Problem of the Month

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Solution by Roman Kogan

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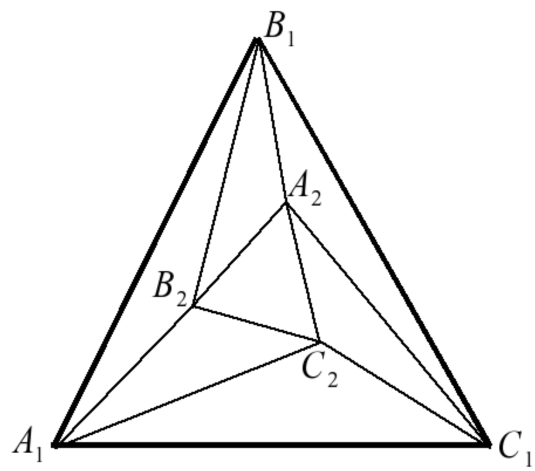
Problem:



Find the minimum and the maximum possible area of the polygon $A_1B_2B_1A_2C_1C_2$ over all locations of the smaller triangle $A_2B_2C_2$ inside the bigger triangle $A_1B_1C_1$.

Solution:

Note: throughout this solution, let $(A_1..A_n)$ denote the area of polygon $A_1..A_n$, for brevity.

Let P denote $A_1B_2B_1A_2C_1C_2$, the polygon in consideration. Let a_1 be the length of A_1C_1, let a_2 be the length of A_2C_2,

Part1: Minimization problem.

From the way P was constructed it folows that it cannot be self-intersecting. Hence the lower bound for the area of P is the area of $A_2B_2C_2$. We shall now show that this lower bound is attainable.

Indeed, let us place $A_2B_2C_2$ so that its orthocenter coincides with the one of $A_1B_1C_1$. We then are able to rotate $A_2B_2C_2$ so that angles $\angle B_2A_1C_2, \angle C_2C_1A_2, \angle A_2B_1B2$ dminish to zero and $(P) = (A_2B_2C_2)$. This intuitive approach, nevertheless, is not a proof; to show that such position is obtainable, we consider the following situation (see Figure 1):

Let three cevians B_1K_1, A_1K_2, C_1K_3 intersect at points A_2', B_2', C_2' and be such that $A_1K_1 = C_1K_2 = B_1K_3 = x$. It is clear, by symmetry, that the trinagle $A_2'B_2'C_2'$ is equilateral. Also $\angle B_2'A_1C_2' = \angle C_2'C_1A_2' = \angle A_2'B_1B'2 = 0$. It is also clear that $A_2'B_2'$, the length of side of the smaller trinangle, depends on the value of x only. Let a denote length $A_2'B_2'$, then $a = f(x)$, a continuous function of x. We observe that $f(0) = a$, and $f(a/2) = 0$. By Intermediate Value Theorem we conclude that for all posssible $y, 0 \leq y \leq a$, there is at leat one value of x such that $f(x) = y$.

Now let's get back to the original problem. We have shown that for all values of a_2 it is possible to draw three cevians so that the triangle formed by their intersection will be an

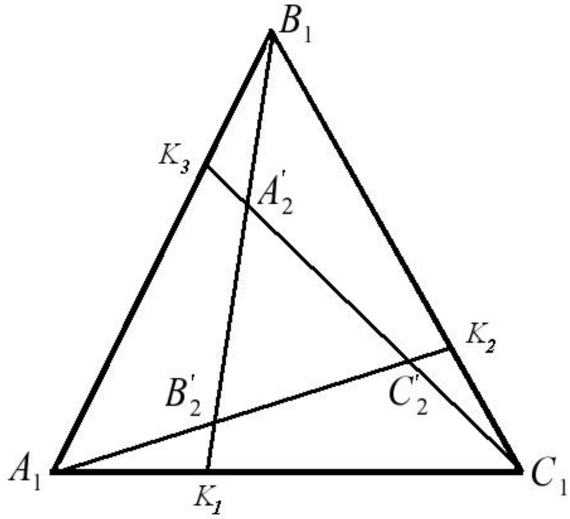


Figure 1: Minimizing position

equilateral traingle with sidelength a_2. Therefore, there will always be some position and rotation of $A_2B_2C_2$ inside $A_1B_1C_1$ such that $\angle B_2A_1C_2 = \angle C_2C_1A_2 = \angle A_2B_1B2 = 0$ and $(P) = (A_2B_2C_2)$.

