Explosion of smoothness from a point to everywhere for conjugacies between diffeomorphisms on surfaces

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Abstract

For diffeomorphisms on surfaces with basic sets, we show the following type of rigidity result: if a topological conjugacy between them is differentiable at a point in the basic set then the conjugacy has a smooth extension to the surface. These results generalize the similar ones of D. Sullivan, E. de Faria, and ours for one-dimensional expanding dynamics.

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1 Introduction

D. Sullivan in [13] states the following rigidity theorem for a topological conjugacy between two expanding circle maps: if the conjugacy is differentiable at a point then the conjugacy is smooth everywhere. In Theorem 1, we prove the corresponding result for diffeomorphisms with basic sets contained in a surface.

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E. de Faria in [2] proves a stronger version of the above result of D. Sullivan, showing that it is sufficient the conjugacy to be uniformly asymptotically affine (uaa) at a point to imply that the conjugacy is smooth everywhere. In [3], it is presented a generalization of this result to a larger class of one-dimensional expanding maps. In Theorem 2, we extend these results to diffeomorphisms f and g defined on surfaces which are topologically conjugated ($h : \Lambda_f \to \Lambda_g$) on their basic sets Λ_f and Λ_g, proving, in particular, that (i) if h is asymptotically affine (aa) at a point in Λ_f with periodic orbit, or (ii) if h is (aa) at a point in Λ_f with dense orbit in Λ_f, or (iii) if h is (uaa) at a point in Λ_f, then h has a $C^{1+Hölder}$ extension to the surface (see the definition of (aa) and (uaa) maps in §1.2).

An interesting feature of the theorems proved in this paper is that they show an unexpected rigidity property for the conjugacy between diffeomorphisms with basic sets contained in a surface, since, in general, the conjugacies between these systems are just Hölder continuous but under the weak assumption of the conjugacy being differentiable at a point, for instance, we show that the conjugacy is smooth everywhere. From a practical point of view these results are also useful. We note that it is easier to check that a map is C^1 at a point than everywhere.

1.1 Smoothness from a point to everywhere

Throughout the paper f is a $C^{1+Hölder}$ diffeomorphism on a surface S and Λ is a basic set, i.e. a compact, topologically transitive, hyperbolic and f-invariant set with a local product structure (see [11]). By $C^{1+Hölder}$ we mean $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. By the Stable Manifold Theorem (see [6]), the local *stable leaves* $\ell^s(x)$ and the local *ustable leaves* $\ell^u(x)$ passing through $x \in \Lambda$ are $C^{1+Hölder}$ embedded 1-dimensional submanifolds of S. We define the *stable leaf segments* $\ell^s_\Lambda(x)$ by $\ell^s(x) \cap \Lambda$ and the *unstable leaf segments* $\ell^u_\Lambda(x)$ by $\ell^u(x) \cap \Lambda$. Given any three distinct points x, y, z either in a stable leaf or in an unstable leaf, the order along the leaf tell us which one of these three points is between the other two. We use this order along the stable leaves $\ell^s(x)$ and along the unstable leaves $\ell^u(x)$ to determine the order along the stable leaf segments $\ell^s_\Lambda(x)$ and along the unstable leaf segments $\ell^u_\Lambda(x)$, respectively.

Definition 1 *The $C^{1+Hölder}$ diffeomorphisms f and g are* topologically conjugate on their basic sets Λ_f and Λ_g *if there is a homeomorphism $h : \Lambda_f \to \Lambda_g$ such that $h \circ f(x) = g \circ h(x)$, and h preserves the order along the stable leaf segments $\ell^s_{\Lambda_f}(x)$ and along the unstable leaf segments $\ell^u_{\Lambda_f}(x)$ for all $x \in \Lambda_f$. If h has a $C^{1+Hölder}$ diffeomorphic extension to an open set containing Λ_f then we say that f and g are $C^{1+Hölder}$* conjugate on their basic sets Λ_f and Λ_g.

