The Complex Hénon Family

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The Hénon map was introduced by the astronomer and applied mathematician Michel Hénon in the 1960's. This is the automorphism of $\mathbf{R}^2$ given by the following formula.

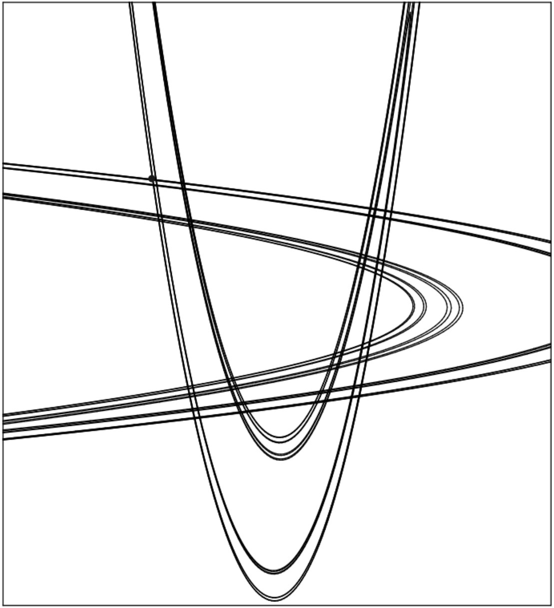
Definition (Hénon Family)

$$f_{c,\delta}(x, y) = (c + \delta y - x^2, -x).$$

The parameter δ is the Jacobian of the map and the map is invertible when $\delta > 0$.
For particular values of δ and c suggested by Hénon the diffeomorphism seem to exhibit a strange attractor. In particular it demonstrates expansion, contraction and folding on many scales.

The Hénon attractor

A Hénon horseshoe



In the 1980's Hubbard suggested that it would be profitable to study the extensions of these polynomial diffeomorphisms to $\mathbf{C}^2$. This is the complex Hénon family:

$$f_{c,\delta} : \mathbf{C}^2 \to \mathbf{C}^2.$$

We allow the coefficients to be real or complex. Thus the parameter space is also $\mathbf{C}^2$.

Hubbard was motivated in part by the successful theory of the family $z \mapsto z^2 + c$ and in part by the prominence of the (real) Hénon family in the field of dynamical systems.

With the family $z \mapsto z^2 + c$ in mind Hubbard defined analogs of Julia sets and filled Julia sets for the complex Hénon family.

Definition

$$K^{\pm} = \{p \in \mathbf{C}^2 : f^n(p) \nrightarrow \infty \text{ as } n \to \pm\infty\}$$

Definition

$$J^{\pm} = \partial K^{\pm}$$

The set J contains all hyperbolic periodic points. The set J^+ contains stable manifolds of points in J. The set J^- contains unstable manifolds of points in J.

Definition

We say that $f_{c,\delta}$ is Axiom A if f is hyperbolic on J.

Remark

In the Axiom A case the non-wandering set of f consists of J and finitely many periodic sink or source orbits.

In one variable complex dynamics the rate of escape function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|$$

plays a role.

Remark

The function G is *subharmonic*.

For Hénon maps there are two rate of escape functions.

Definition

Let

$$G^{\pm}(p) = \lim_{n\to\infty} \frac{1}{d^n} \log^{+} ||f^{\pm n}(p)||.$$

Remark

The functions G^+ and G^- are *pluri-subharmonic* which is to say that its restriction to any one dimensional complex subvariety is subharmonic.

One way to describe the harmonic measure or equilibrium measure (or balanced measure) on a Julia set is as follows where d is the exterior derivative and d^c is the "twisted" exterior derivative rotated using the complex structure.

Definition

$$\mu = dd^c G$$

Question

What is dd^c?

It is a holomorphically invariant version of the in the Laplacian. The Laplacian takes real functions to real functions but is not holomorphically invariant. dd^c takes the smooth real functions h to the two form $\triangle h \, dx \wedge dy$. It can be extended to on operator taking subharmonic functions to measures.

The operator dd^c is defined in any complex manifold so it makes sense in $\mathbf{C}^2$.

