Matings of Cubic Polynomials with a Fixed Critical Point

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Matings: a quick guide

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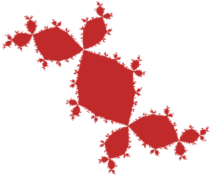

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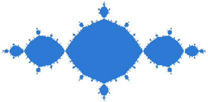

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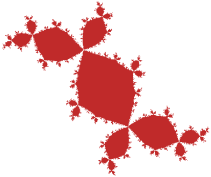

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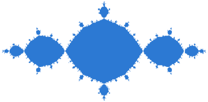

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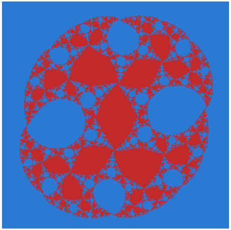

Definitions.

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map.

- The Julia set $J(f)$ is the closure of the set of repelling periodic points of f.
- The Fatou set $F(f)$ is $\widehat{\mathbb{C}} \setminus J(f)$.

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If f is a polynomial

- The point ∞ is a superattracting fixed point.
- The filled Julia set is $K(f) = \{z \in \widehat{\mathbb{C}} \mid f^{\circ n}(z) \nrightarrow \infty\}$, so that $J(f) = \partial K(f)$

Böttcher's theorem and external rays

There exists a map ϕ which is an analytic conjugacy between f on $\widehat{\mathbb{C}} \setminus K(f)$ and the map $z \mapsto z^d$ on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and such that ϕ is asymptotic to the identity at ∞.

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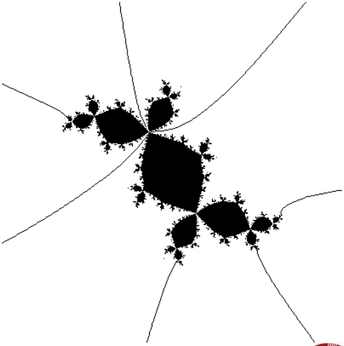
- $R_\theta = \phi^{-1}\{re^{2\pi i\theta} \mid r \in (1, \infty)\}$ is called the external ray of angle θ.
- If $J(f)$ is locally connected, the landing point $\gamma(\theta) = \lim_{r\to 1} \phi^{-1}(re^{2\pi i\theta})$ exists for all θ and belongs to $J(f)$.
- We have the identities $f(R_\theta) = R_{d\theta}$ and $f(\gamma(\theta)) = \gamma(d\theta)$.



Formal Matings

For $i = 1, 2$, let f_i be monic degree d polynomials. Define $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty \cdot e^{2\pi i s} \mid s \in \mathbb{R}/\mathbb{Z}\}$.

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- Extend f_1 and f_2 to the boundary circle at infinity, e.g. $f_1(\infty \cdot e^{2\pi i s}) = \infty \cdot e^{2d\pi i s}$.
- Define $S^2_{f_1,f_2} = \widetilde{\mathbb{C}}_{f_1} \uplus \widetilde{\mathbb{C}}_{f_2} / \{(\infty \cdot e^{2\pi i s}, f_1) \sim (\infty \cdot e^{-2\pi i s}, f_2)\}$.

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- The formal mating is the degree d branched covering $f_1 \uplus f_2 \colon S^2_{f_1,f_2} \to S^2_{f_1,f_2}$ given by
  - $f_1 \uplus f_2 = f_1$ on $\widetilde{\mathbb{C}}_{f_1}$
  - $f_1 \uplus f_2 = f_2$ on $\widetilde{\mathbb{C}}_{f_2}$

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  - Take the disjoint union of K_1 and K_2.
  - $K_1 \amalg K_2$ is the quotient space formed by identifying $\gamma_1(\theta)$ with $\gamma_2(-\theta)$.
  - The maps f_i on the K_i fit together to form a new map $f_1 \amalg f_2$.