Theorem 1 *Let f and g be $C^{1+Hölder}$ diffeomorphisms on surfaces which are topologically conjugate on their basic sets Λ_f and Λ_g. If the conjugacy is differentiable at a point $x \in \Lambda_f$, then f and g are $C^{1+Hölder}$ conjugate on their basic sets Λ_f and Λ_g.*

In Section 2, we give the proof of Theorem 1 which has essentially two parts. In the first part, we transform the problem into two problems of one-dimensional expanding dynamics corresponding to the stable and unstable directions associated to the maps f and g. We do this by constructing $C^{1+Hölder}$ Markov maps $M_{f,s}$, $M_{f,u}$, $M_{g,s}$ and $M_{g,u}$ (with respect to atlases $A_{f,s}$, $A_{f,u}$, $A_{g,s}$ and $A_{g,u}$), which retain the information of the smooth structures along the stable and unstable leaves associated to the diffeomorphisms f and g. Then, we use Theorem 1 in [3] which tell us that there is a $C^{1+Hölder}$ conjugacy ψ_s between $M_{f,s}$ and $M_{g,s}$ and a $C^{1+Hölder}$ conjugacy ψ_u between $M_{f,u}$ and $M_{g,u}$. In the second part, we use these $C^{1+Hölder}$ conjugacies ψ_s and ψ_u between the one-dimensional expanding dynamics together with orthogonal charts to prove that the conjugacy between the diffeomorphisms f and g has a $C^{1+Hölder}$ extension to an open set of the surface.

1.2 (aa) and (uaa) regularities

Here, we present and give some motivation for the definitions of asymptotically affine (aa) and uniformly asymptotically affine (uaa) (or, equivalently, symmetric) functions, that we will use to generalize Theorem 1 (as presented in Theorem 2).

(Uaa) functions are relevant in several distinct mathematical contexts as we point out next by reminding some fundamental results about them. We start noting that by the Beurling-Ahlfors extension theorem every quasisymmetric homeomorphism of $\hat{\mathbf{R}}$ has an extension to a quasiconformal homeomorphism of the upper half plane (we say that a homeomorphism h is quasisymmetric if the modulus of continuity χ_c of h in Definition 2 is just a bounded function). In [5], it is proved that (uaa) (or, equivalently, symmetric) homeomorphisms are the boundary values of quasiconformal homeomorphisms of the upper half plane whose conformal distortion tends to zero at the boundary. (Uaa) homeomorphisms turn out to be precisely those homeomorphisms which have boundary dilatation equal to one, in the sense of Strebel, [12]. In [5], it is also noted that the (uaa) homeomorphisms of a circle comprise the closure, in the quasisymmetric topology, of the real analytic homeomorphisms and this closure contains the set of C^1 diffeomorphisms. Another application of (uaa) functions appear in the following extension of the classic Arnold-Herman-Yoccoz rigidity theorem for diffeomorphisms of the circle: a $C^{1+\text{zigmund}}$ diffeomorphism of the circle with golden rotation number is (uaa) conjugate to the rigid golden rotation (see [4]). Finally, we observe that in [14], it is shown an one-to-one correspondence between (uaa) conjugacy classes of expanding circle maps and complex structures on a solenoidal surface, and moreover that the (uaa) conjugacy classes of (uaa) expanding circle maps form a natural completion of the $C^{1+Hölder}$ conjugacy classes of $C^{1+Hölder}$ expanding circle maps.

As we pass to explain, the definition of an (uaa) function f is a geometric notion consisting in a bound of the ratio distortion for triples of points called the modulus of continuity $\chi(t)$ of f. This bound is slightly weaker than the one satisfied by smooth functions. We recall that if f is $C^{1+\alpha}$ then the modulus of continuity $\chi(t)$ satisfies the inequality $\chi(t) < \mathcal{O}(|t|^\alpha)$, where $0 < \alpha < 1$. The (uaa) regularity is characterized by only demanding that $\chi(t)$ converges to zero when t tends to zero. Hence, the (uaa) regularity arise as a natural limit on the degree $1 + \alpha$ of smoothness of the functions when α tends to 0, instead of the usual C^1 smoothness.

