Exploring the parameter space of an iterated function system

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Joint work with:
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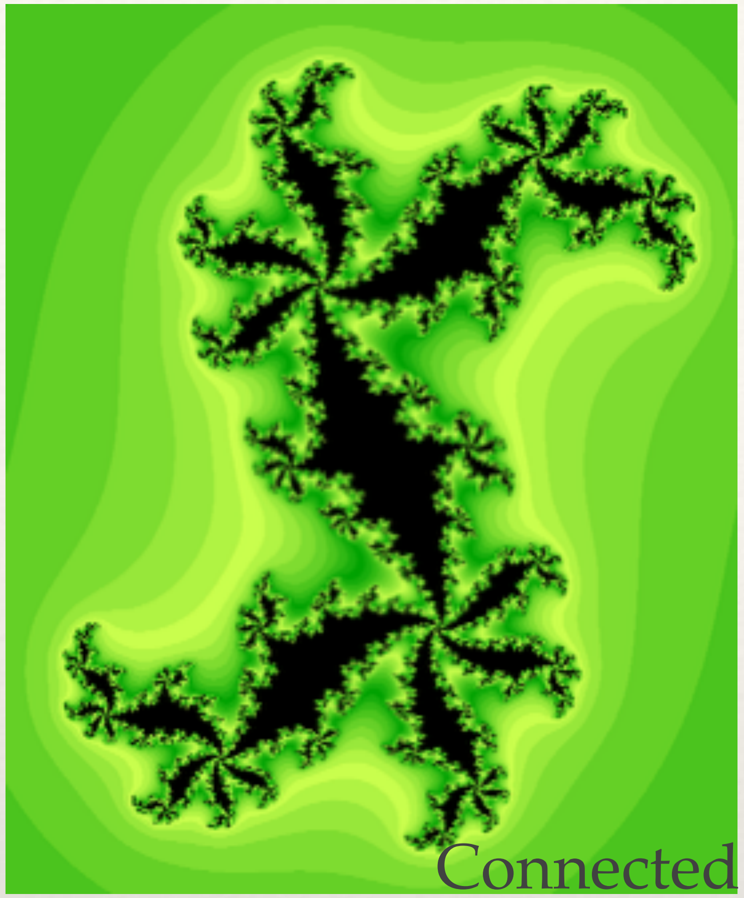
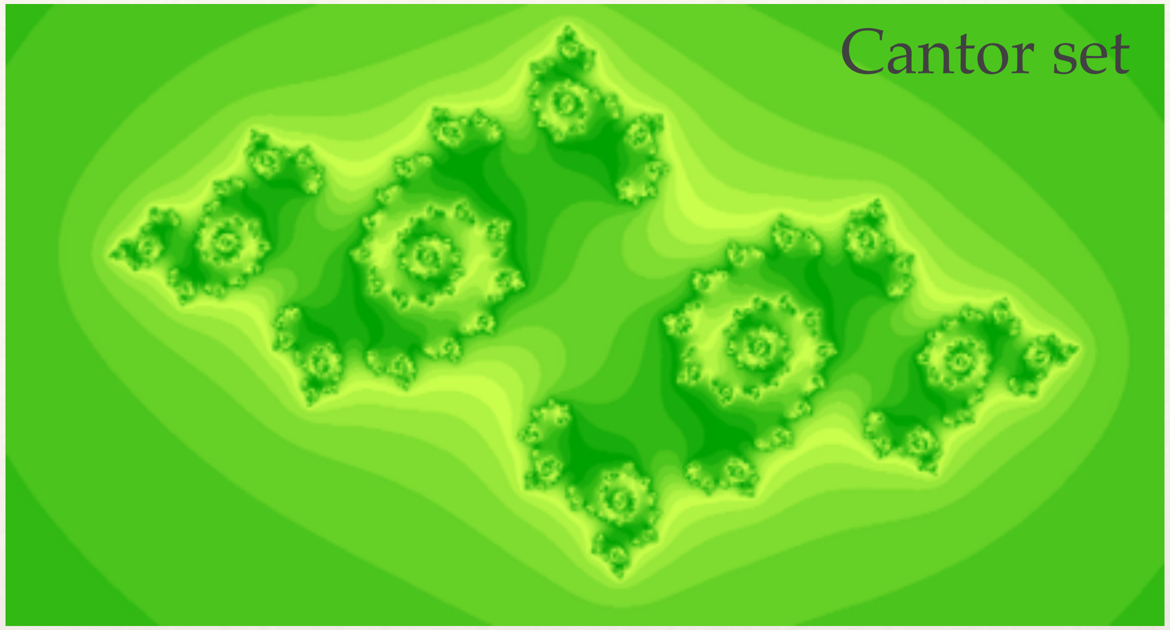
Quadratic polynomials $p_c : z \mapsto z^2 + c$





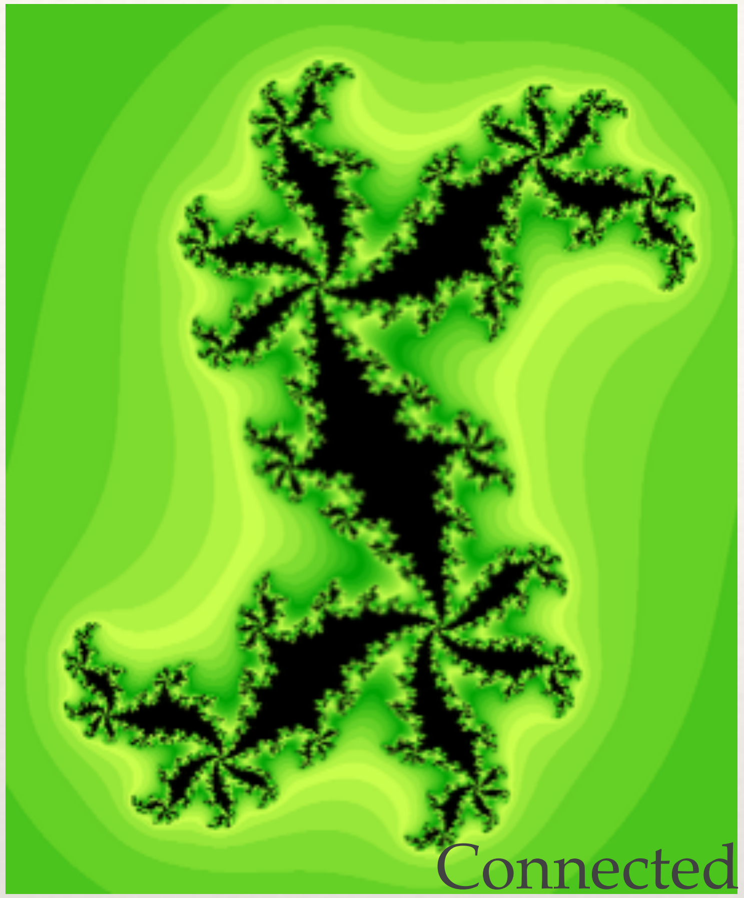
$$\mathcal{M} := \{c \in \mathbb{C} \mid K_c \text{ is connected}\}$$

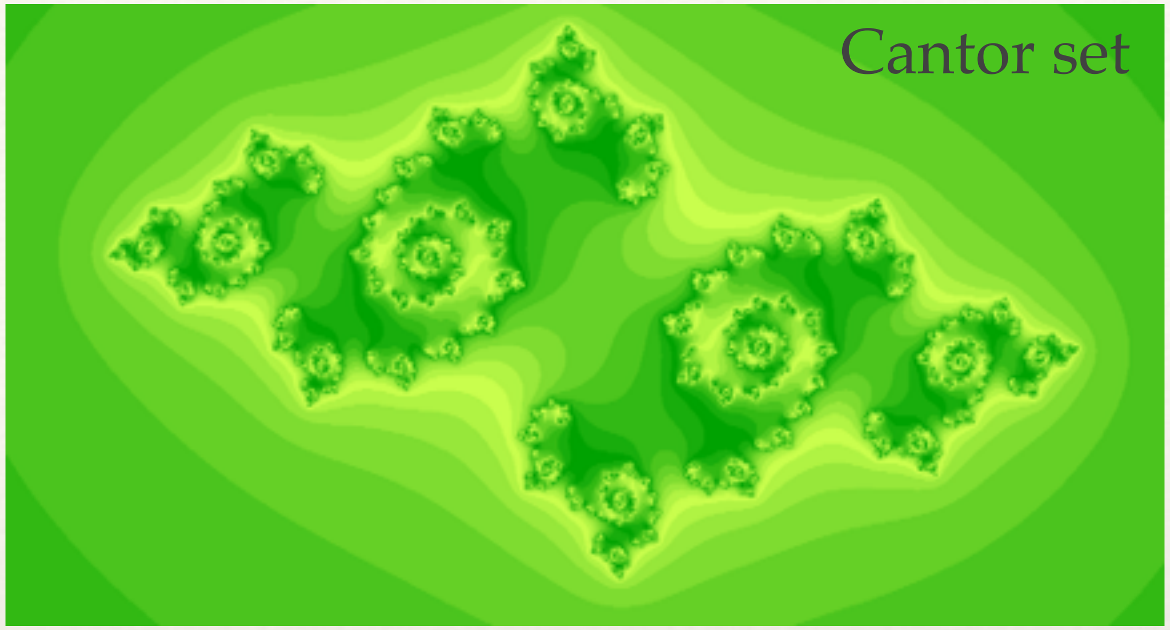
$$\mathcal{M}' := \{c \in \mathbb{C} \mid K_c \ni 0\}$$

$$\mathcal{M}'' := \{c \in \mathbb{C} \mid K_c \text{ is connected and full}\}$$

subsets in parameter space

Quadratic polynomials $p_c : z \mapsto z^2 + c$



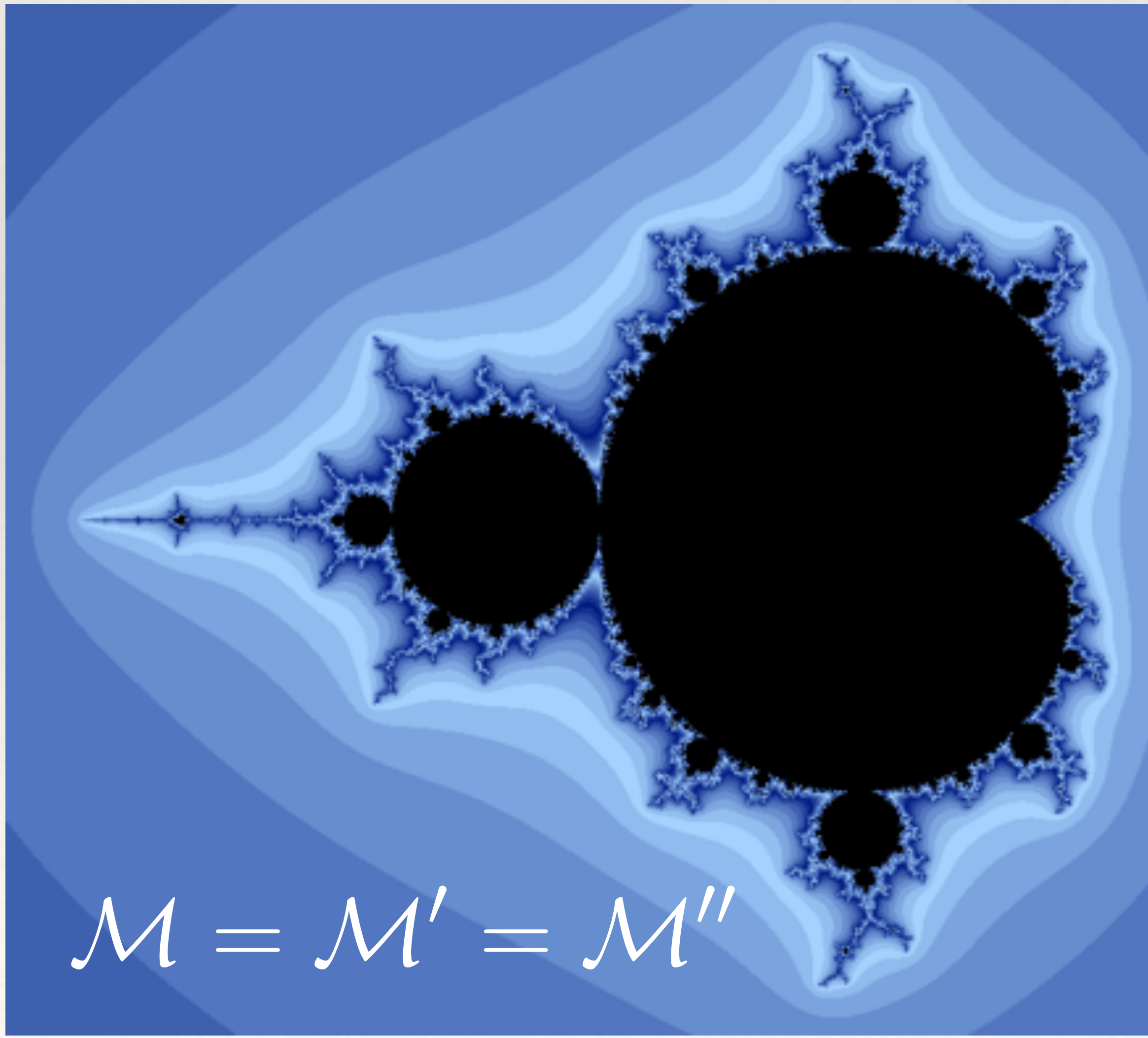


The Mandelbrot set is

- compact
- connected and full
- $\overline{\mathrm{int}(\mathcal{M})} = \mathcal{M}$
- ??? locally connected ???

