Dynamics of polynomial automorphisms of $\mathbb{C}^2$

$$f : \mathbb{C}^2 \to \mathbb{C}^2$$

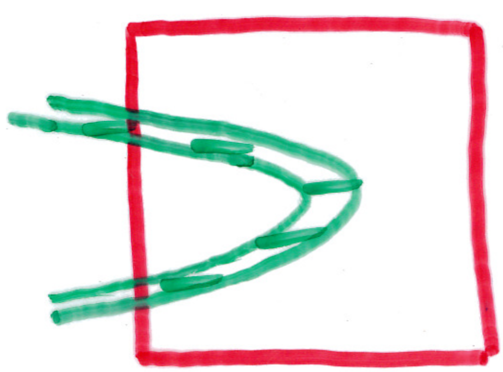
(Generalized) Hénon map

$$f : (x, y) \mapsto (p(x) - b y,\ x)$$

p is a polynomial of deg $d \geq 2$

("algebraic degree" of f)

$b = \text{Jac}\, f = \det Df \equiv \text{const} \neq 0$



$x = p(y)$

thickness $\sim b$

$b \to 0 \Rightarrow f$ degenerates to $x \mapsto p(x)$

1D Polynomial Dynamics

$f: \mathbb{C} \to \mathbb{C}$, $d \geq 2$

$\mathcal{U}(f) = \{z : f^n z \to \infty\}$ basin of ∞

$K(f) = \mathbb{C} \setminus \mathcal{U}(f)$ filled Julia set

$J(f) = \partial K(f) = \partial \mathcal{U}(f)$ Julia set

<u>Thm.</u> $J(f) = cl\{\text{repelling per pts}\}$

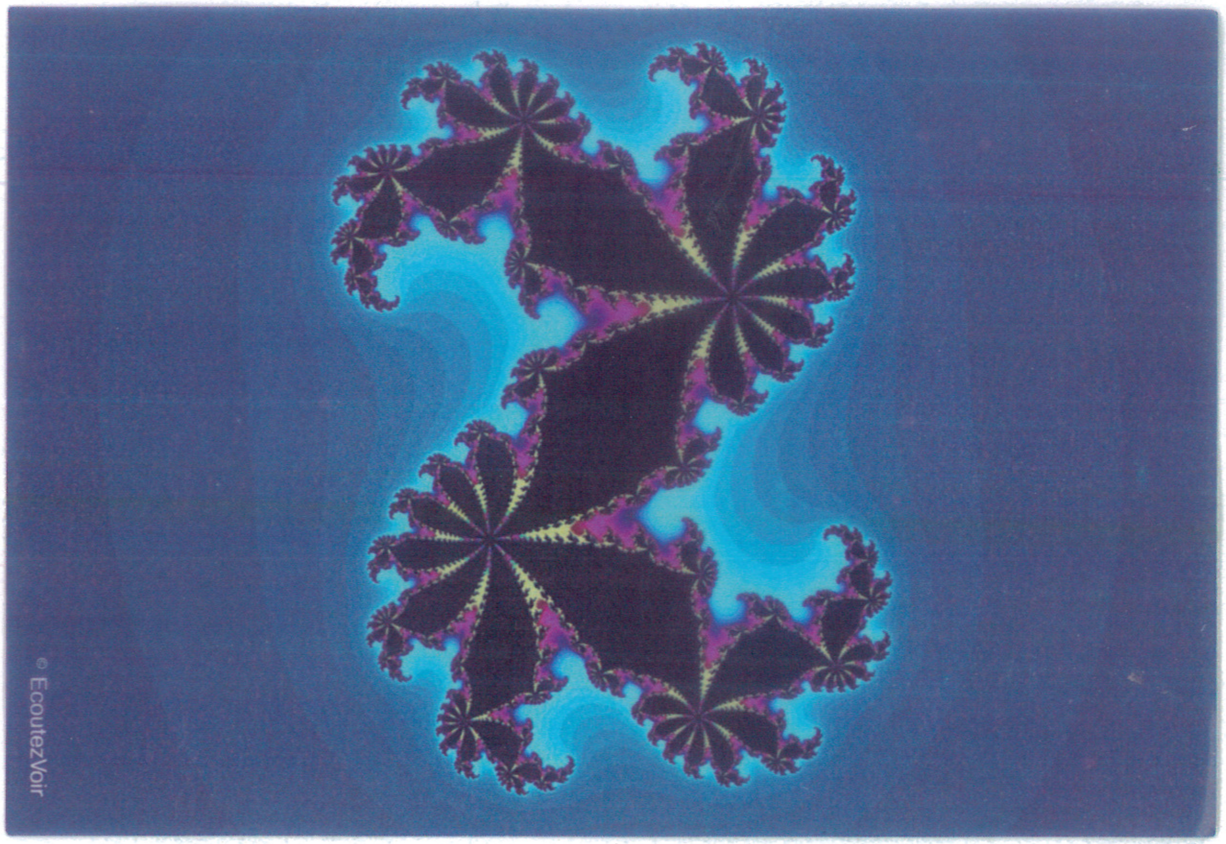
$F(f) = \mathbb{C} \setminus J(f)$ Fatou set