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We say f_1 and f_2 are topologically mateable if this quotient $K_1 \perp\!\!\!\perp K_2$ is a sphere.

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Thurston's Theorem

Let $F\colon \Sigma \to \Sigma$ and $\widehat{F}\colon \widehat{\Sigma} \to \widehat{\Sigma}$ be postcritically finite orientation-preserving branched self-coverings of topological 2-spheres. An equivalence is given by a pair of orientation-preserving homeomorphisms (Φ, Ψ) from Σ to $\widehat{\Sigma}$ such that

- $\Phi|_{P_F} = \Psi|_{P_F}$
- $\Phi \circ F = \widehat{F} \circ \Psi$
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Theorem (Thurston)

Let $F\colon \Sigma \to \Sigma$ be a postcritically finite branched cover with hyperbolic orbifold. Then F is equivalent to a rational map if and only if F has no Thurston obstructions. This rational map is unique up to Möbius transformation.

We say f_1 and f_2 are mateable if $f_1 \perp\!\!\!\perp f_2$ is equivalent to a rational map on $\widehat{\mathbb{C}}$.

Thurston obstructions

Let $F\colon S^2 \to S^2$ be a branched covering and $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ a multicurve in $S^2 \setminus P_F$.

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Definition

The Thurston linear transformation $F_\Gamma\colon \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined by

$$F_\Gamma(\gamma) = \sum_{\gamma' \subset F^{-1}(\gamma)} \frac{1}{\deg(F\colon \gamma' \to \gamma)} [\gamma']_\Gamma$$

where $[\gamma']_\Gamma \in \Gamma$ is isotopic to γ'. Γ is a Thurston obstruction if its leading eigenvalue is greater than or equal to 1.

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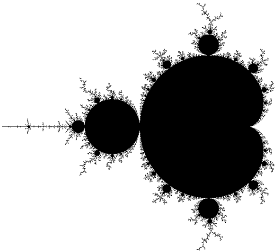
Definition

A multicurve is called a Levy cycle if for $i = 1, 2, \ldots, n$, the curve γ_{i-1} is homotopic (rel P_F) to a component γ'_{i-1} of $F^{-1}(\gamma_i)$ and the map $F\colon \gamma'_i \to \gamma_i$ is a homeomorphism

Quadratic Case

The quadratic (or bicritical) case is reasonably well understood:



Theorem (Rees, Shishikura, Tan)

In the bicritical case, if f_1 and f_2 do not lie in conjugate limbs of $\mathcal{M}$, then $K_1 \perp\!\!\!\perp K_2$ is homeomorphic to S^2 and we can give this sphere a unique conformal structure to make $f_1 \perp\!\!\!\perp f_2$ a holomorphic degree d rational map.

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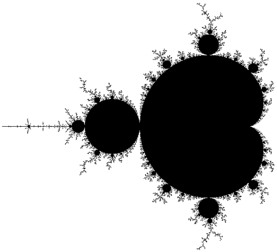

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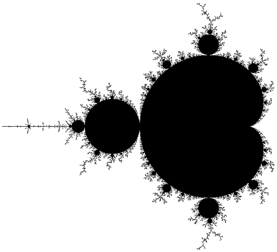

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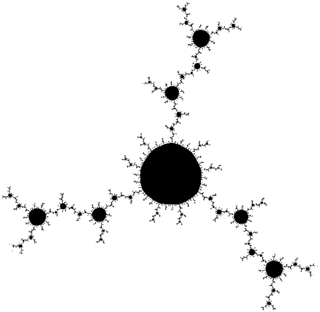
- Essentially, this says that in the quadratic case, the quotient is a sphere if and only if the resulting map is equivalent to a rational map. All obstructions are Levy cycles.
- The mating is obstructed if and only if the two α-fixed points belong to the same ray class.

