# Problem of the Month
# April 2006
# Solution by Itamar Gal

**Problem:**

There are three colleges in a town. Each college has n students. Any student of any college knows n+1 students of the other two colleges. Prove that it is possible to choose a student from each of the three colleges so that all three students would know each other.

**Solution:**

First note that we can reformulate the problem as follows; consider a tri-partite graph of 3-n nodes $G_n := K_{n,n,n}$ where each node has degree n+1. We want to show that this graph contains a $K_3$ subgraph (3-cycle). Next we note that each node v$\varepsilon$ $G_n$ has edges connecting it to one of two partite regions of $G_n$; we define a function f(v) that maps each node v$\varepsilon$ $G_n$ to the lesser of the two numbers of edges connecting v to a partite region (e.g. if v has i edges connected to nodes in region A and j edges connected to nodes in region B then f(v)=min{i,j}). We also define the function σ( $G_n$ ) which maps the graph $G_n$ to the integer min{f(v)|v$\varepsilon$ $G_n$ }.

Notice that σ( $G_n$ )>0 since each node has degree n+1 and each region contains n nodes; therefore each node must contain at least one node in each partite region (e.g. f(v)>0 ∀:v$\varepsilon$ $G_n$ ). Now suppose that σ( $G_n$ )=k and choose a node v1 in the partite region A such that f(v1)=k. Let v2 be one of the k nodes in the partite region B that share an edge with v1. We know that v1 must share an edge with n-k+1 nodes in region C and that v2 must share an edge with at least k nodes in region C. But (n-k+1)+k = n+1 and there are only n nodes in region C, therefore there must be some node v3 in region C that shares an edge with both v1 and v2 so that {v1,v2,v3} form a 3-cycle.▫