the leafwise hyperbolic volume, provides the sought for global measure on $\mathcal{H}$ satisfying property (iii).

Conversely, disintegration of a global measure on $\mathcal{H}$ with property (iii) gives us a transverse measure $\overline{\mu}$ of $\mathcal{H}$ with property (ii). The restriction of $\overline{\mu}$ onto the transversals of $\mathcal{H}$ determined by a metric ρ gives a transverse measure μ_ρ of $\mathcal{A}$. One can easily verify that the family $\{\mu_\rho\}$ is then a parallel transverse conformal stream of dimension δ. □

Remark 2.66. Since the modulus of the transverse measure $\overline{\mu}$ on $\mathcal{H}$ is $\exp[\delta\beta]$, under the action of the vertical flow it transforms as

$$\xi^\tau \overline{\mu} = \exp[-\delta\tau] \cdot \overline{\mu} \,, \qquad \tau \in \mathbb{R} \,. \tag{2.67}$$

On the other hand, the leafwise hyperbolic volume transforms as $\xi^\tau\, vol = \exp[2\tau]\cdot vol$. Therefore, the measure ω^μ (2.65) transforms under ξ as

$$\xi^\tau \omega^\mu = \exp\Big[\tau(2-\delta)\Big]\cdot\omega^\mu\ , \qquad \forall\, \tau\in\mathbb{R}\ .$$

Theorem 2.68. *Let λ and μ be, respectively, a leafwise and a parallel transverse conformal streams of the same dimension on an affine lamination $\mathcal{A}$, and $\boldsymbol{\upsilon} = \lambda\star\mu$ be their product. Then*

$$\overline{\lambda}\star\overline{\mu} = \widetilde{\boldsymbol{\upsilon}}\ ,$$

and the measure $\widetilde{\boldsymbol{\upsilon}}$ on $\mathcal{H}$ is invariant with respect to the vertical flow ξ.

Proof. As it follows from formulas (2.52) and (2.64), for any leafwise metric ρ

$$\overline{\lambda}\star\overline{\mu} = \widetilde{\lambda}_\rho\star\widetilde{\mu}_\rho\ ,$$

whereas by Proposition 2.39 and Theorem 2.57

$$\widetilde{\lambda}_\rho\star\widetilde{\mu}_\rho = \widetilde{\lambda_\rho\star\mu_\rho} = \widetilde{\lambda\star\mu}\ .$$

The ξ-invariance then follows from the definition of the lift $\widetilde{\lambda\star\mu}$ (it can be also directly verified by using formulas (2.53) and (2.67)). □

2.3. **Conformal streams, harmonic measures and harmonic functions.**

2.3.1. *Brownian motion on Riemannian manifolds.* We begin with briefly recalling some basic notions from the theory of Brownian motion on Riemannian manifolds (e.g., see [Gr99]).

Let M be a complete connected Riemannian manifold with the Riemannian volume *vol* and the Laplace–Beltrami operator Δ. Denote by

$$p(t,x,y)\ , \qquad t\in\mathbb{R}_+,\ x,y\in M \tag{2.69}$$

the fundamental solution of the heat equation, and by

$$d\pi_x^t(y) = p(t,x,y)\, d\, vol(y) \tag{2.70}$$

the corresponding transition measures. If M has uniformly bounded sectional curvatures, then it is stochastically complete, i.e., all the measures π_x^t are probability ones. Denote by

$$P^t f(x) = \langle \pi_x^t, f\rangle \tag{2.71}$$

the transition operators determined by the measures π_x^t. Then

$$P^t = e^{t\Delta}\ , \qquad t>0\ , \tag{2.72}$$

so that these operators constitute a semigroup. They determine a time homogeneous Markov process on M with continuous sample paths called the *Brownian motion*. As it follows from formula (2.72), a function f on M is λ-*harmonic* in the usual sense (i.e., $\Delta f = \lambda f$) iff

$$P^t f = e^{t\lambda} f \qquad \text{for all } (\equiv \text{some})\ t>0\ . \tag{2.73}$$

The dual operators $Q^t = (P^t)^*$ act on the space of locally finite measures on M by the formula

$$Q^t\theta = \int \pi_x^t \, d\theta(x) \; . \tag{2.74}$$

A measure ω on M is called λ-*stationary* if

$$Q^t\omega = e^{t\lambda}\omega \qquad \text{for all } (\equiv \text{some}) \; t > 0 \; . \tag{2.75}$$

Since the transition densities (2.69) are symmetric with respect to the coordinates x and y, the operators P^t are self-adjoint in the space $L^2(M, vol)$. Therefore, a measure ω is λ-stationary iff it is absolutely continuous with respect to the Riemannian volume vol, and its density $d\omega/d\,vol$ is a λ-harmonic function.

Remark 2.76. Formulas (2.73) and (2.75) show that λ-harmonic functions and λ-stationary measures are the same for the Brownian motion as a continuous time process and for its *time discretization*, which is a discrete time Markov chain on M with transition probabilities π_x^1, $x \in M$. It is easier to deal with the discretization of the Brownian motion as there we do not have to care about various problems connected with the time continuity.

2.3.2. *λ-harmonic measures of the Brownian motion on laminations.* Let now $\mathcal{L}$ be a (leafwise) Riemannian lamination (see §**1.2.2** for a definition) with stochastically complete leaves. Then the leafwise Brownian motions piece together a Markov process called the *Brownian motion on the lamination $\mathcal{L}$*. Its transition probabilities are still given by formula (2.70), but now it has to be considered leafwise, i.e., the transition density $p = p_L$ and the measure $vol = vol_L$ in the right-hand side of (2.70) are the leafwise ones on the leaf $L = L(x)$ of the point x. Note that the transition probabilities of the leafwise Brownian motion are concentrated on single leaves of $\mathcal{L}$, so that they are *not* absolutely continuous with respect to any measure on $\mathcal{L}$ (unless $\mathcal{L}$ consists of a countable number of leaves). We shall use the same notations P^t (resp., Q^t) for the global transition operators (2.71) on $\mathcal{L}$ (resp., their duals (2.74)).

Lemma 2.77. *Let θ be an absolutely continuous Radon measure on a Riemannian lamination $\mathcal{L}$. Denote by $\Delta_{\theta,vol}(x, y)$, $(x, y) \in \mathsf{graph}\,\mathcal{L}$ its Radon–Nikodym cocycle with respect to the leafwise Riemannian volume vol. Then for any given $t > 0$*

$$\frac{dQ^t\theta}{d\theta}(x) = \int \Delta_{\theta,vol}(x, y) d\pi_x^t(y) \qquad \textit{for } \theta\textit{-a.e. } x \in \mathcal{L} \; . \tag{2.78}$$

More precisely, the measure $Q^t\theta$ is σ-finite iff the right-hand side of (2.78) *is a.e. finite, in which case the measures $Q^t\theta$ and θ are equivalent, and the corresponding Radon–Nikodym derivative is given by formula* (2.78).

Proof. Let us consider the measure

$$d\Pi^t(x, y) = d\theta(x)\, d\pi_x^t(y) \tag{2.79}$$

on $\mathsf{graph}\,\mathcal{L}$. In probabilistic terms the measure Π^t is the joint distribution of the positions of the Brownian motion on $\mathcal{L}$ at times 0 and t provided the initial distribution is θ (we remind however that θ is not necessarily finite). The projections $\iota_{1,2}(\Pi^t)$ of Π^t onto the first and the second coordinates are, respectively

$$\iota_1(\Pi^t) = \theta \ , \qquad \iota_2(\Pi^t) = Q^t\theta \ .$$

By the definition (2.79), $\iota_1(\Pi^t) = \theta$, and the conditional measures of Π^t with respect to the projection ι_1 are π_x^t. Since $\iota_2(\Pi^t) = \iota_1(\check{\Pi}^t)$, where $\check{\Pi}^t = \Sigma(\Pi^t)$ is the "flip" of the measure Π^t (cf. Proposition 2.17), we obtain that the measure

$$Q^t\theta = \iota_2(\Pi^t) = \iota_1(\check{\Pi}^t)$$

is σ-finite iff the integrals

$$\int \frac{d\check{\Pi}^t}{d\Pi^t}(x,y)\, d\pi_x^t(y) \tag{2.80}$$

are θ-a.e. finite, in which case (2.80) is the Radon–Nikodym derivative

$$\frac{d\iota_1(\check{\Pi}^t)}{d\iota_1(\Pi^t)}(x) = \frac{dQ^t\theta}{d\theta}(x) \ .$$

In order to find the Radon–Nikodym derivative $d\check{\Pi}^t/d\Pi^t$ let us also consider the "counting measure"

$$d\Theta(x,y) = d\theta(x)\, d\, vol_{L(x)}(y)$$

on $\mathsf{graph}\,\mathcal{L}$ associated with θ (cf. (2.16) and Proposition 2.17). Since the leafwise transition measures π_x^t are equivalent to the leafwise Riemannian volume, the measures Θ and Π^t are equivalent, and by (2.70)

$$\frac{d\Pi^t}{d\Theta}(x,y) = \frac{d\pi_x^t}{d\, vol_{L(x)}}(y) = p(t,x,y) \ . \tag{2.81}$$

Then by Proposition 2.17 and formula (2.81)

$$\begin{aligned} \frac{d\check{\Pi}^t}{d\Pi^t}(x,y) &= \frac{d\check{\Pi}^t/d\check{\Theta}(x,y)}{d\Pi^t/d\Theta(x,y)} \cdot \frac{d\check{\Theta}}{d\Theta}(x,y) \\ &= \frac{p(t,y,x)}{p(t,x,y)} \cdot \Delta_{\theta,vol}(x,y) = \Delta_{\theta,vol}(x,y) \ . \end{aligned} \tag{2.82}$$

Substituting (2.82) into (2.80) yields the claim. □

Theorem 2.83. *Let ω be a Radon measure on a Riemannian lamination $\mathcal{L}$. Then the following conditions are equivalent:*

(i) $Q^t\omega = e^{\lambda t}\omega$ *for any* $t > 0$*;*
(ii) $Q^t\omega = e^{\lambda t}\omega$ *for some* $t > 0$*;*
(iii) *The measure ω is absolutely continuous (see Definition 2.13), and its Radon–Nikodym cocycle $\Delta_{\omega,vol}(x,y)$, $(x,y) \in \mathsf{graph}\,\mathcal{L}$ with respect to the leafwise Riemannian volume vol is a leafwise λ-harmonic function of the second argument.*

Proof. (i) $\Longrightarrow$ (ii). Trivial.

(ii) $\Longrightarrow$ (iii). Since the transition probabilities π_x^t are equivalent to the leafwise Riemannian volumes, the measure $\omega = e^{-\lambda t} Q^t \omega$ is absolutely continuous (see Definition 2.13). Then by formula (2.78)

$$\int \Delta_{\omega,vol}(x,y) d\pi_x^t(y) = e^{\lambda t} ,$$

for ω-a.e. $x \in \mathcal{L}$, which means that the functions $\Delta_{\omega,vol}(x,\cdot)$ are a.e. λ-harmonic.

(iii) $\Longrightarrow$ (i). Follows immediately from formula (2.78). □

Definition 2.84. A measure ω on a Riemannian lamination $\mathcal{L}$ is called λ-harmonic if it satisfies the conditions of Theorem 2.83. △

Remark 2.85. The 0-harmonic measures in the sense of Definition 2.84 are precisely the "harmonic measures" introduced by Lucy Garnett who was the first to consider the Brownian motion on foliations and proved Theorem 2.83 for compact Riemannian foliations and $\lambda = 0$ [Ga83]. Our proof is completely different and is based on an argument from [Ka98]. Note that Garnett's choice of the term "harmonic" is explained by the fact that by Theorem 2.83 these measures have harmonic leafwise densities. The terminology of Garnett is by now well established, and we shall follow it. However, from the point of view of the general theory of Markov processes it would be more natural to call these measures *stationary* (cf. the above discussion of the Brownian motion on manifolds).

2.3.3. *Transverse conformal streams and λ-harmonic measures.*

Theorem 2.86. *Let $\mathcal{A}$ be an affine lamination and $\mathcal{H} = \mathfrak{H}\mathcal{A}$ its hyperbolization. Then any parallel transverse conformal stream μ on $\mathcal{A}$ of dimension $\delta \geq 0$ determines a λ-harmonic measure ω^μ on $\mathcal{H}$ with $\lambda = \delta(\delta - 2)$.*

Proof. By Theorem 2.63 there is a bijective correspondence between parallel transverse conformal streams μ on $\mathcal{A}$ of dimension δ and absolutely continuous measures ω^μ (2.65) on $\mathcal{H}$ with the Radon–Nikodym cocycle $\exp[\delta\beta]$ with respect to the leafwise hyperbolic volume *vol*. Now, the functions $\exp[\delta\beta(x,\cdot)]$ on a pointed at infinity hyperbolic 3-space are λ-harmonic with $\lambda = \delta(\delta - 2)$ (one can easily check it in the upper half-space model of $\mathbf{H}^3$ by using the explicit formulas (1.15) for the hyperbolic Laplacian), so that the claim follows from Theorem 2.83. □

2.3.4. *Leafwise conformal streams and λ-harmonic functions.* We shall briefly recall the well-known construction which allows one to associate a λ-harmonic function Φ^η on $\mathbf{H}^3$ with $\lambda = \delta(\delta - 2)$ to a conformal stream η of dimension δ on the sphere at infinity $\partial\mathbf{H}^3$ (see [S79]). When applied to a leafwise conformal stream on an affine lamination it gives a leafwise λ-harmonic function.

Any point $h \in \mathbf{H}^3$ determines the *visual metric* ς_h on $\partial\mathbf{H}^3$ obtained by the geodesic projection of the spherical metric on the unit tangent sphere at h onto $\partial\mathbf{H}^3$, see §**5.3.1**.

Proposition 2.87. *Let η be a conformal stream on the affine plane $\mathcal{P} = \mathcal{P}_q \cong \partial\mathbf{H}^3 \setminus \{q\}$ associated with a pointed at infinity hyperbolic space $(\mathbf{H}^3, q)$. If for any ($\equiv$ certain) point $h \in \mathcal{H}$ the total mass of the measure $\eta_h \equiv \eta_{\varsigma_h}$ determined by the stream η and the metric ς_h is finite, then the function*

$$\Phi^\eta(h) = \eta_h(\partial\mathbf{H}^3)$$

on $\mathbf{H}^3$ is λ-harmonic with $\lambda = \delta(\delta - 2)$.

Proof. Take a reference point $o \in \mathbf{H}^3$. Then by the definition of a conformal stream

$$\frac{d\eta_h}{d\eta_o}(z) = \left(\frac{\varsigma_h}{\varsigma_o}(z)\right)^\delta , \qquad \forall\, z \in \mathcal{P} ,$$

so that

$$\Phi^\eta(h) = \int_{\mathcal{P}} \left(\frac{\varsigma_h}{\varsigma_o}(z)\right)^\delta d\eta_o(z) .$$

whereas by Proposition 5.32

$$\log \frac{\varsigma_h}{\varsigma_o}(z) = \beta_z(o,h) ,$$

and the assertion follows from the λ-harmonicity of $\exp[\delta\beta(o,h)]$ in h (cf. the proof of Theorem 2.86). □

The leafwise application of Proposition 2.87 now gives

Theorem 2.88. *Let $\mathcal{A}$ be a $\mathbb{C}$-lamination, $\mathcal{H} = \mathfrak{H}\mathcal{A}$ be its hyperbolization, and λ be a leafwise conformal stream on $\mathcal{A}$. If for any point $h \in \mathcal{H}$ the total mass of the measure λ_h on the leaf $L(\mathfrak{p}h)$ of the lamination $\mathcal{A}$ determined by the stream λ and the visual metric ς_h is finite, then*

$$\Phi^\lambda(h) = \lambda_h(L(\mathfrak{p}h))$$

is a leafwise λ-harmonic function on $\mathcal{H}$ with $\lambda = \delta(\delta - 2)$.

Remark 2.89. In terms of the Euclidean metrics ε_h on the leaves of $\mathcal{A}$ (see Proposition 1.38) the function Φ^λ takes the form

$$\Phi^\lambda(h) = \int_{\mathcal{P}} \left(\frac{\varsigma_h}{\varepsilon_h}(z)\right)^\delta d\ell_h(z) ,$$

where ℓ_h is the Lebesgue area measure determined by the Euclidean metric ε_h; see §**5.3** for a geometric interpretation of the ratio of the metrics ς_h and ε_h.

2.4. **Measures and streams on quotient laminations.** As we have already mentioned in §**2.2.2**, the notion of a conformal stream is not very interesting without additional symmetry assumptions. If F is an automorphism of a lamination $\mathcal{L}$, then it acts both on leafwise and transverse conformal streams by formula (2.48), and similarly to Proposition 2.49 we have

Proposition 2.90. *A leafwise (resp., transverse) conformal stream λ (resp., μ) on a conformal lamination $\mathcal{L}$ is F-invariant iff, respectively,*

$$\mathrm{Jac}_{\lambda_\rho} F = \|DF\|_\rho^\delta ,$$
$$\mathrm{Jac}_{\mu_\rho} F = \|DF\|_\rho^{-\delta}$$

for any ($\equiv$ some) conformal metric ρ.

Let now $\mathcal{A}$ be an affine lamination. All the constructions in §**2.2** and §**2.3** are invariant with respect to the group of automorphisms of $\mathcal{A}$. More precisely,

Theorem 2.91. *Let G be a group of automorphisms of an affine lamination $\mathcal{A}$, and let λ and μ be, respectively, a G-invariant leafwise conformal stream of dimension δ_λ and a G-invariant parallel transverse conformal stream of dimension δ_μ. Then*

(i) *The leafwise measure* $\overline{\lambda}$*, the transverse measure* $\overline{\mu}$*, the* $\delta_\mu(\delta_\mu - 2)$*-harmonic measure* $\omega^\mu = vol \star \widetilde{\mu}$*, and the* $\delta_\lambda(\delta_\lambda - 2)$*-harmonic function* Φ^λ *on the hyperbolization* $\mathcal{H} = \mathfrak{H}\mathcal{A}$ *are all* G*-invariant;*

(ii) *If* $\delta_\lambda = \delta_\mu$*, then the global measure* $\boldsymbol{\upsilon} = \lambda \star \mu$ *on* $\mathcal{A}$ *and the associated* ξ*-invariant global measure* $\overline{\lambda} \star \overline{\mu} = \widetilde{\boldsymbol{\upsilon}}$ *on* $\mathcal{H}$ *are also* G*-invariant;*

(iii) *If the group* G *acts properly discontinuously on the hyperbolization* $\mathcal{H}$ *(but not necessarily on the lamination* $\mathcal{A}$ *itself!), then the* $\delta_\mu(\delta_\mu - 2)$*-harmonic measure* $\omega^\mu = vol \star \widetilde{\mu}$ *and the* $\delta_\lambda(\delta_\lambda - 2)$*-harmonic function* Φ^λ *on* $\mathcal{H}$ *descend, respectively, to a* $\delta_\mu(\delta_\mu - 2)$*-harmonic measure* $\omega^\mu_{\mathcal{M}}$ *and a* $\delta_\lambda(\delta_\lambda - 2)$*-harmonic function* $\Phi^\lambda_{\mathcal{M}}$ *on the quotient lamination* $\mathcal{M} = \mathcal{M}_G = \mathcal{H}/G$*. Moreover, if* $\delta_\lambda = \delta_\mu$*, then the measure* $\widetilde{\boldsymbol{\upsilon}}$ *descends to a* ξ*-invariant global measure* $\widetilde{\boldsymbol{\upsilon}}_{\mathcal{M}}$ *on* $\mathcal{M}$*.*

3. Laminations associated with rational maps

3.1. **Construction of the affine lamination.** We will proceed with briefly recalling the construction of laminations (affine and hyperbolic) associated with a rational endomorphism f of the Riemann sphere $\overline{\mathbb{C}}$ [LM97]. Below we shall always deal with the same f and often omit it from our notations.

3.1.1. *The leaf space.* Let $\mathcal{N} = \mathcal{N}_f$ be the space of *backward trajectories*

$$\widehat{z} = \{\dots, z_{-1}, z_0\}$$

of f, where $fz_{-n} = z_{-n+1}$, and denote by $\widehat{f} : \mathcal{N} \to \mathcal{N}$ the *natural extension* of f obtained from the coordinate-wise action of f on $\mathcal{N}$, i.e.,

$$\widehat{f}\widehat{z} = \{\dots, fz_{-2}, fz_{-1}, fz_0\} = \{\dots, z_{-1}, z_0, fz_0\} .$$

The space $\mathcal{N}$ is compact in the product topology (below we shall refer to this topology as *turbulent*), and $\widehat{f}$ is a homeomorphism of $\mathcal{N}$. The projection

$$\pi : \widehat{z} = \{\dots, z_{-2}, z_{-1}, z_0\} \mapsto z_0 \tag{3.1}$$

from $\mathcal{N}$ onto $\overline{\mathbb{C}}$ semi-conjugates $\widehat{f}$ to f. For $n \in \mathbb{Z}$ let

$$\pi_n(\widehat{z}) = \pi \circ \widehat{f}^n(\widehat{z}) = z_n .$$

A point $\widehat{z} = \{\dots, z_{-1}, z_0\} \in \mathcal{N}$ is called *regular* if starting from a certain integer $N = N(\widehat{z})$ there exist neighbourhoods U_{-n}, $n \geq N$, of z_{-n} in $\overline{\mathbb{C}}$ such that f maps univalently U_{-n} onto U_{-n+1}. The *regular part* $\mathcal{R} \subset \mathcal{N}$ consists of all such points. The path connected component of $\mathcal{R}$ containing a point $\widehat{z} \in \mathcal{N}$ will be called the *leaf* of $\widehat{z}$ and denoted $L(\widehat{z})$. The projections π_{-n}, $n \geq N$, determine a conformal ($\equiv$ Riemann surface) structure on $L(\widehat{z})$, and the map $\widehat{f}$ acts conformally between the leaves. However, in general the conformal structure on the leaf $L(\widehat{z})$ does *not* depend continuously on $\widehat{z}$ (in the turbulent topology of $\mathcal{N}$), so that the arising leaf structure does not make $\mathcal{R}$ a Riemann surface lamination. In the absence of this continuity we call $\mathcal{R}$ just a *leaf space*.

A backward orbit $\widehat{\zeta} = \{\zeta_{-n}\}_{n\in\mathbb{N}}$ is called *critical* if it passes through the *critical set* $C(f)$, in which case $\pi(\widehat{\zeta}) = \zeta_0$ belongs to the *postcritical set* $C_\infty(f)$ (see the list of definitions in §**0.1**). The restriction of the projection π onto a single leaf $L(\widehat{z})$ branches at critical backward orbits. Given a path $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$ with endpoints $z, \zeta \in \overline{\mathbb{C}}$, any backward trajectory $\hat{z} = \{\dots, z_{-1}, z_0 = z\}$ can be analytically continued along this path (a *lift* of ϑ to $\mathcal{N}$). We say that the resulting trajectory $\hat{\zeta} = \{\dots, \zeta_{-1}, \zeta_0 = \zeta\}$ is obtained from $\hat{z}$ by the *holonomy along* ϑ. Thus, we have the holonomy transformation

$$H_\vartheta : \pi^{-1}(z) \to \pi^{-1}(\zeta). \tag{3.2}$$

Obviously, it is continuous. Note also that if a lift of ϑ is *regular*, i.e., if it is contained in $\mathcal{R}$, then it is contained in a single leaf $L(\hat{z})$.

It is proven in [LM97] that any leaf of $\mathcal{R}$ is either a Herman ring, or is conformally equivalent to either the hyperbolic plane $\mathbf{H}^2$ or the parabolic plane $\mathbb{C}$. In the latter two cases the leaves are called *hyperbolic* and *parabolic*, respectively. Any parabolic leaf has a unique affine structure compatible with its conformal structure.

In the case when f is a Chebyshev polynomial or a Lattès rational map, there is a special invariant parabolic leaf, $L_0 \equiv L(\hat{\alpha})$, corresponding to the postcritical fixed point α. This leaf is isolated in the lamination which we are constructing (see [LM97, §5.4]). For this reason, it is convenient to remove it from the space. Let $\mathcal{A}^n$ be the subset of $\mathcal{R}$ consisting of all parabolic leaves, except for the special leaf L_0. The set $\mathcal{A}^n$ is $\hat{f}$-invariant, and $\hat{f}$ acts affinely between the parabolic leaves. Once again, the leafwise affine structures do not have to be transversely continuous in the topology of $\mathcal{N}$.

3.1.2. *The laminar topology.* Let us consider the "universal" space $\mathcal{U}$ of all non-constant meromorphic functions on $\mathbb{C}$ with the metrizable topology of uniform convergence on compact sets (cf. below §**3.3.4**). The space $\mathcal{U}$ is foliated into the orbits of the right action $\varphi \mapsto \varphi \circ A$ of the group Aff of complex affine maps $A : \mathbb{C} \to \mathbb{C}$. The map f acts on $\mathcal{U}$ on the left as $\varphi \mapsto f \circ \varphi$. Let

$$\mathcal{K} = \bigcap_{n \geq 0} f^n(\mathcal{U})$$

be the "global attractor" of f in $\mathcal{U}$, and

$$\hat{\mathcal{K}} = \left\{ \hat{\varphi} = \{\varphi_{-n}\}_{n \geq 0} : \varphi_{-n} \in \mathcal{U},\ \varphi_{-n+1} = f \circ \varphi_{-n} \right\}$$

be its "natural extension" (the inverse limit of the system $\dots \xrightarrow{f} \mathcal{K} \xrightarrow{f} \mathcal{K}$). The set $\hat{\mathcal{K}}$ is still naturally a leaf space with leaves being the orbits of the right action of the affine group. Then factorizing $\hat{\mathcal{K}}$ with respect to the right action of the multiplicative group $\mathbb{C}^* \subset$ Aff gives the *universal orbifold affine lamination* $\boldsymbol{\mathcal{A}}$ associated with the map f. We shall call the topology on $\boldsymbol{\mathcal{A}}$ *laminar* (recall that it is determined by the uniform convergence of meromorphic functions on compact sets; therefore it is perhaps not always locally compact, cf. the discussion in §**3.1.7**). The natural lift of f to $\hat{\mathcal{K}}$ is an affine automorphism which will be also denoted by $\hat{f}$.

There is a laminar embedding of $\mathcal{A}^n$ into $\boldsymbol{\mathcal{A}}$ which is equivariant with respect to the action of $\hat{f}$. Indeed, if

$$\psi : (L(\hat{z}), \hat{z}) \to (\mathbb{C}, 0) \tag{3.3}$$

is an affine chart on the leaf $L(\hat{z})$ of a point $\hat{z} \in \mathcal{A}^n$, then the sequence of meromorphic functions

$$\varphi_{-n} = \pi_{-n} \circ \psi^{-1} : (\mathbb{C}, 0) \to (\overline{\mathbb{C}}, z_{-n}) , \qquad n \geq 0 , \tag{3.4}$$

is an element of $\mathcal{K}$. Any other affine chart (3.3) has the form

$$\psi' = \lambda\psi\ , \qquad \lambda \in \mathbb{C}^*\ ,$$

which means that $\widehat{z}$ determines a well-defined sequence in $\boldsymbol{\mathcal{A}}$. Thus, we have an embedding

$$\iota : \mathcal{A}^n \to \boldsymbol{\mathcal{A}}\ . \tag{3.5}$$

Put

$$\mathcal{A}^l = \iota\mathcal{A}^n \subset \boldsymbol{\mathcal{A}}\ .$$

We emphasize that although $\mathcal{A}^n$ and $\mathcal{A}^l$ are identified via the embedding ι as sets, their topologies are, generally speaking, different (the laminar topology is *a priori* finer than the turbulent one, i.e., it has more open sets), which is why we use the superscripts n and l referring to the turbulent topology coming from the *n*atural extension $\mathcal{N}$ and to the laminar one coming from the universal affine *l*amination $\boldsymbol{\mathcal{A}}$, respectively.

One can write down an explicit formula for the normalized meromorphic function $\varphi = \varphi_0$ (3.4) representing a point $\widehat{z} = \{z_{-n}\}_{n\in\mathbb{N}} \in \mathcal{A}^n$. Let the number R_{-n} be determined by the condition that the disk $\mathbb{D}(z_{-n}, R_{-n})$ is mapped by f^n onto a domain of spherical area 1 (counted with multiplicity). For a point $z \in \overline{\mathbb{C}}$, let

$$\rho_{-n} : (\overline{\mathbb{C}}, 0) \to (\overline{\mathbb{C}}, z_{-n})$$

denote a rotation of the Riemann sphere moving 0 to z_{-n}. The map ρ_{-n} is defined up to pre-composition with rotations $u \mapsto e^{2\pi i\theta}u$.

Lemma 3.6. *There exists the limit*

$$\varphi(u) = \lim_{n\to\infty} f^n(\rho_{-n}(R_{-n}u))\ , \tag{3.7}$$

where the convergence is uniform on compact sets after selecting an appropriate sequence of rotations ρ_{-n} (so that φ is naturally well-defined up to pre-composition with rotations $u \mapsto e^{2\pi i\theta}u$).

Proof. This is a version of [LM97, Lemma 4.7], which differs in the normalization of φ and the way of writing the formula. The proof is the same. □

Lemma 3.8. *The inclusion $\iota : \mathcal{A}^n \to \boldsymbol{\mathcal{A}}$ is Borel.*

Proof. Since the expression under the limit (3.7) is a meromorphic function depending continuously on $\widehat{z}$, the dependence of φ on $\widehat{z}$ is Borel. Now, the inclusion $\iota : \mathcal{A}^n \to \boldsymbol{\mathcal{A}}$ is given by the map

$$\widehat{z} \mapsto \{\varphi_{-n}\}_{n\in\mathbb{N}},$$

where φ_{-n} represents $\widehat{f}^{-n}\widehat{z}$. As every component of this map is Borel, the map is Borel as well. □

Finally, by taking the closure of $\mathcal{A}^n$ in the laminar topology of $\boldsymbol{\mathcal{A}}$ we make leafwise affine structures continuous and obtain an (orbifold) affine lamination $\mathcal{A}$. The restriction of the $\widehat{f}$-action from $\boldsymbol{\mathcal{A}}$ onto $\mathcal{A}$ provides us with an affine automorphism on $\mathcal{A}$, which continuously extends the $\widehat{f}$-action on $\mathcal{A}^l$ (and will still be denoted as $\widehat{f}$). We shall denote points in $\mathcal{A}$ by $\boldsymbol{z}$, reserving the notation $\widehat{z}$ for the points of the natural extension $\mathcal{N}$. Denote by $L_{\mathcal{A}}(\boldsymbol{z})$ the leaf of the affine lamination passing through a point $\boldsymbol{z} \in \mathcal{A}$.

Alongside with the embedding $\iota : \mathcal{A}^n \to \mathcal{A}$ (3.5) there is also a natural projection

$$\wp : \mathcal{A} \to \mathcal{A}^l \cong \mathcal{A}^n$$

equivariant with respect to the $\widehat{f}$-action. The composition

$$\pi \circ \wp : \mathcal{A} \to \overline{\mathbb{C}}$$

extends the projection $\pi : \mathcal{A}^l \to \overline{\mathbb{C}}$ (3.1) and is also denoted by π. [When it can lead to confusion, we will specify the notation as $\pi_{\mathcal{N}}$ or $\pi_{\mathcal{A}}$.] In terms of the sequences of meromorphic functions $\widehat{\varphi}$ it takes the form

$$\pi : \widehat{\varphi} = \{\varphi_{-n}\} \mapsto \varphi_0(0) .$$

3.1.3. *The Julia sets.* The *Julia set* $J = J(f) \subset \overline{\mathbb{C}}$ can be lifted via the projection π to all the spaces $\mathcal{N}, \mathcal{A}^n, \mathcal{A}^l$ and $\mathcal{A}$ under consideration. Accordingly, we will use the following notations:

- $\widehat{\mathcal{J}} = \pi_{\mathcal{N}}^{-1} J \subset \mathcal{N}$, for the natural extension of J;
- $\mathcal{J}^n = \widehat{\mathcal{J}} \cap \mathcal{A}^n$, for its affine part supplied with the turbulent topology;
- $\mathcal{J}^l = \iota(\mathcal{J}^n) \subset \mathcal{A}^l$, for its affine part supplied with the laminar topology;
- and finally: $\mathcal{J} = \mathrm{cl}(\mathcal{J}^l) = \pi_{\mathcal{A}}^{-1}\mathcal{J} \subset \mathcal{A}$.

