# Classification of Thurston maps with parabolic orbifolds

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## Main results

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*There exists an algorithm $\mathcal{A}$ which does the following. Let $f$ and $g$ be marked Thurston maps and assume that every element of the canonical geometrization of $f$ has hyperbolic orbifold. The algorithm $\mathcal{A}$, given the combinatorial descriptions of $f$ and $g$, outputs $1$ if $f$ and $g$ are Thurston equivalent and $0$ otherwise.*

# Thurston maps

**Definition**

A (marked) *Thurston map* is a pair $(f, P_f)$ where $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-preserving branched self-cover of $\mathbb{S}^2$ of degree $d_f \geq 2$ and $P_f$ is a finite forward invariant set that contains all critical values of $f$.

In particular, the branched cover $f$ must be postcritically finite.

# Thurston equivalence

## Definition

Two Thurston maps $f$ and $g$ are combinatorially equivalent if and only if there exist two homeomorphisms $h_1, h_2 : \mathbb{S}^2 \to \mathbb{S}^2$ such that the diagram

$$\begin{tikzcd}
(\mathbb{S}^2, P_f) \arrow[r, "h_1"] \arrow[d, "f"'] & (\mathbb{S}^2, P_g) \arrow[d, "g"'] \\
(\mathbb{S}^2, P_f) \arrow[r, "h_2"] & (\mathbb{S}^2, P_g)
\end{tikzcd}$$

commutes, $h_1|_{P_f} = h_2|_{P_f}$, and $h_1$ and $h_2$ are homotopic relative to $P_f$.

## Thurston's theorem

**Theorem (Thurston's Theorem )**

*A postcritically finite branched cover* $f : \mathbb{S}^2 \to \mathbb{S}^2$ *(except* $(2, 2, 2, 2)$*-maps) is either Thurston-equivalent to a rational map* $g$ *(which is then necessarily unique up to conjugation by a Möbius transformation), or* $f$ *has a Thurston obstruction.*

# Thurston matrix and obstructions

**Definition**

Denote by $\mathcal{C}$ the set of all homotopy classes of essential simple closed curves. Define the Thurston linear operator $M\colon \mathbb{R}^{\mathcal{C}} \to \mathbb{R}^{\mathcal{C}}$ by setting

$$M(\gamma) = \sum_{f(\gamma_i)=\gamma} \frac{1}{\deg f|_{\gamma_i}} \gamma_i.$$

Every multicurve $\Gamma$ has its associated *Thurston matrix* $M_\Gamma$ which is the restriction of $M$ to $\mathbb{R}^\Gamma$.

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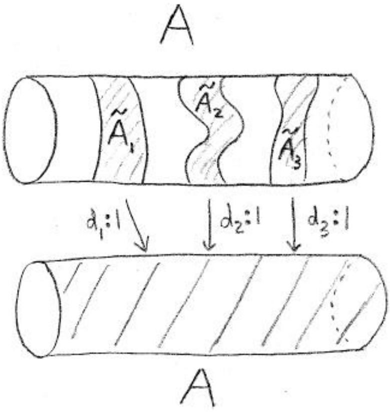
### Definition

Since all entries of $M_\Gamma$ are non-negative real, the leading eigenvalue $\lambda_\Gamma$ of $M_\Gamma$ is also real and non-negative. A multicurve $\Gamma$ is a *Thurston obstruction* if $\lambda_\Gamma \geq 1$.

## An example of Thurston obstruction




For a rational map, we must have $\sum 1/d_i < 1$.

# Levy cycles

## Definition

A *Levy cycle* is a multicurve

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_{n-1}\}$$

such that each $\gamma_i$ has a nontrivial preimage $\gamma_i'$, where the topological degree of $f$ restricted to $\gamma_i'$ is 1 and $\gamma_i'$ is homotopic to $\gamma_{(i-1) \mod n}$ rel $Q$. A Levy cycle is *degenerate* if each $\gamma_i'$ bounds a disk $D_i$ such that the restriction of $f$ to $D_i$ is a homeomorphism and $f(D_i)$ is homotopic to $D_{(i+1) \mod n}$ rel $Q$.

