

The forcing relation for horseshoe braid types

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Abstract

This paper presents evidence for a conjecture concerning the structure of the set of braid types of periodic orbits of Smale's horseshoe map, partially ordered by Boyland's forcing order. The braid types are partitioned into totally ordered subsets, which are defined by parsing the symbolic code of a periodic orbit into two segments, the *prefix* and the *decoration*: the set of braid types of orbits with each given decoration is totally ordered, the order being given by the unimodal order on symbol sequences. The conjecture is supported by computer experiment, by proofs of special cases, and by intuitive argument in terms of pruning theory.

1 Introduction

This paper presents strong evidence for a conjecture concerning the order in which periodic orbits can appear in the creation of Smale's horseshoe [Sma67]. Since any $C^{1+\epsilon}$ surface diffeomorphism with positive topological entropy has horseshoes in some iterate [Kat80], an understanding of the mechanism of horseshoe creation provides insight into the mechanism of transitions to positive entropy for general surface diffeomorphisms. As such, this problem has received a good deal of attention over the last 20 years.

The conjecture is stated in terms of Boyland's *forcing order* [Boy84] on the set of braid types of periodic orbits of the horseshoe (see Section 2.1). Loosely stated, the periodic orbit P *forces* the periodic orbit Q if an orbit of the same type as Q must be present in the dynamics of any homeomorphism which has an orbit of the same type as P .

The conjecture is based on a parsing of the symbolic code of each horseshoe periodic orbit into two segments, the *prefix* and the *decoration* (see Section 3.1). Writing \mathcal{D}_w for the

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family of all periodic orbits with a given decoration w , the main claims of the conjecture are:

- a) Each family \mathcal{D}_w is totally ordered by the forcing order, and this order coincides with the unimodal order on symbol sequences.
- b) All of the orbits in each given family \mathcal{D}_w have the same topological train track type (see Section 3.3).
- c) The forcing order between different families \mathcal{D}_w can be understood in terms of homoclinic orbits associated to the families.

As will be discussed in more detail below, Thurston’s classification theorem for isotopy classes of surface homeomorphisms [Thu88], used in conjunction with a *train track algorithm* such as that of Bestvina and Handel [BH95], makes it theoretically possible — if practically very time consuming — to decide whether and how two orbits are related by the forcing order. Despite this attractive theoretical background, it has proved very difficult to describe the global structure of the forcing order. The conjecture presented here makes the calculation of the order nearly trivial within families: simply compare their symbol sequences using the unimodal order. Moreover, it gives a global description of how the set of horseshoe braid types is organized, information that could not be obtained by comparing braids pairwise.

As the terminology suggests, this work has connections with braid theory. In Section 2.2 it is explained how the conjecture, if proved, would also provide an efficient partial solution to the conjugacy problem for cyclic unimodal permutation braids.

Some well established background to the problem is presented in Section 2: braid types, Boyland’s forcing order, Thurston’s classification theorem for surface homeomorphisms, and the notion of the height of a periodic orbit of the horseshoe. Section 3 is more directly concerned with the conjecture: it describes how symbolic codes are parsed into prefix and decoration, summarizes some well known results arising from the symmetry of the horseshoe and its inverse, and introduces the notion of topological train track types. The main conjecture is stated in Section 3.4. Section 4 is concerned with evidence for the conjecture: proofs of special cases, computational evidence, and intuitive arguments.

2 Background

2.1 Braid types and the Thurston Classification

Braid types were introduced by Boyland [Boy84] as an algebraic specification of periodic orbits of surface homeomorphisms: the braid type of a periodic orbit P of a surface homeomorphism $f: S \rightarrow S$ is essentially the isotopy class of f relative to P . For the sake of

simplicity, the definition given here is restricted to orientation-preserving homeomorphisms of the disk, which is the case of interest in this paper. The definition makes sense for arbitrary invariant sets, not just for periodic orbits: in Section 2.2 it is extended to certain homoclinic orbits.

Definitions: Let D^2 be the unit disk in the plane and let $f: D^2 \rightarrow D^2$ and $g: D^2 \rightarrow D^2$ be orientation-preserving homeomorphisms having periodic orbits P and Q respectively. If P (respectively Q) lies on ∂D^2 , then extend f (respectively g) as a homeomorphism over an exterior collar of D^2 (and use the same notation D^2 to denote the collared disk). Then (P, f) and (Q, g) have the same braid type if there is an orientation-preserving homeomorphism $h: D^2 \rightarrow D^2$ with $h(P) = Q$ such that f and $h^{-1} \circ g \circ h$ are isotopic relative to P . This defines an equivalence relation on the set of all pairs (P, f) : the equivalence class of (P, f) is denoted $\text{bt}(P, f)$, the *braid type* of the periodic orbit P of f .

The set of all braid types of periodic orbits of orientation-preserving homeomorphisms of the disk is denoted BT. Clearly two periodic orbits with the same braid type must have the same period: the set of all braid types of period n orbits of orientation-preserving homeomorphisms of the disk is denoted BT_n . Given an orientation-preserving homeomorphism $f: D^2 \rightarrow D^2$, write

$$\text{bt}(f) = \{\text{bt}(P, f) : P \text{ is a periodic orbit of } f\}.$$

The term *braid type* is appropriate because the group of isotopy classes of orientation-preserving homeomorphisms of the n -punctured disk is isomorphic to the n -braid group B_n modulo its centre. This isomorphism induces a canonical bijection between BT_n and the set of conjugacy classes in $B_n/Z(B_n)$, which provides a convenient way of representing braid types.

One of the main endeavours in this area is to understand which braid types must necessarily coexist with a given braid type: this is formalized by Boyland's forcing order on BT [Boy84].

Definition: The *forcing order* \leq on BT is defined as follows: if $\beta, \gamma \in \text{BT}$, then $\gamma \leq \beta$ if and only if for all homeomorphisms $f: D^2 \rightarrow D^2$, $\beta \in \text{bt}(f) \implies \gamma \in \text{bt}(f)$. If $\gamma \leq \beta$, then one says that β *forces* γ .

It is obvious that \leq is reflexive and transitive; its antisymmetry was proved by Boyland [Boy94]:

Theorem 2.1 (Boyland) \leq is a partial order on BT.

The main tool for understanding the structure of this partially ordered set is Thurston's classification theorem for isotopy classes of surface homeomorphisms [Thu88]: this will be stated here for orientation-preserving homeomorphisms of the punctured disk only.

Definitions: Let A be a finite subset of $D^2 \setminus \partial D^2$. A simple closed curve in $D^2 \setminus A$ is *essential* if it bounds a disk containing more than one but fewer than all of the points of A . An isotopy class α of orientation-preserving homeomorphisms of (D^2, A) is *reducible* if there exists an element $f: (D^2, A) \rightarrow (D^2, A)$ of α and a finite *reducing* collection of pairwise disjoint and non-homotopic essential simple closed curves which are permuted by f . A *pseudo-Anosov* homeomorphism $\phi: (D^2, A) \rightarrow (D^2, A)$ is one for which there exists a number $\lambda > 1$ and a pair $(\mathcal{F}_s, \mu_s), (\mathcal{F}_u, \mu_u)$ of transverse measured foliations of D^2 such that $\phi(\mathcal{F}_s, \mu_s) = (\mathcal{F}_s, \lambda^{-1}\mu_s)$ and $\phi(\mathcal{F}_u, \mu_u) = (\mathcal{F}_u, \lambda\mu_u)$. The foliations can have a finite number of singular points where they each have $p \neq 2$ prongs, but 1-pronged singularities can only occur at points of A and on ∂D^2 .

The idea is that if an isotopy class α is reducible, then one can cut along the reducing curves and study the action of α on the simpler pieces into which the punctured disk is divided. Thurston's classification theorem provides a canonical representative of each irreducible isotopy class.

Theorem 2.2 (Thurston) *Let α be an irreducible isotopy class of orientation-preserving homeomorphisms of (D^2, A) . Then exactly one of the following occurs:*

- a) α contains a finite order homeomorphism ϕ (i.e. $\phi^n = \text{id}$ for some $n > 0$: this implies that ϕ is conjugate to a rigid rotation of D^2).
- b) α contains a pseudo-Anosov homeomorphism ϕ .

Moreover, if α is a reducible isotopy class, then it does not contain a pseudo-Anosov homeomorphism.

The irreducible isotopy class α is said to be *finite order* or *pseudo-Anosov* according as it contains a finite order or pseudo-Anosov homeomorphism. The detailed properties of pseudo-Anosov homeomorphisms will not be used here: the important property from a dynamical point of view is that they have minimal dynamics within their isotopy class. In particular [FLP79, Hal91],

Theorem 2.3 *Let ϕ be a pseudo-Anosov homeomorphism of (D^2, A) and let $f: (D^2, A) \rightarrow (D^2, A)$ be isotopic to ϕ . Then*

- a) $h(\phi) \leq h(f)$.
- b) $\text{bt}(\phi) \subseteq \text{bt}(f)$.