All these Julia sets are closed in the corresponding spaces. If we wish to emphasize the dependence on f, we will write $\widehat{\mathcal{J}}_f$, $\mathcal{J}_f^n$, etc.

3.1.4. *The dual fibration.* For a point $\boldsymbol{z} \in \mathcal{A}$ with $\pi(\boldsymbol{z}) = z \in \overline{\mathbb{C}}$ let

$$\overline{\mathcal{T}}(\boldsymbol{z}) \equiv \overline{\mathcal{T}}_z = \pi^{-1} z \subset \mathcal{A}$$

be the fiber of the projection $\mathcal{A} \to \overline{\mathbb{C}}$ passing through $\boldsymbol{z}$. In the case when $\boldsymbol{z} \in \mathcal{A}^l$, we will also consider the corresponding fiber in $\mathcal{A}^l$:

$$\mathcal{T}(\boldsymbol{z}) \equiv \mathcal{T}_z = \overline{\mathcal{T}}_z \cap \mathcal{A}^l.$$

Clearly, $\overline{\mathcal{T}}_z$ is the closure of $\mathcal{T}_z$ in the laminar topology. We will call these fibers *dual* (to the corresponding laminations). The associated partitions of $\mathcal{A}$ and $\mathcal{A}^l$ into the dual fibers will be referred to as the *dual fibrations*. Note that we are slightly abusing terminology here: these fibrations are *not* locally trivial over $\overline{\mathbb{C}}$.

The dual fibration is clearly forward invariant:

$$\widehat{f}\mathcal{T}_z \subset \mathcal{T}_{fz} \qquad \forall\, z \in \overline{\mathbb{C}} .$$

For $n \in \mathbb{N}, \zeta = z_{-n}$, we let

$$\mathcal{T}^n(\boldsymbol{z}) \equiv \mathcal{T}^n_\zeta = \widehat{f}^n \mathcal{T}_\zeta \subset \mathcal{T}(\boldsymbol{z}), \tag{3.9}$$

and similarly for $\overline{\mathcal{T}}^n(\boldsymbol{z}) \equiv \overline{\mathcal{T}}^n_\zeta$, see Fig. 6. Below we shall often refer to the sets $\overline{\mathcal{T}}^n(\boldsymbol{z})$ or $\mathcal{T}^n(\boldsymbol{z})$ as the rank n *cylinders* in $\overline{\mathcal{T}}(\boldsymbol{z})$ (resp., in $\mathcal{T}(\boldsymbol{z})$).

For a given $\boldsymbol{z} \in \mathcal{A}^l$ the cylinders $\mathcal{T}^n(\boldsymbol{z})$ are closed and open in the fiber $\mathcal{T}(\boldsymbol{z})$, and form a basis of its turbulent topology. If n is fixed, then they form a finite partition of $\mathcal{T}(\boldsymbol{z})$, and these *cylinder partitions* increase as n grows (i.e., the next partition is a refinement of the previous one). If $\pi(\boldsymbol{z})$ is not postcritical, then $\mathcal{T}(\boldsymbol{z})$ consists of precisely d^n rank n cylinders. Analogous statements are valid for the fibers $\overline{\mathcal{T}}(\boldsymbol{z})$ as well.

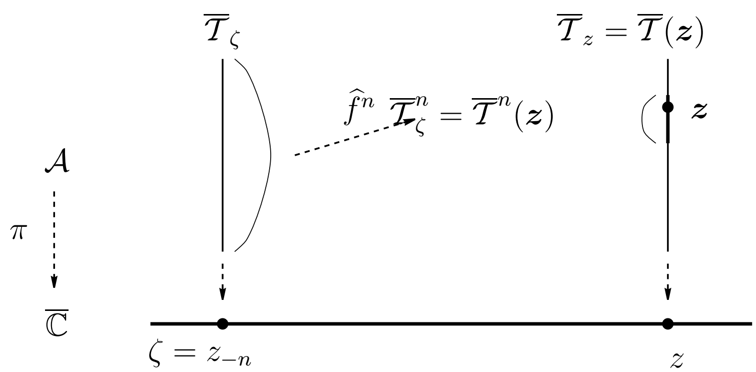

FIGURE 6

The cylinder partitions of a fiber $\mathcal{T}(\boldsymbol{z})$, $\boldsymbol{z} \in \mathcal{A}^l$ separate its points, and Lemma 3.8 yields:

Corollary 3.10. *Given a point $\boldsymbol{z} \in \mathcal{A}^l$, any Borel measure on the fiber $\mathcal{T}(\boldsymbol{z})$ is uniquely determined by its values on the cylinders $\mathcal{T}^n(\boldsymbol{\zeta})$, $\boldsymbol{\zeta} \in \mathcal{T}(\boldsymbol{z})$.*

3.1.5. *Flow boxes.* The presence of the dual fibration is a very important feature of the affine lamination $\mathcal{A}$ which allows one to define a special class of flow boxes adapted to the dual fibration.

The base of affine (orbifold) flow boxes of $\mathcal{A}$ described in [LM97] has an additional property that their transversals belong to vertical fibers and that the orbifold group commutes with the projection π. Such flow boxes will be called *product* flow boxes. A product flow box $\mathcal{B}$ admits a family of *dual holonomies* $V_{\mathcal{B}}$ which map one local leaf to another by sliding along vertical fibers. These dual holonomies commute with the dynamics. Note that the dual holonomies match on the intersection of any two product flow boxes, so that in the sequel we can omit the subscript $\mathcal{B}$ in the notation of the dual holonomy.

The base of flow boxes of the lamination $\mathcal{A}^l$ can be selected even in a more special way [LM97, §7.5]. For these boxes $\psi : \mathcal{B} \to B \times T$, $\mathcal{B} \subset \mathcal{A}^l$, the local leaves $B_t \cong B$ properly cover some topological disc $D \subset \overline{\mathbb{C}}$ with degree $b \geq 1$ and branching over at most one point $\xi \in D$. Then for each fiber $\mathcal{T}_z$, $z \in V \setminus \{\xi\}$ the intersection $\mathcal{T}_z \cap \mathcal{B}$ is a union of b disjoint transversals T_{ζ_i}, $i = 1, \dots, b$, where $\zeta_i \in B$ are the preimages of z under the covering map $B \to D$. We will call such local charts and the associated transversals T_ζ *standard*. In the case when $b = 1$ the flow box $\mathcal{B}$ will be called *univalent.*

If $\widehat{z} = \{z_{-n}\}_{n \in \mathbb{N}} \in \mathcal{A}^l$ is not a branched point on its leaf $L(\widehat{z})$, then the laminar topology at $\widehat{z}$ has a basis consisting of univalent flow boxes of the following type: for a neighborhood $U \ni z_0$ admitting a univalent pullback $\{U_{-n}\}_{n \in \mathbb{N}}$ along the backward orbit $\{z_{-n}\}$ and a number $m \in \mathbb{N}$ let

$$\begin{aligned}\mathcal{O}_m(\hat{z}, U) = \Big\{ \hat{\zeta} \in \mathcal{A}^l : \zeta_{-m} \in U_{-m}, \text{ and} \\ U \text{ admits a univalent pullback along } \hat{\zeta} \Big\}. \end{aligned} \tag{3.11}$$

We put $\mathcal{O}(\hat{z}, U) \equiv \mathcal{O}_0(\hat{z}, U)$.

Remark 3.12. We have considered two different notions of holonomy on $\mathcal{A}^l$: the holonomy H_ϑ along a path $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$ (see (3.2)), and the laminar holonomy (coming from the laminar structure of $\mathcal{A}^l$). These two notions are easily seen to match. Indeed, consider a standard flow box $\mathcal{B}$ over $D \subset \overline{\mathbb{C}}$ with branching degree b, and take a path $\vartheta \subset D \setminus C_\infty$ with endpoints z, ζ. Then the holonomy H_ϑ along this path maps $\mathcal{T}_z \cap \mathcal{B}$ onto $\mathcal{T}_\zeta \cap \mathcal{B}$ in such a way that each of b transversals composing $\mathcal{T}_z \cap \mathcal{B}$ is mapped onto some transversal of $\mathcal{T}_\zeta \cap \mathcal{B}$, so that the path holonomy induces the laminar holonomy on standard flow boxes. Conversely, the laminar holonomy between two standard transversals in a standard flow box is always induced by the map H_ϑ for an appropriately chosen ϑ.

3.1.6. *The hyperbolic lamination.* Applying the hyperbolization functor $\mathfrak{H}$ described in §**1.2.3** to the $\mathbb{C}$-lamination $\mathcal{A}$ we obtain an (orbifold pointed at infinity) $\mathbf{H}^3$-lamination $\mathcal{H} = \mathfrak{H}\mathcal{A}$. The extension of the action of $\hat{f}$ to $\mathcal{H}$ will be also denoted $\hat{f}$ (it acts by isometries between the leaves of $\mathcal{H}$). The action of $\hat{f}$ on $\mathcal{H}$ is properly discontinuous [LM97, Proposition 6.2]. Hence the quotient

$$\mathcal{M} = \mathcal{M}_f = \mathcal{H}/\hat{f} \tag{3.13}$$

is Hausdorff and inherits the structure of an (orbifold pointed at infinity) hyperbolic lamination. Countably many leaves of $\mathcal{M}$ (the cyclic quotients of the $\hat{f}$-periodic leaves of $\mathcal{H}$) are either (orbifold) solid tori, or (orbifold) cusps (see §**1.1.3**). All the other leaves of $\mathcal{M}$ are isomorphic to the hyperbolic space $\mathbf{H}^3$.

The vertical flow ξ and the horosphere foliation on $\mathcal{H}$ descend to a quotient flow (also denoted by ξ) and the associated horosphere foliation on $\mathcal{M}$. Note that the action of ξ on $\mathcal{H}$ is in a sense dual to the dynamical action of $\hat{f}$ (see [LM97, §6] and the discussion in §**5** below).

Remark 3.14. As in the case of Kleinian groups, the proper discontinuity is the point of introducing the 3-dimensional extension, which allows one to take the quotient and to construct a nice geometric object. As it follows from Proposition 4.6 below, the $\hat{f}$-action is never properly discontinuous on the 2-dimensional affine lamination $\mathcal{A}$. However, this action is totally discontinuous on the *Fatou set* $\mathcal{F} = \mathcal{A} \setminus \mathcal{J}$, so that one can quotient it. It adds a 2-dimensional boundary to "geometrically finite ends" of $\mathcal{M}$ (cf. §**5**).

3.1.7. *The germ topology.*

Definition 3.15. The rational function f (and the corresponding laminations $\mathcal{A}, \mathcal{H}$) is *tame* if the laminations $\mathcal{A}$ and $\mathcal{H}$ are locally compact, and *wild* otherwise. △

Interesting tame examples are given by Feigenbaum maps or, more generally, by maps with minimal postcritical set: in this case $\mathcal{A} = \mathcal{A}^n$. Another class of tame examples are maps with non-recurrent critical points on the Julia set ("geometrically finite maps").

However, we do not know whether the laminations $\mathcal{A}$ and $\mathcal{H}$ are always tame. This is the reason why, along with the turbulent and the laminar topologies on $\mathcal{A}$, we will also consider an intermediate *germ topology*. We shall first define it on the space $\mathcal{U}$ of meromorphic functions.

Definition 3.16. A sequence of functions $f_n \in \mathcal{U}$ converges to a function $f \in \mathcal{U}$ in the germ topology if there exists a neighborhood $D \ni 0$ such that φ_n converges to φ uniformly on D. △

Therefore, the completion of the space of meromorphic functions in this topology is the space of germs of analytic functions at 0. Although the germ topology is not metrizable, it is compatible with the natural complete uniform structure (coming from the linear structure on the space of meromorphic functions), and we shall use the notation $\mathsf{germ}(\varphi_n, \psi_n) \to 0$ for convergence with respect to this uniform structure.

Definition 3.17. A set of meromorphic functions is *normal* at 0 if it is normal on some neighborhood $D \ni 0$. △

Obviously, the notions introduced in Definition 3.16 and Definition 3.17 carry over to the space $\mathcal{U}/\mathbb{C}^*$ and further to $\mathcal{A}$ (see §**3.1.2**). The following Lemma makes precise the intuitive view of the dual fibers as "stable submanifolds" of $\widehat{f}$.

Lemma 3.18. *Let $\mathcal{X}, \mathcal{Y} \subset \mathcal{A}$ be two normal subsets. Take two points $\boldsymbol{z}, \boldsymbol{\zeta} \in \mathcal{Y}$ on the same fiber $\overline{\mathcal{T}}_z$, $z \in J$. Assume that the leaves $L(\boldsymbol{z})$ and $L(\boldsymbol{\zeta})$ have the same branching over z, and let $\widehat{f}^{n_k}\boldsymbol{z} \in \mathcal{X}$. Then uniformly over $\mathcal{Y}$*

$$\mathsf{germ}(\widehat{f}^{n_k}\boldsymbol{z}, \widehat{f}^{n_k}\boldsymbol{\zeta}) \to 0 \ .$$

Proof. (cf. the proof of [LM97, Proposition 8.6]). Let $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ be represented by sequences $\{\varphi_{-n}\}_{n\in\mathbb{N}}$ and $\{\psi_{-n}\}_{n\in\mathbb{N}}$ of meromorphic functions, $\varphi(0) = \psi(0)$. Since $\widehat{f}^{n_k}\boldsymbol{z} \in \mathcal{X}$, where $\mathcal{X}$ is normal, there exists a disk $D \ni 0$ and a sequence of scaling factors $\lambda_k \in \mathbb{C}^*$ such that the functions

$$\Phi_k \equiv f^{n_k} \circ \varphi_0 \circ \lambda_k$$

form a normal family on D. Since $z \in J$, the family of iterates f^{n_k} is not normal near z. Hence $\lambda_k \to 0$.

By the hypothesis φ_0 and ψ_0 have the same order of vanishing at 0, and, moreover, we may assume that φ_0 and ψ_0 are normalized in such a way that the leading coefficients of their Taylor expansions at 0 are equal. Then

$$\psi_0 \circ \lambda_k = \varphi_0 \circ \lambda_k \circ h_k \ ,$$

where h_k are conformal maps converging to identity uniformly on any $D' \Subset D$. Applying f^{n_k} we obtain

$$\Psi_k \equiv f^{n_k} \circ \psi_0 \circ \lambda_k = \Phi_k \circ h_k.$$

Since the family Φ_k is normal, the spherical distance between $\Phi_k(z)$ and $\Psi_k(z))$ tends to 0 uniformly on D', and this convergence is also uniform over a normal at 0 family of functions φ_0.

Take now any $m \in \mathbb{N}$. Then

$$f^{n_k} \circ \varphi_{-m} = f^{n_k - m} \circ \varphi_0 \ ,$$

and similarly for ψ. Since $\mathcal{X}$ is normal, for appropriate scaling factors $\lambda_{m,k}$ the family of functions

$$\Phi_{m,k} = f^{n_k - m} \circ \varphi_0 \circ \lambda_{m,k}$$

is normal on some disk $D_m \ni 0$. Repeating the above argument we conclude that the distance between $\Phi_{m,k}$ and $\Psi_{m,k}$ goes to 0 as $k \to \infty$ uniformly on any $D'_m \Subset D_m$ (and uniformly over a normal at 0 family of functions φ_{-m}). This proves the claim. $\square$

3.2. **The Busemann and basic cocycles of a rational map.**

3.2.1. *Dynamical formula for the basic cocycle.* For simplicity we shall use the notation

$$\boldsymbol{z}_n = \widehat{f}^n \boldsymbol{z} \;, \qquad n \in \mathbb{Z} \;,$$

for the $\widehat{f}$-orbit of a point $\boldsymbol{z} \in \mathcal{A}$. Recall that $\widehat{f}$ denotes both the action on $\mathcal{A}$ and on its extension $\mathcal{H}$. Iterating (1.66) we obtain

Lemma 3.19. *Let $\sigma : \mathcal{A} \to \mathcal{H}$ be a section of the fiber bundle $\mathfrak{p} : \mathcal{H} \to \mathcal{A}$. Then for any $n > 0$*

$$\begin{aligned}\beta_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) - \beta_\sigma(\boldsymbol{z}_{-n}, \boldsymbol{\zeta}_{-n}) &= \sum_{k=1}^{n} \log \frac{\|D\widehat{f}(\boldsymbol{z}_{-k})\|_\sigma}{\|D\widehat{f}(\boldsymbol{\zeta}_{-k})\|_\sigma} \\ &= \log \frac{\|D\widehat{f}^n(\boldsymbol{z}_{-n})\|_\sigma}{\|D\widehat{f}^n(\boldsymbol{\zeta}_{-n})\|_\sigma} = \log \frac{\|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma}{\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma} \;.\end{aligned}$$

Definition 3.20. A section $\sigma : \mathcal{A} \to \mathcal{H}$ is called *special* (for the lack of a better term) if

(i) $\widehat{f}^{-1}$ is locally uniformly contracting in the metric ρ_σ, i.e.,

$$\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma \to 0 \;, \qquad n \to \infty \;,$$

and the convergence is locally uniform;

(ii) The cocycle β_σ is uniformly continuous with respect to ρ_σ, i.e., for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) < \delta \implies \beta_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) < \varepsilon \qquad \forall\, (\boldsymbol{z}, \boldsymbol{\zeta}) \in \mathsf{graph}\, \mathcal{A} \;.$$

$\triangle$

Remark 3.21. For the needs of further applications condition (ii) above can be replaced with the following stronger condition:

(ii′) The cocycle β_σ is *Lipschitz continuous* with respect to ρ_σ, i.e., there is a constant $C > 0$ such that

$$|\beta_\sigma(\boldsymbol{z}, \boldsymbol{\zeta})| \le C \rho_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) \qquad \forall\, (\boldsymbol{z}, \boldsymbol{\zeta}) \in \mathsf{graph}\, \mathcal{A} \;.$$

Note that by passing from the section σ to the rescaled section $\sigma'_{\boldsymbol{z}} = C\sigma_{\boldsymbol{z}}$, we may always assume that $C = 1$.

An example of a special section (actually, it will satisfy the stronger Lipschitz condition (ii′)) will be constructed in §**3.3** below.

Theorem 3.22. *If σ is a special section, then for any $(\boldsymbol{z},\boldsymbol{\zeta}) \in \mathsf{graph}\,\mathcal{A}$*

$$\beta_\sigma(\boldsymbol{z},\boldsymbol{\zeta}) = \sum_{k=1}^{\infty} \log \frac{\|D\widehat{f}(\boldsymbol{z}_{-k})\|_\sigma}{\|D\widehat{f}(\boldsymbol{\zeta}_{-k})\|_\sigma} , \tag{3.23}$$

and the convergence of the series is locally uniform.

Proof. Condition (i) of Definition 3.20 implies that

$$\rho_\sigma(\boldsymbol{z}_{-n},\boldsymbol{\zeta}_{-n}) \underset{n\to\infty}{\longrightarrow} 0 \qquad \forall\, (\boldsymbol{z},\boldsymbol{\zeta}) \in \mathsf{graph}\,\mathcal{A} .$$

Therefore,

$$\Big|\beta_\sigma(\boldsymbol{z}_{-n},\boldsymbol{\zeta}_{-n})\Big| \to 0$$

by condition (ii), and Lemma 3.19 yields the claim. □

Remark 3.24. The right-hand side of formula (3.23) measures the limit distortion of the metric ρ_σ by $\widehat{f}$ along its backward trajectories.

Remark 3.25. Formula (3.23) is closely related to the Anosov–Sinai formula for the leafwise density of an absolutely continuous invariant measure m on the unstable foliation [AS67], [Le81]. This relation becomes explicit in the one-dimensional case when the above densities determine an invariant affine structure on the unstable foliation. Then the basic cocycle β_σ can be identified with the Radon–Nikodym cocycle of m, see §**2.1.3**.

3.2.2. *Non-triviality of the Busemann cocycle.* Since the map $\widehat{f}$ acts as isometry between pointed at infinity hyperbolic leaves of $\mathcal{H}$, the leafwise Busemann cocycle β is $\widehat{f}$-invariant. Hence, it descends to a cocycle on the quotient $\mathcal{M} = \mathcal{H}/\widehat{f}$ (3.13), which will be also denoted by β.

This cocycle is clearly non-trivial on the (countably many) leaves of $\mathcal{M}$ associated with repelling periodic orbits (solid tori; see §**1.1.3** and §**3.1.6**). Let us remove these leaves from $\mathcal{M}$ and denote by $\mathcal{M}'$ what is left (so that all the leaves of $\mathcal{M}'$ are pointed at infinity hyperbolic 3-spaces). Correspondingly, denote by $\mathcal{H}' \subset \mathcal{H}$ the preimage of $\mathcal{M}'$ under the factorization map $\mathcal{H} \to \mathcal{M} \cong \mathcal{H}/\widehat{f}$, and by $\mathcal{A}'$ the associated subset of $\mathcal{A}$.

The next result easily follows from the existence of special sections (Theorem 3.30).

Theorem 3.26. *The Busemann cocycle on $\mathcal{M}'$ is non-trivial in the Borel category.*

Proof. (cf. the proof of Theorem 5.26). Triviality of the Busemann cocycle on $\mathcal{M}'$ means that there exists a Borel $\widehat{f}$-invariant function φ on $\mathcal{H}'$ such that

$$\beta(\boldsymbol{h}_1,\boldsymbol{h}_2) = \varphi(\boldsymbol{h}_2) - \varphi(\boldsymbol{h}_1) . \tag{3.27}$$

In other words, φ determines a global $\widehat{f}$-invariant "hyperbolic height" on $\mathcal{H}'$. Take a special section σ, and put

$$\Phi(\boldsymbol{z}) = \varphi\Big(\sigma(\boldsymbol{z})\Big) , \qquad \boldsymbol{z} \in \mathcal{A}' .$$

Then, as it follows from (1.67), (3.27) and $\widehat{f}$-invariance of the function φ, for any $\boldsymbol{z} \in \mathcal{A}'$

$$\begin{aligned}\log \|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma &= \beta\Big(\sigma(\widehat{f}^{-n}(\boldsymbol{z})), \widehat{f}^{-n}(\sigma(\boldsymbol{z}))\Big) \\ &= \varphi\Big(\widehat{f}^{-n}(\sigma(\boldsymbol{z}))\Big) - \varphi\Big(\sigma(\widehat{f}^{-n}(\boldsymbol{z}))\Big) \\ &= \varphi\Big(\sigma(\boldsymbol{z})\Big) - \varphi\Big(\sigma(\boldsymbol{z}_{-n})\Big) = \Phi(\boldsymbol{z}) - \Phi(\boldsymbol{z}_{-n}) \ .\end{aligned}$$

Therefore, by Definition 3.20(i)

$$\Phi(\boldsymbol{z}) - \Phi(\boldsymbol{z}_{-n}) \underset{n\to\infty}{\longrightarrow} -\infty \qquad \forall\, \boldsymbol{z} \in \mathcal{A}' \ , \tag{3.28}$$

Now take any $\widehat{f}$-invariant Borel probability measure θ concentrated on $\mathcal{A}'$ (for instance, the balanced measure $\boldsymbol{\kappa}$, see §**4.1.1**). Then by the Poincaré Recurrence Theorem for θ-a.e. $\boldsymbol{z} \in \mathcal{A}'$ there exists an infinite sequence $\{n_i\}$ such that $\Phi(\boldsymbol{z}) - \Phi(\boldsymbol{z}_{-n_i}) \geq 0$, which contradicts (3.28). □

Corollary 3.29. *There exists no Borel Euclidean structure on $\mathcal{M}'$.*

3.3. **An example of a special section.** We shall now give an example of a section σ for which the conditions of Definition 3.20 are satisfied and the "dynamical formula" (3.23) holds. The rest of §**3.3** will be devoted to a proof of the following

Theorem 3.30. *For any rational function f the fiber bundle $\mathfrak{p} : \mathcal{H} \to \mathcal{A}$ has a special section.*

3.3.1. *Normalization of meromorphic functions.* Recall (see §**3.1.2**) that any global affine chart

$$\psi : (L_{\mathcal{A}}(\boldsymbol{z}), \boldsymbol{z}) \to (\mathbb{C}, 0) \tag{3.31}$$

on the leaf $L_{\mathcal{A}}(\boldsymbol{z})$ determines the Euclidean structure $\varepsilon(\psi)$ and a meromorphic function

$$\varphi = \pi \circ \psi^{-1} : (\mathbb{C}, 0) \xrightarrow{\psi^{-1}} (L_{\mathcal{A}}(\boldsymbol{z}), \boldsymbol{z}) \xrightarrow{\pi} (\overline{\mathbb{C}}, z) \ , \qquad z = \pi(\boldsymbol{z}) \ . \tag{3.32}$$

Any other affine chart ψ' (3.31) has the form

$$\psi' = \lambda\psi \ , \qquad \lambda \in \mathbb{C}^* \ ,$$

and the corresponding Euclidean structure $\varepsilon(\psi')$ and the meromorphic function $\varphi' = \pi \circ \psi'^{-1}$ are

$$\varepsilon(\psi') = |\lambda|\, \varepsilon(\psi) \ , \qquad \varphi'(w) = \varphi(w/\lambda) \ . \tag{3.33}$$

Thus, choosing the value $\sigma(\boldsymbol{z})$ of a section $\sigma : \mathcal{A} \to \mathcal{H}$ at the point $\boldsymbol{z}$ amounts to "normalizing" the family of functions $\{\varphi(w/\lambda)\}$, $\lambda \in \mathbb{C}^*$ (cf. Lemma 3.6).

Given a meromorphic function $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ and a subset $X \subset \mathbb{C}$ let

$$I(\varphi, X) = \int_X \|D\varphi\|\ \mathsf{eucl}$$

be the spherical area of the image φX counted with multiplicity, where the norm of the differential $D\varphi$ is taken with respect to the Euclidean metric on $\mathbb{C}$ and the spherical one on $\overline{\mathbb{C}}$. If $X = \mathbb{D}(z, r)$ is the disk of radius r centered at a point $z \in \mathbb{C}$, then we use the notation

$$I(\varphi, z, r) \equiv I\Big(\varphi, \mathbb{D}(z, r)\Big) \ .$$

By $R(\varphi, z)$ denote the radius uniquely determined by the relation

$$I\Big(\varphi, z, R(\varphi, z)\Big) = 1 \ .$$

If $z = 0$, we shall usually omit it from these notations. Also, when it is clear from the context, we shall often omit the function φ.

Obviously, for the rescaled function $\varphi'(w) = \varphi(w/\lambda)$, $\lambda \in \mathbb{C}^*$

$$I(\varphi', r) = I\left(\varphi, \frac{r}{|\lambda|}\right) \ , \qquad R(\varphi') = |\lambda| R(\varphi) \ . \tag{3.34}$$

3.3.2. *Construction of a special section.* For any affine chart (3.31) and the corresponding meromorphic function (3.32) define a Euclidean structure

$$\sigma_{\boldsymbol{z}} = \frac{\varepsilon(\psi)}{R(\varphi)} \tag{3.35}$$

on the leaf $L_{\mathcal{A}}(\boldsymbol{z})$. By (3.33) and (3.34) it does not depend on the choice of ψ, which provides us with a section $\sigma : \mathcal{A} \to \mathcal{H}$. In other words, we take on $L_{\mathcal{A}}(\boldsymbol{z})$ such an affine disk D centered at $\boldsymbol{z}$ that the spherical area of $\pi(D)$ (counted with multiplicity) equals to 1, and then define $\sigma(\boldsymbol{z})$ as the Euclidean structure on $L_{\mathcal{A}}(\boldsymbol{z})$ for which the radius of D is 1. Since the topologies on $\mathcal{A}$ and $\mathcal{H}$ are induced by uniform convergence of meromorphic functions on compact sets, the section σ is continuous.

The cocycle β_σ associated with the section σ (3.35) can be also calculated as follows. Take an arbitrary affine chart

$$\psi : (L_{\mathcal{A}}, \boldsymbol{z}, \boldsymbol{\zeta}) \to (\mathbb{C}, z, \zeta) \ ,$$

and let φ be the corresponding meromorphic function (3.32). Then

$$\sigma_{\boldsymbol{z}} = \frac{\varepsilon(\psi)}{R(\varphi, z)} \ , \qquad \sigma_{\boldsymbol{\zeta}} = \frac{\varepsilon(\psi)}{R(\varphi, \zeta)} \ ,$$

and

$$\beta_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) = \log \frac{\sigma_{\boldsymbol{z}}}{\sigma_{\boldsymbol{\zeta}}} = \log \frac{R(\varphi, \zeta)}{R(\varphi, z)} \ .$$

3.3.3. *Proof of Theorem 3.30.* Now Theorem 3.30 would follow from

Lemma 3.36. *The above constructed section* σ (3.35) *is special.*

Proof. We shell check two conditions of Definition 3.20.

(i). Take an affine chart (3.31) and the corresponding meromorphic function (3.32). For an integer $n > 0$ let

$$\psi_{-n} = \psi \circ \widehat{f}^n : (L_{\mathcal{A}}(\boldsymbol{z}_{-n}), \boldsymbol{z}_{-n}) \xrightarrow{\widehat{f}^n} (L_{\mathcal{A}}(\boldsymbol{z}), \boldsymbol{z}) \xrightarrow{\psi} (\mathbb{C}, 0)$$

and

$$\varphi_{-n} = \pi \circ \psi_{-n}^{-1} = \pi_{-n} \circ \psi^{-1} \ .$$

Then

$$\widehat{f}^n\Big(\varepsilon(\psi_{-n})\Big) = \varepsilon(\psi) \ ,$$

so that

$$\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma = \frac{(\widehat{f}^n\sigma)_{\boldsymbol{z}}}{\sigma_{\boldsymbol{z}}} = \frac{R(\varphi)}{R(\varphi_{-n})} \ .$$

Now, by the Shrinking Lemma (see [LM97, Appendix 2])

$$\operatorname{diam} \varphi_{-n}(D) \underset{n\to\infty}{\longrightarrow} 0$$

for any disk $D \subset \mathbb{C}$, so that

$$I(\varphi_{-n}, r) \to 0 \qquad \forall\, r > 0 \ .$$

Therefore, $R(\varphi_{-n}) \to \infty$.

Let $D \Subset D'$ and a meromorphic function $\widetilde{\varphi}$ (associated with a leaf of $\mathcal{A}$) be uniformly close to φ on D'. Then the functions $\widetilde{\varphi}_{-n}$ have uniformly bounded degree of branching on D' (by the definition of the topology on $\mathcal{A}$, see [LM97, §7.3]). By the Shrinking Lemma, $\operatorname{diam} \widetilde{\varphi}_{-n}(D) \to 0$ uniformly with respect to $\widetilde{\varphi}$. This yields locally uniform contraction.

(ii). We will show that the cocycle β_σ is Lipschitz continuous with respect to the metric ρ_σ (see Remark 3.21). Take an affine chart

$$\psi : (L_{\mathcal{A}}, \boldsymbol{z}, \boldsymbol{\zeta}) \to (\mathbb{C}, z, \zeta) \ ,$$

let $\varphi = \pi \circ \psi^{-1}$ and $R(z) \equiv R(\varphi, z)$. Then by the definition of the functional R neither of the disks $B(z, R(z))$ and $B(\zeta, R(\zeta))$ can be contained in the other one, so that

$$\Big| R(z) - R(\zeta) \Big| \le |z - \zeta| \ .$$

Dividing by $R(z)$, we obtain

$$\left| 1 - \frac{\sigma_{\boldsymbol{z}}}{\sigma_{\boldsymbol{\zeta}}} \right| \le |\boldsymbol{z} - \boldsymbol{\zeta}|_{\sigma_{\boldsymbol{z}}} \ ,$$

where $|\boldsymbol{z} - \boldsymbol{\zeta}|_{\sigma_{\boldsymbol{z}}}$ is the distance between the points $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ in the Euclidean structure $\sigma_{\boldsymbol{z}}$. Since σ is continuous, it implies that for any fixed $\boldsymbol{z}$ and any $\varepsilon > 0$

$$|\beta_\sigma(\boldsymbol{z}, \boldsymbol{\zeta})| = \left| \log \frac{\sigma_{\boldsymbol{z}}}{\sigma_{\boldsymbol{\zeta}}} \right| \le (1+\varepsilon)\, \rho_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}) \tag{3.37}$$

for all $\boldsymbol{\zeta}$ sufficiently close to $\boldsymbol{z}$.