Answer: $min(P) = (A_2B_2C_2)$

Part 2: Maximization problem

The proof will procced as follows: first, we show that (P) is invariant for all translations of $A_2B_2C_2$; second, we maximize (P) by choosing corresponding rotation; and then we shall find the numerical value of $max(P)$.

Lemma 1:(P) depends only on rotation of $A_2B_2C_2$, and is not changed no matter how $A_2B_2C_2$ is translated inside $A_1B_1C_1$.

Proof: Let B_2C_2 make angle α with A_1C_1. By symmetry, other sides of $A_2B_2C_2$ will make the same angle with corresponding sides of $A_1B_1C_1$ (See Figure 2).

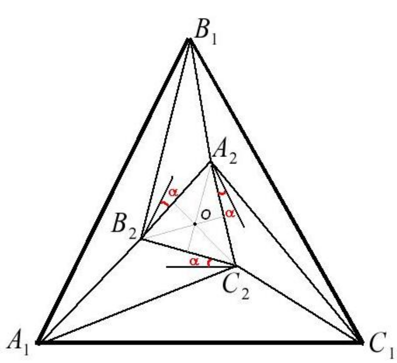


Figure 2: Some rotation of $A_2B_2C_2$

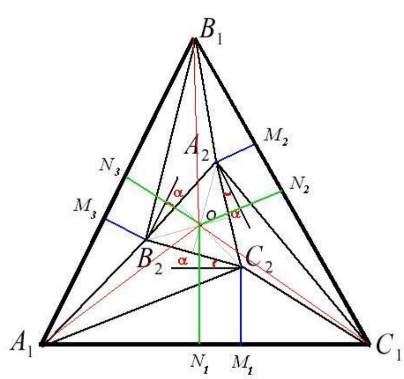


Figure 3: Proof of Lemma 1

Now let O be the orthocenter of $A_2B_2C_2$. Let us draw altitudes ON_1, ON_2, ON_3 and

C_2M_1, A_2M_2, B_2M_3 onto A_1C_1, C_1B_1, B_1A_1, correspondingly.

We first observe that $ON_1 + ON_2 + ON_3 = \frac{(A_1B_1C_1)}{a_1}$ and is therefore invariant. Indeed, $(A_1B_1C_1) = (A_1OC_1) + (C_1OB_1) + (B_1OA_1) = a_1(ON_1 + ON_2 + ON_3)$ Secondly, we observe that $C_2M_1 + A_2M_2 + B_2M_3 = ON_1 + ON_2 + ON_3 - c$, where $c = g(\alpha)$. This follows from the fact that $C_2M_1 = ON_1 - OC_2(sin(\alpha + \frac{\pi}{6}))$; using symmetry we conclude that $g(\alpha) = 3(OC_2)(sin(\alpha + \frac{\pi}{6}))$.

(Note that since O is the orthocenter of $A_2B_2C_2$, rotation will not affect lengths of altitudes from O.)

Now we use the fact that $(P) = (A_1B_1C_1) - \frac{a_1}{2}(C_2M_1 + A_2M_2 + B_2M_3)$. From this we have: $(P) = (A_1B_1C_1) - \frac{a_1}{2}(ON_1 + ON_2 + ON_3) + \frac{a_1}{2}(g(\alpha)) = \frac{a_1}{2}(g(\alpha))$.

This proves Lemma 1.

Now we are facing a simple maximization problem: we have to maximize $(P) = \frac{3}{2}a_1(OC_2)(sin(\alpha + \frac{\pi}{6}))$. Clearly, if $a_2 \leq \frac{a_1}{2}$, then maximum exists and is obtainable at $\alpha = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$. This value of α is obtainable, for if we place $A_2B_2C_2$ inside $A_1B_1C_1$ so that their orthocenters coincide, the circumcircle of $A_2B_2C_2$ will be entirely inside $A_1B_1C_1$. Geometrically, let O be the common orthocenter of both triangles, and let A_1O, B_1O, C_1O intersect B_1C_1, C_1A_1, A_1B_1 at K_3, K_2, K_1 correspondingly (See Figure 4). Then

$$a_2 \leq \frac{a_1}{2} \Leftrightarrow OA_2 \leq \frac{OA_1}{2} \Rightarrow OA_2 \leq OK_1,$$

which implies that we can vary the angle of rotation α without restrictions.

