STABLY CHAOTIC RATIONAL VECTOR FIELDS ON $\mathbb{CP}^n$.

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Abstract: We construct an open set $\mathcal{U}$ of rational foliations of arbitrarily fixed degree $d \geq 2$ by curves in $\mathbb{CP}^n$ such that any foliation $\mathcal{F} \in \mathcal{U}$ has a finite number of singularities and satisfies the following chaotic properties. *Minimality*: any leaf (curve) is dense in $\mathbb{CP}^n$. *Ergodicity*: any Lebesgue measurable subset of leaves has *zero* or total Lebesgue measure. *Entropy*: the topological entropy is strictly positive even far from singularities. *Rigidity*: if $\mathcal{F}$ is conjugate to some $\mathcal{F}' \in \mathcal{U}$ by a homeomorphism close to the identity, then they are also conjugate by a projective transformation.

The main analytic tool employed in the construction of these foliations is the existence of several pseudo-flows in the closure of pseudo-groups generated by perturbations of elements in $\mathrm{Diff}(\mathbb{C}^n, 0)$ on a fixed ball.

Introduction

Let X be a polynomial vector field on $\mathbb{C}^n$ namely,

$$X = P_1(\underline{z})\frac{\partial}{\partial z_1} + \cdots + P_n(\underline{z})\frac{\partial}{\partial z_n}, \quad \underline{z} \in \mathbb{C}^n.$$

The complex integral curves of X define a regular holomorphic foliation by Riemann Surfaces away from the common zero set of the P_i. This foliation admits an analytic extension to a (singular) holomorphic foliation, with singular set of complex codimension ≥ 2, on $\mathbb{CP}^n$: in the affine chart $\underline{z}$, after dividing the P_i by their common factor, we can suppose that the set of their common zeroes has codimension at least 2; in any other chart of $\mathbb{CP}^n$, the vector field X is rational and, after multiplication by a convenient polynomial, becomes polynomial again. Conversely, it turns out that any regular one dimensional holomorphic foliation on $\mathbb{CP}^n \setminus \mathcal{S}$ (where $\mathcal{S}$ is an analytic set of codimension ≥ 2) is obtained as above (see [Si],p.243).

The (*projective*) *degree* d of $\mathcal{F}$ is the number of tangencies of $\mathcal{F}$ with a generic hyperplane. Alternatively, d is the smallest integer for which, after dividing by the common factor, the vector field X has the form

$$X = X_0 + X_1 + \cdots + X_d + H_d \cdot (z_1\frac{\partial}{\partial z_1} + \cdots + z_n\frac{\partial}{\partial z_n}),$$

where X_i denotes the i^{th} homogeneous component of X and H_d either vanishes identically or is a homogeneous polynomial of degree d. The foliation $\mathcal{F}$ may equivalently be defined as the radial projection of the one dimensional foliation $\widetilde{\mathcal{F}}$ defined in $\mathbb{C}^{n+1}$ by a degree d homogeneous vector field $\widetilde{X}$ given by

$$\widetilde{X} = \sum_{i=0}^{n} H_i(Z_0, \underline{Z})\frac{\partial}{\partial Z_i}.$$

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In the main affine chart $\underline{z} = \underline{Z}/Z_0$, the coefficients of the polynomial vector field X defined on $\mathbb{C}^n$ are given by $P_i(\underline{z}) = H_i(1, \underline{z}) - z_i H_0(1, \underline{z})$, which explains the "radial type" of the homogeneous component of degree $d+1$ of X.

Note that two polynomial vector fields X and X' define the same foliation if and only if $X' = \lambda X$ with $\lambda \in \mathbb{C}^*$. Therefore the set $\mathcal{F}^d(\mathbb{CP}^n)$ consisting of one dimensional holomorphic foliations of degree $\leq d$ on $\mathbb{CP}^n$ is a Zariski open subset of the complex projective space of dimension

$$(d+n+1)\frac{(d+n-1)!}{d!(n-1)!} - 1.$$

Consider $\mathcal{F}^d(\mathbb{CP}^n)$ equipped with the natural topology arising from the identification above. A central question in the study of these foliations is to find dynamic properties (e.g. density of leaves, ergodicity and so on) which are satisfied for "most of" the foliations in $\mathcal{F}^d(\mathbb{CP}^n)$. In his report to the Helsinki Conference (cf. [Il2,p.823]), Il'yashenko made some conjectures concerning the global dynamical behavior of "most of" these foliations. The purpose of the present work is to provide a partial (or local) affirmative answer to his conjectures by proving the following theorem:

Main Theorem. *For any $d \geq 2$, there exists a non empty open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^n)$ such that any element $\mathcal{F} \in \mathcal{U}$ has a finite number of singularities and is "chaotic", i.e. satisfies:*

- **Minimality:** *each leaf is dense in $\mathbb{CP}^n$;*
- **Ergodicity:** *any measurable set of leaves has zero or total Lebesgue measure;*
- **Entropy:** *the geometric entropy of the regular foliation induced by $\mathcal{F}$ after deleting small balls around singularities is strictly positive;*
- **Rigidity:** *there exists a neighborhood $\mathcal{V}$ of the identity in the space of homeomorphisms $\Phi : \mathbb{CP}^n \to \mathbb{CP}^n$ such that if another foliation $\mathcal{F}' \in \mathcal{U}$ is topologically conjugate to $\mathcal{F}$ by some $\Phi \in \mathcal{V}$, then $\mathcal{F}$ and $\mathcal{F}'$ are also conjugate by a projective change of coordinates.*

For the definition of the geometric entropy of a foliation we refer to [Gh,La,Wa],p.110. In any case we shall recall this definition at the beginning of section 6.

Even for $n = 2$ this result is new, as far as ergodicity and topological rigidity are concerned. In high dimensions ($n \geq 3$) no example of a foliation having all leaves dense was previously known (cf. below).

It is interesting to remark that, since $\dim_{\mathbb{C}} PGL(n, \mathbb{C}) < \dim_{\mathbb{C}} \mathcal{F}^d(\mathbb{CP}^n)$, the topological rigidity implies structural instability, i.e. the foliations in question are approximated by non-topologically equivalent foliations.

The rest of the introduction is devoted to setting up the main ideas involved in the proof of our theorem as well as situating it with regard to previous work.

Recall that $\mathcal{F}^d(\mathbb{CP}^n)$ has been equipped with a natural topology through its identification with a Zarisky open set of a suitable complex projective space. A property will be said to be *generic* in $\mathcal{F}^d(\mathbb{CP}^n)$ if it is satisfied for an *open subset of $\mathcal{F}^d(\mathbb{CP}^n)$ having total Lebesgue measure*. For instance, it is easy to check that a generic foliation has exactly $\frac{d^{n+1}-1}{d-1}$ singularities (in particular, $\mathcal{S}$ is finite).

In dimension $n = 2$, a great amount of work has been devoted to the dynamical behavior of foliations in $\mathcal{F}^d(\mathbb{CP}^2)$ *which are tangent to a projective line*, say the line L_∞ at infinity; let us denote by $\mathcal{F}^d(\mathbb{C}^2) \subset \mathcal{F}^d(\mathbb{CP}^2)$ the class of these foliations. A combination of remarkable results due to M.O. Hudai-Verenov and mainly to Yu.Il'yashenko in the 70's yields the theorem below:

Theorem ([HV],[Il]). *For any $d \geq 2$, there is a set $\mathcal{A}^d \subset \mathcal{F}^d(\mathbb{C}^2) \subset \mathcal{F}^d(\mathbb{CP}^2)$ such that any foliation $\mathcal{F}$ belonging to $\mathcal{A}^d$ satisfies the following:*

- **(almost) Minimality:** *each leaf (apart from the invariant line at infinity) is dense in $\mathbb{C}^2$;*
- **Ergodicity:** *any measurable set of leaves has full or null Lebesgue measure;*
- **Rigidity:** *there exists an open neighborhood $\mathcal{U}$ of $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{C}^2)$ and a neighborhood $\mathcal{V}$ of the identity in the space of homeomorphisms $\Phi : \mathbb{CP}(2) \to \mathbb{CP}(2)$ such that, if a foliation $\mathcal{F}' \in \mathcal{U}$ is topologically conjugate to $\mathcal{F}$ by some $\Phi \in \mathcal{V}$, then $\mathcal{F}$ and $\mathcal{F}'$ are also conjugate by an affine change of coordinates.*

Furthermore $\mathcal{A}^d \subset \mathcal{F}^d(\mathbb{C}^2) \subset \mathcal{F}^d(\mathbb{CP}^2)$ contains a relatively open dense set (of total Lebesgue measure) of $\mathcal{F}^d(\mathbb{C}^2)$.

The fact that the class $\mathcal{A}^d$ can be considered open inside $\mathcal{F}^d(\mathbb{C}^2)$ is due to Shcherbakov (cf. [Sh1] and [Sh2]).

Nonetheless a *generic* element of $\mathcal{F}^d(\mathbb{CP}^n)$ ($n \geq 2$) possesses *no algebraic invariant curve* (cf. [LN,So]). Thus the subclass of $\mathcal{F}^d(\mathbb{CP}^2)$ consisting of foliations admitting an algebraic invariant curve is *very small*. In particular the class $\mathcal{A}^d$ has empty interior and null Lebesgue measure *inside* $\mathcal{F}^d(\mathbb{CP}^2)$. In other words the results above fail to provide *an open set* of foliations (with fixed degree) exhibiting "chaotic" behavior (a foliation will be called chaotic if it is minimal, ergodic, has positive entropy and is topologically rigid). However, despite being "small", the class $\mathcal{F}^d(\mathbb{C}^2)$ is interesting in its own right and, restricted to this class, the above mentioned results are rather precise. In dimension $n = 2$ further improvements have also been made as one can check in [Sh1], [GM,OB], [GM] and [LN,Sa,Sc], always with a strong additional hypothesis such as "tangent to an algebraic curve".

In [Mj], B.Mjuller constructed an open set of minimal foliations in $\mathcal{F}^d(\mathbb{CP}^2)$ (precisely, the foliations constructed are almost minimal, i.e. excluding a finite number of algebraic leaves, the remaining leaves are dense: however the main result of [LN,So] asserts that there is an open dense set of foliation, with fixed degree, having no algebraic invariant curve, thus intersecting these two open sets we obtain an open set of minimal foliations). B.Mjuller has also obtained examples of foliations in $\mathbb{CP}^3$ tangent to the projective plane at infinity but having all leaves dense in the affine part. Actually only recently B.Wirtz announced a new construction of stably minimal foliations *in dimension 2* having positive entropy. The analogous questions concerning ergodicity and topological rigidity were not addressed as far as we know.

The original approach of Il'yashenko to study elements of $\mathcal{F}^d(\mathbb{C}^2)$ is based on studying the holonomy group $\mathrm{Hol}(L_\infty)$ of the invariant line at infinity L_∞. Indeed $\mathrm{Hol}(L_\infty)$ is in general a "large" (e.g. non solvable) subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ whose dynamics can be well understood. Furthermore, since the complement of L_∞ is Stein (in fact isomorphic to $\mathbb{C}^2$), every leaf of the foliation $\mathcal{F}$ in question must accumulate on L_∞ so that it is captured by the dynamics of $\mathrm{Hol}(L_\infty)$. In this way it is possible to derive global properties of $\mathcal{F}$ from the local dynamics of $\mathrm{Hol}(L_\infty)$.

The generalization of this approach, however, involves some difficulties. First, for studying *generic* foliations (even when $n = 2$), one should be able to handle leaves having only "small" (e.g. cyclic) holonomy groups. In fact, since generic foliations have no algebraic invariant curve, dealing with "small" holonomy groups will be necessary for instance to obtain an open set of chaotic foliations.

On the other hand, for dimensions greater than 2, there is another additional difficulty: the holonomy groups involved, regardless of being "small" or "large", are subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ ($n \geq 2$, or more generally, pseudo-groups acting on the unit ball of $\mathbb{C}^n$) whose study is much harder than the 1-dimensional case.

In this paper we study, first, the dynamics of certain pseudo-groups acting on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ without a common fixed point which are obtained as "perturbations" of subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$. These pseudo-groups will later embody pseudo-groups generated by the holonomy groups of several leaves taken together. Their dynamics is essentially investigated through its affine part. Indeed, under some assumptions, we prove that the pseudo-groups approximate affine "pseudo-flows" as if they were a non-discrete subgroup of a Lie group (sections 2, 3, 4). Using these "pseudo-flows" it is easy to discuss the corresponding dynamics (sections 5, 6).

After realizing the pseudo-groups considered above in the "holonomy groupoid" of a foliation $\mathcal{F}$ in $\mathbb{CP}^{n+1}$, we shall get a good control of the dynamics of $\mathcal{F}$ in a certain region of $\mathbb{CP}^{n+1}$. Unfortunately there is no an argument like the "maximum principle" available (since the complement of a projective line in $\mathbb{CP}^{n+1}$, $n \geq 2$, is not Stein) in order to ensure that the leaves of $\mathcal{F}$ intersect this region. We then use an "induction trick" to deduce the global behavior of $\mathcal{F}$ by means of these local data (section 7).

Finally we should say that our techniques of producing flows associated to pseudo-groups acting on $\mathbb{B}^n$ also works for subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ as was recently observed by S.Lamy who adapted our method to $\mathrm{Diff}(\mathbb{C}^2, 0)$ in [La]. As a complement, we give the generalization of his result to the higher dimensional case. This may be of interest for studying foliations in $\mathcal{F}^d(\mathbb{CP}^n)$ possessing an algebraic invariant curve. Despite forming a "small" class these foliations are also important and they might permit a more detailed picture of "generic" properties (as it happens for $\mathbb{CP}^2$ in view of Hudai-Verenov and Il'yashenko's results). Yet, for these global questions, the techniques developed in section 7 might also be useful.

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1- Preliminary constructions within linear group of $\mathbb{C}^n$.

This section may be viewed as an illustration of some classical ideas which are going to be generalized to the non-linear context of pseudo-groups on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ in the next sections (§2, 3, 4, 5). In any case, some of these linear results will be needed latter. We are going to prove that, close to the identity $I \in GL(n, \mathbb{C})$, two generic matrices A and B generate a *large* subgroup G, in particular accumulating the whole of $SL(n, \mathbb{C})$.

Recall that two complex numbers $\lambda, \mu \in \mathbb{C}^*$ may generate a dense or a discrete subgroup Λ of $\mathbb{C}^*$ depending on their multiplicative $\mathbb{Z}$-dependence. In particular, dense groups as well as discrete groups occur arbitrarily close to any choice of the generators. In other words, groups like Λ cannot be stably (persistently) discrete (resp. dense, non-discrete) under pertubation of the generators. Nonetheless one should notice that for "most" (in the measure sense) choices of the generators, the resulting group is dense in $\mathbb{C}^*$.

Let us now consider subgroups $G \subset GL(n, \mathbb{C})$, $n \geq 2$, generated by two matrices A and B. Due to the determinant projection $det(G) = \{\lambda = det(A)\ ;\ A \in G\}$, which is the subgroup of $\mathbb{C}^*$ generated by the scalars $\lambda = \det(A)$ and $\mu = \det(B)$, it follows again that A, B cannot "stably" generate a dense subgroup of $GL(n, \mathbb{C})$. On the other hand, the so-called Schottky groups give examples of groups G whose projectivization $\widehat{G} = \{\widehat{C}\ ;\ C \in G\}$, being the subgroup of $PGL(n, \mathbb{C})$ generated by the projective transformations $\widehat{A} = Proj(A)$ and $\widehat{B} = Proj(B)$, is stably discrete (or persistently discrete): any subgroup of $PGL(n, \mathbb{C})$ generated by matrices A', B' close to A, B, respectively, is discrete as well. On the other hand the classical Zassenhaus Lemma (which holds for any finite-dimensional Lie group) ensures the existence of a neighborhood U of the identity such that any non-nilpotent subgroup G admitting a finite generating set contained in U is not discrete. Clearly this statement enables us to find examples of groups $G \subseteq PGL(n, \mathbb{C})$ which are persistently non-discrete.

Let us equip $GL(n, \mathbb{C})$ with the distance

$$dist(M, N) = \|M - N\| = \sup_{\|\underline{z}\|=1} \|M\underline{z} - N\underline{z}\| .$$

The key ingredient of Zassenhaus Lemma may be presented as follows:

Lemma 1.0. *There exist constants $\varepsilon_0, C_0 > 0$ such that any given matrices A, B satisfy*

$$\|[A, B] - I\| \leq C_0 \cdot \|A - I\| \cdot \|B - I\|,$$

provided that A, B are ε_0-close to the identity I.

Proof. The differentiable map $(A, B) \mapsto [A, B]$ equals I on $GL(n, \mathbb{C}) \times \{I\}$ and on $\{I\} \times GL(n, \mathbb{C})$. Hence its differential at (I, I) vanishes identically. Taylor Formula gives the desired inequality. $\square$

Equip $(GL(n, \mathbb{C}))^2$ with the product distance arising from the distance in $GL(n, \mathbb{C})$ introduced above.

Corollary 1.1. *There is $\varepsilon_1 > 0$ and an open set $\mathcal{U}_1$ (having total Lebesgue measure) of the ε_1-neighborhood of (I, I) in $GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) = (GL(n, \mathbb{C}))^2$ $(n \geq 2)$ with the following property: if a pair of matrices (A, B) belongs to $\mathcal{U}_1$, then A, B generate a non-discrete subgroup G of $GL(n, \mathbb{C})$ (i.e. the closure $\overline{G}$ of G contains a non-trivial real one-parameter group).*

Proof. Choose $\varepsilon_1 \leq \varepsilon_0$ such that $C_0\varepsilon_1 < 1/2$. Thus the sequence of iterated brackets

$$B_0 = B, \quad B_{k+1} = [A, B_k] = AB_kA^{-1}B_k^{-1} \quad \text{for} \quad k \in \mathbb{N}$$

converges to the identity at least as $\varepsilon_1/2^k$.

Now, assume that A has only simple eigenvalues and choose linear coordinates through which A is diagonal. Clearly, BAB^{-1} is also diagonal if and only if B permutes the eigendirections of A. We claim that, if ε_1 has been chosen small enough, then B cannot permute non trivially these directions (B has to be diagonal simultaneously with A). In particular, $[A,B]=I$ if and only if B is diagonal and, also, $[A,B]$ is a diagonal if and only if B is so. By induction, the sequence B_k is non trivial, i.e. $B_k \neq I$ for every k.

Let us now prove the claim. If $\varepsilon_1 < 1/n$, then any matrix B which is ε_1-close to the identity satisfies $|tr(B)-n|<1$. On the other hand, given a basis of unit vectors v_i which are parallel to the permuted n directions, then clearly (these vectors are close to one another and) we have by assumption $B(v_i)=\lambda_i \cdot v_{\sigma(i)}$ for a permutation σ and scalars λ_i which are ε_1-close to 1. In the basis (v_i), the matrix B (may lie far from the identity but) has the following special form: on each line and each column, all coefficients are zero except one which equals λ_i. The number of λ_i appearing along the diagonal is equal to the number of unpermuted directions. If this number is not n, say $i_0 \leq n-2$, then $|tr(B)-i_0| < i_0 \cdot \varepsilon_1$ which gives a contradiction.

We have proven that the group G is non discrete as long as A has only simple eigenvalues and at least one eigendirection which is not shared with B. These conditions on A and B are Zariski open and hence open and of total measure. According to Cartan's Theorem, the closure, $\overline{G}$, of G is a Lie group which, being not discrete, must have a non-trivial Lie algebra. The lemma is proved. □

It is convenient to remind the reader how to construct a non-trivial real one-parameter group (also called a *real flow*) contained in $\overline{G}$ (so that we can dispense with Cartan's Theorem). This construction may be outlined as follows: for a suitable sequence $N_k \in \mathbb{N}$ (necessarily tending to $+\infty$), the renormalized matrices $C_k = B_k^{N_k}$ have distance to the identity upper and lower bounded by positive constants, say $\frac{\varepsilon_1}{10} < C_k < \varepsilon_1$. Hence, maybe passing to a subsequence, it will converge to some $C \in GL(n,\mathbb{C})$ which is the time-one map of the one-parameter family $C^t = \lim_{k\to\infty} B_k^{[t\cdot N_k]}$, $t \in \mathbb{R}$, where $[\,\cdot\,]$ stands for the integral part.