If $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ are not close, join them with an almost geodesic curve ϑ of Riemannian length at most $(1+\varepsilon)\, \rho_\sigma(\boldsymbol{z}, \boldsymbol{\zeta})$. Subdivide ϑ into small pieces by points $\boldsymbol{z}_i = \vartheta(t_i)$ such that (3.37) is satisfied for any two consecutive points. Since

$$\sum \rho_\sigma(\boldsymbol{z}_i, \boldsymbol{z}_{i+1}) \le \mathsf{length}(\vartheta) \le (1+\varepsilon)\rho_\sigma(\boldsymbol{z}, \boldsymbol{\zeta}),$$

(3.37) is satisfied for $\boldsymbol{z}$, $\boldsymbol{\zeta}$ as well, with the constant $(1+\varepsilon)^2$. As $\varepsilon > 0$ is arbitrary, the section σ satisfies the Lipschitz condition with the constant $C = 1$. □

Remark 3.38. Instead of the functional $I(\varphi, z, r)$ (the spherical area of the image $\varphi B(z, r)$ taken with *multiplicity*) used in the definition of the section (3.35) we might just take the simple spherical area without worrying about the multiplicities. However, we prefer to take multiplicities into account for several reasons: the arising functional I is a natural characteristic of a meromorphic function which is used, for example, in the Nevanlinna theory; the functional I has a better geometric interpretation as it comes from pulling back the spherical area to the leaves of $\mathcal{A}$; finally, we need the functional I for Lemma 4.42 and Lemma 4.46 below anyway.

Remark 3.39. It might be tempting to use the normalization $\varphi'(z) = 1$ instead of the above one. It would work fine in the hyperbolic case. In general, however, it is not well-defined at the critical points of φ and the corresponding section σ is not continuous. This would have caused many technical troubles in what follows.

3.3.4. *Metrization of* $\mathcal{A}$. The above normalization allows one to endow the lamination $\mathcal{A}$ with a natural metrizable uniform structure. Let us go through the construction of $\mathcal{A}$ from §3.1.2. We will use the same notation, dist, for various metrics which appear along the way.

The space of meromorphic functions, being a topological vector space, possesses a natural uniform structure. Let us endow it with some metric compatible with this uniform structure. For instance, given two meromorphic functions ϕ and ψ, let

$$\operatorname{dist}_n(\varphi, \psi) = \sup_{|z| \le n} \varsigma(\varphi(z), \psi(z)) ,$$

where ς is the spherical metric on $\overline{\mathbb{C}}$, and let

$$\operatorname{dist}(\varphi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \operatorname{dist}_n(\varphi, \psi)$$

(note that the diameter of the metric dist is 1). This induces a metric on the space of normalized non-constant meromorphic functions

$$\mathcal{U}^0 = \{\varphi \in \mathcal{U} : \ R(\varphi) = 1\} .$$

Let us represent the group $\mathbb{C}^* \subset \mathrm{Aff}$ as the direct product $\mathbb{R}_+^* \times S^1$, where $\mathbb{R}_+^*$ is the group of scalings and S^1 is the group of rotations. The group $\mathbb{R}_+^*$ acts properly and free on $\mathcal{U}$, so that we have a principal $\mathbb{R}_+^*$-bundle $\mathcal{U} \to \mathcal{U}/\mathbb{R}_+^*$. This bundle is, in fact, trivial, as it has a global section associating to $\varphi \bmod \mathbb{R}^*$ the normalized representative $\varphi(e^t z) \in \mathcal{U}_0$. Thus, $\mathcal{U}/\mathbb{R}_+^* \approx \mathcal{U}_0$, and we can transfer the metric from $\mathcal{U}_0$ to $\mathcal{U}/\mathbb{R}_+^*$.

The unit circle $S^1 \subset \mathrm{Aff}$ acts (isometrically) on $\mathcal{U}^0$, and the above metric descends to the quotient $\mathcal{U}^0/S^1 \approx \mathcal{U}/\mathbb{C}^*$ by taking the minimum-distance between two orbits. It induces a metric on the global attractor $\mathcal{K}/\mathbb{C}^*$. It can be now lifted to the natural extension $\widehat{\mathcal{K}}/\mathbb{C}^*$ as

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \operatorname{dist}(\varphi_{-n} \bmod \mathbb{C}^*, \psi_{-n} \bmod \mathbb{C}^*) .$$

Since $\mathcal{A} \subset \widehat{\mathcal{K}} \bmod \mathbb{C}^*$, we obtain a desired metric on $\mathcal{A}$.

Remark 3.40. The reader can check that all the metrics obtained in such a way determine the same uniform structure on $\mathcal{A}$. Of course, this construction works for the space $\mathcal{H}$ as well (just skip the step of factorization by S^1).

3.4. **Dual basic cocycle.**

3.4.1. *Forward expansion.* Let us consider the leafwise Riemannian metric $\rho \equiv \rho_\sigma$ associated with the special section σ constructed in §**3.3**. We shall show that the forward iterates of $\widehat{f}$ expand ρ on the Julia set.

Lemma 3.41. *The map $\widehat{f} : \mathcal{A} \to \mathcal{A}$ is (locally uniformly) forward expanding on the Julia set $\mathcal{J}$ with respect to the Riemannian metric ρ, i.e., $\|D\widehat{f}^n(\boldsymbol{z})\|_\rho \to \infty$ as $n \to +\infty$ locally uniformly in $\boldsymbol{z} \in \mathcal{J}$.*

Proof. By the classical expanding property of f on its Julia set J (for example, see [Ly86, Theorem 1.15]), for any disk $\mathbb{D}(z, \varepsilon)$ with $z \in J$ there exists a number N depending on ε only such that the spherical area of $f^n \mathbb{D}(z, \varepsilon)$ is greater than 1 for all $n \geq N$.

Let $\varphi_{\boldsymbol{z}} : (\mathbb{C}, 0) \to (\overline{\mathbb{C}}, z)$ be the normalized meromorphic function representing the point $\boldsymbol{z}$, where $z = \pi(\boldsymbol{z})$. Obviously, there is a neighborhood $\mathcal{U} = \mathcal{U}(\boldsymbol{z}) \subset \mathcal{A}$ such that the family of functions $\varphi_{\boldsymbol{\zeta}}$ representing points $\boldsymbol{\zeta} \in \mathcal{U}$ form a normal family on $\mathbb{D}$. Hence for any $r \in (0, 1)$ there exists $\varepsilon = \varepsilon(r) > 0$ such that $\varphi_{\boldsymbol{\zeta}}(\mathbb{D}_r) \supset \mathbb{D}(\zeta, \varepsilon)$ for all $\boldsymbol{\zeta} \in \mathcal{U}$ (where $\zeta = \pi(\boldsymbol{\zeta})$).

By the above expanding property, for $\boldsymbol{\zeta} \in \mathcal{U} \cap \mathcal{J}$ and $n \geq N$, the spherical area of $\pi(\widehat{f}^n \circ \varphi_{\boldsymbol{\zeta}}(\mathbb{D}_r))$ is greater than 1. Therefore, $\|D\widehat{f}^n(\boldsymbol{\zeta})\|_\rho > 1/r$, and the conclusion follows. ∎

3.4.2. *Vertical distortion.* We will now study how the dual holonomy V (see §**3.1.5**) distorts the Riemannian metric ρ. The notation dist below stands for any metric on $\mathcal{A}$ compatible with its uniform structure, see §**3.3.4**.

Lemma 3.42. *Consider a product flow box $\mathcal{B} \cong B \times T$ and the dual holonomy $V : B_t \to B_\tau$ between two leaves of $\mathcal{B}$. Then*

$$\log \frac{\|D\widehat{f}^n(\boldsymbol{z})\|_\rho}{\|D\widehat{f}^n(V\boldsymbol{z})\|_\rho} \leq C \cdot \operatorname{dist}(t, \tau) \leq C \;, \qquad \boldsymbol{z} \in \mathcal{J} \cap B_t \;, \tag{3.43}$$

where $C = C(\mathcal{B})$ depends on $\mathcal{B}$ only, and $C(\mathcal{B}) \to 0$ as $\operatorname{diam} \mathcal{B} \to 0$.

Proof. Consider the normalized meromorphic functions φ and ψ representing the points $\boldsymbol{z}$ and $\boldsymbol{\zeta} = V(\boldsymbol{z})$ respectively, where $\varphi(0) = \psi(0) = z \equiv \pi(\boldsymbol{z})$. Then the point $\widehat{f}^n\boldsymbol{z}$ is represented by the normalized function $\widehat{f}^n \circ \varphi \circ \lambda_n$, where $\lambda_n = \|D\widehat{f}^n(\boldsymbol{z})\|_\rho^{-1} \to 0$ by Lemma 3.41. Consider the disk $D_n \subset L(\boldsymbol{z})$ of radius λ_n centered at $\boldsymbol{z}$. Since $\lambda_n \to 0$, the disks D_n eventually belong to the local leaf of $\mathcal{B}$ containing $\boldsymbol{z}$. By the Schwarz Lemma, the dual holonomy is Lipschitz with respect to the leafwise Riemannian metric ρ on $\mathcal{B}$. Hence the sets $\Delta_n = V(D_n) \subset L(\boldsymbol{\zeta})$ are trapped between the round disks of radii $C\lambda_n$ and $C^{-1}\lambda_n$ centered at $\boldsymbol{\zeta}$, where the constant C depends only on $\mathcal{B}$. Hence $I(\widehat{f}^n \circ \psi,\, C\lambda_n) \geq 1$, while $I(\widehat{f}^n \circ \psi,\, C^{-1}\lambda_n) \leq 1$. It follows that

$$C^{-1}\lambda_n^{-1} \leq \|D\widehat{f}^n(\boldsymbol{\zeta})\|_\rho \leq C\lambda_n^{-1}. \tag{3.44}$$

Moreover, the Lipschitz constant C of V goes to 0 as $\operatorname{dist}(t, \tau) \to 0$, so that (3.44) is equivalent to the desired estimate (3.43). ∎

Let $\|DV\|_\rho$ stand for the norm of the derivative of the dual holonomy with respect to the Riemannian metric ρ.

Lemma 3.45. *Consider a product flow box $\mathcal{B}$, and take two points $\boldsymbol{z} \in \mathcal{B} \cap \mathcal{J}$ and $\boldsymbol{\zeta} = V(\boldsymbol{z}) \in \mathcal{B} \cap \mathcal{J}$ in the same dual fiber of $\mathcal{B}$. Let V_n be the dual holonomy such that $V_n(\widehat{f}^n \boldsymbol{z}) = \widehat{f}^n \boldsymbol{\zeta}$. Then $\|DV_n\|_\rho \to 0$, uniformly over the choice of points $\boldsymbol{z}$ and $\boldsymbol{\zeta}$.*

Proof. Consider the disk $D^n_{\boldsymbol{z}} \subset L(\boldsymbol{z})$ centered at $\boldsymbol{z}$ such that $\pi(\widehat{f}^n D^n_{\boldsymbol{z}})$ has spherical area 1 counted with multiplicity and the analogous disk $D^n_{\boldsymbol{\zeta}}$ for $\boldsymbol{\zeta}$. By the above expanding property (see Lemma 3.41) the radius of $D^n_{\boldsymbol{z}}$ goes to 0 as $n \to \infty$. By the Koebe Distortion Theorem, $V(D^n_{\boldsymbol{z}})$ is a small oval with a small distortion. Hence it is trapped between the two copies of the disk $D^n_{\boldsymbol{\zeta}}$ scaled by the factors $\lambda_n > 1$ and λ_n^{-1}, respectively, where $\lambda_n \to 1$ uniformly over the choice of points $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ under consideration.

Consider now the uniformization $\gamma^n_{\boldsymbol{z}} : (\mathbb{C}, 0) \to (L(\boldsymbol{z}), \boldsymbol{z})$ of the leaf $L(\boldsymbol{z})$ such that $\gamma^n_{\boldsymbol{z}}(\mathbb{D}) = D^n_{\boldsymbol{z}}$, and the analogous uniformization $\gamma^n_{\boldsymbol{\zeta}}$ for $\boldsymbol{\zeta}$. Then the uniformizations $\psi^n_{\boldsymbol{z}} = \widehat{f}^n \circ \gamma^n_{\boldsymbol{z}}$ and $\psi^n_{\boldsymbol{\zeta}} = \widehat{f}^n \circ \gamma^n_{\boldsymbol{\zeta}}$ of the leaves $L(\widehat{f}^n \boldsymbol{z})$ and $L(\widehat{f}^n \boldsymbol{\zeta})$ are normalized. Hence $\|DV_n\|_\rho = |\varphi_n'(0)|$, where φ_n is determined from the diagram: $V_n \circ \psi^n_{\boldsymbol{z}} = \psi^n_{\boldsymbol{\zeta}} \circ \varphi_n$, which is equivalent to: $V \circ \gamma^n_{\boldsymbol{z}} = \gamma^n_{\boldsymbol{\zeta}} \circ \varphi_n$.

It follows that $\varphi_n(\mathbb{D})$ is trapped between the disks of radius λ_n and λ_n^{-1} centered at 0. By the Schwarz Lemma, $\lambda_n^{-1} \le |\varphi_n'(0)| \le \lambda_n$, and we are done. □

3.4.3. *Definition of the dual basic cocycle.* We are now ready to introduce the forward basic cocycle α_ρ associated with the metric ρ.

Theorem 3.46. *Consider a product flow box $\mathcal{B}$ and take points $\boldsymbol{z} \in \mathcal{B} \cap \mathcal{J}$ and $\boldsymbol{\zeta} = V(\boldsymbol{z}) \in \mathcal{B} \cap \mathcal{J}$ in the same dual fiber of $\mathcal{B}$. Let $\boldsymbol{z}_n = \widehat{f}^n \boldsymbol{z}, \boldsymbol{\zeta}_n = \widehat{f}^n \boldsymbol{\zeta}$. Then*

$$\|DV\|_\rho(\boldsymbol{z}) = \prod_{n=0}^{\infty} \frac{\|Df(\boldsymbol{z}_n)\|_\rho}{\|Df(\boldsymbol{\zeta}_n)\|_\rho} \tag{3.47}$$

is well-defined and continuous.

Proof. Consider the dual holonomies V_n from Lemma 3.45. Then $\widehat{f}^n \circ V = V_n \circ \widehat{f}^n$ and hence

$$\|DV\|_\rho = \frac{\|D\widehat{f}^n(\boldsymbol{z})\|_\rho}{\|D\widehat{f}^n(\boldsymbol{\zeta})\|_\rho} \, \|DV_n\|_\rho \, .$$

By Lemma 3.45, the expression in the right-hand side locally uniformly converges to the infinite product in the right-hand side of (3.47), and the conclusion follows. □

Thus, the expression (3.47) determines a locally continuous cocycle $\alpha_\rho(\boldsymbol{z}, \boldsymbol{\zeta})$ which will be called the *dual basic cocycle*.

3.5. **Euclidean laminations.** An affine lamination is called *Euclidean* if its affine structure can be refined to a Euclidean one, which means that there exists a continuous family of leafwise Euclidean structures consistent with the leafwise affine structures, see §**1.3.2**. Etienne Ghys has shown that non-Euclidean affine laminations (even smooth foliations) exist [Gh97] (in contrast with the negatively curved case [Ca93]; see also the discussions in [Gh99] and in §**5.6** below). We shall now prove that the dynamical affine lamination $\mathcal{A} = \mathcal{A}_f$ is *never* Euclidean, except for very special cases.

Recall that a rational function f is called *postcritically finite* if the forward orbits of all critical points are finite. To any postcritically finite map one can associate a *Thurston* *orbifold* O_f with underlying space $\mathbb{C}$ and singularities at the postcritical points (see [Th], [DH93]). Its Euler characteristic is non-positive: $\chi(O_f) \le 0$. The orbifold O_f is called *hyperbolic* if $\chi(O_f) < 0$ and *parabolic* otherwise. The parabolic case is very special: up to a Möbius conjugacy, only power functions $z \mapsto z^d$, Chebyshev polynomials, and Lattès rational maps have parabolic orbifolds.

By [LM97, Proposition 7.6], the lamination $\mathcal{A}_f$ is minimal, i.e., all its leaves are dense. [Note that this claim is true without any restrictions on f as we have already taken care of the Chebyshev and Lattès cases by removing the isolated leaves from the corresponding laminations, see §**3.1.1**.]

Theorem 3.48. *The affine lamination $\mathcal{A} \equiv \mathcal{A}_f$ is Euclidean if and only if the function f is postcritically finite and the Thurston orbifold $O \equiv O_f$ is parabolic.*

Proof. Postcritically finite rational maps with parabolic orbifolds can be characterized by the following property:

> If $f(z) = f(\zeta)$, then the local degrees of f at z and ζ are the same, except if one of these points is a postcritical fixed point.

In terms of the lamination $\mathcal{A}$ it means that if $\pi(\boldsymbol{z}) = \pi(\boldsymbol{\zeta})$, then the leaves $L(\boldsymbol{z})$ and $L(\boldsymbol{\zeta})$ have the same degree of branching over $\overline{\mathbb{C}}$ at the points $\boldsymbol{z}$ and $\boldsymbol{\zeta}$, respectively. [If one of these points is an orbifold point, then one should count the degree of branching in the corresponding orbifold local chart.] Moreover, by the definition of the Thurston orbifold O this degree coincides with the weight of the point $\pi(\boldsymbol{z}) = \pi(\boldsymbol{\zeta})$ in O.

On the other hand, if O is parabolic, then it carries a Euclidean orbifold structure. By the previous remark on the equality of the degrees of branching and the orbifold weights, one can lift this structure from O to the lamination $\mathcal{A}$. This shows that $\mathcal{A}$ is Euclidean once O is such.

Conversely, assume that $\mathcal{A}$ is Euclidean. Let $\|D\widehat{f}(\boldsymbol{z})\|$ stand for the norm of $\widehat{f} : L(\boldsymbol{z}) \to L(\widehat{f}\boldsymbol{z})$ measured with respect to the corresponding Euclidean structures. It is constant in the leafwise direction and continuous in the transverse direction. By minimality, it is constant on the whole lamination $\mathcal{A}$:

$$\|D\widehat{f}(\boldsymbol{z})\| \equiv \lambda\ . \tag{3.49}$$

Since $\widehat{f}$ has a repelling periodic point, $\lambda > 1$.

Let us now show that the leafwise Euclidean structure is invariant with respect to the dual holonomy V, i.e., that for any univalent standard flow box $\mathcal{B} \cong B \times T$ and any two points $\boldsymbol{z}, \boldsymbol{\zeta} \in T$, the dual holonomy $V \equiv V_{\mathcal{B}}$ between the local leaves $L_{\mathcal{B}}(\boldsymbol{z})$ and $L_{\mathcal{B}}(\boldsymbol{\zeta})$ is isometric at $\boldsymbol{z}$:

$$\|DV(\boldsymbol{z})\| = 1. \tag{3.50}$$

Assume first that $\boldsymbol{z} \in \mathcal{J}$ is a repelling periodic point of period p. Then the sequence $\boldsymbol{\zeta}_k = \widehat{f}^{pk}\boldsymbol{\zeta}$ converges to $\boldsymbol{z}$ by Lemma 3.18. By transverse continuity of the Euclidean structure $\|DV^k(\boldsymbol{z})\| \to 1$, where V^k is the dual holonomy between the leaves $L_{\mathcal{B}}(\boldsymbol{z})$ and $L_{\mathcal{B}}(\boldsymbol{\zeta}_k)$. Since the dual holonomy commutes with the dynamics, (3.49) implies $\|DV(\boldsymbol{z})\| = \|DV^k(\boldsymbol{z})\|$. This yields (3.50).

As repelling periodic points are dense in the Julia set by Proposition 4.6 below, we conclude that relation (3.50) is valid for any non-branched $\boldsymbol{z} \in \mathcal{J}$.

It follows that any two points $\boldsymbol{z} \in \mathcal{J}$ and $\boldsymbol{\zeta} \in \mathcal{J}$ in the same dual fiber have the same degree of branching. Indeed, if the degree b of $\boldsymbol{z}$ were greater than the degree b' of $\boldsymbol{\zeta}$, then the dual holonomy $V : L(\boldsymbol{z}) \to L(\boldsymbol{\zeta})$ near $\boldsymbol{z}$ would behave as $z \mapsto z^{b/b'}$, and we would have $\|DV(\boldsymbol{u})\| \to 0$ as $\boldsymbol{u} \to \boldsymbol{z}$ in contradiction with (3.50).

Thus, for any two points $\boldsymbol{z} \in \mathcal{J}$ and $\boldsymbol{\zeta} \in \mathcal{J}$ in the same fiber, the dual holonomy $L(\boldsymbol{z}) \to L(\boldsymbol{\zeta})$ is locally well defined, univalent and isometric at $\boldsymbol{z}$. We will show that the same property holds for points $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ outside $\mathcal{J}$ as well.

First notice that it is enough to show that for any two points $\boldsymbol{z} \in \mathcal{J}$ and $\boldsymbol{\zeta} \in \mathcal{J}$ from the same fiber the dual holonomy $V : L(\boldsymbol{z}) \to L(\boldsymbol{\zeta})$ is locally isometric. Indeed, then V is isometric at any point $\boldsymbol{u} \in L(\boldsymbol{z})$ to which it can be analytically continued. But then it admits a conformal continuation to the whole leaf $L(\boldsymbol{z})$ (for the same reason as above: otherwise it would hit a branched or critical point where $\|DV(\boldsymbol{u})\|$ would either blow up or vanish).

Assume by contradiction that the dual holonomy is not locally isometric somewhere on $\mathcal{J}$. Since periodic points are dense in $\mathcal{J}$ (Proposition 4.6), it must happen near one of them. Denote it by $\boldsymbol{a}$. Passing, if necessary, to an iterate of $\widehat{f}$, we may assume that $\widehat{f}(\boldsymbol{a}) = \boldsymbol{a}$. Let $L \equiv L(\boldsymbol{a})$.

Let us consider the level set

$$\mathcal{Z} = \{\boldsymbol{z} \in L(\boldsymbol{a}) : \|DV(\boldsymbol{z})\| = 1\} .$$

Near $\boldsymbol{a}$, it is a union of b real analytic curves $\mathcal{Z}_i$ meeting at $\boldsymbol{a}$ at the angle π/b (where b is the local degree of $V'(\boldsymbol{z})$ at $\boldsymbol{a}$).

Moreover, $\mathcal{J} \cap L \subset \mathcal{Z}$. Let us show that in fact $\mathcal{J} \cap L \subset \mathcal{Z}_i$ for some i. To this end take a point $\boldsymbol{\zeta} \neq \boldsymbol{a} \in \mathcal{J}$ near $\boldsymbol{a}$ and such that $\widehat{f}^n \boldsymbol{\zeta} \in \mathcal{T}(\boldsymbol{a})$ for some $n \in \mathbb{N}$. The leafwise Julia set $\mathcal{J} \cap L$ near $\boldsymbol{\zeta}$ belongs to some local branch $\mathcal{Z}_j$ of $\mathcal{Z}$, hence $\mathcal{J} \cap L$ near $\widehat{f}^n \boldsymbol{\zeta}$ is contained in $\widehat{f}^n \mathcal{Z}_j$. Passing from $\widehat{f}^n \boldsymbol{\zeta}$ to $\boldsymbol{a}$ by the dual holonomy, we conclude that $\mathcal{J} \cap L$ must belong to some analytic curve passing through $\boldsymbol{a}$. Hence it must belong to some local branch $\mathcal{Z}_i$.

Clearly, $\mathcal{Z}_i$ must be invariant under $\widehat{f}^{-1}$. But the map $\widehat{f}|L$ is linear. Hence $\mathcal{Z}_i$ is a straight interval and $\widehat{f}'(\boldsymbol{a})$ is real. Iterating $\mathcal{Z}_i$ forward, we conclude that $\mathcal{J} \cap L$ is globally contained in a straight line on the leaf L. Since this leaf is dense in $\mathcal{A}$, the same is true on any other leaf.

Therefore, the local dual holonomy $V : L \to L(\boldsymbol{\zeta})$ near $\boldsymbol{a}$ carries the interval $\mathcal{Z}_i$ to another straight interval. Hence the restriction of V' onto $\mathcal{Z}_i$ has a constant argument (with respect to any Euclidean charts on the leaves). Since $|V'(z)| = 1$ on $\mathcal{Z}_i$, we conclude that $V' \equiv \text{const}$ on $\mathcal{Z}_i$. By the uniqueness theorem, V is a local isometry, contradicting our assumption.

Thus, the dual holonomies are global isometries on all leaves. It follows that all the leaves of $\mathcal{A}$ have the same degree of branching over any point of $\mathbb{C}$. As we have mentioned above, it is the characteristic property of rational maps which are post-critically finite with parabolic Thurston orbifold. In fact, since the dual holonomies are isometric, the

Euclidean structure on $\mathcal{A}$ can be correctly pushed down to an orbifold Euclidean structure on the orbifold O. $\square$

4. Measures on laminations associated with rational maps

4.1. **The balanced measures.**

4.1.1. *The balanced measure on Julia sets.*

Definition 4.1. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. A probability measure κ on the Julia set $J = J(f)$ of f is called *balanced* if it is f-invariant, and for κ-a.e. $z \in J$ the conditional measure of κ on the set $f^{-1}(z)$ is uniform. Equivalently, κ is balanced if

$$\mathrm{Jac}_\kappa f \equiv d \ ,$$

where $d = \deg f$ is the degree of f (see Definition 2.1). $\triangle$

Theorem 4.2 ([Br65], [Ly93])**.** *Any rational map f has a unique balanced measure κ. Moreover,* $\operatorname{supp} \kappa = \mathrm{J}$*, and the preimages of any point $z \in J$ (excluding, possibly, two exceptional points) are equidistributed with respect to κ:*

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{\zeta : f^n \zeta = z} \delta_\zeta = \kappa,$$

where the limit is taken with respect to the weak topology of the space of probability measures on $\overline{\mathbb{C}}$.

Being f-invariant, the balanced measure κ can be uniquely lifted to an $\widehat{f}$-invariant measure $\widehat{\kappa}$ on the natural extension $\widehat{J}$ (e.g., see [CFS82]). The measure $\widehat{\kappa}$ is called the natural extension of κ. Although in general $\widehat{J}$ is *not* contained in $\mathcal{A}^n$, the following weaker statement still holds.

Lemma 4.3. *The measure $\widehat{\kappa}$ is supported by the set $\mathcal{J}^n \subset \mathcal{N}$.*

Proof. The balanced measure has entropy $\log d$ [Ly93]. Hence it has a positive Lyapunov exponent [Ru78], and we can apply the argument from [LM97, p. 37]. $\square$

Since the embedding $\iota : \mathcal{A}^n \to \mathcal{A}$ is Borel (Lemma 3.8), the measure $\boldsymbol{\kappa} \equiv \iota(\widehat{\kappa})$ is a Borel measure on $\mathcal{J}^l = \iota(\mathcal{J}^n)$. Recall that the closure of $\mathcal{J}^l$ in the laminar topology is the Julia set $\mathcal{J} = \pi_{\mathcal{A}}^{-1} J$ of the affine lamination $\mathcal{A}$.

Lemma 4.4. *The support of the measure $\boldsymbol{\kappa}$ is the whole Julia set $\mathcal{J}$.*

Proof. Let X be the subset of $\mathcal{J}^l$ obtained from $\mathcal{J}^l$ by removing the set of branched or asymptotically periodic orbits $\widehat{z} = \{z_{-n}\}_{n\in\mathbb{N}}$ (the former means that one of the points z_{-n} is critical; the latter means that the backward orbit $\widehat{z}$ tends to a periodic cycle). They occupy countably many leaves and dual fibers. Hence X is dense in $\mathcal{J}^l$, and in its turn $\mathcal{J}^l$ is dense in $\mathcal{J}$, so that it is sufficient to show that

$$\boldsymbol{\kappa}(\mathcal{V}) = \widehat{\kappa}(\mathcal{V}) > 0$$

for any $\mathcal{A}^l$-neighbourhood $\mathcal{V}$ of any point $\widehat{z} \in X$.

By [Ly93, Proposition 4], for any $\varepsilon > 0$ there exists a number $l = l(\epsilon)$ with the following property:

More than $(1-\varepsilon)d^k$ of the inverse branches of f^{-k} are univalent on Ω for any topological disk $\Omega \subset \overline{\mathbb{C}}$ with $\Omega \cap C_l = \varnothing$ and any $k \in \mathbb{N}$ (recall that C_l stands for the l-postcritical set of f, see the list of notations in §**0.1**).

Therefore, for any such Ω,

$$\widehat{\kappa}\Big(\mathcal{O}(\widehat{z}, \Omega)\Big) \geq (1-\varepsilon)\kappa(\Omega) \ , \tag{4.5}$$

where $\mathcal{O}(\widehat{z}, \Omega)$ is the flow box defined in (3.11).

Since $\widehat{z}$ is not critical, univalent boxes $\{\mathcal{O}_m(\widehat{z}, V)\}$ form a local topological basis at $\widehat{z}$. Take such a box, and denote by $\dots, V_{-1}, V_0 \equiv V$ the univalent pullbacks of V along the backward orbit $\widehat{z} = \{\dots, z_{-1}, z_0 \equiv z\}$. By the Shrinking Lemma (see [LM97, Appendix 2]), $\operatorname{diam} V_{-n} \to 0$. On the other hand, since $\widehat{z}$ is not asymptotically periodic, there exists a $\delta > 0$ such that

$$\operatorname{dist}(z_{-n(k)}, C_l) > \delta \qquad \text{along a sequence} \quad n(k) \to \infty \ ,$$

and therefore

$$V_{-n(k)} \cap C_l = \varnothing \qquad \text{for sufficiently big } k \ .$$

Then by $\widehat{f}$-invariance of $\widehat{\kappa}$ and by (4.5)

$$\widehat{\kappa}\Big(\mathcal{O}_{n(k)}(\widehat{z}, V)\Big) = \widehat{\kappa}\Big(\mathcal{O}(\widehat{f}^{-n(k)}\widehat{z}, V_{-n(k)})\Big) \geq (1-\varepsilon)\kappa(V_{-n(k)}) > 0 \ .$$

But

$$\mathcal{O}_m(\widehat{z}, V) \supset \mathcal{O}_{n(k)}(\widehat{z}, V) \qquad \text{for} \quad n(k) \geq m \ .$$

Hence $\widehat{\kappa}\Big(\mathcal{O}_m(\widehat{z}, V)\Big) > 0$. □

Proposition 4.6. *The map $\widehat{f}$ is topologically transitive on the Julia set $\mathcal{J}$, and repelling periodic points are dense in $\mathcal{J}$.*

Proof. Take two unbranched orbits $\widehat{z} = \{\dots, z_{-1}, z_0\}$ and $\widehat{\zeta} = \{\dots, \zeta_{-1}, \zeta_0\}$ in $\mathcal{J}^l$ such that the first one is not asymptotically periodic, and two standard univalent neighbourhoods $Q_m(\widehat{z}, V)$ and $Q_n(\widehat{\zeta}, U)$. Let $\{V_{-s}\}_{s\in\mathbb{N}}$ (resp., $\{U_{-t}\}_{t\in\mathbb{N}}$) be the pullbacks of V (resp., U) along $\widehat{z}$ (resp., $\widehat{\zeta}$). We will show that there exists $l \in \mathbb{N}$ such that

$$\widehat{f}^{-l} Q_m(\widehat{z}, V) \cap Q_n(\widehat{\zeta}, U) \neq \varnothing \ . \tag{4.7}$$

Take a neighborhood $\widetilde{U} \Subset U$ of ζ_0 and a number $\varepsilon < \kappa(\widetilde{U})$. By Theorem 4.2, eventually more than εd^k of f^k-preimages of any point of J belong to $\widetilde{U}$. Take a big $l \in \mathbb{N}$. We saw in the proof of Lemma 4.3 that there is $s \geq m$ such that $V_{-s} \cap C_l = \varnothing$. If l is sufficiently big, then by [Ly93, Proposition 4], there exists more than $(1-\varepsilon)d^k$ f^k-preimages of z_{-s} such that the corresponding inverse branches φ_k of f^{-k} admit a single-valued extension to V_{-s}. Hence one of these preimages must belong to $\widetilde{U}$. Moreover, if k is sufficiently big, then by the Shrinking Lemma $\varphi_k V_{-s} \subset U$.