Definition 2 *The local homeomorphism $\phi : I \subset \mathbf{R} \to \mathbf{R}$ is* uniformly asymptotically affine (uaa) at a point $x \in I$ *if, for all $c \geq 1$, there is a continuous function $\chi_c : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ satisfying $\chi_c(0) = 0$ such that for all points $y_1, y_2, y_3 \in I$ with $c^{-1} \leq (y_3 - y_2)/(y_2 - y_1) \leq c$, we have*

$$\left| \log \frac{\phi(y_2) - \phi(y_1)}{\phi(y_3) - \phi(y_2)} \frac{y_3 - y_2}{y_2 - y_1} \right| < \chi_c(\max\{|y_3 - x|, |y_1 - x|\}). \qquad (1)$$

We call χ_c the *modulus of continuity of ϕ*. The left hand-side of (1) is called the *ratio distortion* of ϕ at the points y_1, y_2 and y_3.

The local homeomorphism $\phi : I \to \mathbf{R}$ is *(uaa)* if ϕ is (uaa) at every point $x \in I$ with modulus of continuity χ_c not depending upon the point x.

We say that $\phi : I \to \mathbf{R}$ is *asymptotically affine (aa) at a point $x \in I$* if ϕ satisfies inequality (1) in the case where $y_2 = x$.

The classical definition of an (uaa) (or, equivalently, symmetric) function ϕ is given by taking $c = 1$. Here, we consider in the definition all $c \geq 1$ because I does not have to be an interval. For instance I can be a Cantor set. However, we note that these two conditions are equivalent if I is an interval (see Remark 1 in [3]).

Definition 3 *The homeomorphism* $h : \Lambda_f \to \Lambda_g$ *is* (aa) *(resp.* (uaa)*)* at a point $x \in \Lambda_f$ *if* $h|\ell^s_{\Lambda_f}(x)$ *and* $h|\ell^u_{\Lambda_f}(x)$ *are (aa) (resp. (uaa)) at the point* x*. The homeomorphism* $h : \Lambda_f \to \Lambda_g$ *is* (aa) in a set $X \subset \Lambda_f$ *if, for every* $x \in X$*,* $h|\ell^s_{\Lambda_f}(x)$ *and* $h|\ell^u_{\Lambda_f}(x)$ *are (aa) at* x*, and the modulus of continuity does not depend upon the point* $x \in X$*.*

A *generating set* $\mathcal{G}$ is a subset of Λ_f with the property that

$$\Lambda_f = \mathrm{cl}\left(\{f^n(a) : a \in \mathcal{G} \ \text{ and } \ n \geq 0\}\right).$$

A *sub-orbit* is a subset $\{f^{n_i}(p) : i \in \mathbf{Z}\}$ of Λ_f, where $p \in \Lambda_f$ and $(n_i)_{i \in \mathbf{Z}}$ is an increasing sequence of integers.

Theorem 2 *Let* f *and* g *be* $C^{1+H\ddot{o}lder}$ *diffeomorphisms on surfaces with basic sets* Λ_f *and* Λ_g*, and topologically conjugate by a homeomorphism* $h : \Lambda_f \to \Lambda_g$*. (i) If* h *is (aa) in a sub-orbit then* f *and* g *are* $C^{1+H\ddot{o}lder}$ *conjugate on their basic sets* Λ_f *and* Λ_g*; (ii) If* h *is (aa) in a generating set then* f *and* g *are* $C^{1+H\ddot{o}lder}$ *conjugate on their basic sets* Λ_f *and* Λ_g*; (iii) If* h *is (uaa) at a point in* Λ_f *then* f *and* g *are* $C^{1+H\ddot{o}lder}$ *conjugate on their basic sets* Λ_f *and* Λ_g*.*

We would like to point out that the previous conditions used in the previous theorem correspond to very natural and simple dynamical objects. An example of a generating set $\mathcal{G}$ is a point with dense orbit; and an example of a sub-orbit is a point with periodic orbit.

The proof of Theorem 2 follows in the same way as the proof of Theorem 1.

2 Properties of basic sets

The proof of Theorem 1 involves several properties of basic sets that we will recall in this section.