Brolin's Green function

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n z|$$

- $G \geq 0$, subharmonic, $\equiv 0$ on $K(f)$
- $G(fz) = d\,G(z)$.

Eremenko-Levin: dynamical appl-s of the Wiman & Denjoy-Carleman-Ahlfors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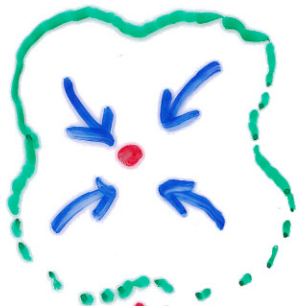
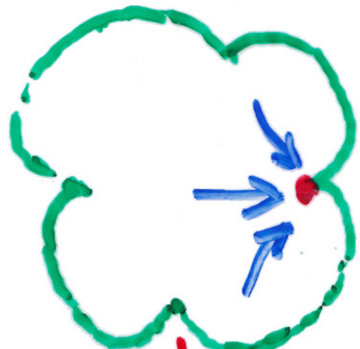
Dynamics on the Fatou set

Classification of periodic comp-s

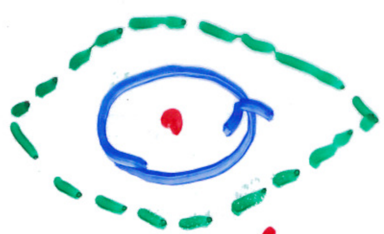
(Fatou – Julia – Siegel)



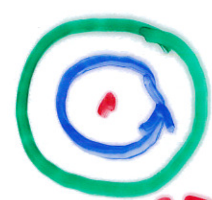
attracting basin



parabolic basin



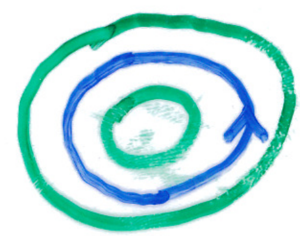
$\approx$



Siegel disk

$z \mapsto e^{2\pi i \theta} z$

Rk: For rational maps, $\exists$ also Herman rings



No Wandering Domains Thm

(Sullivan)