Here $h(f)$ denotes the topological entropy of f [AKM65]. Property b) is the one which is useful in understanding the structure of (BT, \leq) . Since the Thurston classification is invariant under conjugation, each braid type β can be classified as reducible, finite order, or pseudo-Anosov, according to the isotopy class of $f: (D^2, P) \rightarrow (D^2, P)$, where $\text{bt}(P, f) = \beta$. A finite order braid type β can be realised by a rigid rotation, and hence only forces itself and the braid type of a fixed point. If β is a pseudo-Anosov braid type, then let $\phi_\beta: (D^2, A) \rightarrow (D^2, A)$ be a representative pseudo-Anosov homeomorphism. Then

Corollary 2.4 *Let $\beta \in \text{BT}$ be pseudo-Anosov. Then*

$$\{\gamma \in \text{BT}: \gamma \leq \beta\} = \text{bt}(\phi_\beta).$$

Several algorithms (e.g. [BH95, BGN93, BGN95, FM93, Los93, dCH01]) have been presented which, given as input an isotopy class of orientation-preserving homeomorphisms of (D^2, A) , determine whether it is reducible, finite order, or pseudo-Anosov, and provide a set of reducing curves or an explicit construction of a pseudo-Anosov homeomorphism in the isotopy class in the reducible and pseudo-Anosov cases respectively. In principle, these make it possible to calculate whether or not $\gamma \leq \beta$ for any $\beta, \gamma \in \text{BT}$. If β is pseudo-Anosov, the output of the algorithm is an *invariant train track* (see Section 3.3) which enables one to enumerate all of the periodic orbits of ϕ_β of the appropriate period, and test each in turn to determine whether or not its braid type is γ . In practice this takes a very long time: the tests involve solving the conjugacy problem in the braid group. Moreover, the ability to carry out such local calculations does not provide any insight into the global structure of (BT, \leq) .

2.2 The height of a periodic orbit of the horseshoe

It is assumed that the reader is familiar with Smale's horseshoe map [Sma67], with the standard procedure for introducing symbolic dynamics on its non-wandering set, and with the unimodal order on the resulting symbol space. This material can be found in many standard texts on dynamical systems (e.g. [Dev89]). In this paper symbolic dynamics in $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ will be used on a standard model $F: D^2 \rightarrow D^2$ of the horseshoe as depicted in Figure 1: the symbol F will always denote the horseshoe map, and $k: \Lambda \rightarrow \Sigma_2$ will denote the homeomorphism which conjugates the shift map $\sigma: \Sigma_2 \rightarrow \Sigma_2$ to $F|_\Lambda: \Lambda \rightarrow \Lambda$, where Λ is the set of points whose (past and future) orbits lie entirely in the square S . If $x \in \Lambda$, then $k(x) \in \Sigma_2$ is called the *itinerary* of x . The definitions and results summarized in this section can be found in [Hal94a].

A period n orbit P of F will be described by its *code* $c_P \in \{0, 1\}^n$, which is given by the first n symbols of the itinerary of its rightmost point p : thus

$$k(p) = \overline{c_P} \cdot \overline{c_P}$$

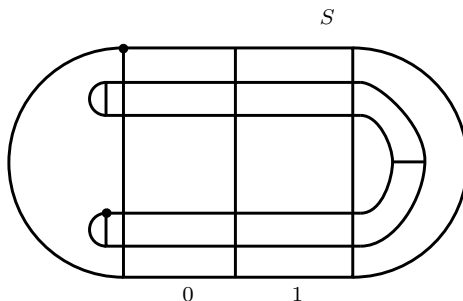


Figure 1: Symbolic dynamics for the horseshoe

(here and throughout, an overbar denotes infinite repetition, and a \cdot appears before the zeroth symbol of an element of Σ_2). For example, the period 5 periodic orbit which contains the point with itinerary $\overline{01001}$ has code 10010. Put another way, the word $c_P \in \{0,1\}^n$ is the code of a period n orbit of the horseshoe if and only if the semi-infinite sequence $\overline{c_P}$ is strictly greater than $\sigma^i(\overline{c_P})$ in the unimodal order for $1 \leq i < n$. This paper is also concerned with homoclinic orbits H to the fixed point with code 0: such an orbit will be described by its *core* c_H , which is the non-zero segment in the itineraries of its points (so, for example, the homoclinic orbit which contains the point with itinerary $\overline{011} \cdot \overline{0010}$ has core 11001). In the remainder of the paper, ‘homoclinic’ will always mean homoclinic to this fixed point. The braid type $\text{bt}(P, F)$ of a periodic orbit P of the horseshoe will be denoted $\text{bt}(P)$; and the notation $P \geq Q$ will be used as an abbreviation for $\text{bt}(P) \geq \text{bt}(Q)$. Similarly it is possible to define the *homoclinic braid type* $\text{hbt}(H)$ of a homoclinic orbit of the horseshoe: two homoclinic orbits H and H' of homeomorphisms f and g have the same homoclinic braid type if there is an orientation-preserving homeomorphism $h: D^2 \rightarrow D^2$ with $h(H) = H'$ such that f is isotopic to $h^{-1} \circ g \circ h$ relative to H . The notation $H \geq H'$ means that every homeomorphism of the disk which has a homoclinic orbit of homoclinic braid type $\text{hbt}(H)$ also has one of homoclinic braid type $\text{hbt}(H')$. Let $\text{HS} = \text{bt}(F)$ denote the set of braid types of periodic orbits of the horseshoe, and HHS the set of homoclinic braid types of the horseshoe. The aim of this paper is to describe the structure of the partially ordered set (HS, \leq) : this gives information about the way in which periodic orbits are created in parameterized families of homeomorphisms leading to the creation of a horseshoe. According to Conjecture 1, this problem reduces to that of understanding the relation \leq on HHS : the set of non-finite order elements of HS can be partitioned into totally ordered families on which the order is well understood; and there is a bijection between HHS and the set of families, such that the ordering of braid types in two different families can be easily determined provided it is known how the corresponding homoclinic braid types are

related by \leq . Note that if $\text{HS}^* = \text{HS} \cup \text{HHS}$, then \leq extends in a natural way to a relation on HS^* . A consequence of Conjecture 1 is that this extended relation is also a partial order.

It is well known that two periodic orbits P and Q of the horseshoe whose codes c_P and c_Q differ only in their final symbol have the same braid type. Thus, for example, the two orbits with codes 10010 and 10011 have the same braid type; the code of either one of these orbits is often written $c_P = 1001_1^0$ to reflect the fact that the distinction between the two is unimportant in so far as (HS, \leq) is concerned. However, this is not the only way in which two horseshoe orbits can have the same braid type: the conjecture also gives necessary and sufficient conditions (which may or may not be verifiable in practice) for two periodic orbits to have the same braid type. In another language, this provides an efficient partial solution to the conjugacy problem for cyclic unimodal permutation braids.

The remainder of this section is devoted to describing the *height* $q(P)$ of a horseshoe periodic orbit P of period greater than 1. The height is a invariant of braid type with values in $(0, 1/2] \cap \mathbb{Q}$ which plays a central rôle in the conjecture. The description is rather complicated at first sight: a program for computing heights of horseshoe orbits can be found at [Hal], and motivation for the definition is given in [Hal94a].

Algorithm

Let P be a horseshoe periodic orbit which is not a fixed point, with code $c = c_P$. If the semi-infinite sequence \bar{c} does not contain the word 010, then change the final symbol of c from 1 to 0. Now write

$$\bar{c} = 10^{\kappa_1} 1^{\mu_1} 0^{\kappa_2} 1^{\mu_2} \dots,$$

where each $\kappa_i \geq 0$, each μ_i is either 1 or 2, and $\mu_i = 1$ only if $\kappa_{i+1} > 0$ (thus κ_i and μ_i are uniquely determined by c). For each $r \geq 1$, define

$$I_r(c) = \left(\frac{r}{2r + \sum_{i=1}^r \kappa_i}, \frac{r}{(2r-1) + \sum_{i=1}^r \kappa_i} \right],$$

and let $s \geq 1$ be the least integer such that either $\mu_s = 1$ or $\bigcap_{i=1}^{s+1} I_i(c) = \emptyset$. Write $\bigcap_{i=1}^s I_i(c) = (x, y]$. Then

$$q(P) = \begin{cases} x & \text{if } \mu_s = 2 \text{ and } w < z \text{ for all } w \in I_{s+1}(c) \text{ and } z \in \bigcap_{i=1}^s I_i(c) \\ y & \text{if } \mu_s = 1, \text{ or } \mu_s = 2 \text{ and } w > z \text{ for all } w \in I_{s+1}(c) \text{ and } z \in \bigcap_{i=1}^s I_i(c). \end{cases}$$

Notice that some μ_i is equal to 1 (since \bar{c} contains the word 010), and hence the algorithm terminates.