## Algorithm for finding Thurston obstructions

**Theorem (Bonnot, Braverman, Yampolsky)**

*There exists an algorithm which for any Thurston map $f$ with hyperbolic orbifold outputs either an obstruction or an equivalent rational map.*

**Proof.**

- Enumerate all possible multicurves and start checking if any of them is an obstruction for $f$ one-by-one.
- List all (finitely many) rational maps that *could* be equivalent to $f$. List all homeomorphisms classes and check whether any of them realizes equivalence one-by-one.

$\square$

# Decidability of combinatorial equivalence

### Conjecture

*There exists an algorithm which can produce a combinatorial equivalence between two Thurston maps or say that they are not equivalent.*

Main Theorem 3 is a partial resolution of this conjecture.

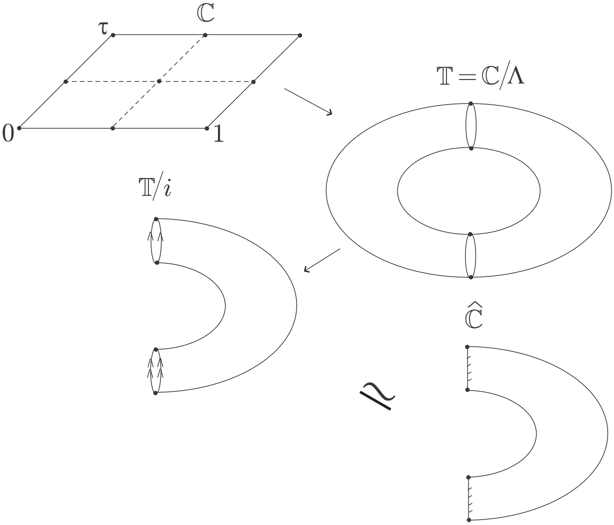
## $(2,2,2,2)$-**maps**

We will refer to a Thurston map that has orbifold with signature $(2,2,2,2)$ simply as a $(2,2,2,2)$-*map*. An orbifold with signature $(2,2,2,2)$ is a quotient of a torus $T$ by an involution $i$; the four fixed points of the involution $i$ correspond to the points with ramification weight 2 on the orbifold. The corresponding branched cover $P : T \to \mathbb{S}^2$ has exactly 4 simple critical points which are the fixed points of $i$. It follows that a $(2,2,2,2)$-map $f$ can be lifted to a covering self-map $\hat{f}$ of $T$.

An orbifold with signature $(2,2,2,2)$ has a unique affine structure of the quotient $\mathbb{R}^2/G$ where

$$G = < z \mapsto z+1, z \mapsto z+i, z \mapsto -z > .$$

# $(2,2,2,2)$-maps



$\mathbb{C}$
$\tau$
0
1
$\mathbb{T} = \mathbb{C}/\Lambda$
$\mathbb{T}/i$
$\widehat{\mathbb{C}}$
$\approx$

## Classification of $(2,2,2,2)$-maps

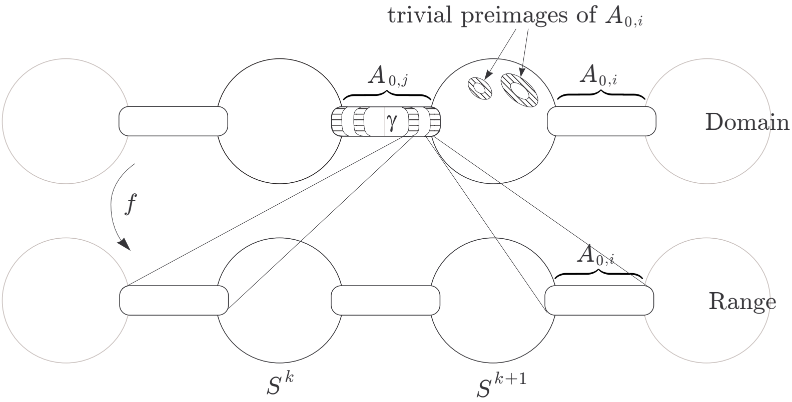
### Theorem (Main Theorem 2)

