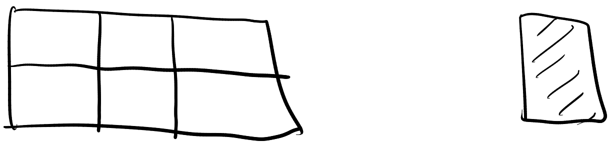
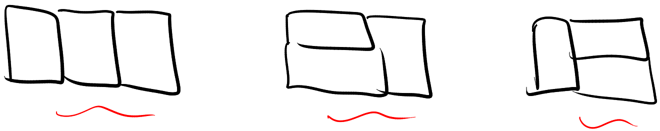
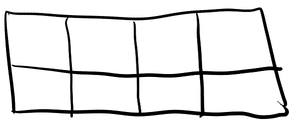
§. Tiling Grids by Dominos.

- 



$d_3 = 3$ ways!

- 

$d_4 = 5$ ways.

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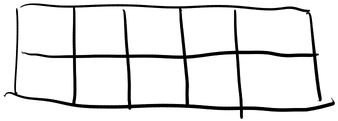


- - - -

d_5
$\shortparallel$
8 ways.

~~8~~ ~~9~~ ~~A~~ - .

•

2

7

- - -

d_7
$\shortparallel$
21 ways

√ •

2

20

- - - - 30 ?50 100 200 .

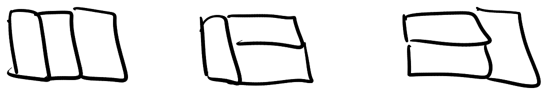
T F —

d_{20}
$\shortparallel$
~11000

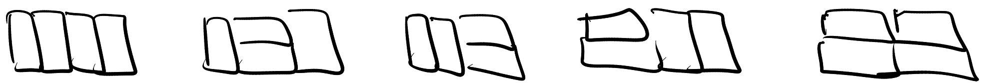
d_3	d_4	d_5	d_6	d_7	$\cdots$	d_{20}	
$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$		$\shortparallel$	$\cdots$
3	5	8	13	21	$\cdots$	~ 11000	

<u>Goal</u>: Show that $d_{n+2} = d_{n+1} + d_n$.

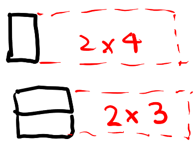
<u>2x3</u>



<u>2x4</u>



<u>2x5</u>

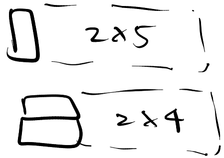


5

3

$=$ 8

2×6



8

5

13

1 2 3 5 8 13 21 34 55 89 144 ...

$$1+2+3+\dots+n = \frac{n(n+1)}{2}$$

Claim

$$d_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

$\boxed{\text{proof 1}}$

$$d_{n+2} = d_{n+1} + d_n$$

$d_0 = 1$

$d_1 = 1$

$d_2 = 2$

$d_3 = 3$

(Step 1) We can find α, β s.t.

$$\boxed{(d_{n+2} - \beta d_{n+1}) = \alpha (d_{n+1} - \beta d_n)} \quad (*)$$

$$\Updownarrow$$

Let's find them

$$d_{n+2} = (\alpha+\beta) d_{n+1} + (-\alpha\beta) d_n$$

$\Rightarrow$ Solve $\begin{cases} \alpha + \beta = 1 \\ \alpha\beta = -1 \end{cases}$

$$\Rightarrow \quad \alpha = \frac{1+\sqrt{5}}{2}$$

$$\beta = \frac{1-\sqrt{5}}{2}$$

(step 2): Cancellation of recursion !

$$
\begin{aligned}
d_n - \beta d_{n-1} &= \alpha^1 \left(\cancel{d_{n-1} - \beta d_{n-2}} \right) \\
\alpha \quad \cancel{d_{n-1} - \beta d_{n-2}} &= \cancel{\alpha^{1+1} \left(d_{n-2} - \beta d_{n-3} \right)} \\
\cancel{\alpha^2 \quad d_{n-2} - \beta d_{n-3}} &= \cancel{\alpha^{1+2} \left(d_{n-3} - \beta d_{n-4} \right)} \\
\alpha^3 \quad & \vdots \\
\vdots \quad & \\
+) \quad \alpha^{n-2} \quad \cancel{d_2 - \beta d_1} &= \alpha^{1+(n-2)} \left(d_1 - \beta d_0 \right)
\end{aligned}
$$

$$d_n - \beta d_{n-1} = \alpha^{n-1}(d_1 - \beta d_0)$$

$$= \alpha^{n-1}(\underline{1-\beta})$$

$$= \alpha^n$$

$$\Rightarrow \quad d_n - \beta d_{n-1} = \alpha^n$$

$$d_n - \beta d_{n-1} = \alpha^n$$

$$\beta \quad \beta\left(d_{n-1} - \beta d_{n-2}\right) = \alpha^{n-1} \cdot \beta$$

$$\beta^2 \qquad \vdots \qquad \beta^2$$

$$+) \; \beta^{n-1}\left(d_1 - \beta d_0\right) = \alpha^1 \cdot \beta^{n-1}$$

$$d_n - \beta^n d_0 = \alpha^n + \alpha^{n-1}\beta + \alpha^{n-2}\beta^2 + \dots + \alpha^1 \beta^{n-1}$$

$$d_0 = 1$$

$$d_n = \alpha^n\beta^0 + \alpha^{n-1}\beta^1 + \alpha^{n-2}\beta^2 + \cdots + \alpha^2\beta^{n-2} + \alpha^1\beta^{n-1} + \alpha^0\beta^n.$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

#

Q Just how large is d_{100}?

$$d_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

$\sqrt{5} = 2.24\ldots$

$n = 100$

positive

0.0000000000000000000016…

$$d_{100} = \frac{1}{\sqrt{5}} \cdot \phi^{101}$$

$\phi = 1.618\ldots$

$$\left(\frac{1-2.24}{2} \right)^{n+1} = (-0.62)^{n+1} \longrightarrow 0$$

as $n \to \infty$.