Example: Let P be the period 17 orbit with code 10011011001011010. Then $\kappa_1 = 2$, $\mu_1 = 2$, $\kappa_2 = 1$, $\mu_2 = 2$, $\kappa_3 = 2$, and $\mu_3 = 1$. Thus $I_1 = (1/4, 1/3]$, $I_2 = (2/7, 2/6]$, and

$I_3 = (3/11, 3/10]$. Since $\mu_3 = 1$, the algorithm terminates with $\bigcap_{i=1}^3 I_i = (2/7, 3/10]$, and hence $q(P) = 3/10$.

The following theorem is a summary of the relevant results from [Hal94a]. Given $q = m/n \in (0, 1/2]$ in lowest terms, define $c_q \in \{0, 1\}^{n+1}$ by

$$c_q = 10^{\kappa_1} 1^2 0^{\kappa_2} 1^2 \dots 1^2 0^{\kappa_m} 1,$$

where

$$\kappa_i = \begin{cases} \lfloor n/m \rfloor - 1 & \text{if } i = 1 \\ \lfloor in/m \rfloor - \lfloor (i-1)n/m \rfloor - 2 & \text{if } 2 \leq i \leq m \end{cases}$$

(here $\lfloor x \rfloor$ denotes the greatest integer which does not exceed x). The words c_q are palindromic: that is, $\kappa_i = \kappa_{m+1-i}$ for all i .

Theorem 2.5 *Let P and Q be periodic orbits of the horseshoe.*

- a) *If P and Q have the same braid type, then $q(P) = q(Q)$.*
- b) *If $P \geq Q$, then $q(P) \leq q(Q)$.*
- c) *Let $q(P) = m/n$ in lowest terms. Then P has period n if and only if it has finite order braid type: in this case, F is isotopic rel. P to a rigid rotation through $2\pi m/n$. Otherwise, the period of P is at least $n + 2$, and c_P starts with the word $c_{m/n}$.*

In fact, $q(P)$ has a dynamical interpretation: it is the left hand endpoint of the rotation interval of P . The definition of the height can be extended to all sequences which contain the word 010: this extension will be used in the statement of Lemma 3.1.

Definitions: Let \mathcal{C} denote the subset of $\{0, 1\}^{\mathbb{N}}$ consisting of sequences which contain the word 010. The *height* $q(c)$ of an element $c \in \mathcal{C}$ is defined by $q(c) = 1/2$ if c does not begin $c = 10\dots$, and by the above algorithm otherwise.

The function $q: \mathcal{C} \rightarrow (0, 1/2] \cap \mathbb{Q}$ is order-reversing with respect to the unimodal order on Σ_2 and the usual order on \mathbb{Q} .

3 The conjecture

3.1 Prefix and Decoration

Let P be a periodic orbit of the horseshoe. If P is a fixed point, then clearly it forces only the fixed point braid type. If P is not a fixed point, then it has a well-defined height $q(P) = m/n \in (0, 1/2]$, written in lowest terms. If P has period n then, by Theorem 2.5 c),

it has finite order braid type with rotation number m/n . In fact, Holmes and Williams showed [HW85] that for each $m/n \in (0, 1/2)$, there is exactly one pair of periodic orbits with this finite order braid type $\alpha_{m/n}$: one has code given by the first n symbols of $c_{m/n}$, and the other has the same code with the final 1 changed to a 0. There is just one periodic orbit of braid type $\alpha_{1/2}$, namely the period 2 orbit with code 10.

These orbits will be ignored in the remainder of the paper: it is obvious what they force, and it is known which other orbits force them ($P \geq \alpha_q$ if and only if q is in the rotation interval of P by a theorem of Boyland [Boy92], and an algorithm for computing rotation intervals of horseshoe orbits is given in [Hal94a]).

The code of any other periodic orbit P can be written in the form

$$c_P = c_{q(P)}v$$

for some word v of length at least 1, by Theorem 2.5 c).

Definitions: Let P be a period N orbit of the horseshoe which is not of finite order braid type, with height $q = q(P) = m/n$. The *prefix* of P is the word c_q . The *decoration* of P is defined to be \star if $N = n + 2$, and to be the element w of $\{0, 1\}^{N-n-3}$ such that

$$c_P = c_q {}^0 w_1 {}^0$$

if $N \geq n + 3$. The notation $P = P_q^w$ means that P is a periodic orbit of height q and decoration w . The choice of 0 or 1 for the final symbol of c_P is almost always unimportant, but the choice of the symbol before the decoration can be significant. Where it is, the periodic orbits with codes $c_q {}^0 w_1 {}^0$ and $c_q {}^1 w_1 {}^0$ will be denoted $P_q^w(0)$ and $P_q^w(1)$ respectively.

Example: Let P be the period 17 orbit with code $c_P = 10011011001011010$. Then $q(P) = 3/10$ as shown in the previous example. Hence P has prefix $10011011001 = c_{3/10}$ and decoration 1101. Thus $P = P_{3/10}^{1101}$. Where the distinction is important, P could be denoted $P_{3/10}^{1101}(0)$.

Hence every periodic orbit which is not of finite order braid type can be written P_q^w , where $q = q(P)$ and w is the decoration of P . This means precisely that the orbit has one of the four codes $c_q {}^0 w_1 {}^0$, unless $w = \star$, in which case it has one of the two codes $c_q {}^0$. It is convenient to extend this notation to homoclinic orbits, writing P_0^w for the homoclinic orbits with core $1_1 {}^0 w_1 {}^0 1$ (and P_0^\star for the homoclinic orbits with core $1_1 {}^0 1$). This description omits only those homoclinic orbits whose core has length 2 or less: these are precisely the homoclinic orbits of *translation homoclinic braid type*, which force no other homoclinic or periodic braid types. In the next subsection it is shown that the four homoclinic orbits described by the symbol P_0^w all have the same homoclinic braid type. Likewise, one of the

claims of Conjecture 1 below is that all of the (four or fewer) periodic orbits described by the symbol P_q^w have the same braid type.

Only certain heights q are compatible with each given decoration: since the code c_P of a period N orbit P corresponds to its rightmost point, it must be *maximal*: that is, $\sigma^i(\overline{c_P})$ must be strictly less than $\overline{c_P}$ in the unimodal order for all integers i with $1 \leq i < N$. The set of heights compatible with a given decoration w is described by the following lemma. The proof of this lemma is omitted: it is long and technical, depending on special properties of the words c_q . These properties lie at the heart of the significance of the separation into prefix and decoration.

Lemma 3.1 *let w be a decoration, and define $q_w \in (0, 1/2] \cap \mathbb{Q}$ by $q_\star = 1/2$ and*

$$q_w = \min_{0 \leq i \leq k+2} q(\sigma^i(\overline{10w0}))$$

if $w \in \{0, 1\}^k$. Then each of the four codes $c_{q_1^0} w_1^0$ (or each of the two codes $c_{q_1^0}$ when $w = \star$) is maximal of height q when $0 < q < q_w$, and none is maximal of height q when $q_w < q \leq 1/2$.

Example: When $q = q_w$, some, none, or all of the codes $c_{q_1^0} w_1^0$ may be maximal. For example, let $w = \star$: then $q_w = 1/2$. Since $c_{1/2} = 101$, the two codes concerned are 1011 (which is maximal), and 1010 (which is not). Thus the periodic orbit P with code $c_P = 1011$ is the only periodic orbit with height $1/2$ and decoration \star . If $w = 0$ then $q_w = 1/4$, and $c_{1/4} = 10001$. Of the four codes 10001101, 10001001, 10001101 and 10001000, only the last is not maximal. If $w = 110$ then $q_w = 1/3$, and $c_{1/3} = 1001$. All of the four codes 100111101, 100101101, 100111100 and 100101100 are maximal. On the other hand, if $w = 10011010$ then $q_w = 1/3$, but none of the four codes 10011100110101, 10010100110101, 10011100110100 and 10010100110100 is maximal.

The point of the phrase ‘of height q ’ in the statement of this lemma is that $c_{q_1^0} w_1^0$ may be a maximal code for $q > q_w$, but with a different height (and hence a different decoration). For example, if $w = \emptyset$ then $q_w = 1/3$. Let $q = 2/5 > q_w$: then $c_q 1_1^0 = 1011011_1^0$ is maximal, but has height $3/8$, and is the code of a finite order orbit (and hence has no decoration).

A program for computing q_w can be found at [Hal].