Other obstructions

However, there exist other obstructions in higher degrees: Consider the following



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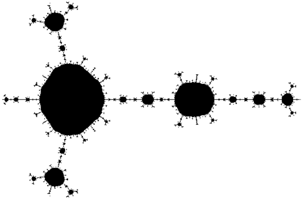

Both polynomials are in $\mathcal{S}_3$. The quotient is a sphere, but the mating is not a rational map.

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Levy cycles and external rays

There is a close link between Levy cycles and loops of external rays in the formal mating.

Theorem (Tan 1992, Shishikura-Tan 2000)

Let $F = f \perp\!\!\!\perp g$.

- *Each Levy cycle* Γ *for* F *corresponds to a unique periodic cycle of ray classes (the "limit set"). In particular, if* Γ *is not a degenerate Levy cycle, then each ray class contains a closed loop.*
- *If a periodic ray class contains a closed loop then each boundary curve of a tubular neighbourhood generates a Levy cycle.*

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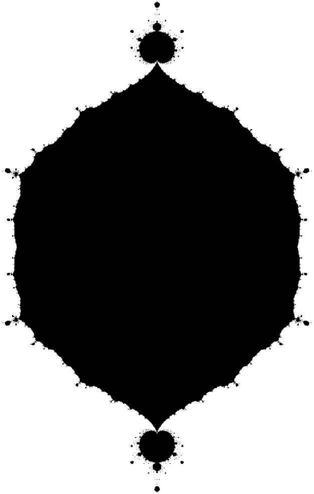
The space $\mathcal{S}_1$.



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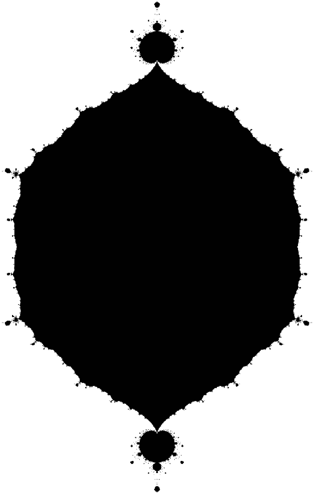
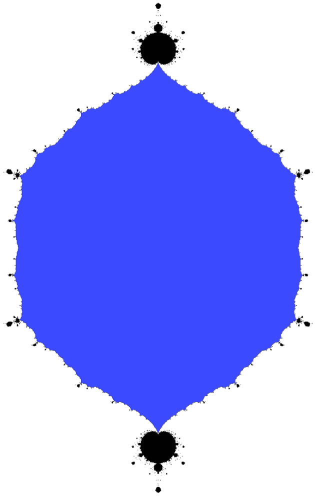


There is only one escape region in $\mathcal{S}_1$. In particular this means that the intersection of $\mathcal{S}_1$ with the connectedness locus is combinatorially a tree.

This is not true for $\mathcal{S}_2, \mathcal{S}_3 \ldots$

Limbs in $\mathcal{S}_1$

Like the Mandelbrot set, $\mathcal{S}_1$ has a main hyperbolic component $\mathcal{H}_0$. Its centre is $z \mapsto z^3$.



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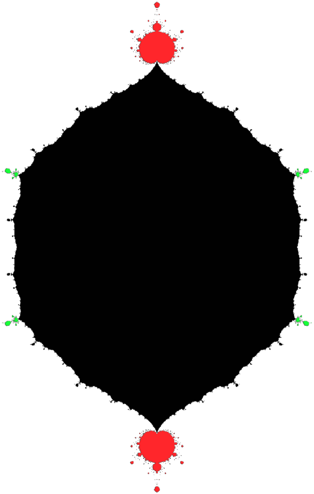
Attached to this component are various limbs.