Using the preceding statements, it is rather easy to ensure that, under further generic assumptions on the matrices A, B, the closure $\overline{G}$ is as large as possible, namely it maps onto $PGL(n,\mathbb{C})$. This will follow (Corollary 1.3) from:

Lemma 1.2. *There exists an open subset $\mathcal{U}_2$ of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ having total Lebesgue measure and such that, if $(A,B) \in \mathcal{U}_2$, then the only real vector subspaces $E \subset \mathfrak{gl}(n,\mathbb{C})$ invariant under conjugation by A and B, (s.t. $AEA^{-1} = BEB^{-1} = E$) either are homothetic, $E \subset \mathbb{C}\cdot I$, or do contain $\mathfrak{sl}(n,\mathbb{C})$.*

Proof. In fact, we prove that, under generic assumptions on A and B, any non homothetic (and in particular non zero) matrix $M \in \mathfrak{gl}(n,\mathbb{C})$ along with its conjugates by A and B generate a subspace E over $\mathbb{R}$ which contains $\mathfrak{sl}(n,\mathbb{C})$ (E being closed by definition).

We first impose the condition of Corollary 1.1, namely that A has only simple eigenvalues $\lambda_1, \ldots, \lambda_n$ and choose linear coordinates in which A is diagonal. The action of A by conjugation on $\mathfrak{gl}(n,\mathbb{C})$,

$$\mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C}) \; ; \; M \mapsto AMA^{-1},$$

is linear diagonal with eigenvalues $\lambda_{i,j} = \lambda_i/\lambda_j$ in the Kronecker basis $(\delta_{i,j})$ of the space of matrix. Let us also request that the λ_i's are pairwise distinct in norm and in argument (thus $\lambda_{i,j} = \lambda_i/\lambda_j$ does not belong to $\mathbb{R}$ for $i \neq j$). Furthermore no relation, other than the obvious one, of the type $|\lambda_i\lambda_j| = |\lambda_{i'}\lambda_{j'}|$ is verified. Finally we suppose that B (resp. B^{-1}) takes the n eigendirections of A to the complement of the n invariant hyperplanes of A (in the coordinate in which A is represented by a diagonal matrix, the preceding condition means that neither B nor B^{-1} have a vanishing entry). Obviously the pairs (A,B) fulfilling the conditions above define a "full measure" open set of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$.

Notice that the action of B by conjugation is characterized by

$$B\delta_{i_0,j_0}B^{-1} = (b_{i,i_0} \cdot \widetilde{b}_{j_0,j})$$

where $B = (b_{i,j})$ and $B^{-1} = (\widetilde{b}_{i,j})$.

We clearly have $\lambda_{i,j} = 1$ if and only if $i = j$. Denote by Δ_A and $\Delta_A^{\perp}$ the complementary subspaces of $\mathfrak{gl}(n,\mathbb{C})$ respectively consisting of diagonal matrices and matrices with zero entries on the diagonal. Denote by Π_A and $\Pi_A^{\perp}$ the respective projections.

Claim: E contains a matrix M_1 ($M_1 \neq (0)$) with vanishing entries on the diagonal.

Proof of the Claim. If M is not diagonal then it is enough to set $M_1 = AMA^{-1} - M$ which obviously satisfies our needs. Let us therefore suppose that M is diagonal *but non-homothetic*. In this case the matrix $M_0 = BMB^{-1}$ cannot be diagonal. Indeed suppose by contradiction that M_0 is also diagonal. Since M is not homothetic, it has at least 2 distinct eigenvalues and hence non-trivial eigenspaces which are direct sums of eigendirections of A. Because M, M_0 are diagonal, it follows that B permutes these eigenspaces. Therefore B is block-triangular (maybe after a permutation of the eigendirections of A) which gives us a contradiction since all the entries of B are different from *zero*. We conclude that M_0 is not diagonal. Now the claim follows from setting $M_1 = BM_0B^{-1} - M_0$.

Notice that $M_1 \in \Delta_A^{\perp} \cap E$ and we can write $M_1 = \sum_{(i,j)\in S} m_{i,j}\delta_{i,j}$ where S is the subset of $\{(i,j) \; ; \; i,j \in 1,\dots,n \quad \text{and} \quad i \neq j\}$ corresponding to the pairs (i,j) for which $m_{i,j} \neq 0$. As we have seen S is not empty.

Let us now consider the sequence of conjugates $A^kM_1A^{-k}$ ($k \in \mathbb{N}$). Note that this sequence tends *in direction* towards the complex line $\mathbb{C} \cdot \delta_{i_0,j_0}$ where (i_0,j_0) is the pair (i,j) corresponding to the $\lambda_{i,j}$ of maximum norm (note that the $\lambda_{i,j}$'s are distinct in norm for $i \neq j$). Recalling that no $\lambda_{i,j}$ is real (for $i \neq j$), it results that the complex line $\mathbb{C} \cdot \delta_{i_0,j_0}$ is completely accumulated by the sequence of "conjugate real lines" $\mathbb{R} \cdot A^kM_1A^{-k}$ and thus it is generated over $\mathbb{R}$ by their different limits.

Summarizing, we have proved that $E \subseteq \mathfrak{gl}(n,\mathbb{C})$ contains the complex line through some matrix $M_2 = \delta_{i_0,j_0}$ ($i_0 \neq j_0$) (and hence the complex subspace generated by the conjugates of this line under A, B). We shall deduce from these last conditions that $\mathfrak{sl}(n,\mathbb{C})$ is contained in E.

Since B and B^{-1} have no vanishing entry, the matrix $M_3 = BM_2B^{-1} = (b_{i,i_0}.\widetilde{b}_{j_0,j})$ also has all entries different from *zero*. Therefore $M_4 = AM_3A^{-1} - M_3$ has vanishing entries precisely on the diagonal. Hence the conjugates $A^k M_4 A^{-k}$ ($k = 1, \dots, n^2 - n$) are linearly independent and generate over $\mathbb{C}$ the whole $\Delta_A^{\perp}$. Finally, in order to verify that $\mathfrak{sl}(n,\mathbb{C}) \subseteq E$, it is enough to show that the mapping from $\Delta_A^{\perp} \subset E$ to $GL(n,\mathbb{C})$ given by $\widetilde{M} \mapsto \Pi_A(B\widetilde{M}B^{-1})$ has rank $n-1$. Actually it suffices to consider the restriction of this mapping to the space generated by $\delta_{1,2}, \dots, \delta_{1,n}$ (which is clearly contained in $\Delta_A^{\perp}$). Indeed note that the corresponding $(n-1) \times n-$matrix $(b_{i,1}.\widetilde{b}_{j,i})_{i=1,\dots,n; j=2,\dots,n}$, whose columns are the coefficients of $\Pi_A(B\delta_{1,j}B^{-1})$ in the basis $(\delta_{i,j})$ of Δ_A, is such that the subdeterminant $\det(b_{i,1} \cdot \widetilde{b}_{j,i})_{i,j=2,\dots,n}$ equals $\frac{b_{1,1} \cdot \cdots \cdot b_{n,1}}{\det(B)}$ which is not *zero* in view of the preceding assumptions on B. This accomplishes the proof of the lemma. $\square$

Corollary 1.3. *There is an open subset $\mathcal{U}_3$ with total Lebesgue measure in the ε_3-neighborhood of (I,I) in $GL(n,\mathbb{C}) \times GL(n,\mathbb{C}) = (GL(n,\mathbb{C}))^2$, $\varepsilon_3 > 0$, such that the subgroup G generated by any $(A,B) \in \mathcal{U}_3$ contains a dense subgroup of $SL(n,\mathbb{C})$. In fact the closure $\overline{G}$ of G has the form*

$$\overline{G} = \Lambda \times SL(n,\mathbb{C}),$$

where $\Lambda \subset \mathbb{C}^*$ is a closed subgroup of scalar matrices.

Proof. Set $\mathcal{U}_3 = \mathcal{U}_1 \cap \mathcal{U}_2$ so that any $(A,B) \in \mathcal{U}_3$ generates a non discrete subgroup $G \subseteq GL(n,\mathbb{C})$ (Corollary 1.1) whose closure $\overline{G}$, and hence whose associated Lie algebra $\mathfrak{G}$, are both invariant by A and B. Since the non trivial element of $\mathfrak{G}$ were constructed by means of commutators, it clearly belongs to $\mathfrak{sl}(n,\mathbb{C})$ (and hence is not homothetic). Therefore Lemma 1.2 implies that $\mathfrak{sl}(n,\mathbb{C}) \subseteq \mathfrak{G}$ and thus $SL(n,\mathbb{C}) \subseteq \overline{G}$. Since $SL(n,\mathbb{C})$ is simple (i.e. it has only trivial normal subgroups), we conclude that $[G,G]$ is dense in $SL(n,\mathbb{C})$. Finally a simple argument involving the obvious short exact sequence shows that $\overline{G}/SL(n,\mathbb{C})$ can be identified with $\Lambda = \sqrt[n]{\det(\overline{G})}$. Clearly, $\mathcal{U}_3$ is open and has total Lebesgue measure in the ε_1-neighborhood of $(I,I) \in GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$. $\square$

If Λ is moreover neither real nor contained in the unit circle, then we say that G is *rich*. This property is persistent (stable) and generic for matrices close to I as follows from:

Corollary 1.4. *The subset $\mathcal{U}_4 \subset (GL(n,\mathbb{C}))^2$ of those (A,B) which generate a rich subgroup $G \subseteq GL(n,\mathbb{C})$ is an open set. Furthermore the intersection of $\mathcal{U}_4$ with a suitable neighborhood of (I,I) has total Lebesgue measure in this neighborhood.*

In this sense, rich subgroups are the largest stable subgroups of $GL(n,\mathbb{C})$. Similarly, the subsets $\mathcal{U}_4' \subset (SL(n,\mathbb{C}))^2$ and $\mathcal{U}_4'' \subset (PGL(n,\mathbb{C}))^2$ of those pairs which generate a dense subgroup of $SL(n,\mathbb{C})$ (resp. $PGL(n,\mathbb{C})$) are open sets and have "full measure" in a neighborhood of (I,I).

Proof. Denoting by $\widehat{C}$ the projectivization of a matrix $C \in GL(n\mathbb{C})$, we clearly have:

$$\begin{cases} \mathcal{U}_4' = \{(A,B) \in (SL(n,\mathbb{C}))^2 \ ; \ (\widehat{A},\widehat{B}) \in \mathcal{U}_4''\}, \\ \mathcal{U}_4 = \{(A,B) \in (GL(n,\mathbb{C}))^2 \ ; \ (\widehat{A},\widehat{B}) \in \mathcal{U}_4'', \ \det(A) \notin \mathbb{R} \cup \mathbb{S}^1 \quad (\text{resp.} \det(B))\}. \end{cases}$$

Conversely

$$\mathcal{U}_4'' = \widehat{\mathcal{U}_4} = \widehat{\mathcal{U}_4'}.$$

We easily deduce from Corollary 1.3 that all the above mentioned sets contain a "full measure" open subset of the respective ε_3-ball of (I, I). Thus, only the part involving stability (i.e. the fact that these sets are open) needs further comments. Clearly it is enough to show that only one of them is open, for instance $\mathcal{U}_4''$. If the group $\widehat{G}$ generated by $(\widehat{A}, \widehat{B})$ is dense in $PGL(n, \mathbb{C})$, one can find words $(\widehat{A}', \widehat{B}')$ on the generators which "persistently generate" a dense subgroup of $PGL(n, \mathbb{C})$ (since $\mathcal{U}_4''$ contains an open set). Because these words depend smoothly on the generators, one immediately concludes the desired stability. □

Finally, we derive the n-dimensional generalization of [La]'s Proposition 4:

Corollary 1.5. *The subset $\mathcal{U}_5 \subset (GL(n, \mathbb{C}))^2$ of those (A, B) which generate a dense subgroup $G \subset GL(n, \mathbb{C})$ has total Lebesgue measure in a neighborhood of (I, I).*

Proof. If we denote by $\mathcal{U}_5^1$ the subset formed by those $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* = (\mathbb{C}^*)^2$ which generate a dense subgroup Λ of $\mathbb{C}^*$, then $\mathcal{U}_5$ does contain

$$\{(A, B) \in \mathcal{U}_3 \ ; \ (det(A), det(B)) \in \mathcal{U}_5^1\}.$$

The corollary is therefore established since $\mathcal{U}_5^1$ has total Lebesgue measure in $(\mathbb{C}^*)^2$. □

2- Pseudo-groups on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$.

In what follows $\underline{z} = (z_1, \dots, z_n)$ stands for the usual variable of $\mathbb{C}^n$ and $\|\underline{z}\|$ for the usual norm. Denote by $\mathbb{B}_r^n$ the ball of radius $r > 0$ centered at the origin of $\mathbb{C}^n$ and set $\mathbb{B}^n = \mathbb{B}_1^n$ where $\mathbb{B}_r^n = \{\underline{z} \ ; \ \|\underline{z}\| < r\}$; for a given mapping $f : \mathbb{B}_r^n \to \mathbb{C}^n$, $\|f\|_r$ stands for $\sup_{\underline{z} \in \mathbb{B}_r^n} \|f(\underline{z})\|$.

By definition a *pseudo-group* G on $\mathbb{B}^n$ will be any collection of biholomorphic transformations $f : U \to V = f(U)$ within the ball, $U, V \subset \mathbb{B}^n$, which is stable under restriction of the domain of definition (if $W \subset U$ then $f|_W \in G$), inversion $((f^{-1} : V \to U) \in G)$ and composition (if $(g : V \to W) \in G$ then $(g \circ f : U \to W) \in G$). The pseudo-group G *generated by* injective holomorphic transformations $f_1, \dots, f_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ will be the smallest pseudo-group on $\mathbb{B}^n$ containing the $f_k|_{f_k^{-1}(f_k(\mathbb{B}^n) \cap \mathbb{B}^n)}$. In practice, we shall mainly work with elements $f \in G$ restricted to sub-balls $\mathbb{B}_r^n$.

A natural topology on G is given by uniform convergence on compact sets. Namely a sequence $f_k : U_k \to V_k$, $k \in \mathbb{N}$, is said to converge to an element $f : U \to V$ in G if for any compact subset $K \subset U$, K is contained in U_k provided that $k >> 0$, and the sequence $f_k|_K$ restricted to K converges uniformly to $f|_K$. We can then talk about the closure $\overline{G}$ of a pseudo-group G. Actually we often consider elements h in $\overline{G}$, which are defined on some sub-ball $\mathbb{B}_r^n$ and, obtained as uniform limit of elements $f_k \in G$ on $\mathbb{B}_r^n$, i.e. $\lim_{k\to\infty} \|f - f_k\|_r = 0$. Clearly, if $\overline{G}$ acts transitively on $\mathbb{B}^n$, then G acts minimally on $\mathbb{B}^n$; the converse need not be true.

Let v denote a real vector field defined on an open set $U \subset \mathbb{B}^n$ and consider the pseudo-flow $\varphi_v^t : U_t \to V_t$, $t \in \mathbb{R}$ and $U_t, V_t \subset U$, obtained by integration of v (note that U_t might be empty for $|t| >> 0$). Clearly, this pseudo-flow is a pseudo-group of holomorphic transformations if and only if v is the real part of a complex holomorphic vector field X (i.e.

φ_v^t is the restriction to real time of the complex one parameter pseudo-group φ_X^T, $T \in \mathbb{C}$). More explicitly, in coordinates $\underline{z} = (z_1, \ldots, z_n)$ with $z_i = x_i + \sqrt{-1} \cdot y_i$, one has:

$$X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i} \longleftrightarrow v = \sum_{i=1}^{n} \Re e(f_i) \frac{\partial}{\partial x_i} + \Im m(f_i) \frac{\partial}{\partial y_i}$$

where the f_i's are holomorphic functions of $\underline{z}$, $f_i = \Re e(f_i) + \sqrt{-1} \cdot \Im m(f_i)$. In this case, the pseudo-flow φ_v^t is a continuous path with respect to the topologies in question. Conversely, it will be proved latter (Lemma 6.5) that any germ at $0 \in \mathbb{R}$ of continuous homomorphism from the additive group $(\mathbb{R}, +)$ to a pseudo-group of holomorphic transformations must be as above (i.e. it can be represented by the pseudo-flow of a vector field like v).

This suggests to consider the *real* Lie pseudo-algebra (which will be often referred to simply as the Lie algebra) $\mathfrak{G}$ associated to a closed pseudo-group $\overline{G}$ as the collection $\mathfrak{G}(U)$, for every open set $U \subset \mathbb{B}^n$, of the set of holomorphic vector fields X defined on U whose corresponding real pseudo-flow $\varphi_X^t : U_t \to V_t$ ($t \in \mathbb{R}$ small) is entirely contained in $\overline{G}$. In other words, given a pseudo-group G on $\mathbb{B}^n$, the Lie pseudo-algebra $\mathfrak{G}$ associated to its closure $\overline{G}$ consists of the complex holomorphic vector fields X defined on some $U \subset \mathbb{B}^n$ possessing the following property: every local diffeomorphism induced by the corresponding real pseudo-flow φ_X^t, $t \in \mathbb{R}$ fixed, can be uniformly approximated by a sequence h_k of elements of G on any compact subset of U_t (on which φ_X^t is defined).

It is easy to check that $\mathfrak{G}$ inherits a structure of a presheaf of real Lie algebras, i.e. the set of vector fields in $\mathfrak{G}(U)$ is stable under restrictions and any $\mathfrak{G}(U)$ is a real Lie algebra with respect to the Lie brackets of vector fields. Finally, each $\mathfrak{G}(U)$ is closed by uniform convergence on compact subsets and the entire collection $\mathfrak{G}$ is invariant under $\overline{G}$ (i.e. if $f : U \to V$ belongs to $\overline{G}$ then $\mathfrak{G}(V) = f_* \mathfrak{G}(U)$).

Remark. There is also an analogous definition for the *complex* Lie algebra $\mathfrak{G}_{\mathbb{C}}$ of $\overline{G}$: a holomorphic vector field X defined on U will belong to the complex Lie algebra of $\overline{G}$ (or to $\mathfrak{G}_{\mathbb{C}}(U)$) if the pseudo-group $\varphi_X^T : U_T \to V_T$ ($T \in \mathbb{C}$ small in norm) is entirely contained in $\overline{G}$. It is important to point out that the real Lie algebra of a closed pseudo-group $\overline{G}$ as above does not necessarily comes from a complex Lie algebra. Actually given a holomorphic vector field X such that its real pseudo-flow φ_X^t is contained in $\overline{G}$ (for $t \in \mathbb{R}$ small), there is no reason why the complex iteration φ_X^{it} should be also in $\overline{G}$ (for $t \in \mathbb{R}$ small). Thus it may happen that a closed pseudo-group $\overline{G}$ has *trivial* complex Lie algebra but *non-trivial* real Lie algebra. For instance, the pseudo-group generated on the Poincaré disc by two generic automorphisms is a pseudo-group of conformal transformations with closure $\overline{G} = PSL(2, \mathbb{R})$ and its Lie (pseudo-)algebra $\mathfrak{G} = \mathfrak{sl}(2, \mathbb{R})$ is a totally real submanifold of the complex manifold $\mathfrak{sl}(2, \mathbb{C})$. On the other hand note that the "realification" of the complex Lie algebra of $\overline{G}$ is always contained in the real Lie algebra $\mathfrak{G}_{\mathbb{R}}$ of $\overline{G}$. In particular, if $\mathfrak{G}_{\mathbb{C}}$ is non-trivial then so is $\mathfrak{G}_{\mathbb{R}}$. In this paper, we shall only deal with the real Lie algebra of $\overline{G}$.

Clearly, if the pseudo-flows belonging to $\overline{G}$ act transitively on $\mathbb{B}^n$ (we also say that $\mathfrak{G}$ acts transitively on $\mathbb{B}^n$), then G acts minimally on $\mathbb{B}^n$; the converse need not be true. Actually the transitivity of the Lie algebra $\mathfrak{G}$ associated to a pseudo-group has many other consequences for the dynamics of the pseudo-group as it will be shown in §6. This way of associating vector fields to pseudo-group has already been introduced in [Sh], [Na], [Reb] and [Be,Li,Lo] in order to study the dynamics of certain pseudo-groups in dimension 1.