For each decoration w , let \mathcal{D}_w denote the set of all periodic and homoclinic orbits of the horseshoe with decoration w : thus

$$\mathcal{D}_w = \{P_q^w : 0 \leq q \leq q_w\},$$

where the symbol \leq denotes that $q = q_w$ is possible for some decorations, but not for others.

Convention: In what follows, whenever the symbol P_q^w is used it is assumed that the height q is compatible with the decoration w .

Summarizing the results of this section: the union of the sets \mathcal{D}_w is precisely the set of all periodic and homoclinic orbits of the horseshoe less the periodic orbits of finite order braid type and the homoclinic orbits of translation homoclinic braid type. One of the main claims of Conjecture 1 is that each \mathcal{D}_w is totally ordered by the forcing order \leq .

3.2 Reversing horseshoe codes

Let P be a period N orbit of the horseshoe with code c_P , and let \widehat{c}_P denote the reverse of the code c_P . Let \widehat{P} be the period N orbit containing the point of itinerary $\widehat{c}_P \cdot \widehat{c}_P$. The code $c_{\widehat{P}}$ is therefore a cyclic permutation of \widehat{c}_P .

Example: Let P be the periodic orbit with code 1001011. Then \widehat{P} is the periodic orbit which contains the point with itinerary $\overline{1101001} \cdot \overline{1101001}$: thus it has code 1001110.

Let β_P be the N -braid representing the period N orbit P which is obtained from the natural suspension of the horseshoe (see [Hal94a] for more details): thus $\text{bt}(P)$ is represented by the conjugacy class of $\beta_P Z(B_N)$ in $B_N/Z(B_N)$. Given $\beta \in B_N$, let $\widehat{\beta}$ be the element of B_N obtained by writing β in terms of the standard Artin generators σ_i of B_N , and then reversing the order of these generators (this is a well defined operation, since the relations between the Artin generators are symmetric under order reversal). It is a well-known consequence of the symmetry of the horseshoe map F and its inverse that $\beta_{\widehat{P}} = \widehat{\beta}_P$, where β'_P is the braid obtained by looking at the natural suspension of P from the right rather than from the front of the horseshoe (and is thus conjugate to β_P). That is,

Lemma 3.2 *If $\text{bt}(P)$ is represented by the braid β_P , then $\text{bt}(\widehat{P})$ is represented by the braid $\widehat{\beta}_P$. In particular, if two horseshoe orbits P and Q have the same braid type, then \widehat{P} and \widehat{Q} have the same braid type.*

Because the braids β and $\widehat{\beta}$ close to the same knot, it is in general difficult to determine whether or not they are conjugate. In particular, traditional dynamical invariants cannot distinguish between them by the following lemma, which follows easily from the previous one and the uniqueness of pseudo-Anosov representatives of pseudo-Anosov isotopy classes.

Lemma 3.3 *Suppose that P is a horseshoe periodic orbit of pseudo-Anosov braid type, and let ϕ_P be a pseudo-Anosov homeomorphism in the isotopy class of $F: (D^2, P) \rightarrow (D^2, P)$. Then \widehat{P} also has pseudo-Anosov braid type, and*

$$\phi_{\widehat{P}} = h^{-1} \circ \phi_P^{-1} \circ h$$

for some orientation-reversing homeomorphism $h: (D^2, \widehat{P}) \rightarrow (D^2, P)$.

The authors know of no horseshoe orbit P for which P and \widehat{P} have distinct braid types, nor of any proof that P and \widehat{P} always have the same braid type (there are no counterexamples up to period 9). They would be grateful to hear from anyone who can resolve this problem. (In the language of braid theory, the question is whether or not there exist cyclic unimodal permutation braids β which are not conjugate to $\widehat{\beta}$.) It is proved in [Hal94a] that $q(P) = q(\widehat{P})$ for all horseshoe orbits P : this fact will be used in Section 3.4.

The fact, mentioned earlier, that two periodic orbits P and Q whose codes differ only in their final symbol have the same braid type follows from the observation that their natural braid representatives β_P and β_Q are equal. The same observation shows that for any decoration $w \neq \star$, the two homoclinic orbits with cores $11w_1^0 1$ have the same homoclinic braid type, as do the two with cores $10w_1^0 1$. By the same reversal construction as for periodic orbits, the fact that the orbits with cores $11\widehat{w}_1^0 1$ have the same homoclinic braid type means that the orbits with cores $1_1^0 w 1 1$ have the same homoclinic braid type: hence all four homoclinic orbits with cores $1_1^0 w_1^0 1$ have the same homoclinic braid type. That is,

Lemma 3.4 *Let w be a decoration. If $w \neq \star$ then the four homoclinic orbits represented by P_0^w all have the same homoclinic braid type. Likewise, the two homoclinic orbits represented by P_0^\star have the same homoclinic braid type.*

According to Conjecture 1, all of the (four or fewer) periodic orbits represented by P_q^w also have the same braid type.

3.3 Topological train track types

Let P be a periodic orbit of the horseshoe of pseudo-Anosov braid type. A *Bestvina-Handel train track for P* is a pair (G, g) , where G is a connected finite graph without valence one or two vertices embedded in $D^2 \setminus P$ and $g: G \rightarrow G$ is a graph map such that

- a) G has a *peripheral subgraph* P , consisting of a loop around each point of P , and g restricts to a homeomorphism $P \rightarrow P$. There is exactly one vertex of G on each component of P .
- b) The subgraph T of G consisting of edges not in P is a tree. This implies that there exists a retraction $r: D^2 \setminus P \rightarrow G$.
- c) $g: G \rightarrow G$ sends vertices to vertices and is homotopic to $r \circ F|_G: G \rightarrow G$.
- d) $g: G \rightarrow G$ is *efficient*: that is, every iterate $g^n: G \rightarrow G$ is locally injective away from the vertices of G .

In [BH95], Bestvina and Handel give an algorithm for constructing such a train track for any periodic orbit P of pseudo-Anosov braid type (the condition that each component of P contains only one vertex of G can be ensured when P is a single periodic orbit by starting the algorithm with a graph which satisfies this condition). The pseudo-Anosov homeomorphism

ϕ in the isotopy class of $F: (D^2, P) \rightarrow (D^2, P)$ is semi-conjugate to $g: G \rightarrow G$: indeed, a Markov partition for ϕ can be constructed by thickening up the edges of G into Markov boxes with non-negative widths and lengths given by the row and column eigenvectors corresponding to the maximal eigenvalue of the transition matrix of $g: G \rightarrow G$, providing a construction of the invariant measured foliations of ϕ .

This paper is concerned with an equivalence relation on horseshoe periodic orbits of pseudo-Anosov braid type which corresponds to their train tracks being ‘essentially the same’. Since two periodic orbits of different periods have different numbers of loops in the peripheral subgraphs of their train tracks, the first step is to restrict attention to the trees obtained when the edges of the peripheral subgraph are removed.

Let P be a horseshoe periodic orbit of pseudo-Anosov braid type, and let (G, g) be a Bestvina-Handel train track for P . The corresponding *reduced train track* for P is the pair (T, t) , where T is the tree embedded in $D^2 \setminus P$ whose edges are the non-peripheral edges of G and which has no valence two vertices (i.e. any vertices of G which become valence two vertices of T are deleted), and $t: T \rightarrow T$ is the tree map obtained from $g: G \rightarrow G$ by restricting to T and deleting peripheral loops in image edge-paths. Note that t is locally injective away from vertices of T and preimages of points where the peripheral loops of G were attached. An *initial edge* of T is one whose counterpart in G has one end (its *free end*) on the peripheral loop surrounding the leftmost point of P .

Let P and Q be two horseshoe periodic orbits of pseudo-Anosov braid type. Write $P \triangleright Q$ if P and Q have reduced train tracks (T_P, t_P) and (T_Q, t_Q) respectively, each with only one initial edge (denoted e_P and e_Q respectively), such that there is an orientation-preserving homeomorphism $\psi: (D^2, P) \rightarrow (D^2, Q)$ sending T_P onto T_Q , with

- a) $t_Q(x) = \psi \circ t_P \circ \psi^{-1}(x)$ for all $x \in T_Q \setminus e_Q$.
- b) There is an embedding $\theta: e_Q \rightarrow e_P$ sending the non-free end of e_Q to itself such that $t_Q(x) = \psi \circ t_P \circ \psi^{-1} \circ \theta(x)$ for all $x \in e_Q$.

Intuitively, the reduced train track of Q is the same as the reduced train track of P , except that the image of its initial edge has been shortened. It is therefore clear that if $P \triangleright Q$ then $P \geq Q$. P and Q are said to *have the same topological train track type* if either $P \triangleright Q$ or $Q \triangleright P$.

In particular, if P and Q have the same topological train track type then the invariant foliations of the corresponding pseudo-Anosov homeomorphisms have the same number of interior singularities with each number $v > 2$ of prongs, and the prongs are permuted in the same way by the actions of the pseudo-Anosovs.