hyperbolic components

(dis)continuity

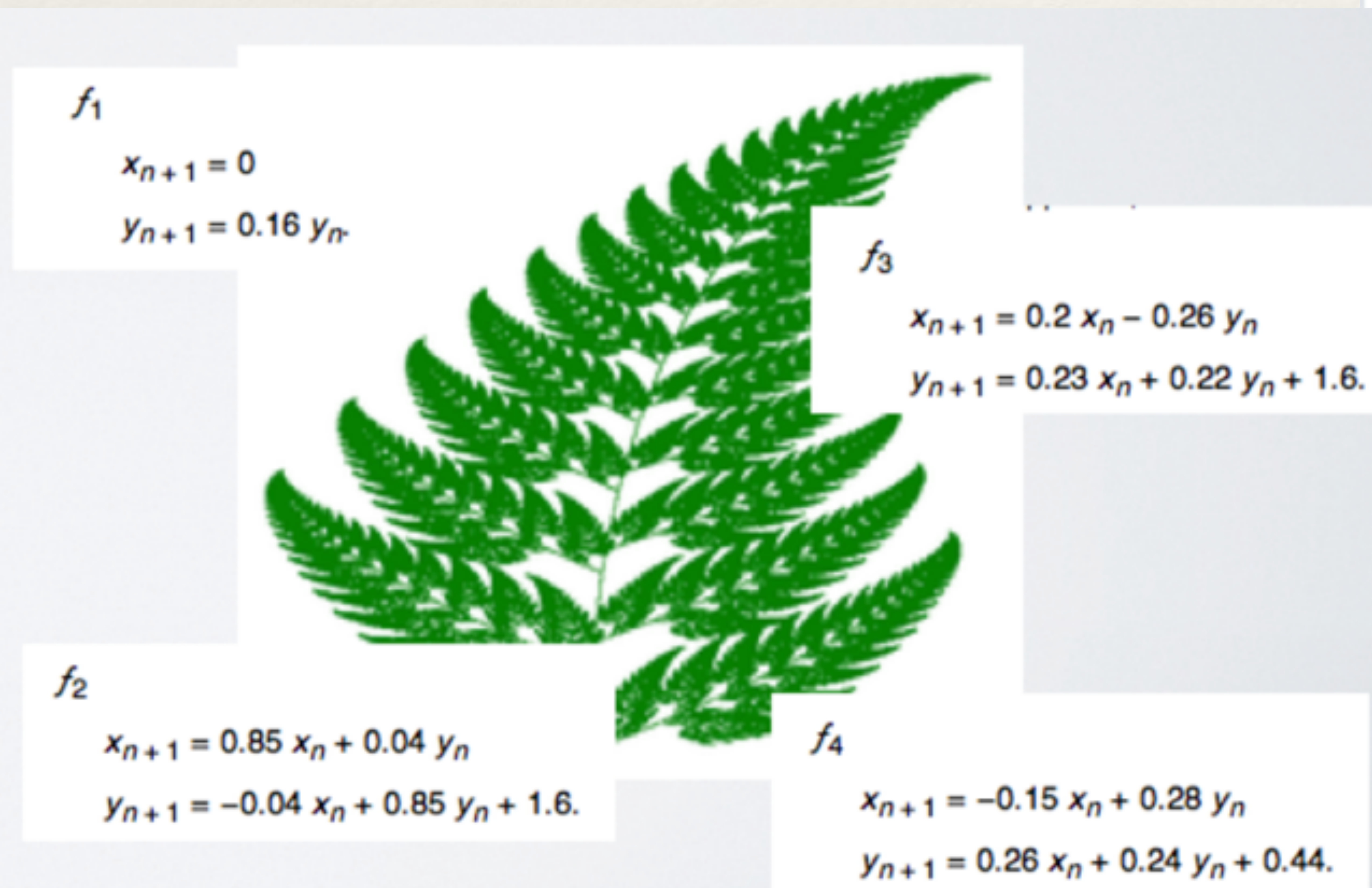


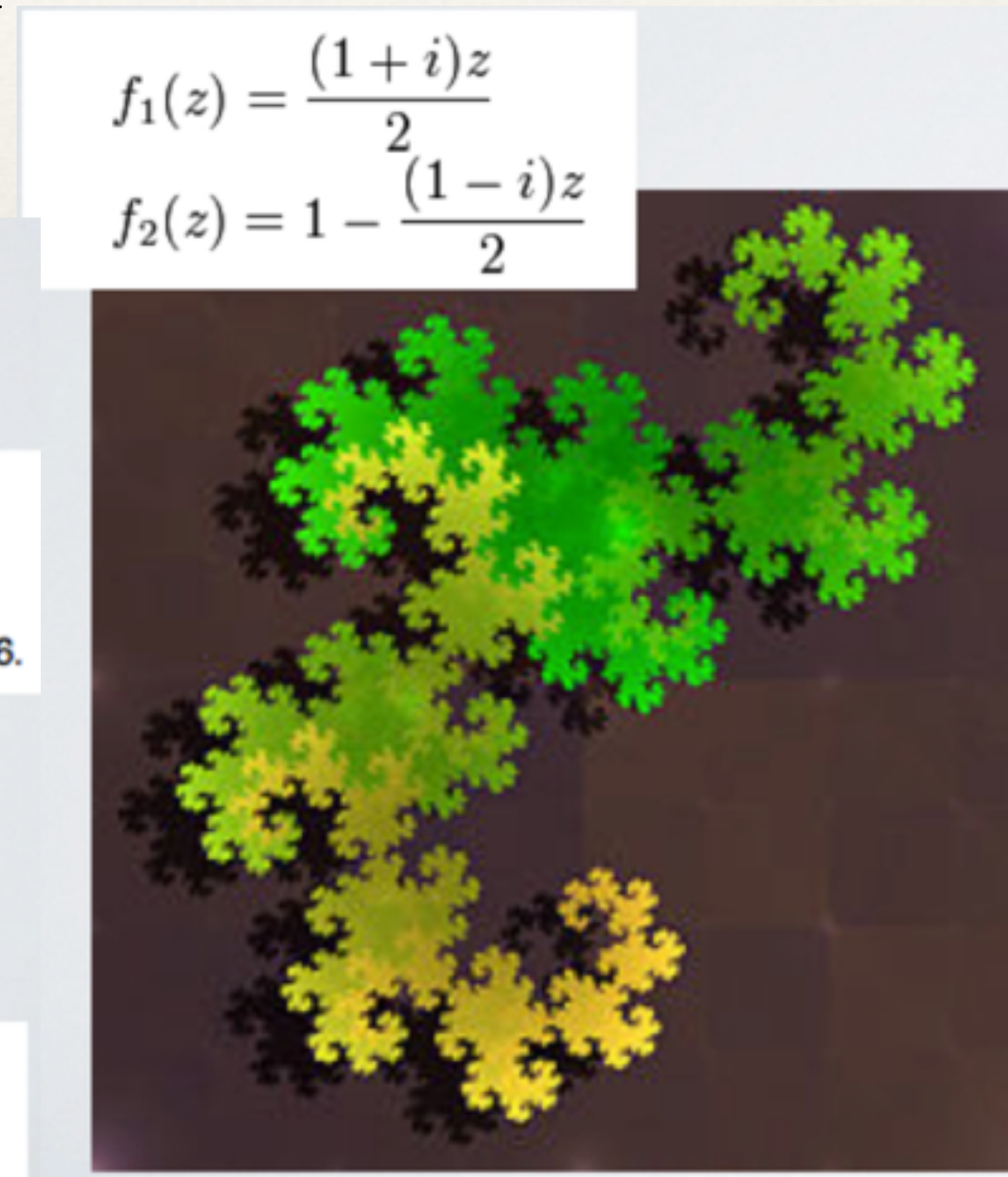
Iterated Function Systems

X is a complete metric space

$\{f_1, \ldots, f_n\}$ contractions $X \to X$

$\exists$ nonempty compact attractor $\Lambda \subseteq X$





Parameterized family of similarities

$$f_c, g_c : \mathbb{C} \to \mathbb{C}$$

$$f_c : z \mapsto cz + 1 \quad g_c : z \mapsto cz - 1, \quad 0 < |c| < 1.$$

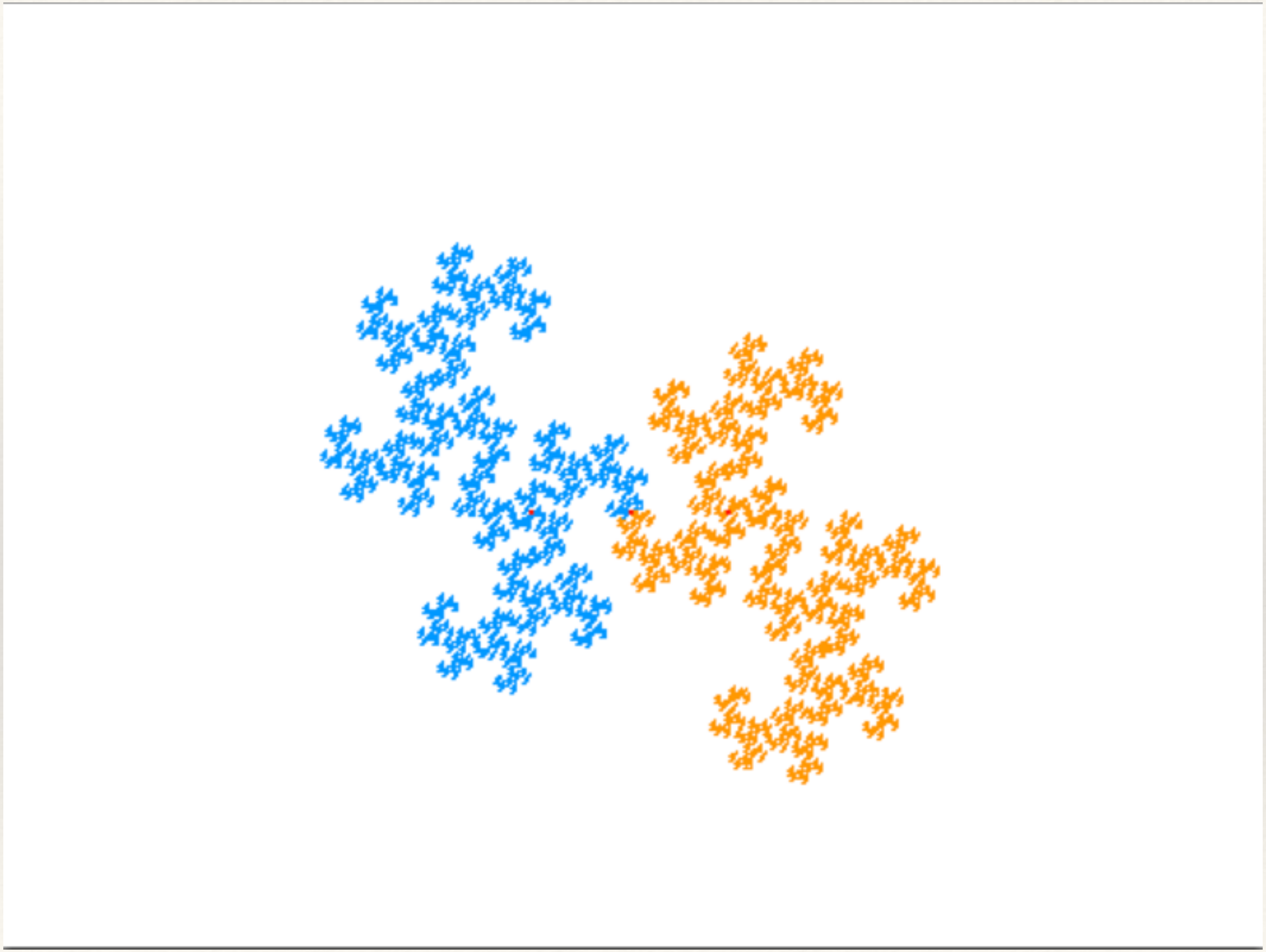
Remark: coordinates

Λ_c is the closure of the set of fixed points of $< f_c, g_c >$.