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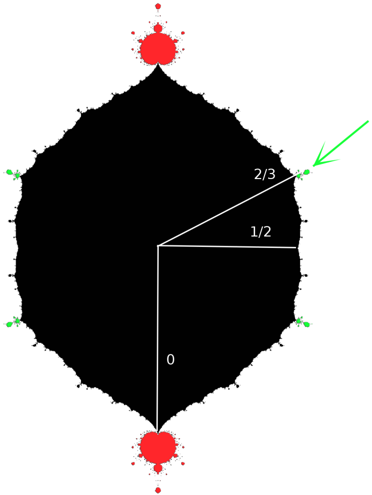
Limbs are characterised by the existence of α-periodic cycles. We will look at an example, one of the $\frac{2}{3}$-limbs in $\mathcal{S}_1$.



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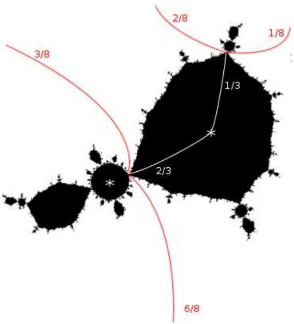
Denote by U the Fatou component containing the fixed critical point a. Maps in the $\frac{p}{q}$-limb have a “dynamical limb” containing the free critical point attached to the landing point of the internal ray of angle $\frac{p}{q}$ in U.



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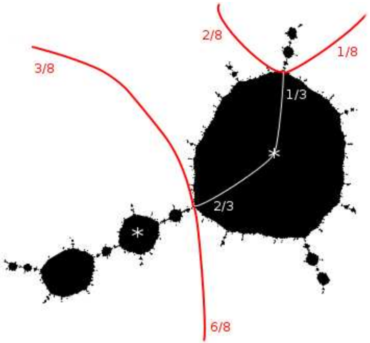

This gives us a distinguished periodic cycle which we call the α-periodic cycle. Furthermore, the angles of the external rays landing at this periodic cycle persist in the limb.

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Topological Mating

Let f_1, f_2 be postcritically finite polynomials in $\mathcal{S}_1$ with filled Julia sets K_1 and K_2 respectively.

Question

When is the quotient space $K_1 \perp\!\!\!\perp K_2$ a sphere (when are f_1 and f_2 topologically mateable)?

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Recall that for quadratics, the ray classes contained loops precisely when the fixed points α_1 and α_2 belong to the same ray class.

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Let f_1 be the period 2 map in the $\frac{2}{3}$-limb.

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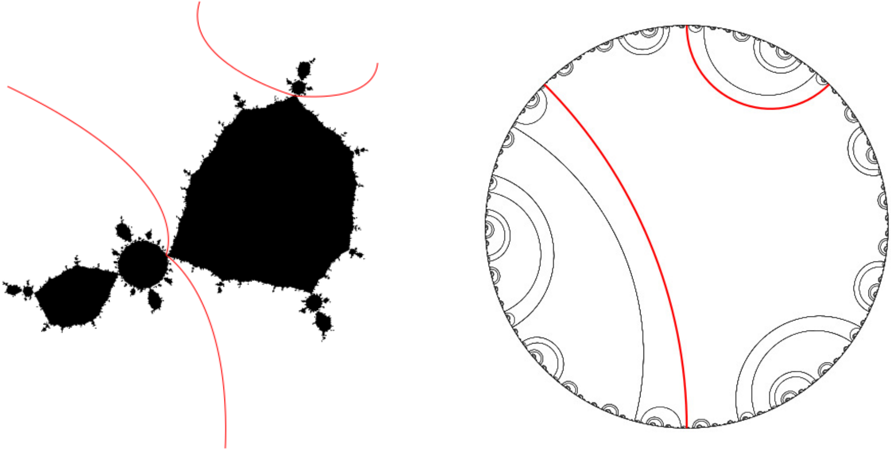
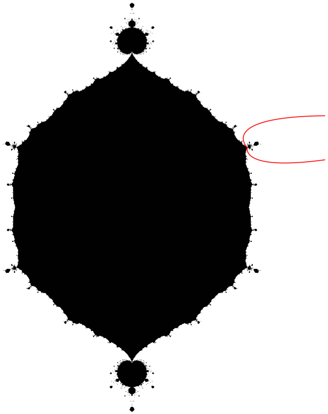


Example

This map resides here in parameter space. . .