Note that Bestvina-Handel train tracks, and hence reduced train tracks, are not unique. In particular (cf. the discussion in section 3.3 of [BH95]), the train track graph can have vertices which don’t correspond to singularities of the measured foliations, or whose valence is different from the number of prongs at the corresponding singularity. In this paper train

tracks will always be chosen so that this is not the case: thus the invariant foliations of a pseudo-Anosov homeomorphism $\psi: (D^2, P) \rightarrow (D^2, P)$ with reduced train track (T, t) have a one-pronged singularity at each point of P ; an interior v -pronged singularity for each vertex of T of valence $v \geq 3$; and a boundary singularity at which the number of prongs depends on the period of P . Such train tracks can always be found (see [FM93], for example).

A convenient notation for describing a reduced train track (T, t) is to start at the free end of the initial edge of T and move around T in the positive direction, numbering edges sequentially as they are encountered and listing each edge in the order in which it is encountered; and then listing in turn the image edge paths. Thus, for example, for the topological train track type corresponding to $w = 1$ in Table 1, T would be described by the list 1223445531, and t by (1223, 4, 553, 1, 2). A program which takes as input a decoration, and returns the corresponding topological train track type in this format, can be found at [Hal]. Note that, in contrast to standard Markov partition conventions, the image of the initial edge need only intersect the interior of, rather than cover, the first edge in its image edge-path. If P and Q have the same topological train track type (say $P \triangleright Q$), then the descriptions of the trees T_P and T_Q are equal, while those of the tree maps t_P and t_Q differ only in that the image edge path of the initial edge of Q is obtained from that of P by deleting $i \geq 0$ initial symbols.

3.4 Statement of the conjecture

Given two decorations w_1 and w_2 , write $w_1 \sim w_2$ if $\text{hbt}(P_0^{w_1}) = \text{hbt}(P_0^{w_2})$; and write $w_1 \succeq w_2$ if $\text{hbt}(P_0^{w_1}) \geq \text{hbt}(P_0^{w_2})$ (notice that $\text{hbt}(P_0^w)$ is well-defined by Lemma 3.4). Recall that \mathcal{D}_w denotes the set of all periodic and homoclinic orbits which have decoration w and that in the notation P_q^w it is assumed that the height q is compatible with the decoration w (i.e. $q \leq q_w$).

Conjecture 1 *Let w and w' be decorations. Then*

- a) $\text{bt}(P_q^w) = \text{bt}(P_{q'}^{w'})$ if and only if $q = q'$ and $w \sim w'$.
- b) If $0 < q < q_w$ then P_q^w has pseudo-Anosov braid type.
- c) All of the periodic orbits in $\{P_q^w : 0 < q < q_w\}$ have the same topological train track type.
- d) The family \mathcal{D}_w is totally ordered by \leq , with $P_q^w \leq P_r^w$ if and only if $q \geq r$.
- e) If $q < q'$ and $w \succeq w'$ then $P_q^w \geq P_{q'}^{w'}$.

The five parts of this conjecture have each been stated in full without common hypotheses because there are different types and amounts of evidence for the different statements, and it is therefore desirable to be able to treat them separately. Notice in particular that part e) implies part d), that part c) depends on part b), and that if part a) is false then the statements of the other parts would have to be changed to reflect this.

Conjecture 1 could also be stated for periodic orbits only, leaving out the homoclinic orbits P_0^w . The reason for their inclusion is that they appear naturally when the problem is considered in terms of pruning theory (Section 4.3), which was the original motivation for the conjecture.

The conjecture addresses two problems: if P and Q are horseshoe periodic orbits, is it true a) that $\text{bt}(P) = \text{bt}(Q)$, and b) that $P \leq Q$? (Being able to answer b) in general provides an answer to a), since $\text{bt}(P) = \text{bt}(Q)$ if and only if $P \leq Q$ and $Q \leq P$.) It does this by rephrasing them in terms of the same questions for homoclinic orbits, which are not in general any easier to answer than the original question. Nevertheless, if the conjecture can be proved it will add considerably to our understanding of the problem in two distinct ways:

- a) On the theoretical level, it provides a better understanding of the global structure of (HS, \leq) : the partial order has been ‘factored’ into a total order (within families with fixed decoration) and the partial order on decorations. Figure 2 provides a schematic illustration: each periodic orbit of the horseshoe is parameterised by its height q and decoration w . Each family of orbits with a given \sim -equivalence class of decorations is represented by a vertical line: the orbits in such a family are created monotonically from bottom to top as a horseshoe is created. The complication in (HS, \leq) has been shuffled away in this figure by imagining that \preceq is a total order (perhaps we have restricted to those periodic orbits corresponding to a given chain of decorations) — it is assumed that $w_1 \succeq w_2$ whenever w_1 is to the left of w_2 . Thus progress through a given family to height q implies progress through all families to its right to at least height q , and hence a partially formed horseshoe can be represented by an upward sloping line through the families (as depicted in the figure).
- b) On a practical level, showing that two periodic orbits $P_q^{w_1}$ and $P_q^{w_2}$ of relatively small period have the same braid type implies that $P_{q'}^{w_1}$ and $P_{q'}^{w_2}$ have the same braid type for all $0 < q, q' \leq q_{w_1} = q_{w_2}$. For example, it can be shown directly (by constructing a conjugacy between the corresponding braids) that the periodic orbits $P_{1/3}^{10}$ and $P_{1/3}^{01}$ with codes 10011101 and 10011011 have the same braid type. It follows from the conjecture that the orbits with codes $c_{q_1}^0 10_1^0$ and $c_{q_1}^0 01_1^0$ have the same braid type for all q with $0 < q \leq q_{10} = 1/3$.

Under the assumption that part c) of the conjecture is true, Table 1 shows the topological train track types (including images) corresponding to decorations of lengths 3 or less.

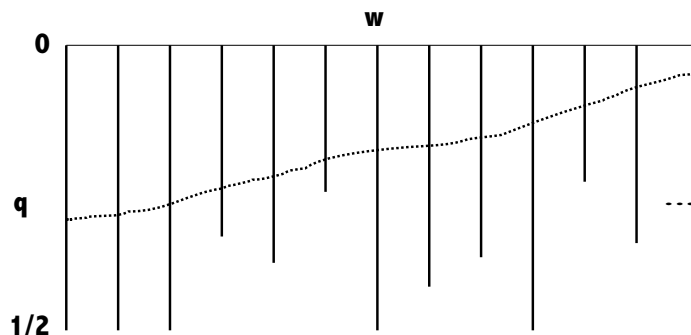


Figure 2: A schematic illustration of the conjecture

Decorations which are equivalent under \sim are shown in the same row. The train tracks have been drawn in such a way that the free endpoint of the initial edge and its preimage are connected by a horizontal line.

The question, posed in Section 3.2, of whether there exists a horseshoe orbit P whose braid type is distinct from that of \widehat{P} is, under the assumption that part a) of the conjecture is true, very closely related to the question of whether there exists a decoration w which is not \sim -equivalent to its reverse \widehat{w} . To see why this is so, note first that $q_w = q_{\widehat{w}}$ for all decorations w . For by Lemma 3.1, q_w is the height of the periodic orbit containing the point with itinerary $\overline{10w0}$, and as stated in Section 3.2, this is the same as the height of the orbit containing the reverse itinerary $\overline{0\widehat{w}01}$: applying Lemma 3.1 again, this is equal to $q_{\widehat{w}}$. Now let $P = P_q^w$ be any horseshoe orbit with $q < q_w$. Then P has code $c_{q1}^0 w_1^0$, and so \widehat{P} contains the point with itinerary $\overline{{}_1^0 \widehat{w}_1^0 c_q}$. Since c_q is palindromic and $q < q_w = q_{\widehat{w}}$, \widehat{P} has code $c_{q1}^0 \widehat{w}_1^0$, i.e. $\widehat{P} = P_q^{\widehat{w}}$. Hence P and \widehat{P} have the same braid type if and only if $w \sim \widehat{w}$.

4 Evidence for the conjecture

The current evidence for the conjecture is of three types: proofs of some special cases, computational evidence for low period orbits with short decorations, and an intuitive justification using pruning theory [dC99, dCH01].