2D Polynomial Dynamics

$f: \mathbb{C}^2 \to \mathbb{C}^2, \ (x,y) \mapsto (p(x) - by, \ x)$

Standing Assumption: $|b| < 1$

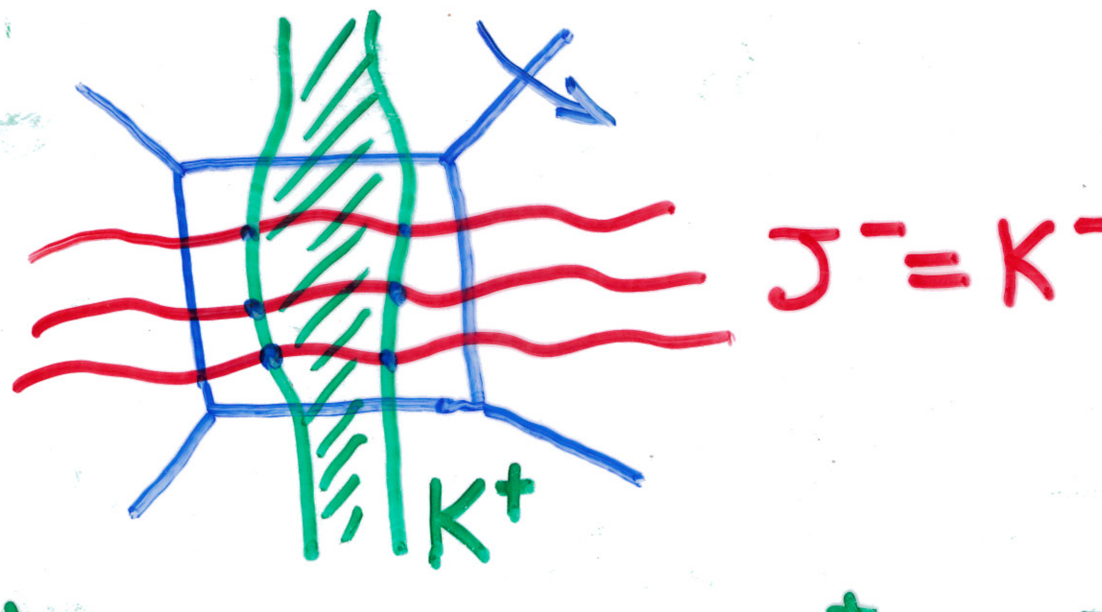
(dissipation)

$U^{\pm} = \{z : f^n z \to \infty, \ n \to \pm\infty\}$

basins of ∞

$K^{\pm} = \mathbb{C}^2 \setminus U^{\pm}$ filled Julia sets

$J^{\pm} = \partial K^{\pm}$ Julia sets



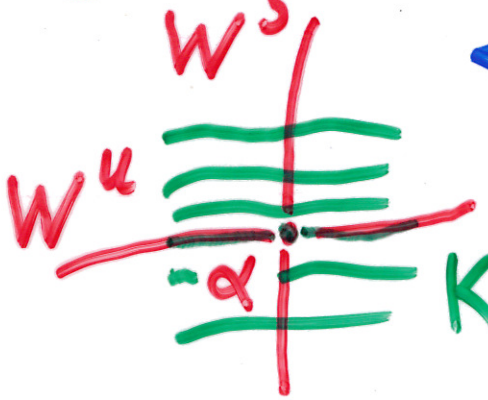
$K = K^+ \cap K^-$, $\quad J = J^+ \cap J^-$

$J^* = \overline{\mathrm{Per}^s(f)} \subset J$, saddle pts: $|\lambda_1| < 1 < |\lambda_2|$

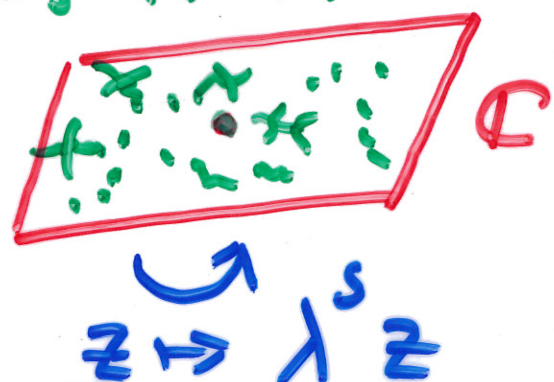
Green functions:

$$G^{\pm}(z) = \lim_{n \to \pm\infty} \frac{1}{d^n} \log^+ \| f^n z \|$$

- $G^{\pm} \geq 0$, vanish on $K^{\pm}$
- pluri-subharmonic (harm on $U^{\pm}$)
- $G^{\pm}(f^{\pm 1} z) = d\, G^{\pm}(z)$



$\{G^- = 0\} = K^- \cap W^s$



$G^- | W^s$ is a subharmonic f-n of order $\sigma = \dfrac{\log d}{|\log |\lambda^s||}$

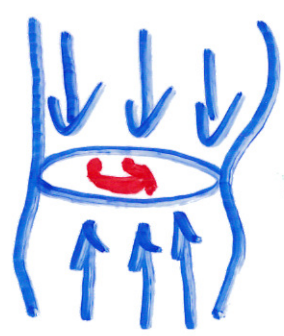
If $|\lambda^s| < d^{-2}$ then $\sigma < 1/2$, and by the Wiman Thm All comp-s of $K^- \cap W^s$ are bounded