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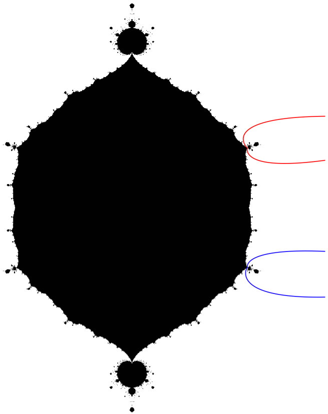

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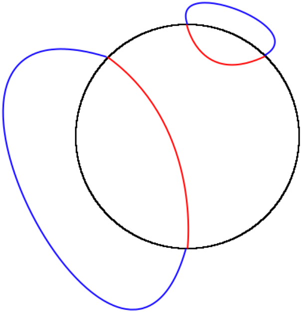

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This map resides here in parameter space...
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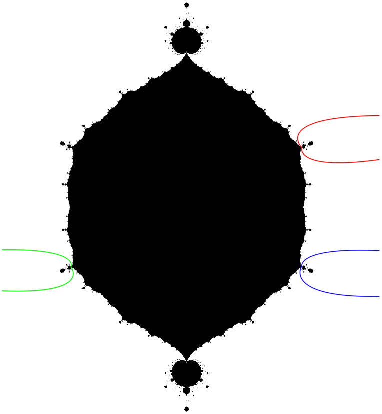
...as is the mating with this complementary map.



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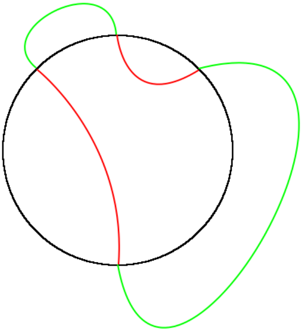

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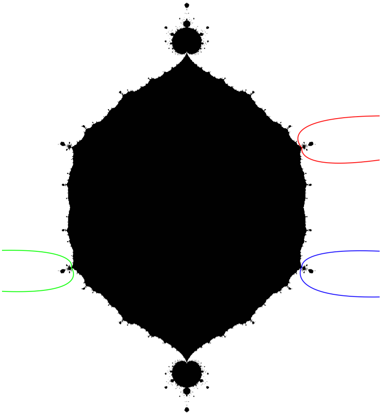

In both cases, the α-cycles are in the same ray class(es) and these ray classes contain loops.

Characterisation of topological obstructions

Conjecture

The mating $f_1 \perp\!\!\!\perp f_2$ is topologically obstructed if and only if one of the following occurs.

- f_1 and f_2 lie in conjugate limbs.
- f_1 and f_2 lie in complementary limbs.

Clearly all limbs have a conjugate limb. But when does a limb have a complementary limb?

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Clearly all limbs have a conjugate limb. But when does a limb have a complementary limb?

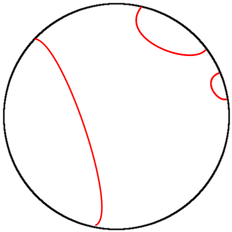
Conjecture

Let $\mathcal{C}_t \subset \mathcal{S}_1$ be a limb. Then $\mathcal{C}_t$ has a complementary limb if and only if t has a non-zero rotation number under the map $t \mapsto 2t$ on $\mathbb{R}/\mathbb{Z}$.

Here t represents the internal angle of the limb with respect to the type A component.

Sketch Proof

A typical example looks like this.



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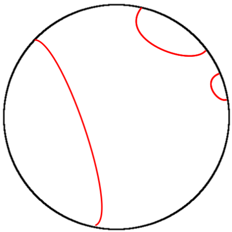

Notice the gaps between each dynamical limb also form a rotational set, corresponding to a limb with rotation number $\frac{p}{q}$.