Altogether, we conclude that there exists a sequence of points

$$u^k = \varphi_k(z_{-s}) \in U \cap f^{-k} z_{-s}, \quad k \geq k_0,$$

such that the inverse branches φ_k admit an extension to V_{-s}, and $\varphi_k V_{-s} \subset U$. Consider a point

$$\widehat{u}^k = \{\dots, u^k_{-1}, u^k_0 \equiv u^k\} \in Q_n(\widehat{\zeta}, U)$$

such that $u^k_{-t} \in U_{-t}$, $t \in \mathbb{N}$. Obviously, $\widehat{f}^{k+s}\widehat{u}^k \in Q_m(\widehat{z}, \widetilde{V})$, and (4.7) follows.

Since standard univalent boxes form a basis of neighborhoods for a dense subset of points in $\mathcal{J}$ (and since $\mathcal{J}$ is complete), (4.7) immediately implies topological transitivity of $\widehat{f}$ on $\mathcal{J}$.

Now take $\widehat{\zeta} = \widehat{z}$, $m = n$ and $U \Subset V$, and let $\widehat{w} = \{\dots, w_{-1}, w_0\}$ be the corresponding intersection point in (4.7). Let $\{\dots, W_{-1}, W_0\}$ be the pullbacks of V along $\widehat{w}$, and let $\psi_k : V \to W_{-k}$ be the corresponding inverse branches of f^{-k}. If l is sufficiently big then by the Shrinking Lemma, $\psi_l V \Subset V$. By the Schwarz Lemma, ψ_l has an attracting fixed point a in V. Moreover, for this point, $\psi_k(a) \in W_{-k} = V_{-k}$ for $k = 0, 1, \dots, l$. Hence $\widehat{a} \in Q_m(\widehat{z}, V)$, where $\widehat{a}$ is the periodic lift of a to $\mathcal{A}^l$.

Thus, any standard univalent box contains a repelling periodic point. Therefore, such points are dense in $\mathcal{A}$. □

4.1.2. *The transverse balanced measure.*

Definition 4.8. A transverse measure $m = \{m_T\}$ of the lamination $\mathcal{A}$ is called a *transverse balanced measure* if it satisfies the following properties:

(i) The family $\{m_T\}$ is holonomy invariant, i.e.,

$$\Delta_m \equiv 1 \ ;$$

(ii) $\widehat{f} m_T = d \cdot m_{\widehat{f}T}$ for any transversal T, i.e.,

$$\operatorname{Jac}_m \widehat{f} \equiv d \ ;$$

△

In what follows, we shall use the simplified notation $m(X) \equiv m_T(X)$ for subsets of a given transversal T.

Theorem 4.9. *The lamination $\mathcal{A}$ has a transverse balanced measure m.*

Proof. For any point $z \in J \setminus C_\infty$, the preimage $f^{-1}(z)$ consists of precisely d points. Denote by m_z the uniform measure on the set of backward orbits starting from z, i.e., for any $u \in f^{-n}z$, let

$$m_z\Big\{\widehat{\zeta} \in \pi^{-1}(z) : \zeta_{-n} = u\Big\} = 1/d^n \ .$$

Recall that $\mathcal{T}_z$ stands for the affine part of $\pi^{-1}(z)$.

Claim. Let $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$ be a path joining some points z and ζ. Then the holonomy

$$H_\vartheta : \pi^{-1}(z) \to \pi^{-1}(\zeta) \tag{4.10}$$

transforms $\mathcal{T}_z$ to $\mathcal{T}_\zeta$ modulo a set of zero m_z-measure.

For proving this assertion we will show that for m_z-a.e. $\widehat{z} \in \mathcal{T}_z$, the lift of ϑ to $\mathcal{N}$ starting at $\widehat{z}$ is contained in $\mathcal{R}$. Then it will be actually contained in the leaf $L(\widehat{z})$ and the conclusion will follow.

Let is consider a simply connected neighborhood $V_l \subset \overline{\mathbb{C}} \setminus C_l$ of the path ϑ. Let $\mathcal{X}_l$ be the set of backward orbits $\widehat{\zeta} \in \pi^{-1}(z)$ such that V_l does not admit a univalent pullback along $\widehat{\zeta}$. By [Ly93, Proposition 4], $m_z(\mathcal{X}_l) \le Cd^{-l}$, with a constant C independent of l. Hence

$$m_z\left(\bigcap_{l=0}^{\infty} \mathcal{X}_l\right) = 0.$$

But if $\widehat{z}$ does not belong to the above intersection, then the lift of the path ϑ starting at $\widehat{z}$ is contained in $\mathcal{R}$. Therefore, this property holds for almost all $\widehat{z} \in \mathcal{T}_z$, and the claim is proven. $\triangle$

We will now show that the uniform measures m_z on the fibers $\pi^{-1}(z)$ of $\mathcal{N}$, $z \in \overline{\mathbb{C}} \setminus C_\infty$, are actually supported on $\mathcal{T}_z \subset \mathcal{A}^n$ and can be promoted to a balanced measure on $\mathcal{A}$.

Take a path $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$ joining two points z and ζ. Since the holonomy (4.10) is a bijection between the sets of cylinders of a given rank over z and ζ, and all these cylinders have equal measures, we conclude that $H_\vartheta(m_z) = m_\zeta$. Together with the above assertion this implies: If m_z is supported on $\mathcal{T}_z$ for some point $z \in \overline{\mathbb{C}} \setminus C_\infty$, then the same property holds for any $z \in \overline{\mathbb{C}} \setminus C_\infty$.

But for κ-a.e. $z \in J$, the measure m_z coincides with the conditional measures of $\widehat{\kappa}$. Indeed, the balanced property of κ easily implies that the conditional measures of $\widehat{\kappa}$ are equidistributed on the cylinders, which uniquely determines a Borel measure on $\pi^{-1}(z)$. By Lemma 4.3, the balanced measure $\widehat{\kappa}$ is supported on $\mathcal{A}^n$. Thus, m_z is supported on $\mathcal{T}_z$ for κ-a.e. z, and hence for all $z \in \overline{\mathbb{C}} \setminus C_\infty$.

Now, by Lemma 3.8, the measures m_z can be transferred to a Borel measures on the fibers $\mathcal{T}_z$ endowed with the laminar topology (which will still be denoted as m_z). Obviously the transferred measures will also be invariant under the holonomies H_ϑ along paths $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$.

Next, let us promote m to a transverse measure on $\mathcal{A}^l$. Take a standard flow box $\mathcal{B}$ in $\mathcal{A}^l$ over a disk $D \subset \overline{\mathbb{C}}$ (see §**3.1.5**). The measure m_z on $\mathcal{T}_z$, $z \in D \setminus C_\infty$, induces measures on the transversals of $\mathcal{B}$ contained in $\mathcal{T}_z$. Since the holonomy along a path $\vartheta \subset \overline{\mathbb{C}} \setminus C_\infty$ permutes these transversals, we obtain a holonomy invariant measure defined on a dense set of transversals of $\mathcal{B}$. This measure can obviously be extended to a holonomy invariant measure on $\mathcal{B}$, and any two such extensions match on the intersection of the corresponding boxes.

To complete the construction, we need to extend the measure to an arbitrary transversal T in a flow box $\mathcal{B}$ of the lamination $\mathcal{A}$. Cover $\mathcal{B} \cap \mathcal{A}^l$ with a union of standard flow boxes $\mathcal{B}_i$. The transverse measure m can be transferred to the pieces $T \cap \mathcal{B}_i$. By the holonomy invariance, these assignments match on the intersections, and determine a measure on T. By Lemma 4.3 this measure is concentrated on $\mathcal{A}^l$.

By the construction, this measure is holonomy invariant. It also satisfies property (ii) of Definition 4.8 since this property is satisfied for the measures m_z on fibers $\pi^{-1}(z)$ and obviously carries through all the steps of the construction (i.e., restriction to a dense set of vertical transversals, extension by holonomy invariance to all vertical transversals, and further extension to arbitrary transversals by taking their slices). $\square$

Remark 4.11. The above discussion is closely related to the discussion in [Su97] of the transverse invariant measure on $\mathcal{N}$. See also [BLS93, §4] for a related discussion of the holonomy invariant measures for Hénon maps.

Remark 4.12. We do not know whether the measure m constructed in Theorem 4.9 is the *unique* balanced measure on the lamination $\mathcal{A}$. However, as it follows from [Su97], it is the unique balanced measure concentrated on $\mathcal{A}^l$. Another closely related question is whether properties (i) and (ii) in Definition 4.8 are equivalent.

4.1.3. *The global balanced measure.*

Definition 4.13. A measure θ on the lamination $\mathcal{H}$ is called a *global balanced measure* if it is absolutely continuous with the Radon–Nikodym cocycle

$$\Delta_{\theta,vol} \equiv 1 \ , \tag{4.14}$$

with respect to the leafwise hyperbolic volume vol, and

$$\widehat{f}\theta = d \cdot \theta \ . \tag{4.15}$$

$\triangle$

Proposition 4.16. *There is an affine one-to-one correspondence between transverse balanced measures of the lamination $\mathcal{A}$ (see Definition 4.8) and global balanced measures on the lamination $\mathcal{H}$.*

Proof. Let $\widetilde{m}$ be the lift of a transverse balanced measure m from $\mathcal{A}$ to $\mathcal{H}$ (see Definition 2.38). Then by Definition 4.8(i) and Proposition 2.36 the measure $\widetilde{m}$ is holonomy invariant, and by Definition 4.8(ii) it has the property that

$$\widehat{f}\widetilde{m} = d \cdot \widetilde{m} \ . \tag{4.17}$$

Define the global measure θ on $\mathcal{H}$ as

$$\theta = vol \star \widetilde{m} \ ,$$

where vol is the leafwise hyperbolic volume on $\mathcal{H}$. Since the map $\widehat{f}$ preserves vol, Proposition 2.20 and formula (4.17) imply that θ is a global balanced measure.

Conversely, the disintegration of a global balanced measure θ (see Proposition 2.20) gives a holonomy invariant transverse measure $\widetilde{m}$ on $\mathcal{H}$ satisfying (4.17), which further projects to a transverse balanced measure on $\mathcal{A}$. $\square$

4.2. **Equidistribution of leaves.**

4.2.1. *Convergence of measures.* We will show that the leaves of $\mathcal{A}$ are uniformly equidistributed with respect to the transverse balanced measure m constructed in Theorem 4.9. Let us consider a relatively compact domain Δ on a leaf L of the affine lamination $\mathcal{A}$ and a fiber $\overline{\mathcal{T}}_z$, $z \in \overline{\mathbb{C}}$. For $n \in \mathbb{N}$, let $\eta^n_{\Delta,z}$ denote the discrete probability measure on $\overline{\mathcal{T}}_z$ which assigns equal masses to the intersection points of $\widehat{f}^n\Delta$ with the fiber $\overline{\mathcal{T}}_z$, i.e.,

$$\eta^n_{\Delta,z}(A) = \frac{\mathsf{card}[\widehat{f}^n\Delta \cap A]}{\mathsf{card}[\widehat{f}^n\Delta \cap \overline{\mathcal{T}}_z]} \ , \qquad A \subset \overline{\mathcal{T}}_z \ .$$

Denote by $\widetilde{\kappa} = \widetilde{\kappa}_L$ the infinite Radon measure on the leaf L with $\operatorname{supp}\widetilde{\kappa}_{\mathrm{L}} = \mathcal{J} \cap \mathrm{L}$ obtained by pulling back the balanced measure κ via the projection π, i.e., $\widetilde{\kappa}$ is obtained by integrating (by the measure κ) the counting measures on the preimages of π:

$$\widetilde{\kappa}(\Delta) = \int_{\overline{\mathbb{C}}} \mathsf{card}[\Delta \cap \overline{\mathcal{T}}_z] \, d\kappa(z) \ , \qquad \Delta \subset L \ .$$

To define the weak topology on the space of measures, we will consider the following space $\mathcal{S}$ of *test functions*. A function $h : \mathcal{A} \to \mathbb{R}$ belongs to $\mathcal{S}$ if its support is normal and it is uniformly continuous with respect to the germ uniform structure (recall the definitions from §**3.1.7**). Of course, functions of class $\mathcal{S}$ are continuous. Moreover, when f is tame (i.e., $\mathcal{A}$ is locally compact), the class $\mathcal{S}$ consists of continuous functions with compact support.

Let us say that a sequence of measures μ_n on $\overline{\mathcal{T}}_z$ *weakly converges* to a measure μ if

$$\langle h, \mu_n \rangle \to \langle h, \mu \rangle$$

for any test function $h \in \mathcal{S}$.

Theorem 4.18. *Let Δ be a domain on a leaf $L \subset \mathcal{A}$ with $0 < \widetilde{\kappa}(\Delta) < \infty$ and $\widetilde{\kappa}(\partial\Delta) = 0$. Then for any non-exceptional point $z \in \overline{\mathbb{C}}$ the measures $\eta^n_{\Delta,z}$ weakly converge to m_z as $n \to \infty$.*

Proof. Step 1. It is enough to verify this statement locally, taking for Δ a small leafwise neighbourhood of an arbitrary point $\boldsymbol{u} \in \mathcal{A}$. Since $\widetilde{\kappa}(\Delta) > 0$, we may assume without loss of generality that $\boldsymbol{u} \in \mathcal{J}$. Suppose first that

$$\boldsymbol{u} \in \mathcal{J}^l \equiv \mathcal{J} \cap \mathcal{A}^l \ ,$$

and include Δ into a standard flow box $\mathcal{B} \cong B \times T$, so that $\Delta = \Delta(\boldsymbol{u}) \cong B$ is the local leaf of $\boldsymbol{u}$, and $T \subset \mathcal{T}(\boldsymbol{u})$. Recall that the local leaves $\Delta(\boldsymbol{v}) \cong B$, $\boldsymbol{v} \in T$ of $\mathcal{B}$ properly cover a topological disk $D \subset \overline{\mathbb{C}}$ with degree $b \geq 1$ and a single branched point $\xi \in D$ (see §**3.1.5**). Then $\kappa(D) > 0$ and $\kappa(\partial D) = 0$ by the definition of the measure $\widetilde{\kappa}$. By Lemma 4.4 $\boldsymbol{\kappa}(\mathcal{B}) > 0$, so that also $m(T) > 0$.

Fix a point $\boldsymbol{z} \in \mathcal{A}$, and let

$$\pi(\boldsymbol{z}) = z \equiv z_0 \ , \qquad \wp(\boldsymbol{z}) = \{z_{-m}\}_{m\in\mathbb{N}} \in \mathcal{A}^l \ ,$$

where π and $\wp$ are the projections of $\mathcal{A}$ onto $\overline{\mathbb{C}}$ and $\mathcal{A}^l$, respectively, see §**3.1.2**. For simplicity put $\Pi = \Pi^0 \equiv \overline{\mathcal{T}}(\boldsymbol{z})$, and denote by $\Pi^k \equiv \overline{\mathcal{T}}^k(\boldsymbol{z})$ the transverse cylinders over $\boldsymbol{z}$ (see (3.9)). $\triangle$

Step 2. In the course of the proof we will have to deal with all local leaves $\Delta(\boldsymbol{v})$, $\boldsymbol{v} \in T$ rather than just with the single local leaf $\Delta = \Delta(\boldsymbol{u})$. Denote by

$$\Delta_n(\boldsymbol{v}) = \widehat{f}^n\Delta(\boldsymbol{v}) \ , \qquad \Delta_n = \Delta_n(\boldsymbol{u}) = \widehat{f}^n\Delta$$

the forward iterations of the local leaves of $\mathcal{B}$ and count the intersection points of $\Delta_n(\boldsymbol{v})$ with the cylinder Π^k, $k \geq 0$.

Let $n \geq k$. If

$$\boldsymbol{\zeta} \in \widehat{f}^{-n}\Big(\Delta_n(\boldsymbol{v}) \cap \Pi^k\Big) = \Delta(\boldsymbol{v}) \cap \widehat{f}^{-n}\Pi^k \ ,$$

i.e.,

$$\boldsymbol{\zeta} \in \Delta(\boldsymbol{v}) \ , \qquad \widehat{f}^n\boldsymbol{\zeta} \in \Pi^k \ , \tag{4.19}$$

then

$$\zeta \equiv \pi(\boldsymbol{\zeta}) \in D \ , \qquad f^{n-k}\zeta = z_{-k} \ . \tag{4.20}$$

Conversely, any point $\zeta \in D$ satisfying (4.20) can be lifted to b points $\boldsymbol{\zeta}$ satisfying (4.19). [Strictly speaking, this is true only for the points $\zeta \in D \setminus \{\xi\}$, whereas the point ξ

lifts to the unique point $\boldsymbol{v}$. However, the corresponding correction term in (4.21) below is negligible for our subsequent calculations, so that without loss of generality we may always assume $\zeta \neq \xi$]. It follows that

$$\mathsf{card}[\Delta_n(\boldsymbol{v}) \cap \Pi^k] = \mathsf{card}[\Delta_n \cap \Pi^k] = b \cdot \mathsf{card}[\zeta \in D : f^{n-k}\zeta = z_{-k}] \tag{4.21}$$

is independent of $\boldsymbol{v} \in T$ for $n \geq k$ (see Fig. 7 where for simplicity we assume $b = 1$).

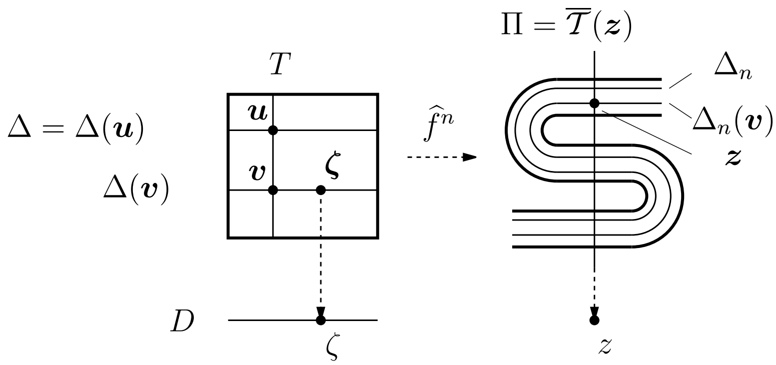

FIGURE 7

On the other hand, since $\kappa(\partial D) = 0$, by Theorem 4.2

$$\mathsf{card}[\zeta \in D : f^{n-k}\zeta = z_{-k}] \sim \kappa(D) \cdot d^{n-k} .$$

Hence

$$\begin{aligned} \mathsf{card}[\Delta_n \cap \Pi^k] &\sim b \cdot \kappa(D) \cdot d^{n-k} \\ &= \widetilde{\kappa}(\Delta) \cdot d^{n-k} = \widetilde{\kappa}(\Delta) \cdot d^n \cdot m(\Pi^k) . \end{aligned} \tag{4.22}$$

In particular,

$$\mathsf{card}[\Delta_n \cap \Pi] \sim \widetilde{\kappa}(\Delta) \cdot d^n . \tag{4.23}$$

Dividing (4.22) by (4.23), we obtain:

$$\eta^n_{\boldsymbol{v}}(\Pi^k) \underset{n\to\infty}{\longrightarrow} m(\Pi^k) , \tag{4.24}$$

where $\eta^n_{\boldsymbol{v}} \equiv \eta^n_{\Delta(\boldsymbol{v}),z}$. Moreover, by Theorem 4.2 this convergence is uniform over $\boldsymbol{v} \in T$. $\triangle$

Step 3. The family of probability measures $\eta^n_{\boldsymbol{v}}$, $\boldsymbol{v} \in T$ on Π determines for any n a Markov transition kernel P_n from T to Π. Denote by

$$m_n = (m|_T)P_n = \int_T \eta^n_{\boldsymbol{v}} \, dm(\boldsymbol{v})$$

the measure on Π which is the image of the restriction of the measure m onto T under the kernel P_n. It is easy to see that m_n is proportional to the restriction $\widetilde{m}_n$ of the measure m onto $\widehat{f}^n\mathcal{B} \cap \Pi$:

$$m_n = \gamma_n \widetilde{m}_n , \qquad \text{where} \quad \gamma_n = \frac{d^n}{\mathsf{card}[\Delta_n \cap \Pi]} , \tag{4.25}$$

and, as it follows from (4.23),

$$\gamma_n \underset{n\to\infty}{\longrightarrow} \frac{1}{\widetilde{\kappa}(\Delta)} < \infty \ . \tag{4.26}$$

Indeed, $\mathcal{B} \cap \widehat{f}^{-n}\Pi$ consists of $\mathsf{card}[\Delta_n \cap \Pi]$ disjoint transversals T_i of the flow box $\mathcal{B}$, see Fig. 8. Then

$$m_n = \frac{1}{\mathsf{card}[\Delta_n \cap \Pi]} \sum_i \widehat{f}^n(m|_{T_i}) \ .$$

On the other hand, by Definition 4.8 (ii),

$$\widehat{f}^n(m|_{T_i}) = d^n \cdot m|_{\widehat{f}^n(T_i)} \ .$$

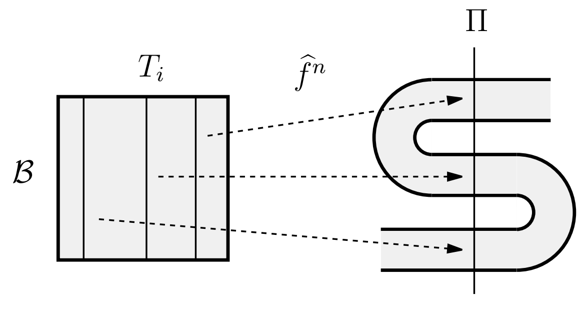

FIGURE 8

The kernels P_n can be considered as "transfer operators" from $L^1(\Pi, m)$ to $L^1(T, m)$:

$$P_n\chi(\boldsymbol{v}) = \langle \chi, \eta^n_{\boldsymbol{v}} \rangle = \frac{1}{\mathsf{card}[\Delta_n \cap \Pi]} \sum_{\boldsymbol{\zeta}\in\Delta_n(\boldsymbol{v})\cap\Pi} \chi(\boldsymbol{\zeta}) \ .$$

By (4.25) for any non-negative $\chi \in L^1(\Pi, m)$

$$\|P_n\chi\| = \int_T \langle \chi, \eta^n_{\boldsymbol{v}} \rangle \, dm(\boldsymbol{v}) = \langle \chi, m_n \rangle = \gamma_n \langle \chi, \widetilde{m}_n \rangle \le \gamma_n \langle \chi, m \rangle = \gamma_n \|\chi\| \ ,$$

so that (4.26) implies that the $L^1 \to L^1$ norms of P_n are uniformly bounded. On the other hand, if χ is a cylinder function, then by (4.24)

$$P_n\chi \to \langle \chi, m \rangle \tag{4.27}$$

uniformly and hence in L^1. As cylinder functions are dense in L^1, we conclude that (4.27) holds for any $\chi \in L^1(\Pi, m)$. $\triangle$

Step 4. By Lemma 3.18 for any test function $\chi \in \mathcal{S}$

$$|P_n\chi(\boldsymbol{v}) - P_n\chi(\boldsymbol{u})| \underset{n\to\infty}{\longrightarrow} 0 \tag{4.28}$$

uniformly in $\boldsymbol{v} \in T$, whereas, by (4.27),

$$\int |P_n\chi(\boldsymbol{v}) - \langle \chi, m \rangle| \, dm(\boldsymbol{v}) \to 0 \ ,$$

which implies that

$$\langle \chi, \eta_{\boldsymbol{u}}^n \rangle = P_n \chi(\boldsymbol{u}) \to \langle \chi, m \rangle \ .$$

$\triangle$

Step 5. Finally, if $\boldsymbol{u} \in \mathcal{J} \setminus \mathcal{J}^l$, then we can approximate Δ by a local leaf $\Delta' \subset \mathcal{A}^l$ in some flow box around $\boldsymbol{u}$. Then the claim follows from the fact that the local leaves Δ and Δ' are forward asymptotic and from uniformity of convergence in (4.24) and (4.28). $\square$

Remark 4.29. Theorem 4.18 can be also interpreted as saying that the transverse balanced measure m is obtained from the sequence of leafwise Følner sets $\widehat{f}^n\Delta$ (see [Pl75]).

4.2.2. *Convergence of currents.* Along with test functions, we can consider *test forms* on $\mathcal{A}$ supported on a normal set and uniformly continuous with respect to the germ uniform structure. We will use the same notation $\mathcal{S}$ for the class of test forms. Such a class determines the associated weak topology on the space of currents, see (2.31). Below in this Section by convergence of currents we shall always mean weak convergence with respect to the class $\mathcal{S}$.

Theorem 4.30. *Let Δ be a domain on a leaf $L \subset \mathcal{A}$ with $0 < \widetilde{\kappa}(\Delta) < \infty$ and $\widetilde{\kappa}(\partial\Delta) = 0$. Then*

$$\frac{1}{d^n}\,[\widehat{f}^n\Delta] \to \widetilde{\kappa}(\Delta)\,[m] \ ,$$

where m is the transverse balanced measure on $\mathcal{A}$ constructed in Theorem 4.9.

Proof. Take a product flow box $\mathcal{C} \cong K \times C$, where C is identified with a fixed local leaf C_{k_0}, $k_0 \in K$. We may assume that all transversals $K_{\boldsymbol{z}}$, $\boldsymbol{z} \in C_0$ are contained in the corresponding dual fibers $\overline{\mathcal{T}}(\boldsymbol{z})$.

$$\frac{1}{d^n} \int_{\widehat{f}^n\Delta} h \to \widetilde{\kappa}(\Delta) \int_K dm(k) \int_{C_k} h \ , \tag{4.31}$$

where C_k is the local leaf passing through the point $k \in K$. Take an affine area form on C_0. By the transversals of $\mathcal{C}$ it carries over to a global leafwise form ω on local leaves of $\mathcal{C}$. Denote by h' the density of h with respect to ω. Obviously, the function h' belongs to the function space $\mathcal{S}$. In terms of the function h'

$$\int_{\widehat{f}^n\Delta} h = \int_C \langle h', \lambda_{\Delta,\boldsymbol{z}} \rangle\, \omega(\boldsymbol{z}) \ ,$$

where $\lambda_{\Delta,\boldsymbol{z}}$ is the counting measure on the intersection $\widehat{f}^n\Delta \cap K_{\boldsymbol{z}}$, and by Theorem 4.18

$$\frac{1}{d^n}\lambda_{\Delta,\boldsymbol{z}} \to \widetilde{\kappa}(\Delta) \cdot m|_{K_{\boldsymbol{z}}} \tag{4.32}$$

in the weak topology induced by the space of functions $\mathcal{S}$. Integration of (4.32) by the form ω yields the claim. $\square$

Remark 4.33. The reader should compare Theorem 4.30 with analogous results of Bedford–Smillie [BS91, Theorem 3] and Fornaess–Sibony [FS92, Theorem 1.6] for polynomial automorphisms of $\mathbb{C}^2$.

4.3. **Critical exponent.**

4.3.1. *Definition of the critical exponent.* Recall that m stands for the transverse balanced measure on $\mathcal{A}$ constructed in Theorem 4.9. Let us consider the transverse measure $\mu^{\delta,n} = \{\mu_T^{\delta,n}\}$ defined as

$$\mu^{\delta,n} = \widehat{f}^n(\|D\widehat{f}^n\|_\sigma^{-\delta} m) = \|D\widehat{f}^{-n}\|_\sigma^{\delta} \cdot \widehat{f}^n(m) \ , \tag{4.34}$$

where, as in §**3.2**, $\|D\widehat{f}\|_\sigma$ denotes the norm of $D\widehat{f}$ measured with respect to the leafwise Riemannian metric ρ_σ (§**1.1.4**) associated with the special section σ (3.35). This measure assigns the mass

$$\mu^{\delta,n}(X) = \mu_T^{\delta,n}(X) = \int_{\widehat{f}^{-n}X} \|D\widehat{f}^n(\boldsymbol{z})\|_\sigma^{-\delta} \, dm(\boldsymbol{z})$$

to a Borel subset X of a transversal T. In view of Definition 4.8(ii) formula (4.34) can be also rewritten as

$$\mu^{\delta,n} = d^n \|D\widehat{f}^{-n}\|_\sigma^{\delta} \cdot m \ , \tag{4.35}$$

so that in the definition of the measures $\mu^{\delta,n}$ the "big" factor d^n (arising from the expanding action of $\widehat{f}$ on m) is "compensated" by the "small" factor $\|D\widehat{f}^{-n}\|_\sigma^{\delta}$ (cf. Definition 3.20).

Definition 4.36. For a transversal T consider the *"statistical sum"*

$$\Xi_T(\delta) = \sum_{n\in\mathbb{N}} \|\mu_T^{\delta,n}\| \ ,$$

and denote by $\delta_{\rm cr}(T)$ the *critical exponent* of this statistical sum separating the convergent and divergent cases. It is well-defined because by Definition 3.20 $\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma \to 0$ locally uniformly. △

Until the end of §**4** *we shall assume that* f *is tame*, i.e., $\mathcal{A}$ and $\mathcal{H}$ are locally compact. Then these laminations have many compact transversals.

Lemma 4.37. *For any precompact transversal* S *and any other transversal* T

$$\Xi_S(\delta) \le C\,\Xi_T(\delta) \qquad \text{and} \qquad \delta_{\rm cr}(S) \le \delta_{\rm cr}(T) \ ,$$

where the constant C *depends on* S *and* T *only.*

Proof. Clearly, it is enough to prove the first inequality. If S is covered with a finite number of transversals S_i, then

$$\Xi_S(\delta) \le \sum_i \Xi_{S_i}(\delta) \ .$$

As $\overline{S}$ is compact, it is therefore sufficient to check the inequality

$$\Xi_U(\delta) \le C\,\Xi_T(\delta)$$

for some neighborhood $U \subset \overline{S}$ of any point $\boldsymbol{z} \in \overline{S}$.

Since the lamination is minimal, a sufficiently small neighborhood U can be mapped by a holonomy H onto some neighborhood $V \subset T$. Then by the Koebe Distortion Theorem (e.g., see [LM97, p. 86])

$$\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma \asymp \|D\widehat{f}^{-n}(H\boldsymbol{z})\|_\sigma \ , \qquad \boldsymbol{z} \in U \ .$$

Therefore, since m is holonomy invariant,

$$\Xi_U(\delta) \asymp \Xi_V(\delta) \le \Xi_T(\delta) \ .$$

$\square$

Corollary 4.38. *For all precompact transversals T the value $\delta_{\rm cr}(T)$ is the same.*

Definition 4.39. The common value $\delta_{\rm cr} = \delta_{\rm cr}(T)$ is called the *critical exponent* of the map f. The map f (or the corresponding laminations $\mathcal{A}$ and $\mathcal{H}$) is said to be of *divergent type* if $\Xi_T(\delta_{\rm cr}) = \infty$ for any precompact transversal T (by Lemma 4.37 this property is independent of the choice of the transversal). Otherwise the map and the corresponding laminations are of *convergent type*. $\triangle$

Remark 4.40. The critical exponent can be also defined in terms of the global balanced measure θ on the lamination (see §**4.1.3**) without an explicit use of transversals.

Theorem 4.41. *$\delta_{\rm cr} \le 2$.*

To prove this result, we will need a few lemmas.

4.3.2. *An area estimate for meromorphic functions.* Let $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ be a meromorphic function. We shall use the functional I and the function $R : z \mapsto R(\varphi, z)$ introduced in §**3.3.1**.