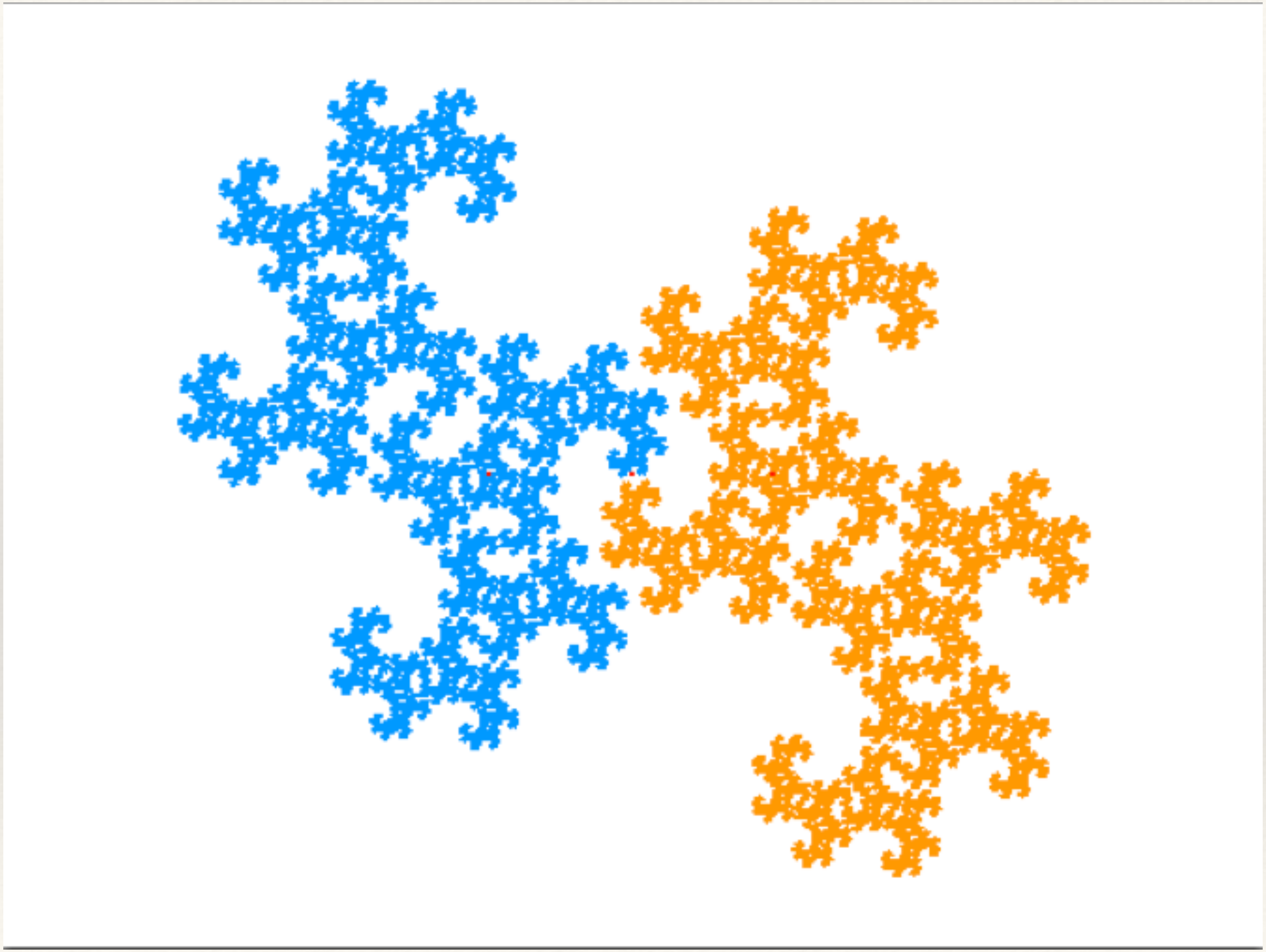
Drawing the limit set

$$\Lambda_c = \bigcap_n \bigcup_{w \in G_n} w(D)$$

The dynamical plane



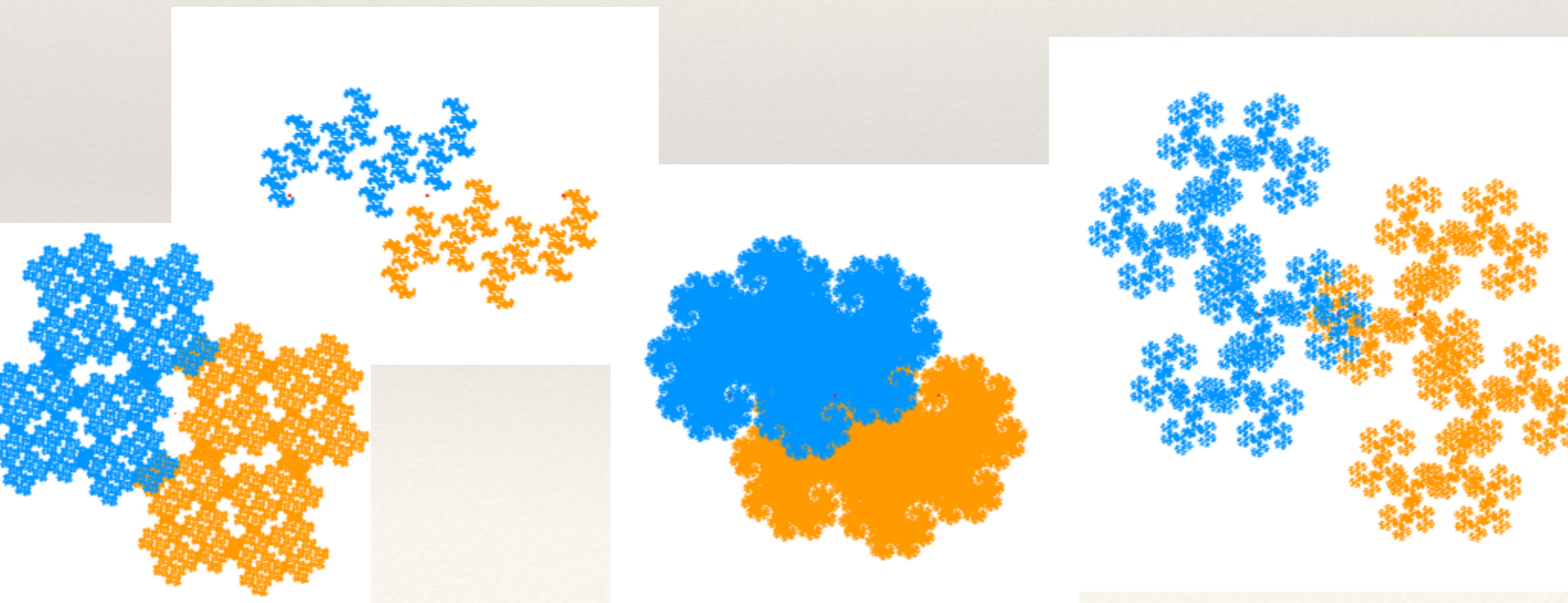
Certifying limit set is disconnected



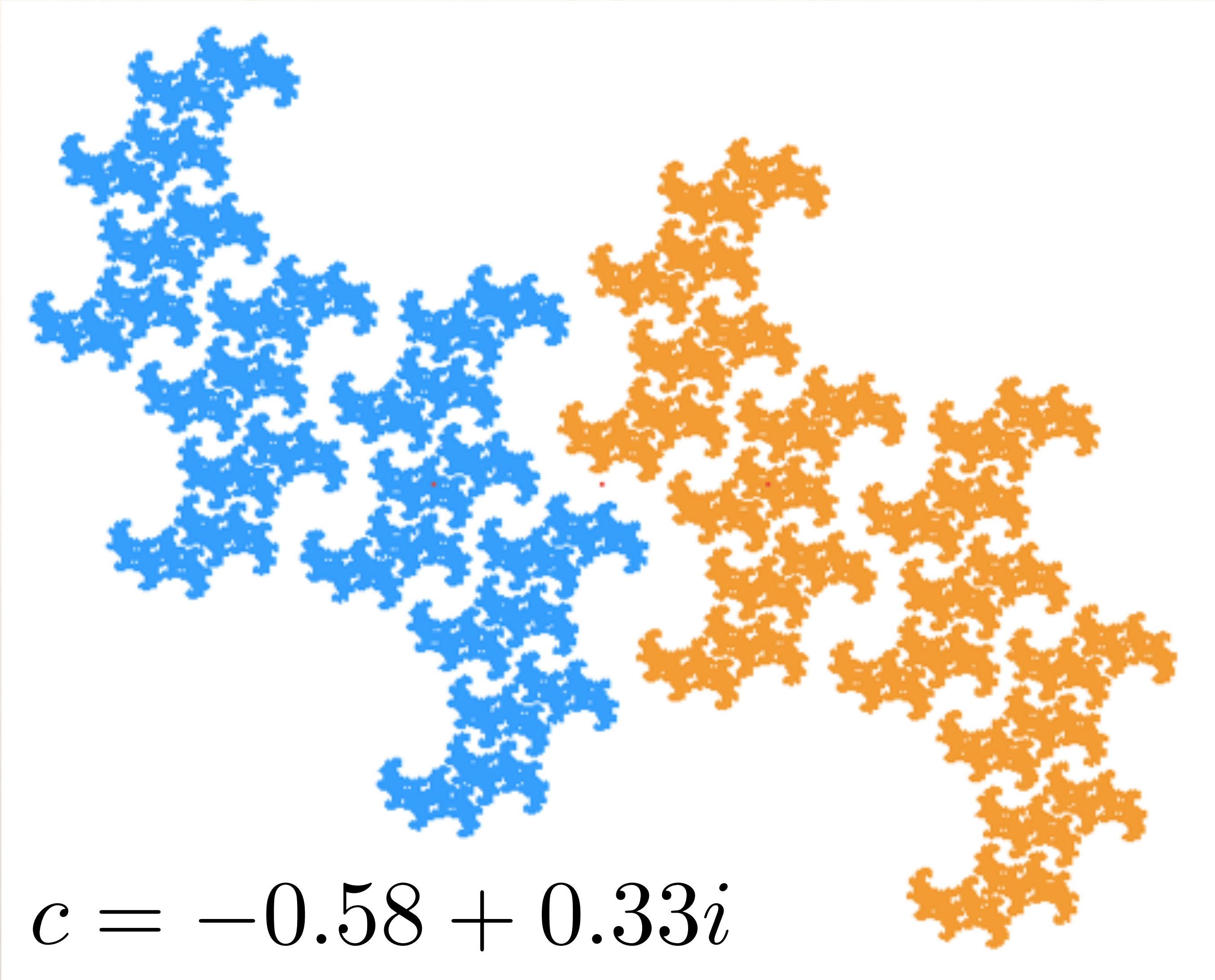
The Limit Set: topology

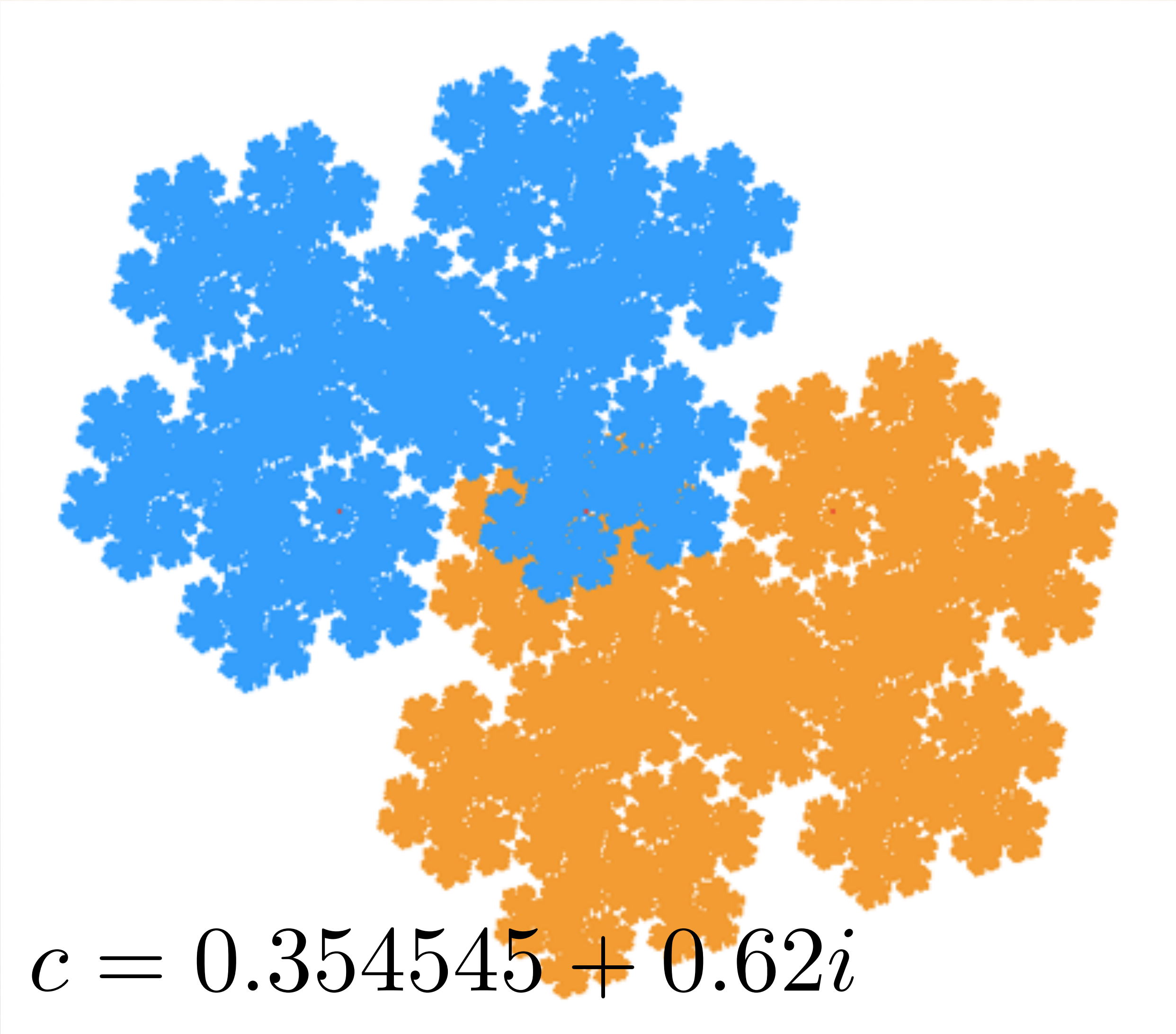
- Λ_c connected iff $f_c(\Lambda_c) \cap g_c(\Lambda_c) \neq \emptyset$
- Λ_c disconnected iff $\exists$ disk D so that $f_c(D), g_c(D) \subseteq D$, and $f_c(D) \cap g_c(D) = \emptyset$

$\Longrightarrow$ **Dichotomy:** Λ_c is connected, or Λ_c is a Cantor set



“Critical point”





Fixed point of f_c is $\alpha_c := 1/(1-c)$

Fixed point of g_c is $\beta_c := -1/(1-c)$

Center of symmetry $(\alpha_c + \beta_c)/2 = 0$

Parameter space

$$f_c : z \mapsto cz + 1 \quad g_c : z \mapsto cz - 1, \quad 0 < |c| < 1.$$

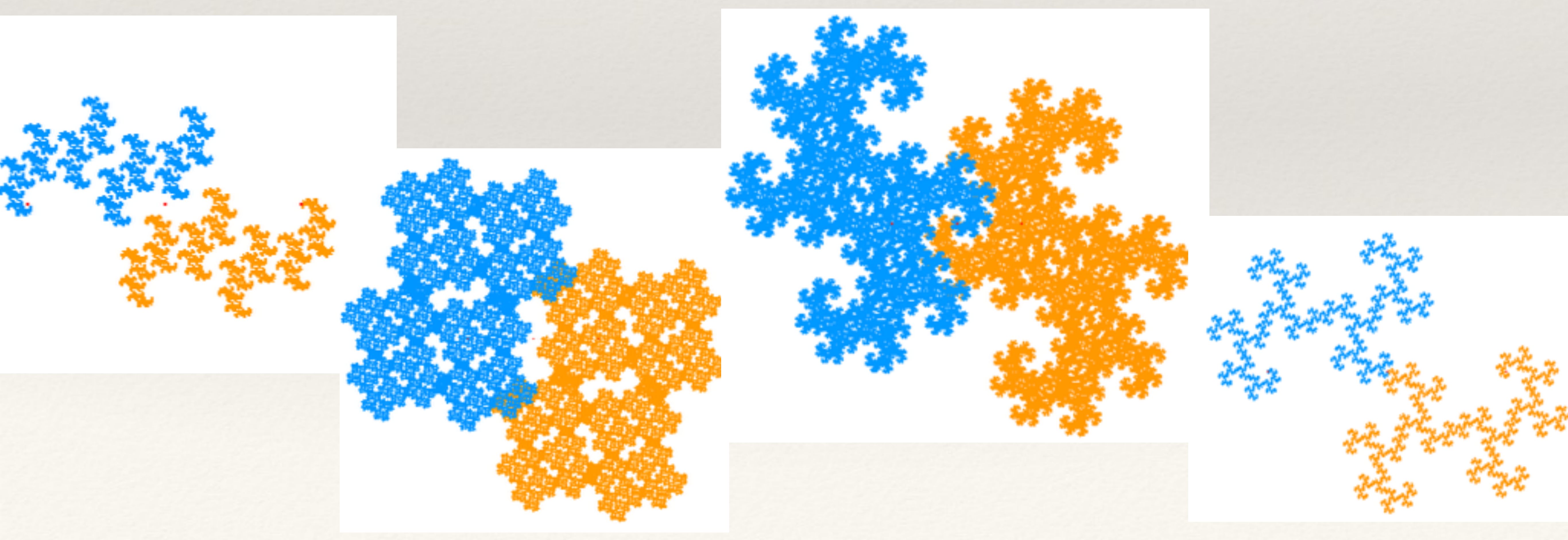
Three subsets of interest:

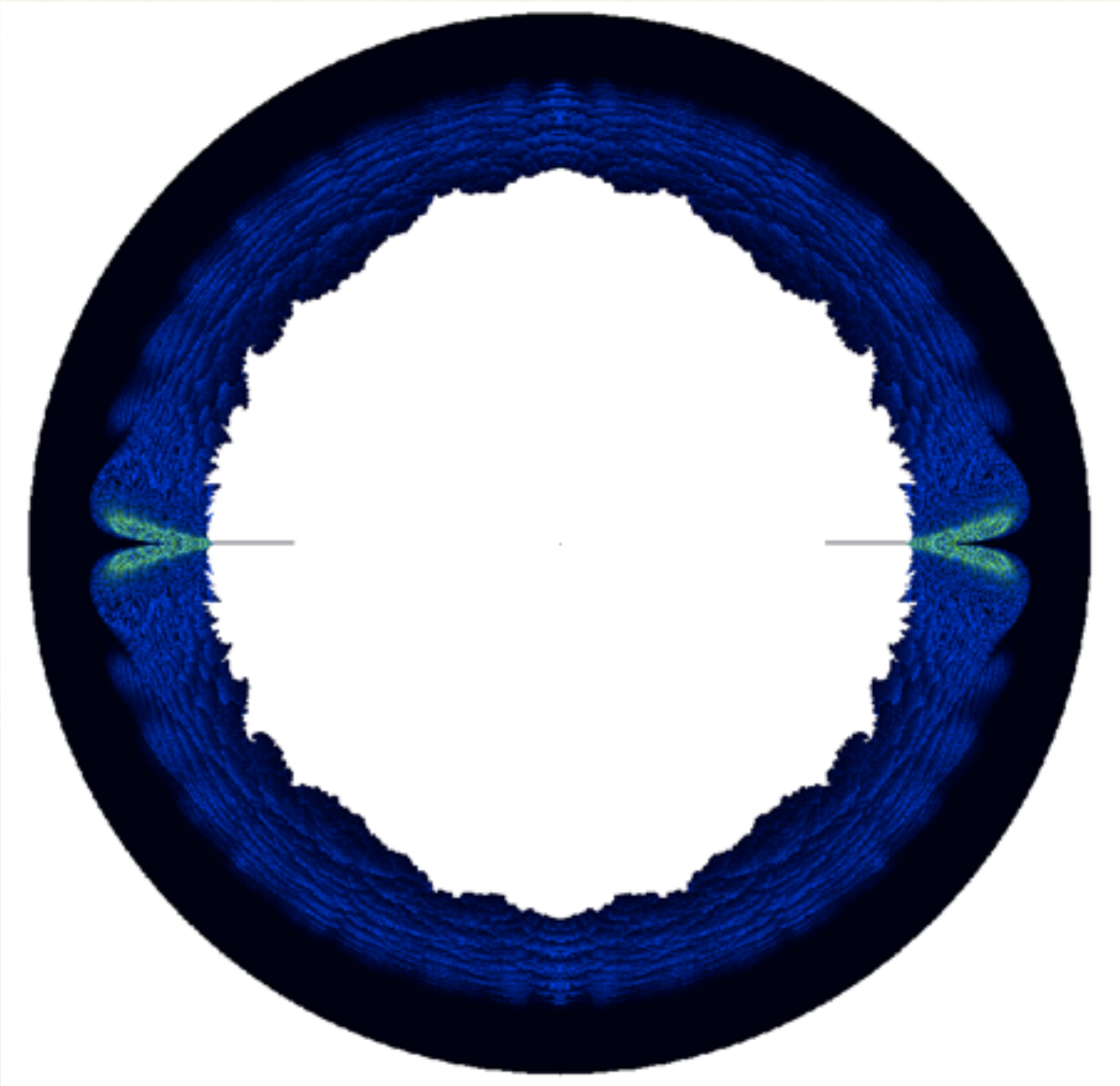
- $M := \{c \in \mathbb{D}^* \mid \Lambda_c \text{ is connected}\}$
- $M' := \{c \in \mathbb{D}^* \mid \Lambda_c \ni 0\}$, and
- $M'' := \{c \in \mathbb{D}^* \mid \Lambda_c \text{ is connected and full}\}$.

M, M', and M'' are closed

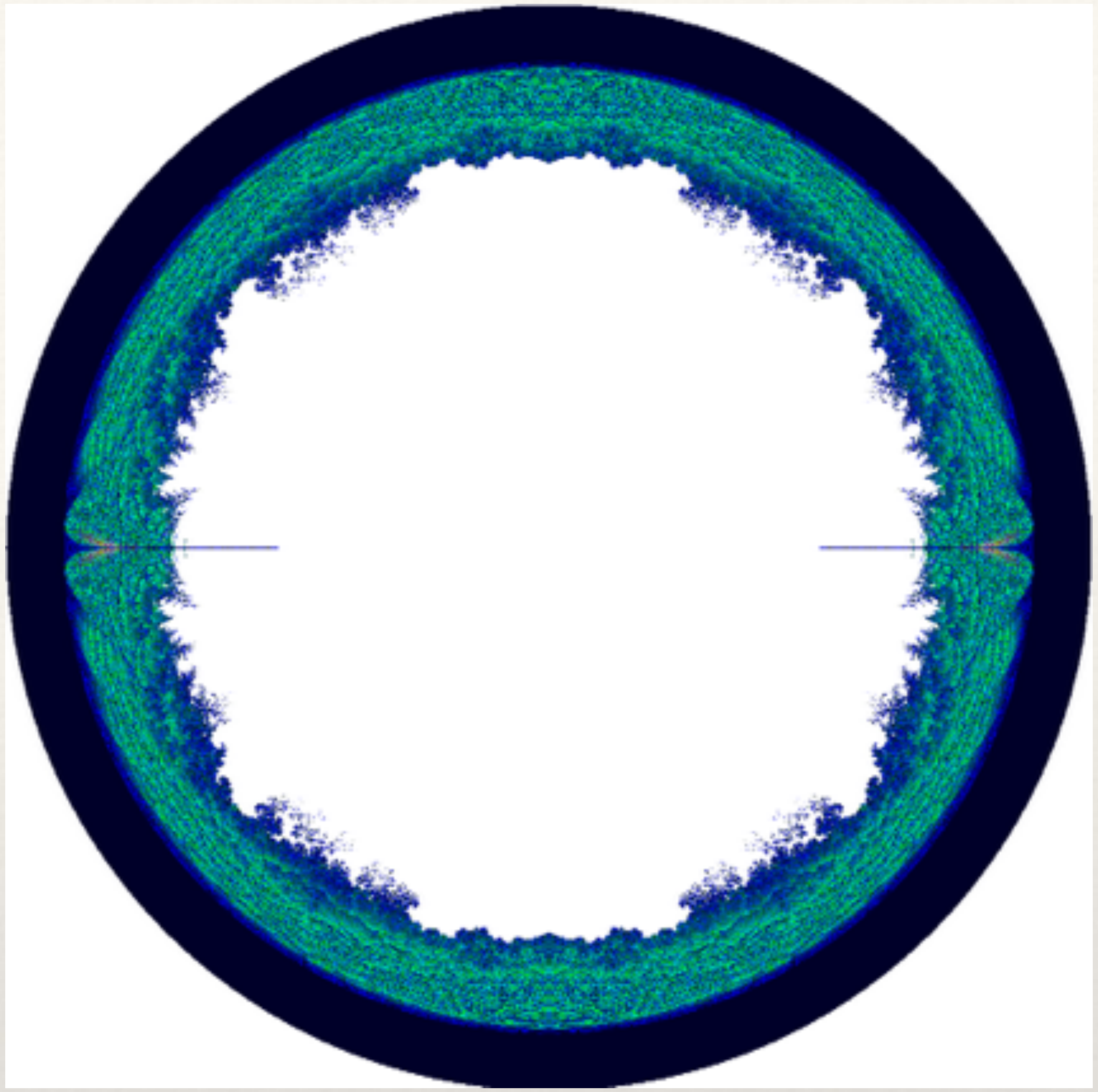
$M'' \subsetneq M' \subsetneq M$

continuity





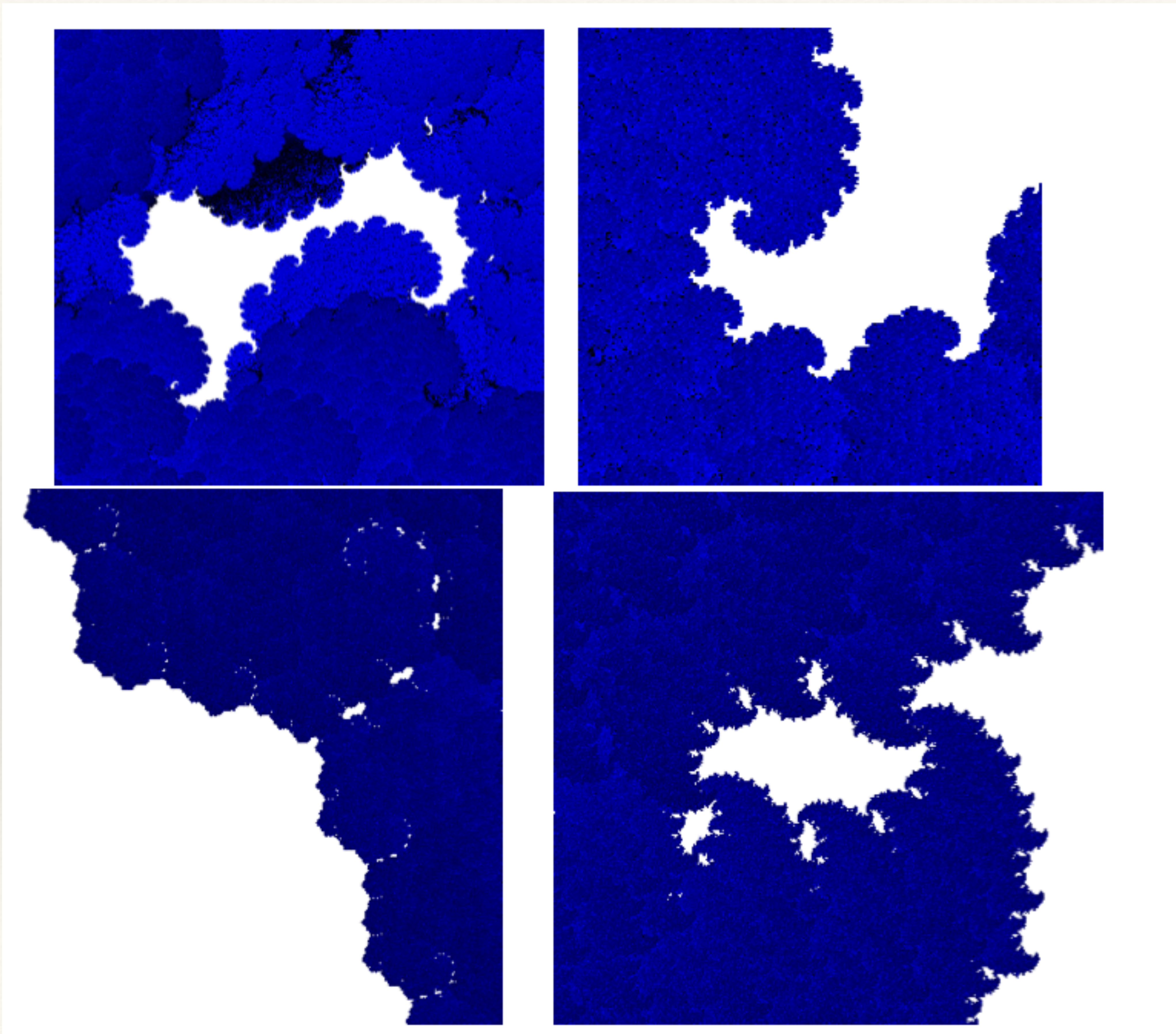
$M := \{c \in \mathbb{D}^* \mid \Lambda_c \text{ is connected}\}$