The purpose of the next two sections is to provide *sufficient* conditions for a pseudo-group G, consisting of holomorphic transformations within the ball $\mathbb{B}^n$, to have non-trivial (real) Lie algebra. Our main result is:

Proposition 2.0. *Let $f_0 : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ be a contracting homothety, $f_0(\underline{z}) = \lambda \cdot \underline{z}$, $0 < |\lambda| < 1$ and g_0 be the identity map on $\mathbb{B}^n$. If λ is sufficiently close to 1 (i.e. f_0 close to identity), then there exists $0 < \varepsilon_\lambda < |\lambda - 1|$ such that any ε_λ perturbations $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ of f_0 and g_0 which have no common fixed point anymore, i.e.*

$$\|f - f_0\|_1 \ , \ \|g - g_0\|_1 < \varepsilon_\lambda \text{ and } f(\underline{z}) = g(\underline{z}) = \underline{z} \text{ for no } \underline{z} \in \mathbb{B}^n,$$

generate a pseudo-group G whose Lie algebra $\mathfrak{G}(\mathbb{B}^n)$ contains at least some non trivial vector field X (there is X in $\mathfrak{G}$ which does not vanish identically).

3- Catching non discrete pseudo-groups.

Let us first derive sufficient conditions on $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ so that the pseudo-group they generate on $\mathbb{B}^n$ contains a sequence of non trivial elements converging to the identity Id uniformly on some ball $\mathbb{B}^n_r$. We begin with an analogous, in our context, of Lemma 1.0 which can be found in [Gh].

Lemma 3.0. *Assume that ε_0, $0 < \varepsilon_0 < \frac{1}{4}$, is fixed and consider mappings $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ which are ε_0-close to the identity. Then the commutator $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$ induces a mapping $\mathbb{B}^n_{1-4\varepsilon_0} \hookrightarrow \mathbb{B}^n$ belonging to the pseudo-group generated by f, g. Furthermore the estimate below does hold for any $0 < \tau < 1 - 4\varepsilon_0$*

$$\|[f, g] - Id\|_{1-4\varepsilon_0-\tau} \leq \frac{2}{\tau}\|f - Id\|_1 \|g - Id\|_1 \, .$$

In [Gh], the left side of the corresponding inequality is $\sup\Big(\| f - id \|^2_{\mathbb{B}^n}, \| g - Id \|^2_{\mathbb{B}^n}\Big)$. Nonetheless our sharper inequality follows from similar arguments as the reader can check below.

Proof. Denote by Δ_f (resp. Δ_g) the variation $f - Id$ (resp. $g - Id$) which , by assumption, is bounded by ε_0 on $\mathbb{B}^n$. Clearly $f \circ g$ is well defined as mapping $\mathbb{B}^n_{1-2\varepsilon_0} \hookrightarrow \mathbb{B}^n$ and can be written as

$$f \circ g = Id + \Delta_f + \Delta_g + (\Delta_f \circ g - \Delta_f).$$

Cauchy Formula applied on small disks of radius τ, $0 < \tau < 1$, gives the following bound for the partial derivatives

$$\left\|\frac{\partial \Delta_f}{\partial z_i}\right\|_{1-\tau} \leq \frac{1}{\tau}\|\Delta_f\|_1 \, .$$

There is also an analogous estimate for R_g. By integration, one concludes that

$$\|\Delta_f \circ g - \Delta_f\|_{1-2\varepsilon_0-\tau} \leq \sup_i \left\|\frac{\partial \Delta_f}{\partial z_i}\right\|_{1-\varepsilon_0-\tau} \|\Delta_g\|_{1-2\varepsilon_0} \leq \frac{1}{\tau}\|\Delta_f\|_1 \|\Delta_g\|_1,$$

where $\tau < 1 - 2\varepsilon_0$. Therefore $f \circ g - g \circ f$, which is also well defined on $\mathbb{B}^n_{1-2\varepsilon_0}$, verifies

$$\|f \circ g - g \circ f\|_{1-2\varepsilon_0-\tau} \leq \|\Delta_f \circ g - \Delta_f\| + \|\Delta_g \circ f - \Delta_g\| \leq \frac{2}{\tau}\|\Delta_f\|_1\|\Delta_g\|_1.$$

On the other hand, $(g \circ f)^{-1}$ takes the ball of radius $1 - 4\varepsilon_0 - \tau$ to the interior of the ball of radius $1 - 2\varepsilon_0 - \tau$, so that $[f, g] - Id = (f \circ g - g \circ f) \circ (g \circ f)^{-1}$ satisfies

$$\|[f, g] - Id\|_{1-\tau-4\varepsilon_0} \leq \frac{2}{\tau}\|\Delta_f\|_1\|\Delta_g\|_1 \, .$$

□

In [Gh], this lemma is used for proving the convergence of a variant of the derived sequences h_k to the identity.

Corollary 3.1 (Ghys). *There is $\varepsilon_1 > 0$ such that if $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ are ε_1-close to the identity, then any sequence $h_k \in Z_k$, $k \in \mathbb{N}$, contained in the "derived" sequence of sets, which is inductively defined by*

$$Z_0 = \{f, f^{-1}, g, g^{-1}\} \quad \textit{and} \quad Z_{k+1} = \{[h_k, h'_k]; h_k, h'_k \in Z_k\},$$

is well defined on $\mathbb{B}^n_{1/2}$ as element of the pseudo-group G and converges to the identity uniformly on this sub-ball.

Proof. We prove that any $h_k \in Z_k$ is well defined on the ball of radius $r_k = \frac{1}{2} + \frac{1}{4.2^k}$ and uniformly bounded from the identity by $\epsilon_k = \epsilon_0/2^k$. The inequality of Lemma 3.0,

$$\|[h_k, h'_k] - Id\|_{r_k-4\epsilon_k-\tau_k} \leq \frac{2\epsilon_k}{\tau_k}\epsilon_k \ ,$$

inductively yields the desired estimates with $\tau_k = \epsilon_k/4$ and $\epsilon_k = 1/4(4 + 1/4)2^k$. □

Nevertheless, it is a hard task to verify the non triviality of such sequence. The non solvability of G (when it makes sense) is not enough in general. Moreover, the existence of sequences accumulating uniformly the identity within G is still not sufficient to guarantee the existence of pseudo-flows, as it is shown by the next example.

Example 3.2. For arithmetical reasons (see [Be,Ce,L-N], Cor.4.2, p.262), the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ generated by

$$f(z) = z/(1 - z) \quad \text{and} \quad g(z) = z/\sqrt{1 - z^2},$$

where the determination $\sqrt{1} = 1$ is chosen, is free (as group of germs fixing $0 \in \mathbb{C}$). In particular, any sequence h_k constructed as in Corollary 3.1 is non trivial. On the other hand, since f, g are tangent to identity at their common fixed point 0, they become well defined on the unit disc $\mathbb{B}^1$ and arbitrarily close to the identity after conjugation by a contracting homothety. By virtue of Corollary 3.1, the sequences h_k are non trivial sequences accumulating the identity on a neighborhood of 0 (within the pseudo-group G generated by

f and g). On the other hand, it follows from the composition and inversion rules of formal power series that the Taylor coefficients of any word in f and g are integers, as well as for f and g. Therefore the pseudo-group G cannot accumulate a non trivial one parameter pseudo-group uniformly on a neighborhood of 0! Nevertheless, Nakai's theorem (see [Na]) asserts the existence of (many!) such pseudo-flows in the closure of G at the neighborhood of any point $z_0 \neq 0$ sufficiently close to 0.

Contrasting with the finite dimensional case (see §1), Lemma 3.0 is clearly not sufficient (without further assumptions) to imply that the sequences $h_k \in S_k$, $k \in \mathbb{N}$, contained in the "central" sequence of sets

$$S_0 = \{f, f^{-1}, g, g^{-1}\} \quad \text{and} \quad S_{k+1} = \{[h_0, h_k]; h_0 \in S_0, h_k \in S_k\},$$

are well defined as elements of the pseudo-group G and, furthermore, converge uniformly towards the identity. Indeed, if one takes for f, g translations arbitrarily close to the identity, then we clearly always find an integer k_0 for which any word $h_{k_0} \in S_{k_0}$ has empty domain of definition. Although this counterexample is somehow trivial (any S_k consists of the identity transformation for $k > 0$) a small "non nilpotent" perturbation of it, within the affine group, will have the same property and will provide serious obstructions to the existence of a Zassenhaus Lemma for pseudo-groups.

In order to ensure that the pseudo-group G is not discrete, our idea is to require also that f is a contraction. So, instead of dealing with the sequence $g_0 = g$, $g_{k+1} = [f, g_k]$ whose common domain of definition of the elements is shrinking, we will be able to "restore" (i.e. "re-enlarge") domains, little by little, working with an alternate sequence of the type $h_0 = g$, $h_{k+1} = f^{-N}[f, h_k]f^N$. This approach will be suceeded only if the distortion of f can be bounded. More precisely, if we denote by $0 < \lambda_- \leq \lambda_+$ the lower and upper bounds for directional derivatives of f given by

$$\lambda_- = \inf_{\|v\|=1, \|\underline{z}\|<1} \|\frac{\partial}{\partial t} f(\underline{z} + tv)|_{t=0}\| \quad \text{and} \quad \lambda_+ = \sup_{\|v\|=1, \|\underline{z}\|<1} \|\frac{\partial}{\partial t} f(\underline{z} + tv)|_{t=0}\| ,$$

then we will require that

$$(*) \quad 0 < (\lambda_+)^2 < \lambda_- \leq \lambda_+ < 1 .$$

The assumption above is strong but sufficient for our purpose. The reader may already notice that perturbations f of $f_0(\underline{z}) = \lambda.\underline{z}$ considered in Proposition 2.0 satisfy this requirement. This $(*)$ condition will also imply that f can be linearized by holomorphic change of coordinates at a neighborhood of its fixed point $\underline{0}$ (Poincaré's Theorem, see Lemma 3.5).

Lemma 3.3. *There is a $\varepsilon_3 > 0$ such that, if $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ are ε_3-close to the identity with f fixing $\underline{0}$ and satisfying $(*)$, there exists a (uniform) $N \in \mathbb{N}$, depending only on f, such that the sequence of one-to-one holomorphic mappings inductively given by*

$$g_0 = g \quad \textit{and} \quad g_{k+1} = f^{-N} \circ [f, g_k] \circ f^N$$

is well defined on $\mathbb{B}^n_{1-2\varepsilon_3}$ as element of the pseudo-group G. Moreover this sequence converges uniformly towards the identity when k tends to infinity.

Further conditions, needed to guarantee in addition that $g_k \neq Id$ for every k, will be given later on.

Proof. Fix $0 < \varepsilon_3 < \varepsilon_0$ so that Lemma 3.0 holds. Since $\|f^N\|_1 \leq \lambda_+^N$, for $N \in \mathbb{N}$ large enough, one has

$$\|f^N\|_1 = 1 - 4\varepsilon_3 - \tau, \quad \text{for some} \quad \tau \in]0, 1-4\varepsilon_3[\,,$$

so that f^N contracts the unit ball into the ball $\mathbb{B}^n_{1-4\varepsilon_3-\tau} \subseteq \mathbb{B}^n_{1-4\varepsilon_3}$. In this smaller ball we can apply Lemma 3.0 to obtain

$$\|[f,g] - Id\|_{1-4\varepsilon_3-\tau} \leq \frac{2\varepsilon_3}{\tau}\|g - Id\|_2 \,.$$

Hence

$$\|[f,g] \circ f^N - f^N\|_1 \leq \frac{2\varepsilon_3}{\tau}\|g - Id\|_1 \,.$$

Suppose first that we are allowed to iterate N times f^{-1} from $[f,g] \circ f^N(\mathbb{B}^n)$. Then one has the estimate below

$$\|f^{-N} \circ [f,g] \circ f^N - Id\|_1 \leq \frac{2\varepsilon_3}{\tau\lambda_-^N}\|g - Id\|_1 \,.$$

Since $\lambda_+^2 < \lambda_-$ and $\lambda_-^N \leq \|f^N\|_1 = 1 - 4\varepsilon_3 - \tau$, it results that

$$\|g_{k+1} - Id\|_1 \leq \frac{2\varepsilon_3}{\tau(1-4\varepsilon_3-\tau)^2}\|g_k - Id\|_1 \,.$$

The coefficient $c(\tau) = \frac{2\varepsilon_3}{\tau(1-4\varepsilon_3-\tau)^2}$ attains its minimum for $\tau_0 = \frac{1}{3}(1-4\varepsilon_3)$ which yields $c(\tau_0) = \frac{\varepsilon_3}{2}\left(\frac{3}{1-4\varepsilon_3}\right)^3$. If ε_3 was chosen small enough then $c(\tau_0) < 1$.

In order to deduce the proof of Lemma 3.3 from these estimates, we have to justify first that we can choose N so that $\tau(N) = 1 - 4\varepsilon_3 - \|f^N\|_1 \sim \tau_0$ and then $c(\tau(N)) \sim c(\tau_0) < 1$, and secondly that, without loss of generality, f^{-1} can be supposed well defined (as element of the pseudo-group) on the whole ball of radius $1+\varepsilon_3$ (so that the iterations above make sense).

First justification: for $\tau_1 = \frac{1}{2}(1-4\varepsilon_3)$, ε_3 can be chosen so that we still have $c(\tau_1) = 2\varepsilon_3\left(\frac{2}{1-4\varepsilon_3}\right)^3 < 1$. Let us verify that we can choose N so that $\tau_0 \leq \tau(N) \leq \tau_1$, i.e. $\frac{1}{2}(1-4\varepsilon_3) \leq 1 - 4\varepsilon_3 - \tau(N) \leq \frac{2}{3}(1-4\varepsilon_3)$. Since $\lambda_+^2 < \lambda_- \leq 1 - 4\varepsilon_3 - \tau(N) \leq \lambda_+$, it is enough to find $N \in \mathbb{N}$ satisfying

$$\frac{1}{2}\log\left(\frac{1}{2}(1-4\varepsilon_3)\right) \leq N \cdot \log(\lambda_+) \leq \log\left(\frac{2}{3}(1-4\varepsilon_3)\right) \,.$$

Since $|\lambda_+ - 1| \leq \varepsilon_3$, it is obvious that, as long as ε_3 is very small, $\log(\lambda_+)$ is close to 0 so that there exists the desired integer N.

Second justification: starting with f and g satisfying the assumptions of Lemma 3.3, it is clearly that f^{-1} is well defined on $\mathbb{B}^n_{1-\varepsilon_3}$ as element of the pseudo-group as well as f and g. After a homothety of ratio $1-2\varepsilon_3$, so that the sub-ball of the statement is now the unit ball, our pseudo-group is actually defined on the ball of radius $\frac{1}{1-2\varepsilon_3} > 1$ and f, g and f^{-1} become well defined as element of G. Finally they are also $\widetilde{\varepsilon}_3$-close to the identity on the intermediate ball of radius $1 + \widetilde{\varepsilon}_3$ with $\widetilde{\varepsilon}_3 = \frac{\varepsilon_3}{1-2\varepsilon_3}$. □

Remark. The condition $(*)$ means that the distortion coefficient

$$\delta = \frac{\log(\lambda_-)}{\log(\lambda_+)}$$

which is always ≥ 1 for a uniform contraction, is actually bounded by 2. The preceding proof may be re-arranged so that it is possible to replace $(*)$ by $\delta < \delta_0$ for a fixed $\delta_0 >> 0$. Nonetheless notice that ε_3 depends on this bound and asymptotically $\varepsilon_3(\delta_0) \sim 1/\delta_0$ so that one certainly cannot avoid any hypothesis about distortion.

The sequence g_k constructed in Lemma 3.3 is non trivial under generic assumptions on f and g, due to:

Lemma 3.4. *Let $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be as in Lemma 2.3 and consider their differentials at $\underline{0}$*

$$A = D_{\underline{0}} f \quad \text{and} \quad B = D_{\underline{0}} g \ .$$

The sequence g_k constructed in Lemma 3.3 is non trivial, i.e. $g_k \neq Id$ for every k, provided that one of the following conditions holds:

(i) *$g(\underline{0}) \neq \underline{0}$;*
(ii) *$g(\underline{0}) = \underline{0}$, A has only simple eigenvalues and $[A, B] \neq I$ (or equivalently A and B are not simultaneously diagonalizable);*
(iii) *$g(\underline{0}) = \underline{0}$, $[A, B] = I$ and $[f, g] \neq Id$ (or equivalently f and g are not simultaneously linearizable).*

Before proving Lemma 3.4, let us complete its statement with the following lemma.

Lemma 3.5. *Suppose that $f : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$, $f(\underline{0}) = \underline{0}$, satisfies $(*)$. Then f is linearizable by a holomorphic change of coordinates. In other words, there exists a holomorphic embedding $\Phi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ fixing $\underline{0}$ such that*

$$\Phi^{-1} \circ f \circ \Phi(\underline{z}) = A \cdot \underline{z} \ .$$

Moreover, Φ is unique up to right composition by a matrix commuting with A.

It follows that, in case (iii) of Lemma 2.4, g almost never commutes with f.

Proof of Lemma 3.5. Denote by $\lambda_1, \dots, \lambda_n$ the spectrum of the linear part A of f. Modulo a permutation of indices, one has

$$0 < \lambda_- \leq |\lambda_1| \leq \cdots \leq |\lambda_n| \leq \lambda_+ < 1.$$

Furthermore $(*)$-condition clearly implies that

$$0 < |\lambda_n|^2 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 \ .$$

In particular, there exists no Poincaré resonance between these eigenvalues, i.e.

$$|\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}| < |\lambda_{i_0}|$$

as long as $k \geq 2$. On the other hand, this spectrum obviously belongs to the Poincaré domain since it consists only of contractions. Hence, due to Poincaré Theorem (see [Ar,Il], p.72), f is linearizable by a holomorphic germ of diffeomorphism Φ at $\underline{0}$. Clearly, Φ is well defined up to right composition by germ of diffeomorphism commuting with A and, in particular, after composition by a convenient homothety, we can assume that Φ is defined on $\mathbb{B}^n$. Finally the formal part of the proof of Poincaré Theorem relies on the fact that absence of resonance implies absence of non linear germ of diffeomorphism Ψ commuting with A. □

Proof of Lemma 3.4. Case (i): if g does not fix $\underline{0}$, then $g^{-1}(\underline{0}) \neq \underline{0}$ is not fixed by f, i.e. $g^{-1} \circ f^{-1}(\underline{0}) = g^{-1}(\underline{0}) \neq f^{-1} \circ g^{-1}(\underline{0})$, which implies that $[f,g](\underline{0}) \neq \underline{0}$ and hence $f^{-N} \circ [f,g] \circ f^N(\underline{0}) \neq \underline{0}$. Using induction we see that $g_k(\underline{0}) \neq \underline{0}$ for every k.

Case (ii): since the linear part of $[f,g]$ is given by $[A,B]$, this case promtly follows from the proof of Corollary 1.1.

Case (iii): if g does not commute with f, it follows that $[f,g]$, whose linear part is given by $[A,B] = I$, should be a (non trivial) map which is tangent to the identity. In the coordinate given by Lemma 3.5 where $f = A$ is linear, it is clear that $[f,g]$ is still a (non trivial) map tangent to the identity and thus it is not linear. Employing again Lemma 3.5, it follows that $[f,g]$ does not commute with $A = f$ and proof follows by induction. □

In the next section we shall work through the coordinate Φ given by Lemma 3.5 and hence deal only with the linear contraction A and the sequence $\{h_k = \Phi^{-1} \circ g_k \circ \Phi\}$ (which is converging to the identity as well). All these mappings can be supposed defined on $\mathbb{B}^n$ without loss of generality (just compose Φ to the right with a convenient homothety). Clearly, any pseudo-flow uniformly accumulated at the neighborhood of $\underline{0}$ by words in A and h_k will give rise to a pseudo-flow in the closure of G which, after conjugation by a convenient iterate of the contraction f, can be supposed defined on the whole ball $\mathbb{B}^n$ as well.

4- Catching pseudo-flows in the closure of non discrete pseudo-groups.

In this section, for the sake of notations, we shall make no distinction between A thought of as a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} ,$$

or as the induced map $\mathbb{B}^n \hookrightarrow \mathbb{C}^n$. Recall that A is supposed to satisfy

$$(*) \quad 0 < |\lambda_n|^2 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 .$$

Furthermore $h_k : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ will be a sequence of one-to-one holomorphic mappings converging to the identity uniformly on the whole ball $\mathbb{B}^n$. Note that we do not need anymore the fact that A is close to the identity throughout the sequel.