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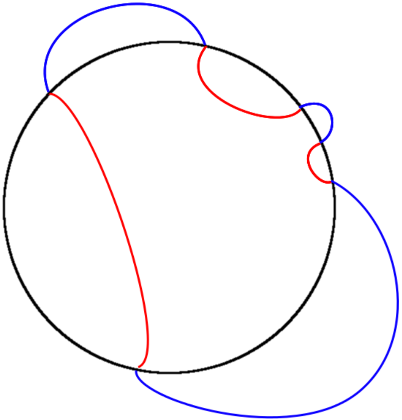

Notice the gaps between each dynamical limb also form a rotational set, corresponding to a limb with rotation number $\frac{p}{q}$. We “fill” these gaps with dynamical limbs

Sketch Proof

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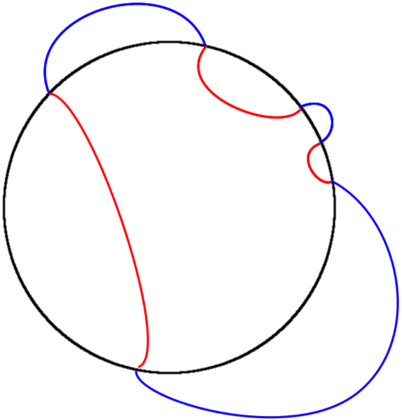

Notice the gaps between each dynamical limb also form a rotational set, corresponding to a limb with rotation number $\frac{p}{q}$. We "fill" these gaps with dynamical limbs which correspond to the conjugate limb in $\mathcal{S}_1$ which has rotation number $-\frac{p}{q}$.

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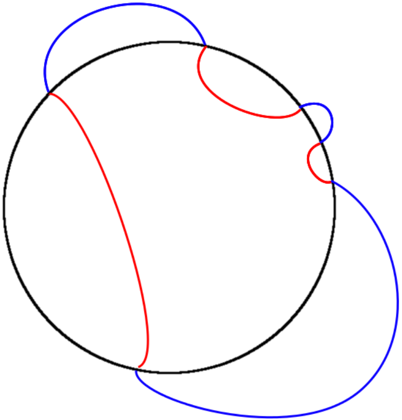

Notice the gaps between each dynamical limb also form a rotational set, corresponding to a limb with rotation number $\frac{p}{q}$. We “fill” these gaps with dynamical limbs which correspond to the conjugate limb in $\mathcal{S}_1$ which has rotation number $-\frac{p}{q}$.

If the limb does not have a rotation number, no such pairing of limbs exists.

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Main Theorem

Theorem

Let f_1, f_2 be postcritically finite polynomials in $\mathcal{S}_1$ with α-periodic cycles $\langle \alpha_1 \rangle$ and $\langle \alpha_2 \rangle$ respectively. Then the mating is obstructed if and only if $[\langle \alpha_1 \rangle] = [\langle \alpha_2 \rangle]$ and this ray class contains a loop.

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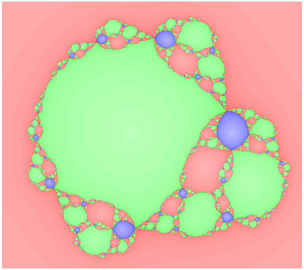
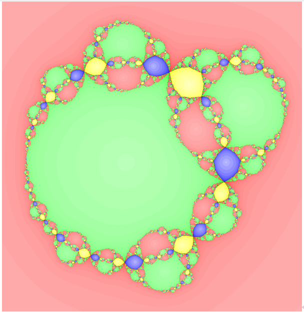
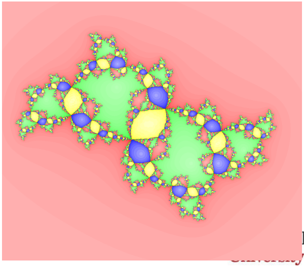

Some general results for obstructions

To study Thurston obstructions, we need a couple of lemmas.

