Problem of the Month
March 2007
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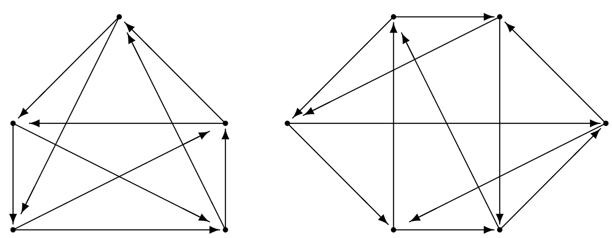
We can easily check that the statement is true for $n = 5$ and $n = 6$ in the following directed graphs:



Assume that the statement is true for some $n > 4$. Then $\exists$ a directed graph, call it G_n, such that in G_n, one can travel from every node to every other node in at most two steps.

Now, we construct a new directed graph G_{n+2} using G_n in the following way:

- Add in two nodes, X and Y to G_n.
- From every node in G_n, connect an edge to X.
- Connect an edge from X to Y.
- Connect an edge from Y to every node in G_n.

Obviously, for the nodes in G_n, one can travel from every node to every other node in at most two steps. Moreover, by construction, from every node in G_n, one can travel to X in one step and to Y in two steps (by first traveling to X). Also one can travel from Y to any node in G_n in one step and from X to G_n in two steps (by traveling to Y first). Finally, one can easily travel from X to Y in one step, and Y to X in two steps (by first traveling to any node in G_n). Therefore, in G_{n+2}, one can travel from every node to every other node in at most two steps.

As we have both $n = 5$ and $n = 6$ as our base cases, by induction, the statement holds true for any $n > 4$. QED