Lemma 4.42. *There exists an absolute constant C such that for any compact set $X \subset \mathbb{C}$ with $I(X) > 1$*

$$\int_X \frac{\mathsf{eucl}}{R^2} \le C \cdot I(U) \ ,$$

where

$$U = \{z : \operatorname{dist}(z, X) < \operatorname{diam}(X)\} \ ,$$

and eucl *is the standard area form on $\mathbb{C}$.*

Proof. By the Besikovich Covering Lemma (see [Ma95]) it is possible to cover X with a finite number of disks

$$D_i = \mathbb{D}\Big(z_i, \tfrac{1}{2} R(z_i)\Big) \ , \qquad z_i \in X \ , \quad 1 \le i \le K \ ,$$

such that the family of twice bigger disks

$$D_i^{\times 2} = \mathbb{D}\Big(z_i, R(z_i)\Big)$$

have intersection multiplicity at most N, with an absolute constant N. Since $I(X) > 1$,

$$R(z_i) < \operatorname{diam}(X) \ ,$$

and

$$D_i^{\times 2} \subset U \ .$$

Therefore,

$$K = \sum I\left(D_i^{\times 2}\right) \le N \cdot I(U) \ . \tag{4.43}$$

On the other hand,

$$\int_X \frac{\mathsf{eucl}}{R^2} \le \sum_i \int_{D_i} \frac{\mathsf{eucl}}{R^2} = \sum_i \frac{\mathsf{eucl}(D_i)}{R(\zeta_i)^2} = \frac{\pi}{4} \sum_i \frac{R(z_i)^2}{R(\zeta_i)^2} \,, \tag{4.44}$$

where $\zeta_i \in D_i$ are selected by the Mean Value Theorem (the function R is continuous, even 1-Lipschitz, see Lemma 3.36). Note that

$$\mathbb{D}\Big(\zeta_i, \tfrac{1}{2}R(z_i)\Big) \subset D_i^{\times 2} \,,$$

so that

$$I\Big(\mathbb{D}\Big(\zeta_i, \tfrac{1}{2}R(z_i)\Big)\Big) \le I\left(D_i^{\times 2}\right) = 1 \,,$$

and therefore

$$R(\zeta_i) \ge \frac{1}{2} R(z_i)$$

by the definition of the function R. Incorporating this into (4.44), we conclude that

$$\int_X \frac{\mathsf{eucl}}{R^2} \le \pi K \,,$$

where K is the number of the disks D_i. Combining the latter inequality with (4.43) yields the claim. □

4.3.3. *Finiteness of the total area of the affine lamination.* Recall that area_σ denotes the leafwise area form of the metric ρ_σ (§**1.1.4**) associated with the special section σ (3.35), see §**1.1.7**. Uniformizing a leaf L, we can transfer the metric and the corresponding area form to $\mathbb{C}$. Somewhat abusing notations, we will use the same letters for the transferred objects. With this convention, if $\varphi : \mathbb{C} \to \overline{\mathbb{C}}$ is a meromorphic function associated with a leaf L (see §**3.1.2**), then by (3.35)

$$\rho_\sigma = \frac{1}{R(\varphi, z)} |dz|, \qquad \mathsf{area}_\sigma = \frac{\mathsf{eucl}}{R(\varphi, z)^2}.$$

In these terms Lemma 4.42 takes the form of the estimate

$$\mathsf{area}_\sigma(X) \le C \cdot I(U) \,. \tag{4.45}$$

Combining the leafwise form area_σ and the transverse balanced measure m constructed in Theorem 4.9 gives a global measure

$$area_\sigma = \mathsf{area}_\sigma \star m$$

on $\mathcal{A}$. Recall that by definition (2.30)

$$area_\sigma(\mathcal{B}) = \int_T \mathsf{area}_\sigma(B_t)\, dm(t)$$

for any flow box $\mathcal{B} \approx B \times T$.

Lemma 4.46. *$area_\sigma(\mathcal{A}) < \infty$.*

Proof. Choose a Euclidean disk D in a leaf L in such a way that

(i) The 3 times bigger concentric disk $D^{\times 3}$ is univalent (i.e, the projection $\pi : L \to \overline{\mathbb{C}}$ is univalent on $D^{\times 3}$);

(ii) The disk D satisfies conditions of Theorem 4.30, i.e., $\widetilde{\kappa}(D) > 0$.

Then by (4.45)

$$\mathsf{area}_\sigma(\widehat{f}^n D) \le C \cdot I\left(\widehat{f}^n D^{\times 3}\right) .$$

By definition, $I\left(\widehat{f}^n D^{\times 3}\right)$ is the spherical area of the projection $\pi\left(\widehat{f}^n D^{\times 3}\right)$ counted with multiplicity. Condition (i) implies that the multiplicity of the projection π restricted to $\widehat{f}^n D^{\times 3}$ is bounded by d^n. Therefore,

$$\frac{1}{d^n}\,\mathsf{area}_\sigma(\widehat{f}^n D) \le 4\pi C = C' , \qquad \forall\, n \in \mathbb{N} .$$

This formula can be rewritten as

$$\left\langle \mathsf{area}_\sigma, \frac{1}{d^n}[\widehat{f}^n D]\right\rangle \le C' , \qquad \forall\, n \in \mathbb{N} . \tag{4.47}$$

where $[\widehat{f}^n D]$ is the integration current over $\widehat{f}^n D$.

In view of condition (ii), Theorem 4.30 implies that $\frac{1}{d^n}[\widehat{f}^n D]$ weakly converges to $\widetilde{\kappa}(D)[m]$, where $[m]$ is the Ruelle–Sullivan current determined by the transverse measure m (see Definition 2.32). By a standard truncation argument it yields the inequality

$$area_\sigma(\mathcal{A}) < \frac{C'}{\widetilde{\kappa}(D)} .$$

Indeed, let ω be a truncation of the form area_σ obtained by multiplying it by a bump function supported on some compact subset of $\mathcal{A}$. Then (4.47) and Theorem 4.30 imply that

$$\langle \omega, [m]\rangle \le \frac{C'}{\widetilde{\kappa}(D)} .$$

But, since $\mathcal{A}$ is locally compact,

$$area_\sigma(\mathcal{A}) = \langle \mathsf{area}_\sigma, [m]\rangle = \sup_\omega \langle \omega, [m]\rangle ,$$

where the supremum is taken over all truncations of area_σ. □

4.3.4. *Volume estimates.* Recall the notation $b_\sigma(\boldsymbol{h})$ (1.34) for the relative hyperbolic height of a point $\boldsymbol{h} \in \mathcal{H}$ with respect to a section σ. As it follows from $\widehat{f}$-invariance of the basic cocycle,

$$b_{\widehat{f}.\sigma}(\widehat{f}\boldsymbol{h}) = b_\sigma(\boldsymbol{h}) , \tag{4.48}$$

where $\widehat{f}.\sigma$ denotes the result of the action of $\widehat{f}$ (1.64) on the section σ. Let

$$\mathcal{H}_\sigma^- = \left\{\boldsymbol{h} \in \mathcal{H} : b_\sigma(\boldsymbol{h}) < 0\right\}$$

be the part of the hyperbolic lamination under the graph of the section σ.

Denote by

$$vol_\sigma = \widetilde{area_\sigma} = \widetilde{\mathsf{area}}_\sigma \star \widetilde{m}$$

the lift of the measure $area_\sigma$ to $\mathcal{H}$ (see Definition 2.34 and Proposition 2.39), where $\widetilde{m}$ is the lift of the transverse measure m (see Definition 2.38). For any real ε let

$$vol_\sigma^\varepsilon = \exp[\varepsilon b_\sigma] \cdot vol_\sigma = \mathsf{vol}_\sigma^\varepsilon \star \widetilde{m} ,$$

where

$$\mathsf{vol}_\sigma^\varepsilon = \exp[\varepsilon b_\sigma] \cdot \widetilde{\mathsf{area}}_\sigma .$$

In other words, if $\ell^\varepsilon = \{\ell^\varepsilon_{\boldsymbol{z}}\}$, $\boldsymbol{z} \in \mathcal{A}$ is the family of measures on the vertical geodesics $\mathfrak{p}^{-1}(\boldsymbol{z})$ with densities $\exp[\varepsilon b_\sigma]$ with respect to the hyperbolic length $\ell = \{\ell_{\boldsymbol{z}}\}$, then

$$vol^\varepsilon_\sigma = \int \ell^\varepsilon_{\boldsymbol{z}}\, d\, area_\sigma(\boldsymbol{z}) \,. \tag{4.49}$$

Note that as it follows from Proposition 1.58, $\mathsf{vol}^\varepsilon_\sigma$ and vol^ε_σ for $\varepsilon = -2$ are independent of σ and coincide with the leafwise hyperbolic volume vol and with the global balanced measure $\theta = vol \star m$, respectively.

Lemma 4.50. *$vol^\varepsilon_\sigma(\mathcal{H}^-_\sigma) < \infty$ for any $\varepsilon > 0$.*

Proof. By formula (4.49)

$$vol^\varepsilon_\sigma(\mathcal{H}^-_\sigma) = \int_{-\infty}^0 e^{\varepsilon t}\, dt \cdot area_\sigma(\mathcal{A}) = \frac{1}{\varepsilon}\, area_\sigma(\mathcal{A}) < \infty \,.$$

□

For $\boldsymbol{z} \in \mathcal{A}$, let $D(\boldsymbol{z}) = D_\sigma(\boldsymbol{z}) \subset L_{\mathcal{A}}(\boldsymbol{z})$ stand for the disk which is centered at $\boldsymbol{z}$ and has radius 1 with respect to the Euclidean structure $\sigma_{\boldsymbol{z}}$, and let

$$I(\boldsymbol{z}) = \{\boldsymbol{h} \in \mathfrak{p}^{-1}(\boldsymbol{z}) : -1 < b_\sigma(\boldsymbol{h}) < 0\} \,,$$
$$W(\boldsymbol{z}) = \bigcup_{\boldsymbol{\zeta} \in D(\boldsymbol{z})} I_{\boldsymbol{\zeta}} \subset L_{\mathcal{H}}(\boldsymbol{z}) \,,$$

see Fig. 9.

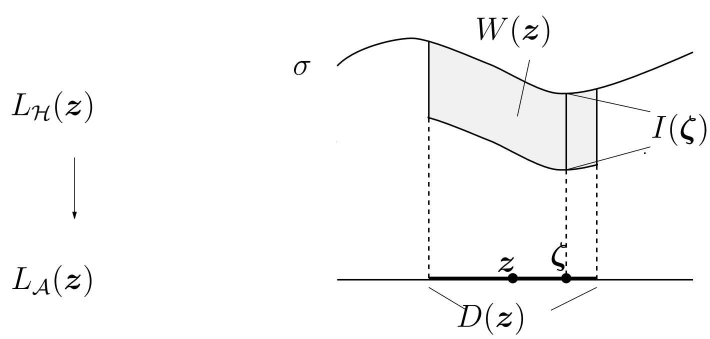

FIGURE 9

Lemma 4.51. *Locally uniformly on $\boldsymbol{z} \in \mathcal{A}$*

$$\mathsf{vol}^\varepsilon_\sigma\Big(\widehat{f}^{-n} W(\boldsymbol{z})\Big) \asymp \|D\widehat{f}^{-n}(\boldsymbol{z})\|^{2+\varepsilon}_\sigma \,.$$

Proof. First notice that by formula (1.67) for any point $\boldsymbol{\zeta} \in D(\boldsymbol{z})$

$$\beta\Big(\sigma(\boldsymbol{\zeta}_{-n}), \widehat{f}^{-n} \circ \sigma(\boldsymbol{\zeta})\Big) = \log \|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma$$

(as before we use the notation $\boldsymbol{\zeta}_n = \widehat{f}^n \boldsymbol{\zeta}$ for the orbit of the point $\boldsymbol{\zeta}$). Therefore, formula (4.48) shows that the interval $\widehat{f}^{-n} I(\boldsymbol{\zeta})$ is the result of the translation ($\equiv$ action of the vertical flow ξ) of the interval $I(\boldsymbol{\zeta}_{-n})$ by $\log \|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma$ (see Fig. 10, where we assume that $\log \|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma < 0$, in accordance with Definition 3.20).

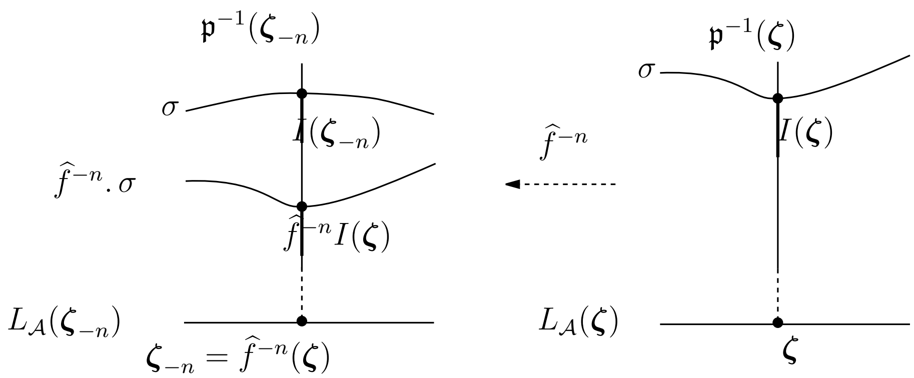

FIGURE 10

Therefore,

$$\ell^\varepsilon\Big(\widehat{f}^{-n} I(\boldsymbol{z})\Big) = \|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma^\varepsilon \cdot \ell^\varepsilon\Big(I(\boldsymbol{z})\Big) \ . \tag{4.52}$$

On the other hand,

$$\mathsf{area}_\sigma\Big(\widehat{f}^{-n} D(\boldsymbol{z})\Big) \asymp \|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma^2$$

locally uniformly, because by Theorem 3.22 and Lemma 3.36 the ratio

$$\frac{\|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma}{\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma}$$

is locally uniformly bounded and bounded away from 0 on $\boldsymbol{z} \in \mathcal{A}, \boldsymbol{\zeta} \in D(\boldsymbol{z})$ and $n \in \mathbb{N}$. Integrating the function (4.52) by area_σ over $\widehat{f}^{-n} D(\boldsymbol{z})$ yields the claim. □

We are now ready to prove the theorem.

4.3.5. *Proof of Theorem 4.41.* Given a precompact transversal T of the lamination $\mathcal{A}$, let us consider the flow box

$$\mathcal{B} = \bigcup_{\boldsymbol{z} \in T} W(\boldsymbol{z}) \subset \mathcal{H} \ .$$

Then by Lemma 4.51

$$\begin{aligned} vol_\sigma^\varepsilon(\widehat{f}^{-n}\mathcal{B}) = \int_{\widehat{f}^{-n}T} \mathsf{vol}_\sigma^\varepsilon\Big(\widehat{f}^{-n} W(\boldsymbol{z}_n)\Big)\, dm(\boldsymbol{z}) &\asymp \int_{\widehat{f}^{-n}T} \|D\widehat{f}^{-n}(\boldsymbol{z}_n)\|_\sigma^{2+\varepsilon}\, dm(\boldsymbol{z}) \\ &= \int_{\widehat{f}^{-n}T} \|D\widehat{f}^{n}(\boldsymbol{z})\|_\sigma^{-2-\varepsilon}\, dm(\boldsymbol{z}) \ . \end{aligned}$$

Summing up we obtain

$$\Xi_T(2+\varepsilon) \asymp \sum_{n \in \mathbb{N}} vol_\sigma^\varepsilon(\widehat{f}^{-n}\mathcal{B}) \ .$$

But the latter sum is finite, since

- The box $\mathcal{B}$ is wandering [LM97, Proposition 6.2];
- Its backward iterates eventually belong to $\mathcal{H}_\sigma^-$ by [LM97, Lemma 6.1];
- $vol_\sigma^\varepsilon(\mathcal{H}_\sigma^-) < \infty$ by Lemma 4.50.

□

4.4. **Transverse conformal stream and λ-harmonic measure.**

4.4.1. *The formulations.*

Theorem 4.53. *The lamination $\mathcal{A}$ carries an $\widehat{f}$-invariant parallel transverse conformal stream μ of dimension $\delta_{\rm cr}$.*

Applying Theorem 2.86, we immediately obtain

Theorem 4.54. *The hyperbolic lamination $\mathcal{H}$ carries an $\widehat{f}$-invariant λ-harmonic measure ω with $\lambda = \delta_{\rm cr}(\delta_{\rm cr}-2)$, which descends to a λ-harmonic measure on the quotient hyperbolic lamination $\mathcal{M}$.*

4.4.2. *Proof of Theorem 4.53.* We shall use Proposition 2.90 and Theorem 2.61. Namely, we shall fix a special section $\sigma : \mathcal{A} \to \mathcal{H}$ (see Lemma 3.36), and construct for this section a transverse measure μ_σ satisfying conditions of the above Propositions with $\delta = \delta_{\rm cr}$. In other words, the Jacobian of $\widehat{f}$ with respect to μ_σ has to be

$$\mathrm{Jac}_{\mu_\sigma} \widehat{f} = \|D\widehat{f}\|_\sigma^{-\delta_{\rm cr}} , \tag{4.55}$$

and the modulus of μ_σ has to be

$$\Delta_{\mu_\sigma} = \exp[\delta_{\rm cr}\beta_\sigma] . \tag{4.56}$$

Step 1. Let us first consider the transverse measures

$$\mu^{\delta,n} = \widehat{f}^n\Big(\|D\widehat{f}^n\|_\sigma^{-\delta} m\Big) = \|D\widehat{f}^{-n}\|_\sigma^\delta \cdot \widehat{f}^n(m) = d^n \|D\widehat{f}^{-n}\|_\sigma^\delta \cdot m$$

defined in §**4.3.1**, and find their moduli and the Jacobian of the map $\widehat{f}$ with respect to these measures. By holonomy invariance of the measure m and formula (2.7) the modulus of $\mu^{\delta,n}$ is

$$\Delta_{\delta,n}(\boldsymbol{z},\boldsymbol{\zeta}) = \frac{\|D\widehat{f}^{-n}(\boldsymbol{\zeta})\|_\sigma^\delta}{\|D\widehat{f}^{-n}(\boldsymbol{z})\|_\sigma^\delta} . \tag{4.57}$$

The Jacobian of the map $\widehat{f}$ with respect to the source measure $\mu^{\delta,n}$ and the target measure $\mu^{\delta,n+1}$ is

$$\begin{aligned} \mathrm{Jac}_{\delta,n} \widehat{f}(\boldsymbol{z}) &= \frac{d\widehat{f}^{-1}\mu^{\delta,n+1}}{d\mu^{\delta,n}}(\boldsymbol{z}) = \frac{d\widehat{f}^n(\|D\widehat{f}^{n+1}\|_\sigma^{-\delta} m)}{d\widehat{f}^n(\|D\widehat{f}^n\|_\sigma^{-\delta} m)}(\boldsymbol{z}) \\ &= \frac{\|D\widehat{f}^{n+1}\|_\sigma^{-\delta}(\boldsymbol{z}_{-n})}{\|D\widehat{f}^n\|_\sigma^{-\delta}(\boldsymbol{z}_{-n})} = \|D\widehat{f}(\boldsymbol{z})\|_\sigma^{-\delta} . \end{aligned} \tag{4.58}$$

△

Step 2. For $\delta > \delta_{\rm cr}$, let us sum up the measures $\mu^{\delta,n}$:

$$\mu^\delta = \sum_{n=0}^\infty \mu^{\delta,n} .$$

By Definition 4.36 and Lemma 4.37, the measure μ^δ is locally finite, and

$$\|\mu^\delta_T\| = \Xi_T(\delta)$$

for any precompact transversal T. Fix now a compact transversal Q, and normalize the measures μ^δ by letting

$$\mu^\delta = \frac{\mu^\delta}{\Xi_Q(\delta)} ,$$

so that

$$\|\mu^\delta_Q\| = 1 .$$

Consider first the divergent case, i.e., assume that $\Xi_Q(\delta_{\rm cr}) = \infty$. Take a weak limit point μ_Q of the family $\{\mu^\delta_Q\}$ as $\delta \to \delta_{\rm cr}$ (instead of passing to a subsequence $\delta_k \to \delta_{\rm cr}$ and further taxing our notations we shall assume for simplicity that the family $\{\mu^\delta_Q\}$ itself is convergent). We claim that the family μ^δ then weakly converges on *any* precompact transversal, and the resulting transverse measure satisfies conditions (4.55) and (4.56). △

Step 3. Formula (4.55) easily follows from (4.58). Indeed, take a precompact transversal T. Then

$$\mu^{\delta,n+1}_{\widehat{f}T} = \widehat{f}\Big(\|D\widehat{f}\|_\sigma^{-\delta} \cdot \mu^{\delta,n}_T\Big) .$$

Summing over n and normalizing, we obtain that for any $\delta > \delta_{\rm cr}$

$$\mu^\delta_{\widehat{f}T} - \frac{1}{\Xi_Q(\delta)}\mu^{\delta,0}_{\widehat{f}T} = \widehat{f}\Big(\|D\widehat{f}\|_\sigma^{-\delta} \cdot \mu^\delta_T\Big) .$$

Since by the divergence assumption the second term in the left-hand side is vanishing as $\delta \to \delta_{\rm cr}$, the desired property follows. △

Step 4. Let us now check formula (4.56). The crucial point of the proof is the uniform convergence of the moduli $\Delta_{\delta,n}$ of the measures $\mu^{\delta,n}$ (4.57) to the sought for cocycle $\exp[\delta\beta_\sigma]$, which follows from Theorem 3.22.

Let $H : Q \to T$ be a holonomy map from Q onto another transversal T. Denote by

$$\varphi_{\delta,n}(\boldsymbol{\zeta}) = \Delta_{\delta,n}(H^{-1}\boldsymbol{\zeta}, \boldsymbol{\zeta})$$

and

$$\varphi(\boldsymbol{\zeta}) = \exp\Big[\delta_{\rm cr}\beta_\sigma(H^{-1}\boldsymbol{\zeta}, \boldsymbol{\zeta})\Big]$$

the functions on T determined by the moduli of the measures $\mu^{\delta,n}$ and by the cocycle $\exp[\delta_{\rm cr}\beta_\sigma]$, respectively. Then for any $\delta > \delta_{\rm cr}$

$$\begin{aligned}
\left\|\mu_T^\delta - \varphi\cdot H\mu_Q^\delta\right\| &= \frac{1}{\Xi_Q(\delta)}\left\|\mu_T^\delta - \varphi\cdot H\mu_Q^\delta\right\| = \frac{1}{\Xi_Q(\delta)}\left\|\sum_{n=0}^\infty\Big(\mu_T^{\delta,n} - \varphi\cdot H\mu_Q^{\delta,n}\Big)\right\| \\
&\le \frac{1}{\Xi_Q(\delta)}\sum_{n=0}^\infty\left\|\mu_T^{\delta,n} - \varphi\cdot H\mu_Q^{\delta,n}\right\| \\
&= \frac{1}{\Xi_Q(\delta)}\sum_{n=0}^\infty\left\|\varphi_{\delta,n}\cdot H\mu_Q^{\delta,n} - \varphi\cdot H\mu_Q^{\delta,n}\right\| \\
&\le \frac{1}{\Xi_Q(\delta)}\sum_{n=0}^\infty \|\varphi_{\delta,n}-\varphi\|_\infty\cdot\|\mu_Q^{\delta,n}\| \\
&= \frac{\sum_{n=0}^\infty \|\varphi_{\delta,n}-\varphi\|_\infty\cdot\|\mu_Q^{\delta,n}\|}{\sum_{n=0}^\infty \|\mu_Q^{\delta,n}\|} \ .
\end{aligned}$$

Here $\|\varphi_{\delta,n}-\varphi\|_\infty$ denotes the sup-norm of the difference $(\varphi_{\delta,n}-\varphi)$ over T, which by Theorem 3.22 tends to 0 uniformly on δ near $\delta_{\rm cr}$ as $n\to\infty$. Since for any n

$$\|\mu_Q^{\delta,n}\| \underset{\delta\to\delta_{\rm cr}}{\longrightarrow} \|\mu_Q^{\delta_{\rm cr},n}\| < \infty \ ,$$

and the series $\Xi_Q(\delta_{\rm cr}) = \sum_n \|\mu_Q^{\delta_{\rm cr},n}\|$ diverges, we obtain that

$$\left\|\mu_T^\delta - \varphi\cdot H\mu_Q^\delta\right\| \underset{\delta\to\delta_{\rm cr}}{\longrightarrow} 0 \ .$$

Therefore,

$$\mu_T^\delta \underset{\delta\to\delta_{\rm cr}}{\longrightarrow} \mu_T = \varphi\cdot H\mu_Q \ .$$

△

Step 5. In the convergence case we apply the Patterson regularization procedure. In our setting it looks as follows. Instead of the measures $\mu^{\delta,n}$ (4.34) we use the modified measures

$$\mathring{\mu}^{\delta,n} = \widehat{f}^n\Big(\varphi(\|D\widehat{f}^n\|_\sigma)\|D\widehat{f}^n\|_\sigma^{-\delta}m\Big) \ ,$$

where the function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ has the following properties:

- $\lim_{t\to\infty}\varphi(t) = \infty$,
- $\lim_{t\to\infty}\varphi(t)/t^\varepsilon = 0 \qquad \forall\,\varepsilon>0$,
- $\lim_{t\to\infty}\varphi(Ct)/\varphi(t) = 1 \qquad \forall\,C>0$,
- $\mathring{\Xi}_Q(\delta_{\rm cr}) \equiv \sum \|\mathring{\mu}_T^{\delta_{\rm cr},n}\| = \infty$.

Then $\mathring{\Xi}_Q(\delta)<\infty$ for any $\delta>\delta_{\rm cr}$, and we may consider the normalized measures

$$\mathring{\mu}^\delta = \frac{1}{\mathring{\Xi}_Q(\delta)}\sum_{n=0}^\infty \mathring{\mu}^{\delta,n} \ .$$

Taking a weak limit point of the measures $\mathring{\mu}^\delta$ we can proceed in the same way as in the divergent case. △

Remark 4.59. Although the measures μ^δ in the above construction are equivalent to the balanced transverse measure m, obviously the limit conformal transverse measure μ does not have to be equivalent to m. If the basic class b is non-trivial, then for any conformal transverse measure and any balanced transverse measure the cohomology classes of their moduli are different, and therefore these measures are necessarily singular. Therefore, for all maps described by Theorem 3.48 conformal transverse and balanced transverse measures are singular.

Remark 4.60. The harmonic measure ω can also be constructed directly in terms of global measures on the hyperbolic lamination $\mathcal{H}$. Namely, the balanced transverse measure m gives rise to the global balanced measure θ on $\mathcal{H}$ whose Radon–Nikodym cocycle (with respect to the leafwise hyperbolic volume) is trivial but which is not invariant under the dynamics: $\widehat{f}\theta = d \cdot \theta$ (see §**4.1.3**). Applying to this measure the Patterson method we construct a measure ω on $\mathcal{H}$ with dual properties: it is $\widehat{f}$-invariant but its Radon–Nikodym cocycle is non-trivial, $\Delta_\omega = \exp[\delta\beta]$. [There are no $\widehat{f}$-invariant measures on $\mathcal{H}$ with a trivial Radon–Nikodym cocycle.]

4.5. **Leafwise conformal streams.**

Theorem 4.61. *The lamination $\mathcal{A}$ carries an $\widehat{f}$-invariant leafwise conformal stream λ.*

4.5.1. *The lift of a conformal measure from the Julia set.* One way of proving Theorem 4.61 consists in considering a continuous δ-conformal measure $\eta \equiv \eta_\varsigma$ on the Julia set of a rational function f [S83]. The measure η can be lifted via the projection $\pi : \mathcal{A} \to \overline{\mathbb{C}}$ to a leafwise Radon measure $\widetilde{\eta}$ and then transferred in a δ-conformal way into a δ-conformal leafwise stream λ. Namely, given a leafwise conformal metric ρ, let

$$\lambda_\rho = \left(\frac{\rho}{\widetilde{\varsigma}}\right)^\delta \widetilde{\eta} \,, \tag{4.62}$$

where $\widetilde{\varsigma}$ is the pullback of the spherical metric to the leaf L. [This metric has isolated singularities but they do not matter since the measure η is assumed to be continuous.]

More functorially, we consider the f-invariant δ-conformal stream $d\eta_\rho \otimes d\rho^{-\delta}$ on the sphere $\overline{\mathbb{C}}$ corresponding to $\eta \equiv \eta_\varsigma$, and naturally lift it to the $\widehat{f}$-invariant leafwise δ-conformal stream λ (4.62) on $\mathcal{A}$.

In the case when there exists an invariant "Gibbs measure" equivalent to the conformal measure η, the lifting procedure can be also described as follows:

Proposition 4.63 (cf. Ledrappier [Le84] and Lemma 4.3)**.** *Let υ be an f-invariant measure equivalent to a δ-conformal measure η on $J(f)$, and let $\widehat{\upsilon}$ be its lift to the natural extension. Then its leafwise conditional measures determine an $\widehat{f}$-invariant leafwise δ-conformal stream on the lamination $\mathcal{A}$.*

4.5.2. *An intrinsic construction of a leafwise conformal stream.* We shall now give an intrinsic proof of Theorem 4.61 which does not make use of the construction of an f-invariant conformal stream on the Julia set $J(f)$ [S83] or of Proposition 4.63 above. It is dual to the construction of the $\widehat{f}$-invariant parallel transverse conformal stream in Theorem 4.53 from §**4.4**.

First we shall define the forward critical exponent $\gamma_{\rm cr}$ of $\mathcal{A}$ dual to the (backward) critical exponent $\delta_{\rm cr}$ constructed in §**4.3.1**. Given a number $\gamma \geq 0$, let us consider the following leafwise measure:

$$\lambda^{\gamma,n} = \widehat{f}^{-n}(\|D\widehat{f}^{-n}\|_\rho^\gamma \cdot \widetilde{\kappa}) ,$$

where $\widetilde{\kappa}$ is the leafwise lift of the balanced measure κ on $J(f)$ (see §**4.2.1**), and the norm is measured with respect to the Riemannian metric $\rho = \rho_\sigma$ associated with the special section σ from §**3.3**. If D is a Borel subset of a leaf L, then by definition

$$\lambda^{\gamma,n}(D) = \int_{\widehat{f}^n D} \|D\widehat{f}^{-n}(\boldsymbol{z})\|_\rho^\gamma \, d\widetilde{\kappa}(\boldsymbol{z}) .$$

Given a leafwise bounded domain $D \subset L$ which meets the Julia set $\mathcal{J}$, let us now consider the following *forward Poincaré series*:

$$\Theta_D(\gamma) = \sum_{n\in\mathbb{N}} \lambda^{\gamma,n}(D) . \tag{4.64}$$

Define $\gamma_{\rm cr}(D)$, the *forward critical exponent*, as the one which separates divergent and convergent values of γ. It is well defined because of the expanding property of $\widehat{f}$ from Lemma 3.41.

Lemma 4.65. *The critical exponent $\gamma_{\rm cr}(D)$ is independent of the choice of D.*

Proof. Lemma 3.42 implies that the critical exponent is preserved under the dual holonomy: $\gamma_{\rm cr}(D) = \gamma_{\rm cr}(V(D))$ for any leafwise domain $D \subset B_t$ in a product flow box $\mathcal{B}$.

Furthermore, $\gamma_{\rm cr}(\widehat{f}^n D) = \gamma_{\rm cr}(D)$, and $\gamma_{\rm cr}(D') \leq \gamma_{\rm cr}(D)$ if $D' \subset D$. It follows that given a saddle periodic point $\widehat{\alpha} \in \mathcal{J}$ (i.e., the lift of a repelling periodic point $\alpha \in J$), $\gamma_{\rm cr}(D)$ is the same for all leafwise domains $D \subset L(\widehat{\alpha})$. Thus, this exponent can be attributed to the point $\widehat{\alpha}$ itself: $\gamma_{\rm cr}(\widehat{\alpha}) \equiv \gamma_{\rm cr}(D)$ for any domain D as above.

Let us show now that $\gamma_{\rm cr}(\widehat{\alpha}) = \gamma_{\rm cr}(\widehat{\beta})$ for any two saddle points $\widehat{\alpha}$ and $\widehat{\beta}$. Take a flow box $\mathcal{B}$ containing $\widehat{\alpha}$. Since the lamination $\mathcal{A}$ is minimal the leaf $L(\widehat{\beta})$ passes through this flow box. Hence there is a domain $\Delta \subset L(\widehat{\beta})$ containing a local leaf D of $\mathcal{B}$. Let V be the dual holonomy moving D to the local leaf of α. Then

$$\gamma_{\rm cr}(\beta) = \gamma_{\rm cr}(\Delta) \geq \gamma_{\rm cr}(D) = \gamma_{\rm cr}(V(D)) = \gamma_{\rm cr}(\alpha) ,$$

and the opposite inequality holds by symmetry. Denote the common critical exponent for all saddle points by $\gamma_{\rm cr}$.