*Let $f$ be a $(2,2,2,2)$-map (with extra marked points) such that the associated matrix is hyperbolic. Then either $f$ is equivalent to a quotient of an affine map or $f$ admits a degenerate Levy cycle.*
*Furthermore, in the former case the affine map is defined uniquely up to conjugacy.*

### Corollary

*There exists an algorithm which for any $(2,2,2,2)$-map $f$ with hyperbolic matrix outputs either a degenerate Levy cycle or an equivalent quotient of an affine map.*

# Pilgrim's decomposition of a Thurston map



## Characterization of Canonical Thurston Obstructions

### Theorem

*The canonical obstruction* $\Gamma$ *is a unique minimal Thurston obstruction with the following properties.*

- *If the first-return map* $F$ *of a cycle of components in* $\mathcal{S}_\Gamma$ *is a* $(2, 2, 2, 2)$*-map, then every curve of every simple Thurston obstruction for* $F$ *has two postcritical points of* $f$ *in each complementary component and the two eigenvalues of* $\hat{F}_*$ *are equal or non-integer.*
- *If the first-return map* $F$ *of a cycle of components in* $\mathcal{S}_\Gamma$ *is not a* $(2, 2, 2, 2)$*-map or a homeomorphism, then there exists no Thurston obstruction of* $F$*.*

## Computing Canonical Obstructions

**Theorem**

*There exists an algorithm which for any Thurston map $f$ finds its canonical obstruction $\Gamma_f$.*

**Proof.**

1. Run the BBY algorithm to get an obstruction $\Gamma$.
2. Decompose $f$ along $\Gamma$.
3. Check conditions of the previous theorem. Either they are satisfied or we can construct an obstruction within one of the decomposition pieces.
4. Once we have found a maximal obstruction we check the conditions of the characterization theorem for all of its subsets.

□

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**Definition**

Let $f$ be a $(2,2,2,2)$-map and let $z$ be an $f$-periodic point with period $n$. Fix a universal cover $F$ of $f$ and take a point $\tilde{z}$ in the fiber of $z$. If $z \notin P$, we define the *Nielsen index* $\mathrm{ind}_{F,n}(\tilde{z})$ to be the unique element $g$ of the orbifold group $G$ such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of $z$ is defined up to pre-composition with the symmetry around $z$.

# Nielsen theory

## Definition

Let $f$ be a $(2,2,2,2)$-map and let $z$ be an $f$-periodic point with period $n$. Fix a universal cover $F$ of $f$ and take a point $\tilde{z}$ in the fiber of $z$. If $z \notin P$, we define the *Nielsen index* $\mathrm{ind}_{F,n}(\tilde{z})$ to be the unique element $g$ of the orbifold group $G$ such that $F^n(\tilde{z}) = g \cdot \tilde{z}$. If $z \in P$ then the Nielsen index of $z$ is defined up to pre-composition with the symmetry around $z$.

## Definition

Let $f$ be a $(2,2,2,2)$-map and let $z_1, z_2$ be $f$-periodic points with period $n$. We say that $z_1$ and $z_2$ are in the same *Nielsen class of period $n$* if there exists a universal cover $F_n$ of $f^n$ and points $\tilde{z}_1, \tilde{z}_2$ in the fibers of $z_1, z_2$ respectively, such that both $\tilde{z}_1$ and $\tilde{z}_2$ are fixed by $F_n$.

## Strategy of the proof

- A map $f$ admits a degenerate Levy cycle if and only if there exist two distinct periodic points in $P_f$ in the same Nielsen class.
- If there are points in the same Nielsen class, one can find a curve that separates them from other marked points which will generate a degenerate Levy cycle.
- If all points have distinct Nielsen indexes, they define a conjugacy between $f$ and the appropriate quotient of an affine map on $Q$. It can be shown that in the absence of Levy cycles such a conjugacy can be promoted to a combinatorial equivalence on the whole sphere.