2.1 Rectangles

Let ρ be a $C^{1+H\ddot{o}lder}$ Riemannian metric on S and d the distance on S determined by ρ. Since Λ_f has a local product structure, there exist constants $\xi, \xi' > 0$ such that, for every $x, y \in \Lambda_f$ with $d(x, y) < \xi'$, the *bracket* $[x, y]_{\xi,\xi'} = \ell^s(x, \xi) \cap \ell^u(y, \xi)$ is a single point contained in Λ_f, where

$$\ell^s(x, \varepsilon) = \{y \in S : d(f^n(x), f^n(y)) < \varepsilon, \text{ for all } n \geq 0\}$$

and

$$\ell^u(x, \varepsilon) = \{y \in S : d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \text{ for all } n \geq 0\}.$$

A *rectangle* $R = R^f$ is a subset of Λ_f which is closed under the bracket, i.e for every $x, y \in R$, the bracket $[x, y]_{\xi,\xi'}$ is contained in R. A rectangle R is *proper* if R is the closure of its interior in Λ_f. A *stable spanning leaf segment* $\ell^s_R(x)$ contained in a proper rectangle R is the union of a stable leaf segment $\ell^s_{\Lambda_f}(x)$ with its endpoints and satisfying the property that $[x, y]_{\xi,\xi'} \in \ell^s_R(x)$ for every $y \in R$. Similarly, we define the *unstable spanning leaf segment* $\ell^u_R(x)$, replacing $\ell^s_{\Lambda_f}(x)$ by $\ell^u_{\Lambda_f}(x)$. Let $\partial\ell^\tau_R(x)$ be the set consisting of the endpoints of $\ell^\tau_R(x)$, and let $\ell^\tau_R(x) \setminus \partial\ell^\tau_R(x)$ be denoted by int $\ell^\tau_R(x)$ for $\tau \in \{s, u\}$. By the local product structure of Λ_f, for every proper rectangle R and for every $x \in R$, the *interior of* R is $\text{int}R = \{[y, z]_{\xi,\xi'} : y \in \ell^u_R(x) \text{ and } z \in \ell^s_R(x)\}$, and the *boundary of* R is

$$\partial R = \{[y, z]_{\xi,\xi'} : (y \in \partial\ell^u_R(x) \text{ and } z \in \ell^s_R(x)) \text{ or } (y \in \ell^u_R(x) \text{ and } z \in \partial\ell^s_R(x))\}.$$

Note that the definitions of int$R$ and ∂R do not depend upon $x \in R$. A *τ-side of* R is a τ-spanning leaf segment $\ell^\tau_R(x)$ contained in the boundary of R for $\tau \in \{s, u\}$. A *corner of* R is an endpoint of a side of R.

2.2 Basic holonomies

Let τ be equal to s or u, and τ' be the opposite of τ. Given a proper rectangle R and two points $x, y \in R$, we denote by $\Theta : \ell^\tau_R(x) \to \ell^\tau_R(y)$ the *basic holonomy* given by $\Theta(z) = \ell^{\tau'}_R(z) \cap \ell^{\tau}_R(y)$ for every $z \in \ell^{\tau}_R(x)$. From Theorem 2.1 in [7], we get the following result.

Lemma 1 *Each basic holonomy $\Theta : \ell^{\tau}_R(x) \to \ell^{\tau}_R(y)$ is a $C^{1+H\ddot{o}lder}$ diffeomorphism, i.e Θ has a $C^{1+H\ddot{o}lder}$ extension $\tilde{\Theta} : \ell^{\tau}(x) \to \ell^{\tau}(y)$ to the leaves $\ell^{\tau}(x)$ and $\ell^{\tau}(y)$ such that $\ell^{\tau}_R(x) \subset \ell^{\tau}(x) \cap \Lambda_f$ and $\ell^{\tau}_R(y) \subset \ell^{\tau}(y) \cap \Lambda_f$.*

2.3 Markov partition

By Theorem 3.12 in page 79 of [1], the basic set Λ_f has a *Markov partition* given by a collection $\mathcal{M} = \{R_1, \ldots, R_m\}$ of proper rectangles with the following properties: (i) $\mathrm{int} R_i \bigcap \mathrm{int} R_j = \emptyset$, if $i \neq j$; (ii) $\Lambda_f = \bigcup_{i=1}^m R_i$; (iii) if $x \in R_i$ and $f(x) \in R_j$, then

(a) $f\left(\ell^s_{R_i}(x)\right) \subset \ell^s_{R_j}(f(x))$ and $f^{-1}\left(\ell^u_{R_j}(f(x))\right) \subset \ell^u_{R_i}(x)$;

(b) $f\left(\ell^u_{R_i}(x)\right) \bigcap R_j = \ell^u_{R_j}(f(x))$ and $f^{-1}\left(\ell^s_{R_j}(f(x))\right) \bigcap R_i = \ell^s_{R_i}(x)$.