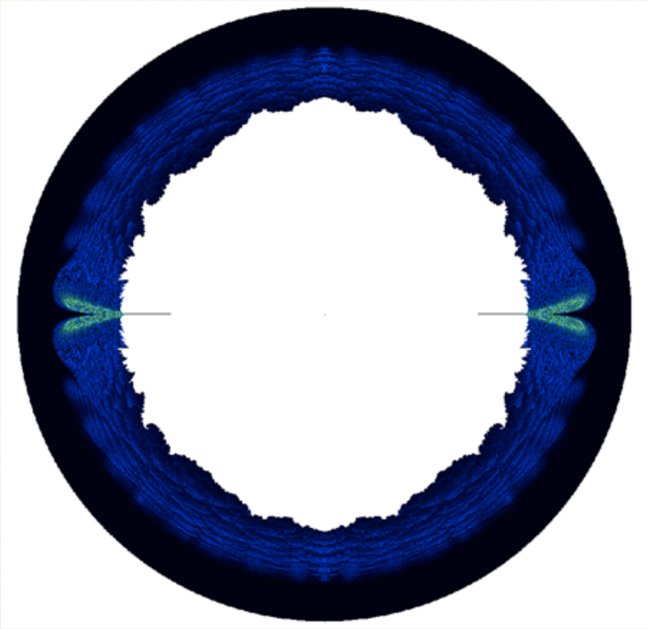


$M' := \{c \in \mathbb{D}^* \mid \Lambda_c \ni 0\}$

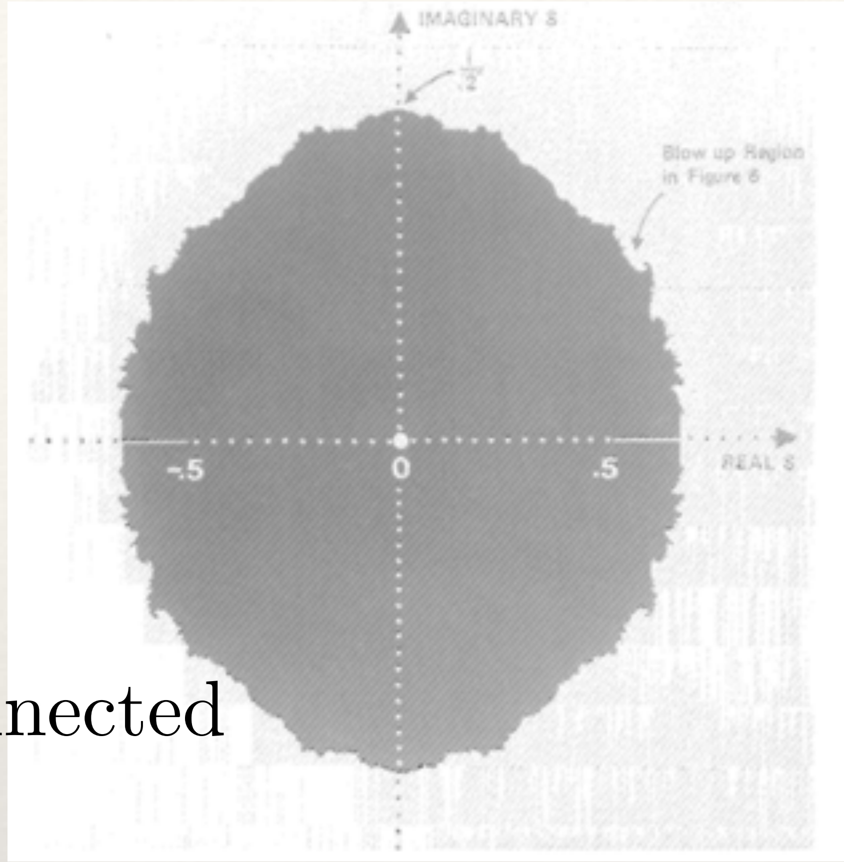
https://github.com/dannycalegari/schottky

Zoom into M





Historical remarks



- 1985, Barnsley-Harrington; M has real whiskers
- 1988, 1993, Bousch; M connected and locally connected
- 2002, Bandt; M is NOT full - there is at least one exotic component of the complement

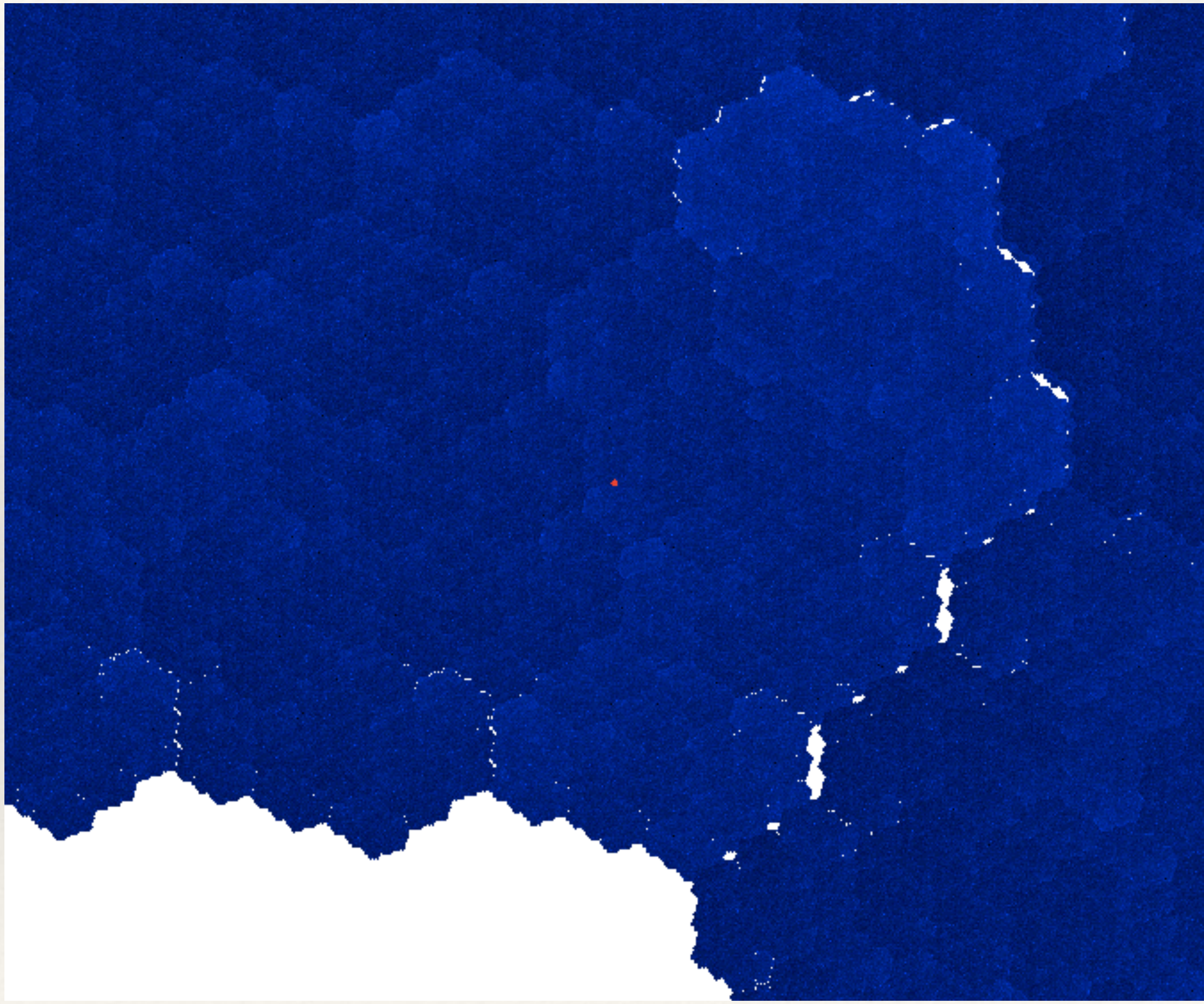
Conjecture. $\mathrm{int}(M)$ is dense away from $\mathbb{R}$

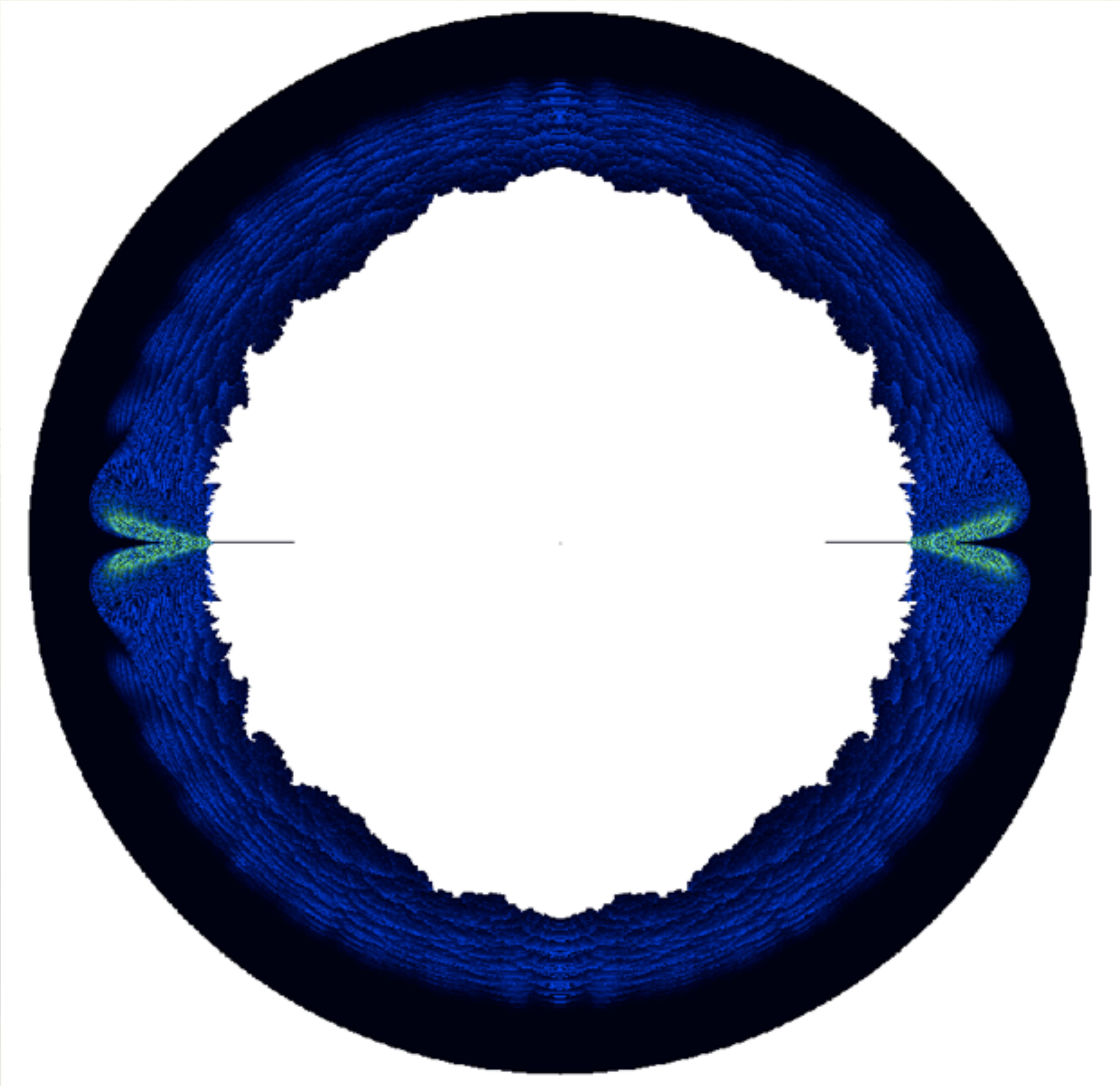
- 2003 Solomyak-Xu; M has dense interior in a neighborhood of $i\mathbb{R}$

Theorem. The interior of M is dense away from $\mathbb{R}$.

Theorem. There are infinitely many connected components of the complement of M.

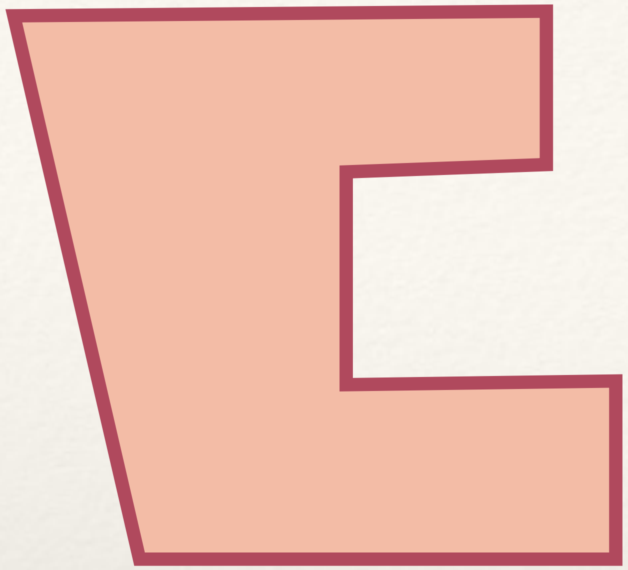
Understand interior points



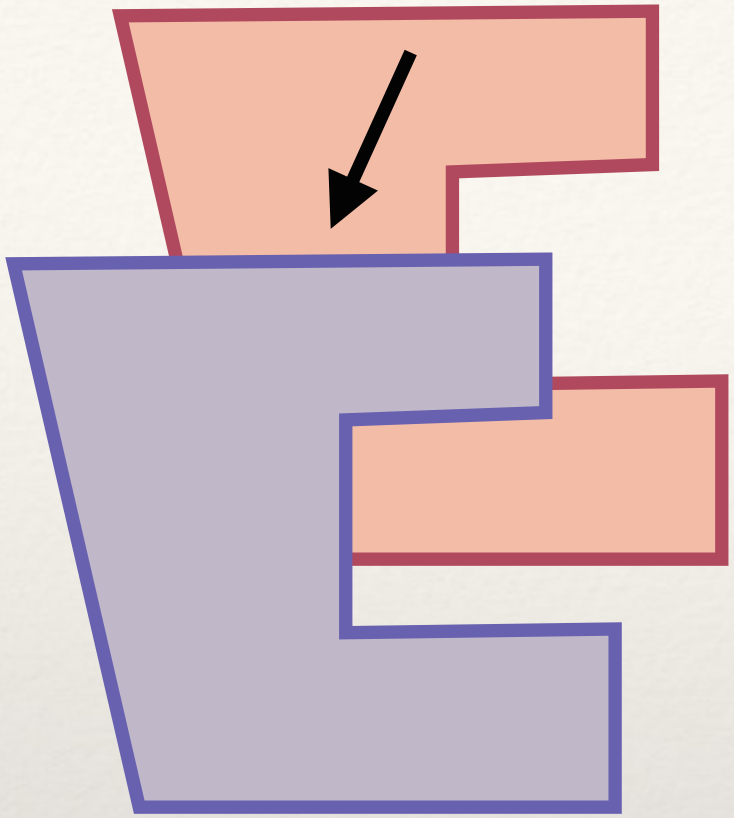


Proposition. (Bousch) M contains the annulus $\left\{\frac{1}{\sqrt{2}} \leqslant |c| < 1\right\}$.

