Tropical Complex Dynamics

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Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree ≥ 2.

A Fatou component is a connected component of the Fatou set $F(f)$.

No Wandering Domain Theorem. (Sullivan) Every Fatou component is eventually periodic.

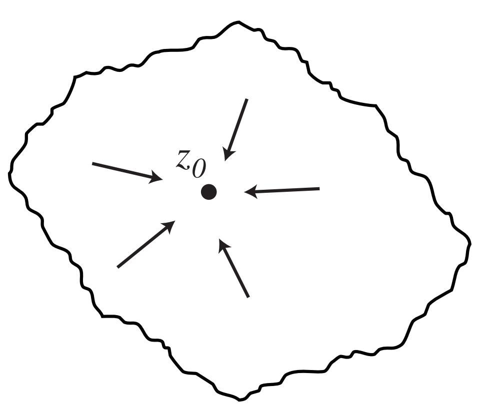
Classification Theorem. (Sullivan) If U is a periodic Fatou component ($f^p : U \to U$), then U is one of the following types:

(AB) Attracting basin orbits in U are attracted under f^p to an attracting periodic point in U;

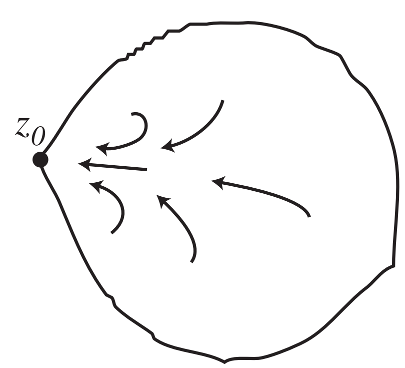
(PB) Parabolic basin orbits in U are attracted under f^p to a parabolic periodic point on ∂U;

(SD) Siegel disk $f^p|_U$ is holomorphically conjugate to $z \mapsto e^{2\pi i\alpha} z$ on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ ($\alpha \in \mathbb{R} \setminus \mathbb{Q}$);

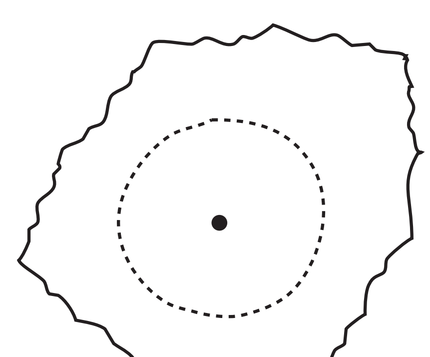
(HR) Herman rings $f^p|_U$ is holomorphically conjugate to $z \mapsto e^{2\pi i\alpha} z$ on $A_r = \{z \in \mathbb{C} : r < |z| < 1\}$ ($0 < r < 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$).

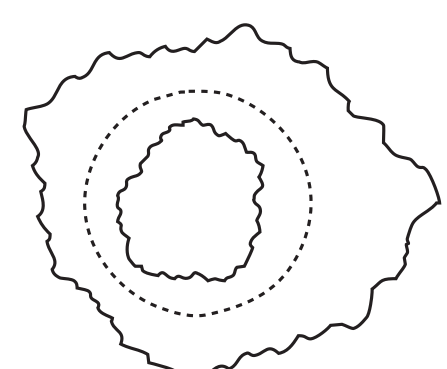


attracting or superattracting basin



parabolic basin

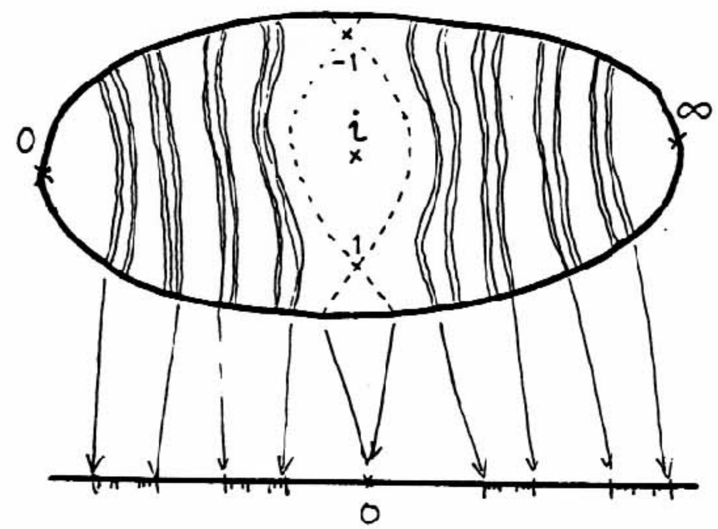

Siegel disk

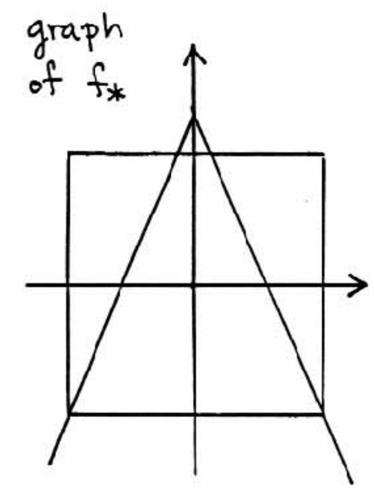

Herman ring

Goal

For a rational map with (multiply connected) Fatou components, define a tree and a piecewise linear map on it, which encode the dynamics on Fatou components. The tree is also related to the limit of quasiconformal deformation

$$f(z) = c\left(\frac{z}{1+z^2}\right)^3$$





Application of the tree map:

Surgery construction (How to design), and limit of degeneration.

Connectivity of Julia set for Newton's method of a polynomial.

Determination of configuration of Herman rings.