4.1 Proofs of special cases

The simplest case, in which $w = \star$, was treated in [Hal94a], where it was shown that parts a), b), c), and d) of the conjecture hold for this decoration. (Note that, since \star is not equivalent to any other decoration, in this case part a) says that the pair of periodic orbits

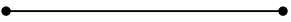
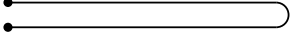

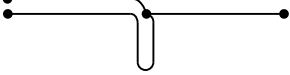

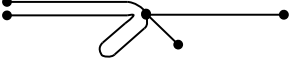
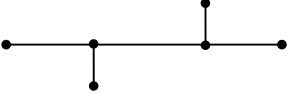
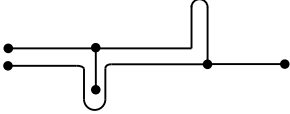

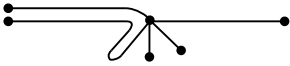
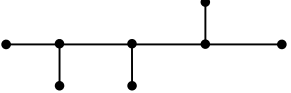
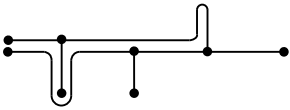
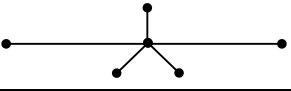
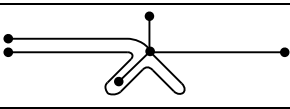
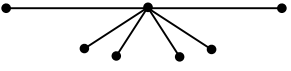
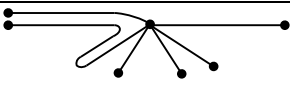
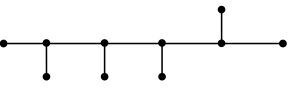
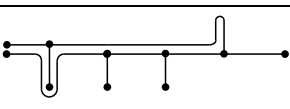
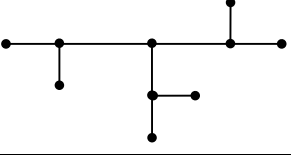
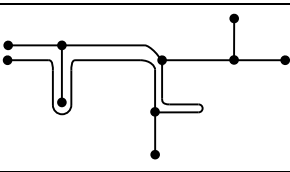
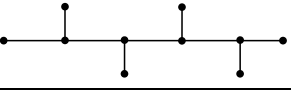
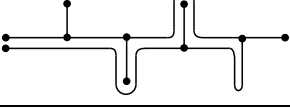
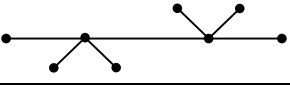
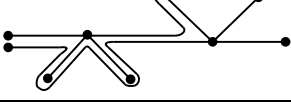
| w | q_w | Train track | Image |
|---------------|-------|---|--|
| * | 1/2 |  |  |
| \emptyset | 1/3 |  |  |
| 0 | 1/4 |  |  |
| 1 | 1/2 |  |  |
| 00 | 1/5 |  |  |
| 01, 10 | 1/3 |  |  |
| 11 | 2/5 |  |  |
| 000 | 1/6 |  |  |
| 001, 100 | 1/4 |  |  |
| 110, 010, 011 | 1/3 |  |  |
| 101 | 1/2 |  |  |
| 111 | 1/2 |  |  |

Table 1: Topological train track types for short decorations

P_q^* are the only horseshoe periodic orbits of their braid type for each $q \in \mathbb{Q} \cap (0, 1/2)$.) Using the techniques of this paper, it doesn't seem hard to show that parts b), c), and d) of the conjecture hold for any given decoration w , simply by calculating the appropriate isotopy classes and showing that they leave invariant train tracks of the given type. For example, parts b), c), and d) are proved in [Hal94b] for decorations of the form $w = 1^{2i-1}$, where $i \geq 1$ is an integer: it is also shown there that for each $q < q_w = 1/2$, the four orbits $P_q^{1^{2i-1}}$ are the only horseshoe orbits of their braid type (the topological train track types which arise are discussed later in this section). However, treating individual cases in this way seems a rather pointless endeavour: the challenge in proving the conjecture is to develop techniques which are applicable when no information about topological train track type is available.

Part e) of the conjecture only makes sense when a family of decorations is considered simultaneously. In [dCH] it will be shown that all five parts of the conjecture hold when attention is restricted to the set $\{w_q : q \in (0, 1/2) \cap \mathbb{Q}\}$ of decorations, where w_q is the word obtained from c_q by deleting the initial symbols 10 and the final symbols 01 (it satisfies $q_{w_q} = q$). In this case, the forcing order can be simply expressed:

$$P_r^{w_q} \geq P_{r'}^{w_{q'}} \quad \text{if and only if} \quad [r, q] \supseteq [r', q'].$$

The conjecture says nothing about the particular topological train track types corresponding to particular decorations: however, studying Table 1 and its extensions to longer decorations makes it very tempting to produce conjectures, the simpler of which seem relatively easy to prove by brute force calculations of the action of the appropriate isotopy classes on trial train tracks. Thus, for example, in [Hal94b] it is shown that if $w = 1^{2i-i}$ for some integer $i \geq 1$, then the reduced train track has a single period 2 orbit of valence $i + 2$ vertices (see $w = 1$ and $w = 111$ in Table 1). That is,

$$T = 12233 \dots (i+1)(i+1)(i+2)(i+3)(i+3)(i+4)(i+4) \dots (2i+3)(2i+3)(i+2)1,$$

and

$$t = (12233 \dots (i+1)(i+1)(i+2), (i+3), (i+4), \dots, \\ (2i+2), (2i+3)(2i+3)(i+2), 1, 2, 3, \dots, (i+1))$$

Likewise in [dCH] it will be shown that if $q = m/n \in (0, 1/2)$, and the length $n - 3$ decoration w_q is obtained from c_q by deleting the initial 10 and the final 01, then T has a single fixed valence n vertex, and the action of t on non-initial edges of T is rotation by m/n (see $w = \emptyset, 0, 00, 11$ and 000 in Table 1, corresponding to $q = 1/3, 1/4, 1/5, 2/5$ and $1/6$ respectively).

Examining the topological train track types corresponding to $w = 1, 10$ and 100 also leads to an obvious conjecture about the case $w = 10^n$, which can be proved with a lot

of work but not much difficulty. A more interesting conjecture, however, is that if P is a period n horseshoe orbit with code c_P , and if w is obtained from c_P by deleting the final symbol, then the corresponding topological train track type has a single period n orbit of valence 3 vertices, and the braid type of this periodic orbit is that of P . This conjecture remains unproved.

4.2 Computational evidence

The main tool which the authors have used for computational investigation of Conjecture 1 is an implementation by the first author of the Bestvina-Handel train track algorithm. Using this, parts b), c), and d) of the conjecture have been checked directly for all periodic orbits whose decoration is of length 8 or less, and whose height has denominator 10 or less. Specifically, it was verified that for each such decoration w

- a) All of the orbits $P_q^w(0)$ and $P_q^w(1)$ with $q < q_w$ have pseudo-Anosov braid type and the same topological train track type.
- b) The pseudo-Anosov representatives of the orbits $P_q^w(0)$ and $P_q^w(1)$ have the same topological entropy for each $q < q_w$.
- c) For each $q_1 < q_2 < q_w$, the pseudo-Anosov representative of the orbit $P_{q_1}^w(0)$ has greater topological entropy than that of the orbit $P_{q_2}^w(0)$.

This check involved calculating train tracks for 15566 periodic orbits. Naturally there is no constraint other than time preventing the continuation of this search: the program used is available on the web [Hal]. Part a) is much harder to check computationally however, since determining whether or not two decorations w and w' are equivalent is essentially the same problem as the conjugacy problem in the braid group, and as such becomes impractical for quite short decorations. The authors do not have any good computational approach to part e) of the conjecture.

4.3 Pruning theory

Pruning is a mechanism for destroying dynamics of surface homeomorphisms in a controlled manner. The following account avoids some technical details: a full description can be found in [dC99, dCH01].

Let $F: D^2 \rightarrow D^2$ be a homeomorphism (which here will always be the horseshoe). A *pruning region* for F is an F -invariant open set \mathcal{R} such that there exists an isotopy supported in \mathcal{R} which destroys the dynamics there: that is, if $F_{\mathcal{R}}$ denotes the homeomorphism obtained at the end of the isotopy, then every point of \mathcal{R} is wandering under $F_{\mathcal{R}}$. Since the isotopy is supported in \mathcal{R} , $F_{\mathcal{R}}$ is equal to F outside \mathcal{R} .

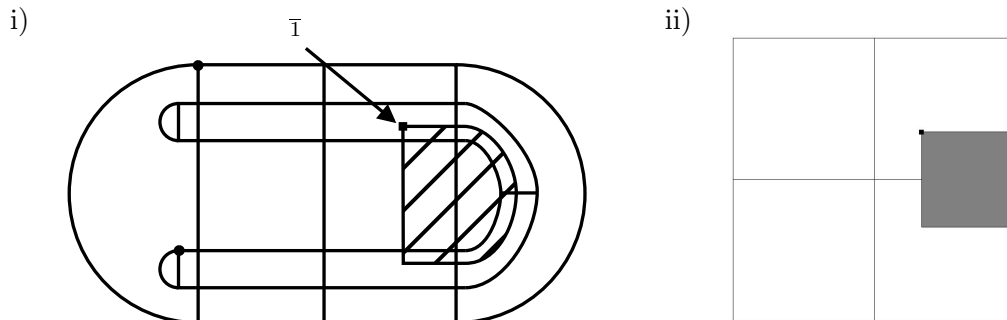


Figure 3: A pruning disk in the horseshoe

Example: Consider the disk shown in Figure 3 i), which is bounded by segments of the stable and unstable manifolds of the fixed point of the horseshoe with code 1. Then $\mathcal{R} = \bigcup_{n \in \mathbb{Z}} F^n(\text{Int}(D))$ is a pruning region for F (see [dC99] for the construction of the isotopy in this case).