Definition

$$\mu^{\pm} = dd^c G^{\pm}$$

$\mu^{\pm}$ are currents. We can think of them as transverse measures which assign a measure to holomorphic transversals. They are analogous to the Margulis transverse invariant measure in hyperbolic dynamics and they are equal to Marguis measure in the Axiom A Hénon case.

Definition

$$\mu = \mu^+ \wedge \mu^-$$

In the Axiom A case μ is the Bowen measure which is the unique measure of maximal entropy.

Theorem (Bedford-Lyubich-S)

The measure μ is the unique measure of maximal entropy. It describes the distribution of periodic points. The support of μ is the closure of the set of periodic saddle points.

Definition Hubbard forgot to make

Definition

J^* is the closure of hyperbolic periodic points.

Alternatively J^* is the support of μ or the Shilov boundary of K.

The potential theory techniques in multidimensional complex dynamics have been extensively developed and applied to a wide range systems. That is nice but we should not lose sight of the fact that most problems for the Hénon family remain unsolved. Presumably potential theory is one of *many* tools which will be useful in attacking these problems.

How should one approach this family of examples? Here are some possible questions:

- Give many concrete examples of structurally stable maps.
- Give a computationally effective way to tell if a particular parameter corresponds to an Axiom A diffeomorphism and construct a combinatorial model of it.
- Describe concrete mechanisms which produce instability.
- Understand the global behavior of structural stability and instability.
- What is the topology of the horseshoe locus?

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There are two simple ways that stability can fail in one complex dimension: Misiurewicz parameters and parabolic parameters. I will describe one dimensional examples of both types of maps and then look at two dimensional analogs.

Example

An example of parabolic behavior in dimension 1 is the map $z \mapsto z^2 + 1/4$ where the map has a parabolic fixed point.

In this case the Julia set (the cauliflower) is a topological circle and the map is topologically conjugate on J to the hyperbolic maps $z \mapsto z^2 + c$ for $c < 1/4$. The maps with $c > 1/4$ are not conjugate.

Definition

A point $p \in \mathbf{C}^2$ which is periodic of period n is semi-parabolic if $Df^n(p)$ has eigenvalues λ and μ where λ is a root of unity and $|\mu| < 1$.

A map f_{c_0} with a parabolic periodic point gives rise to a curve of diffeomorphisms $f_{c,\delta}$ with semi-parabolic fixed points where $c = c(\delta)$ and $c_0 = c(0)$.

Theorem (Radu)

Given a map f_{c_0} with a parabolic periodic point there is a constant $\delta_0 > 0$ so that the diffeomorphisms $f_{c,\delta}$ are topologically conjugate on their Julia sets for $|\delta| < \delta_0$.

In their lectures here Milnor and Hubbard described the phenomenon of parabolic implosion. This phenomenon shows that the Julia sets J_c and filled Julia sets K_c do not vary continuously with the parameter c. This phenomenon occurs in the Hénon family as well.

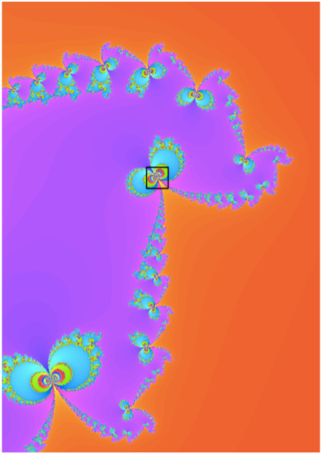
Theorem (Bedford-S-Ueda)

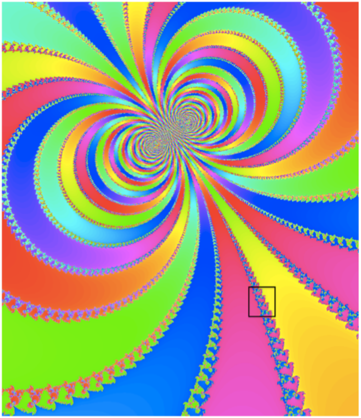
At at a parameter value with a semi-parabolic fixed point with eigenvalues 1 and δ the sets J^*, J, J^+, K and K^+ vary discontinuously with the parameters.