Wandering Julia components.

cf. S. 1987, 1989, McMullen-DeMarco-Pilgrim, Thurston-Douady-Hubbard, Cui-Tan Lei, J. Kiwi (arithmetic dynamics of Puiseux series field), M. Arfeux

Sample problems

Problem (Beardon): Let $p \geq 3$ be an integer. Construct a rational map f which has a Fatou component U which is p-connected (the complement has p connected component). Such a component must be strictly pre-periodic. Give an explicit form of f and prove this property without surgery (by an elementary method). What is the lowest degree of such a rational map?

§11.7. A Finitely Connected Component of F (Beardon's book)

If P is a polynomial, then every component of $F(P)$ is simply connected or infinitely connected. If R is rational, then R may have Herman rings and these are doubly connected. Using quasiconformal mappings, I.N. Baker has established (implicitly) the existence of components of a Fatou set of any given connectivity: here, we give an explicit example (suggested by Shishikura) of a Fatou set with a component of finite connectivity greater than two.

Let

$$R(z) = \frac{z^2(1 + t^{12}z^3)}{(1 - t^4 z)(1 - tz)^3}, \qquad t > 0.$$

We shall show that if t is sufficiently small, then $F(R)$ *has a component of connectivity* 3 *or* 4 (it seems probable that with a little more work, one could compute the connectivity exactly, but this is not the main point of the example).

Problem: Construct a rational map f which has an Herman ring of period two. Find an explicit form of f and draw its picture.

Start with f_0 (say $z^2 + c$) with period two SD cycle $\{z_0, z_1\}$. Choose a, b close to z_0 and take $f(z) = z^2 \dfrac{z-a}{z-b} + c$. Adjust a, b so that (rescaled) return map near z_0 looks like a map with a SD. (Moving frame method.)

Tropical Geometry

For $x, y \in \mathbb{R}$, define new operations:

$$x \text{ “}+\text{” } y := \max\{x, y\}, \quad x \text{ “}\times\text{” } y := x + y$$

interpretation: let $x = \log_T \tilde{x}$, etc. with $T > 1$.

$$x \text{ “}\times\text{” } y = \log_T(\tilde{x}\tilde{y}), \quad x \text{ “}+\text{” } y = \lim_{T\to\infty} \frac{\log(T^x + T^y)}{\log T} = \lim_{T\to\infty} \log_T(\tilde{x} + \tilde{y})$$

Algebraic geometry $\longrightarrow$ Piecewise linear geometry

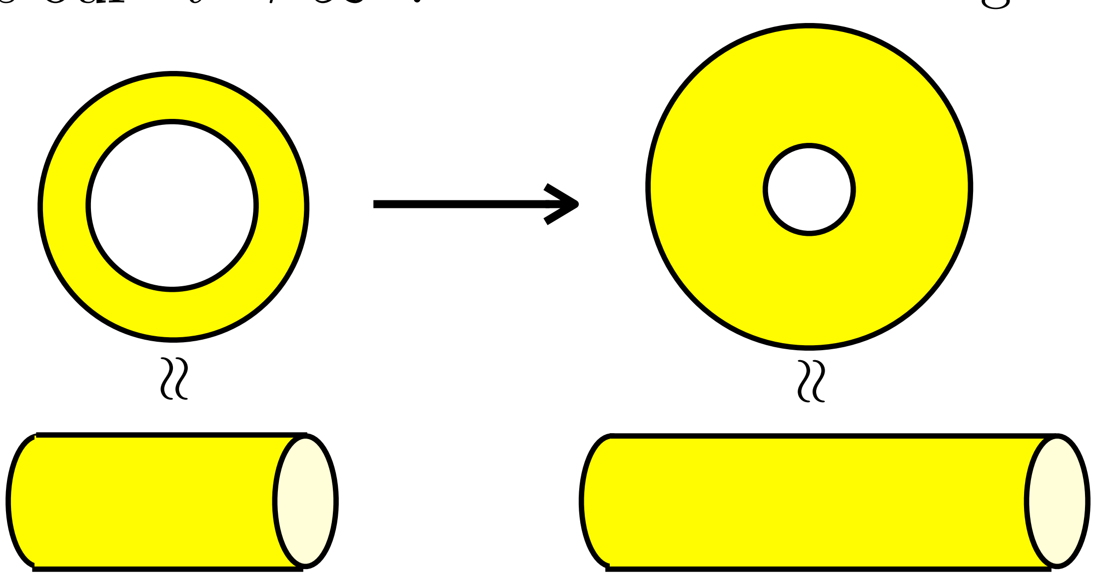
“Tropical Complex Dynamics”

Piecewise linear dynamics defined on a tree, which is related to a certain limit operation (t-stretching qc-deformation)

WARNING: Not Hubbard tree; Not interesting when the Julia set is connected, e.g. PCF maps

What is our “$t \to \infty$”?

t-stretching of annuli



Annuli

An annulus A is a doubly connected domain in $\widehat{\mathbb{C}}$.

There is a conformal map $\varphi_A : A \to \{z \in \mathbb{C} | r < |z| < R\}$, and its *modulus* is defined by

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{R}{r}.$$

$\operatorname{mod}(A)$ is conformal invariant; if $f : A_1 \to A_2$ is a conformal covering of degree k then $\operatorname{mod}(A_2) = k \operatorname{mod}(A_1)$. Moreover,

Grötzsch inequality: Let $A_1, A_2, \ldots$ be disjoint annuli contained in an annulus A. If A_i's are essential (π_1 injects), then