As in this example, pruning regions are usually presented as saturations of open sets under the dynamics. Such a generating open set is called a *pruning front*, and these will be used to describe pruning regions in what follows.

Pruning fronts are themselves unions of *pruning disks* which are, roughly speaking, open disks bounded by segments of the stable and unstable manifolds of (possibly different) periodic points, as in the example above. The figures shown in the remainder of this section all contain a square, which represents the non-wandering set of the horseshoe (after collapsing gaps in the Cantor set). Pruning disks will be depicted with shaded boxes: together they make up the pruning front, which in turn yields the pruning region under saturation. The schematic representation of the pruning disk in the above example is shown in Figure 3 ii).

A general pruning disk can be specified by the horizontal and vertical coordinates of its edges. The simplest type of pruning disk is a *vertical pruning disk*, which extends all the way from the bottom to the top of the square, and from some point up to the right of the square. A vertical pruning disk can be specified by the horizontal coordinate of its left edge, which will be referred to as its *horizontal coordinate*.

In [dCH01], the Bestvina-Handel algorithm is recast in the language of pruning to give an alternative algorithmic proof of Thurston's classification theorem for surface homeomorphisms. Given a periodic orbit P (or indeed any finite invariant set) of the horseshoe, this algorithm yields a *maximal pruning region* $\mathcal{R} = \mathcal{R}(P)$ for F relative to P with the property that, after collapsing wandering domains, $F_{\mathcal{R}}$ is the Thurston representative in the isotopy

class of $F: (D^2, P) \rightarrow (D^2, P)$. The pruning region is maximal relative to P in the sense that no further dynamics can be destroyed by pruning without destroying the periodic orbit P itself. The same algorithm appears to work for homoclinic orbits of the horseshoe: a proof that this is the case might be obtained by combining the techniques of [dCH01] with those of [Hul00].

Note that maximal pruning regions are not unique. The reason is that there are different regions of the dynamics of the horseshoe which are conjugate to each other: in carrying out the algorithm, one has to decide which of two conjugate regions should be destroyed. Different choices lead to homeomorphisms $F_{\mathcal{R}_1}$ and $F_{\mathcal{R}_2}$ which, though conjugate, are described by different pruning regions. This phenomenon will be illustrated in Example d) below. This non-uniqueness is probably the main cause of difficulty in proving the conjectures in this paper.

The following conjecture, if proved, would provide the main technical ingredient for proving parts d) and e) of Conjecture 1. It uses the notation \mathcal{D}_w^s to denote the family of orbits $\{P_q^w(s): 0 < q \leq q_w\}$, where $s \in \{0, 1\}$.

Conjecture 2 *Let w and w' be decorations with $w \succeq w'$, and consider (any particular choices of) the homoclinic orbits P_0^w and $P_0^{w'}$. Then there exist $s, s' \in \{0, 1\}$ and maximal pruning regions \mathcal{R}_w and $\mathcal{R}_{w'}$ for F relative to P_0^w and $P_0^{w'}$ respectively, such that:*

- a) $\mathcal{R}_{w'}$ is disjoint from $\mathcal{D}_{w'}^{s'}$ and \mathcal{R}_w is disjoint from both \mathcal{D}_w^s and $\mathcal{D}_{w'}^{s'}$.
- b) Let $q < q_w$ and $q' < q_{w'}$. Then a maximal pruning region for the periodic orbit $P_q^w(s)$ (respectively $P_{q'}^{w'}(s')$) can be obtained by adding to \mathcal{R}_w (respectively $\mathcal{R}_{w'}$) the saturation of the vertical pruning disk with horizontal coordinate $\overline{c_q s w_1^0}$ (respectively $\overline{c_{q'} s' w_1^0}$).

These statements could be rephrased in terms of equivalent decorations. For example, in a) above, instead of saying that it is possible to find \mathcal{R}_w disjoint from \mathcal{D}_w^s , it could be said that given *any* maximal pruning region \mathcal{R}_w for F relative to P_0^w , there exists a decoration $v \sim w$ and $s \in \{0, 1\}$ such that \mathcal{R}_w is disjoint from all orbits in the family \mathcal{D}_v^s . The point is that a maximal pruning relative to P_0^w may destroy some orbits of the relevant braid types, but cannot destroy all of them.

If Conjecture 2 holds, and it is true that $P_q^w(0)$ and $P_q^w(1)$ have the same braid type for all q and w with $q < q_w$ (which would be implied by Conjecture 1 a)), then Conjecture 1 e) follows (and hence so also does Conjecture 1 d)). For suppose $q < q'$ and $w \succeq w'$, and let \mathcal{R}_w , $\mathcal{R}_{w'}$, s , and s' be as given by Conjecture 2. The fact that $q < q'$ implies that $\overline{c_q s w_1^0}$ is greater than $\overline{c_{q'} s' w_1^0}$ in the unimodal order, and hence that the vertical pruning disk \mathcal{V} with horizontal coordinate $\overline{c_q s w_1^0}$ contains no point in the periodic orbit $P_{q'}^{w'}(s')$. By part a) of the conjecture \mathcal{R}_w is also disjoint from $P_{q'}^{w'}(s')$, and hence by part b) there is a maximal pruning region for $P_q^w(s)$ which is disjoint from $P_{q'}^{w'}(s')$. Thus the pseudo-Anosov

representative of the braid type of $P_q^w(s)$ contains a periodic orbit of the braid type of $P_q^{w'}(s')$: that is, $P_q^w \geq P_q^{w'}$.

The motivation for this conjecture is again computational: it seems very natural once one has calculated maximal pruning regions for homoclinic and periodic orbits with many different decorations. The following examples are illustrative of such computations.

Examples

- a) Let $w = \emptyset$. Figure 4 i) depicts the homoclinic orbit P_0^\emptyset (with core 1001), together with a maximal pruning front \mathcal{F}^\emptyset (which generates \mathcal{R}^\emptyset under saturation). In Figure 4 ii), the periodic orbits $P_{1/4}^\emptyset$ and $P_{2/7}^\emptyset$ (with codes 1000100 and 1001100100) are shown (depicted \bullet and \circ respectively), together with a maximal pruning front $\mathcal{F}_{1/4}^\emptyset$ relative to $P_{1/4}^\emptyset$, which is precisely \mathcal{F}^\emptyset together with a vertical pruning disk with horizontal coordinate $\overline{1000100}$. Since $2/7 > 1/4$, the rightmost point of $P_{2/7}^\emptyset$ lies to the left of this vertical pruning disk, and hence no point of $P_{2/7}^\emptyset$ falls into $\mathcal{F}_{1/4}^\emptyset$. It follows that $P_{1/4}^\emptyset$ forces $P_{2/7}^\emptyset$.

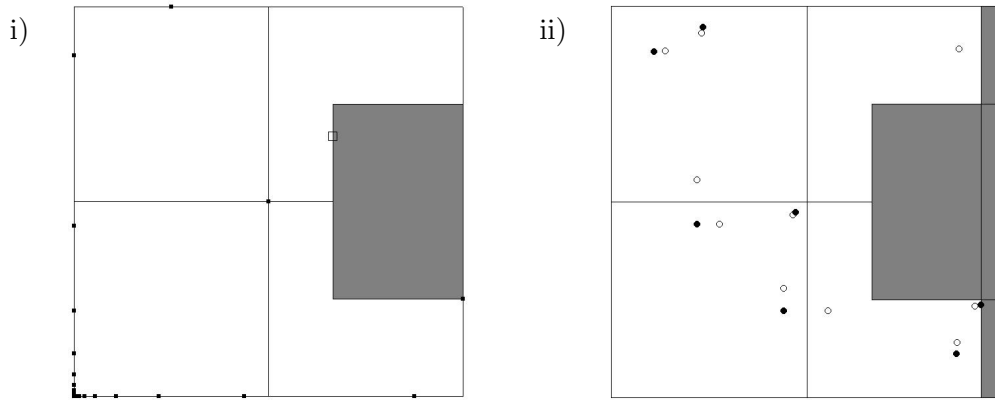


Figure 4: Maximal pruning fronts for the decoration \emptyset

The fixed point of code 1 is also depicted (with \square) in Figure 4 i). The fact that \mathcal{F}^\emptyset has this fixed point on its boundary accounts for the topological train track type corresponding to the decoration \emptyset having a fixed valence 3 vertex (see Table 1).