Finally, take any bounded leafwise domain D on $\mathcal{A}$ intersecting the Julia set $\mathcal{J}$. Chop it into pieces D_i contained in product flow boxes $\mathcal{B}_i$. Then

$$\gamma_{\rm cr}(D) = \max_i \gamma_{\rm cr}(D_i),$$

where the maximum is taken over the domains D_i meeting the Julia set $\mathcal{J}$ (since the others do not contribute to the forward Poincaré series). Each of the corresponding flow boxes $\mathcal{B}_i$ contains a saddle point $\widehat{\alpha}_i$ (Proposition 4.6), and therefore $\gamma_{\rm cr}(D_i) = \gamma_{\rm cr}(\alpha_i) = \gamma_{\rm cr}$. $\square$

We can now apply the Patterson method to construct a leafwise conformal stream at the critical exponent. Taking a (regularized if necessary) limit $\sum_{n\in\mathbb{N}} \lambda^{\gamma,n}$ as $\gamma \searrow \gamma_{\rm cr}$ from above, we obtain a leafwise measure λ_ρ which is $\gamma_{\rm cr}$-conformal under the dynamics and the vertical holonomy:

$$\frac{d\widehat{f}^{-1}\lambda_\rho}{d\lambda_\rho} = \|D\widehat{f}\|_\rho^\delta , \qquad \frac{dV^{-1}\lambda_\rho}{d\lambda_\rho} = \exp[\gamma_{\rm cr}\alpha_\rho] ,$$

where α_ρ is the basic dual cocycle on $\mathcal{A}$. Since the cocycle is locally continuous, so is the measure. Hence it determines a $\gamma_{\rm cr}$-conformal stream (this construction is exactly dual to the construction of the transverse stream in §**4.4**, so that we omit the details).

4.5.3. *Forward and backward critical exponents.* It is important to know whether the forward and backward critical exponents coincide. If so then we have a transverse δ-conformal stream μ and a leafwise δ-conformal stream λ on $\mathcal{A}$ with the same exponent $\delta = \delta_{\rm cr} = \gamma_{\rm cr}$. Integrating one against the other (see Theorem 2.57) we obtain an $\widehat{f}$-invariant global Radon measure $\boldsymbol{\upsilon} = \lambda\star\mu$ on $\mathcal{A}$ (perhaps infinite). This measure can be viewed as the "Gibbs measure" of $\widehat{f}$ corresponding to the potential $-\delta\log\|D\widehat{f}\|_\rho$ with "zero pressure" (see, e.g., the survey [EL90, Ch. 3] for the concepts involved); "zero pressure" means that δ is the critical exponent of the Poincaré series, which does not depend on the choice of a conformal metric ρ.

Pushing $\boldsymbol{\upsilon}$ down to the Riemann sphere $\overline{\mathbb{C}}$ via the projection $\pi : \mathcal{A} \to \overline{\mathbb{C}}$, we obtain an invariant measure υ on the Julia set $J(f)$. This is the "Gibbs measure" of f corresponding to the potential $-\delta\log\|Df\|$ with "zero pressure". Then the streams λ and μ can be recovered from υ as its conditional measures (cf. Proposition 4.63 and Theorem 4.9, respectively). *This gives us a new way of constructing conformal and Gibbs measures.*

The Gibbs measure $\boldsymbol{\upsilon} = \lambda\star\mu$ on $\mathcal{J}$ can be suspended to a measure $\widetilde{\boldsymbol{\upsilon}} = \overline{\lambda}\star\overline{\mu} = \widetilde{\lambda\star\mu}$ on $\mathcal{M}$ invariant under the vertical flow ξ (Theorem 2.68), which further descends to the measure $\widetilde{\boldsymbol{\upsilon}}_{\mathcal{M}}$ on the quotient $\mathcal{M} = \mathcal{H}/\widehat{f}$ (Theorem 2.91). This measure is supported on the "curtain" over the Julia set $\mathcal{J}$, i.e., on the union of the vertical geodesics terminating at $\mathcal{J}$ modulo the $\widehat{f}$-action. One can view this measure as "weakly hyperbolic" with horospheres serving for unstable leaves and standard transversals serving for the "local stable leaves". For rational functions satisfying *Axiom A* (i.e., such that critical points are attracted to attracting cycles), the measure $\widetilde{\boldsymbol{\upsilon}}_{\mathcal{M}}$ was considered in [BFU]. In this case the vertical flow ξ is hyperbolic and $\widetilde{\boldsymbol{\upsilon}}_{\mathcal{M}}$ is its unique measure of maximal entropy.

4.5.4. *Convex cocompact case.* Let us finally show that the forward and backward exponents coincide in the convex co-compact case. Recall from [LM97, §8] that the *convex core* $\mathcal{C}$ of the lamination $\mathcal{H}$ is defined as the leafwise convex core of the Julia set $\mathcal{J}$. It is invariant under $\widehat{f}$, so that the quotient $\mathcal{C}/\widehat{f}$ is well-defined. By definition, this is the convex core of the hyperbolic lamination $\mathcal{M} = \mathcal{H}/\widehat{f}$. A rational function f is called *convex co-compact* if $\mathcal{C}/\widehat{f}$ is compact. It is equivalent to compactness of the Julia set $\mathcal{J}$ ([LM97, Proposition 8.5]). Moreover, convex co-compact functions can be dynamically characterized by the property that all critical points $c \in J(f)$ are non-recurrent and there are no parabolic points [LM97, Theorem 8.1].

Theorem 4.66. *If $\mathcal{M}$ is convex co-compact, then $\delta_{\rm cr} = \gamma_{\rm cr}$.*

Proof. (sketch). Since the Julia set $\mathcal{J}$ is compact and the whole lamination $\mathcal{A}$ is locally compact, $\mathcal{J}$ can be covered with finitely many precompact flow boxes $\mathcal{B}_i \cong B_i \times T_i$. By Lemma 4.37,

$$\Xi_S(\delta) \asymp \Xi_T(\delta) \tag{4.67}$$

for any two transversals S and T of these boxes (with the constants dependent only on δ). Integrating (4.67) over the leafwise measures $\widetilde{\kappa}$, we conclude that the backward Poincaré series $\Xi_T(\delta)$ are all comparable with the series

$$\sum_{n=0}^{\infty} \int_{\widehat{f}^{-n}\mathcal{J}} \|D\widehat{f}^n(\boldsymbol{z})\|_\rho^{-\delta} d\boldsymbol{\kappa}(\boldsymbol{z}) = \sum_{n=0}^{\infty} \int_{\mathcal{J}} \|D\widehat{f}^n(\boldsymbol{z})\|_\rho^{-\delta} d\boldsymbol{\kappa}(\boldsymbol{z}) .$$

Similarly, the forward Poincaré series $\Theta_D(\delta)$ (4.64) are comparable with the series

$$\sum_{n=0}^{\infty} \int_{\mathcal{J}} \|D\widehat{f}^{-n}(\boldsymbol{z})\|_\rho^{\delta} d\boldsymbol{\kappa}(\boldsymbol{z}) .$$

But since the measure $\boldsymbol{\kappa}$ is $\widehat{f}$-invariant, the last two series are equal. □

4.6. **Sullivan's Riemann surface laminations.** The above discussion (reduced by 1 in dimension) also applies to Sullivan's Riemann surface lamination associated with a C^2 circle diffeomorphism $f : \mathbb{T} \to \mathbb{T}$ (see [LM97, §11], [MS93]). The natural extension $\widehat{f} : \mathcal{N} \to \mathcal{N}$ of such a diffeomorphism is a one-dimensional affine lamination (the unstable lamination of $\widehat{f}$). It is endowed with the leafwise Riemannian metric ρ lifted from the circle $\mathbb{T}$.

Applying our constructions, we can endow $\mathcal{N}$ with $\widehat{f}$-invariant 1-conformal transverse and leafwise streams μ and λ. The product $\boldsymbol{\upsilon} = \lambda \star \mu$ of these streams is the invariant Gibbs measure which is absolutely continuous on the unstable lamination. This measure can also be constructed by lifting the absolutely continuous invariant measure υ of f (the Gibbs measure corresponding to the potential $-\log|f'|$).

By means of the hyperbolization functor the affine lamination $\mathcal{N}$ extends to a (pointed at infinity) hyperbolic two-dimensional lamination $\mathcal{H}^2$ whose leaves are hyperbolic planes supplied with the hyperbolic action of $\widehat{f}$. The Sullivan lamination $\mathcal{M}^2$ is obtained by taking the quotient $\mathcal{H}^2/\widehat{f}$.

According to Theorem 2.86, the transverse stream μ lifts to a harmonic measure ω^μ on $\mathcal{H}^2$ (with the eigenvalue $\lambda = 0$). The Radon–Nikodym cocycle of this measure is equal to $\exp[\beta]$, where β is the Busemann cocycle on $\mathcal{H}^2$. This measure is $\widehat{f}$-invariant and hence descends to a harmonic measure $\omega^\mu_{\mathcal{M}}$ on $\mathcal{M}^2$.

On the other hand, the quotient of the Gibbs measure $\boldsymbol{\upsilon}$ is the measure $\boldsymbol{\upsilon}_{\mathcal{M}}$ of maximal entropy of the vertical flow ξ on $\mathcal{M}^2$ (see §**2.4**). In fact, in this situation both constructions lead to the same result: $\omega^\mu_{\mathcal{M}} = \boldsymbol{\upsilon}_{\mathcal{M}}$.

The Riemann surface lamination $\mathcal{M}^2$ is similar in many respects to the unit tangent bundle US of a compact Riemannian surface S of (variable) negative curvature (cf. the Appendix §**5**). Uniformizing the leaves of the (weak) stable foliation, we turn US into a two-dimensional hyperbolic foliation. The "universal covering" of this foliation is an $\mathbf{H}^2$-fibration over the circle S^1 with pointed at infinity fibers ($\equiv$ leaves). The hyperbolic structure on the leaves induces the affine structure on the punctured circle at infinity.

This endows $S^1 \times S^1 \setminus \text{diag}$ with the structure of a one-dimensional affine lamination. Moreover, the fundamental group $G = \pi_1(S)$ naturally acts on this lamination by leafwise affine transformations.

Remark 4.68. One can extend this discussion further to more general one-dimensional C^2 maps, which are allowed to have critical points, cf. §**2.1.3**.

4.7. **Problems.** In [LM97, §10] there is a list of problems on the structure of the laminations associated with rational maps. Below we will formulate a few more problems motivated by the results of this paper.

1. Tameness. Our construction of conformal streams and harmonic measures works in the case of tame (i.e., locally compact) laminations $\mathcal{A}_f$ and $\mathcal{H}_f$. There are many tame laminations but there are wild laminations as well (e.g., when the regular leaf space $\mathcal{R}_f$ has a hyperbolic leaf). Give a dynamical criterion of tameness. Are there transverse conformal streams on wild $\mathcal{A}_f$ and harmonic measures on wild $\mathcal{H}_f$?

2. Uniqueness and ergodic theory. Study the problem of uniqueness of conformal streams and harmonic measures on the laminations in question. Study ergodic properties of the holonomy pseudo-group with respect to the conformal stream. Study ergodic properties of the vertical flow and the horosphere foliation with respect to the harmonic measure. Study when these flows and foliations are conservative/dissipative.

3. Critical exponents. Give conditions for coincidence of the forward and backward critical exponents. Relate these critical exponents to the other critical exponents associated with f (see [S83], [DU91a]). Relate them to the Hausdorff dimension of the Julia set and the set of conical limit points.

4. Finiteness Problem. Give a dynamical criterion of finiteness of the harmonic measure on the convex core of $\mathcal{H}_f$. Is it so in the "geometrically finite case" (when all critical points on the Julia set are non-recurrent, parabolic cycles are allowed)? Is it so in the Collet-Eckmann case?

5. Holonomy vs. *dynamics.* The definitions of balanced and conformal measures include the transformation rules under both holonomy and dynamics. Are they equivalent?

5. Appendix. Laminations associated with Kleinian groups

As we have already mentioned in the Introduction, the main incentive for the construction of the affine lamination associated with a rational map was an attempt to better understand the existing parallelism between two branches of the conformal dynamics: the Kleinian and the rational dynamics. In order to show how the theory of laminations encompasses both these fields we shall now describe the affine laminations associated with Kleinian groups. We show that the well-known Patterson measures for Kleinian groups admit a natural interpretation both as leafwise and transverse G-invariant conformal streams for these laminations.

5.1. **Foliations associated with the hyperbolic space.**

5.1.1. *Tautological foliations.* The simplest building blocks of an affine lamination are the standard affine planes. As we have seen in §**1**, such a plane arises as the punctured visibility sphere $\mathcal{P}_q \cong \partial\mathbf{H}^3 \setminus \{q\}$ of a pointed at infinity hyperbolic space $(\mathbf{H}^3, q)$. Conversely, the hyperbolization functor $\mathfrak{H}$ allows one to recover the space $(\mathbf{H}^3, q)$ from $\mathcal{P}_q$. Varying the boundary points $q \in \partial\mathbf{H}^3$ we obtain a family of affine planes $\mathcal{P}_q$, $q \in \partial\mathbf{H}^3$.

Definition 5.1. The *tautological* $\mathbb{C}$*-foliation* $\mathring{\mathcal{A}}$ is the foliation of the space

$$\partial^2\mathbf{H}^3 = \partial\mathbf{H}^3 \times \partial\mathbf{H}^3 \setminus \mathrm{diag} = \bigcup_{\mathrm{q}\in\partial\mathbf{H}^3} \mathcal{P}_{\mathrm{q}} \times \{\mathrm{q}\} \tag{5.2}$$

with leaves $\mathcal{P}_q$. [In fact, it is a $\mathbb{C}$-fibration over the Riemann sphere $\overline{\mathbb{C}}$.] Its hyperbolization $\mathring{\mathcal{H}} = \mathfrak{H}\mathring{\mathcal{A}}$ is called the *tautological pointed at infinity hyperbolic foliation*. The total space of $\mathring{\mathcal{H}}$ is

$$\mathbf{H}^3 \times \partial\mathbf{H}^3 = \bigcup_{q\in\partial\mathbf{H}^3} \mathbf{H}^3 \times \{q\}$$

and the leaves are pointed at infinity hyperbolic spaces $(\mathbf{H}^3, q)$. The Busemann cocycle on $\mathring{\mathcal{H}}$ is

$$\beta\Big((h_1, q), (h_2, q)\Big) = \beta_q(h_1, h_2)\ . \tag{5.3}$$

△

5.1.2. *Parameterizations of the unit tangent bundle.* Denote the *unit tangent bundle* of the hyperbolic space by $U\mathbf{H}^3$ with the canonical projection

$$p : U\mathbf{H}^3 \to \mathbf{H}^3\ ,$$

and let $\gamma = \{\gamma^\tau\}_{\tau\in\mathbb{R}}$ be the *geodesic flow* on $U\mathbf{H}^3$. The endpoints of the geodesic determined by a tangent vector $v \in U\mathbf{H}^3$ are denoted $\gamma^{\pm\infty}(v) \in \partial\mathbf{H}^3$. By

$$\mathrm{Hor}(v) = \mathrm{Hor}_{\gamma^\infty(v)}\Big(p(v)\Big) = \{h \in \mathbf{H}^3 : \beta_{\gamma^\infty(v)}\Big(p(v)v, h\Big) = 0\}$$

we denote the horosphere centered at the point $\gamma^\infty(v)$ and passing through the point $p(v)$. Clearly,

$$\mathrm{Hor}(v_1) = \mathrm{Hor}(v_2) \qquad \Longleftrightarrow \qquad \mathrm{Hor}(\gamma^\tau v_1) = \mathrm{Hor}(\gamma^\tau v_2)\ \forall\, \tau \in \mathbb{R}\ ,$$

so that the formula

$$\gamma^\tau\, \mathrm{Hor}(v) = \mathrm{Hor}(\gamma^\tau v)\ , \qquad \tau \in \mathbb{R} \tag{5.4}$$

determines an action of the geodesic flow on the space $\mathrm{Hor}(\mathbf{H}^3)$.

There are two natural parameterizations of the space $U\mathbf{H}^3$ (see Fig. 11).

Proposition 5.5. *The map*

$$v \mapsto \Big(p(v), \gamma^\infty(v)\Big) \tag{5.6}$$

from $U\mathbf{H}^3$ to the space $\mathbf{H}^3 \times \partial\mathbf{H}^3$ is a diffeomorphism. For any $(h, q_+) \in \mathbf{H}^3 \times \partial\mathbf{H}^3$ the associated vector $v \in U\mathbf{H}^3$ is the directing vector of the geodesic ray issued from the point h in the direction q_+.

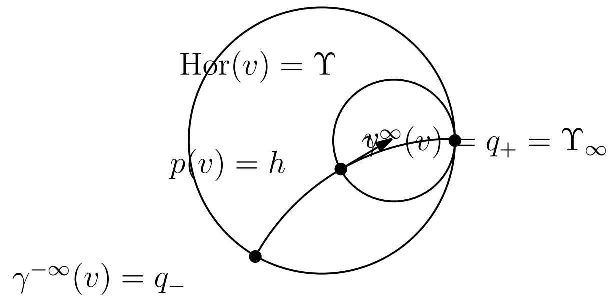

FIGURE 11

Proposition 5.7. *The map*

$$v \mapsto \Big(\gamma^{-\infty}(v), \mathrm{Hor}(v)\Big) \tag{5.8}$$

from $U\mathbf{H}^3$ to the space

$$\partial\mathbf{H}^3 \times \mathrm{Hor}(\mathbf{H}^3) \setminus \{(q,\Upsilon) : q = \Upsilon_\infty\} = \bigcup_{\Upsilon\in\mathrm{Hor}(\mathbf{H}^3)} \mathcal{P}_{\Upsilon_\infty} \times \{\Upsilon\} \tag{5.9}$$

is a diffeomorphism. For any $q_- \in \partial\mathbf{H}^3, \Upsilon \in \mathrm{Hor}(\mathbf{H}^3)$ with $q_- \neq \Upsilon_\infty$ the associated vector $v \in U\mathbf{H}^3$ is the tangent vector to the geodesic joining q_- with Υ_∞ at the point of its intersection with the horosphere Υ.

5.1.3. *Weakly stable and strongly stable foliations.* Recall the definitions of two natural foliations associated with the geodesic flow on $U\mathbf{H}^3$:

Definition 5.10. Two vectors $v_1, v_2 \in U\mathbf{H}^3$ belong to the same leaf of the *weakly stable foliation* $\mathcal{W}^s$ of the geodesic flow if

$$\limsup_{t\to+\infty} \mathrm{dist}(\gamma^t v_1, \gamma^t v_2) < \infty\ , \tag{5.11}$$

and to the same leaf of the *strongly stable foliation* $\mathcal{W}^{ss}$ if

$$\lim_{t\to+\infty} \mathrm{dist}(\gamma^t v_1, \gamma^t v_2) = 0\ , \tag{5.12}$$

where dist denotes the natural metric on $U\mathbf{H}^3$. △

Condition (5.11) means that $\gamma^\infty(v_1) = \gamma^\infty(v_2)$. Therefore,

Proposition 5.13. *The identification* (5.6) *establishes an isomorphism between the foliations $\overset{\circ}{\mathcal{H}}$ and $\mathcal{W}^s$ and conjugates the vertical flow on $\overset{\circ}{\mathcal{H}}$ with the geodesic flow.*

Condition (5.12) means that $\mathrm{Hor}(v_1) = \mathrm{Hor}(v_2)$ (this is why $\mathcal{W}^{ss}$ is also often called *horosphere foliation*). Consequently,

Proposition 5.14. *Under the identification* (5.8) *the foliation $\mathcal{W}^{ss}$ is isomorphic to the foliation of the space* (5.9) *with the leaves $\mathcal{P}_{\Upsilon_\infty} \times \{\Upsilon\}$.*

Corollary 5.15. *The foliation $\mathcal{W}^{ss}$ is a Euclidean foliation.*

Proof. Indeed, the leaves of $\mathcal{W}^{ss}$ can be identified with horospheres in $\mathbf{H}^3$, so that they are endowed with the natural Euclidean structures induced by the hyperbolic metric. □

In terms of the coordinates (q_-, Υ) (5.8) the geodesic flow on $U\mathbf{H}^3$ takes the form

$$\gamma^\tau(q_-, \Upsilon) = (q_-, \gamma^\tau \Upsilon) , \tag{5.16}$$

so that it acts by affine laminar automorphisms of $\mathcal{W}^{ss}$ mapping concentric horospheres one onto the other. The foliation $\mathring{\mathcal{A}}$ is the result of factorization of $\mathcal{W}^{ss}$ by the geodesic flow (so that the space of $\mathring{\mathcal{A}}$ has one dimension less than the space of $\mathcal{W}^{ss}$). This is the reason why the leaves of $\mathring{\mathcal{A}}$ carry just a natural affine structure (the "scale" having been lost as a result of the action of the geodesic flow).

The total space of the hyperbolization $\mathfrak{H}\mathcal{W}^{ss}$ is

$$\bigcup_{\Upsilon \in \mathrm{Hor}(\mathbf{H}^3)} \mathfrak{H}\mathcal{P}_{\Upsilon_\infty} \times \{\Upsilon\} = \mathbf{H}^3 \times \mathrm{Hor}(\mathbf{H}^3) . \tag{5.17}$$

The Euclidean structure on a horosphere $\Upsilon \in \mathrm{Hor}(\mathbf{H}^3)$ determined by a point $h \in \mathbf{H}^3$ is the image of the induced hyperbolic metric on $\mathrm{Hor}_{\Upsilon_\infty}(h)$ under the vertical flow

$$\xi_q^s : \mathrm{Hor}_{\Upsilon_\infty}(h) \to \Upsilon , \qquad s = \beta_{\Upsilon_\infty}(h, \Upsilon) .$$

The Busemann cocycle on $\mathfrak{H}\mathcal{W}^{ss}$ is

$$\beta\Big((h_1, \Upsilon), (h_2, \Upsilon)\Big) = \beta_{\Upsilon_\infty}(h_1, h_2) . \tag{5.18}$$

Remark 5.19. Note the difference between the actions of the vertical and the geodesic flows on $\mathfrak{H}\mathcal{W}^{ss}$. The vertical flow acts leafwise by the formula

$$\xi^\tau(h, \Upsilon) = \left(\xi^\tau_{\Upsilon_\infty} h, \Upsilon\right) ,$$

whereas the action of the geodesic flow on $U\mathbf{H}^3$ (5.16) induces its action on $\mathfrak{H}\mathcal{W}^{ss}$

$$\gamma^\tau(h, \Upsilon) = \left(h, \gamma^\tau \Upsilon\right) \tag{5.20}$$

by isometries between leaves.

5.2. **Laminations associated with Kleinian groups.**

5.2.1. *Definitions and basic properties.* The roles of two factors $\partial\mathbf{H}^3$ in the definition (5.2) of the total space $\partial^2\mathbf{H}^3$ of the foliation $\mathring{\mathcal{A}}$ are quite different: the first one (the "leafwise direction") is indispensable if we want to have a $\mathbb{C}$-lamination, whereas nothing prevents us from replacing the second one (the "transverse direction") with an arbitrary subset of $\partial\mathbf{H}^3$. Therefore, for any subset $X \subset \partial\mathbf{H}^3$ the space

$$\mathcal{A}_X = \bigcup_{q \in X} \mathcal{P}_q \times \{q\}$$

is endowed with a lamination structure (this is not a foliation unless X is a submanifold of $\partial\mathbf{H}^3$).

Recall that the *limit set* $\Lambda = \Lambda(G) \subset \partial\mathbf{H}^3$ of a Kleinian group G is defined as the closure (in the visibility compactification $\mathbf{H}^3 \cup \partial\mathbf{H}^3$) of any given orbit Go, $o \in \mathbf{H}^3$ (the result does not depend on the choice of o). The limit set is either finite (and consists of not more than 2 points) or uncountable. In the latter case the group G is called *non-elementary*. The action of a non-elementary group on its limit set is minimal (e.g., see [GH55, Chapter 13]).

Definition 5.21. Let G be a Kleinian group. The lamination $\mathcal{A}_G = \mathcal{A}_{\Lambda(G)}$ is called the *affine lamination associated with the group* G. The corresponding *hyperbolic lamination* $\mathcal{H}_G = \mathfrak{H}\mathcal{A}_G$ *associated with the group* G is the product lamination of the total space $\mathbf{H}^3 \times \Lambda(G)$. $\triangle$

Since the limit set is G-invariant, the group G acts on $\mathcal{A}_G$ by laminar affine maps and on $\mathcal{H}_G$ by laminar isometries. Moreover, the action of G on $\mathbf{H}^3$ (and, therefore, on $U\mathbf{H}^3 \cong \mathbf{H}^3 \times \partial\mathbf{H}^3$) is properly discontinuous, so that it is also properly discontinuous on $\mathcal{H}_G \cong \mathbf{H}^3 \times \Lambda(G) \subset U\mathbf{H}^3$. Denote by $\mathcal{M}_G \subset UM$ the corresponding quotient hyperbolic lamination (here $M = \mathbf{H}^3/G$ is the quotient hyperbolic manifold associated with the group G).

Remark 5.22. If the action of G on $\mathbf{H}^3$ is not free, then $\mathcal{M}_G$ is an orbifold lamination.

Remark 5.23. Although the quotient hyperbolic lamination $\mathcal{M}_G$ always makes sense, the quotient of the affine lamination $\mathcal{A}_G$ by the action of the group G is *not* well-defined for non-elementary groups G (precisely as in the case of the laminations associated with rational maps, see Remark 3.14). Indeed, since the actions of G and of the vertical flow on $\mathcal{H}_G$ commute, if the quotient $\mathcal{A}_G/G$ is well-defined, then necessarily $\mathcal{M}_G = \mathcal{H}_G/G$ coincides with the hyperbolization $\mathfrak{H}(\mathcal{A}_G/G)$. However, the latter is impossible. For, since G is non-elementary, there is a hyperbolic element $g \in G$ (e.g., see [GH55, Chapter 13]). It implies that the vertical flow on $\mathcal{M}_G$ ($\equiv$ the geodesic flow on $U\mathbf{H}^3/G$) has a periodic orbit covered by the axis of g, so that the condition formulated in Remark 1.80 can not be satisfied.

In spite of a number of similarities, there are also some differences between the properties of the laminations $\mathcal{A}_G, \mathcal{H}_G, \mathcal{M}_G$ associated with Kleinian groups and those of the laminations $\mathcal{A}_f, \mathcal{H}_f, \mathcal{M}_f$ associated with rational maps (see §**3.1**).

Remark 5.24. Clearly, if the group G is non-elementary, then the laminations $\mathcal{A}_G$ and $\mathcal{H}_G$ are never minimal (unlike the laminations $\mathcal{A}_f$ and $\mathcal{H}_f$).

Remark 5.25. Yet another difference is that the fiber bundle $\mathfrak{p} : \mathcal{H}_G \to \mathcal{A}_G$ always admits a parallel section, i.e., the lamination $\mathcal{A}_G$ is always Euclidean (unlike the affine laminations $\mathcal{A}_f$, cf. §**3.5**). Indeed, the Busemann cocycle on $\mathcal{H}_G$ is given by formula (5.3), where $q \in \Lambda$ and $h_1, h_2 \in \mathbf{H}^3$, so that it can be trivialized, for example, by the function $\varphi(q, h) = \beta_q(o, h)$, where $o \in \mathbf{H}^3$ is a fixed reference point.

5.2.2. *Non-triviality of the Busemann cocycle.* If the group G is non-elementary, then the Busemann cocycle of the lamination $\mathcal{M}_G$ is non-trivial for an obvious reason: it is non-trivial for the leaves in $\mathcal{M}_G$ corresponding to fixed points of the hyperbolic elements in G. Denote by $\Lambda_0 = \Lambda_0(G) \subset \Lambda$ the set of all such fixed points. It is not hard to see (cf. formula (5.27) below) that even if we discard the set Λ_0 (i.e., remove from $\mathcal{M}_G$ all leaves with non-trivial Busemann cocycle), then the Busemann cocycle of the lamination

$$\mathcal{M}'_G = \mathcal{H}'_G/G , \qquad \mathcal{H}'_G = \mathbf{H}^3 \times \Big(\Lambda(G) \setminus \Lambda_0(G)\Big)$$

is non-trivial in the continuous cohomology. We shall prove a stronger result (cf. Theorem 3.26) :

Theorem 5.26. *For any non-elementary Kleinian group G the Busemann cocycle of the lamination $\mathcal{M}'_G$ is non-trivial in the Borel cohomology.*

Proof. Triviality of the Busemann cocycle on $\mathcal{M}'_G$ is equivalent to existence of a G-invariant Borel function φ on $\mathcal{H}'_G$ trivializing the Busemann cocycle:

$$\beta_q(h_1, h_2) = \varphi(h_2, q) - \varphi(h_1, q) . \tag{5.27}$$

Our proof that this is impossible is based on the theory of random walks on the group G.

Fix for convenience a reference point $o \in \mathbf{H}^3$ (its choice is irrelevant for what follows) and take a probability measure μ on the group G such that the first moment $\sum_{g\in G} \operatorname{dist}(o, go)\, \mu(g)$ is finite and $\mu(g) > 0$ for all $g \in G$. Denote by μ^∞ the product measure on the space G^∞ of sequences $\boldsymbol{g} = (g_1, g_2, \dots)$. Every $\boldsymbol{g} \in G^\infty$ gives rise to the (random) sequence $h_n = h_n(\boldsymbol{g}) = g_1 g_2 \dots g_n o \in \mathbf{H}^3$ which μ^∞-a.e. has the following properties (see [Ka00b]):

(i) There exists a limit
$$h_\infty = h_\infty(\boldsymbol{g}) = \lim_{n\to\infty} h_n \in \partial\mathbf{H}^3 ,$$
and the image ν of the measure μ^∞ under the map $\boldsymbol{g} \mapsto h_\infty(\boldsymbol{g})$ is purely non-atomic.

(ii) If U denotes the Bernoulli shift $(g_1, g_2, \dots) \mapsto (g_2, g_3, \dots)$ in the space (G^∞, μ^∞), then
$$h_\infty(U\boldsymbol{g}) = g_1^{-1} h_\infty(\boldsymbol{g}) .$$

(iii) There exists a number $l = l(\mu) > 0$ (the same for a.e. $\boldsymbol{g} \in G^\infty$) such that
$$\frac{1}{n} \operatorname{dist}(o, h_n) \underset{n\to\infty}{\longrightarrow} l .$$

(iv) The distance between h_n and the geodesic ray joining the points $o \in \mathbf{H}^3$ and $h_\infty \in \partial\mathbf{H}^3$ is $o(n)$.