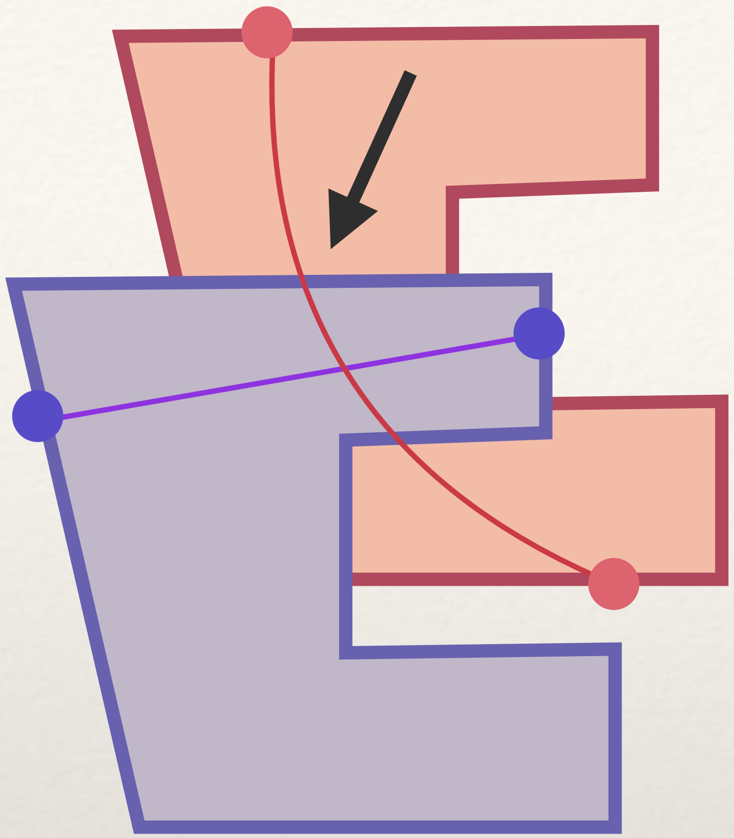
(Non)convexity



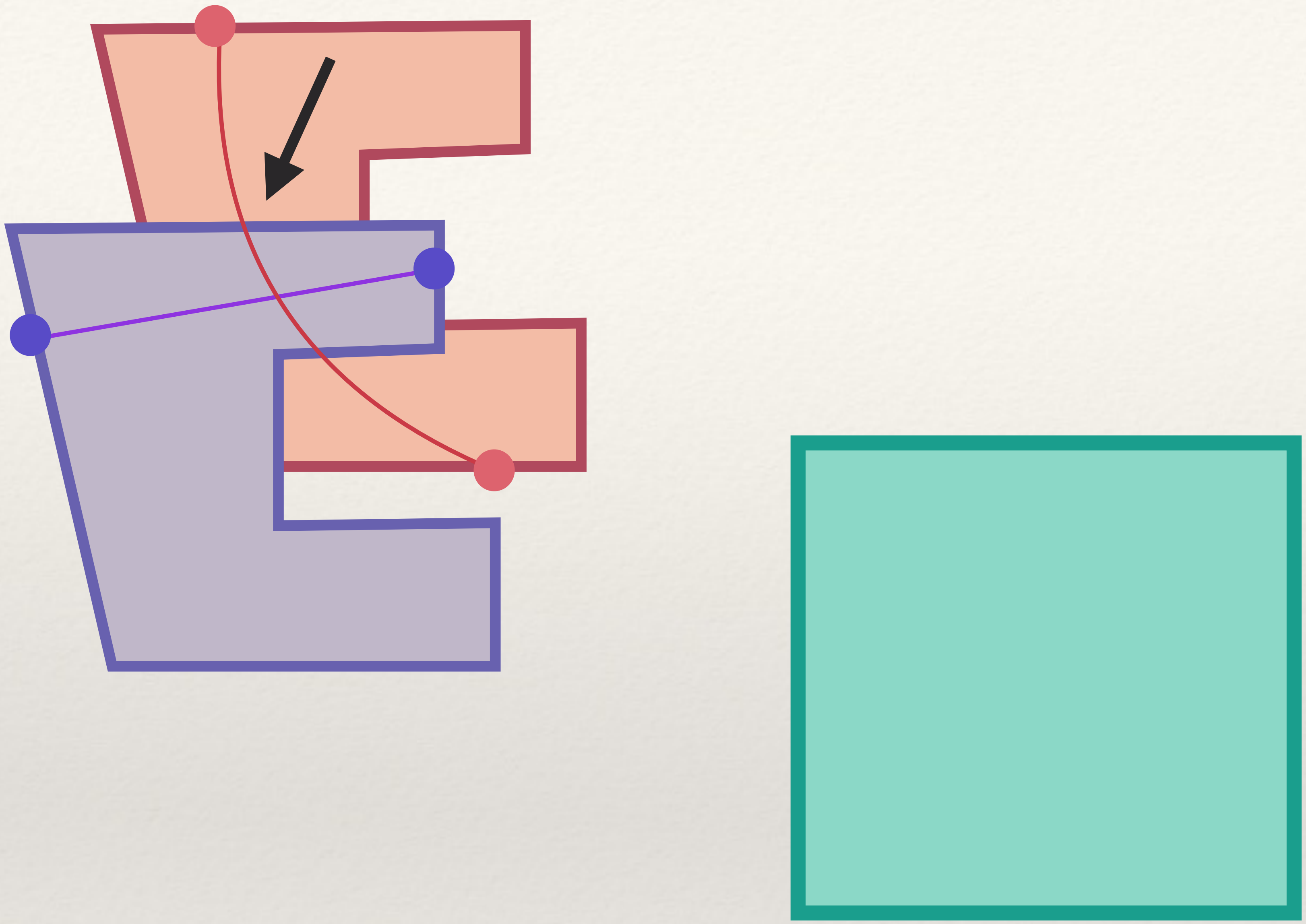
(Non)convexity



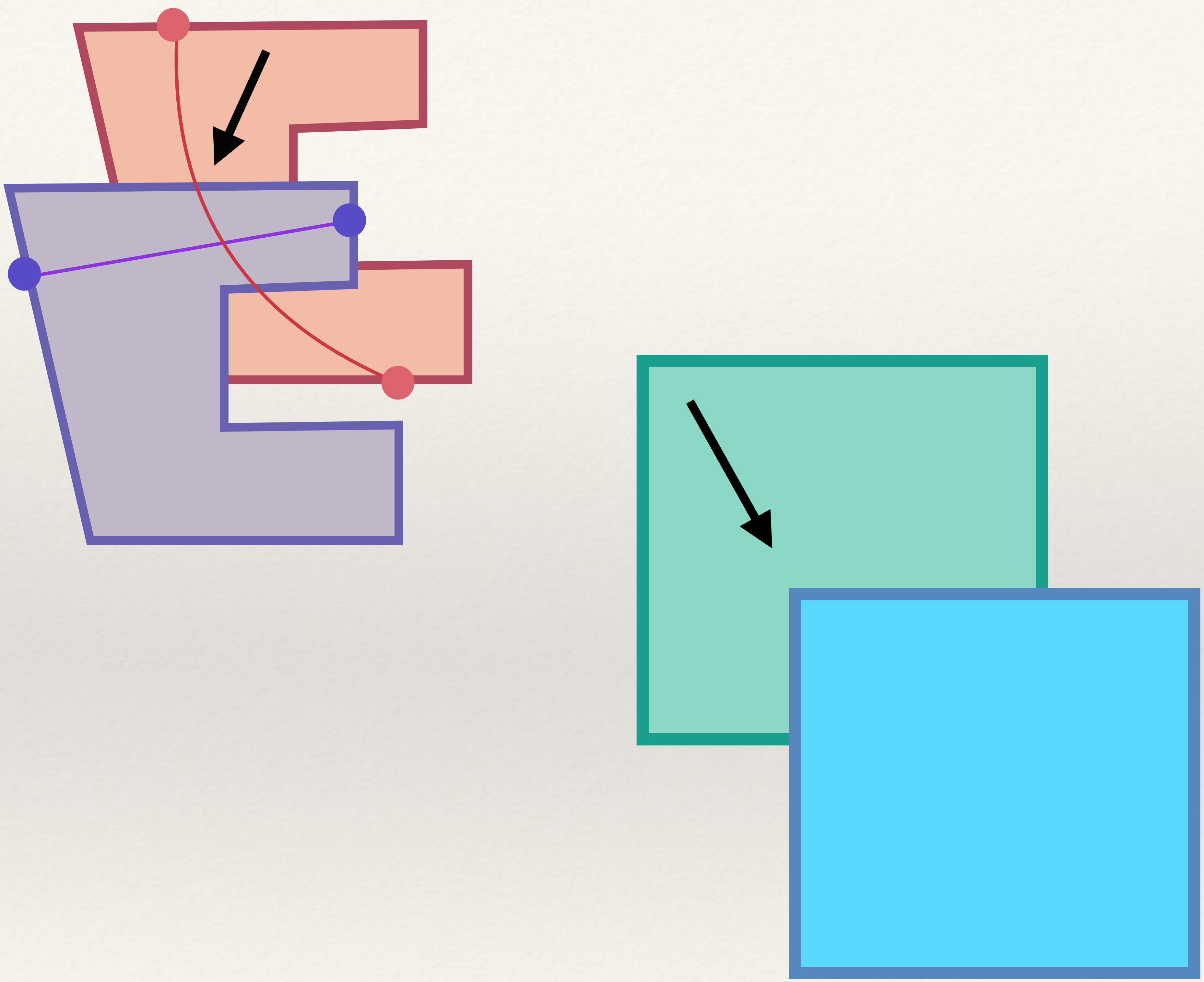
(Non)convexity



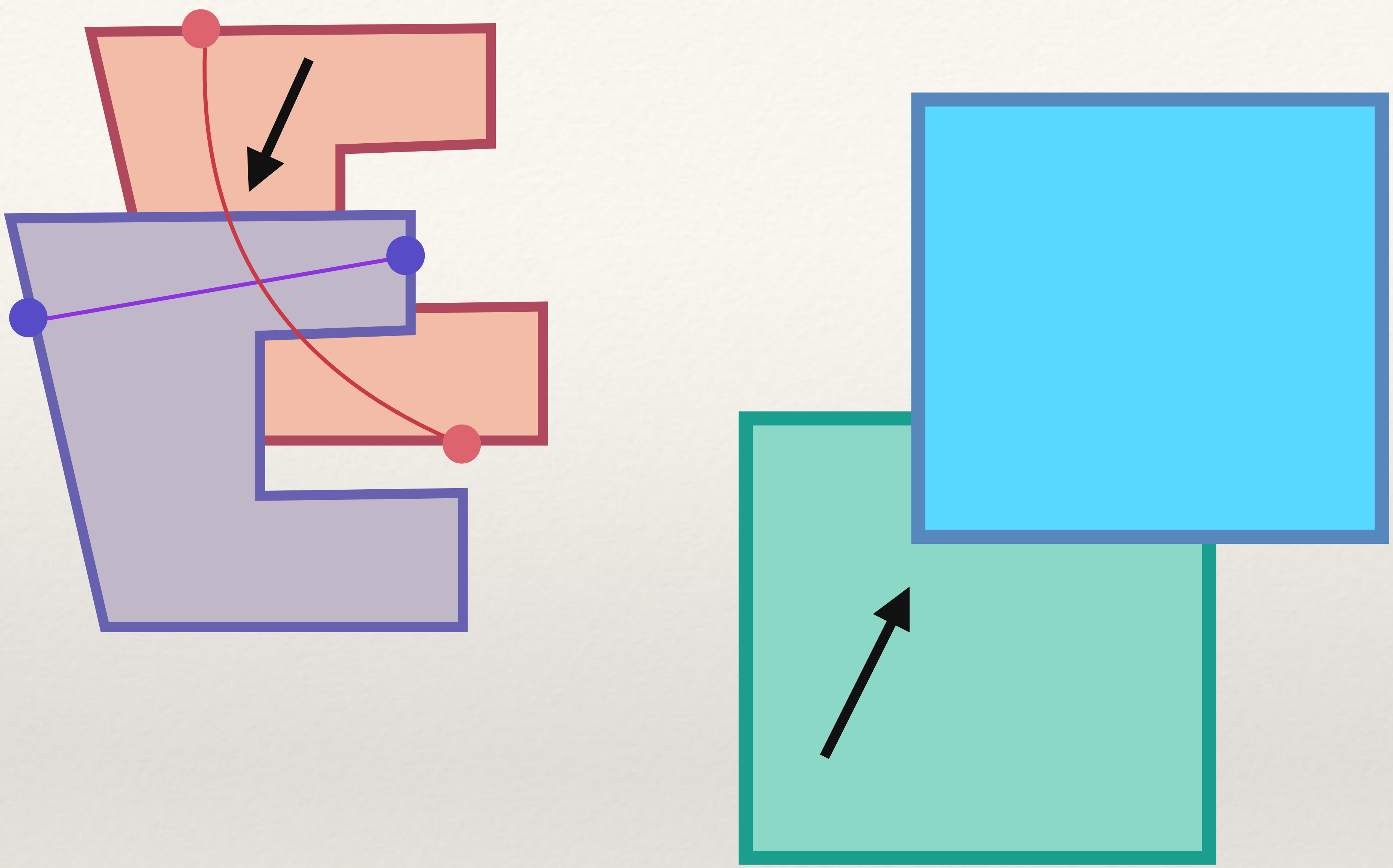
(Non)convexity

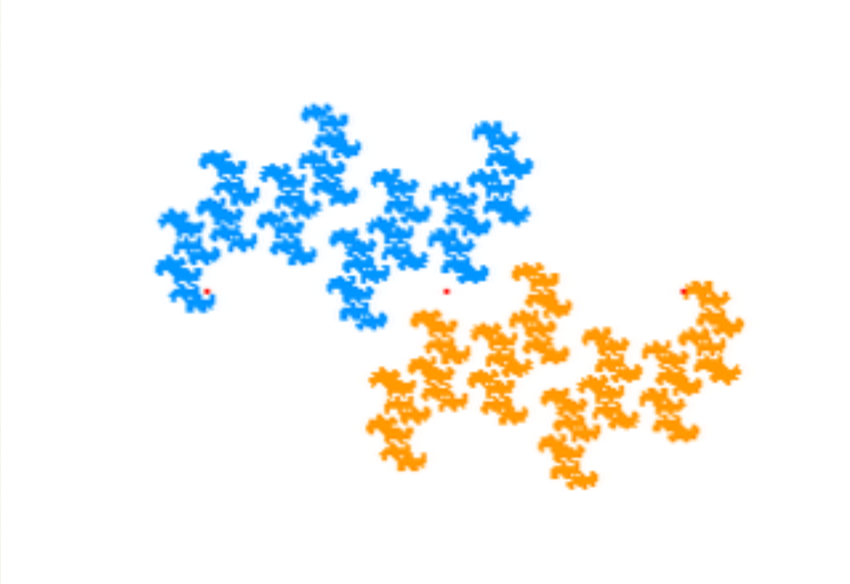


(Non)convexity



(Non)convexity



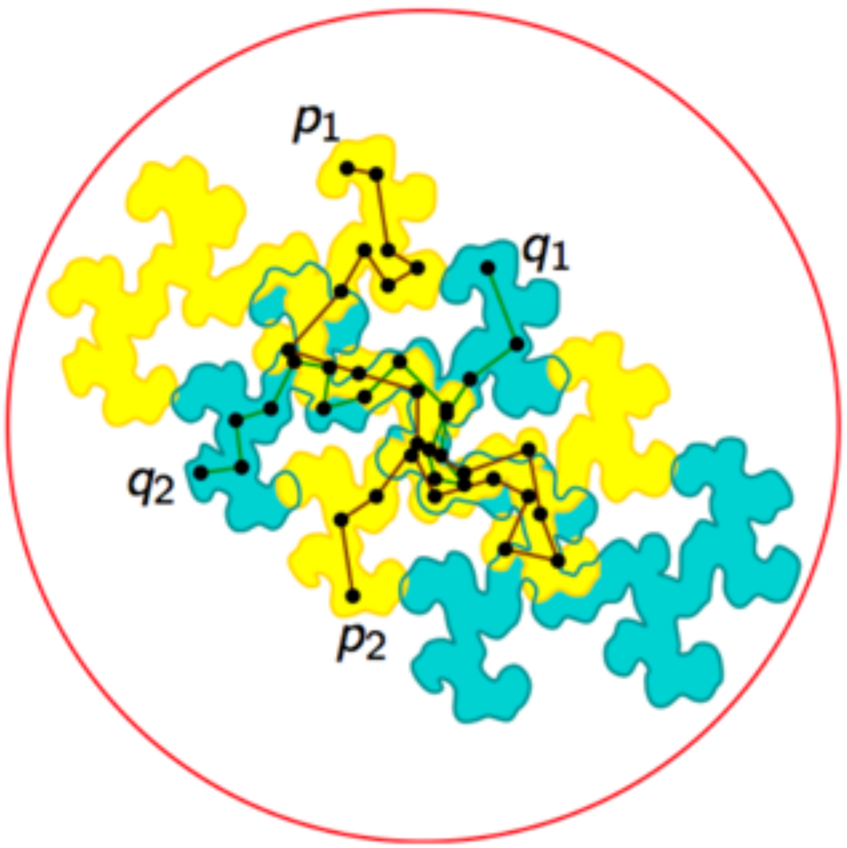


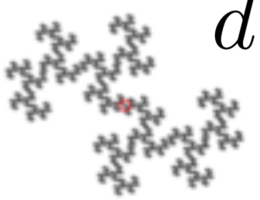
Traps

Lemma. If $d(f_c(\Lambda_c), g_c(\Lambda_c)) \leqslant \delta$, then $N_{\delta/2}(\Lambda_c)$ is path-connected.

Corollary. If $u \in G_n$, then $N_{|c|^n\delta/2}(u(\Lambda_c))$ is path-connected.

Definition. A *trap* is a pair $u, v \in G_n$, $u = f_c u'$ and $v = g_c v'$, so that $N_{|c|^n\delta/2}(u(\Lambda_c))$ and $N_{|c|^n\delta/2}(v(\Lambda_c))$ cross transversely.





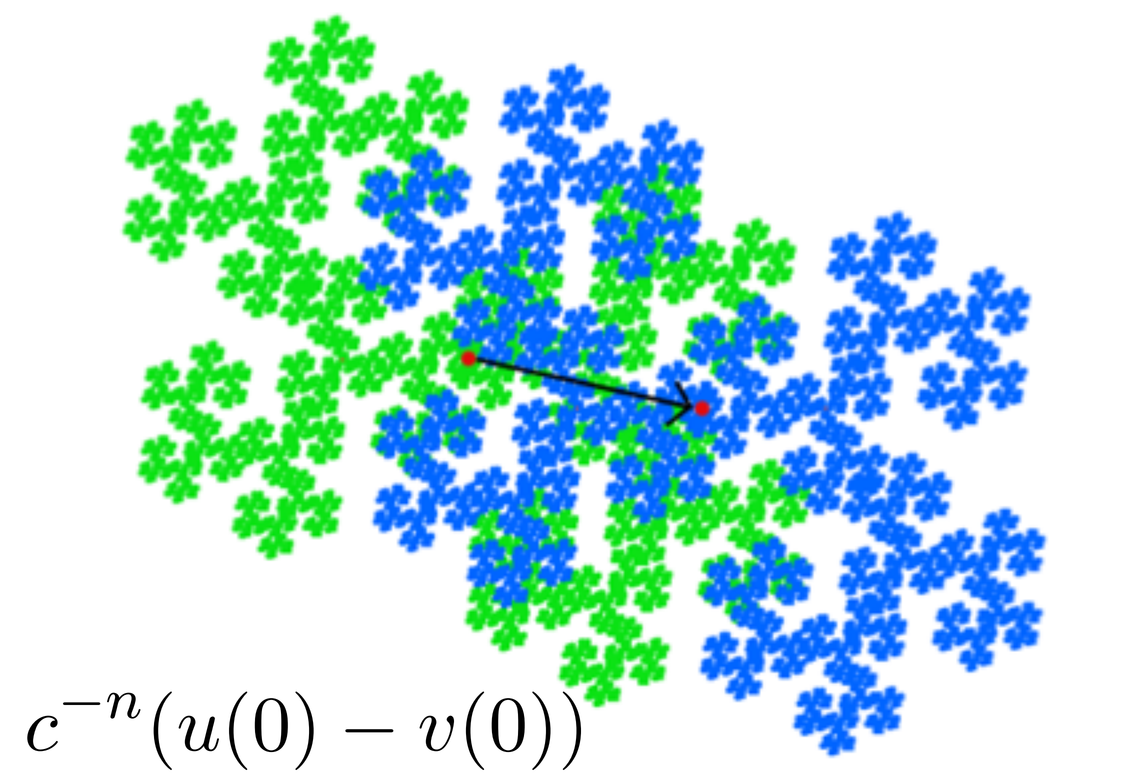
$$d(f_c(\Lambda_c), g_c(\Lambda_c)) \leqslant d(u(\Lambda_c), v(\Lambda_c))$$

$$d(u(\Lambda_c), v(\Lambda_c)) \leqslant |c|^n d(f_c(\Lambda_c), g_c(\Lambda_c))$$

$$\implies d(f_c(\Lambda_c), g_c(\Lambda_c)) = 0$$

$$\implies c \in M$$

In fact, $c \in \text{int}(M)$.