The last condition means that $f(R_i)$ goes across R_j just once. The proper rectangles $R_i \in \mathcal{M}$ are called *Markov rectangles*.

3 From two to one-dimensional dynamics

We will use the properties of the basic sets presented in the previous section to pass from two-dimensional dynamics to one-dimensional expanding dynamics. We do it by constructing $C^{1+H\ddot{o}lder}$ Markov maps on train tracks.

3.1 Train tracks

Let $T^{\tau} = T^{\tau}_f$ be the set of all τ'-leaf spanning segments $\ell^{\tau'}_R(x)$ for all $R \in \mathcal{M}$ and for all $x \in R$, where we identify two of these τ'-leaf spanning segments $\ell^{\tau'}_R(x)$ and $\ell^{\tau'}_R(y)$ if $\mathrm{int}\ell^{\tau'}_R(x) \cap \mathrm{int}\ell^{\tau'}_R(y) \neq \emptyset$. The set T^{τ} is a train track. Let $\pi_{f,\tau} : \Lambda_f \to T^{\tau}$ be the projection which associates to a point $x \in \Lambda_f$

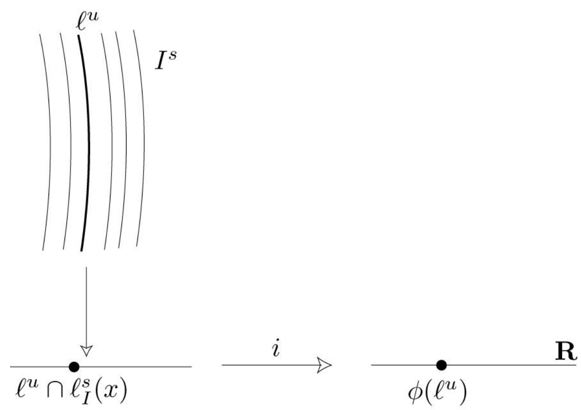

Figure 1: The chart ϕ on T^s

the spanning leaf segment (or segments) $\ell \in T^\tau$ which contains x. We note that, for every $x \in \mathrm{int}R$, the projection $\pi_{f,\tau}(x)$ is a single point in T^τ. For a point x contained in a τ-side of a Markov rectangle the projection $\pi_{f,\tau}(x)$ can consist in more than one point.

3.2 Atlas on train tracks

We say that $I \subset T^\tau$ is a *segment of* T^τ *associated to a leaf segment* ℓ^τ_I (not intersecting the τ-boundary of a Markov rectangle) if and only if (i) for every $x \in \ell^\tau_I$ there exists a leaf $\ell^{\tau'} \in I$ such that $\ell^{\tau'} \cap \ell^\tau_I \neq \emptyset$ and (ii) for every $\ell^{\tau'} \in I$, $\ell^{\tau'} \cap \ell^\tau_I \neq \emptyset$. A *chart* $\phi : I \to \mathbf{R}$ *on a segment* I is defined by $\phi(\ell^{\tau'}) = i\left(\ell^{\tau'} \cap \ell^\tau_I\right)$, where $i : \ell^\tau_I \to \mathbf{R}$ is a homeomorphism onto its image which preserves the local order of the points in ℓ^τ_I. The map $\phi : I \to \mathbf{R}$ defined by $\phi(\ell^{\tau'}) = i\left(\ell^{\tau'} \cap \ell^\tau_I\right)$ is a *chart on* T^τ (see Figure 1).