Classification of periodic Fatou components

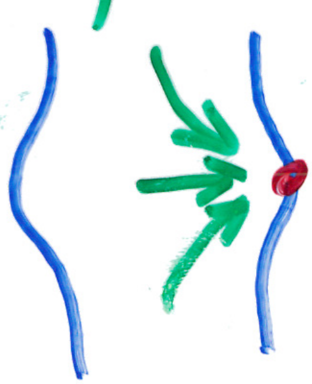
$f : D \circlearrowleft$; D is recurrent if $\exists z \in D$ s.t. $f^{n_k} z \to \alpha \in D$

Thm (BS, FS) Recurrent comp-s are either attracting basins or basins of rotation hol curves

Problem: Do Herman rings exist?



Thm (Han Peters + L) For moderately dissipative maps, $(|b| < d^{-2})$ any non-recurrent component is a parabolic basin



Possible new phenomenon:

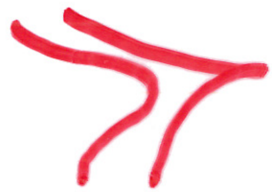
rank 1 limits $f^{n_k}|_D \to \varphi$,

$\varphi : D \to \Lambda \subset \partial D$

hol curve

Elements of the proof:

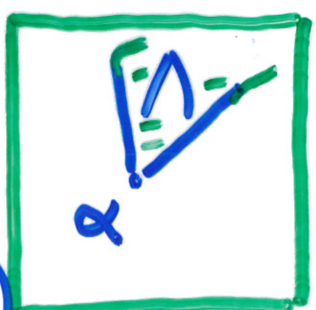
- Λ is non-singular



- $\exists$ Λ is an invariant unbounded domain in the stable manifold of a saddle α

- $\Lambda \subset K \Rightarrow G^- |_\Lambda \equiv 0$ contradicting the Wiman thm.

$\mathbb{C} = W^s(\alpha)$



J-Stability in D1

f_λ - a hol family over Λ

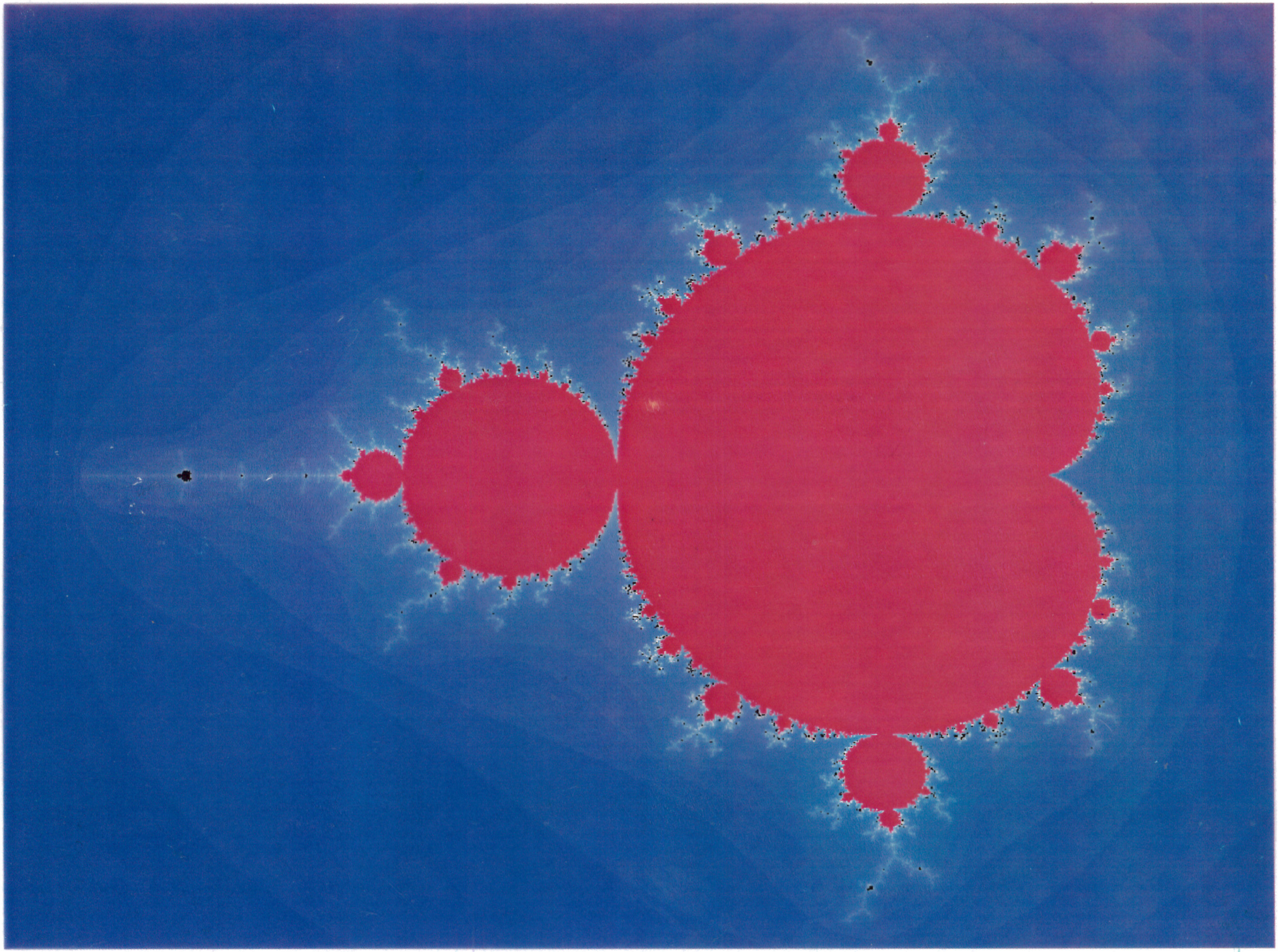
$\lambda_0 \in \Lambda$ is J-stable if $\forall_\lambda$ close to λ_0, we have