Combination of (iii) and (iv) implies that μ^∞-a.e.

$$\frac{1}{n}\beta_{h_\infty}(h_0, h_n) \to l . \tag{5.28}$$

Assuming that (5.27) is satisfied, put

$$\Phi(\boldsymbol{g}) = \varphi(o, h_\infty(\boldsymbol{g})) .$$

As it follows from (i) above, $\nu(\Lambda_0) = 0$, so that the function Φ is μ^∞-a.e. well-defined. Then

$$\begin{aligned}\beta_{h_\infty}(o, h_n) &= \varphi(h_n, h_\infty) - \varphi(o, h_\infty) = \varphi(g_1 g_2 \dots g_n o, h_\infty) - \varphi(o, h_\infty) \\ &= \varphi(o, g_n^{-1} \dots g_2^{-1} g_1^{-1} h_\infty) - \varphi(o, h_\infty) = \Phi(U^n \boldsymbol{g}) - \Phi(\boldsymbol{g}) .\end{aligned}$$

Since U preserves the measure μ^∞, (5.28) would be then impossible by the Poincaré recurrence theorem, which gives the sought for contradiction. □

Corollary 5.29. *The lamination $\mathcal{M}_G$ admits no Borel Euclidean structure.*

Corollary 5.30. *There is no Borel G-equivariant map assigning to every point $q \in \Lambda(G) \setminus \Lambda_0(G)$ a horosphere centered at q.*

5.3. Metrics on the Riemann sphere.

5.3.1. *Visual metrics.*

Definition 5.31. The *visual* (*spherical, angular*) metric ς_h, $h \in \mathbf{H}^3$ on the sphere $\partial\mathbf{H}^3$ is the unique Riemannian metric invariant with respect to all isometries of $\mathbf{H}^3$ which fix h and normalized to have curvature 1. The distance between two points $q_-, q_+ \in \partial\mathbf{H}^3$ in the metric ς_h is just

$$\varsigma_h(q_-, q_+) = \angle_h(q_-, q_+) \ ,$$

i.e., the angle between these points "as seen from the point h" (more rigorously, the angle between the directing vectors of the geodesic rays joining h with q_- and q_+, see Fig. 12). $\triangle$

Clearly, the assignment $h \mapsto \varsigma_h$ is equivariant with respect to the group $\mathrm{Iso}(\mathbf{H}^3)$ of isometries of $\mathbf{H}^3$:

$$g\varsigma_h = \varsigma_{gh} \qquad \forall\, h \in \mathbf{H}^3,\ g \in \mathrm{Iso}(\mathbf{H}^3) \ .$$

The following is well-known (e.g., see [Ka00a]):

Proposition 5.32. *All metrics ς_h, $h \in \mathbf{H}^3$ are pairwise conformally equivalent, and*

$$\frac{\varsigma_{h_2}}{\varsigma_{h_1}}(q) = \exp\Big[\beta_q(h_1, h_2)\Big] \ , \qquad \forall\, h_1, h_2 \in \mathbf{H}^3,\ q \in \partial\mathbf{H}^3 \ .$$

5.3.2. *Cut length.* Given two points $q_- \neq q_+ \in \partial\mathbf{H}^3$ and a point $h \in \mathbf{H}^3$ denote by $l_h(q_-, q_+)$ the length of the segment of the geodesic (q_-, q_+) cut out by the horospheres passing through h and centered at the points q_- and q_+. We shall call $l_h(q_-, q_+)$ the *cut length*. Clearly,

$$l_h(q_-, q_+) = \beta_{q_-}(h, o) + \beta_{q_+}(h, o) \tag{5.33}$$

for any point o lying on the geodesic (q_-, q_+) (see Fig. 12). Therefore,

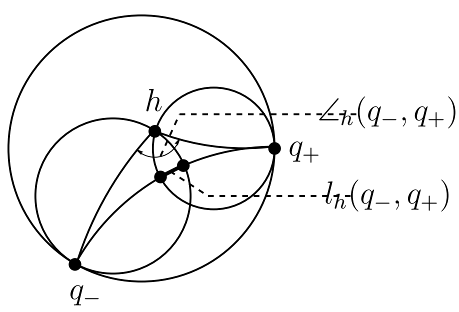

FIGURE 12

Proposition 5.34 ([Ka90])**.** *For any $h_1, h_2 \in \mathbf{H}^3$ and $(q_-, q_+) \in \partial\mathbf{H}^3$*

$$l_{h_1}(q_-, q_+) - l_{h_2}(q_-, q_+) = \beta_{q_-}(h_1, h_2) + \beta_{q_-}(h_1, h_2) \ .$$

We shall now calculate the value of $l_h(q_-, q_+)$. Recall that $\varepsilon_{h,q}$ denotes the Euclidean metric on the punctured sphere $\mathcal{P}_q = \partial\mathbf{H}^3 \setminus \{q\}$ which is the image of the induced hyperbolic metric on the horosphere $\mathrm{Hor}_q(h)$ (we add the subscript q to the notation from (1.39) where the point $q \in \partial\mathbf{H}^3$ was assumed fixed). By $\mathfrak{p}_q$ is denoted the projection of the hyperbolic space $\mathbf{H}^3$ onto the plane $\mathcal{P}_q$, i.e., $\mathfrak{p}_q h$ is the endpoint on $\partial\mathbf{H}^3$ of the geodesic passing through q and $h \in \mathbf{H}^3$ (see §**1.1.1**).

Proposition 5.35. *For any $h \in \mathbf{H}^3$ and $q_- \neq q_+ \in \partial\mathbf{H}^3$*

$$l_h(q_-, q_+) = \log\Big[1 + \Big[\varepsilon_{h,q_+}(q_-, \mathfrak{p}_{q_+}h)\Big]^2\Big] = -2\log\sin\left[\frac{1}{2}\varsigma_h(q_-, q_+)\right] .$$

Proof. Denote by (z, t) and $(\zeta, 0)$ the coordinates of the points h and q_-, respectively in the upper half-space model (q_+ being the point at infinity). Then the geodesic joining q_- and q_+ is just the vertical lime passing through the point $(\zeta, 0)$. The horosphere $\mathrm{Hor}_{q_+}(h)$ is the horizontal plane passing through the point h, and the horosphere $\mathrm{Hor}_{q_-}(h)$ is a Euclidean sphere passing through h and tangent to the boundary plane at the point q_- (see Fig. 13). One can easily see that the Euclidean radius r of this sphere satisfies the relation

$$r^2 = (r - t)^2 + |z - \zeta|^2 ,$$

whence

$$2r = \frac{t^2 + |z - \zeta|^2}{t} . \tag{5.36}$$

The value of $l_h(q_-, q_+)$ is the ratio of Euclidean heights of the points of intersection of the horospheres $\mathrm{Hor}_{q_\pm}(h)$ with the geodesic (q_-, q_+), i.e.,

$$\begin{aligned} l_h(q_-, q_+) &= \log\frac{2r}{t} = \log\frac{t^2 + |z - \zeta|^2}{t^2} = \log\left[1 + \left(\frac{|z - \zeta|}{t}\right)^2\right] \\ &= \log\Big[1 + \Big[\varepsilon_{h,q_+}(q_-, \mathfrak{p}_{q_+}h)\Big]^2\Big] . \end{aligned}$$

On the other hand, the angle $\alpha = \angle_h(q_-, q_+)$ coincides with the angle between the radii of the Euclidean sphere $\mathrm{Hor}_{q_-}(h)$ joining its center with the points h and q_- (see Fig. 13). Therefore, by (5.36),

$$\sin\alpha/2 = \frac{\sqrt{t^2 + |z - \zeta|^2}}{2r} = \frac{t}{\sqrt{t^2 + |z - \zeta|^2}} ,$$

whence

$$l_h(q_-, q_+) = -2\log\sin\alpha/2 = -2\log\sin\left[\frac{1}{2}\varsigma_h(q_-, q_+)\right] .$$

$\square$

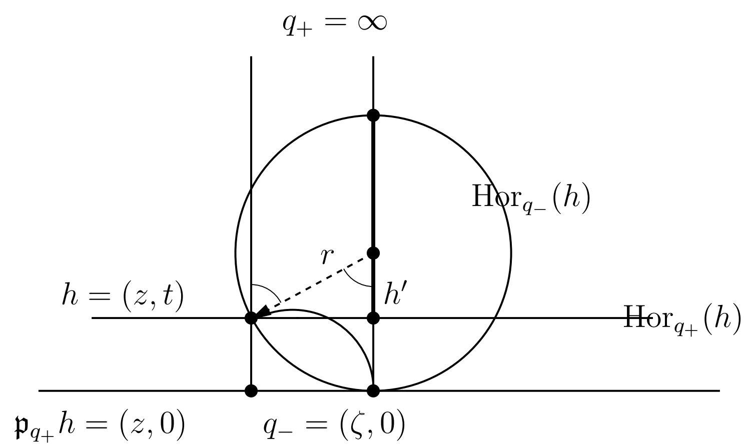

FIGURE 13

5.3.3. *Comparison of Euclidean and visual metrics.*

Proposition 5.37. *For any $h \in \mathbf{H}^3$ and $q_- \neq q_+ \in \partial\mathbf{H}^3$*

$$\frac{\varepsilon_{h,q_+}}{\varsigma_h}(q_-) = \frac{1}{2}\exp\Big[l_h(q_-,q_+)\Big] \,. \tag{5.38}$$

Proof. Clearly (see Fig. 13),

$$\varsigma_h(q_-, \mathfrak{p}_{q_+}h) = \angle_h(q_-, \mathfrak{p}_{q_+}h) = \pi - \angle_h(q_-, q_+) = \pi - \angle_h(q_-, q_+) \,.$$

Therefore, by Proposition 5.35

$$1 + \Big[\varepsilon_{h,q_+}(q_-, \mathfrak{p}_{q_+}h)\Big]^2 = \frac{1}{\cos^2\Big[\frac{1}{2}\varsigma_h(q_-, \mathfrak{p}_{q_+}h)\Big]} \,.$$

Letting the point q_- tend to $\mathfrak{p}_{q_+}h$ we obtain

$$\frac{\varepsilon_{h,q_+}}{\varsigma_h}(\mathfrak{p}_{q_+}h) = \frac{1}{2}$$

in perfect keeping with (5.38), because $l_h(\mathfrak{p}_{q_+}h, q_+) = 0$.

Now, for an arbitrary point $q_- \in \mathcal{P}_{q_+}$ let h' be the intersection of the geodesic (q_-, q_+) with the horosphere $\mathrm{Hor}_{q_+}(h)$, so that $\beta_{q_+}(h,h') = 0$ and $\beta_{q_-}(h,h') = l_h(q_-,q_+)$ (see Fig. 13). Then $\varepsilon_{h',q_+} = \varepsilon_{h,q_+}$, and by Proposition 5.32

$$\begin{aligned}\frac{\varepsilon_{h,q_+}}{\varsigma_h}(q_-) &= \frac{\varepsilon_{h',q_+}}{\varsigma_h}(q_-) = \frac{\varepsilon_{h',q_+}}{\varsigma_{h'}}(q_-) \cdot \frac{\varsigma_{h'}}{\varsigma_h}(q_-) \\ &= \frac{1}{2}\exp\Big[\beta_{q_-}(h,h')\Big] = \frac{1}{2}\exp\Big[l_h(q_-,q_+)\Big] \,.\end{aligned}$$

□

5.4. **Conformal streams and invariant measures of the geodesic flow.**

5.4.1. *The Patterson–Sullivan stream.*

Definition 5.39. A *Patterson–Sullivan stream* of a Kleinian group G is a G-invariant conformal stream ν (of some dimension $\delta > 0$) on the limit set $\Lambda(G)$. $\triangle$

One can easily verify that a Patterson–Sullivan stream ν is determined just by its values $\nu_h = \nu_{\varsigma_h}$ on the visual metrics. More precisely,

Proposition 5.40. *A Patterson–Sullivan stream ν is uniquely determined by the family $\{\nu_h\}$, $h \in \mathbf{H}^3$ of finite positive measures on $\Lambda(G)$ (not necessarily probability ones!) satisfying the following properties:*

(i) *The measures ν_h are all pairwise equivalent, and their Radon–Nikodym derivatives are*
$$\frac{d\nu_{h_2}}{d\nu_{h_1}}(q) = \exp\Big[\delta\beta_q(h_1, h_2)\Big] \qquad \forall\, h_1, h_2 \in \mathbf{H}^3,\ q \in \partial\mathbf{H}^3\ ;$$

(ii) *The family $\{\nu_h\}$ is G-invariant, i.e.,*
$$\nu_{gh} = g\nu_h \qquad \forall\, g \in G,\ h \in \mathbf{H}^3\ .$$

Remark 5.41. A finite measure ν_o on $\Lambda(G)$ (where $o \in \mathbf{H}^3$ is a fixed reference point) is sometimes called a Patterson–Sullivan measure (of dimension δ) if it is quasi-invariant under the action of G, and its Radon–Nikodym derivatives are
$$\frac{dg\nu_o}{d\nu_o}(q) = \exp\Big[\delta\beta_q(o, go)\Big]\ .$$
Clearly, such a measure uniquely extends to a family $\{\nu_h\}$ satisfying conditions (i) and (ii) from Proposition 5.40, and therefore to a Patterson–Sullivan stream.

Remark 5.42. If $\Lambda(G) = \partial\mathbf{H}^3$, then the area stream on $\partial\mathbf{H}^3$ (which is obviously G-invariant) is a Patterson–Sullivan stream of dimension $\delta = 2$.

Definition 5.43 ([Pa76], [S79])**.** The *critical exponent* δ_G of a Kleinian group G separates domains of divergence and convergence (with respect to the parameter s) of the *Poincaré series*
$$\sum_{g\in G} \exp\Big[-s\,\mathrm{dist}(o, go)\Big]\ ,$$
where $o \in \mathbf{H}^3$ is a chosen reference point. $\triangle$

Theorem 5.44 ([Pa76], [S79])**.** *Let G be a non-elementary Kleinian group. Then $\delta_G \in (0, 2]$, the dimension δ of any Patterson–Sullivan stream of G satisfies the inequality $\delta \geq \delta_G$, and there exists a Patterson–Sullivan stream of dimension δ_G.*

Proposition 5.45. *There is a natural one-to-one correspondence between Patterson–Sullivan streams ν of a Kleinian group G and G-invariant leafwise conformal streams $\lambda = \lambda(\nu)$ of the same dimension on the lamination $\mathcal{A}_G$ concentrated on the leafwise limit sets.*

Proof. Obviously, any Patterson–Sullivan stream ν lifts to a G-invariant leafwise conformal stream $\lambda = \lambda(\nu)$ on $\mathcal{A}_G$ of the same dimension concentrated on the leafwise limit sets. Conversely, a leafwise conformal stream λ on $\mathcal{A}_G$ concentrated on leafwise limit sets is a continuous map $q \mapsto \lambda_{L(q)}$ from Λ (which parameterizes the leaves of $\mathcal{A}_G$) to the space of conformal streams on Λ, and G-invariance of λ means that this map is equivariant. Minimality of the action of G on Λ implies that $\lambda_{L(q)}$ must be constant, i.e., λ determines a single Patterson–Sullivan stream ν. $\square$

On the other hand, the hyperbolization $\mathcal{H}_G = \mathfrak{H}\mathcal{A}_G$ is the product lamination of the space $\mathbf{H}^3 \times \Lambda$. Therefore, the standard transversals of $\mathcal{H}_G$ can be parameterized by points $h \in \mathbf{H}^3$. By Proposition 5.40 the Patterson–Sullivan stream gives rise to a G-equivariant assignment of measures on Λ to points $h \in \mathbf{H}^3$. Hence, it determines a transverse measure on $\mathcal{H}_G$, and by property (i) from Proposition 5.40 the modulus of this measure is $\exp[\delta\beta]$. Conversely, any G-invariant transverse measure of $\mathcal{H}_G$ with modulus $\exp[\delta\beta]$ restricted to standard transversals satisfies the conditions of Proposition 5.40, and therefore determines a Patterson–Sullivan stream of dimension δ. Thus, Theorem 2.63 implies:

Proposition 5.46. *There is a natural one-to-one correspondence between Patterson–Sullivan streams ν of a Kleinian group G and G-invariant parallel transverse conformal streams $\mu = \mu(\nu)$ of the same dimension on the lamination $\mathcal{A}_G$.*

5.4.2. *Geodesic currents and invariant measures of the geodesic flow.* Recall that the total space of the lamination $\mathcal{H}_G$ is $\mathbf{H}^3 \times \Lambda(G) \subset \mathbf{H}^3 \times \partial\mathbf{H}^3 \cong U\mathbf{H}^3$ and that the vertical flow on $\mathcal{H}_G$ is just the restriction of the geodesic flow. The measures on $\partial^2\mathbf{H}^3$ which are G-invariant are called *geodesic currents of the quotient manifold $M = \mathbf{H}^3/G$* (e.g., see [Bo86]). The reason is that they naturally induce 1-currents on the geodesic foliation of UM. In measure theoretic terms the lift $\widetilde{\boldsymbol{v}}$ of any such measure $\boldsymbol{v}$ to $U\mathbf{H}^3$ (obtained by integrating the geodesic length against $\boldsymbol{v}$, cf. Definition 2.38) is invariant both with respect to the action of G and the geodesic flow on $U\mathbf{H}^3$, i.e., determines an invariant measure $\widetilde{\boldsymbol{v}}_M$ of the geodesic flow on the quotient manifold M (e.g., see [F73]).

Proposition 5.45 and Proposition 5.46 in combination with Theorem 2.91 yield

Theorem 5.47. *Let ν be a Patterson–Sullivan stream of a Kleinian group G. Let $\lambda = \lambda(\nu)$ and $\mu = \mu(\nu)$ be the corresponding leafwise and transverse conformal streams on the lamination $\mathcal{A}_G$. Then $\boldsymbol{v}(\nu) = \lambda \star \mu$ is a geodesic current of M concentrated on $\Lambda \times \Lambda \setminus \mathrm{diag}$.*

Corollary 5.48. *A Patterson–Sullivan stream ν of a Kleinian group G determines an invariant measure $\widetilde{\boldsymbol{v}}_{\mathcal{M}} = \widetilde{\boldsymbol{v}}_{\mathcal{M}}(\nu)$ of the geodesic flow on the quotient lamination $\mathcal{M}_G \subset UM$.*

5.4.3. *Other constructions of geodesic currents.* A geodesic current associated with a Patterson–Sullivan stream ($\equiv$ with a Patterson–Sullivan measure) was first constructed by Sullivan [S79] as

$$d\boldsymbol{v}^{[1]}(q_-, q_+) = \frac{d\nu_o(q_-)\, d\nu_o(q_+)}{|q_- - q_+|^{2\delta}} , \tag{5.49}$$

where the hyperbolic space $\mathbf{H}^3$ is realized in the Poincaré model as the Euclidean ball of radius 1 centered at the point o, and $|q_- - q_+|$ denotes the Euclidean chordal distance between points q_- and q_+ from the unit sphere.

Kaimanovich [Ka90] considered the measure

$$d\boldsymbol{v}^{[2]}(q_-, q_+) = \exp\Big[\delta l_h(q_-, q_+)\Big] \cdot d\nu_h(q_-)\, d\nu_h(q_+) . \tag{5.50}$$

By Proposition 5.34 it does not depend on the choice of $h \in \mathbf{H}^3$, in particular, it is G-invariant, i.e., is a geodesic current. The advantage of the intrinsic approach of Kaimanovich (as well as of the construction below) is that it works for an arbitrary negatively curved manifold (not necessarily of constant curvature) as well and produces an invariant measure of the geodesic flow called *Margulis–Sullivan–Kaimanovich* measure. In the compact case it leads to the *Margulis measure* ($\equiv$ the maximal entropy invariant measure of the geodesic flow).

We shall now describe a new functorial construction of an invariant measure of the geodesic flow on the quotient manifold $M = \mathbf{H}^3/G$ associated with a Patterson–Sullivan stream. This construction is direct (one does not have to consider first the corresponding geodesic current) and completely symmetric with respect to the stable and unstable directions. It is based on the fact that the geodesic projection (1.11) is a conformal isomorphism between the horosphere $\mathrm{Hor}(v)$ (endowed with the Euclidean metric ε_v induced from $\mathbf{H}^3$) and the punctured sphere $\mathcal{P}_q = \partial\mathbf{H}^3 \setminus \{q_+\}$ (see Fig. 11). Therefore, the Patterson–Sullivan stream ν carries over from $\partial\mathbf{H}^3$ to the horosphere $\mathrm{Hor}(v)$ (see Remark 2.42), where it assigns a Radon measure ν_v to the metric ε_v. Since the metric ε_v is parallel, i.e., the same for all vectors from the strongly stable leaf $\mathcal{W}^{ss}(v)$ of the vector v, we obtain that the Patterson–Sullivan stream ν determines a leafwise measure ν^{ss} on the strongly stable foliation $\mathcal{W}^{ss}$ of $U\mathbf{H}^3$ (if ν is the area stream, then this is just the leafwise Euclidean area). Denote by ν^{su} the analogous leafwise measure on the strongly unstable foliation $\mathcal{W}^{su}$, and let ℓ be the Lebesgue measure along the trajectories of the geodesic flow. We claim that

$$d\ell \times d\nu^{ss} \times d\nu^{su} \tag{5.51}$$

is an invariant measure of the geodesic flow on $U\mathbf{H}^3$, which is also G-invariant, i.e., descends to an invariant measure of the geodesic flow on the quotient manifold.

First note that such a "local" definition makes sense because all the measures involved in (5.51) are obviously quasi-invariant with respect to the corresponding holonomies (see below for an explicit form of the associated Radon–Nikodym derivatives). Since the Euclidean leafwise metric on the strongly stable (resp., strongly unstable) foliation is uniformly exponentially contracted (resp., expanded) by the geodesic flow (see §**5.1.2**), the δ-covariance of the definition of the measure ν^{ss} (resp., ν^{su}) immediately implies that it is also uniformly contracted (resp. expanded) by the flow:

$$\frac{d(\gamma^{-\tau}\nu^{ss})}{d\nu^{ss}} = \exp[-\delta\tau]\ , \qquad \frac{d(\gamma^{-\tau}\nu^{su})}{d\nu^{su}} = \exp[\delta\tau]\ , \tag{5.52}$$

which is, in fact, the characteristic property of the measure (5.51). In turn, formulas (5.52) immediately imply invariance of the measure (5.51) with respect to the geodesic flow, whereas its G-invariance follows from G-invariance of the Patterson–Sullivan stream ν and functoriality of the whole construction.

Remark 5.53. The original construction of Margulis from [Ma70] is also based on using a uniformly contracting leafwise measure on the strongly stable foliation, although he obtains this measure in a completely different way.

We shall now recast the previous construction in terms of geodesic streams. Denote by ν_{h,q_+} the measure on $\partial\mathbf{H}^3$ assigned by the Patterson–Sullivan stream ν to the conformal

Euclidean metric ε_{h,q_+} on $\partial\mathbf{H}^3$ induced from the stable horosphere $\mathrm{Hor}(v)$ (see §**5.3.2**). In the same way the measure ν_{h,q_-} is determined by the Euclidean metric on the unstable horosphere $\mathrm{Hor}(-v)$ (as before, we use the notations from Fig. 11). Then, as it follows from (5.52), the product measure $\boldsymbol{v}_{q_-,q_+} = \nu_{h,q_+} \otimes \nu_{h,q_-}$ is the same for any choice of the point h on the geodesic (q_-, q_+). Now fix $(q_-^0, q_+^0) \in \Lambda \times \Lambda \setminus \mathrm{diag}$, and denote by $\boldsymbol{v}^{[3]}$ the measure on $\Lambda \times \Lambda \setminus \mathrm{diag}$ from the common measure class of the measures $\boldsymbol{v}_{q_-,q_+}$ determined by the relation

$$\frac{d\boldsymbol{v}^{[3]}}{d\boldsymbol{v}_{q_-^0,q_+^0}}(q_-,q_+) = \frac{d\boldsymbol{v}_{q_-,q_+}}{d\boldsymbol{v}_{q_-^0,q_+^0}}(q_-,q_+) \,. \tag{5.54}$$

Clearly, $\boldsymbol{v}^{[3]}$ does not depend on the choice of (q_-^0, q_+^0). Moreover, by G-invariance of the Patterson–Sullivan stream ν the map $(q_-, q_+) \mapsto \boldsymbol{v}_{q_-,q_+}$ is G-equivariant, whence the measure $\boldsymbol{v}^{[3]}$ is also G-invariant, i.e., a geodesic current.

Take a point h^0 on the geodesic (q_-^0, q_+^0), then by Proposition 5.32, formula (5.33) and Proposition 5.37 the density in the right-hand side of formula (5.54) takes the form

$$\begin{aligned}
\frac{d\boldsymbol{v}_{q_-,q_+}}{d\boldsymbol{v}_{q_-^0,q_+^0}}(q_-,q_+) &= \frac{d\nu_{h,q_+}}{d\nu_{h^0,q_+^0}}(q_-)\cdot\frac{d\nu_{h,q_-}}{d\nu_{h^0,q_-^0}}(q_+) \\
&= \frac{d\nu_{h,q_+}}{d\varsigma_h}(q_-)\cdot\frac{d\varsigma_{h^0}}{d\nu_{h^0,q_+^0}}(q_-)\cdot\frac{d\varsigma_h}{d\varsigma_{h^0}}(q_-)\cdot\frac{d\nu_{h,q_-}}{d\varsigma_h}(q_+)\cdot\frac{d\varsigma_{h^0}}{d\nu_{h^0,q_-^0}}(q_+)\cdot\frac{d\varsigma_h}{d\varsigma_{h^0}}(q_+) \\
&= \exp\Big[\delta\Big(l_{h^0}(q_-,q_+) - l_{h^0}(q_-,q_+^0) - l_{h^0}(q_-^0,q_+)\Big)\Big] \\
&= \exp\Big[-\delta\mathcal{R}(q_-^0,q_-,q_+^0,q_+)\Big] \,,
\end{aligned}$$

where

$$\mathcal{R}(q_1,q_2,q_3,q_4) = \lim\Big[\mathrm{dist}(h_1,h_3) + \mathrm{dist}(h_2,h_4) - \mathrm{dist}(h_1,h_4) - \mathrm{dist}(h_2,h_3)\Big]$$

denotes the *cross ratio* of the points $q_1, q_2, q_3, q_4 \in \partial\mathbf{H}^3$ [Ot92] (in the definition of $\mathcal{R}$ the points $h_i \in \mathbf{H}^3$ converge in the visibility compactification to the respective points q_i).

5.4.4. *Comparison of geodesic currents.*

Theorem 5.55. *For a given Patterson–Sullivan stream ν the geodesic current $\boldsymbol{v}$ from Theorem 5.47 and the geodesic currents $\boldsymbol{v}^{[1]}$* (5.49)*, $\boldsymbol{v}^{[2]}$* (5.50)*, $\boldsymbol{v}^{[3]}$* (5.54) *are all the same (up to a normalizing multiplier).*

Proof. $\boldsymbol{v}^{[1]}$ vs. $\boldsymbol{v}^{[2]}$. Since

$$|q_- - q_+| = 2\sin\left[\frac{1}{2}\angle_o(q_-,q_+)\right] \,,$$

by Proposition 5.35

$$l_o(q_-,q_+) = -2\log\sin\left[\frac{1}{2}\angle_o(q_-,q_+)\right] = -2\log\frac{|q_- - q_+|}{2} = 2\log 2 - 2\log|q_- - q_+| \,,$$

whence $\boldsymbol{v}^{[2]} = 2^{2\delta}\boldsymbol{v}^{[1]}$. △

$\boldsymbol{v}^{[2]}$ vs. $\boldsymbol{v}^{[3]}$. By formula (5.54) for any point h from the geodesic (q_-, q_+)

$$d\boldsymbol{v}^{[3]}(q_-,q_+) = d\boldsymbol{v}_{q_-,q_+}(q_-,q_+) = d\nu_{h,q_+}(q_-)\,d\nu_{h,q_-}(q_+) \,,$$

whence $\boldsymbol{\upsilon}^{[3]} = 2^{-2\delta}\boldsymbol{\upsilon}^{[2]}$ by Proposition 5.37. $\triangle$

$\boldsymbol{\upsilon}$ *vs.* $\boldsymbol{\upsilon}^{[2]}$. By Theorem 2.57 the measure $\lambda \star \mu = \lambda_\sigma \star \mu_\sigma$ does not depend on the choice of the section σ of the fiber bundle $\mathfrak{p} : \mathcal{H}_G \to \mathcal{A}_G$. Therefore, we may fix such a section σ and do all the computations just for this section. Fix a point $o \in \mathbf{H}^3$, and let $\sigma : \mathcal{A}_G \to \mathcal{H}_G$ be the section determined by the condition that the leafwise metrics ρ_σ are the visual metrics ς_o. Then the leafwise measures λ_σ are just the measures ν_o: in the coordinates (q_-, q_+) on $\mathcal{A}_G$

$$d\lambda_\sigma^{q_+}(q_-) = d\nu_o(q_-)$$

(recall that $\mathcal{A}_G = \Lambda \times \partial\mathbf{H}^3 \setminus \mathrm{diag}$, so that q_+ is the transverse coordinate, and q_- is the leafwise coordinate).

Let us now consider the transverse measure μ_σ. Due to the product structure of $\mathcal{A}_G$, we can just look at the restrictions $\mu_\sigma^{q_-}$ of μ_σ onto the standard transversals $T_{q_-} = \{(q_-, q_+),\ q_+ \in \Lambda \setminus \{q_-\}\}$. By Theorem 2.63

$$d\mu_\sigma^{q_-}(q_+) = d\nu_h(q_+) = \exp\Big[\delta\beta_{q_+}(o,h)\Big]\, d\nu_o(q_+)\ ,$$

where $\sigma(q_-, q_+) = (h, q_+) \in \mathcal{H}_G$, so that the point $h \in \mathbf{H}^3$ is determined by two conditions:

(i) It lies on the geodesic (q_-, q_+), because $\sigma(q_-, q_+)$ has to belong to the fiber of the bundle $\mathfrak{p}$ over the point (q_-, q_+);

(ii) It belongs to the section σ, i.e.,

$$\frac{\varepsilon_{h,q_+}}{\varsigma_o}(q_-) = 1\ .$$

By Proposition 5.37

$$\frac{\varepsilon_{o,q_+}}{\varsigma_o}(q_-) = \frac{1}{2}\exp\Big[l_o(q_-, q_+)\Big]\ ,$$

on the other hand,

$$\frac{\varepsilon_{o,q_+}}{\varepsilon_{h,q_+}}(q_-) = \exp\Big[\beta_{q_+}(o,h)\Big]\ ,$$

whence

$$\beta_{q_+}(o,h) = l_o(q_-, q_+) - \log 2\ ,$$

which means that $\boldsymbol{\upsilon} = \lambda_\sigma \star \mu_\sigma = 2^{-\delta}\boldsymbol{\upsilon}^{[2]}$. $\square$

Remark 5.56. In the case when $\Lambda = \partial\mathbf{H}^3$ and ν is the area stream on $\partial\mathbf{H}^3$ all these geodesic currents are proportional to the *Liouville geodesic current* (which corresponds to the Liouville invariant measure of the geodesic flow).

5.4.5. *The Patterson–Sullivan streams, harmonic measures and functions.* By the general results from §**2.3** (Theorem 2.86 and Theorem 2.88, respectively) the parallel transverse conformal stream $\mu(\nu)$ and the leafwise conformal stream $\lambda(\nu)$ on $\mathcal{A}_G$ determined by a Patterson–Sullivan stream ν of dimension δ (Proposition 5.46 and Proposition 5.45, respectively) give rise to a G-invariant λ-harmonic measure ω^ν and to a G-invariant leafwise λ-harmonic function Φ^ν on the hyperbolic lamination $\mathcal{H}_G = \mathfrak{H}\mathcal{A}_G$ (with $\lambda = \delta(\delta - 2)$). In terms of the measures ν_h assigned by the stream ν to the visual metrics ς_h they are expressed as

$$d\omega^\nu(h,q) = d\,vol(h) d\nu_h(q)\ , \qquad \Phi^\nu(h,q) = \|\nu_h\|\ .$$

It is well-known (see [S79]), that if the group G is convex cocompact, then it has a unique Patterson–Sullivan stream of dimension δ_G, and that Φ^μ descends to a unique λ-harmonic function on the quotient manifold $M = \mathbf{H}^3/G$. By using the arguments from [Ga83] (also see [Ka88]) one can prove

Theorem 5.57. *If the Kleinian group G is convex cocompact, and ν is the unique Patterson–Sullivan stream of dimension δ_G, then ω^ν is the unique G-invariant λ-harmonic measure of the lamination $\mathcal{H}_G$ which descends to a unique λ-harmonic measure of the quotient lamination $\mathcal{M}_G \cong \mathcal{H}_G/G$.*

5.5. **New lines in the dictionary.** In order to summarize the above discussion, let us list some correspondences between various objects associated with affine laminations determined by Kleinian groups and rational maps (here, as always, $\lambda = \delta(\delta - 2)$).