We say that the charts $\phi : I \to \mathbf{R}$ and $\psi : J \to \mathbf{R}$ on T^τ are $C^{1+\alpha}$ compatible if the overlap map $\psi \circ \phi^{-1} : \phi(I \cap J) \to \psi(I \cap J)$ has a $C^{1+\alpha}$ diffeomorphic extension to $\mathbf{R}$, where $\alpha > 0$. A $C^{1+Hölder}$ *atlas* $\mathcal{A}^\tau$ *on* T^τ consists on a finite set of charts on T^τ which cover all small segments of T^τ and any two of them are $C^{1+\alpha}$ compatible with $C^{1+\alpha}$ bounded norm, for some $\alpha > 0$.

Let $I \subset T^\tau$ be a segment of T^τ associated to a leaf segment ℓ^τ_I. Let $\tilde{\ell}^\tau_I$

be a leaf containing ℓ_I^τ and $c : (-1,1) \to \tilde{\ell}_I^\tau$ a $C^{1+\alpha}$ diffeomorphism given by the Stable Manifold Theorem applied to f. We say that $\mathcal{A}_f^\tau$ is an *atlas on T^τ determined by f* if $\mathcal{A}_f^\tau$ is a set consisting of charts $\phi_f : I \to \mathbf{R}$ given by $\phi_f(\ell^{\tau'}) = c^{-1}\left(\ell^{\tau'} \cap \ell_I^\tau\right)$.

3.3 Markov maps

The $C^{1+Hölder}$ diffeomorphism f determines Markov maps $M_{f,s} : T^s \to T^s$ and $M_{f,u} : T^u \to T^u$ such that the following diagrams commute:

$$\begin{array}{ccc} \Lambda_f & \xrightarrow{f^{-1}} & \Lambda_f \\ \downarrow{\scriptstyle \pi_{f,s}} & & \downarrow{\scriptstyle \pi_{f,s}} \\ T^s & \xrightarrow{M_{f,s}} & T^s \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda_f & \xrightarrow{f} & \Lambda_f \\ \downarrow{\scriptstyle \pi_{f,u}} & & \downarrow{\scriptstyle \pi_{f,u}} \\ T^u & \xrightarrow{M_{f,u}} & T^u \end{array}$$

The Markov partition $\{R_1, \ldots, R_m\}$ of f determines the Markov partition $\{I_1^\tau, \ldots, I_m^\tau\}$ of $M_{f,\tau}$, where $I_i^\tau = \bigcup_{x \in R_i} \ell_{R_i}^{\tau'}(x)$ for every $i = 1, \ldots, m$.

A Markov map $M_{f,\tau}$ is $C^{1+Hölder}$ with respect to an atlas $\mathcal{A}_f^\tau$ if (i) for every charts $\phi : I \to \mathbf{R}$ and $\psi : J \to \mathbf{R}$ in $\mathcal{A}_f^\tau$ such that the composition $\psi \circ M_{f,\tau} \circ \phi^{-1} : \phi(I) \to \psi(J)$ is a homeomorphism and has a $C^{1+\alpha}$ extension $M_{\phi,\psi}$ to $\mathbf{R}$ with uniformly bounded $C^{1+\alpha}$ norm; and (ii) there exist $c, \lambda > 0$ such that for all possible compositions $M_{\phi_n,\phi_{n-1}} \circ \ldots \circ M_{\phi_1,\phi_0}$ we have

$$\| M_{\phi_n,\phi_{n-1}} \circ \ldots \circ M_{\phi_1,\phi_0} \|_{C^1} > c\lambda^n. \tag{2}$$

Lemma 2 *If f is a $C^{1+Hölder}$ diffeomorphism of a compact surface with a basic set then the atlas A_f^τ determined by f is $C^{1+Hölder}$ and $M_{f,\tau}$ is a $C^{1+Hölder}$ Markov map with respect to the atlas A_f^τ, for $\tau \in \{s,u\}$.*

Proof: Let us prove Lemma 2 in two parts. In the first part we prove that the overlap maps for charts in A_f^τ are $C^{1+Hölder}$ and so A_f^τ is a $C^{1+Hölder}$ atlas. In the second part we prove that the Markov map $M_{f,\tau}$ is $C^{1+Hölder}$ with respect to A_f^τ.