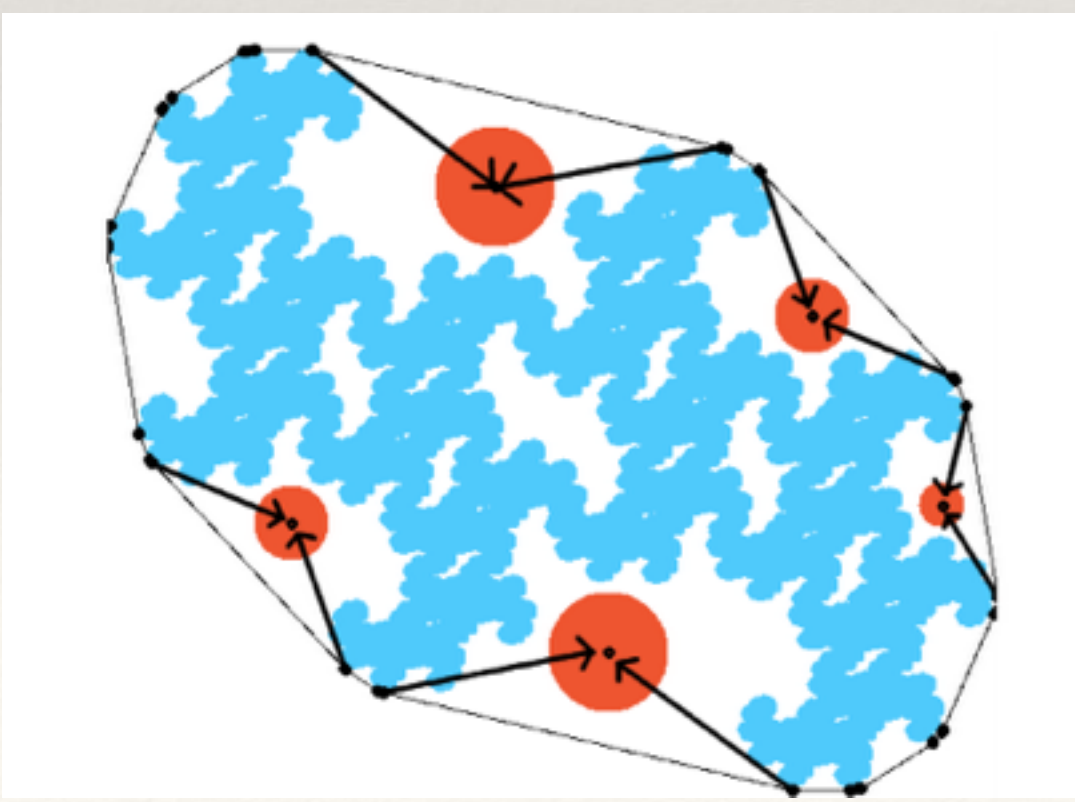


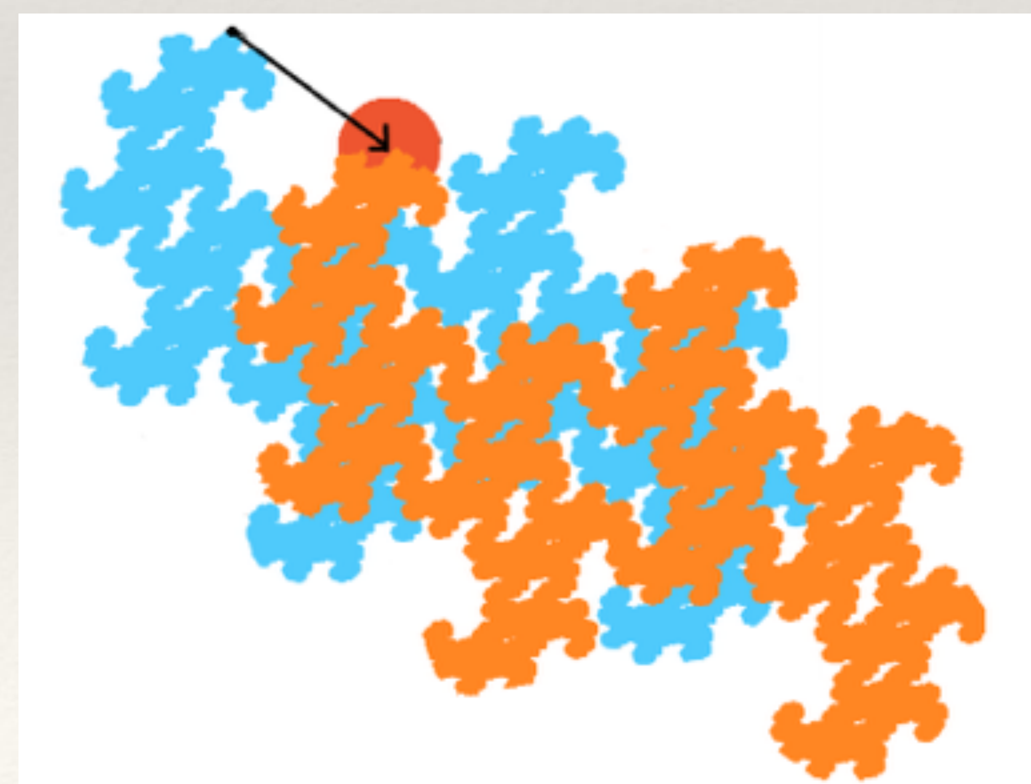
Finding traps

Find balls $B_1, \ldots, B_k$ so that if there *were* words $u, v, \in G_n$ with

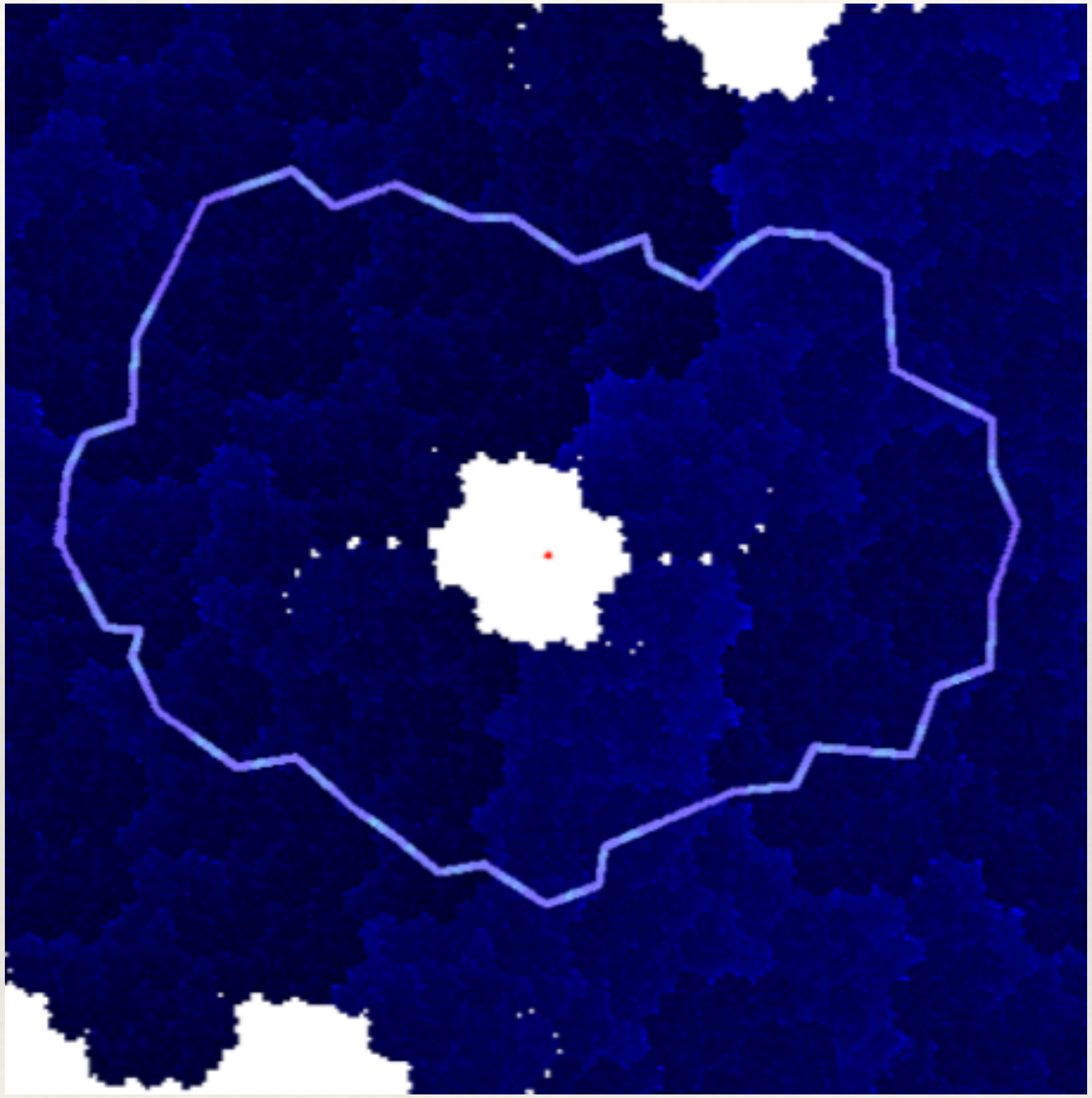
$$c^{-n}(u(0) - v(0)) \in B_i \text{ for some } i$$

then there would be a trap for Λ_c.