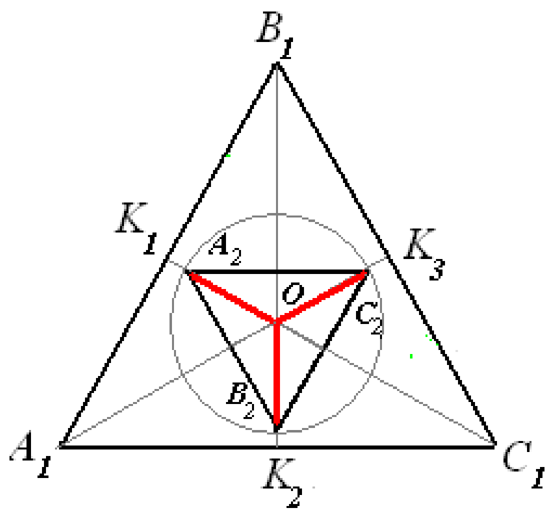


Figure 4: Rotation is unrestricted

In such case, $max(P) = 3a_1(OC_2)$. Using simple euclidian geomery, we establish that $OC_2 = \frac{2}{3}\frac{\sqrt{3}}{2}a_2 = \frac{a_2}{\sqrt{3}}$. Therefore, $max(P) = a_1a_2\frac{\sqrt{3}}{2}$ if $a_2 \leq \frac{a_1}{2}$.

However, this must not be necesserily be the case; it may happen that $a_2 > \frac{a_1}{2}$. We find $max(P)$ in this case by observing that the upper bound of (P) is $(A_1B_1C_1)$ (since P lies entirely within $A_1B_1C_1$) and showing that this upper bound is obtainable.

Consider Figure 5. Let us place A_2, B_2, C_2 on sides of $A_1B_1C_1$ so that $A_1B_2 = C_1C_2 = B_1A_2 = x$. Then $A_2B_2C_2$ is an eqilateral triangle, and $a_2 = (A_2B_2) = f(x)$, a continuous

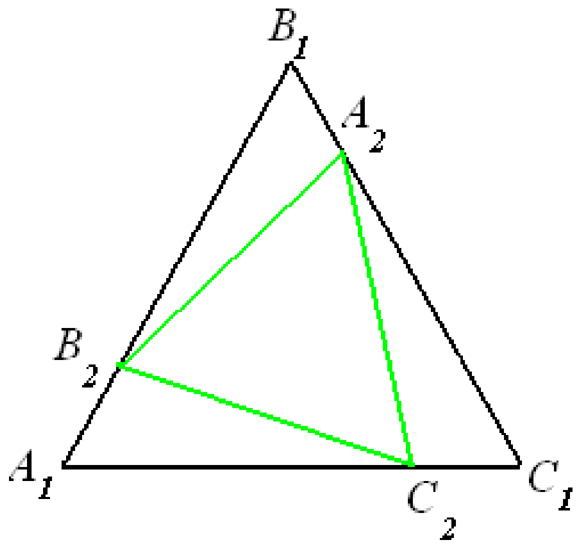


Figure 5: Upper bound reached

function of x. Note that $(P) = (A_1B_1C_1)$ in such position of $A_2B_2C_2$. Now since $f(x)$ is strictly decreasing on $[0, \frac{a_1}{2})$ and is symmetric around $\frac{a_1}{2}$, we conclude that $min(f(x)) = f(\frac{a_1}{2}) = \frac{a_1}{2}$ and $max(f(x)) = f(0) = a_1$. By Intermediate Value Theorem, f(x) takes all values in $[\frac{a_1}{2}, a_1]$ We have therefore shown that for all y in $[\frac{a_1}{2}, a_1]$, if $a_2 = y$, then the upper bound $(P) = (A_1B_1C_1)$ is reachable by placing vertices of $A_2B_2C_2$ on sides of $A_1B_1C_1$. Now our solution is complete.

Answer:

$$max(P) = \begin{cases} a_1a_2\frac{\sqrt{3}}{2} = (A_1C_1)(A_2C_2)\frac{\sqrt{3}}{2}, & \text{if } a_2 \leq \frac{a_1}{2} \\ (A_1B_1C_1), & \text{if } a_2 > \frac{a_1}{2} \end{cases}$$