$$\begin{array}{ccc} J(f_{\lambda_0}) & \xrightarrow[f_{\lambda_0}]{} & J(f_{\lambda_0}) \\ h_\lambda \downarrow \wr & & \wr \downarrow h_\lambda \\ J(f_\lambda) & \xrightarrow[f_\lambda]{} & J(f_\lambda) \end{array}$$

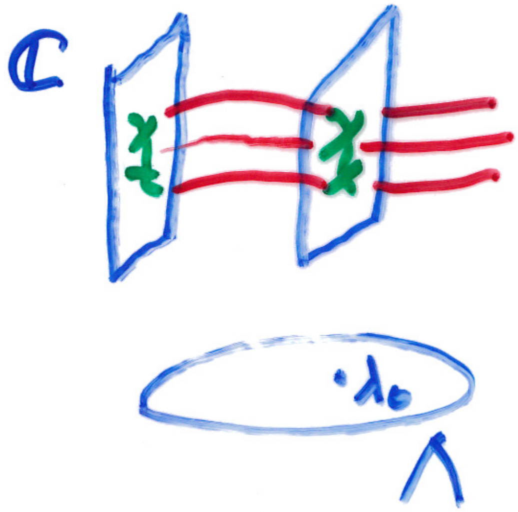
$\mathcal{P}_d$ - space of polynomials of deg d

Thm (Mañé-Sad-Sullivan, L)
The set of J-stable $f \in \mathcal{P}_d$ is open and dense. Its complement $\mathcal{B}$ (the bifurcation locus) is the closure of parabolic maps $f \in \mathcal{P}_d$

For $\mathcal{P}_1 = \{z^2+c\}$, $\mathcal{B} = \partial M$, M-Mandelbrot



Holomorphic motions



$h_\lambda : X_0 \to X_\lambda \subset \mathbb{C}$

- $h_{\lambda_0} = id$
- h_λ is 1-to-1
- $h_\lambda(z)$ hol in $\lambda \in \Lambda$.

λ-Lemma. Holomorphic motion of any set $X_0 \subset \mathbb{C}$ extends to a holomorphic motion of $\overline{X_0}$. The maps $h_\lambda : \overline{X_0} \to \overline{X_\lambda}$ are homeos.