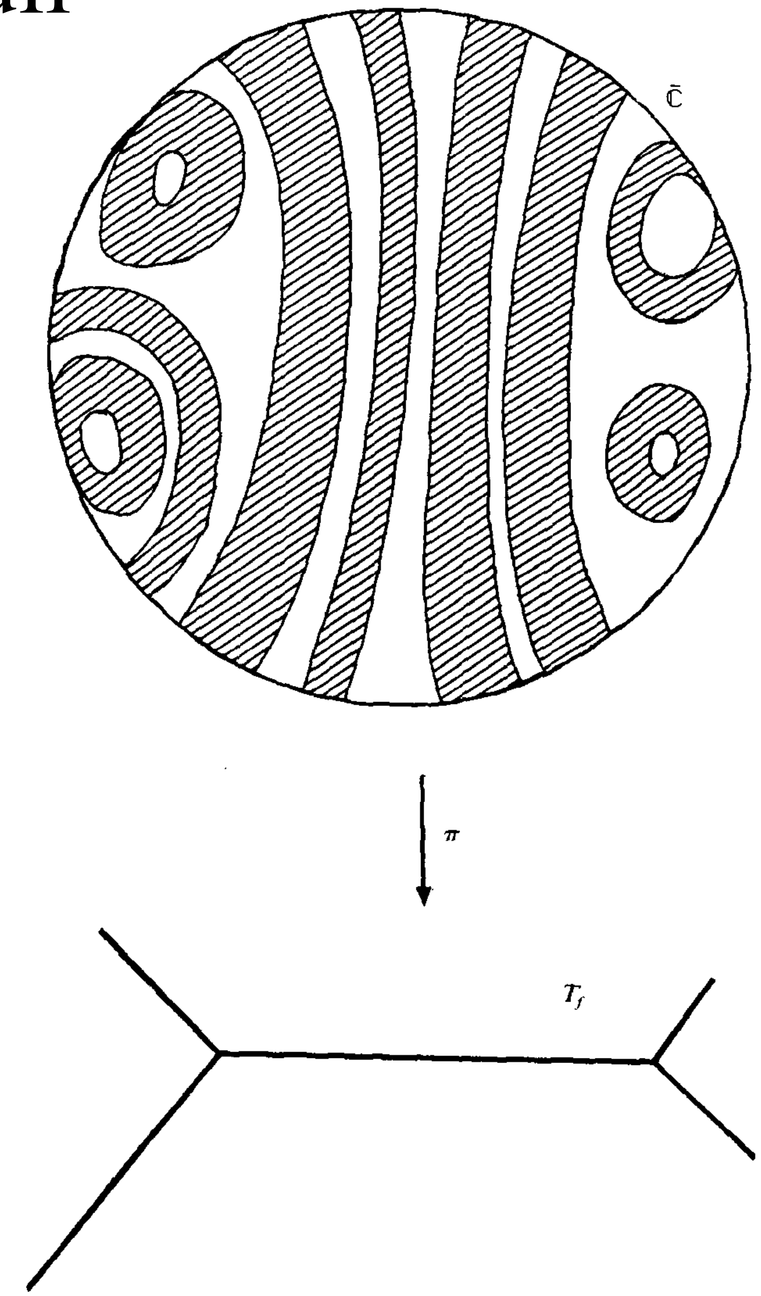
$$\sum_i \operatorname{mod}(A_i) \leq \operatorname{mod}(A).$$

An annulus A is foliated by "circles" $\varphi_A^{-1}(r)$.

For $x, y \in \widehat{\mathbb{C}}$, let $A[x, y]$ be the union of the circles that separate x and y.

Define $\operatorname{mod} \emptyset = 0$.

Tree from disjoint annuli



Suppose that $\mathcal{A}$ is a collection of disjoint annuli in $\widehat{\mathbb{C}}$.

Define $d = d_{\mathcal{A}} : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ by

$$d(x, y) = \sum_{A \in \mathcal{A}} \operatorname{mod} A[x, y].$$

Grötzsch inequality implies that

$$d(x, z) \leq d(x, y) + d(y, z).$$

Let $T = T_{\mathcal{A}} = \widehat{\mathbb{C}} / \sim_{\mathcal{A}}$, where $x \sim_{\mathcal{A}} y \Leftrightarrow d(x, y) = 0$.

"circle" $\longrightarrow$ a point, annulus $A \in \mathcal{A} \longrightarrow$ a segment,
a complementary component of $\cup \mathcal{A} \longrightarrow$ a point.

Fact: $T_{\mathcal{A}}$ is a tree and $d(\cdot, \cdot)$ is a (geodesic) metric on $T_{\mathcal{A}}^{finite}$, where $T_{\mathcal{A}} = T_{\mathcal{A}}^{finite} \sqcup T_{\mathcal{A}}^{\infty}$ with $T_{\mathcal{A}}^{\infty} = \{x \in \widehat{\mathbb{C}} : d(x, y) = \infty \text{ for } y \neq x\}$. The image of circles are dense.

pf. Jordan's curve theorem.

Tree associated with a rational map

For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$ with super attracting basins (SAB), attracting basins (AB), Siegel disks (SD), Herman rings (HR), one can associate a collection $\mathcal{A}_f$ of disjoint annuli as follows. (Not for parabolic basins)

Each of SAB, AB, SD, HR is (eventually) conjugate to a model dynamics:

$$z \mapsto z^k \ (k \geq 2); \quad z \mapsto \lambda z \ (0 < |\lambda| < 1); \quad z \mapsto e^{2\pi i \alpha} z \ (\theta \in \mathbb{R} \setminus \mathbb{Q}).$$

The family of concentric circles is preserved by the model dynamics. After removing circles which intersect the grand orbits of critical points, and taking all inverse images, one obtains a collection $\mathcal{A}_f$ of disjoint annuli in the Fatou set.

Define the tree $T = T_f = T_{\mathcal{A}_f}$ as before.

The map f induces a map $F = F_f : T_f \to T_f$.