In general, as shown by example 3.2, the fact that the h_k's converge uniformly to the identity is not sufficient to derive pseudo-flows. For instance, the natural strategy would consist of considering some sequences of iterates $\varphi_k = (h_k)^{N_k}$ for a suitable sequence of integers $N_k \to \infty$, so that φ_k remains say ε-close to the identity and $\varepsilon/2$ far from the identity on $\mathbb{B}^n_{1-\varepsilon}$. Modulo passing to a subsequence, Montel Theorem asserts that φ_k converges to a

(ε-close to the identity) transformation φ uniformly on compact subsets. By construction, φ is the time-one-map of the real pseudo-flow φ^t given, for each $t \in [0,1]$, by uniform convergence on any compact subset of a convenient subsequence of $\varphi_k^t = (h_k)^{[t \cdot N_k]}$ where $[\cdot]$ stands for the integral part. The only problem, occuring for instance in example 3.2, is that $\{(h_k)^{N_k}\}$ may converges to the identity on any compact subset of $\mathbb{B}^n_{1-\varepsilon}$, so that φ^t coincides with the identity. Later on (see Proposition 4.6), we shall give useful additional sufficient conditions on h_k in order to avoid such phenomenon, but before, we propose an alternate strategy to construct pseudo-flows by using again the contraction A so that the proof of Proposition 2.0 can be quickly deduced.

Lemma 4.0. *Let A be a diagonal matrix satisfying $(*)$ and $h_k : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be a sequence of holomorphic mappings uniformly tending to Id. Suppose moreover that one of the following conditions is satisfied:*

(i) *$h_k(\underline{0}) \neq \underline{0}$ for all k;*
(ii) *$h_k(\underline{0}) = \underline{0}$ for every k, $|\lambda_1| < \cdots < |\lambda_n|$ and the linear part $D_{\underline{0}} h_k$ is not lower triangular for any k.*

Then, for any $\varepsilon > 0$, modulo passing to a subsequence of h_k, there exists a sequence of positive integers N_k such that the renormalized sequence

$$A^{-N_k} \circ h_k \circ A^{N_k}$$

is well defined on the ball $\mathbb{B}^n_{1-2\varepsilon}$ for $k >> 0$ and converges uniformly to a non trivial affine transformation h_∞ which is ε-close to the identity on $\mathbb{B}^n_{1-2\varepsilon}$.

Proof. This Lemma relies on the following remark. The action of A by conjugation increases the affine part of h_k because of (i) or (ii) while it decreases its non linear part because of $(*)$. Let us make precise and prove these facts. Let

$$h_k = h_k^{Aff} + h_k^{\geq 2} ,$$

where h_k^{Aff} denotes the affine part of h_k and $h_k^{\geq 2}$ stands for the remaining non linear terms. The sequence of affine mappings h_k^{Aff} converges to the identity as one can see from estimating their coefficients by means of Cauchy Formula. Hence the sequence $h_k^{\geq 2}$ formed by the higher order terms goes to *zero* and satisfies

$$\|h_k^{\geq 2}(\underline{z})\| \leq |\underline{z}|^2 \cdot \|h_k^{\geq 2}\|_1$$

for any $\underline{z} \in \mathbb{B}^n$ and any k. Clearly $(*)$ condition implies:

$$\|A^{-1} \circ h_k^{\geq 2} \circ A(\underline{z})\| \leq \frac{|\lambda_n|^2}{|\lambda_1|} |\underline{z}|^2 \cdot \|h_k^{\geq 2}\|_1 < \|h_k^{\geq 2}\|_1 \ .$$

On the other hand, if $h^{Aff}(\underline{z}) = T + C \cdot \underline{z}$ is an affine transformation where $T = (t_i)_i$ denotes the translation part and $C = (c_{i,j})_{i,j}$ denote the linear part, then

$$A^{-1} \circ h^{Aff} \circ A = A^{-1}T + A^{-1}CA = (\frac{t_i}{\lambda_i})_i + (\frac{\lambda_j}{\lambda_i} c_{i,j})_{i,j},$$

so that condition (i), as well as condition (ii), implies that at least one of the Taylor coefficients of $A^{-N} \circ h_k^{Aff} \circ A^N$ increases exponentially when $N \to +\infty$. Of course $\|A^{-N} \circ h_k^{Aff} \circ A^N\|_{\mathbb{B}^n}$ increases as well.

Equip $\mathrm{Aff}(\mathbb{C}^n)$ with the metric induced by $\|\cdot\|_1$ and, for $\varepsilon > 0$ small enough, denote by U_ε the ε-neighborhood of Id in this Lie group. The action by conjugation of A on $\mathrm{Aff}(\mathbb{C}^n)$ fixes Id so that there exists an open neighborhood $V_\varepsilon \subset U_\varepsilon$ of the Id in $\mathrm{Aff}(\mathbb{C}^n)$ such that $A^{-1}V_\varepsilon A$ remains in U_ε.

For k large enough h_k^{Aff} belongs to V_ε. For N sufficiently large, we have seen that $A^{-N} \circ h_k^{Aff} \circ A^N$ lies on the complement of U_ε, and hence away from V_ε. Thus, if one defines N_k as the smallest positive integer for which $A^{-N_k} \circ h_k^{Aff} \circ A^{N_k}$ does not belong to V_ε, the sequence of affine mappings $A^{-N_k} \circ h_k^{Aff} \circ A^{N_k}$ will remain in the relatively compact annulus $U_\varepsilon \setminus V_\varepsilon$. Up to choosing a subsequence, the sequence $A^{-N_k} \circ h_k^{Aff} \circ A^{N_k}$ converges towards some affine transformation h_∞^{Aff} uniformly on $\mathbb{B}^n$. Clearly h_∞^{Aff} is ε-close to the identity but lies on the complement of V_ε so that it is not trivial.

Therefore one gets

$$\begin{gathered}\|A^{-N_k} \circ h_k \circ A^{N_k} - h_\infty^{Aff}\|_1 \leq \\ \leq \|A^{-N_k} \circ h_k^{Aff} \circ A^{N_k} - h_\infty^{Aff}\|_1 + \|A^{-N_k} \circ h_k^{\geq 2} \circ A^{N_k}\|_1 \\ \leq \|A^{-N_k} \circ h_k^{Aff} \circ A^{N_k} - h_\infty^{Aff}\|_1 + \|h_k^{\geq 2}\|_1 \,.\end{gathered}$$

The proposition immediately follows from the estimates above. □

Remark 4.1. Here, if we replace $(*)$ condition by

$$0 < |\lambda_n|^\delta < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 \ ,$$

for some $\delta > 2$, then one should truncate $h_k = h_k^{<\delta} + h_k^{\geq\delta}$ where $h_k^{<\delta}$ denote the Taylor jet of order $\delta - 1$ of h_k and $h_k^{\geq\delta}$ the remaining higher order terms which satisfy

$$\|h_k^{\geq\delta}(\underline{z})\| \leq |\underline{z}|^\delta \cdot \|h_k^{\geq\delta}\|_1 \ .$$

Then the same proof shows that, under condition (i) or (ii), for any $\varepsilon > 0$, some subsequence of the type $A^{-N_k} \circ h_k \circ A^{N_k}$ tends uniformly to a polynomial transformation $h_\infty^{<\delta}$ of order $\delta - 1$. But $h_\infty^{<\delta}$ does not lie anymore on a Lie group contrary to the affine case above, and the next arguments cannot immediately be adapted.

Corollary 4.2. *Let A and h_k as in Lemma 4.0. Then there exists some non trivial affine pseudo-flow in the closure of the pseudo-group generated by A and the h_k.*

Proof. Applying Lemma 4.0 to a sequence $\varepsilon_k \to 0$, we construct a non trivial sequence h_k^{Aff} of affine transformations tending to the identity uniformly on some fixed ball, say $\mathbb{B}^n$, which is in the closure of the pseudo-group. Then the usual strategy to derive one parameter subgroups of non discrete Lie groups works here.

Equip again $\mathrm{Aff}(\mathbb{C}^n)$ with the metric induced by $\|\cdot\|_1$. For $\varepsilon > 0$ small enough, the ε-neighborhood U_ε of Id in this Lie group is diffeomorphic to a neighborhood of zero of the

Lie algebra via the exponential map and hence is such that any element $h^{Aff} \in U_\varepsilon$ escapes after a finite number of iterations, namely there is $N \in \mathbb{N}$ such that $(h^{Aff})^N \notin U_\varepsilon$. Also there exists an open neighborhood $V_\varepsilon \subset U_\varepsilon$ of the Id in $\mathrm{Aff}(\mathbb{C}^n)$ such that any $h^{Aff} \in V_\varepsilon$ satisfies $h^{Aff} \circ h^{Aff} \in U_\varepsilon$.

For k large enough h_k^{Aff} belongs to V_ε. Define N_k as the smallest positive integer for which $(h_k^{Aff})^{N_k}$ does not belong to V_ε, so that this renormalized sequence of affine mappings remains in the relatively compact annulus $U_\varepsilon \setminus V_\varepsilon$. Modulo passing to a subsequence, the sequence $(h_k^{Aff})^{N_k}$ tends uniformly to some affine transformation h_∞^{Aff} on $\mathbb{B}^n$ which is ε-close to identity and lies on the complement of V_ε so that it does not coincide with the identity. By construction, h_∞^{Aff} is the time-one-map of the real pseudo-flow φ^t given, for each $t \in [0,1]$, by uniform convergence on any compact subset of an appropriate subsequence of $\varphi_k^t = (h_k^{Aff})^{[t \cdot N_k]}$ where $[\cdot]$ stands for the integral part. □

Proof of Proposition 2.0. First, choose $|\lambda - 1| \leq \varepsilon_0$ with $\varepsilon_0 < \varepsilon_3$ given by Lemma 3.3. For a given $0 < \tau < 1$, there clearly exists $0 < \varepsilon_\lambda < |\lambda - 1|$ such that, for any ε_λ-perturbation f of f_0, any directional derivative of f remains $|\lambda-1|^2$ close to λ on some ball $\mathbb{B}^n_{1-\tau}$. Modulo rescaling ε_0, this implies that f satisfies condition $(*)$ on $\mathbb{B}^n_{1-\tau}$. Thus we can apply lemmas 3.3 and 3.4 to f and g and derive some non trivial sequence $g_{k+1} = f^{-N} \circ [f,g] \circ f^N$ converging to the identity uniformly on a sub-ball and satisfying $g_k(\underline{0}) \neq \underline{0}$. On the other hand, f is linear through the coordinate Φ given by Lemma 3.5. If the linear part A of f is diagonalizable, we can suppose, without loss of generality, that $\Phi^{-1} \circ f \circ \Phi = A$ is diagonal and satisfies $(*)$. Moreover $h_k = \Phi^{-1} \circ g_k \circ \Phi$ converges to the identity uniformly on $\mathbb{B}^n$ and satisfies condition (i) of Corollary 4.2 which finishes the proof.

To accomplish the proof of Proposition 2.0 in full generality, it remains to show that proofs of Lemma 4.0 and Corollary 4.2 in the case (i) also hold when A is no more diagonal but has Jordan blocks. This is rather easy and left to the reader. □

Remark 4.3. It is possible to improve the estimates of Lemma 3.3 and 4.0 so that we may ensure that the vector field constructed is actually non singular at $\underline{0}$, and even is a translation. This can be carried out by writing $g = T \circ \tilde{g}$, where T stands for the translation part of g and $\tilde{g}$ is the remaining part fixing $\underline{0}$. We then consider the same decomposition for $g' = [A,g] = T' \circ \tilde{g}'$ with regard to the following formula

$$[A, T \circ \tilde{g}] = [A,T] \circ [T,[A,\tilde{g}]] \circ [A,\tilde{g}] \ .$$

It is possible to manage these terms so that the central double bracket $[T,[A,\tilde{g}]]$ becomes neglectable and $T' \sim [A,T]$ and $\tilde{g}' \sim [A,\tilde{g}]$. So, considering the action by conjugation of A on T' and $\tilde{g}'$, as in Lemma 4.0, we are able to find some sequence $g_{k+1} = A^{-N_K} \circ [A,g] \circ A^{N_K}$ uniformly tending to a translation (maybe passing to a subsequence). Anyway, this would had led us to many more estimates, at least in order to control the domains of definition. In the sequel we shall derive non singular vector fields just by considering conjugations under A and g with additional generic assumptions (needed later) in Proposition 5.2.

In the following, we give complementary results that can be obtained from our work but which are not strictly needed to our Main Theorem. The first one will be interpreted in Corollary 5.3 as an analogous of Proposition 2.0 for the common fixed point case. Some of these statements may be useful for other similar problems.

Proposition 4.4. *There is a $\varepsilon > 0$ such that if $A, B \in Gl(n, \mathbb{C})$ are ε-close to the identity matrices satisfying:*

- *A is diagonal with eigenvalues satisfying $0 < |\lambda_n|^2 < |\lambda_1| < \cdots < |\lambda_n| < 1$,*
- *B is not lower triangular,*

then for any $f, g \in Diff(\mathbb{C}^n, \underline{0})$ with respective linear parts A and B, there exists some ball $\mathbb{B}^n_r$ on which f and g are well defined, one-to-one and the pseudo-group G generated by them has a non trivial linear pseudo-flow in its closure.

Proof. Fix $\varepsilon > 0$ and A, B, f and g as in the statement. Up to homothety, we can suppose that f and g are well defined on the ball $\mathbb{B}^n$ and also arbitrarily close to A and B respectively. In particular, if ε was chosen small enough, f and g are ε_3-close to identity, one-to-one and f is a contraction satisfying $(*)$. Therefore lemmas 3.3 and 3.4 do apply to provide a non trivial sequence g_k converging to the identity uniformly on some sub-ball. Notice that, if the linear part of g_k is not lower triangular for every k, then the proof is finished by using Corollary 4.2 (similarly to the proof of Proposition 2.0 above). In order to check that the linear part of g_k is not lower triangular note that if B is not lower triangular then the same holds for $[A, B]$. Indeed if $[A, B] = T$ were lower triangular, then one would have $B^{-1}(T^{-1}A)B = A$. Employing an argument similar to the one used in the proof of Corollary 1.1 (replacing invariant directions by invariants flags), the last claim implies that B is the product of a lower triangular matrix and a permutation matrix. Actually B will be lower triangular provided that ε is sufficiently small. This gives us the desired contradiction. $\square$

The following may be found in [Gh] (Lemma 2.5).

Lemma 4.5 (Ghys). *Fix ε satisfying $0 < \varepsilon < 1/k$ for a positive integer $k \in \mathbb{N}^*$. Any given transformation $f = Id + \Delta_f : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ $\varepsilon-$close to the identity is such that the power f^k is well defined on $\mathbb{B}^n_{1-k\varepsilon}$ and the estimate below does hold*

$$\|f^k - Id - k\Delta_f\|_{1-k\varepsilon-\tau} \leq \frac{k(k-1)}{2\tau} \left(\|f - Id\|_1\right)^2 ,$$

for any $0 < \tau < 1 - k\varepsilon$.

Proof. It is similar to Lemma 3.0. Let

$$\begin{aligned} f^k &= Id + \Delta_f + \Delta_f \circ f + \Delta_f \circ f^2 + \cdots + \Delta_f \circ f^{k-1} \\ &= Id + k\Delta_f + (k-1)(\Delta_f \circ f - \Delta_f) + (k-2)(\Delta_f \circ f^2 - \Delta_f \circ f) + \cdots \\ &\qquad + (\Delta_f \circ f^{k-1} - \Delta_f \circ f^{k-2}) \end{aligned}$$

On the other hand, by Cauchy Formula:

$$\|\Delta_f \circ f - \Delta_f\|_{1-2\varepsilon-\tau} \leq \frac{1}{\tau}\left(\|\Delta_f\|_1\right)^2 .$$

The result follows. $\square$

Proposition 4.6. *Let $h_k : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be a sequence of injective holomorphic mappings uniformly converging to the identity and consider the sequence of positive scalars $\epsilon_k = \|\Delta_k\|_1$, $\Delta_k = h_k - Id$, which obviously tends to 0. Suppose that the bounded sequence of mappings $\frac{\Delta_k}{\epsilon_k}$ is not uniformly tending to the constant map which equals 0. Then, for any $\tau, \varepsilon > 0$, $\tau + \varepsilon < 1$, there exists a non trivial holomorphic vector field X on the ball $\mathbb{B}^n_{1-\tau}$ (which is $\varepsilon-$close to $\underline{0}$) and a sequence of positive integers $N_k \to \infty$ such that, for any $0 < t < 1$, the associated (time-t) map $\exp(t \cdot X)$ is uniformly approximated on $\mathbb{B}^n_{1-\tau}$ by a subsequence of $(h_k)^{[t \cdot N_k]}$ where $[\cdot]$ stands for the integral part.*

Proof. Fix ε and τ as in the statement. Modulo passing to a subsequence, we can suppose that $\frac{\Delta_k}{\epsilon_k}$ converges to a non constant mapping Δ_∞ uniformly on any compact set. We can also suppose that $\epsilon_k \sim \frac{\varepsilon}{k}$ for infinitely many k after deleting or doubling some element of the sequence. For simplicity assume $\epsilon_k \sim \frac{\varepsilon}{k}$ since the following arguments adapt without any further difficult to the general case.

In view of the preceding assumptions we have $\|k\Delta_k\|_1 = \varepsilon$ (which obviously does not imply that $\|\Delta_\infty\|_1 = \varepsilon$). Therefore Lemma 4.4 gives

$$\|(h_k)^k - Id - k\Delta_k\|_{1-\tau-\varepsilon} \leq \frac{\varepsilon^2}{\tau} .$$

Clearly, $N_k = k$ is a suitable solution to our problem provided that that $\{(h_k)^k\}$ does not tend uniformly to the identity. However the triangular inequality yields

$$\|(h_k)^k - Id\|_{1-\tau-\varepsilon} \geq \|k\Delta_k\|_{1-\tau-\varepsilon} - \|(h_k)^k - Id - k\Delta_k\|_{1-\tau-\varepsilon} ,$$

so that $(h_k)^k$ does not converge to the identity as long as $\|k\Delta_k\|_{1-\tau-\varepsilon}$, which tends to $\|\Delta_\infty\|_{1-\tau-\varepsilon}$, is greater than, say, $2\frac{\varepsilon^2}{\tau}$ for k large enough. Thus setting $\varepsilon_\infty = \|\Delta_\infty\|_{1-\tau-\varepsilon}$ it is sufficient to check that $2\frac{\varepsilon^2}{\tau} < \varepsilon_\infty$. We may rearrange our sequence in order to satisfy this estimate in the following way.

Choose $N >> 0$. Since the subsequence $(h_{kN})^{kN}$ also satisfies $\|kN\Delta_{kN}\|_1 = \varepsilon$ with $kN\Delta_{kN}$ converging to Δ_∞ uniformly on compact subsets, the alternate sequence $(h_{kN})^k$ automatically satisfies $\|k\Delta_{kN}\|_1 = \frac{\varepsilon}{N}$ with $k\Delta_{kN}$ converging to $\frac{\Delta_\infty}{N}$ uniformly on compact subsets. On the other hand,

$$\|\frac{\Delta_\infty}{N}\|_{1-\tau-\varepsilon/N} \geq \|\frac{\Delta_\infty}{N}\|_{1-\tau-\varepsilon} \geq \frac{\varepsilon_\infty}{N} .$$

Applying the above arguments to this new sequence with new data $\tau' = \tau$, $\varepsilon' = \frac{\varepsilon}{N}$ and $\varepsilon'_\infty = \frac{\varepsilon_\infty}{N}$, it results the new inequality $\frac{2\varepsilon^2}{\tau N^2} < \frac{\varepsilon_\infty}{N}$ which is clearly satisfied for N large enough. □

Remark 4.7. We may rather easily derive from the previous results that any pseudo-group containing some linear diagonal contraction A as well as some sequence h_k uniformly tending to identity on $\mathbb{B}^n$ does contain pseudo-flows in its closure. In any case it seems to us that the same should hold avoiding "linear diagonal" assumptions on A and such result could be interesting. Nevertheless, this later statement needs much more work and is somehow out of the subject of our paper.

5- Obtaining several pseudo-flows from a given pseudo-flow in the closure of a pseudo-group.