Kleinian groups	**Rational maps**
Kleinian group G acting on $\overline{\mathbb{C}}$	Rational map f acting on $\overline{\mathbb{C}}$
Limit set $\Lambda = \Lambda(G) \subset \overline{\mathbb{C}}$	The Julia set $J = J(f) \subset \overline{\mathbb{C}}$
Affine $\mathbb{C}$-fibration $\mathcal{A}_G \cong \overline{\mathbb{C}} \times \Lambda \setminus \mathrm{diag}$ over Λ with the diagonal action of G	Affine lamination $\mathcal{A}_f$ with the automorphism $\widehat{f}$
Product structure on $\mathcal{A}_G$	Dual fibration of $\mathcal{A}_f$
Hyperbolic lamination $\mathcal{H}_G = \mathfrak{H}\mathcal{A}_G \cong \mathbf{H}^3 \times \Lambda \setminus \mathrm{diag}$ with the properly discontinuous diagonal action of G	Hyperbolic lamination $\mathcal{H}_f = \mathfrak{H}\mathcal{A}_f$ with the properly discontinuous automorphism $\widehat{f}$
Hyperbolic lamination $\mathcal{M}_G = \mathcal{H}_G/G$	Hyperbolic lamination $\mathcal{M}_f = \mathcal{H}_f/\widehat{f}$
Non-triviality of the Busemann cocycle of $\mathcal{M}_G$ in the Borel cohomology (Theorem 5.26)	Non-triviality of the Busemann cocycle of $\mathcal{M}_f$ in the Borel cohomology (Theorem 3.26)
Critical exponent δ_G	Forward and backward critical exponents δ_{cr} and γ_{cr}
G-invariant parallel transverse conformal stream on $\mathcal{A}_G$ corresponding to a Patterson–Sullivan stream of dimension δ_G of the group G (with Λ considered as the "past" of the geodesic flow)	$\widehat{f}$-invariant parallel transverse conformal stream on $\mathcal{A}_f$ of dimension δ_{cr}
G-invariant leafwise conformal stream on $\mathcal{A}_G$ corresponding to a Patterson–Sullivan stream of dimension δ_G of the group G (with Λ considered as the "future" of the geodesic flow)	$\widehat{f}$-invariant leafwise conformal stream on $\mathcal{A}_f$ of dimension γ_{cr}
Geodesic current ($\equiv$ a G-invariant Radon measure) on $\Lambda \times \Lambda \setminus \mathrm{diag}$ determined by a Patterson–Sullivan stream	The global $\widehat{f}$-invariant measure $\boldsymbol{\upsilon}$ on $\mathcal{A}_f$, which is the product of the leafwise and transverse conformal streams of the same dimension
The invariant measure of the geodesic flow on $U\mathbf{H}^3/G$ determined by a Patterson–Sullivan stream	The natural lift $\widetilde{\boldsymbol{\upsilon}}$ of $\boldsymbol{\upsilon}$ to an invariant measure of the vertical flow on $\mathcal{M}_f$
λ-harmonic measure on $\mathcal{M}_G$ associated with a Patterson–Sullivan stream of dimension δ of the group G	λ-harmonic measure on $\mathcal{M}_f$ associated with a $\widehat{f}$-invariant parallel transverse conformal stream of dimension δ on $\mathcal{A}_f$
λ-harmonic function on $M = \mathbf{H}^3/G$ associated with a Patterson–Sullivan stream of dimension δ of the group G	Leafwise λ-harmonic function on $\mathcal{M}_f$ associated with a $\widehat{f}$-invariant leafwise conformal stream of dimension δ on $\mathcal{A}_f$

5.6. **An example of a non-Euclidean affine foliation.** The first example of an affine foliation $\mathcal{A}$ which is not Euclidean (recall that by Proposition 1.87 it is equivalent to saying that the basic class of $\mathcal{A}$ is non-zero) was given by Ghys [Gh97] (also see [Gh99]) on the base of a construction of non-standard deformations of Fuchsian groups due to Goldman [Go85] and Ghys [Gh87]. Here we shall recast this example making more explicit its relation to the foliations associated with the geodesic flow on $\mathbf{H}^3$.

For an arbitrary Kleinian group G the Busemann cocycle on the hyperbolic foliation $\mathcal{M}_G$ is cohomologically non-trivial (Theorem 5.26); however, $\mathcal{M}_G$ does not correspond to any affine foliation (see Remark 5.23). Recall that $\mathcal{A}_G$ can be considered as a quotient of the strongly stable foliation $\mathcal{W}^{ss}$ of the geodesic flow with respect to the action of the flow. This factorization preserves the leafwise affine structure, but destroys the Euclidean structure. The example exploits the same idea, but, in order to have a properly discontinuous action we replace the "whole" geodesic flow with a one-dimensional representation ($\equiv$ character) of the group G.

5.6.1. *Twisted action.* Take a compact hyperbolic manifold H and put $G = \pi_1(H)$. The actions of G and of the geodesic flow γ on $U\mathbf{H}^3$ commute (note that only the first of these actions is isometric!). We shall now define a new "twisted" action of G on $U\mathbf{H}^3$ by combining the original action of G with the geodesic flow.

> *From now on we shall assume that the first Betti number of G is positive, i.e., the group $\mathrm{Hom}(G,\mathbb{R}) \cong H^1(H,\mathbb{R})$ of additive real-valued characters of G is non-trivial.*

Remark 5.58. The groups $\mathrm{Hom}(G,\mathbb{R})$ and $H^1(\mathbf{H}^3/G,\mathbb{R})$ coincide for any Kleinian group G without parabolic elements. Although there are cocompact Kleinian groups with trivial space $\mathrm{Hom}(G,\mathbb{R})$, it is plausible (according to [VGS00, p. 98]) that if G is a lattice (in particular, if G is cocompact), then it always has the so-called *Millson property*: there exists a finite index subgroup $G' \subset G$ such that $\mathrm{Hom}(G',\mathbb{R})$ is non-trivial. This property has been proved for several classes of lattices.

Definition 5.59. The *twisted action* T_χ of the group G on $U\mathbf{H}^3$ determined by a character $\chi \in \mathrm{Hom}(G,\mathbb{R})$ is

$$T_\chi^g v = g \circ \gamma^{\chi(g)}(v) = \gamma^{\chi(g)} \circ g(v) \ , \tag{5.60}$$

where in the right-hand side $v \mapsto gv$ is the standard action of G on $U\mathbf{H}^3$ (see Fig. 14). We shall also use the same notation T_χ for the action of the group G on the space of horospheres $\mathrm{Hor}(\mathbf{H}^3)$ defined by the formula

$$T_\chi^g \Upsilon = \gamma^{\chi(g)} \circ g(\Upsilon) = g \circ \gamma^{\chi(g)}(\Upsilon) \ ,$$

where γ now stands for the action (5.4) of the geodesic flow on $\mathrm{Hor}(\mathbf{H}^3)$. Then in the coordinates (q,Υ) (5.8) on $U\mathbf{H}^3$

$$T_\chi^g(q,\Upsilon) = (gq, T_\chi^g \Upsilon) \ .$$

△

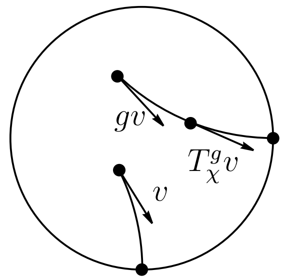

FIGURE 14

5.6.2. *Admissible characters.*

Definition 5.61. A character $\chi \in \mathrm{Hom}(G, \mathbb{R})$ is *admissible* if the action T_χ is free, proper and totally discontinuous. △

One can easily see that all characters sufficiently close to the identity in $\mathrm{Hom}(G, \mathbb{R})$ are admissible. The following complete description is due to Salein [Sa97] (although he deals with hyperbolic surfaces only, his proof based on a criterion of Benoist [Be96] almost *verbatim* carries over to higher dimensions as well).

Definition 5.62. The *stable norm* on $\mathrm{Hom}(G, \mathbb{R})$ is

$$\|\chi\| = \sup_{g \in G} \frac{\chi(g)}{l(g)} ,$$

where

$$l(g) = \min\{\mathrm{dist}(h, gh) : h \in \mathbf{H}^3\} , \qquad g \in G ,$$

denotes the length of the closed geodesic on H associated with the conjugacy class of g in the group G. △

Proposition 5.63. *The action T_χ is admissible iff $\|\chi\| < 1$.*

Remark 5.64. As a motivation for this result note that if $\chi(g) = -l(g)$ for a certain element $g \in G$, then $T^g_\chi v = v$ for any vector $v \in U\mathbf{H}^3$ tangent to the axis of g.

5.6.3. *Absence of Euclidean structures.* The action T_χ preserves the foliation $\mathcal{W}^{ss}$ and its affine (but not Euclidean, unless $\chi = 0$!) structure.

Definition 5.65. For an admissible character $\chi \in \mathrm{Hom}(G, \mathbb{R})$ denote by $\mathcal{B}_\chi$ the affine foliation of the quotient manifold $U\mathbf{H}^3/T_\chi$ obtained by factorizing the strongly stable foliation $\mathcal{W}^{ss}$ by the action T_χ. △

For $\chi = 0$ the foliation $\mathcal{B}_0$ is just the strongly stable foliation of the geodesic flow on $U\mathbf{H}^3/G$, so that it is endowed with a natural leafwise Euclidean structure.

Theorem 5.66. *If $\chi \neq 0$, then the affine foliation $\mathcal{B}_\chi$ does not admit any Borel Euclidean structure.*

Proof. By Proposition 1.87 we have to prove non-triviality of the basic class of $\mathcal{B}_\chi$. In view of Proposition 1.87 it is equivalent to non-triviality of the Busemann cocycle of the hyperbolization $\mathfrak{H}\mathcal{B}_\chi$. Further, $\mathcal{B}_\chi$ is defined as the quotient of the strongly stable foliation $\mathcal{W}^{ss}$ by the action T_χ (5.60), so that we have to prove that the Busemann cocycle (5.18) on the hyperbolization $\mathfrak{H}\mathcal{W}^{ss}$ (see §**5.1.3**) is not cohomological to zero by means of a certain T_χ-invariant function:

$$\begin{aligned} \beta\Big((h_1,\Upsilon),(h_2,\Upsilon)\Big) &= f(h_2,\Upsilon) - f(h_1,\Upsilon) \;, \\ f\Big(T_\chi^g(h,\Upsilon)\Big) &= f(h,\Upsilon) \qquad \forall\, g\in G,\ (h,\Upsilon)\in \mathfrak{H}\mathcal{W}^{ss} \;. \end{aligned} \tag{5.67}$$

We shall prove it under the only assumption that $\chi\neq 0$ (i.e., without requiring that the character χ be necessarily admissible).

Indeed, if (5.67) were satisfied, then, as it follows from formula (5.18) for the Busemann cocycle on $\mathfrak{H}\mathcal{W}^{ss}$, the function f would be expressed as

$$f(h,\Upsilon) = \varphi(\Upsilon) + \beta_{\Upsilon_\infty}(\Upsilon,h) \;, \tag{5.68}$$

where φ is a function on the space of horospheres $\mathrm{Hor}(\mathbf{H}^3)$, and $\beta_{\Upsilon_\infty}(\Upsilon,h)$ denotes the common value $\beta_{\Upsilon_\infty}(h',h)$, $h'\in\Upsilon$.

On the other hand, as it follows from (5.20), the action T_χ on the hyperbolization $\mathfrak{H}W^{ss}$ has the form

$$T_\chi^g(h,\Upsilon) = (gh, T_\chi^g\Upsilon) \;. \tag{5.69}$$

Therefore, using (5.68) and the fact that $(\gamma^\tau\Upsilon)_\infty = \Upsilon_\infty$ for any $\tau\in\mathbb{R}$ and $\Upsilon\in\mathrm{Hor}(\mathbf{H}^3)$, we obtain

$$\begin{aligned} f\Big(T_\chi^g(h,\Upsilon)\Big) = f(gh,T_\chi^g\Upsilon) &= \varphi(T_\chi^g\Upsilon) + \beta_{g\Upsilon_\infty}(T_\chi^g\Upsilon, gh) \\ &= \varphi(T_\chi^g\Upsilon) + \beta_{\Upsilon_\infty}(\gamma^{\chi(g)}\Upsilon, h) \\ &= \varphi(T_\chi^g\Upsilon) + \beta_{\Upsilon_\infty}(\gamma^{\chi(g)}\Upsilon, \Upsilon) + \beta_{\Upsilon_\infty}(\Upsilon,h) \\ &= \varphi(T_\chi^g\Upsilon) - \chi(g) + \beta_{\Upsilon_\infty}(\Upsilon,h) \;. \end{aligned}$$

The latter formula compared with (5.68) implies that f is T_χ-invariant iff the function φ on $\mathrm{Hor}(\mathbf{H}^3)$ satisfies the relation

$$\varphi(T_\chi^g\Upsilon) - \chi(g) = \varphi(\Upsilon) \qquad \forall\, g\in G,\ \Upsilon\in\mathrm{Hor}(\mathbf{H}^3) \;. \tag{5.70}$$

In particular, the function φ has to be invariant with respect to the standard action of the kernel $G_0=\ker\chi$ of the homomorphism χ. On the other hand, the horosphere foliation on the abelian cover $\mathbf{H}^3/G_0$ of the compact manifold $H=\mathbf{H}^3/G$ is ergodic with respect to the smooth measure class, which is equivalent to ergodicity (again with respect to the smooth measure class) of the action of G_0 on the space $\mathrm{Hor}(\mathbf{H}^3)$, see [BL98], [Ka00a]. Therefore, the function φ must be a.e. constant, which is impossible for $\chi\neq 0$ in view of formula (5.70). □

Remark 5.71. Ergodicity of the horosphere foliations was established in [Ka00a] for all abelian covers of any such manifold $H=\mathbf{H}^3/G$ that the geodesic flow on UH is ergodic with respect to the Liouville measure ($\equiv$ the group G is of divergent type), so that the cocompactness assumption in the formulation of Theorem 5.66 can be replaced with this weaker condition.

5.6.4. *Transverse conformal streams.* We shall now briefly describe parallel transverse conformal streams for the foliations $\mathcal{B}_\chi$ without going into details (to be given elsewhere). As it follows from Theorem 2.63, they are in one-to-one correspondence with transverse measures of the foliation $\mathfrak{H}\mathcal{B}_\chi$ with modulus $\exp[\delta\beta]$. Since $\mathfrak{H}\mathcal{B}_\chi$ is the quotient of the foliation $\mathfrak{H}\mathcal{W}^{ss}$ (5.17) by the action T_χ (5.69), it means that there is a one-to-one correspondence between parallel transverse conformal streams for the foliation $\mathcal{B}_\chi$ and T_χ-invariant transverse measures of $\mathfrak{H}\mathcal{W}^{ss}$ with modulus $\exp[\delta\beta]$. Since $\mathfrak{H}\mathcal{W}^{ss} \cong \mathbf{H}^3 \times \mathrm{Hor}(\mathbf{H}^3)$ is a product foliation, and taking into account formula (5.18) for the Busemann cocycle on $\mathfrak{H}\mathcal{W}^{ss}$ we arrive at

Proposition 5.72. *Let $\mathcal{B}_\chi$ be one of the foliations described in §**5.6.3**. Then there is a natural one-to-one correspondence between parallel transverse conformal streams on $\mathcal{B}_\chi$ of dimension δ and families of measures θ_h, $h \in \mathbf{H}^3$ on $\mathrm{Hor}(\mathbf{H}^3)$ such that*

$$\begin{aligned} \log \frac{d\theta_{h_2}}{d\theta_{h_1}}(\Upsilon) &= \delta\beta_{\Upsilon_\infty}(h_1,h_2) \qquad \forall\, h_1,h_2 \in \mathbf{H}^3,\ \Upsilon \in \mathrm{Hor}(\mathbf{H}^3)\ , \\ T_\chi^g \theta_h &= \theta_{gh} \qquad \forall\, h \in \mathbf{H}^3,\ g \in G\ . \end{aligned} \tag{5.73}$$

It turns out that systems of measures satisfying condition (5.73) are intimately connected with a family of invariant measures of the horocycle flow on the homology cover of a compact hyperbolic surface constructed by Babillot and Ledrappier [BL98]. We shall follow a more geometric description of these measures given in [Ba96] and adapt it to our setup by passing from dimension 2 to dimension 3 (which does not change anything in the considerations from [BL98] and [Ba96] modulo substituting "horosphere foliation" for the "horocycle flow").

Fix a character $\chi : G \to \mathbb{R}$. Then there is a unique number $\delta = \delta(\chi) \ge 2$ for which there exists a system (again unique) of finite measures $\nu_h = \nu_h^\chi$, $h \in \mathbf{H}^3$ on $\partial\mathbf{H}^3$ satisfying the relations

$$\begin{aligned} \log \frac{d\nu_{h_2}}{d\nu_{h_1}}(q) &= \delta\beta_q(h_1,h_2) \qquad \forall\, h_1,h_2 \in \mathbf{H}^3,\ q \in \partial\mathbf{H}^3\ , \\ \log \frac{dg\nu_h}{d\nu_{gh}} &= \chi(g) \qquad \forall\, h \in \mathbf{H}^3,\ g \in G\ . \end{aligned}$$

The number δ is the critical exponent of convergence of the generalized Poincaré series

$$\Sigma(s) = \sum_{g\in G} \exp\Big[-s\,\mathrm{dist}(o,go) - \chi(g)\Big]\ , \tag{5.74}$$

where o is a fixed reference point. The series (5.74) diverges for $s = \delta$, and the measures ν_h are the weak limits (as $s \searrow \delta$) of the measures

$$\frac{1}{\Sigma(s)} \sum_{g\in G} \exp\Big[-s\,\mathrm{dist}(h,go) - \chi(g)\Big]\,\delta_{go}\ .$$

Denote by $\widetilde{\nu}_h$ the lifts of the measures ν_h from $\partial\mathbf{H}^3$ to $\mathrm{Hor}(\mathbf{H}^3)$ obtained by integrating the Lebesgue measures on the fibers of the projection $\mathrm{Hor}(\mathbf{H}^3) \to \partial\mathbf{H}^3$ with respect to the measures ν_h. Then, clearly,

$$\begin{aligned}
\log \frac{d\widetilde{\nu}_{h_2}}{d\widetilde{\nu}_{h_1}}(\Upsilon) &= \delta\beta_{\Upsilon_\infty}(h_1,h_2) \qquad \forall\, h_1,h_2 \in \mathbf{H}^3,\ q\in\partial\mathbf{H}^3\ , \\
\log \frac{dg\widetilde{\nu}_h}{d\widetilde{\nu}_{gh}} &= \chi(g) \qquad \forall\, h\in\mathbf{H}^3,\ g\in G\ ,
\end{aligned} \tag{5.75}$$

and the measures $\widetilde{\nu}$ are invariant with respect to the action of the geodesic flow γ on $\mathrm{Hor}(\mathbf{H}^3)$. Thus,

$$T^g_\chi \widetilde{\nu}_h = g\widetilde{\nu}_h = \exp\Big[\chi(g)\Big]\,\widetilde{\nu}_{gh}\ .$$

Now put

$$\theta_h = \varphi\widetilde{\nu}_h\ , \qquad \varphi(h,\Upsilon) = \exp\Big[\beta_{\Upsilon_\infty}(h,\Upsilon)\Big]\ . \tag{5.76}$$

Then by (5.75)

$$\begin{aligned}
\log \frac{d\theta_{h_2}}{d\theta_{h_1}}(\Upsilon) &= \log\frac{\varphi(h_2,\Upsilon)}{\varphi(h_1,\Upsilon)} + \log\frac{d\widetilde{\nu}_{h_2}}{d\widetilde{\nu}_{h_1}}(\Upsilon) \\
&= \beta_{\Upsilon_\infty}(h_2,\Upsilon) - \beta_{\Upsilon_\infty}(h_1,\Upsilon) + \delta\beta_{\Upsilon_\infty}(h_1,h_2) \\
&= (\delta-1)\beta_{\Upsilon_\infty}(h_1,h_2)\ ,
\end{aligned}$$

and

$$\begin{aligned}
\log \frac{dT^g_\chi\theta_h}{d\theta_{gh}}(T^g_\chi\Upsilon) &= \log\frac{\varphi(h,\Upsilon)}{\varphi(gh,T^g_\chi\Upsilon)} + \log\frac{dT^g_\chi\widetilde{\nu}_h}{d\widetilde{\nu}_{gh}}(T^g_\chi\Upsilon) \\
&= \beta_{\Upsilon_\infty}(h,\Upsilon) - \beta_{g\Upsilon_\infty}(gh,T^g_\chi\Upsilon) + \log\frac{dg\widetilde{\nu}_h}{d\widetilde{\nu}_{gh}}(T^g_\chi\Upsilon) \\
&= \beta_{\Upsilon_\infty}(h,\Upsilon) - \beta_{\Upsilon_\infty}(h,\gamma^{\chi(g)}\Upsilon) + \chi(g) \\
&= \beta_{\Upsilon_\infty}(\gamma^{\chi(g)}\Upsilon,\Upsilon) + \chi(g) = 0\ .
\end{aligned}$$

Therefore,

Proposition 5.77. *For a given character $\chi : G \to \mathbb{R}$ the family of measures θ_h, $h\in\mathbf{H}^3$ on* $\mathrm{Hor}(\mathbf{H}^3)$ (5.76) *determines a parallel transverse conformal stream on the foliation $\mathcal{B}_\chi$ of dimension $\delta(\chi)-1$, where $\delta(\chi)$ is the critical exponent of the series* (5.74)*.*

Remark 5.78. In a similar way one can also describe leafwise conformal streams of the foliations $\mathcal{B}_\chi$.

Remark 5.79. By using the methods from [BL98] (also see [ANSS]) one can prove that the foliations $\mathcal{B}_\chi$ are actually *uniquely ergodic* in the sense that for each $\chi\neq 0$ there is a unique number $\delta>0$ such that the space of parallel transverse conformal streams on $\mathcal{B}_\chi$ of dimension δ is non-empty, and there exists a unique (up to a constant multiplier) parallel transverse conformal streams of this dimension δ. In other words, there is a unique δ for which the hyperbolization $\mathfrak{H}\mathcal{B}_\chi$ has a transverse measure with modulus $\exp[\delta\beta]$, and this measure is unique (up to a multiplier). In particular, there are no transverse invariant measures on $\mathfrak{H}\mathcal{B}_\chi$.

Remark 5.80. As it follows from the definition of the stable norm (Definition 5.62),

$$s\ \mathrm{dist}(o, go) + \chi(g) \ge s\ \mathrm{dist}(o, go) - \|\chi\|\, l(g) \ge s\ \mathrm{dist}(o, go) - \|\chi\|\ \mathrm{dist}(o, go)$$
$$= (s - \|\chi\|)\ \mathrm{dist}(o, go)\ ,$$

so that the critical exponent $\delta(\chi)$ of the series (5.74) satisfies the inequality

$$\delta(\chi) \le \delta(0) + \|\chi\| = 2 + \|\chi\|\ .$$

Therefore, the dimension $\delta(\chi) - 1$ of the unique parallel transverse stream on $\mathcal{B}_\chi$ is strictly less than 2 for any non-zero admissible character χ.

Remark 5.81. Since the Laplacian in dimension 2 is conformal, the notion of a harmonic measure (with eigenvalue 0) for laminations of leafwise dimension 2 only requires the presence of a leafwise conformal structure (e.g., see [Gh99] for an intrinsic description of a harmonic measure in these terms). It appears that the laminations $\mathcal{B}_\chi$ have a unique harmonic measure. We shall return to this subject elsewhere.

References

[ANSS] J. Aaronson, B. Nakada, D. Sarig, R. Solomyak. Invariant measures and asymptotics for some skew products. Preprint, 2000.

[AS67] D.V. Anosov, Ya.G. Sinai. Some smooth ergodic systems. Russian Math. Surveys **22** (1967), 103–167.

[Ba96] M. Babillot. Géodésiques et horocycles sur le revêtement d'homologie d'une surface hyperbolique. Sémin. Théor. Spectrale Géom. Chambery–Grenoble. **14** (1996), 89–104.

[Be96] Y. Benoist. Actions propres sur les espaces homogènes réductifs. Ann. of Math. **144** (1996), 315–347.

[BFU] T. Bedford, A. Fisher, M. Urbanski. The scenery flow for hyperbolic Julia sets. Preprint.

[BJ97] C. Bishop, P. Jones. Hausdorff dimension and Kleinian groups. Acta Math. **179** (1997), 1–39.

[BL98] M. Babillot, F. Ledrappier. Geodesic paths and horocycle flow on abelian covers. Proceedings of the International Colloquium on Lie groups and Ergodic Theory (Bombay, 1996, S. G. Dani, ed.), Narosa Publ. House, Bombay, 1998, 1–32.

[BS91] E. Bedford, J. Smillie. Polynomial diffeomorphisms of $\mathbb{C}^2$. II: stable manifolds and recurrence. J. Amer. Math. Soc. **4** (1991), 657–679.

[BLS93] E. Bedford, M. Lyubich, J. Smillie. Polynomial diffeomorphism of $\mathbb{C}^2$. IV: The measure of maximal entropy and laminar currents. Invent. Math. **112** (1993), 77–125.

[Bo86] F. Bonahon. Bouts des variétés hyperboliques de dimension 3. Ann. of Math. **124** (1986), 71–158.

[Br65] H. Brolin. Invariant sets under iteration of rational functions. Arkiv Math. **6** (1965), 103–143.

[BR75] R. Bowen, D. Ruelle. The ergodic theory of Axiom A flows. Invent. Math. **29** (1975), 181–202.

[Ca93] A. Candel. Uniformization theorem for surface laminations. Ann. Scient. Éc. Norm. Sup., 4^e série **26** (1993), 489–516.

[Ca88] Y. Carrière. Un survol de la théorie des variétés affines. Sémin. Théor. Spectrale Géom. Chambery–Grenoble. **6** (1987–1988), 9–22.

[CFS82] I. P. Cornfeld, S.V. Fomin, Ya. G. Sinai. Ergodic Theory. Springer-Verlag, 1982.

[DH93] A. Douady, J. Hubbard. A proof of Thurston's topological characterization of rational maps. Acta Math. **171** (1993), 263–297.

[DU91a] M. Denker, M. Urbanski. On Sullivan's conformal measures for rational maps of the Riemann sphere. Nonlinearity **4** (1991), 365–384.

[DU91b] M. Denker, M. Urbanski. On Hausdorff measures on Julia sets of subexpanding rational maps. Israel J. Math. **76** (1991), 193–214.

[EL90] A. Eremenko, M. Lyubich. The dynamcis of analytic transformations. Leningrad J. Math. **1** (1990), 563–634.

[Ga83] L. Garnett. Foliations, the ergodic theorem and Brownian motions. J. Funct. Anal. **51** (1983), 283–311.

[FM77] J. Feldman, C. C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. Trans. Amer. Math. Soc. **234** (1977), 289–324.

[FS92] J.E. Fornaess, N. Sibony. Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains. Duke Math. J. **65** (1992), 345–380.

[F73] H. Furstenberg. The unique ergodicity of the horocycle flow. Springer Lecture Notes in Math. **318** (1973), 95–115.

[Fu86] D. B. Fuchs. Cohomologies of infinite dimensional Lie algebras. Consultants Bureau, N. Y., 1986.

[Gh87] E. Ghys. Flots d'Anosov dont les feuilletages stables sont différentiables. Ann. Sci. École Norm. Sup. **20** (1987), 251–270.

[Gh97] E. Ghys. Sur l'uniformisation des laminations paraboliques. Progress in Math. (Birkhaüser, Boston) **145** (1997), 73–91.

[Gh99] E. Ghys. Laminations par surfaces de Riemann. Panoramas et Synthèses **8** (1999), 49–95.

[GH55] W. H. Gottschalk, G. A. Hedlund. Topological dynamics. AMS, Providence R.I., 1955.

[GH78] Ph. Griffiths, J. Harris. Principles of Algebraic Geometry. Wiley, N.Y., 1978.

[Go85] W. Goldman. Non-standard Lorentz space forms. J. Diff. Geom. **21** (1985), 301–308.

[Go] W. Goldman. Projective geometry on manifolds. Preprint, 1988.

[Gr99] A. Grigoryan. Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds. Bull. Amer. Math. Soc. **36** (1999), 135–249.

[Ka88] V. A. Kaimanovich. Brownian motion on foliations: entropy, invariant measures, mixing. Funct. Anal. Appl. **22:4** (1988), 326–328.

[Ka90] V. A. Kaimanovich. Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds. Ann. Inst. H. Poincaré, Phys. Théor. **53** (1990), 361–393.

[Ka98] V. A. Kaimanovich. Hausdorff dimension of the harmonic measure on trees. Ergodic Theory Dynam. Systems **18** (1998), 631–660.

[Ka00a] V. A. Kaimanovich. Ergodic properties of the horocycle flow and classification of Fuchsian groups. J. Dynam. Control Systems **6** (2000), 21–56.

[Ka00b] V. A. Kaimanovich. The Poisson formula for groups with hyperbolic properties. Ann. of Math. **152** (2000), 659–692.

[KL98] V. A. Kaimanovich, M. Lyubich. Harmonic measures on hyperbolic laminations. Abstracts AMS **19** (1998), # 934-51-99.

[Le81] F. Ledrappier. Some properties of absolutely continuous invariant measures on an interval. Ergodic Theory Dynam. Systems **1** (1981), 77–94.

[Le84] F. Ledrappier. Quelques propriétés ergodiques des applications rationnelles. C. R. Acad. Sci. Paris **299** (1984), 37–40.

[Ly86] M. Lyubich. The dynamics of rational transforms: the topological picture. Russian Math. Surveys **41** (1986), 43–117.

[Ly93] M. Lyubich. Entropy properities of rational endomorphisms of the Riemann sphere. Ergodic Theory Dynam. Systems **3** (1993), 351–386.

[LM97] M. Lyubich, Y. Minsky. Laminatoins in holomorphic dynamics. J. Diff. Geom. **47** (1997), 17–94.

[Ma70] G. A. Margulis, Certain measures associated with U-flows on compact manifolds. Funct. Anal. Appl. **4**:1 (1970), 55–67.

[Ma95] P. Mattila. Geometry of sets and measure in Euclidean spaces. Cambridge Studies in Advanced Math. **44**, Cambridge Univ. Press, 1995.

[Mi99] Y. Minsky. The classification of punctured torus groups. Ann. of Math. **149** (1999), 559–626.

[Mo68] G. D. Mostow. Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms, Publ. Math. IHES **34** (1968), 53–104.

[MS74] J. Milnor, J. Stasheff. Characteristic classes. Ann. of Math. Studies **76**, Princeton Univ. Press, 1974.

[MS88] C. C. Moore, C. Schochet Global Analysis on Foliated Spaces, Springer, New York, 1988

[MS93] W. de Melo, S. van Strien. One dimensional dynamics. Springer-Verlag, 1993.

[Ot92] J.-P. Otal. Sur la géométrie symplectique de l'espace des géodésiques d'une variété à courbure négative. Rev. Mat. Iberoamericana **8** (1992), 441–456.

[Pa76] S. J. Patterson. The limit set of a Fuchsian group. Acta Math. **136** (1976), 241–273.

[Pl75] J. F. Plante. Foliations with measure preserving holonomy. Ann. of Math. **102** (1975), 327–361.

[Ro49] V.A. Rokhlin. Main notions in measure theory. Math. Sbornik **67** (1949), 107–150.

[Ru78] D. Ruelle. An inequality for the entropy of differentiable maps. Bol. Soc. Brasil. Mat. **9** (1978), 83–87.

[Sa97] F. Salein. Variétés anti-de Sitter de dimension 3 possédant un champ de Killing non trivial. C. R. Acad. Sci. Paris, Sér. I **324** (1997), 525–530.

[Sc83] P. Scott. The geometries of 3-manifolds. Bull. London Math. Soc. **15** (1983), 401–487.

[S79] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. Publ. Math. IHES **50** (1979), 419–450.

[S83] D. Sullivan. Conformal dynamical systems. Springer Lect. Notes in Math. **1007** (1983), 725–752.

[S85] D. Sullivan. Quasiconformal homeomorphisms and dynamics, II: Structural stability implies hyperbolicity for Kleinian groups. Acta Math. **155** (1985), 243 - 260.

[S87] D. Sullivan. Related aspects of positivity in Riemannian geometry. J. Diff. Geom. **25** (1987), 237–351.

[S92] D. Sullivan. Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications **2**: Mathematics into Twenty-first Century (1992).

[Si72] Ya. G. Sinai. Gibbs measures in ergodic theory. Russian Math. Surveys **27** (1972), 21–69.

[Sm81] J. Smillie. An obstruction to the existence of affine structures. Invent. Math. **64** (1981), 411–415.

[Su97] M. Su. Measured solenoidal Riemann surfaces and holomorphic dynamics. J. Diff. Geom. **47** (1997), 170–195.

[Th91] W. Thurston. Three-Dimensional Geometry & Topology. Draft. University of Minnesota Geometry Center. 1991 (first 4 chapters published as Three-Dimensional Geometry and Topology, vol. 1. Princeton Univ. Press, 1997).

[Th] W. Thurston. On the combinatorics of iterated rational maps. Preprint.

[VGS00] E. B. Vinberg, V. V. Gorbatsevich, O. V. Shvartsman. Discrete subgroups of Lie groups. In: "Lie Groups and Lie Algebras II" (A. L. Onishchik, E. B. Vinberg eds.), Springer, Berlin, 2000, 1–124.

[Ve87] A. Verjovsky. A uniformization theorem for holomorphic foliations. Contemp. Math. **58** (1987), 233 - 253.

[U94] M. Urbansky. Rational functions with no recurrent critical points. Ergodic Theory Dynam. Systems **14** (1994), 391–414.

[We58] A. Weil. Introduction à l'étude des variétés kählériennes. Hermann, Paris, 1958.

[Yo89] L.-S. Young. Ergodic theory of differentiable dynamical systems. In: "Real and Complex Dynamics" (B. Branner, P. Hjorth eds.), NATO ASI Series **464** (1989), Kluwer Acad. Publ.

CNRS UMR-6625, IRMAR, Université Rennes-1
E-mail address: `kaimanovuniv-rennes1.fr`

Mathematics Department and IMS, SUNY Stony Brook
E-mail address: `mlyubichmath.sunysb.edu`