Construction of conjugacies: Consider hol motion of repelling per pts, and extend to the closure

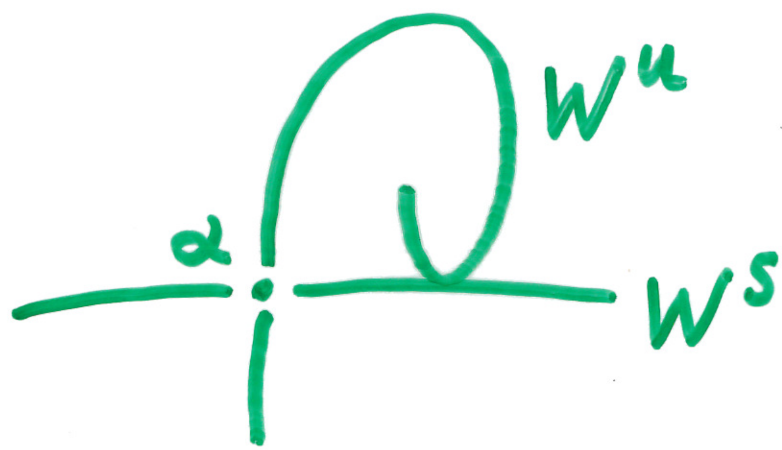
Density of J-stable maps: based upon # attracting cycles $\le$ $\le 2d-2$ (Fatou-Julia)

New phenomena in 2D

Newhouse phenomenon:

$\exists$ parameter domains densely filled with maps that have ∞ many attracting cycles.

Related to Homoclinic tangencies



Palis Conjecture (in real 2D)

Lack of stability is related to Homoclinic tangencies & the Newhouse phenomenon

Palis Conjecture (complex version)

(joint w. Romain Dujardin)

$\mathcal{H}_d$ - the space of complex Hénon maps of deg d

Thm. In the moderately dissipative space $\mathcal{H}_d^{\text{diss}}$, weak J^*-stability + homoclinic tangencies / Newhouse are dense.

Weak J^*-stability:

- All saddles are persistent (move without bifurcations)

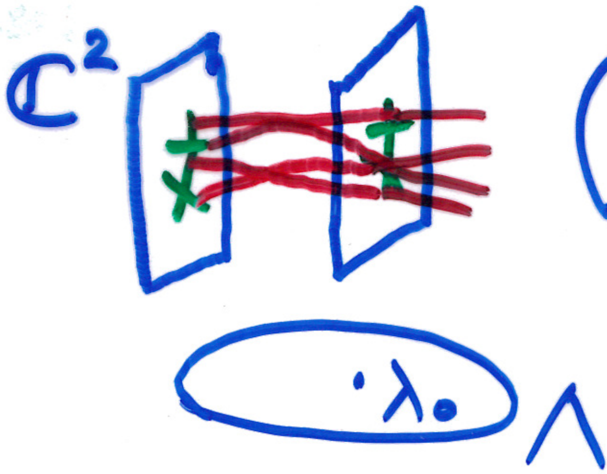
$\Updownarrow$

- J^*_λ depends continuously on λ

$\Updownarrow$

- J^*_λ moves under a branched holom. motion

Branched Holomorphic Motions



(equivariant correspondence)

$J_{\lambda_0} \Longleftrightarrow J_\lambda$

In $\mathbb{C}^2$, hol motions may extend to Branched hol motions.

Hol motion of saddles extends to weak J^*-stability.