In this section, we think of A as a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} ,$$

and we denote by f the induced map $\mathbb{B}^n \hookrightarrow \mathbb{C}^n$ (i.e. $f(\underline{z}) = A\underline{z}$), which is still supposed to satisfy

$$(*) \quad 0 < |\lambda_n|^2 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 \ .$$

Let G be the pseudo-group generated by f and another mapping $g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ whose closure is already supposed to contain the real pseudo-flow generated by some holomorphic vector field X on $\mathbb{B}^n$. Hence, the pseudo-group G has *non-trivial real Lie algebra* $\mathfrak{G}$. The purpose of this section is to show that actually $\mathfrak{G}$ is "large" provided that A along with the linear part $B = D_{\underline{0}}g$ of g satisfy some further generic conditions. Let us be more precise.

Recall that the "richest" subgroups of $GL(n, \mathbb{C})$ that can be stably generated by two matrices A and B have closure of the type $\Lambda \times SL(n, \mathbb{C})$ where Λ is a (closed) group of homotheties containing at least some contraction and some non real element. This class of subgroups is characterized by the fact that the Lie algebra $\mathfrak{G}$ contains $\mathfrak{sl}(n, \mathbb{C})$ and furthermore the determinant projection is neither contained in $\mathbb{S}^1$ nor in $\mathbb{R}$. Also, we could show that the "richest" subgroups of the affine group stably generated by two transformations do contain rich linear groups as well as all the translations. This leads to the following:

Definition 5.0. Given a pseudo-group G of holomorphic transformations within $\mathbb{B}^n$, we say that G has *rich affine part* (resp. *rich linear part*) at $\underline{0}$ if, through an appropriate system of coordinates, we have:

(i) the Lie algebra $\mathfrak{G}$ contains a copy of $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ (resp. $\mathfrak{sl}(n, \mathbb{C})$) near $\underline{0}$,
(ii) the closure $\overline{G}$ contains some non real contracting homothety.

In order to show that the pseudo-group G generated by f and g as above has rich linear or affine part, we now make additional requirements to the pair (A, B), where $B = D_{\underline{0}}g$ denote the linear part of g, namely the requirements used in Lemma 1.2:

$$(**) \quad \begin{cases} 0 < |\lambda_n|^2 < |\lambda_1| < \cdots < |\lambda_n| < 1 \ , \\ \text{the } \lambda_i \text{ are pairwise distinct in argument, none being real,} \\ \text{no relation } |\lambda_i \lambda_j| = |\lambda_{i'} \lambda_{j'}|, \text{ other than the obvious ones, is verified,} \\ \text{neither } B \text{ nor } B^{-1} \text{ admits } \textit{zero} \text{ as entry.} \end{cases}$$

We begin with a continuous analogous of Lemma 4.0 which will be useful in the sequel. In any case it actually shows how simpler are the arguments of the proof when diffeomorphisms are replaced by vector fields.

Lemma 5.1. *Let A be a diagonal matrix satisfying $(*)$, G be a pseudo-group on $\mathbb{B}^n$ containing A and X be a holomorphic vector field defined on $\mathbb{B}^n$ and contained in the closure $\overline{G}$ of*

G. If X does not vanish at $\underline{0}$, then $\overline{G}$ also contains some non trivial translation pseudo-flow on $\mathbb{B}^n$. On the other hand, if X vanishes at $\underline{0}$ but if its linear part is not strictly lower triangular, then $\overline{G}$ also contains some non lower triangular linear pseudo-flow on $\mathbb{B}^n$.

Proof. Suppose first $X(\underline{0}) \neq \underline{0}$. If one decomposes X accordingly to its translation, linear and remaining part, $X = X^0 + X^1 + X^{\geq 2}$, then the action of A by conjugation expands the translation part of X i.e.

$$\|A^{-1}X^0\|_1 \geq \frac{1}{|\lambda_n|}\|X^0\|_1 \ ,$$

faster than the linear part since

$$\|A^{-1}X^1A - Id\|_1 \leq \frac{|\lambda_n|}{|\lambda_1|}\|X^1\|_1 \ .$$

Because of $(*)$ assumption this action also decreases the higher order terms

$$\|A^* X^{\geq 2}\|_1 \leq \frac{|\lambda_n|^2}{|\lambda_1|}\|X^{\geq 2}\|_1 \ .$$

Therefore, there exists a sequence of positive scalars $t_k \in \mathbb{R}^+$, such that the sequence of holomorphic vector field defined on $\mathbb{B}^n$ by $(t_k \cdot (A^k)^* X)$ has translation part of constant (non vanishing) modulus although its linear along with the higher order part uniformly tend to the identity. Hence some subsequence uniformly tends to a translation vector field which, by construction, is contained in the closure of G.

Now, if $X(\underline{0}) = \underline{0}$ but its linear part is not strictly lower triangular, then the action of A is linear diagonal on the entries of X^1, i.e. setting $X^1 = (v^1_{i,j})_{i,j}$ one has

$$A^{-1}X^1A = (\frac{\lambda_j}{\lambda_i}v_{i,j})_{i,j}$$

with eigenvalues of modulus $\frac{|\lambda_j|}{|\lambda_i|} \geq 1$ for upper entries $i \leq j$, while the other terms are decreasing. Now the proof follows as above. □

Proposition 5.2. *Let G be the pseudo-group generated on $\mathbb{B}^n$ by a linear diagonal matrix A and some mapping $g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$. Assume that A along with the linear part B of g satisfy $(**)$. Let X be a non trivial holomorphic vector field defined on $\mathbb{B}^n$ and contained in the closure $\overline{G}$ of G whose linear part M at $\underline{0}$ is not homothetic.*

Then the associated Lie algebra $\mathfrak{G}$ does contain a copy of $\mathfrak{sl}(n,\mathbb{C})$ on $\mathbb{B}^n$. Furthermore if $g(\underline{0}) \neq \underline{0}$, then $\mathfrak{G}$ does also contain all the translations and hence a copy of the affine Lie algebra $\mathfrak{sl}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ on $\mathbb{B}^n$.

Notice that when $\det(A) = \lambda_1 \cdots \lambda_n \notin \mathbb{R}$, G has rich linear (resp. affine) part.

Proof. Suppose first $g(\underline{0}) = \underline{0}$ (and hence $X(\underline{0}) = \underline{0}$). Assumption $(**)$ combined with Lemma 1.2 implies that the linear part X^1 of X along with a finite number of its conjugates under A and B do generate $\mathfrak{sl}(n,\mathbb{C})$ over $\mathbb{R}$ on some ball $\mathbb{B}^n_r$ (on which the iterations A and B

needed by this construction are well defined). Then, by a finite number of additional linear operations over $\mathbb{R}$, we can also find on $\mathbb{B}^n_r$ a collection $X_{i,j}$ of elements of $\overline{G}$ such that the corresponding linear part is the Kronecker matrix $X^1_{i,j} = \delta_{i,j}$. Now, we proceed as in proof of Lemma 5.0 in order to linearize the elements $X_{i,j}$ for which $i \leq j$. Namely one has

$$A^{-k} X^1_{i,j} A^k = \left(\frac{\lambda_j}{\lambda_i} \right)^k \delta_{i,j} \ .$$

On the other hand, $\|(A^k)^* X^{\geq 2}_{i,j}\|_r$ tends to $\underline{0}$ when $k \to +\infty$. Letting $t = \frac{|\lambda_i|}{|\lambda_j|} \leq 1$, the sequence of elements of $\overline{G}$ defined on $\mathbb{B}^n$ by $(t^k \cdot (A^k)^* X_{i,j})$ converges uniformly (maybe passing to an appropriate subsequence) to the linear Kronecker matrix $\delta_{i,j}$. The same construction can be carried out with imaginary Kronecker matrices $\sqrt{-1} \cdot \delta_{i,j}$. Thus $\overline{G}$ already contains on $\mathbb{B}^n_r$ the upper triangular complex Lie sub-algebra of $\mathfrak{sl}(n, \mathbb{C})$. A conjugation by a suitable power of A enables us to suppose that all these vector fields are defined on $\mathbb{B}^n$. In particular, $\overline{G}$ contains any diagonal element of $SL(n, \mathbb{C})$ sufficiently close to identity. We next replace A by some of these elements, say $\tilde{A}$, with eigenvalues $\tilde{\lambda}_i$ pairwise distinct in norm and argument but now satisfying

$$0 < |\tilde{\lambda}_1|^2 < |\tilde{\lambda}_n| \leq \cdots \leq |\tilde{\lambda}_1| < 1 \ .$$

Thus the previous construction shows that $\overline{G}$ contains lower triangular elements of $\mathfrak{sl}(n, \mathbb{C})$ as well.

Suppose now $g(\underline{0}) \neq \underline{0}$ and $X(\underline{0}) \neq \underline{0}$. The translation part of X and its conjugates under A generate a real subspace $E \subset \mathbb{C}^n$ invariant by A. Employing a procedure of renormalization similar to the preocedure explained above, we can suppose that the translations by elements in E also belong to $\overline{G}$. If $E \neq \mathbb{C}^n$, then E contains at least the translation along some coordinate axis (A has only simple eigenvalues) which is taken, after conjugation by B, to a vector field Y whose translation part lies away from any A-invariant hyperplane. Repeating the same arguments, but using Y instead of X, we see that any translation pseudo-flow actually belongs to $\overline{G}$. We can then delete the translation part of g and X and conclude as in the first case.

Finally, suppose $g(\underline{0}) \neq \underline{0}$ and $X(\underline{0}) = \underline{0}$. It is sufficient to show that we can replace X by a convenient conjugate under A and g which does not vanish at $\underline{0}$. Suppose for a contradiction that all such conjugates vanish at $\underline{0}$. Then Lemma 5.0 allows us to suppose that X linear (non homothetic) and, by assumption, $g^* X$ vanishes at $\underline{0}$ but its linear part is given by $B^* X$. Hence, as in the common fixed point case, a finite number of conjugates of X and $g^* X$ under A have linear part generating $\mathfrak{sl}(n, \mathbb{C})$. Employing again Lemma 5.0, we conclude that $\mathfrak{sl}(n, \mathbb{C})$ is contained in $\mathfrak{G}$. Therefore there exists an element $Y \in \mathfrak{sl}(n, \mathbb{C})$ which does not vanish on the direction $g(\underline{0})$ so that $g^* Y$ in turn does not vanish at $\underline{0}$. Now we proceed as in the preceding case. □

The following technical corollary is strictly what is needed for our Main Theorem (along with all the consequences settled in §6). In this corollary ε_λ should be thought of as being much smaller than $|\lambda - 1|$.

Corollary 5.3. *Let $f_0 : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ be a contracting homothety, $f_0(\underline{z}) = \lambda \cdot \underline{z}$, $0 < |\lambda| < 1$ and g_0 be the identity map on $\mathbb{B}^n$. Suppose that λ is sufficiently close to 1 (i.e. f_0 close to identity). Then there exists $0 < \varepsilon_\lambda < |\lambda - 1|$ and some Zariski open subset $\mathcal{U} \subset (GL(n,\mathbb{C}))^2$ such that any ε_λ perturbation $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ of the pair f_0, g_0 with $f(\underline{0}) = \underline{0}$ and derivatives $(D_{\underline{0}} f, D_{\underline{0}} g)$ at $\underline{0}$ lying on $\mathcal{U}$ satisfies:*

- *either $g(\underline{0}) \neq \underline{0}$ and G has large affine part,*
- *or $g(\underline{0}) = \underline{0}$ and G has large linear part.*

Proof. Choose $\mathcal{U} = \mathcal{U}_2$ as in Lemma 1.2 (defined by $(**)$). The first alternative follows from the combination of Proposition 2.0, (which gives existence of pseudo-flows) and Proposition 5.2. In the second alternative, the existence of pseudo-flows follows from Proposition 4.4. Again, we conclude with Proposition 5.1. □

As direct application we can also provide a generalization of Lamy's result, [La], to any dimension:

Corollary 5.4. *Suppose we are given holomorphic transformations $f_1, \dots, f_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ fixing $\underline{0} \in \mathbb{C}^n$. Denote by G the pseudo-group generated by the f_i's. Suppose that the derivatives at $\underline{0}$ of the elements of G generate a subgroup $D_{\underline{0}} G \subset GL(n,\mathbb{C})$ which is rich in the sense of §1. Then G has rich linear part (in the sense of this section). In particular the action of G on the punctured ball $\mathbb{B}^n \setminus \{\underline{0}\}$ is ergodic (w.r.t. Lebesgue) and has all orbits dense.*

Recall that two elements $f, g \in \mathrm{Diff}(\mathbb{C}^n, \underline{0})$ will generate such pseudo-group G with rich linear part provided that their respective linear part $A, B \in GL(n,\mathbb{C})$ are sufficiently close to I and generic (cf. Corollary 1.3).

Proof. First we find elements $A, B \in D_{\underline{0}} G$ whose corresponding mappings $A, B : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ fulfil the assumptions of Corollary 5.3 for a given λ. Consider the corresponding elements $f, g \in G$ ($A = D_{\underline{0}} f$ and $B = D_{\underline{0}} g$). Up to a conjugation of f, g under an appropriate homothety, these mappings are defined on the ball $\mathbb{B}^n$ and sufficiently close to A, B so that they also fulfil the assumptions of Corollary 5.3. □

Corollary 5.5. *Let $f_1, \dots, f_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be as in Corollary 5.4. Then there is a $\varepsilon_G > 0$ such that for any ε_G perturbation $g_1, \dots, g_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ of the previous generators:*

- *either the g_i have no more common fixed point and G has large affine part,*
- *or the g_i have a common fixed point, say $\underline{0}$, and G has large linear part.*

In the first case, the action of G near $\underline{0}$ is ergodic (w.r.t. Lebesgue) and has all orbits dense.

6- Chaotic pseudo-groups.

In this section we are going to complement the results of section 5 by showing that a pseudo-group G defined on $\mathbb{B}^n$ and having large affine part possesses several "chaotic properties". Throughout the sequel, G will be a pseudo-group on $\mathbb{B}^n$ with rich affine part in the sense of Definition 5.0: the closure $\overline{G}$ contains at least some contracting non real homothety as well as the restriction of any elements $f \in SL(n,\mathbb{C}) \ltimes \mathbb{C}^n$ within the ball.

First, from the measure theoretic point of vue, one has:

Property 6.1. *1) There does not exist any σ-finite measure μ on the ball which is preserved by elements of the pseudo-group.*

2) Ergodicity: any Lebesgue measurable subset which is invariant by G has null or total Lebesgue measure.

The first property results from the existence of a contracting homothety (along with the fact that $\underline{0}$ is not fixed by G) although the second part follows from the existence of pseudo-flows acting transitively. Results of measurable rigidity (like in [Reb2]) may also be obtained, but we prefer do not go further in this direction.

A notion of geometric entropy for foliations and pseudo-groups is defined in [Gh,La,Wa]. Let us recall this definition for a foliation.

Let $(M, \mathcal{F}, <.>)$ be a Riemannian manifold equipped with a foliation $\mathcal{F}$. For $x \in M$, we denote by $\exp^{\mathcal{F}} : T_x L_x \to L_x$ the exponential map of the leaf L_x containing x. Let $B_{\mathcal{F}}(0_x, r)$ (resp. $\overline{B}_{\mathcal{F}}(0_x, r)$) be the open (resp. closed) ball of center 0 and radius r contained in $T_x L_x$. Similarly $B_{\mathcal{F}}(x, r)$ (resp. $\overline{B}_{\mathcal{F}}(x, r)$) stands for the open (resp. closed) ball of center x and radius r contained in L_x. Finally given $y \in L_x$, we denote by $d_{\mathcal{F}}(x, y)$ the distance between x, y in L_x while $d_M(x, y)$ stands for the distance between x, y in M.

Given points x, y in M, let $\Omega(x, y, r)$ be the set of continuous maps from $\overline{B}_{\mathcal{F}}(0_x, r)$ to L_y sending 0_x to y. For $f \in \Omega(x, y, r)$ we set:

$$\delta(f) = \sup \{ \, d_M(\exp^{\mathcal{F}} v, f(v)) \, ; \, v \in \overline{B}_{\mathcal{F}}(0_x, r) \, \},$$

$$\delta_r(x, y) = \inf \{ \, \delta(f) \, ; \, f \in \Omega(x, y, r) \, \} + \inf \{ \, \delta(f) \, ; \, f \in \Omega(y, x, r) \, \}.$$

A subset $\Lambda \subset M$ is said $(\mathcal{F}, <.>, r, \varepsilon)-$separated if $\delta_r(x, y) \geq \varepsilon$ for every pair x, y of Λ $(x \neq y)$. Finally we define

$$N(\mathcal{F}, <.>, r, \varepsilon) = \max \{ \, \sharp\Lambda \, ; \, \Lambda \text{ is } (\mathcal{F}, <.>, r, \varepsilon) - \text{separated} \, \},$$

$$h(\mathcal{F}, <.>, \varepsilon) = \limsup \frac{1}{r} \log N(\mathcal{F}, <.>, r, \varepsilon) \text{ and}$$
$$h(\mathcal{F}, <.>) = \lim_{\varepsilon \to 0^+} h(\mathcal{F}, <.>, \varepsilon).$$

The number $h(\mathcal{F}, <.>)$ is called the geometric entropy of $\mathcal{F}$ (relative to $<.>$). It is closely related to the entropy of certain holonomy pseudo-groups of $\mathcal{F}$ (see [Gh,La,Wa] for details). Notice that it is very easy to adapt these definitions to our singular foliations. Actually, since our singularities are isolated and hyperbolic, the foliation is transverse to the boundaries of small balls around the singularities in question.

The first example of a pseudo-group with strictly positive entropy is a Shottky configuration (see [Gh,La,Wa],p.107); such dynamics are obviously contained in the affine case.

Property 6.2. *In case G has rich affine part, G has strictly positive entropy.*

We may expect from generic foliations or pseudo-groups more complicated dynamics than affine dynamics. When $n = 1$ results of [Be,Li,Lo2] yield:

Property 6.3. *If $n = 1$ and G has rich affine part but is not conjugate to a subgroup of Mœbius transformations (which is the case provided that the generators have generic cubic Taylor jet) then any conformal transformation within the unit disc $\mathbb{B}^1$ is uniformly approximated by elements of G. In particular, no differential-geometrico structure but the conformal structure is preserved by G.*

M.Belliart recently worked out a generalization of this property for any dimension $n \geq 2$.

If the previous features are those expected by chaotic dynamics, less expected is the structural instability which immediately results (see Introduction) from the topological rigidity. The remainder part of the section is devoted to the proof of this last property.

Proposition 6.4. *Assume that $G = G_1$ has rich affine part and consider an homeomorphism $h : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ onto its image conjugating G_1 with an holomorphic pseudo-group G_2 within $h(\mathbb{B}^n)$. Then h turns to be either a holomorphic or a anti-holomorphic diffeomorphism.*

We begin our approach to this proposition by showing that h actually conjugates the respective Lie algebras $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Let us consider a holomorphic vector field X_1 belonging to $\mathfrak{G}_1$ and denote by ϕ^t ($t \in \mathbb{R}$) the corresponding pseudo-flow contained in $\overline{G}_1$. The assignement $\rho_1 : t \mapsto \phi^t$ furnishes a pseudo-homomorphism from $(\mathbb{R}, +)$ to $\overline{G}_1$ (i.e. $\rho_1(t_1 + t_2) = \rho_1(t_1) \circ \rho_1(t_2)$ provided that both members are defined). Furthermore ρ_1 is *continuous* for the topologies in question.

Given $t \in \mathbb{R}$, we define a transformation ψ^t by setting $\psi^t = h \circ \phi^t \circ h^{-1}$. Note that ψ^t is defined on $h(\text{Dom } \phi^t)$ where Dom ϕ^t stands for the domain of ϕ^t (recall that ϕ^t is defined within $\mathbb{B}^n$).

Observe that ψ^t, $t \in \mathbb{R}$ fixed, is a holomorphic transformation. Indeed by definition of ϕ^t, there is a sequence of elements $\{\widetilde{f}_{1,k}\} \subset G_1$, where each $\widetilde{f}_{1,k}$ is defined on Dom ϕ^t, converging uniformly to ϕ^t on Dom ϕ^t. Since h is a homeomorphism, we immediately conclude that $\{h \circ \widetilde{f}_{1,k} \circ h^{-1}\}$ converges uniformly towards ψ^t on Dom ψ^t. However $h \circ \widetilde{f}_{1,k} \circ h^{-1}$ belongs to G_2 so that it is holomorphic (for every k). It follows that ψ^t is holomorphic as uniform limit of holomorphic transformations.