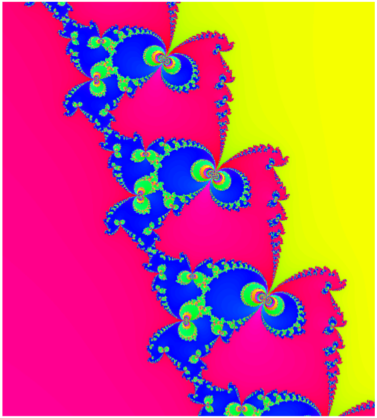
Remark

The remaining sets J^- and K^- vary continuously with the parameter.

Two dimensional parabolic implosion







Example

The Ulam-Von Neuman map $z \mapsto z^2 - 2$ can be thought of either as a real or complex dynamical system. It demonstrates Misiurewicz behavior in 1 dimension.

The critical point 0 is pre-periodic but not periodic, it maps to the fixed point 2. This map is expanding but not uniformly expanding and not structurally stable.

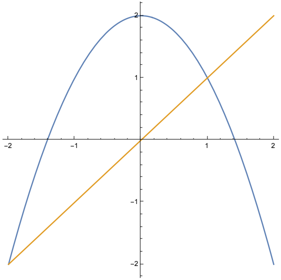
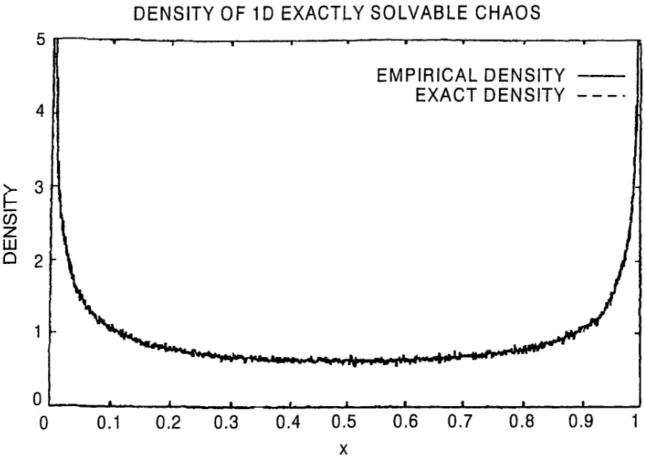
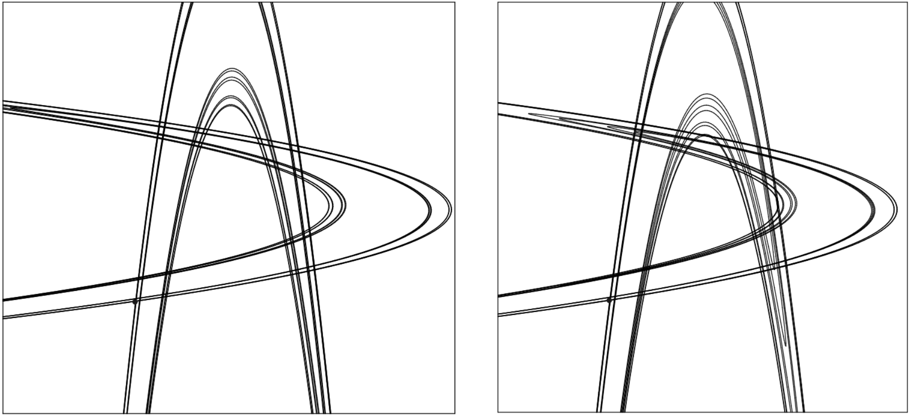


FIG. 7


DENSITY OF 1D EXACTLY SOLVABLE CHAOS
EMPIRICAL DENSITY
EXACT DENSITY
DENSITY
x
0
1
2
3
4
5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

Misiurewicz type behavior



Theorem (Bedford-S)

Let f be a Hénon diffeomorphism on the boundary of the horseshoe locus. Let p be the non-flipping fixed point. Over the set $J - \{p\}$ there are stable and unstable metrics which are uniformly expanding. These metrics blow up at p.

Remark

At each point the unstable metric expands by a factor greater than 2.