Semi-parabolic pt: $\lambda_1 < 1 = \lambda_2$

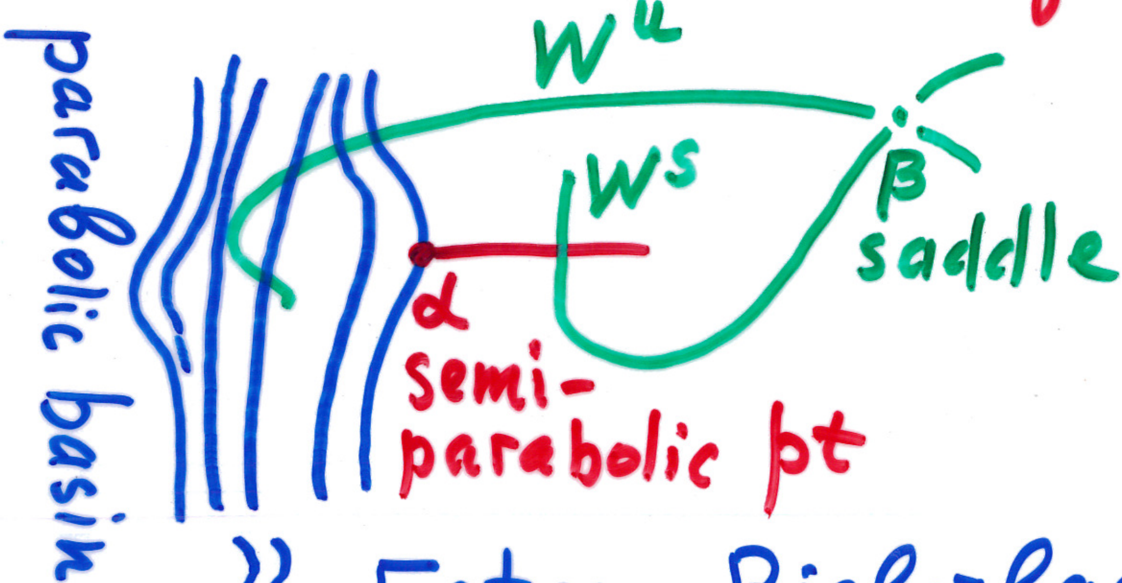
Can be perturbed to attracting pt

If approximated by semi-parabolic again, perturb it; etc.

$\Downarrow$

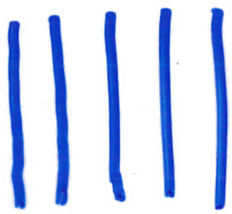
Either produces a stable or Newhouse map