In view of the preceding, the assignement $\rho_2(t) = \psi^t$ gives a pseudo-homomorphism from $(\mathbb{R}, +)$ to $\overline{G}_2$ which is also continuous. The next lemma shows that there exists a unique holomorphic vector field X_2 in $\mathfrak{G}_2$ whose pseudo-flow induces ψ^t for every $t \in \mathbb{R}$ small.

Lemma 6.5. *Let $\psi^t : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$, $t \in]-t_0, t_0[$, $t_0 > 0$, be a pseudo-homomorphism from $(\mathbb{R}, +)$ to a holomorphic pseudo-group of transformations such that*

- *the mapping $(t, \underline{z}) \mapsto \psi^t(\underline{z})$ is continuous,*
- *each mapping $\underline{z} \mapsto \psi^t(\underline{z})$ is holomorphic,*
- *$\psi^{t_1+t_2} = \psi^{t_1} \circ \psi^{t_2}$ whenever it makes sense and $\psi^0 = Id$.*

Then, there exists a holomorphic vector field X on $\mathbb{B}^n$ such that $\psi^t = \exp(tX)$ (whenever it makes sense).

Proof. The proof consists of showing that X is unequivocally defined on compact sets as the uniform limit

$$X = \lim_{t \to 0} \frac{\psi^t - Id}{t}.$$

In fact when the limit above exists, we can recover ψ^t from X by means of the formula

$$\psi^t = \lim_{n\to\infty} \left(Id + \frac{t}{n} X \right)^n .$$

To simplify the notations, let us write

$$\psi^t = Id + \Delta^t \quad \text{and} \quad \|\Delta^t\|_r = \delta_r^t.$$

Since the result is of local nature, it is enough to prove the convergence of X above on some sub-ball, so that we can already suppose (up to a homothety) that the mapping $(t, \underline{z}) \mapsto \psi^t(\underline{z})$ is uniformly continuous and in particular $\delta_1^t \to 0$ when $t \to 0$. Maybe reducing t_0, it is possible to find $\tau > 0$ satisfying $r + \delta_1^{2t} + \tau < 1$ for a given fixed $0 < r < 1$ whenever $t \in]-t_0, t_0[$. Reasoning as in the proof of Lemma 3.0, one gets

$$\|\Delta^{2t} - 2\Delta^t\|_r \leq \frac{1}{\tau} \delta_1^t \delta_r^t .$$

Hence $2\delta_r^t \leq \delta_r^{2t} + \delta_1^t \delta_r^t / \tau$. Choosing t_0 so that $\delta_1^t \leq (1-r)/3$ (with $t \in]-t_0, t_0[$), we can set $\tau = \frac{2}{3}(1-r)$. It follows that

$$2\delta_r^t \leq \delta_r^{2t} + \frac{\delta_r^t}{2} \quad \text{and} \quad \delta_r^t \leq \frac{2}{3}\delta_r^{2t}.$$

In particular $\delta_r^{t/2^n} \leq (\frac{2}{3})^n \delta_r^t$. Let us first obtain a uniform bound for $\frac{\delta_r^{t/2^n}}{t/2^n}$ in order to conclude that the family $\{(\psi^t - Id)/t\}$ is uniformly bounded and thus *compact* (Montel Theorem). Such a bound can effectively be found in a smaller ball. Actually we now assume, without loss of generality by virtue of the last inequality above, that $\delta_1^{t/2^n} \leq (\frac{2}{3})^n \delta_1^t$. Replacing this estimate in the preceding inequalities, it results that

$$2\delta_r^{t/2^{n+1}} \leq \delta_r^{t/2^n} + (\frac{2}{3})^{n+1} \frac{\delta_r^{t/2^{n+1}}}{2} \quad \text{and} \quad \frac{\delta_r^{t/2^{n+1}}}{t/2^{n+1}} \left(1 - \frac{1}{4}(\frac{2}{3})^{n+1} \right) \leq \frac{\delta_r^{t/2^n}}{t/2^n}.$$

Denote by C_n the supremum of $\|\frac{\psi^t - Id}{t}\|_r$ for $| t | \in [t_0/2^{n+1}, t_0/2^n]$. One has

$$\limsup_{t\to 0} \left\| \frac{\psi^t - Id}{t} \right\|_r \leq \limsup_{n\to\infty} C_n \leq \left(\limsup_{n\to\infty} \prod_{k=0}^{n} (1 - \frac{1}{4}(\frac{2}{3})^{n+1}) \right) C_1.$$

Since the right side of the inequality above converges when $n \to \infty$, we conclude that $\{(\psi^t - Id)/t\}$ forms a compact family as required. This accomplishes the proof. □

Proof of Proposition 6.4. First notice that there is a well-defined correspondence σ between vector fields in $\mathfrak{G}_1$ and vector fields in $\mathfrak{G}_2$ obtained by means of Lemma 6.5. Precisely just let $\sigma(X_1) = X_2$ and denote by ϕ^t, ψ^t the associated pseudo-flows, then $\psi^t = h \circ \phi^t \circ h^{-1}$ for every $t \in \mathbb{R}$.

Now consider $2n$ translation (i.e. constant) vector fields $X_{1,1}, \ldots, X_{1,2n}$ in $\mathfrak{G}_1(\mathbb{B}^n)$ which are $\mathbb{R}$-linearly independent on the ball. The neighborhood of any point p possesses a parametrization given by

$$(\mathbb{R}^{2n}, \underline{0}) \to (\mathbb{B}^n, p) \ ; \ (t_1, \ldots, t_{2n}) \mapsto \phi_{1,1}^{t_1} \circ \cdots \circ \phi_{1,2n}^{t_{2n}}(p) \, .$$

Similarly the image under h of the neighborhood in question admits the parametrization

$$(t_1, \ldots, t_{2n}) \mapsto \psi_{1,1}^{t_1} \circ \cdots \circ \psi_{1,2n}^{t_{2n}} \circ h(p) \ .$$

Since h preserves the dimension as well as the commutativity of the pseudo-flows, the corresponding vector fields $X_{2,1}, \ldots, X_{2,2n}$ in $\mathfrak{G}_2$ turn to be also $\mathbb{R}$-linearly independent at $h(p)$. Through these (real) analytic parametrizations, h is the identity mapping at $(\mathbb{R}^{2n}, \underline{0})$ by construction. Hence, h is real analytic at p, and then at any point of the ball $\mathbb{B}^n$ since $\mathfrak{G}_1$ is locally transitive everywhere.

It remains to show that h has complex derivative at any point p. This is a standard argument belonging to Linear Algebra. Denote by $C \in GL(2n, \mathbb{R})$ the differential (derivative) of h at p. Note that h induces a conjugacy between the isotropy subgroups $\overline{G}_1^{\{p\}} = \{\phi \in \overline{G}_1 \ ; \ \phi(p) = p\}$ of $\overline{G}_1$ at p and the corresponding subgroup $\overline{G}_2^{\{h(p)\}}$ of $\overline{G}_2$. In particular, C induces a conjugacy between their linear part (which contains only complex matrices). Because G_1 has rich affine part, $D_p\overline{G}_1^{\{p\}} \in GL(n, \mathbb{C})$ contains at least $SL(n, \mathbb{C})$ and some non real homothety Λ whose image under C is formed by complex matrices of $GL(2n, \mathbb{R})$. Denote by $J \in GL(2n, \mathbb{R})$ the pull back by C of the complex multiplication iI. By construction, $J^2 = -I$ and J does commute with Λ and with any element of $SL(n, \mathbb{C})$. Since J commutes with Λ, it follows first that J is complex. Besides J is diagonal since it belongs to the center of $SL(n, \mathbb{C})$. Finally, since $J^2 = -I$, one has $J = iI$ or $J = -iI$. Hence, C is complex or anti-complex. This accomplishes our proof. □

In order to derive absolute rigidity (i.e. the topological rigidity of the foliation without assuming that we have a parametrized deformation), we will need the following complement:

Proposition 6.6. *Suppose we are given a family $f_j : U_j \hookrightarrow \mathbb{B}^n$ of holomorphic transformations within the ball, $j \in J$, such that the pseudo-group G generated by them has rich affine part on $\mathbb{B}^n$. Consider now some holomorphic deformation $f_{j,T} : U_j \hookrightarrow \mathbb{B}^n$ parametrized by $T \in \mathcal{U} \subset \mathbb{C}^m$: $f_{j,T}(\underline{z})$ is holomorphic in both variables $(T, \underline{z})$ and $f_{j,\underline{0}} = f_j$ for any $j \in J$. Then, up to rescaling $\mathcal{U}$ and up to a holomorphic change of parameters and coordinates of the type*

$$\Phi : \mathcal{U} \times \mathbb{B}^n_\tau \hookrightarrow \mathbb{C}^m \times \mathbb{B}^n \ ; \ (T, \underline{z}) \mapsto (\varphi(T), \phi_{\varphi(T)}(\underline{z})) \ ,$$

there exists some integer $0 \leq r \leq m$ such that the deformation possesses the following property over $\mathcal{U}$:

- *$f_{j,T} = f_{j,T'}$ for any $j \in J$ if and only if $\Pi_r(T) = \Pi_r(T')$ where Π_r stands for the linear projection on r first coordinates ($\Pi_r(t_1, \ldots, t_m) = (t_1, \ldots, t_r)$);*
- *if a homeomorphism $h : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{B}^n$ induces a conjugacy between two sets of generators, $h^{-1} \circ f_{j,T} \circ h = f_{j,T'}$ for any $j \in J$, then $\Pi_r(T) = \Pi_r(T')$.*

Proof. First observe that for small T, the pseudo-group G_T generated by the $f_{i,T}$ on $\mathbb{B}^n$ has still rich affine part. In what follows we are allowed to reduce the domain of T. Choose some element along the deformation f_T such that $f_{\underline{0}}$ fixes some point p close to the origine of $\mathbb{C}^n$ with differential at p having only contracting simple eigenvalues satisfying $(*)$. Modulo a holomorphic change of coordinates of the type $(T, \underline{z}) \mapsto (T, \phi_T(\underline{z}))$, one can assume that f_T is linear diagonal, namely

$$f_T = \begin{pmatrix} \lambda_1(T) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(T) \end{pmatrix} .$$

Here, the space of parameters T have been restricted so that the version of Poincaré Linearization Theorem with holomorphic parameters holds as long as F_T lies on the Poincaré domain without resonance and with simple eigenvalues. Clearly any analytic conjugacy between two such diagonal transformations f_T is necessarily diagonal as well. If $\lambda_1(\underline{0}), \ldots , \lambda_n(\underline{0})$ were chosen sufficiently generic, then no anti-complex matrix can conjugate two such f_T close to $\underline{0} \in \mathbb{C}^m$. Next Proposition 6.4 asserts that any topological conjugacy h between two sets of generators $(f_{j,T})_{j\in J}$ will be holomorphic or anti-holomorphic. However since h induces a conjugacy between the corresponding diagonal matrices f_T, h will finally be $\mathbb{C}$-linear and diagonal. Now, choose an element g_T along the deformation taking $\underline{0} \in \mathbb{B}^n$ to the complement of all the coordinate hyperplanes. Modulo a new change of coordinates of the type $(T, \underline{z}) \mapsto (T, \phi_T(\underline{z}))$ with ϕ_T linear, one can assume that $g_T(\underline{0}) = p$ does not depend anymore on T. Then, clearly, any topological conjugacy h between two sets of generators is necessarily trivial i.e. it must coincide with the identity.

We are now able to distinguish topological classes of conjugacies just by comparing Taylor coefficients of the generators. Actually let us denote by

$$f_{j,T} = \sum_{\underline{k}} a_{j,\underline{k}} (\underline{z} - p_j)^{\underline{k}}$$

the Taylor series of the generators (at some arbitrarily fixed point $p_j \in U_j$), where $\underline{k}$ denote the multi-index $\underline{k} = (k_1, \ldots , k_n) \in \mathbb{N}^n$. In this case $(f_{j,T})_j$ is topologically conjugate to $(f_{j,T'})_j$ if and only if all the holomorphic functions $\Delta_{j,\underline{k}}(T, T') = a_{j,\underline{k}}(T) - a_{j,\underline{k}}(T')$ vanish at (T, T'). After Hilbert Basis Theorem, there exists a finite set $\Delta_1, \ldots , \Delta_K$ generating all these conditions at a neighborhood of some point $T_0 \in \mathcal{U}$. In other words, there exists some germ of holomorphic function $\mathbf{a} : (\mathbb{C}^m, T_0) \to (\mathbb{C}^K, \underline{0})$ such that the sets of generators $(f_{j,T})_j$ and $(f_{j,T'})_j$ do coincide if and only if $\mathbf{a}(T) = \mathbf{a}(T')$. Arbitrarily close to $T_0 \in \mathbb{C}^m$, there exists a point T_1 where the function $\mathbf{a}$ is well defined and regular. Modulo performing a holomorphic change of parameters of the form $(T, \underline{z}) \mapsto (\varphi(T), \underline{z})$ around T_1, $\mathbf{a}$ can be decomposed into $\mathbf{a} = \tilde{\mathbf{a}} \circ \Pi_r$ where r stands for the rank of $\mathbf{a}$ at T_1 and $\tilde{\mathbf{a}}$ is a germ of embedding $(\mathbb{C}^r, \underline{0}) \hookrightarrow (\mathbb{C}^K, \underline{0})$. By construction, near T_1 (and under new coordinates and parametrizations constructed above), the set of generators $(f_{j,T})_j$ fulfils the conclusions of Proposition 6.6. □

7- Construction of stably chaotic rational vector fields.

A holomorphic one dimensional singular foliation $\mathcal{F}$ of degree d on $\mathbb{CP}^n$, $n \geq 2$, $d \geq 1$, is given either by a homogeneous vector field of degree d on $\mathbb{C}^{n+1}$

$$\widetilde{X} = \sum_{i=1}^{n} H_i(\underline{Z}, T)\frac{\partial}{\partial Z_i} + H_{n+1}(\underline{Z}, T)\frac{\partial}{\partial T}$$

(where H_i are homogeneous polynomials of degree d), or, more commonly, by a polynomial vector field

$$X = \sum_{i=1}^{n} P_i(\underline{z})\frac{\partial}{\partial z_i}$$

defined on $\mathbb{C}^n$ viewed as the image of the main affine chart $(\underline{z}) = (\underline{Z}/T)$. Actually the expressions above are related by the equation $P_i(\underline{z}) = H_i(\underline{z}, 1) - z_i H_{n+1}(\underline{z}, 1)$. Since the dimension of the set of homogeneous polynomials of degree d in $n+1$ variables, or equivalently of arbitrary polynomials of degree d in n variables, has dimension $\frac{(d+n)!}{d!n!}$, it follows that the space $\mathcal{F}^d(\mathbb{CP}^n)$ of holomorphic one dimensional singular foliations of degree d on $\mathbb{CP}^n$ is a Zariski open subset of a projective space of dimension

$$(d+n+1)\frac{(d+n-1)!}{d!(n-1)!} - 1.$$

Indeed for a "smaller" Zarisky open set of the mentioned projective space, the singular set of $\mathcal{F}$ consists of $\frac{d^{n+1}-1}{d-1}$ isolated points of $\mathbb{CP}^n$.

Let X be a germ of vector field at $\underline{0} \in \mathbb{C}^n$ with an isolated singularity at $\underline{0}$ and denote by $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ the spectrum of its linear part. We say that X is *hyperbolic* at $\underline{0}$ if none of the quotients λ_k/λ_l is real. A *branch* at $\underline{0}$ is any germ of irreducible analytic curve through $\underline{0}$ which is tangent to X on some neighborhood. It is proved in [LN,So] that, in the hyperbolic case, $\underline{0}$ has exactly n distinct branches $B_0, \dots, B_n$. More precisely, B_k is smooth and tangent to the eigenspace of λ_k at $\underline{0}$.

Theorem [LN,So]. *For any $n, d \geq 2$, there exists a Zariski open subset of foliations $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^n)$ such that:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+n)!}{d!n!}$ hyperbolic singularities and is regular on the complement;*
(ii) *$\mathcal{F}$ has no invariant algebraic curve.*

In particular, any leaf of $\mathcal{F}$ accumulates a non empty set of leaves.

It should be pointed out that the eigenvalues and the corresponding branches depend holomorphically on the parameters (as it follows from the proof given in [LN,So]). For instance, if a branch B_0 at a hyperbolic singular point p_0 of $\mathcal{F}_0$ is parametrized by $\phi : (\mathbb{C}, 0) \to \mathbb{CP}^n$, then there also exists a germ of holomorphic map

$$\widetilde{\phi} : (\mathbb{C}, 0) \times (\mathcal{F}^d(\mathbb{CP}^n), \mathcal{F}_0) \to \mathbb{CP}^n$$

such that $\widetilde{\phi}(., \mathcal{F}_0)$ coincides with $\phi(.)$. Moreover for any $\mathcal{F}$ sufficiently close to $\mathcal{F}_0$, $p = \widetilde{\phi}(0, \mathcal{F})$ is a hyperbolic singularity of $\mathcal{F}$ and $\widetilde{\phi}(., \mathcal{F})$ parametrizes the corresponding branch B of $\mathcal{F}$ through p.

For the sake of notations, we shall work with $\mathbb{CP}^{n+1}$ ($n \geq 1$) instead of $\mathbb{CP}^n$. Let us consider the main affine chart $\mathbb{C}^{n+1}$ of $\mathbb{CP}^{n+1}$, $n \geq 1$, with coordinates $(w, \underline{z})$ where $\underline{z} = (z_1, \ldots, z_n)$ (in order to specialize the first coordinate). For $d \geq 2$, consider also the rational vector field

$$X(M_1, \ldots, M_d) = \frac{\partial}{\partial w} + \sum_{k=1}^{d} \frac{M_k}{w - w_k} \underline{z} \frac{\partial}{\partial \underline{z}},$$

where $w_1, \ldots, w_d \in \mathbb{C}$ are pairwise distinct and $M_1, \ldots, M_d \in M(n, \mathbb{C})$. Here, the notation $M\underline{z}\frac{\partial}{\partial \underline{z}}$ has to be understood as $\sum_{i,j} m_{i,j} z_i \frac{\partial}{\partial z_j}$ where $M = (m_{i,j})$. Denote by $\mathcal{F}(M_1, \ldots, M_d)$ the induced foliation on $\mathbb{CP}^{n+1}$. We want to describe some dynamical features of foliations close to $\mathcal{F}_0 = \mathcal{F}(I, \ldots, I)$ where I denotes the identity matrix of $GL(n, \mathbb{C})$.

Lemma 7.1. *Assume that $M_1, \ldots, M_d \neq (0)$ and $M_1 + \cdots + M_d \neq I$. Then the foliation $\mathcal{F} = \mathcal{F}(M_1, \ldots, M_d)$ has projective degree d, is tangent to the projective line $L_0 : \{\underline{z} = 0\}$ and has $d + 1$ isolated singularities $p_k = (w_k, \underline{0})$, $k = 1, \ldots, d$, and $p_{d+1} = (\infty, \underline{0})$ belonging to L_0. Furthermore in a neighborhood of any singularity p_k, the foliation is defined by a holomorphic vector field whose linear part at p_k is respectively given (in matricial notation) by*

$$\begin{pmatrix} 1 & 0 \\ 0 & M_1 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} 1 & 0 \\ 0 & M_d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & I - M_1 - \cdots - M_d \end{pmatrix}.$$

Finally the hyperplanes $\{w = w_1\}, \ldots, \{w = w_d\}$ and the hyperplane at infinity $\{w = \infty\}$ are all tangent to the foliation and intersect along a degenerate codimension 2 singularity at infinity. There are no other singularity.

Denote by $\mathcal{V}$ the dn^2-dimensional smooth complex submanifold of $\mathcal{F}^d(\mathbb{CP}^{n+1})$ given by the parametrization $(M_1, \ldots, M_d) \mapsto \mathcal{F}(M_1, \ldots, M_d)$.

Proof. The conditions $M_1, \ldots, M_d \neq 0$ and $M_1 + \cdots + M_d \neq I$ imply that the degree d polynomial vector field $(\Pi_{k=1}^{d} w - w_k) \cdot X$, where $X = X(M_1, \ldots, M_d)$, is irreducible. Also the homogeneous part of degree d of the polynomial in question is not tangent to the radial vector field $w\partial/\partial w + \underline{z}\partial/\partial \underline{z}$. Using new projective coordinates $(t = 1/w, \tilde{\underline{z}} = \underline{z}/w)$, the foliation is defined by the rational vector field

$$t\frac{\partial}{\partial t} + (I - \sum_{k=1}^{d} \frac{M_k}{1 - w_k t})\tilde{\underline{z}}\frac{\partial}{\partial \tilde{\underline{z}}}\}.$$

This proves the lemma. □

Consider $\mathcal{F}_1 \in \mathcal{V}$ and let us recall the classical construction of the *holonomy pseudo-group* associated to the special leaf $L_0^* = L_0 \setminus \{p_1, \ldots, p_{d+1}\}$.