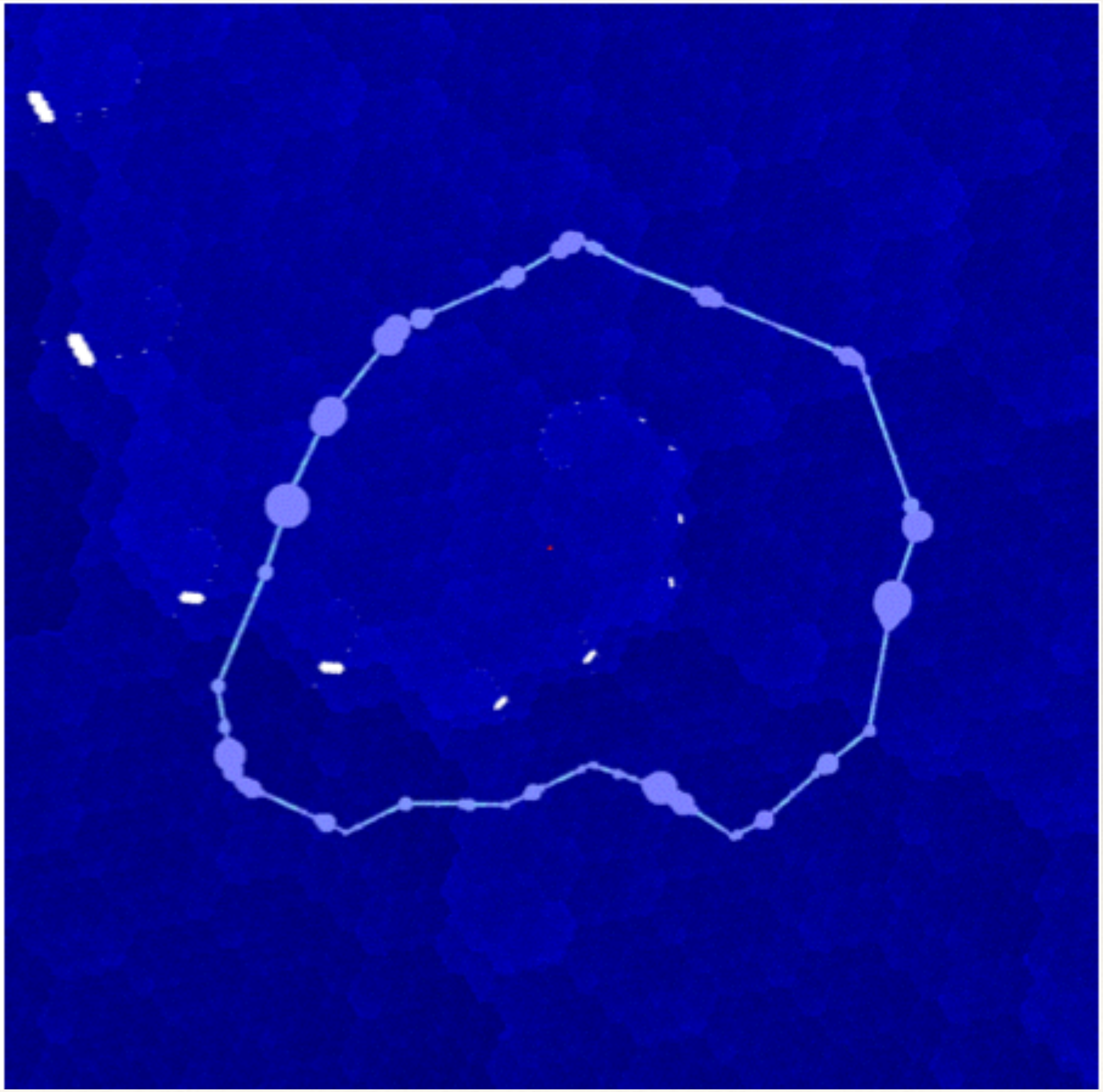




Finding exotic components in the complement

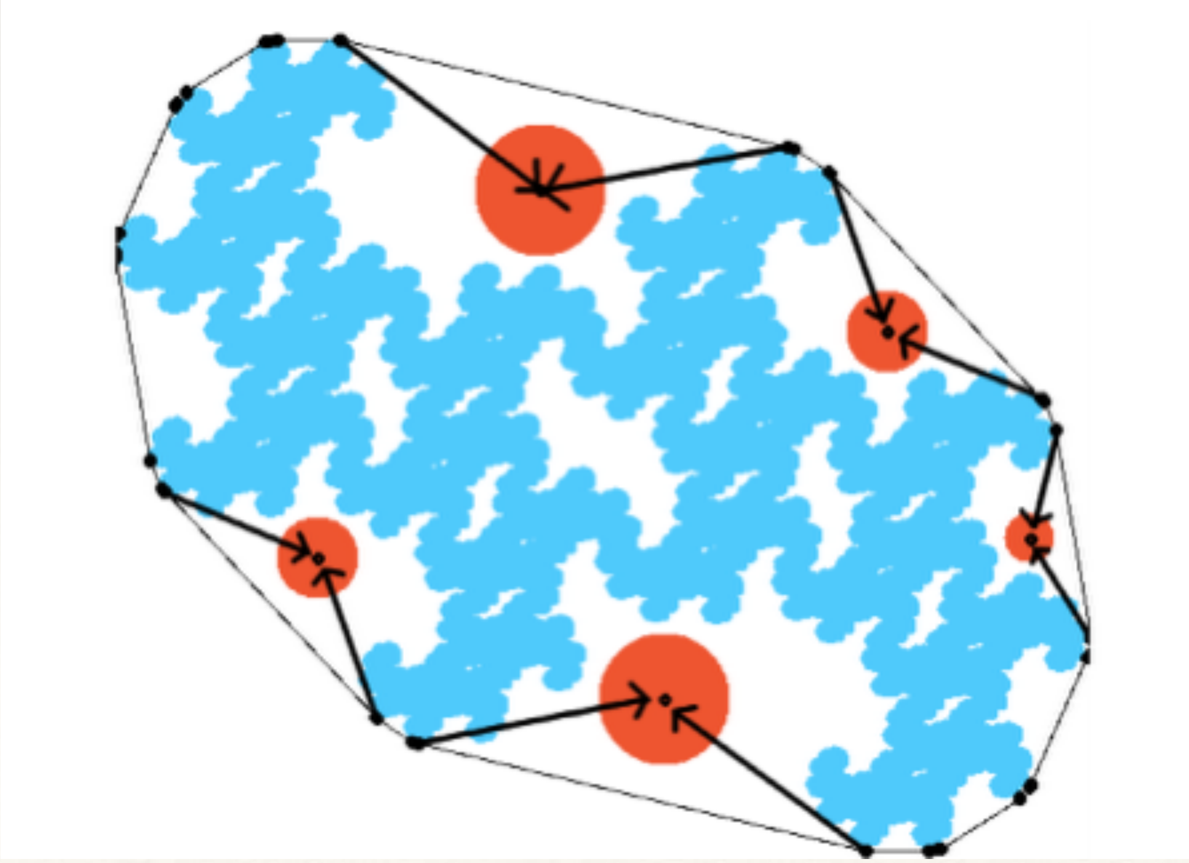


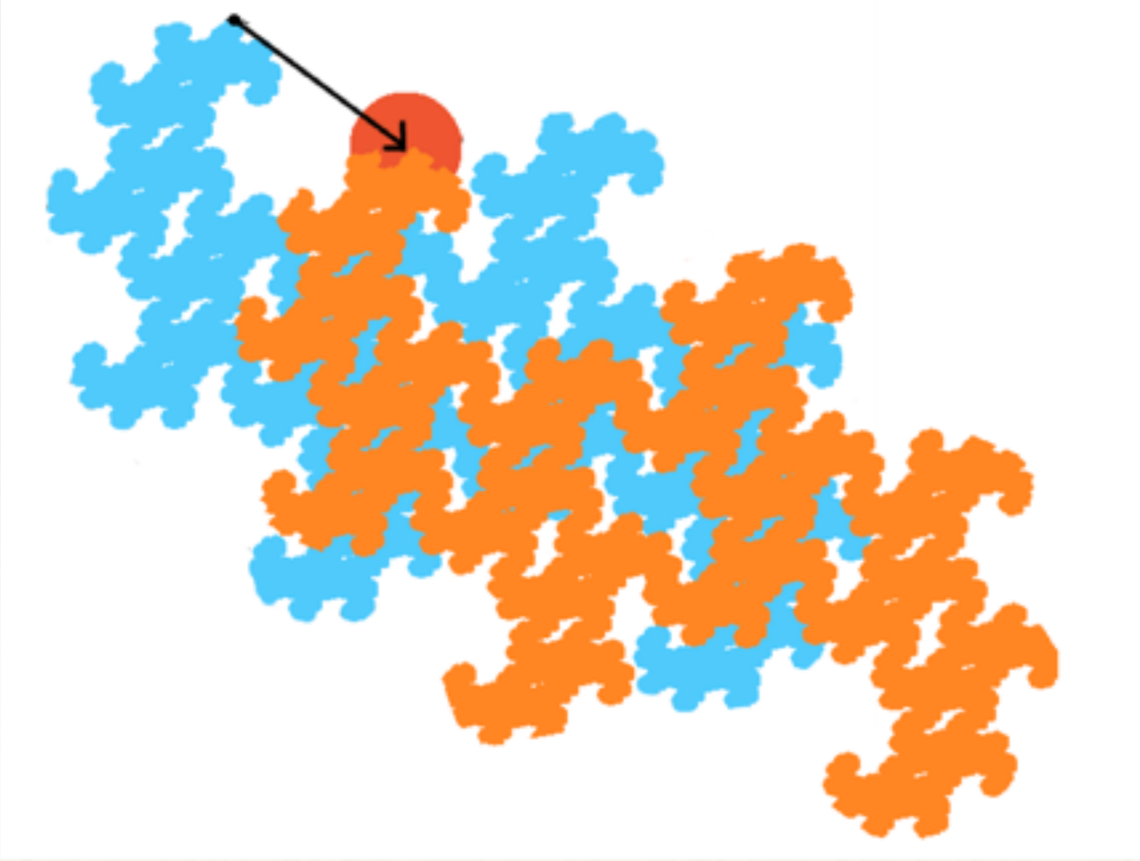
Infinitely many exotic components



Interior is dense away from real axis

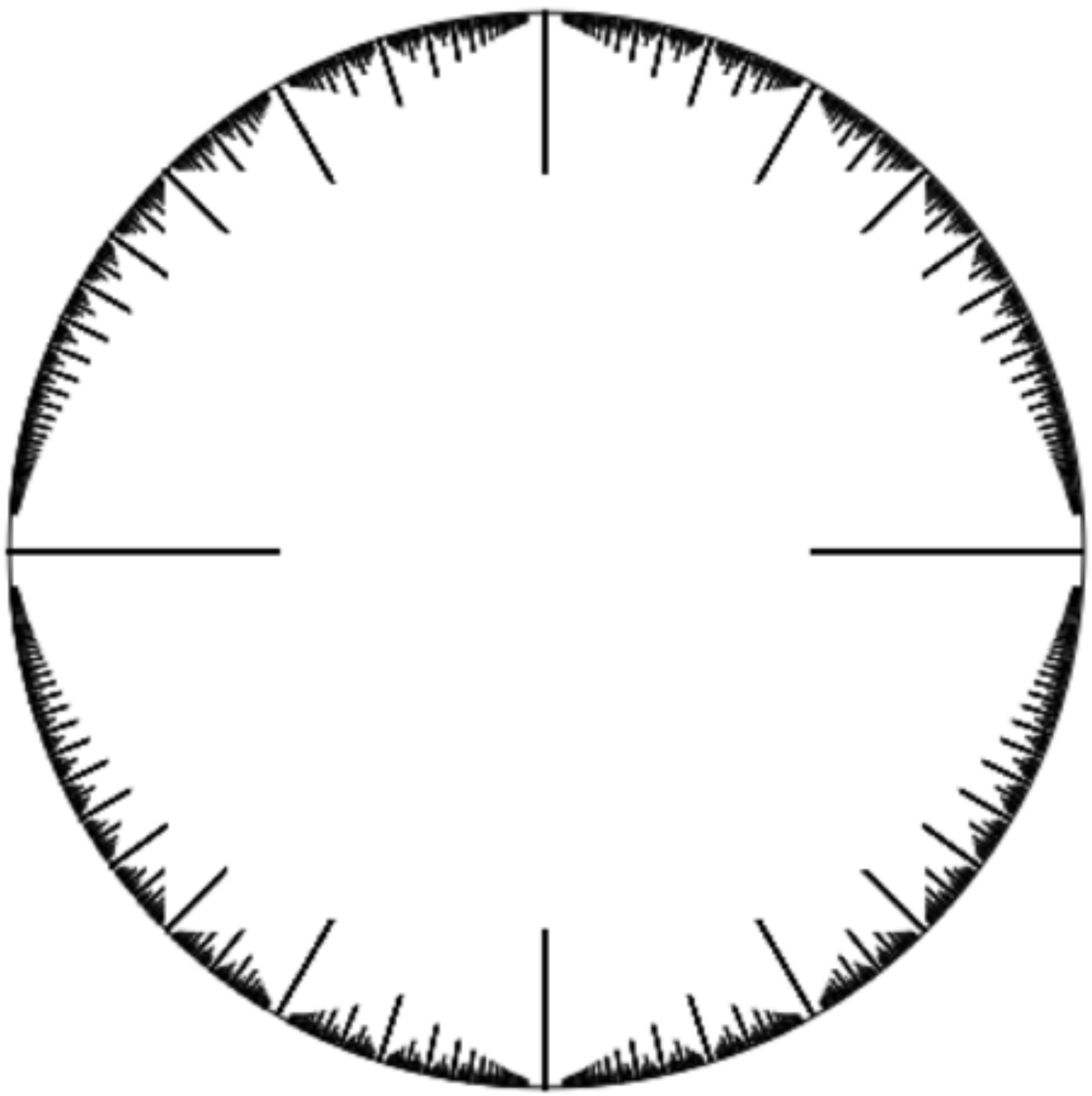
If Λ_c is not convex, then there are trap-like vectors.
Since $c \in M$, $f_c(\Lambda_c) \cap g_c(\Lambda_c) \neq \emptyset$, there are $u, v \in G_n$ with $d(u(0), v(0))$ small, and we can perturb c so that $c^{-n}(u(0) - v(0))$ will be trap-like.

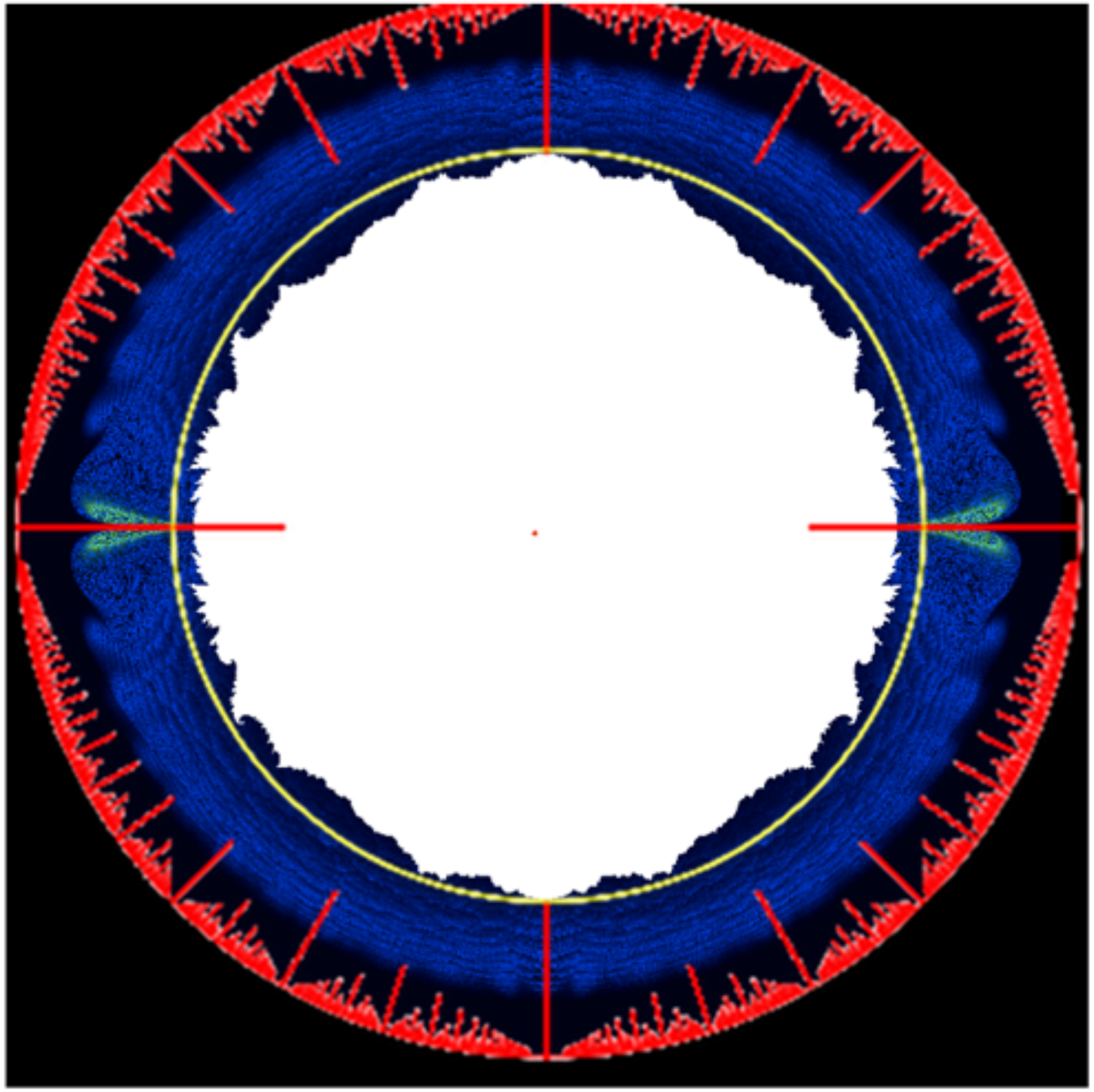




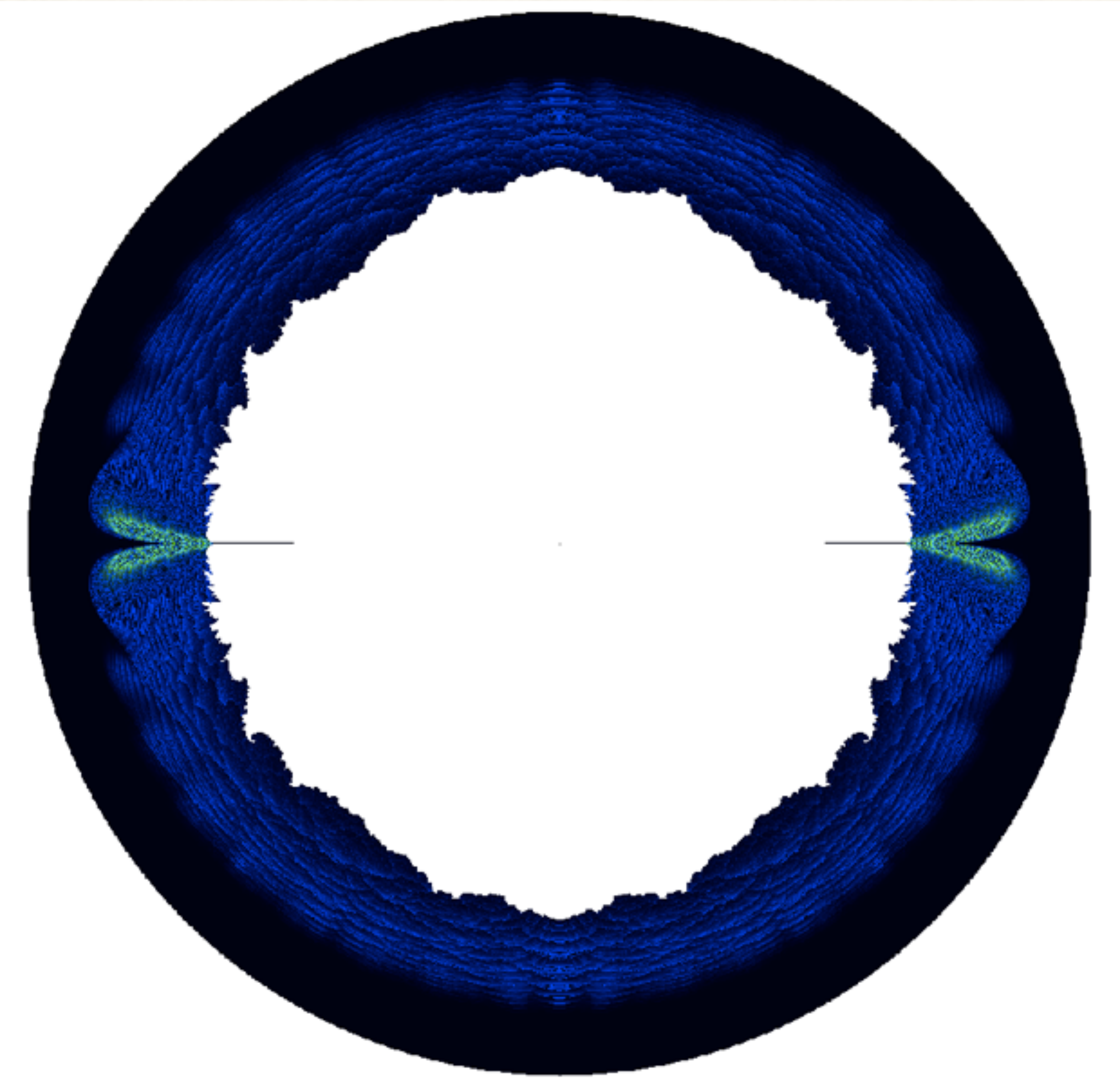
Convex limit sets

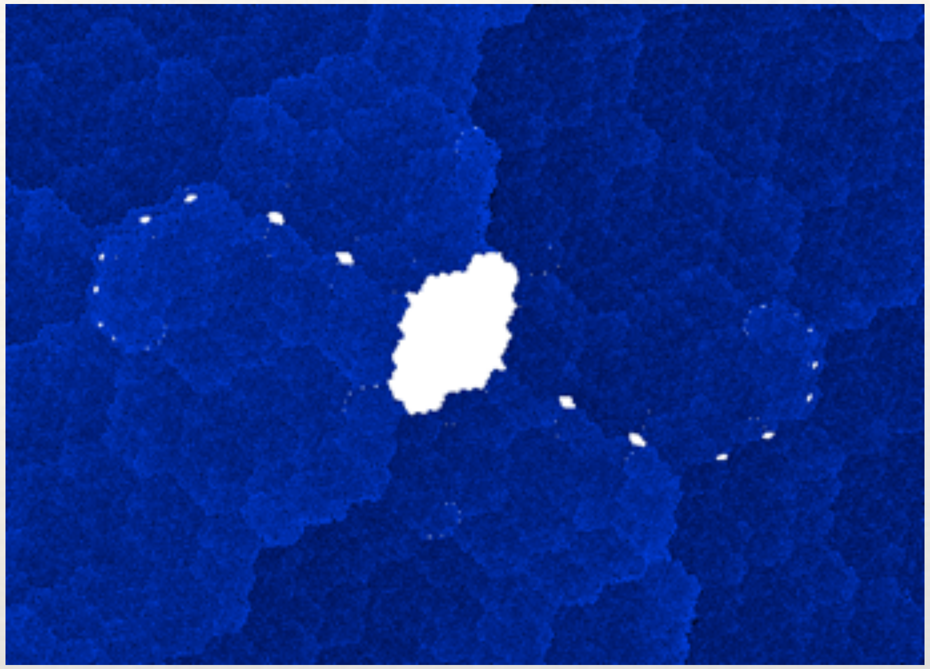
Lemma. Λ_c is convex iff $c = re^{\pi i p/q}$, where $r \geqslant 2^{-1/q}$.