Let I and J be segments associated to leaf segments ℓ_I^τ and ℓ_J^τ, and $\phi : I \to \mathbf{R}$ and $\psi : J \to \mathbf{R}$ be charts in A_f^τ such that $\phi(\ell^{\tau'}) = c_I^{-1}\left(\ell^{\tau'} \cap \ell_I^\tau\right)$

and $\psi(\ell^{\tau'}) = c_J^{-1}\left(\ell^{\tau'} \cap \ell^\tau_J\right)$, where c_I and c_J are $C^{1+H\ddot{o}lder}$ curves given by the Stable Manifold Theorem (see [6]).

Part I: Let us suppose that $I \cap J \neq \emptyset$. The segment $I \cap J$ has leaf segments $\ell^\tau_{I,J} \subset \ell^\tau_I$ and $\ell^\tau_{J,I} \subset \ell^\tau_J$ associated to it, and $\psi \circ \phi^{-1} = c_J^{-1} \circ \theta \circ c_I$ where $\theta : \ell^\tau_{I\cap J} \to \ell^\tau_{J\cap I}$ is a holonomy. By Lemma 1, θ is $C^{1+H\ddot{o}lder}$ and so $c_J^{-1} \circ \theta \circ c_I$ has a $C^{1+H\ddot{o}lder}$ diffeomorphic extension to $\mathbf{R}$. Hence, the atlas $\mathcal{A}^\tau_f$ is $C^{1+H\ddot{o}lder}$.

Part II: Let us suppose that $\psi \circ M_{f,\tau} \circ \phi^{-1} : \phi(I) \to \psi \circ M_{f,\tau}(I)$ is a homeomorphism. Let $\ell^\tau_{M,I} = M_{f,\tau}\left(\ell^\tau_I\right)$ and $\theta : \ell^\tau_{M,I} \to \ell^\tau_J$ be a holonomy. First, we note that $\psi \circ M_{f,\tau} \circ \phi^{-1} = c_J^{-1} \circ \theta \circ f \circ c_I$ and by Lemma 1 $c_J^{-1} \circ \theta \circ f \circ c_I$ has a $C^{1+H\ddot{o}lder}$ diffeomorphic extension to $\mathbf{R}$. Since Λ_f is hyperbolic, we obtain that the Markov map $M_{f,\tau}$ also satisfies inequality (2). ∎

3.4 $C^{1+H\ddot{o}lder}$ conjugacies between Markov maps

Let $h : \Lambda_f \to \Lambda_g$ be the conjugacy between f and g on their basic sets. Given a Markov partition $\mathcal{M}_f = \{R_1, \ldots, R_m\}$ of f, we consider the Markov partition of g given by $\mathcal{M}_g = \{h(R_1), \ldots, h(R_m)\}$. The conjugacy $h : \Lambda_f \to \Lambda_g$ determines the conjugacy $\psi_s : T^s_f \to T^s_g$ between the Markov maps $M_{f,s}$ and $M_{g,s}$, and the conjugacy $\psi_u : T^u_f \to T^u_g$ between the Markov maps $M_{f,u}$ and $M_{g,u}$ such that the following diagrams commute:

$$\begin{array}{ccc} \Lambda_f & \xrightarrow{h} & \Lambda_g \\ \downarrow{\scriptstyle \pi_{f,s}} & & \downarrow{\scriptstyle \pi_{f,s}} \\ T^s_f & \xrightarrow{\psi_s} & T^s_g \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda_f & \xrightarrow{h} & \Lambda_g \\ \downarrow{\scriptstyle \pi_{f,u}} & & \downarrow{\scriptstyle \pi_{f,u}} \\ T^u_f & \xrightarrow{\psi_u} & T^u_g \end{array}$$

Lemma 3 *Let f and g be $C^{1+H\ddot{o}lder}$ diffeomorphisms on surfaces which are topologically conjugate on their basic sets. If the conjugacy between f and g satisfies the hypotheses of Theorem 1 or of Theorem 2 then $M_{f,\tau}$ and $M_{g,\tau}$ are $C^{1+H\ddot{o}lder}$ conjugate for $\tau \in \{s, u\}$.*

Proof: Similarly to Lemma 2 (using that the basic holonomies are $C^{1+Hölder}$), if the conjugacy h is C^1 at a point p then ψ_τ is C^1 at the point $\pi_{f,\tau}(p)$ for $\tau \in \{s,u\}$. Hence, by Theorem 1 in [3], we obtain that ψ_τ has a $C^{1+Hölder}$ extension. ∎

4 From one to two dimensional dynamics

Here, we do the last step of the proof of Theorem 1 which consists in using the $C^{1+Hölder}$ conjugacies between Markov maps, as proved in Lemma 3, to prove the existence of a $C^{1+Hölder}$ diffeomorphic extension of the conjugacy between $C^{1+Hölder}$ diffeomorphisms on surfaces as claimed in theorem 1 and 2.