Lemma

Let Γ be an irreducible multicurve for a brached covering F which is not a removable Levy cycle. Then there exists a disk component of $S^2 \setminus \Gamma$ such that $F^{-1}(D)$ contains a non-disk component.

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Topologically, it is easy to see that such a disk component must contain (at least) two critical values of F.

Lemma

Any connected component of $S^2 \setminus F^{-1}(\Gamma)$ is isotopically contained in a connected component of $S^2 \setminus \Gamma$.

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Matings in $\mathcal{S}_1$

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The first case is a “Newton-like” case, the second a “quadratic-like” case. We will show in both cases that Γ must contain a Levy cycle.

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Case 1: Newton-like

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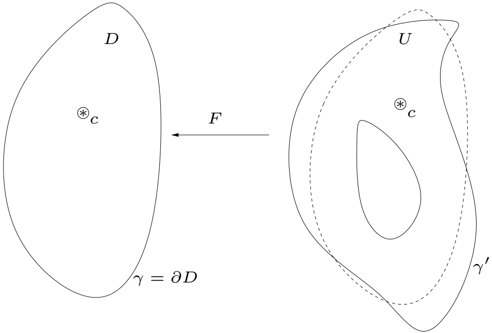

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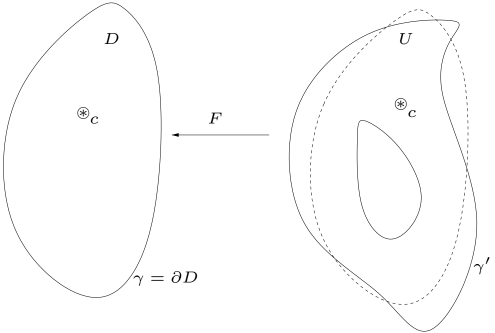

In particular, one component U of $F^{-1}(D)$ must contain c, and so is isotopically contained in D. Hence there is a curve $\gamma \subset F^{-1}(\gamma)$ isotopic to γ such that $F\colon \gamma' \to \gamma$ is a homeomorphism.

Case 2: Quadratic-like

This case has much in common with the case of quadratic matings (Rees-Shishikura-Tan Theorem).

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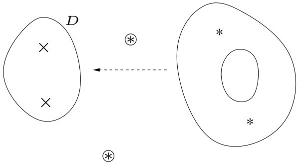

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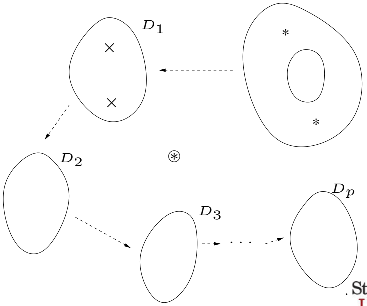

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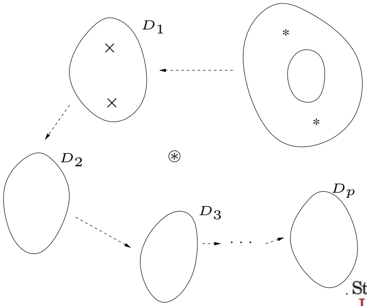
By a generalised “Rees' Lemma”, the set of disk components of $S^2 \setminus \Gamma$ form a sequence $D = D_1, D_2, \ldots, D_p \ldots$ and we can construct a Levy cycle by showing every curve has 3 pre-images.



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1. Every obstruction is a Levy cycle, and so has an associated limit set which is a collection of ray classes.
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Remarks

- The result generalises to the case where the polynomials are of degree d and have a fixed point of degree $d-1$ (see Roesch '07).
- Presumably something similar can be done in the case where the polynomials have two critical points and one is fixed.

Thank you for listening!