Fix a point $p_0 = (w_0, \underline{0}) \in L_0^*$ and note that the affine hyperplane $\{w = w_0\}$ is transversal to $\mathcal{F}_1$. Denote by Σ the embedding

$$\Sigma_r : \mathbb{B}_r^n \hookrightarrow \mathbb{CP}^{n+1}$$

induced by the embedding $\underline{z} \mapsto (w_0, \underline{z})$ of the ball of radius $r > 0$, $\mathbb{B}_r^n$, in $\{w = w_0\} \subset \mathbb{C}^{n+1}$. Since no misunderstanding is possible, we sometimes denote by Σ_r the image $\Sigma_r(\mathbb{B}_r^n) \subset \mathbb{C}^{n+1}$ of the mentioned embedding as well.

Choose a collection $\gamma_1, \dots, \gamma_d : [0,1] \to L_0^*$ of generators for the fundamental group $\pi_1(L_0^*, p_0)$ so that γ_k has index 1 around p_k and 0 around the other singularities ($k = 1, \dots, d$). Next we want to define the return map associated to each singularity p_k. First notice that, given any foliation $\mathcal{F}$ close to $\mathcal{F}_1$ and belonging to $\mathcal{V}$, the vertical hyperplanes $\{w = \text{cte}\}$ are transversal to $\mathcal{F}$ (excepted those hyperplanes over the corresponding singularities on L_0). However for a general foliation (i.e. a foliation which may not belong to $\mathcal{V}$) $\mathcal{F}$ close to $\mathcal{F}_1$, the corresponding transversality condition can only be ensured in a compact part of the hyperplane in question.

In any case, for $\mathcal{F}_1 \in \mathcal{V}$, the desired return maps are defined as follows. Given $R > 0$, the "hypersurfaces" $\{(w, \underline{z}) \ ; \ w = \gamma_k(t), \ \underline{z} \in \mathbb{B}_R^n\}$ are transversal to $\mathcal{F}_1$ for $t \in [0,1]$ and $k = 1, \dots, d$ (indeed R can be chosen arbitrarily large). Hence for a sufficiently small r ($0 < r < R$) and any point $p \in \Sigma_r$, the path γ_k can be lifted in the leaf through p (and w.r.t. the transverse "fibration" by hyperplanes) to a path $\gamma_{k,p}$ verifying $\gamma_{k,p}(0) = p$ and $\gamma_{k,p}(1) \in \Sigma_R$. This allows us to define the *return map f_k around the singularity p_k* (relative to the choices of the homotopy classes $\gamma_1, \dots, \gamma_d$ and r, R) without ambiguity by

$$f_k : \Sigma_r \to \Sigma_R \ ; \ p \mapsto \gamma_{k,p}(1) \, .$$

As it is well-known, the map f_k is holomorphic and one-to-one for r small enough (we shall see later that, if $\mathcal{F}_1 \in \mathcal{V}$, then f_k is defined in the whole hyperplane $\{w = w_0\}$ and is globally one-to-one). Finally let us observe that the preceding construction also applies to any foliation $\mathcal{F}$ close to $\mathcal{F}_1$ (regardless whether or not $\mathcal{F}$ belongs to $\mathcal{V}$) maybe choosing r, R small. In the sequel, we fix r, R so that all the f_k ($k = 1, \dots, d$) are well defined and injective mappings from $\mathbb{B}_r^n$ to $\mathbb{B}_R^n$.

The *holonomy pseudo-group* $G_{\mathcal{F}_1}$ of L_0^* relative to $\mathcal{F}_1$ will be the pseudo-group generated by the return maps f_k on $\mathbb{B}_r^n$. If a leaf L of $\mathcal{F}$ intersects Σ_r at a point p, then the pseudo-orbit of p under $G_{\mathcal{F}_1}$ is contained in $L \cap \Sigma_r$, so that if the pseudo-orbit is dense, then L is dense in a neighborhood of Σ_r. This last remark will be often used to settle the density of given leaves.

As we have observed, maybe choosing r, R small, the previous construction remains valid for any sufficiently small pertubation $\mathcal{F}$ of $\mathcal{F}_1$ since we can control the dependence of the slope of the leaves (w.r.t. the parameters) on compact tubular-neighborhoods (where the foliation is regular) of the γ_k's. Let us point out that, if $\mathcal{F}$ does not belong to $\mathcal{V}$, the γ_k's are not necessarily invariant under $\mathcal{F}$, however our construction does not require this fact. For such a general $\mathcal{F}$, the associated "holonomy pseudo-group" $G_{\mathcal{F}}$, generated by the corresponding maps with parameters

$$f_{k,\mathcal{F}} : \mathbb{B}_r^n \hookrightarrow \mathbb{B}_R^n \, ,$$

may have no common fixed point in $\mathbb{B}_r^n$. In other words $G_{\mathcal{F}}$ will be in fact a pseudo-group generated by "holonomy maps" which are however associated to different leaves (i.e. $G_{\mathcal{F}}$ is contained in the "holonomy groupoid" of $\mathcal{F}$). By some abuse of notation we shall say that $G_{\mathcal{F}}$ is a holonomy pseudo-group. Finally it is clear that the "maps with parameters" $f_{k,\mathcal{F}}$ depend holomorphically on $\mathcal{F}$.

The next proposition allows us to arbitrarily choose the linear part of the initial f_k.

Proposition 7.2. *For $\mathcal{F} \in \mathcal{V}$, the return maps $f_1, \dots, f_d : \mathbb{B}^n_r \hookrightarrow \mathbb{B}^n_R$ are linear, respectively denoted by $A_1, \dots, A_d \in GL(n, \mathbb{C})$. Furthermore the holomorphic mapping from $\mathcal{V}$ to $GL(n, \mathbb{C})$ given by*

$$\begin{cases} \mathcal{F} = \mathcal{F}(M_1, \dots, M_d) & \mapsto \quad (A_1, \dots, A_d)\,; \\ \mathcal{F}_0 = \mathcal{F}(I, \dots, I) & \mapsto \quad (I, \dots, I)\,, \end{cases}$$

is a local diffeomorphism at $\mathcal{F}_0$.

The authors have discovered later that the proposition above is a variant of the famous 1920's Lappo-Danilevskii's affirmative answer to the Riemann-Hilbert problem in the case in which the monodromy representation is close to the identity.

Proof. The return map f_k are clearly obtained by integration of the non-autonomous vector field

$$Y(t) = \sum_{k=1}^{d} \frac{M_k}{\gamma_k(t) - w_k} \underline{z} \frac{\partial}{\partial \underline{z}}$$

along $t \in [0, 1]$. Since this vector field is always in the linear group (for $t \in [0, 1]$), the resulting transformation f_k is linear.

Because the M_k's are close to I, each of the $d + 1$ singularities of the initial vector field $X = X(M_1, \dots, M_d)$ lying on the projective line L_0 are non resonant in the Poincaré domain (see [Ar,Il], p.72) and hence can be linearized by local holomorphic change of coordinates. Using these linearizations, it becomes clear that each f_k is locally conjugate to the (local) holonomy of the corresponding linear vector field. In other words, there is $g_k \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $f_k = g_k^{-1} e^{2i\pi M_k} g_k$. However, since f_k is linear, the previous equation still holds when g_k is replaced by its linear part $D_{\underline{0}} g_k = B_k \in GL(n, \mathbb{C})$. Thus

$$f_k = A_k = B_k^{-1} e^{2i\pi M_k} B_k \quad \text{for a suitable} \quad B_k \in GL(n, \mathbb{C}), \quad k = 1, \dots, d.$$

It should be noted that the transition matrices B_k are generally distinct, each of them depending on all the coefficients (entries) of M_k so that they cannot be explicited. In fact the subgroup of $GL(n, \mathbb{C})$ generated by the return maps A_k cannot be determined even up to conjugation. Actually the restriction of the associated foliation to $\mathbb{C} \times (\mathbb{C}^n \setminus \{\underline{0}\})$ is invariant by homotheties on the variable $\underline{z}$ and therefore induces (via projectivization on $\underline{z}$) a one-dimensional foliation on $\mathbb{C} \times \mathbb{CP}^{n-1}$. When $n = 2$, the resulting foliation on $\mathbb{C} \times \mathbb{CP}^1$ is defined by a Ricatti equation. Finally it has been well-known since Liouville that Ricatti equations hardly ever can be integrated explicitly in terms of its coefficients.

Nevertheless, when only one of the M_k differs from the identity by exactly one of its entries, the A_k are computable as follows. Let $k_0 \in \{1, \dots, d\}$ and $i_0, j_0 \in \{1, \dots, n\}$. Let also $M_k = I$ for $k \neq k_0$ and $M_{k_0} = I + t\delta_{i_0, j_0}$ where δ_{i_0, j_0} stands for the Kronecker matrix. In the main affine chart and away from the poles $\{w = w_k\}$, $k = 1, \dots, d$, the integral curves of the vector field $X(M_1, \dots, M_d)$ are locally parametrized by $w \mapsto (w, \underline{z}(w))$ where the functions $z_i(w)$ satisfy the following system of differential equations:

$$\begin{cases} \frac{dz_i}{dw} & = \quad \sum\limits_{k=1}^{d} \frac{z_i}{w - w_k}, \quad i \neq i_0 \\ \frac{dz_{i_0}}{dw} & = \quad \sum\limits_{k=1}^{d} \frac{z_{i_0}}{w - w_k} + t \frac{z_{j_0}}{w - w_{k_0}}. \end{cases}$$

Beginning with initial data $p = (w_0, \underline{z}(w_0))$, a direct integration gives (for $i \neq i_0$)

$$z_i(w) = z_i(0) \cdot \prod_{k=1}^{d} (\frac{w - w_k}{w_0 - w_k}),$$

and, when $i_0 = j_0$,

$$z_{i_0}(w) = z_{i_0}(0) \cdot (\prod_{k \neq k_0} (\frac{w - w_k}{w_0 - w_k})) \cdot (\frac{w - w_{k_0}}{w_0 - w_{k_0}})^{1+t}.$$

In this last case (i.e. $i_0 = j_0$), we obtain by continuation along γ_k:

$$\begin{cases} A_k & = & I, \quad k \neq k_0 \\ A_{k_0} & = & \begin{pmatrix} I^{i_0-1} & & 0 \\ & e^{2i\pi t} & \\ 0 & & I^{n-i_0} \end{pmatrix} \end{cases}.$$

However, if $j_0 \neq i_0$, then the differential equation satisfied by $z_{i_0}(w)$ becomes

$$\frac{dz_{i_0}}{dw} = \sum_{k=1}^{d} \frac{z_{i_0}}{w - w_k} + t \frac{z_{j_0}(0)}{w - w_{k_0}} \prod_{k=1}^{d} (\frac{w - w_k}{w_0 - w_k}).$$

Replacing $z_{i_0}(w) = c(w) \cdot \prod_{k=1}^{d} (\frac{w-w_k}{w_0-w_k})$ in the last equation with initial value $c(0) = z_{i_0}(0)$, the function $c(w)$ may be computed by a direct integration providing

$$z_{i_0}(w) = [z_{i_0}(0) + t z_{j_0}(0) \log(\frac{w - w_{k_0}}{w_0 - w_{k_0}})] \prod_{k=1}^{d} (\frac{w - w_k}{w_0 - w_k}).$$

Therefore by analytic continuation along γ_k, we obtain:

$$\begin{cases} A_k & = & I, \quad k \neq k_0, \\ A_{k_0} & = & I + 2i\pi t \delta_{i_0, j_0}. \end{cases}$$

These computations mean that the two holomorphic maps

$$\begin{array}{c} (M_1, \ldots, M_d) \mapsto (A_1, \ldots, A_d) \\ (M_1, \ldots, M_d) \mapsto (e^{2i\pi M_1}, \ldots, e^{2i\pi M_d}) \end{array}$$

turn to coincide near $\mathcal{F}_0$ along the coordinate axis (relative to the parametrization given by Kronecker matrices) and then are tangent at $\mathcal{F}_0$. □

Corollary 7.3. *The germ of holomorphic mapping:*

$$\begin{cases} (\mathcal{F}^d(\mathbb{CP}^{n+1}), \mathcal{F}_0) & \to & (GL(n, \mathbb{C}))^d \\ \mathcal{F} & \mapsto & (A_1, \ldots, A_d) = (D_{\underline{0}} f_1, \ldots, D_{\underline{0}} f_d) \end{cases}$$

defined by the return maps is a germ of submersion at $\mathcal{F}_0$. □

The corollary above is an immediate consequence of the Proposition 7.2. These two statements mean that for $\mathcal{F}$ sufficiently close to $\mathcal{F}_0$ within $\mathcal{V}$, or more generally within

$\mathcal{F}^d(\mathbb{CP}^{n+1})$, the linear parts of the d return maps $f_{k,\mathcal{F}}$ may be deformed arbitrarily and independently by a deformation of $\mathcal{F}$.

Notice that the return maps $f_1, \dots, f_d$ of a foliation $\mathcal{F}_1$ in $\mathcal{V}$ are linear and hence globally defined (i.e. we can choose $r = R = \infty$), in particular they are also globally one-to-one. Therefore it is easy to construct, for $n \geq 2$, foliations in $\mathcal{V}$ such that any leaf, apart from L_0 and those contained in the vertical hyperplanes $H_k = \{w = w_k\}$, $k = 1, \dots, d+1$, is everywhere dense. Indeed it is enough to choose $(A_1, A_2) \in \mathcal{U}$ where $\mathcal{U}$ is given by Corollary 1.4.

Now, consider $\mathcal{F}_1 = \mathcal{F}(M_1, \dots, M_d)$ where $M_1 = \cdots = M_d = M$ is diagonal. In this case the associated return maps can be computed by explicitly integration and all of them coincide with the linear map $A = e^{2i\pi M}$. Moreover, we choose $\mathcal{F}_1$ such that A is "weakly" contracting, as needed by Proposition 2.0. Hereafter let us suppose that such a $\mathcal{F}_1$, which in addition is close to $\mathcal{F}_0$ in the domain of submersion given by Corollary 5.3, is fixed. For $\mathcal{F}$ close enough to $\mathcal{F}_1$, the corresponding return maps $f_{k,\mathcal{F}}$ are defined and injective at least on $\mathbb{B}^n$ (i.e. $r = 1$). Then, we can identify the pseudo-groups generated by the return maps with pseudo-groups defined on $\mathbb{B}^n$. In what follows, we shall make no distinction between these two points of view, moreover the pseudo-group in question will be denoted by $G_{\mathcal{F}}$. When $G_{\mathcal{F}}$ acts minimally on $\mathbb{B}^n$, any leaf of $\mathcal{F}$ intersecting the transversal Σ has to be dense in a neighborhood of Σ. In this way, we want to construct a "plug" for producing "local minimality".

Proposition 7.4. *For $\mathcal{F}_1$ as before and $\mathcal{F}$ sufficiently close to $\mathcal{F}_1$, we have the following alternative:*

- *either $G_{\mathcal{F}}$ has a common fixed point (which corresponds to an invariant projective line $L_{0,\mathcal{F}}$ close to L_0),*
- *or $G_{\mathcal{F}}$ accumulates some non trivial (real) pseudo-flow on $\mathbb{B}^n$.*

Proof. The $d+1$ singular points $p_1, \dots, p_{d+1}$ of $\mathcal{F}_1$ along L_0 are hyperbolic. At each of these points p_k, the foliation admits exactly $n+1$ transversal branches. For a sufficiently small ball W_k centered at p_k, denote by $L_k = L_0 \cap W_k$ the local branch contained in L_0.

Given a foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_1$, those $d+1$ hyperbolic singularities correspond to singularities $p_{1,\mathcal{F}}, \dots, p_{d+1,\mathcal{F}}$ of $\mathcal{F}$ (since the initial singularities are "persistent"). The persistent fixed point of the k^{th} return map $f_{k,\mathcal{F}}$ within $\mathbb{B}^n_r$ corresponds to the intersection with $\Sigma_{\mathcal{F}}$ of the leaf which, in W_k, induces the persistent branch $L_{k,\mathcal{F}}$ of $p_{k,\mathcal{F}} \in W_k$ close to L_k. Then, if the unique fixed point of $f_{1,\mathcal{F}}$, is also fixed by the other return maps, this means that the branches $L_{k,\mathcal{F}}$ are parts of a common leaf which turns to be an embedded sphere close to L_0 and hence a projective line.

On the other hand, if one of the return maps $f_{k,\mathcal{F}}$ does not fix anymore the unique fixed point of $f_{1,\mathcal{F}}$, we then apply Proposition 2.0 to $f = f_{1,\mathcal{F}}$ and $g = (f_{k,\mathcal{F}})^{-1} \circ f_{1,\mathcal{F}}$. □

Lemma 7.5. *For any $d \geq 2$, there exists an open subset $\mathcal{W} \subset \mathcal{F}^d(\mathbb{CP}^2)$ approximating $\mathcal{F}_1$ such that any foliation $\mathcal{F} \in \mathcal{W}$ satisfies:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+2)(d+1)}{2}$ hyperbolic singularities and is regular on the complement,*
(ii) *any leaf of $\mathcal{F}$ is dense in $\mathbb{CP}^2$,*
(iii) *the holonomy pseudo-group $G_{\mathcal{F}}$ has large affine part.*

Proof. We keep the notations of the proof of Proposition 5.5. Let us consider the compact part of L_0 given by $L_0 \setminus (\cup_{k=1}^{d+1} L_k)$. We also consider a neighborhood W_0 of $L_0 \setminus (\cup_{i=k}^{d+1} L_k)$ such that any leaf intersecting W_0 will also meet the transversal Σ on which all the return maps are well defined (the existence of W_0 can easily be established, for instance by rescaling a finite covering by trivialization boxes).

Because of the hyperbolicity of $p_{k,\mathcal{F}}$, the horizontal and vertical branches $L_{k,\mathcal{F}}$ and $B_{k,\mathcal{F}}$ in W_k depend holomorphically on $\mathcal{F}$ (where $L_{k,\mathcal{F}_1} = L_0 \cap W_k$ and $B_{k,\mathcal{F}_1} = \{w = w_k\} \cap W_k$). Hence we can assume that $L_{k,\mathcal{F}}$ intersects W_0. Since $p_{k,\mathcal{F}}$ is non resonant in the Poincaré domain, one can also suppose that this singularity is linearizable in the whole W_k for $\mathcal{F}$ close to $\mathcal{F}_1$. Thus any leaf other than $B_{k,\mathcal{F}}$ in W_k will accumulate $L_{k,\mathcal{F}}$ and hence intersect W_0 (and finally meet $\Sigma_{\mathcal{F}}$). Denote by W the neighborhood $W = W_0 \cup (\cup_{k=1}^{d+1} W_k)$ of L_0.

Since $\mathbb{CP}^2 \setminus L_0$ is Stein, any leaf $L \neq L_0$ of $\mathcal{F}$ has to accumulate L_0 and therefore intersects W. If $L \cap W$ consists only of vertical branches $B_{k,\mathcal{F}}$, then its closure $\overline{L}$ of L is a proper analytic subset in the neighborhood W of L_0 and thus it is algebraic (since $\mathbb{CP}^2 \setminus L_0$ is Stein). Summarizing, we have the following alternative: *any leaf L of $\mathcal{F}$ either has algebraic closure or meets Σ.*

Now choose $\mathcal{F}$ in the open set given by [LN,So]. Since there is no invariant algebraic curve, any leaf L of $\mathcal{F}$ meets Σ and thus is captured ("trapped") by the dynamics generated by the return maps $f_{1,\mathcal{F}}, \dots, f_{d,\mathcal{F}}$ on $\Sigma_{\mathcal{F}}$. Furthermore, we are in the second alternative of Proposition 7.4 namely $G_{\mathcal{F}}$ accumulates some non trivial pseudo-flow on $\mathbb{B}^1$. If we choose $\mathcal{F}$ so that, in addition, the linear part of $f_{1,\mathcal{F}}$ is not real at its fixed point, then one can ensure the minimality of $G_{\mathcal{F}}$ on $\mathbb{B}^1$ by Corollary 5.3 and hence of any leaf L in a neighborhhod of Σ. Minimality propagates everywhere in $\mathbb{CP}^2$ because, given any leaf L and any point $p \in \mathbb{CP}^2$ regular for $\mathcal{F}$, denoting by L' the leaf passing through p and by $\gamma(t)$ a path in L' joining $\gamma(0) = p$ to $\gamma(1) \in L' \cap \Sigma$, we see that L must accumulate $\gamma(1)$. By using a simple argument involving flow-boxes along γ, one easily conclude that L accumulates p as well.