Fact: $F : T \to T$ is continuous (w.r.t. the quotient topology) and if $[x, y]$ is an arc in T such that the "full annulus" corresponding to (x, y) does not contain critical points, then $A' = f(A)$ is an annulus, $f : A \to A'$ is a covering map and

$$d(F(x), F(y)) = k \, d(x, y),$$

where k is the covering degree of $f|_A$. (We will denote $DF(z) := k$ for an unbranched point $z \in (x, y)$.)

cf. McMullen-DeMarco-Pilgrim tree for polynomials

Extracting finite skeleton

Sometimes it is more convenient to extract a tree with finite topological type.

Choose a set $X \subset \widehat{\mathbb{C}}$ such that X is disjoint from $\mathcal{A}_f$, $f(X) \subset X$ and X consists of a finite number of connected components.

Example: $X = \{$all non-repelling periodic points$\} \cup$ (union of all the boundaries of Siegel disks and Herman rings). (together with a finite number of inverse images)

Let $\mathcal{A}_{f.X} = \{A \in \mathcal{A}_f |\ A$ separates $X\}$.
Then $T = T_{f,X} = T_{\mathcal{A}_{f,X}}$ and $F = F_{f,X} : T \to T$ is defined as before.

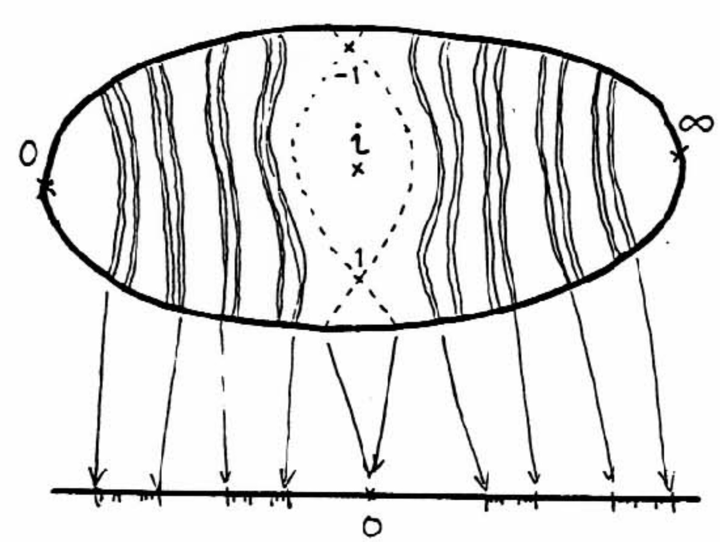
Theorem: Let f, X, $F : T \to T$ be as above. Then T is a tree of finite topological type and F is "piecewise linear" with $DF \in \mathbb{N}$. For a dense set of points the orbit lands on a periodic segment on which F^p is conjugate to $x \mapsto kx$, $x \mapsto x + c$ or id on a half line or a finite segment.

$$f(z) = c\left(\frac{z}{1+z^2}\right)^3$$

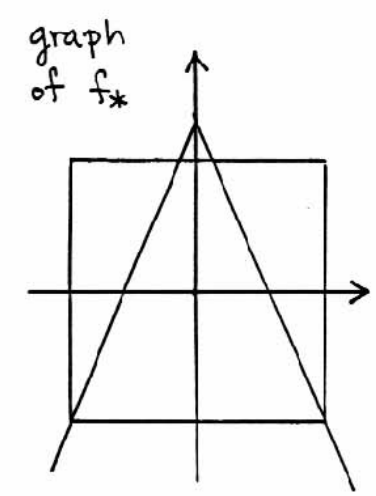
or

$$f(z) = z^m + \frac{\varepsilon}{z^n}$$

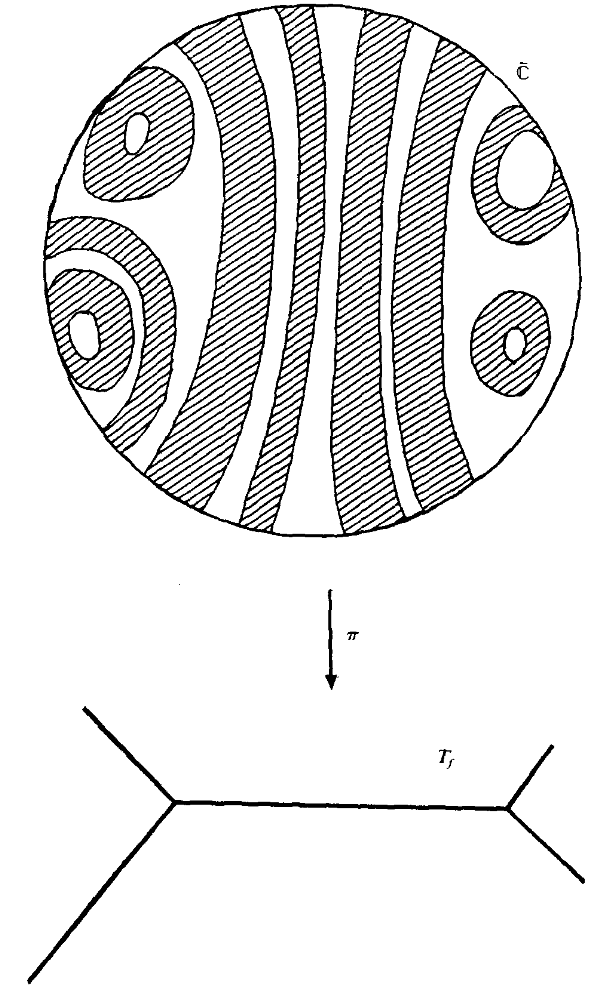
with $1/m + 1/n < 1$
and $|\varepsilon| << 1$



$T = \mathbb{R} \cup \{\pm\infty\}$



Properties of the tree and the tree map



For four points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, if there is a maximal segment J which separates $\pi(z_1), \pi(z_2)$ from $\pi(z_3), \pi(z_4)$, then

$$\mathrm{length}(J) = \sum_{\substack{A \in \mathcal{A}_f \\ A \text{ separates } z_1, z_2 \text{ from } z_3, z_4}} mod(A).$$