Mechanism of creating Homoclinic Tangencies



≀ Fatou-Biebarback domain

$\mathbb{C}^2$



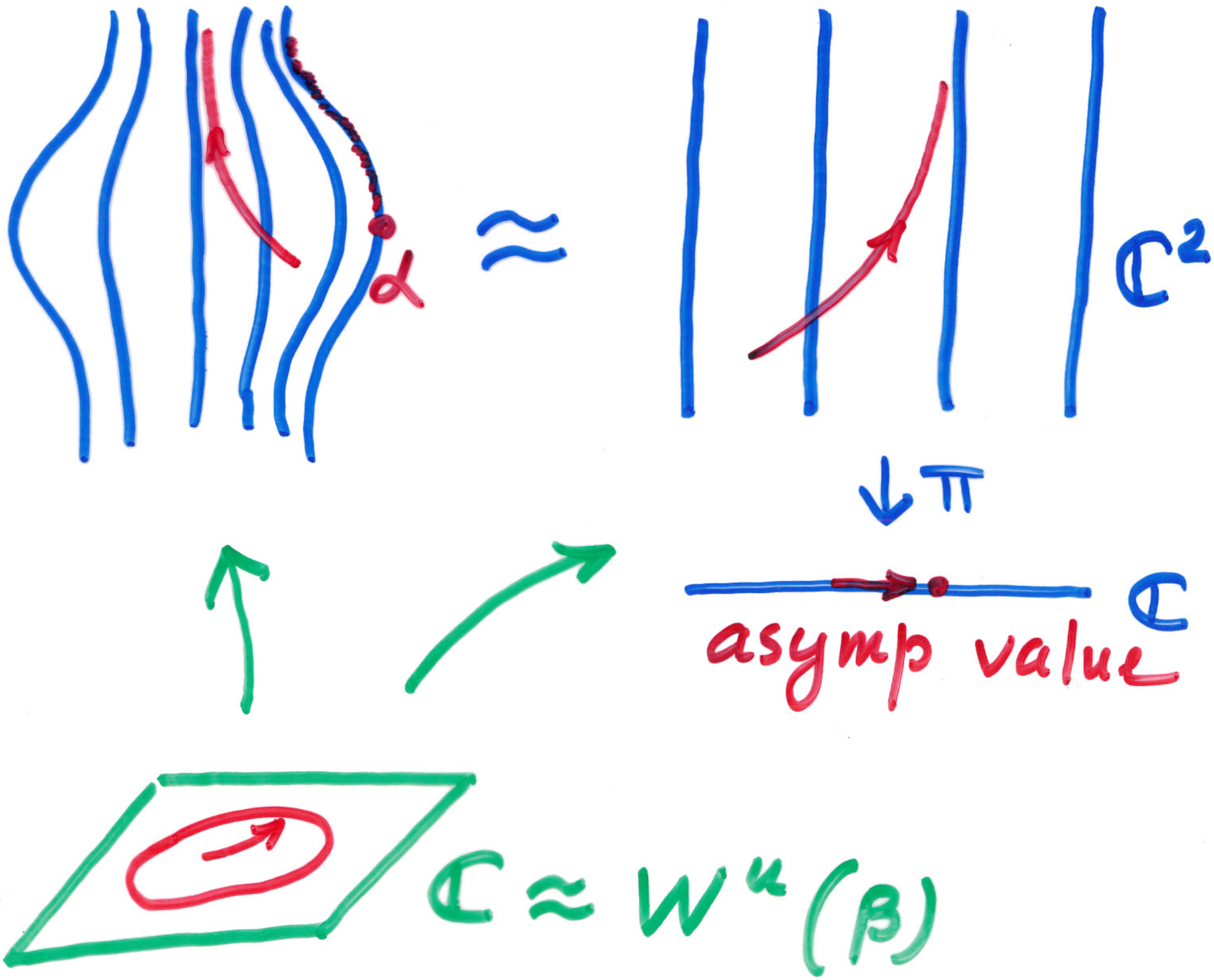
$(x,y) \mapsto (x+1, y)$

stable foliation

(Bedford-Smillie-Ueda)

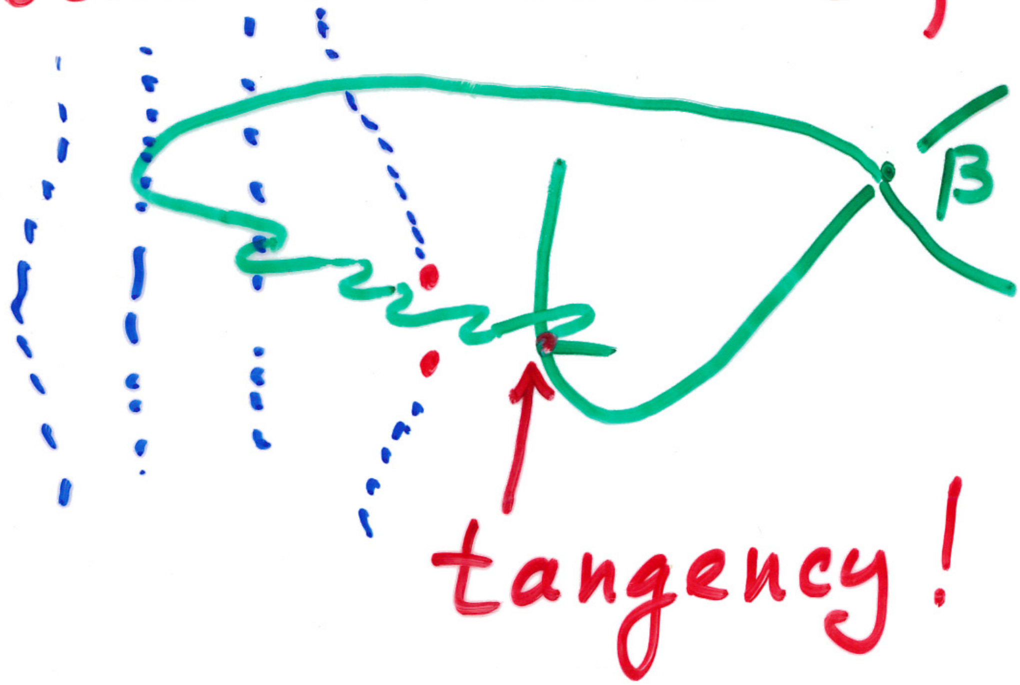
<u>Thm</u> For moderately dissipative maps, $\exists$ a "critical pt" in the parabolic basin. (Dujardin & L)

Uses the Wiman Thm again! also DCA



Contradiction to Wiman!

Semi-Parabolic Bifurcation



(Un tour de valse by Douady & Sentenac)

<u>Thm</u> (D-L) The bifurcation locus $\mathcal{B}$ is the closure of the set of parameters with homoclinic tangencies.