By construction, such a $\mathcal{F}$ lies on an open set of the parameters accumulating $\mathcal{F}_1$. □

The only one reason for which the previous proof cannot be adapted to the general case is that, for $n \geq 2$, the complement of L_0 in $\mathbb{CP}^{n+1}$ is not anymore Stein. Thus we cannot ensure that an arbitrary leaf will accumulate L_0 and hence meet Σ. Hidden behind the recursive proof below (Theorem 7.7) is the idea that, for a foliation in $\mathbb{CP}^{n+1}$ tangent to a projective flag

$$L_0 = H^1 \subset H^2 \subset \cdots \subset H^n$$

(where H^i stands for some $i-$dimensional linear projective space), we can ensure (modulo a few restrictions) that any leaf has to accumulate H^n. Actually the complement of H^n is Stein. On the other hand, if a leaf L is contained in H^n, then it will accumulate H^{n-1} since $H^n \setminus H^{n-1}$ is again Stein. Proceeding inductively we eventually conclude that every leaf accumulates L_0.

However before presenting the argument sketched above, we shall need a last easy lemma:

Lemma 7.6. *Let A be a hyperbolic matrix. Consider $R > r > 0$ such that, for every point $x \in \mathbb{B}_r^n$ ($x \neq 0$) there exists $n_x \in \mathbb{Z}$ such that $A^{n_x}x$ belongs to $\mathbb{B}_R^n \setminus \mathbb{B}_r^n$. Assume that $g : \mathbb{B}_r^n \hookrightarrow \mathbb{B}_R^n$ is sufficiently (C^1) close to A in $\mathbb{B}_r^n$. Then the following holds:*

1. *g has a unique fixed point in $\mathbb{B}_r^n$ which will be denoted by p_g.*

2. *If $x \in \mathbb{B}^n_r$ is distinct from p_g, then there exists $n_x \in \mathbb{Z}$ such that $g^{n_x}(x)$ belongs to $\mathbb{B}^n_R \setminus \mathbb{B}^n_r$.*

Proof. The existence and uniqueness of a fixed point for g is rather well known. Thus we just need to prove assertion 2. One could employ a linearization theorem (e.g. Grobman-Hartman) however we prefer to proceed directly. Note that the differential $D_{p_g}g$ of g at p_g is close to A. Hence $D_{p_g}g$ is hyperbolic and satisfies the same assumption as A (where the origin should be replaced by p_g in the obvious sense). Since g is close to $D_{p_g}g$, we conclude that there is no loss of generality in supposing that p_g coincides with the origin. In other words we can suppose that g fixes the origin $\underline{0}$ and also that the differential of g at $\underline{0}$ is the matrix A.

Therefore let $g(\underline{0}) = \underline{0}$. If g is very close to A then, for every point $x \in \mathbb{B}^n_r \setminus \mathbb{B}^n_{r/2}$, there exists $n_x \in \mathbb{Z}$ such that $g^{n_x}(x)$ belongs to $\mathbb{B}^n_R \setminus \mathbb{B}^n_r$. Next observe that the function $\underline{x} \mapsto 2g(\underline{x}/2)$ is closer than g to A. In particular the orbit of any point in $\mathbb{B}^n_r \setminus \mathbb{B}^n_{r/2}$ under this new function also leaves $\mathbb{B}^n_r$. Nonetheless the last claim means that any point in $\mathbb{B}^n_{r/2} \setminus \mathbb{B}^n_{r/4}$ leaves $\mathbb{B}^n_{r/2}$ under iteration by g. In view of the conclusions above, it results that any point in $\mathbb{B}^n_r \setminus \mathbb{B}^n_{r/4}$ will leave $\mathbb{B}^n_r$ under a suitable power of g. Continuing this procedure, we finally establish the lemma. □

Theorem 7.7. *For any $n \geq 1$ and $d \geq 2$, there exists an open subset $\mathcal{W} \subset \mathcal{F}^d(\mathbb{CP}^{n+1})$ approximating $\mathcal{F}_0$ such that any foliation $\mathcal{F} \in \mathcal{W}$ satisfies:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+n)!}{d!n!}$ hyperbolic singularities and is regular on the complement,*
(ii) *any leaf of $\mathcal{F}$ is dense in $\mathbb{CP}^{n+1}$,*
(iii) *the holonomy pseudo-group $G_{\mathcal{F}}$ has large affine part.*

Proof. Assume that the degree $d \geq 2$ and the dimension $n \geq 2$ are fixed. Let us consider the $\frac{(d+n)!}{d!n!}$-codimensional subspace $\mathcal{F}^d(\mathbb{CP}^{n+1}, H)$ consisting of those foliations of degree d in $\mathbb{CP}^{n+1}$ which are tangent to the horizontal hyperplane $H : \{z_n = 0\}$. We also have the natural restriction map

$$\begin{cases} \mathcal{F}^d(\mathbb{CP}^{n+1}, H) & \rightarrow & \mathcal{F}^d(H \simeq \mathbb{CP}^n) \\ \mathcal{F} & \mapsto & \widehat{\mathcal{F}} = \mathcal{F}|_H. \end{cases}$$

In the sequel, any object relative to $\mathcal{F}$ will be assigned with a hat to denote its restriction to H. For instance $\mathcal{F}_1 \in \mathcal{F}^d(\mathbb{CP}^{n+1}, H)$ (in fact recall that $\mathcal{F}_1 = \mathcal{F}(M, \dots, M)$ with M diagonal) and $\widehat{\mathcal{F}}_1 = \mathcal{F}(\widehat{M}, \dots, \widehat{M})$ is a foliation (in lower dimension) possessing properties similar to those of $\mathcal{F}_1$. We now make our induction assumption namely, *we suppose that arbitrarily close to $\widehat{\mathcal{F}}_1$ in $\mathcal{F}^d(\mathbb{CP}^n)$, we have already constructed a stably chaotic foliation $\widehat{\mathcal{F}}$.* Next consider a foliation $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^{n+1}, H)$, close to $\mathcal{F}_1$, whose restriction $\widehat{\mathcal{F}}$ to H coincides with the mentioned chaotic foliation (the existence of the foliation $\mathcal{F}$ is clear after the characterization of foliations in $\mathbb{CP}^n$ by means of homogeneous polynomial vector fields in $\mathbb{C}^{n+1}$, cf. the introduction or the beginning of the present section). Furthermore by a small pertubation, fixing $\widehat{\mathcal{F}}$, of $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^{n+1}, H)$, one can also assume that $\mathcal{F}$ has only isolated singularities which are hyperbolic. Notice that the importance of our "induction assumption" lies on the fact that every leaf of $\mathcal{F}$ contained in H must accumulate L_0 while H contains no algebraic leaf (in particular L_0 is not anymore a leaf).

We divide our proof into 3 steps. The main difficulty of the proof is to establish the assertion contained in the Step A below:

Step A A *generic* perturbation $\mathcal{F}'$ of $\mathcal{F}$ is such that any leaf L' of $\mathcal{F}'$ is non-algebraic, intersects Σ and is therefore captured ("trapped") by the dynamics of the pseudo-group $G_{\mathcal{F}'}$.

By a *generic* perturbation, we mean that the set of foliations close to $\mathcal{F}$ and satisfying the conclusion above form an open subset of total Lebesgue measure contained in a small neighborhood of $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^{n+1})$.

Denote by p_s the singularities lying on H while s runs over $s = 1, \dots, \frac{d^{n+1}-1}{d-1}$ and let B_s be the corresponding germ of branch which is transversal to H. Notice that any leaf L of $\mathcal{F}$ accumulates H (since $\mathbb{CP}^{n+1} \setminus H$ is Stein); if L accumulates only singular points p_s, then L consists of finitely many branches B_s (in a neighborhood of H) and thus it has algebraic closure. On the other hand, a non-algebraic leaf L must accumulate a leaf L_1 which is dense in H (induction assumption) and therefore both leaves intersect Σ. Hence the following alternative holds: *a given leaf L of $\mathcal{F}$ either is contained in an algebraic curve transversal to H or intersects* Σ. Denote by $L_1, \dots, L_r$ the irreducible invariant algebraic curves of $\mathcal{F}$.

Claim 1: The preceding alternative is verified for every foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^{n+1})$ (i.e. if $\mathcal{F}'$ is sufficiently close to $\mathcal{F}$, then any leaf of $\mathcal{F}'$ either has algebraic closure or intersects $\Sigma_{\mathcal{F}'}$).

In order to demonstrate Claim 1, let p_s denote also the singularities of $\mathcal{F}$ which do not lie on H while s runs over $\frac{d^{n+1}-1}{d-1} + 1, \dots, \frac{d^{n+2}-1}{d-1}$. Consider also a small ball W_s centered at p_s. Finally denote by V_l, for $l = 1, \dots, r$, a suitable tubular neighborhood of the compact part of L_l given by $L_l \setminus \cup_{p_s \in L_l} L_l \cap W_s$.

First we construct a neighborhood U of the compact set $K = \mathbb{CP}^{n+1} \setminus (\bigcup_s W_s \cup \bigcup_l V_l)$ so that, for $\mathcal{F}'$ sufficiently close to $\mathcal{F}$, any leaf intersecting U will also meet Σ. Observe that, for any given point $p \in K$, there is a path γ_p contained in the leaf through p and joining $\gamma_p(0) = p$ to $\gamma_p(1) \in \Sigma$. It is then easy to find a tubular neighborhood U_p of γ_p on which $\mathcal{F}$ is regular and such that any leaf intersecting U_p will meet Σ. Now the required neighborhood U can be obtained by choosing a finite sub-covering from the open covering $\bigcup_{p \in K} U_p$.

Claim 1 is now an easy consequence of the

Claim 2: If V_l was chosen sufficiently small then, for $\mathcal{F}'$ very close to $\mathcal{F}$, any leaf in V_l (excepted a possible persistent invariant curve) will *escape* and intersect K.

Before proving Claim 2, we want to point out that it implies Claim 1. Actually it is enough to observe that a leaf in W_s will necessarily leave W_s (by the maximum principle) and therefore it will meet K or some V_l.

To prove Claim 2, note that any algebraic invariant curve L_l contains at least one singularity P_s which is hyperbolic. Furthermore the associated return map $f_l : \mathbb{B}_r \hookrightarrow \mathbb{B}_R$, defined through a parametrization $i_l : \mathbb{B}_R^n \hookrightarrow \Sigma_l$ of a transversal Σ_l, is also hyperbolic; in other words, f_l has a unique fixed point $\underline{0} = (i_l)^{-1}(\Sigma_l \cap L_l)$ and the corresponding differential $A_l = D_{\underline{0}} f_l$ has no eigenvalue of modulo one (obviously by choosing all the pertubations very small, we can also suppose that f_l is arbitrarily close to $A_l = D_{\underline{0}} f_l$). It results from Lemma 7.6 that f_l as well as any sufficiently small pertubation of f_l possesses the following property: any point of $\mathbb{B}_r^n$ other than the unique (persistent) fixed point reaches $\mathbb{B}_R^n \setminus \mathbb{B}_r^n$ either by positive or negative iteration of f.

The proof of Claim 2 is now easy. Choose a neighborhood V_l of $L_l \setminus \cup_{p_s \in L_l} L_l \cap W_s$ so that, for any small pertubation $\mathcal{F}'$ of $\mathcal{F}$, $(V_l \cap \Sigma_l)$ is contained in $i_l(\mathbb{B}^n_r)$ (which is in turn contained in the "stably repelling" part of the dynamics of f_l in view of the discussion above). Thus $i_l(\mathbb{B}^n_R \setminus \mathbb{B}^n_r)$ is contained in the compact K. Furthermore any leaf in V_l intersects the transversal Σ_l. In particular every leaf excepted the one corresponding to the fixed point $\underline{0}$ ($\simeq p_{s_l}$) of f_l (i.e. the one inducing the "horizontal" branch of p_{s_l}) necessarily "escapes" from V_l and meets K. However the special leaf inducing the "horizontal" branch of $p_{s_l} \simeq \underline{0}$ either is "fixed" by all the other return maps (i.e. $\underline{0}$ is a common fixed point for the return maps) and hence it compactfies into an invariant algebraic curve, or it cuts Σ_l away from $\underline{0}$ and "escapes" too. Claim 2 and Claim 1 are proved.

Step A now follows immediately from the combination of the claim with [LN,So] theorem.

Step B Assume that $\mathcal{F}$ is very close to $\mathcal{F}_1$. In this case, for an open choice in $\mathcal{F}^d(\mathbb{CP}^{n+1})$ of the pertubation $\mathcal{F}'$ of $\mathcal{F}$, the pseudo-group $G_{\mathcal{F}'}$ has large affine part.

Since $\mathcal{F}$ has no algebraic invariant curve within H, it follows from Proposition 7.4 that $G_{\mathcal{F}}$ does not fix any point and hence accumulates some pseudo-flow within $\mathbb{B}^n$. We point out that, even though $\widehat{G}_{\mathcal{F}}$ has large affine part, $G_{\mathcal{F}}$ (and in particular the previous pseudo-flow) preserves the horizontal hypersurface $\mathbb{B}^{n-1} = \mathbb{B}^n \cap \{z_n = 0\}$ corresponding to the transversal $\widehat{\Sigma}_{\mathcal{F}}$ on which $\widehat{G}_{\mathcal{F}}$ acts minimally.

Moreover this translation pseudo-flow persists for small pertubations $\mathcal{F}'$ of $\mathcal{F}$. Let us observe that $\mathcal{F}'$ can be chosen so that $G_{\mathcal{F}'}$ satisfies the assumptions of Corollary 5.3 in order to conclude that it has large affine part; indeed, assumptions $(**)$ needed to apply Corollary 5.3 relie on the genericity of the linear part of $f_{1,\mathcal{F}'}$ and $f_{1,\mathcal{F}'}^{-1} \circ f_{2,\mathcal{F}'}$ at the fixed point of $f_{1,\mathcal{F}'}$. Corollary 7.3 enables us to a find a deformation such that $f_{1,\mathcal{F}'}$ and $f_{1,\mathcal{F}'}^{-1} \circ f_{2,\mathcal{F}'}$ satisfy the desired generic conditions.

Step C: conclusion Let $\mathcal{F}'$ be as in steps A and B; then any leaf L of $\mathcal{F}'$ is "trapped" by the "rich" dynamics of the pseudo-group $G_{\mathcal{F}'}$ on Σ and hence is dense on a neighborhood of this transversal. The density of the leaves propagates to the whole $\mathbb{CP}^{n+1}$ by the same arguments as in the end of the proof of Lemma 7.5. □

Proof of the Main Theorem. Obviously the foliations constructed above lie on an open set $\mathcal{U}$ of foliations which are minimal and ergodic w.r.t Lebesgue (see Property 6.1). Furthermore, Property 6.2 shows that these foliations has strictly positive entropy at least near L_0, even after deleting some balls around the singularities, simply by using the return maps $f_{i,\mathcal{F}}$.

Now, let us prove the rigidity property up to rescaling $\mathcal{U}$. First of all, fix $\mathcal{F}_0 \in \mathcal{U}$ and $W \subset \mathbb{CP}^{n+1}$ a small tubular neighborhood of the singular locus of $\mathcal{F}_0$: W consists of $\frac{d^{n+2}-1}{d-1}$ disjoint balls and we can suppose that $\mathbb{CP}^{n+1}$ is the holomorphic hull of $\mathbb{CP}^{n+1} \setminus W$.

Modulo rescaling $\mathcal{U}$, we can suppose that the singular set of $\mathcal{F}$ remains in W for any $\mathcal{F} \in \mathcal{U}$. Furhtermore there is no loss of generality in supposing the existence of a finite trivialization covering of $\mathbb{CP}^{n+1} \setminus W$ for the smooth foliation $\widehat{\mathcal{F}}$ induced by restriction of $\mathcal{F}$ to $\mathbb{CP}^{n+1} \setminus W$ which depends holomorphically on the parameter $\mathcal{F} \in \mathcal{U}$. Actually such a trivialization can be constructed just by using the compactness of $\overline{\mathbb{CP}^{n+1} \setminus W}$. In particular, the pseudo-group $G_{\widehat{\mathcal{F}}}$ induced by $\widehat{\mathcal{F}}$ on Σ (may be strictly smaller than the pseudo-group

$G_{\mathcal{F}}$ induced by the whole foliation $\mathcal{F}$ but) is generated by a *finite* collection of return maps $f_{j,\mathcal{F}} : U_j \hookrightarrow \mathbb{B}^n$, $U_j \subset \mathbb{B}^n$, $j \in J$ finite, which depend holomorphically both on $\underline{z}$ and on the parameter $\mathcal{F}$. Clearly, $G_{\widehat{\mathcal{F}}}$ does contain at least the dynamics of the d return maps near L_0 considered all along this section. Therefore $G_{\widehat{\mathcal{F}}}$ has rich affine part.

On the other hand, any homeomorphism $H : \mathbb{CP}^{n+1} \to \mathbb{CP}^{n+1}$, sufficiently close to the identity, conjugating two elements $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{U}$ should induce (by a classical argument that can be found in [Il]) a conjugacy between the respective finite sets of return maps $f_{j,\mathcal{F}_0}$ and $f_{j,\mathcal{F}_1}$ by some "restriction" homeomorphism $h : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{B}^n$.

Next we notice that Proposition 6.6 applied to the finite family $(f_{j,\mathcal{F}})_{j\in J}$ implies that (up to rescaling $\mathcal{U}$) the existence of such a conjugacy H guarantees the existence of a smooth holomorphic one parameter family of foliations $\mathcal{F}_t$, $t \in U \subset \mathbb{C}$, joining $\mathcal{F}_0$ to $\mathcal{F}_1$ along with the existence of a smooth holomorphic one parameter family of diffeomorphisms $h^t : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{B}^n$ conjugating $f_{j,\mathcal{F}_0}$ to $f_{j,\mathcal{F}_t}$ for any $j \in J$ and any $t \in U$.

Now, using this fact, we construct a complex two dimensional regular foliation $\mathcal{G}$ on $U \times \mathbb{CP}^{n+1}$ transversal to the vertical hyperplanes $\Pi_t : \{t\} \times \mathbb{CP}^{n+1}$ inducing the given foliation $\mathcal{F}_t$ on Π_t. First, we construct the restriction $\widehat{\mathcal{G}}$ of $\mathcal{G}$ to $\mathbb{CP}^{n+1} \setminus W$; the whole (singular) foliation $\mathcal{G}$ will be deduced from some extension theorem.

The leaves of $\widehat{\mathcal{G}}$ are defined as follows: (t, p) and (t', p') belong to the same leaf if the respective intersections q (resp. q') between $\Sigma_{1/2}$ and the leaf passing through p (resp. p') satisfy $h_t^{-1}(q) = h_{t'}^{-1}(q')$. This is well defined for p and p' close to Σ since the intersections of the corresponding leaves with Σ is defined without ambiguity. Because h_t conjugates the respective return maps generating the respective pseudo-groups of $\widehat{\mathcal{F}_t}$ and $\widehat{\mathcal{F}_0}$ on Σ implies that this way of analytically gluing the one dimensional leaves of the $\mathcal{F}_t$ into 2-dimensional leaves extends analytically to $\mathbb{CP}^{n+1} \setminus W$. It is a classical result that the complex 2-dimensional foliation $\widehat{\mathcal{G}}$ constructed by this way is well defined, regular and holomorphic on $\mathbb{CP}^{n+1} \setminus W$ and satisfies the required property, namely $\widehat{\mathcal{G}}$ is transversal to the vertical hyperplanes Π_t (its restriction to $\mathbb{CP}^{n+1} \setminus W$ induces the given foliation $\widehat{\mathcal{F}_t}$ on Π_t). Now it suffices to use an extension theorem to extend $\tilde{\mathcal{G}}$ to W.

Finally the main result of [GM] (first part of the Rigidity Theorem) asserts that the deformation $\mathcal{F}_t$ is actually projectively trivial. The Main Theorem is proved. □

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