Remark

There is a unique orbit along which stable and unstable manifolds are tangent.

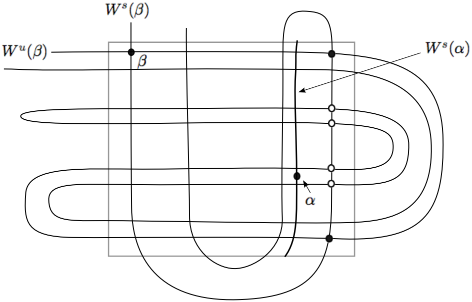
The proof of this result exploits some very tight connections between real diffeomorphisms with maximal entropy and the measure μ. A diffeomorphism on the boundary of the horseshoe locus has the property that the entropy of its restriction to $\mathbf{R}^2$ is maximal. Since μ is the unique measure of maximal entropy it must be supported on $\mathbf{R}^2$. We know that the support of μ is the Shilov boundary of K. It follows that K which is a priori a subset of $\mathbf{C}^2$ is in fact a subset of $\mathbf{R}^2$.

Remark

Just as the Ulam-Von Neuman map is a very special example of a Misurewicz parameter, real maps of maximal entropy are very special examples of Misurewicz type behavior in the Hénon family.

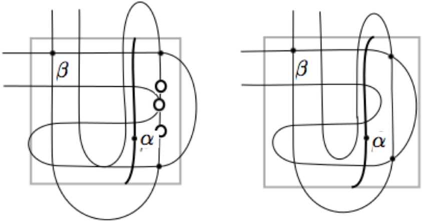
We are also interested in developing techniques which could apply more generally to understanding Misurewicz type behavior in the Hénon family. To this end we have defined a notion of quasi-hyperbolicity which captures some of the properties of these real maximal entropy examples. Quasi-hyperbolicity is defined in terms of a locally bounded area condition for stable and unstable manifolds. On the other hand we are not finished with the horseshoe.

We also have looked at limits of horseshoe maps using tools which have the potential to be used more widely. The idea is to make an assumption analogous to assuming that our parameter is real and lies in the “period 2 wake”. We do not know how to define wakes in general. These assumptions imply that the fixed point labeled α is real, a flipping saddle point and that the stable manifold of β cuts through the box shown.



This allows us to define a coding of points relative to the local stable manifold of α. We define a "right-left" coding of an itinerary, with "0" to the left and "1" on the right. The four white dots have the coding $\overline{0}1010\overline{0}$.

These two schematic pictures show the situation where the number of points with the coding $\overline{0}1010\overline{0}$ is 3 on the left and less than 3 on the right.



Theorem (Bedford-S 2014)

Let f be an orientation-preserving real Hénon map satisfying the a priori parameter restrictions. Consider the collection of (real) points with coding sequence $\overline{0}1010\overline{0}$. There are at most 4 such points, and:

1. *If there are exactly 4 such points, then f is a hyperbolic horseshoe, by which we mean that it is hyperbolic and conjugate to the full 2-shift.*
2. *If there are 3 such points, then f has a quadratic tangency but entropy $\log 2$.*
3. *If there are less than 3 such points, then f has entropy less than $\log 2$.*

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Theorem (Bedford-S 2014)

If f is as in case (ii) above, then f is topologically conjugate to the shift map on $\Sigma_2/\sim$, which is the quotient of the full shift on two symbols $\{a, b\}$, modulo the identification $\bar{a}bab\bar{a} \sim \bar{a}bbb\bar{a}$.

Remark

The problem with proving this is finding a natural map from a 2-shift to J. We cannot construct such a map by coding. The solution is to take advantage of the complex structure and define a landing map from $J^- - J$ to K. This map will not be defined everywhere since certain external rays hit critical points. It turns out that the rays that hit critical points correspond to dyadic angles and thus have two binary codings. If we code the solenoid by the full 2-shift using dyadic expansion then when a ray hits a critical point we have two continuations, a right continuation and a left continuation. These correspond to two different dyadic codings. Thus we get a map from dyadic codings to J. This map is generically two to one. If we mod out by complex conjugation we get the map we seek which is generically one to one.