- b) A similar treatment of the decoration $w = 110$ yields Figure 5. In i) the homoclinic orbit P_0^{110} with core 1011001 is depicted together with a maximal pruning front \mathcal{F}^{110} . Note that in this case the maximal pruning front consists of two pruning disks. The fixed point of code 1 and the period 3 orbit of code 110 are also shown: the fact that they lie on the boundary of \mathcal{F}^{110} accounts for the topological train track type corresponding to

the decoration 110 having four valence three vertices, one fixed and the others lying on a period 3 orbit.

In Figure 5 ii), a maximal pruning front $\mathcal{F}_{1/4}^{110}$ relative to the periodic orbit $P_{1/4}^{110}$ with code 1000101100 is shown: it is exactly \mathcal{F}^{110} together with a vertical pruning disk with horizontal coordinate $\overline{1000101100}$.

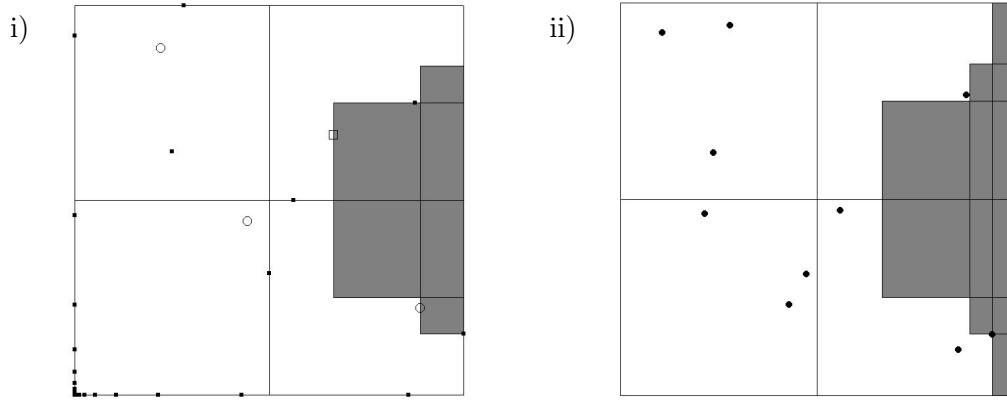


Figure 5: Maximal pruning fronts for the decoration 110

- c) This example illustrates part e) of Conjecture 1. Let $w = 1$. Figure 6 i) depicts the homoclinic orbit P_0^1 (with core 10101, shown as ●), together with a maximal pruning front \mathcal{F}^1 . The homoclinic orbit P_0^\emptyset is also shown (○): since it lies outside \mathcal{F}^1 , it follows that the decoration 1 forces the empty decoration, i.e. $1 \succ \emptyset$. In Figure 6 ii), the periodic orbits $P_{1/4}^1(1)$ and $P_{2/7}^\emptyset(0)$ (with codes 10001111 and 1001100100, respectively) are shown (depicted ● and ○ respectively), together with a maximal pruning front $\mathcal{F}_{1/4}^1$ relative to $P_{1/4}^1$, which is precisely \mathcal{F}^1 together with a vertical pruning disk with horizontal coordinate $\overline{10001111}$. Since $2/7 > 1/4$, the orbit $P_{2/7}^\emptyset$ is disjoint from this vertical pruning disk; and since $1 \succ \emptyset$, it was possible to choose \mathcal{F}^1 to be disjoint from \mathcal{D}_0^\emptyset . Thus $P_{2/7}^\emptyset$ is disjoint from $\mathcal{F}_{1/4}^1$, i.e. $P_{1/4}^1 \geq P_{2/7}^\emptyset$.
- d) The final example illustrates the non-uniqueness of maximal pruning fronts. Let $w = 010$. As noted in Table 1, the three decorations 010, 110, and 011 are all equivalent, i.e. $010 \sim 110 \sim 011$. The simplest maximal pruning front relative to the homoclinic orbit $H = P_0^{010}$ consists of two pruning disks, and is depicted in Figure 7. H itself is not depicted in this figure: instead, the periodic orbit $P_{1/5}^{010}(0)$ is shown. It can be seen that this periodic orbit lies in the pruning region, and hence is destroyed during the pruning isotopy. (In fact the same is true for all of the periodic orbits in \mathcal{D}_{010} .)

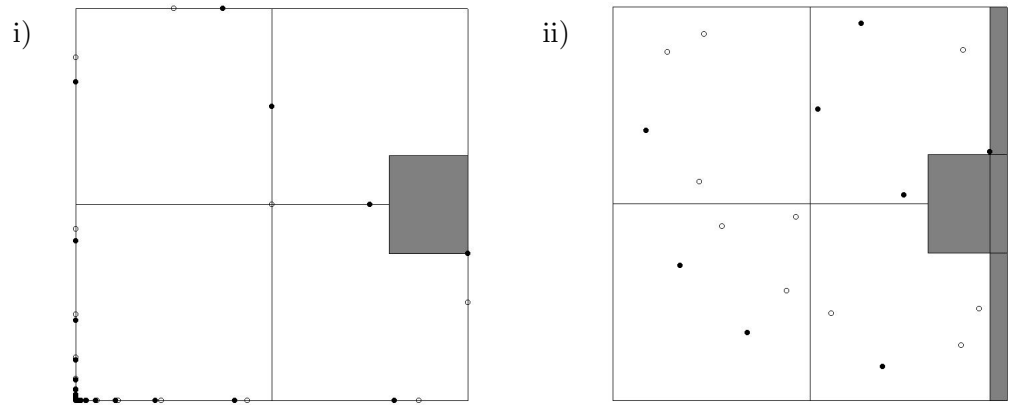


Figure 6: Maximal pruning fronts for the decorations 1 and \emptyset

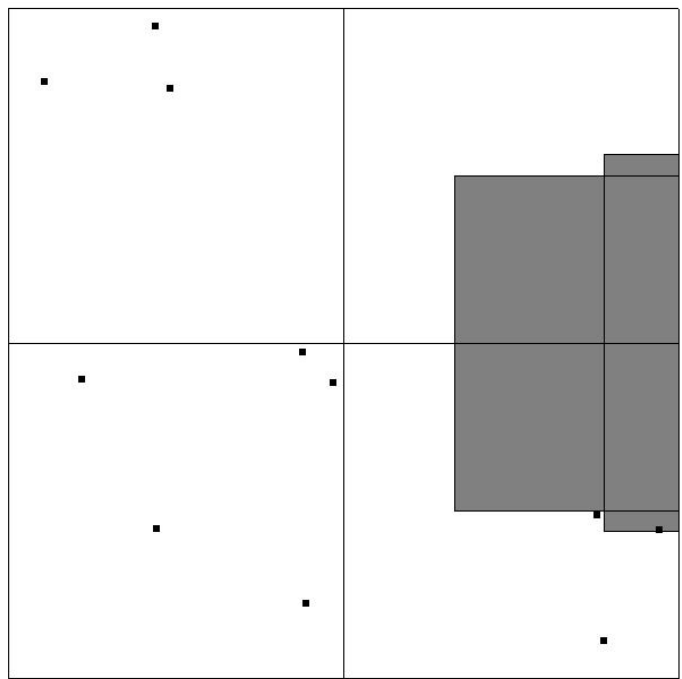


Figure 7: Maximal pruning front relative to P_0^{010}

There are two ways to resolve this problem. The first, following the original formulation of Conjecture 2, is to find a different maximal pruning front which is disjoint from the periodic orbits in \mathcal{D}_{010}^0 . Such a maximal pruning front exists, but it consists of infinitely many pruning disks. The second, following the restatement of Conjecture 2, is to observe that the maximal pruning front of Figure 7 avoids the orbits of the family \mathcal{D}_{110}^0 (although it meets those of \mathcal{D}_{110}^1). Although this is a much simpler resolution in this particular case, it does not appear to be a practical approach in general, since computation of the equivalence relation \sim on decorations seems to be quite intractable.

The non-uniqueness of maximal pruning regions, as shown in Example d), is a major obstacle to proving the conjecture. On the one hand, there are some natural choices for a maximal pruning region relative to P_0^w as one follows the algorithm described in [dCH01]. However these natural pruning regions sometimes contain the family \mathcal{D}_w . If, on the other hand, one tries to obtain a maximal pruning region which avoids this family, the construction may require infinitely many steps, and may indeed cease to be algorithmic. The alternative approach, to find an equivalent family \mathcal{D}_v which is untouched by the natural maximal pruning region, would require a far deeper understanding of the relation \sim than seems possible at present.

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