There are finite number of segments which are periodic ($F^p(J) \subset J$) and the return maps are

the identity (corresponding to SD or HR);

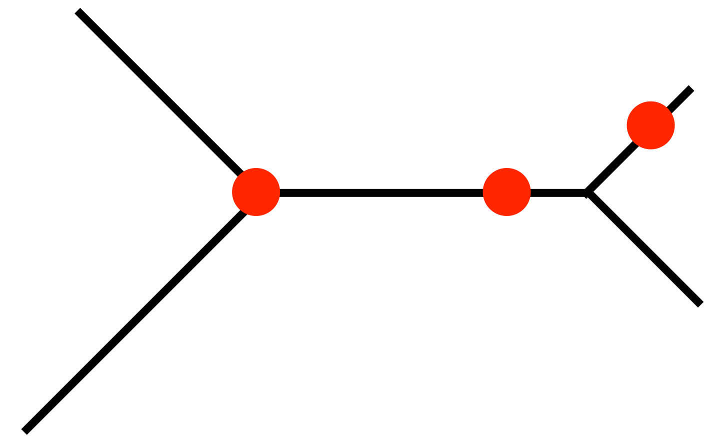
conjugate to $x \mapsto x + c$ on $[0, +\infty)$ ($c > 0$, corresponding to AB);

conjugate to $x \mapsto kx$ on $[0, +\infty)$ ($k > 1$ integer, corresponding to SAB).

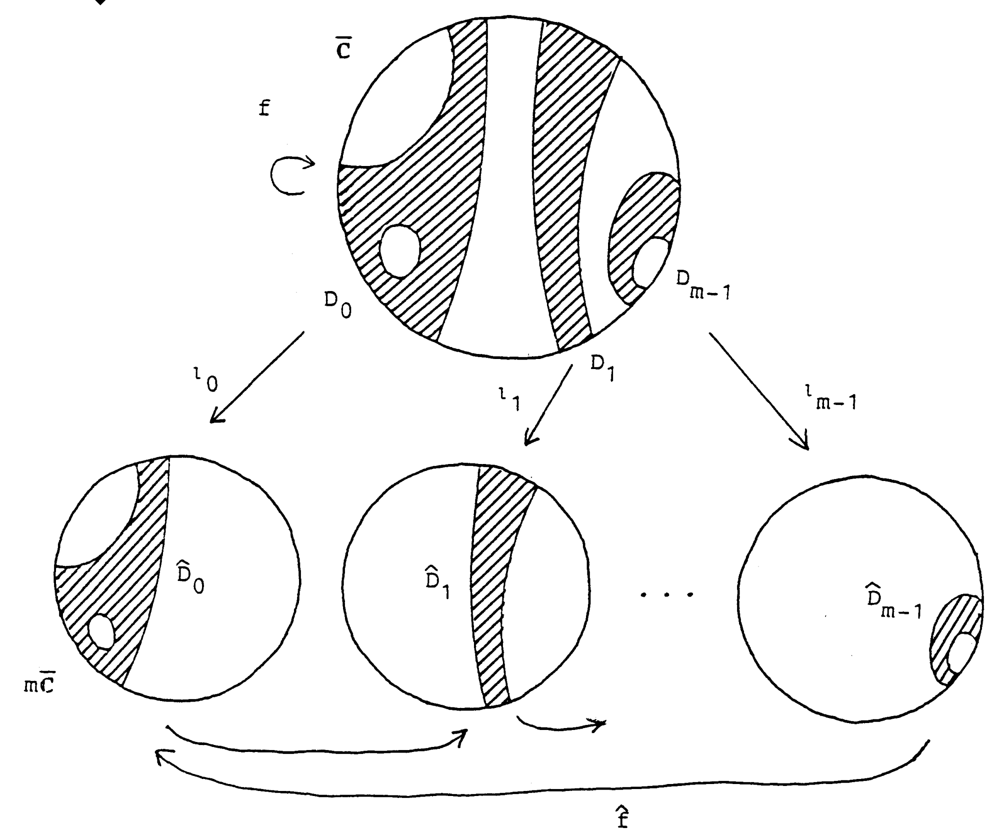
The inverse image of these points are open and dense.

Local models for the periodic points on the tree

Since each complementary component of $\mathcal{A}_f$ is collapsed to a point, we lose the information on the dynamics on this component. It can be recovered by defining local models via surgery.



For a periodic cycle $x_0, x_1, \ldots, x_{p-1}$ of $F : T \to T$ ($x_p = x_0$), cut the sphere along circles sufficiently close to x_i's and replace the map by a simpler one outside the corresponding domains, we obtain via surgery a cycle of rational maps which have marked points corresponding to the branches such that local degree coincides with DF. This cycle of rational maps are called the local model for $\{x_i\}$.



Consequences: If x_0 is a fixed point and its local model had degree ≥ 2, then $\pi^{-1}(x_0)$ must contain a weak-repelling fixed point (parabolic with multiplier 1 or repelling).

Then $\pi^{-1}(x)$ contains at least $k_1 + k_2$ critical points.

A folding costs two critical points!

Local models can be constructed as the limit of stretching qc-deformation: need to take different normalization for each point of the cycle. (the map viewed via “moving frame”)

Quasiconformal surgery

Surgery: Cut and Paste objects to create a new object

Analytic functions are rigid because of the theorem of unicity, hence the straightforward cut and paste is not possible.

Quasiconformal surgery is a technique to avoid this difficulty:
first construct a non-holomorphic mapping, then change the conformal structure so that the map becomes holomorphic with respect to the new conformal structure.

Tools: theory of quasiconformal mappings and the Measurable Riemann Mapping Theorem.