4.1 Proof of theorems 1 and 2

For every $x \in \Lambda_f$, let R^f be a rectangle containing x in its interior. Let $\ell^s_f(x)$ and $\ell^u_f(x)$ be stable and unstable leaves with the property that $\overline{\ell^s_f(x)} \cap \Lambda_f = \ell^s_{R^f}(x)$ and $\overline{\ell^u_f(x)} \cap \Lambda_f = \ell^u_{R^f}(x)$. By the Stable Manifold Theorem, there are $C^{1+Hölder}$ curves $c_{f,s} : (-1,1) \to \ell^s_f(x)$ and $c_{f,u} : (-1,1) \to \ell^u_f(x)$ with $c^{-1}_{f,s}(x) = c^{-1}_{f,u}(x) = 0$. Let us denote by $i_f : \mathrm{int}R^f \to \mathbf{R}^2$ the orthogonal map given by $i_f(z) = \left(c^{-1}_{f,s}([x,z]), c^{-1}_{f,u}([z,x])\right)$, for every $z \in \mathrm{int}R^f$. Similarly, for $R^g = h(R^f)$ we define as above $C^{1+Hölder}$ curves $c_{g,s} : (-1,1) \to \ell^s_g(h(x))$ and $c_{g,u} : (-1,1) \to \ell^u_g(h(x))$ and the orthogonal map $i_g : \mathrm{int}R^g \to \mathbf{R}^2$. Since the Markov maps $M_{f,\tau}$ and $M_{g,\tau}$ are $C^{1+Hölder}$ conjugate then $c^{-1}_{g,s} \circ h \circ c_{f,s}$ and $c^{-1}_{g,u} \circ h \circ c_{f,u}$ have $C^{1+Hölder}$ diffeomorphic extensions $\hat{h}_s : \mathbf{R} \to \mathbf{R}$ and $\hat{h}_u : \mathbf{R} \to \mathbf{R}$. Hence, the map $i_g \circ h \circ i_f^{-1}$ has a $C^{1+Hölder}$ diffeomorphic extension $H : \mathbf{R}^2 \to \mathbf{R}^2$ given by $H(z,w) = \left(\hat{h}_s(z), \hat{h}_u(w)\right)$.

Since S_f and S_g are $C^{1+Hölder}$ manifolds, there are $C^{1+Hölder}$ atlas $\mathcal{S}_f$ on S_f and $\mathcal{S}_g$ on S_g consisting of charts with $C^{1+Hölder}$ overlap maps. Using

Proposition 5.4 in [8], the orthogonal map i_f extends to a chart $\hat{i}_f$ on an open set $\hat{U}_f \subset S_f$ containing x and which is contained in the smooth atlas $\mathcal{S}_f$. Similarly, the orthogonal map i_g extends to a chart $\hat{i}_g$ on an open set $\hat{U}_g \subset S_g$ containing $h(x)$ and which is contained in the smooth atlas $\mathcal{S}_g$.

We choose an open set $U_f \subset \hat{U}_f$ of S_f containing x and small enough such that $U_f \cap R^f = U_f \cap \Lambda_f$ and $H(U_f) \subset \hat{U}_g$. Hence, for every $x \in \Lambda_f$ the map $h|(U_f \cap \Lambda_f)$ has a $C^{1+H\ddot{o}lder}$ diffeomorphic extension to U_f given by $\hat{i}_g^{-1} \circ H \circ \hat{i}_f$. Therefore, using partitions of the unity (see Lemma 5.6 in [8]) the map $h : \Lambda_f \to \Lambda_g$ has a $C^{1+H\ddot{o}lder}$ diffeomorphic extension to an open set U_f of S_f containing Λ_f. ∎

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