Applications:

- Sullivan: Used quasiconformal deformation to prove No Wandering Domain Thm
- Douady-Hubbard: Polynomial-like mappings
- S.: General theory,
  - Sharp estimate on the number of non-repelling cycles, Fatou components

Surgery construction

Given a piecewise linear map $F : T \to T$ on a tree and local models for periodic "singular points" with certain conditions, one can do a quasiconformal surgery to construct a rational map.
Idea is to replace edges of the tree by cylinders and "singular points" by small spheres (Plumbing construction). If the constructed map is holomorphic outside regions which are transient, by a surgery principle, one can change the conformal structure to obtain a rational map.

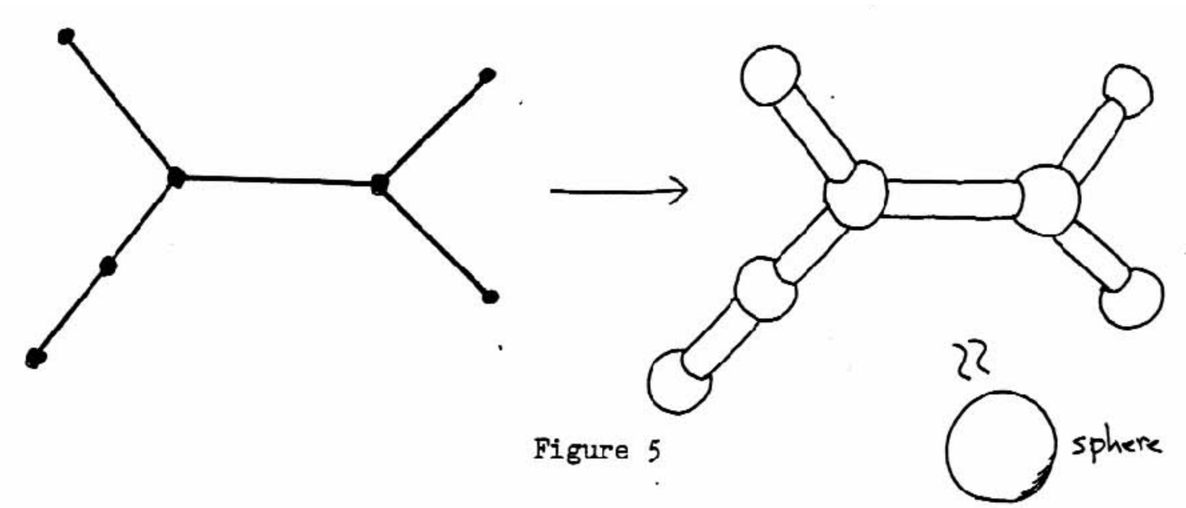
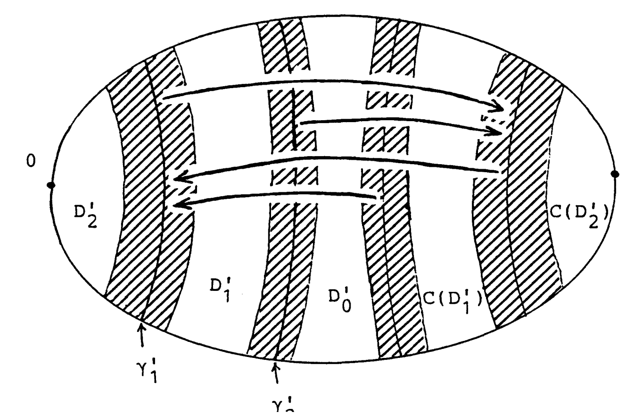
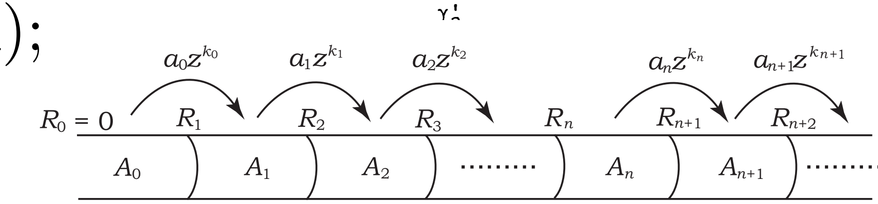


Figure 5



Examples:
Siegel disk $\leftrightarrow$ Herman rings surgery with period > 1;
p-connected Fatou component for $3 \le p < \infty$ (Beardon's book);
Pilgrim-Tan Lei;
Wandering annulus for a transcendental entire function;
P-connected wandering Julia component.



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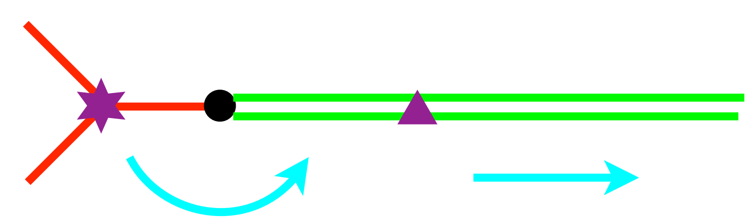


Applications

Theorem: If P is a polynomial of degree ≥ 2, the its Newton's method

$$N_P(z) = z - \frac{P(z)}{P'(z)}$$

has connected Julia set.

Theorem: If a rational map f of degree ≥ 2 has disconnected Julia set, then there must be at least two weak-repelling fixed points which are separated by the Fatou set.

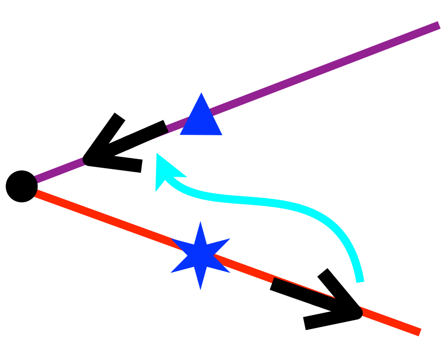
Idea of Proof: Find two "repelling" fixed points on the tree.

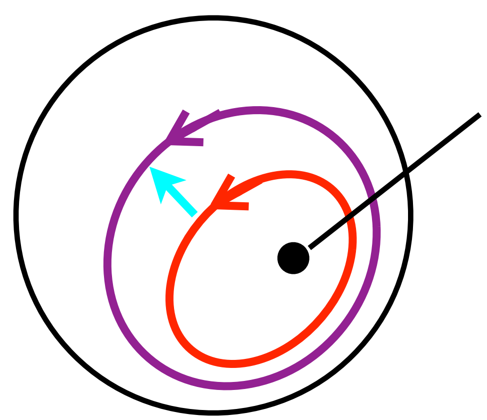
First, by a surgery perturbation, reduce to the case where there are no parabolic basins.

There is a (Julia) fixed point on the tree T. Since a fixed point in the Julia set has dense inverse orbits in J_f, The tree map cannot be injective (After choosing a suitable finite set X.) In other words, there must be some "folding" on the tree.

Analyze the branches around the fixed point and possible types of folding.

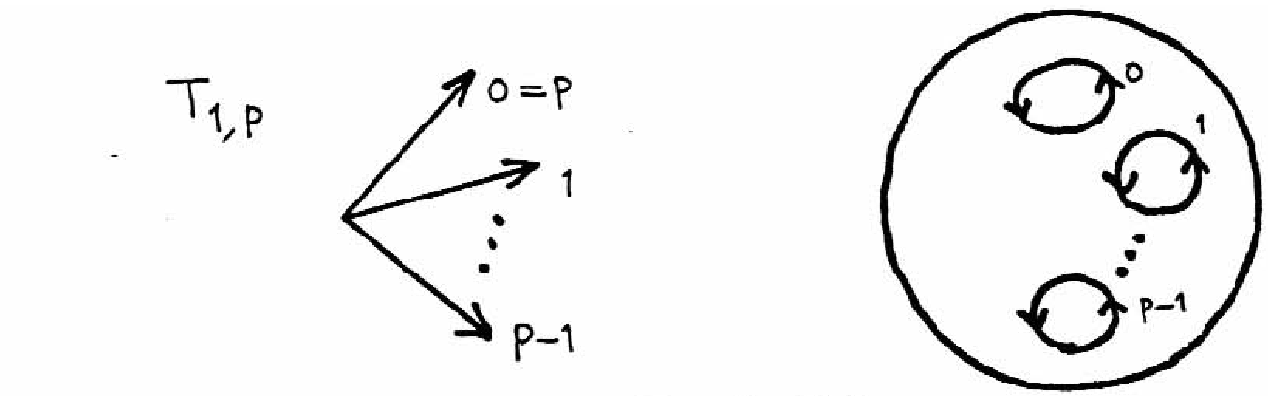
The simplest case:





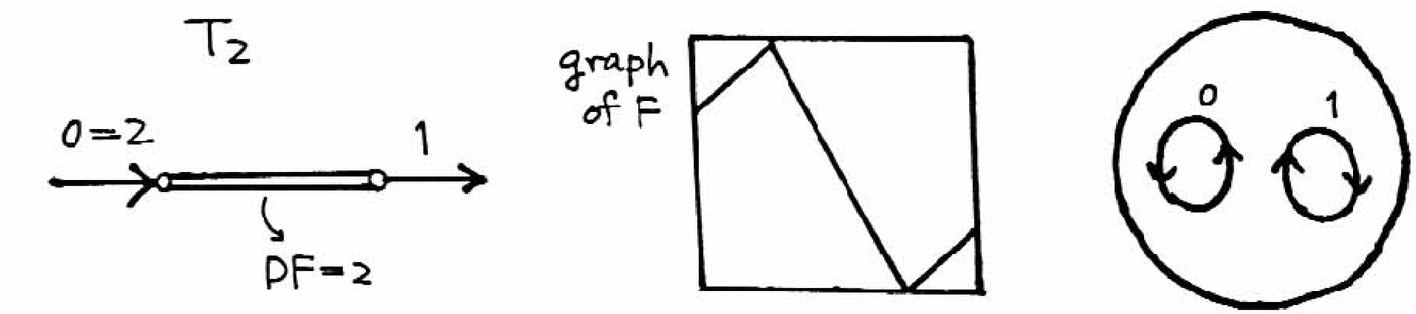
There must be at least one "weakly repelling fixed point" here by Fatou (after a surgery).

Configuration of Herman rings



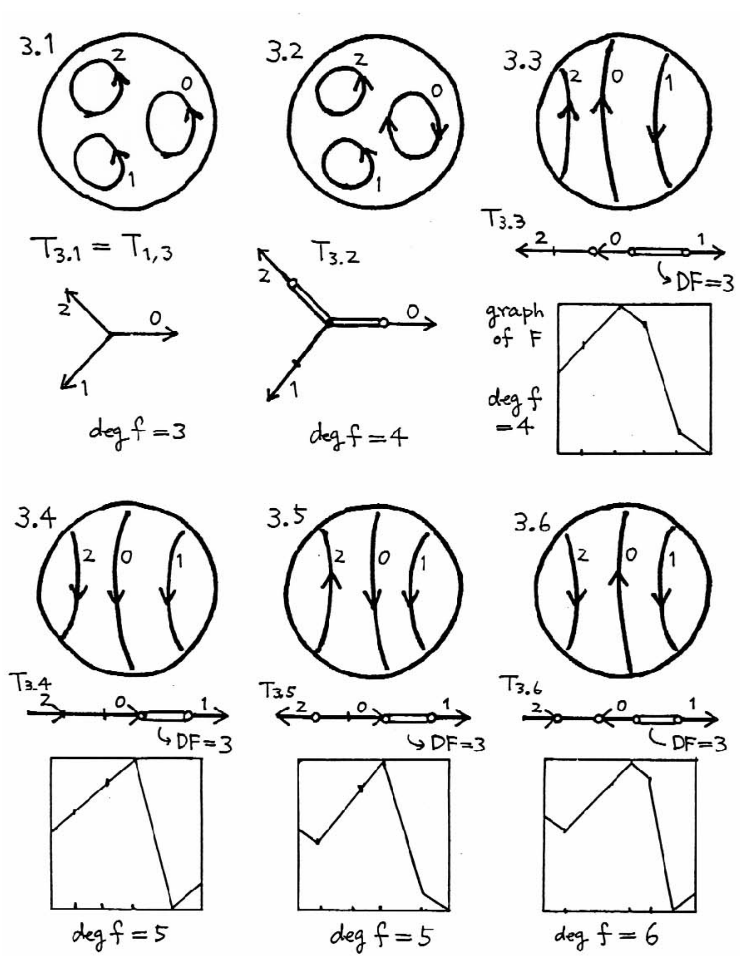
Example 1(p)

realized
by deg 3 map



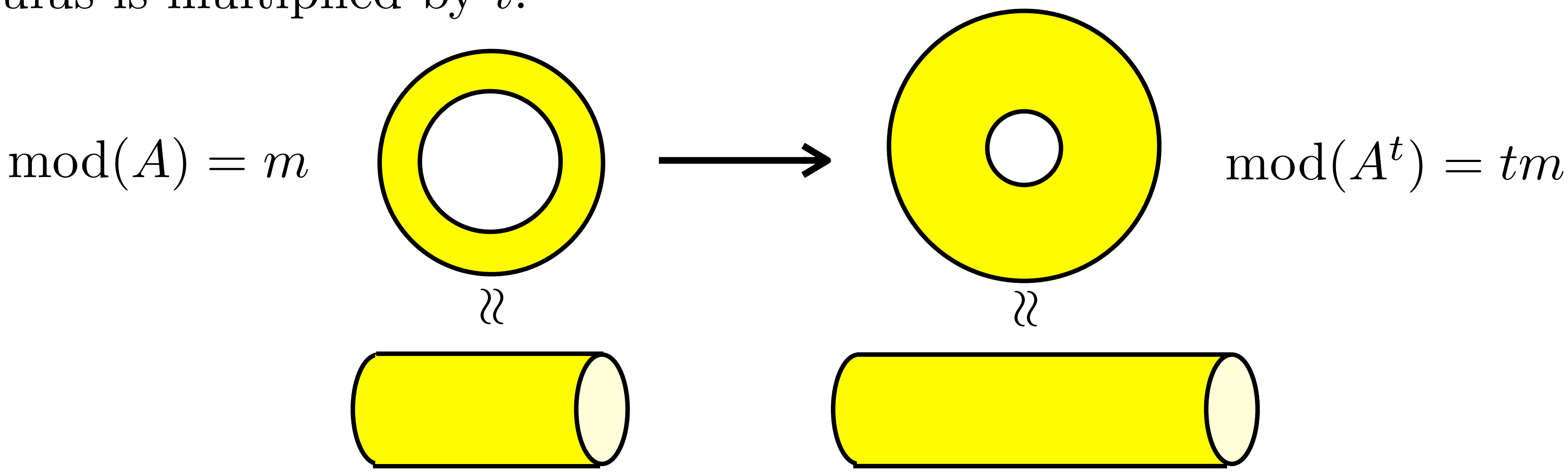
Example 2

realized
by deg 4 map
not deg 3



Limit of quasiconformal deformation

t-stretching deformation of an annulus A: Change the conformal structure so that the modulus is multiplied by t.



How the annulus look like depends strongly on the normalization (choice of $0, 1, \infty$). For example, if $1, \infty$ in one boundary component of A and 0 in the other component, then the component of 0 has size $\approx \exp(-2\pi tm)$ as $t \to \infty$. (cf. correspondence in Tropical geometry)

The family $\mathcal{A}_f$ can be simultaneously t-stretched so that the resulting conformal structure is compatible with a new rational map f_t. This is the t-stretching deformation of f.

The local models can be obtained as the limit of t-stretching deformation after suitable normalizations.

Question: Does f_t degenerate in the way it is supposed be? For example, asymptotic behavior of the cross ratio of four points should be determined by the length of edges that separate the four points.

If Yes, it helps us to find where these rational maps are in the parameter space. Cf. Kiwi's Piuseux series dynamics.

Problem: Suppose A an annulus and $\mathcal{A}$ is a collection of disjoint sub-annuli of A. For $A_i \in \mathcal{A}$, let A_i^t denote t-stretching of A_i. Apply t-stretching to all $A_i \subset A$ and obtain a new (measurable) conformal structure for A. Let A^t be A with this conformal structure. Show

$$mod(A^t) \sim \sum_{\substack{A_i \in \mathcal{A} \\ A_i \text{ is essential in A}}} t\; mod(A_i) \quad \text{as } t \to \infty.$$

If one can show this, it is expected that for four points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, if there is a maximal segment J separating z_1, z_2 from z_3, z_4, then

$$\frac{1}{2\pi} \log CR(z_1^t, z_2^t, z_3^t, z_4^t) \sim t \text{ length}(J) \text{ as } t \to \infty,$$

where CR denotes a certain cross ratio.

From this, one should be able to deduce how the t-stretching f_t degenerates and one can find the asymptotics of f_t, i.e. where they are in the parameter space.

The asymptotics can be deduced from the surgery construction.

Note: The above claim is true when $\mathcal{A}$ is finite.

Thank you!