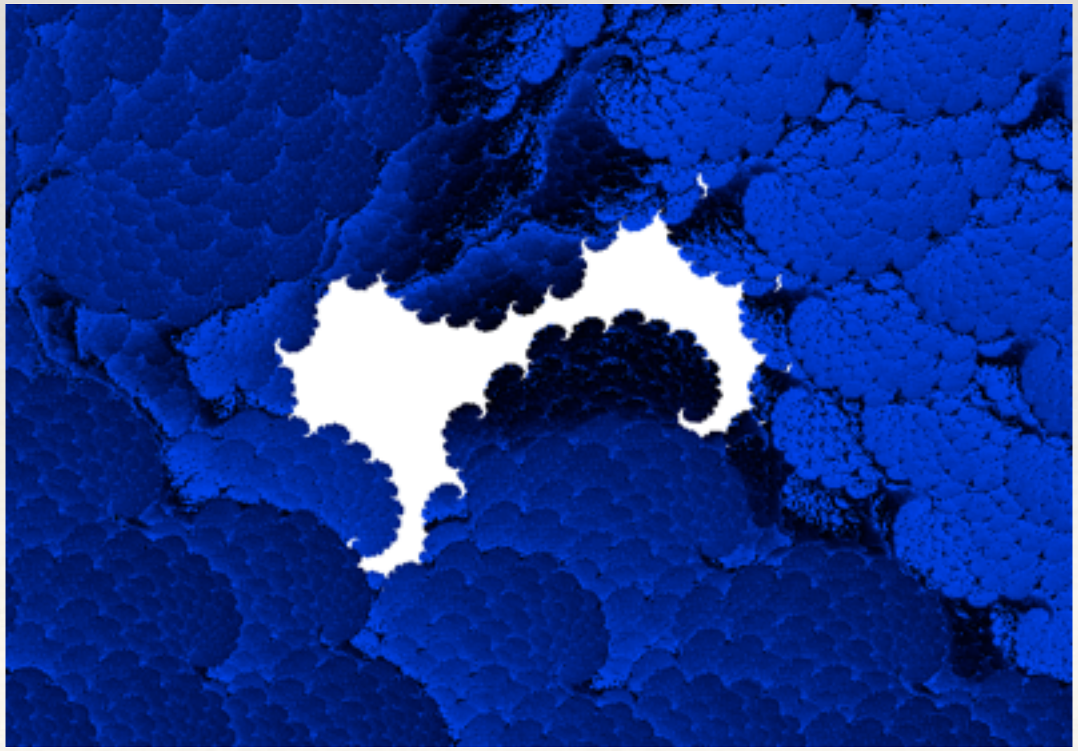


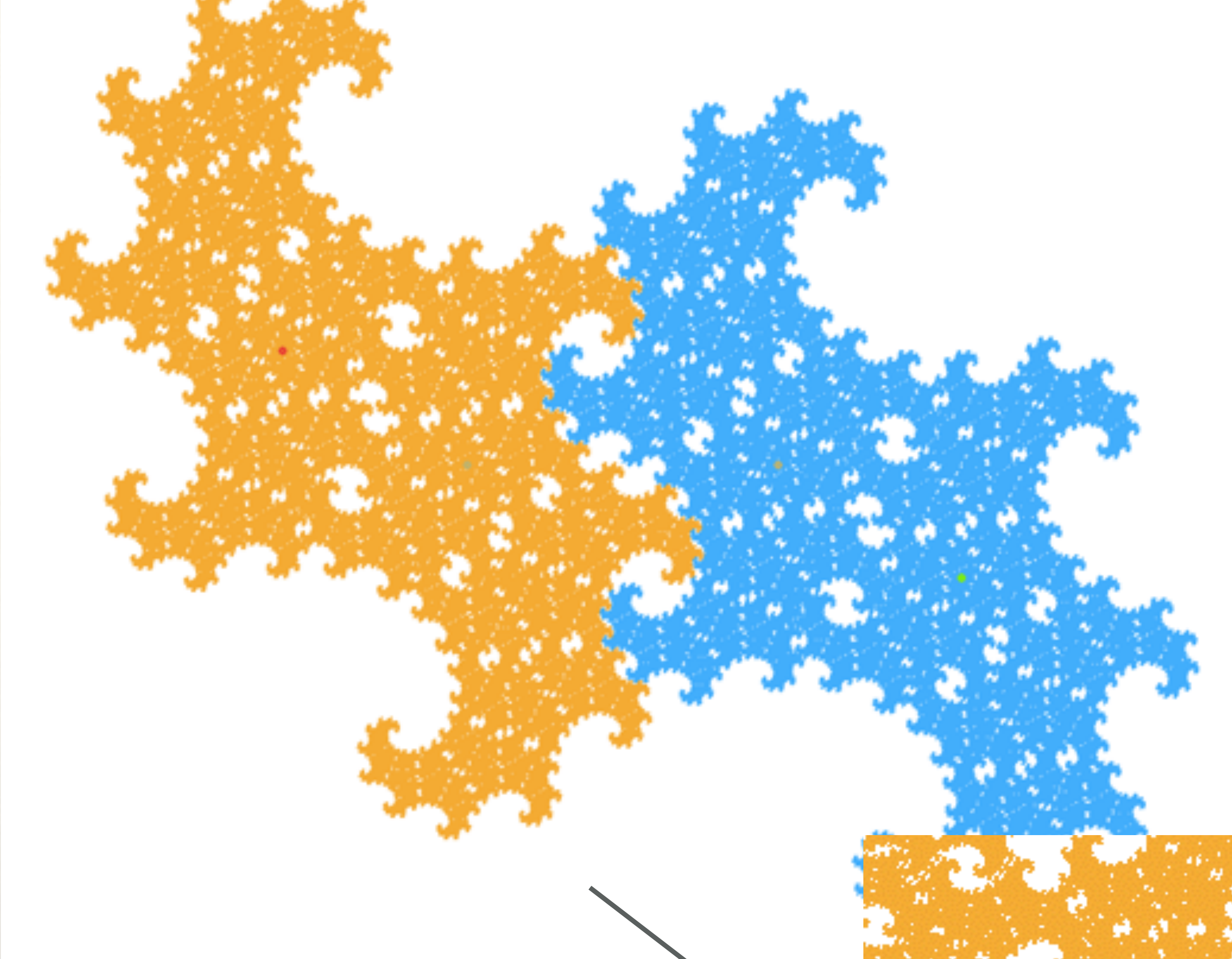


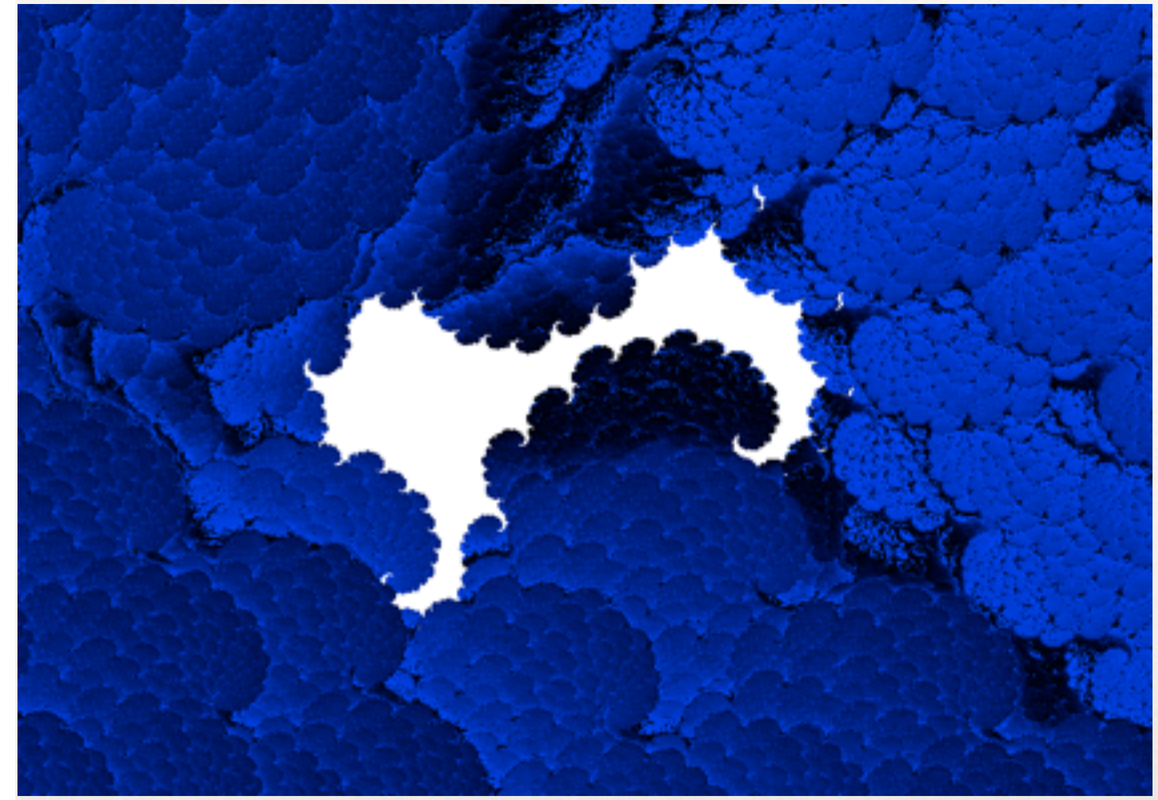
Goal: Classify connected components

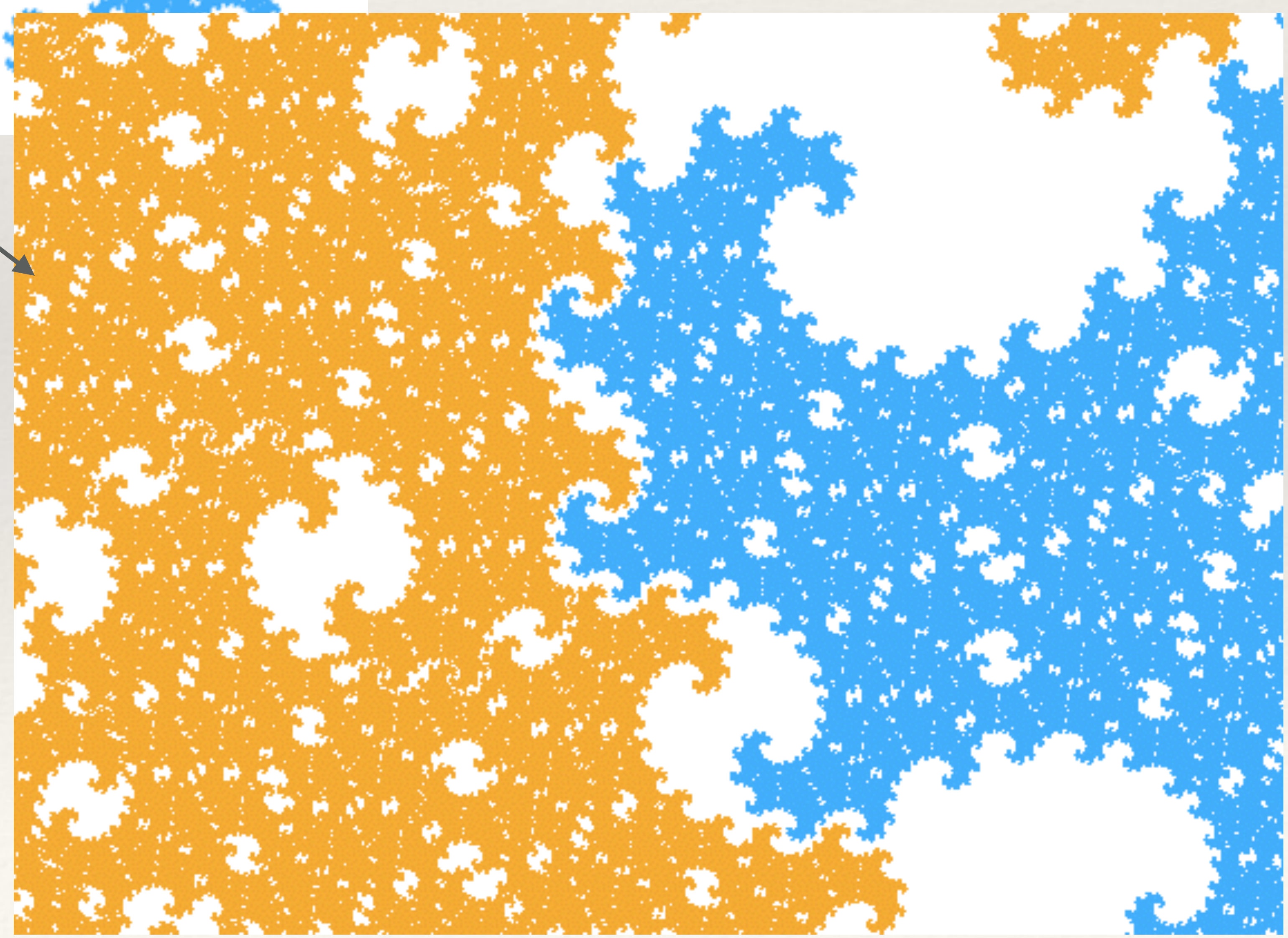




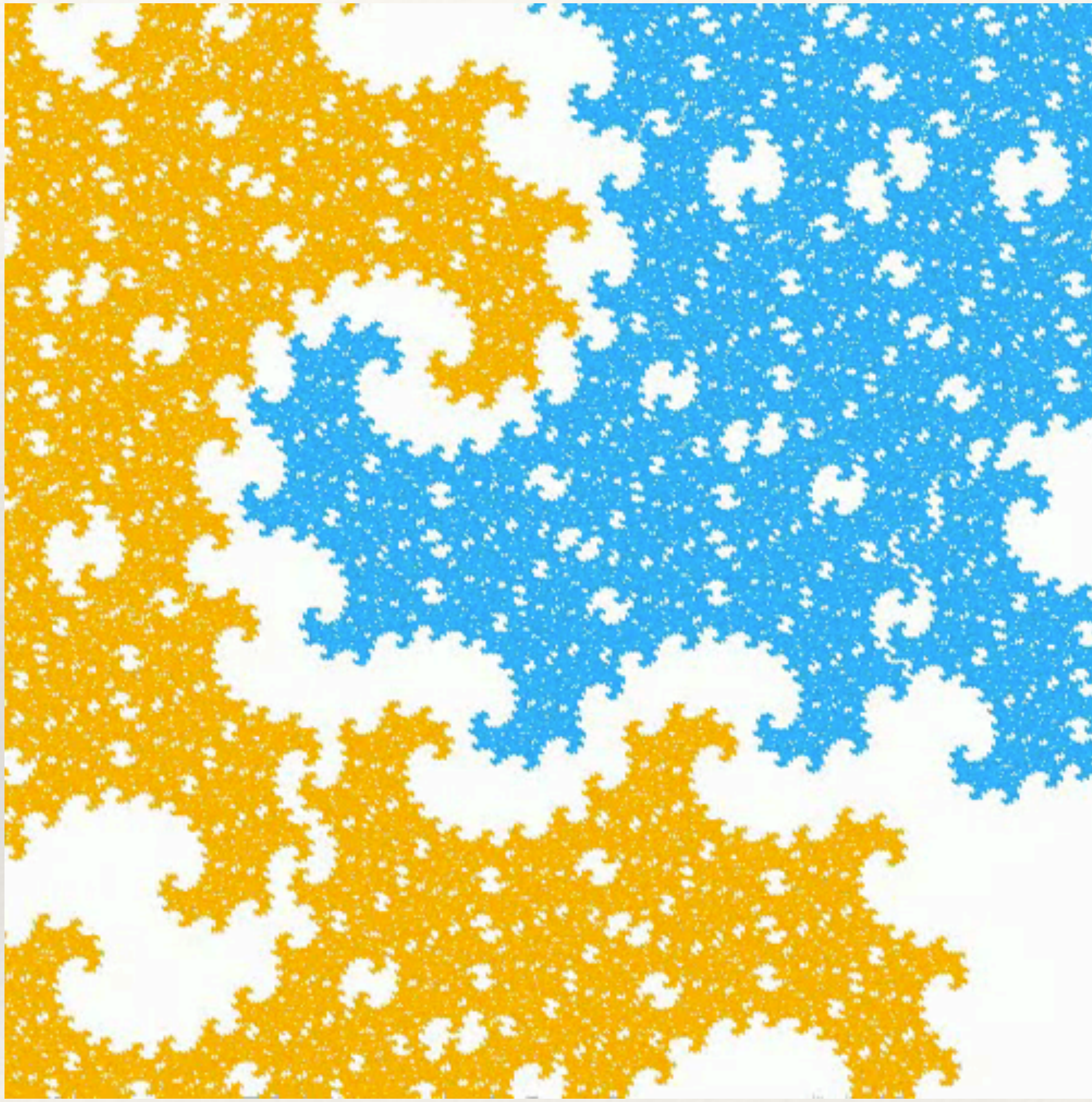
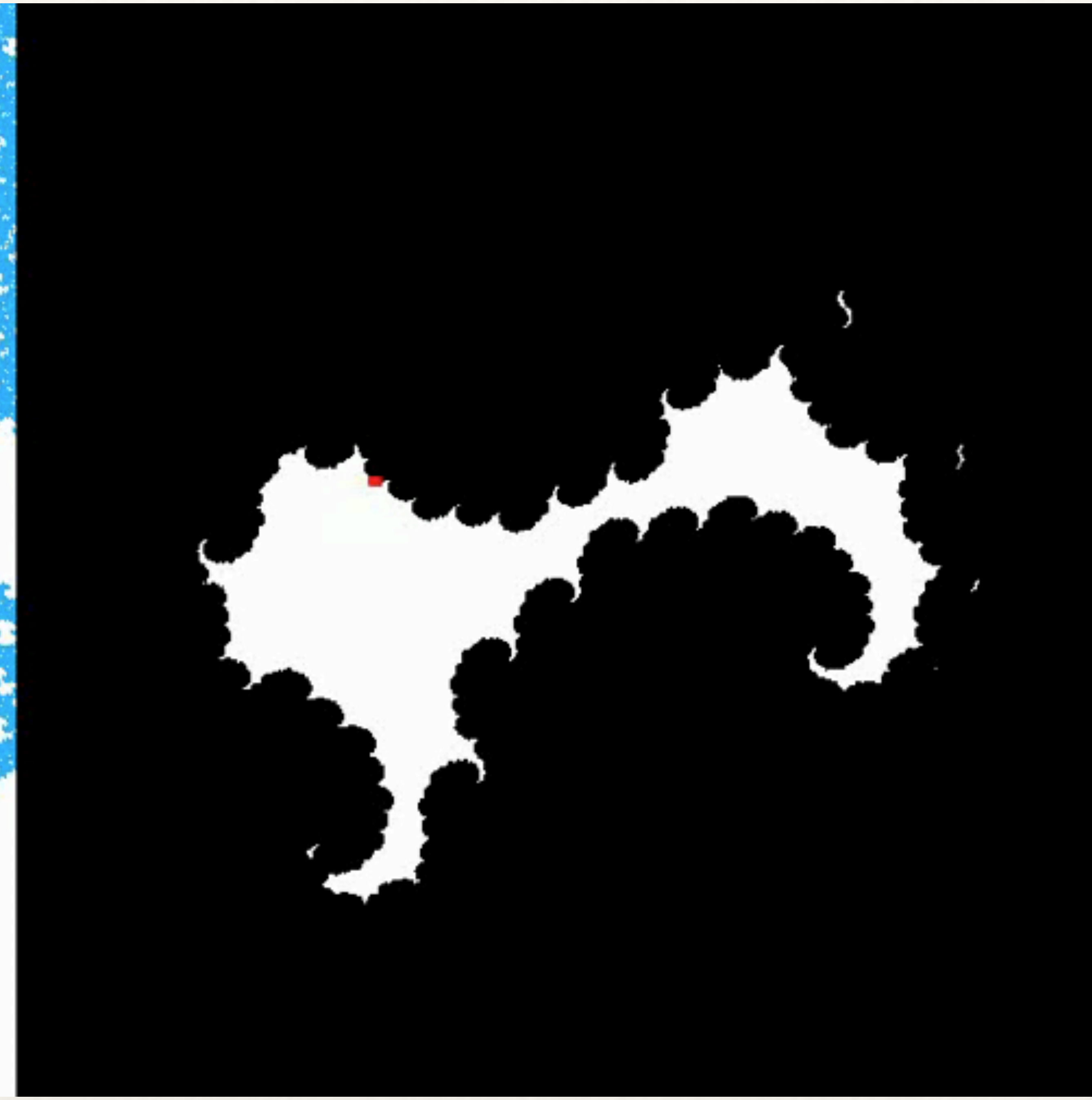








locking
gears

Polynomials and power series

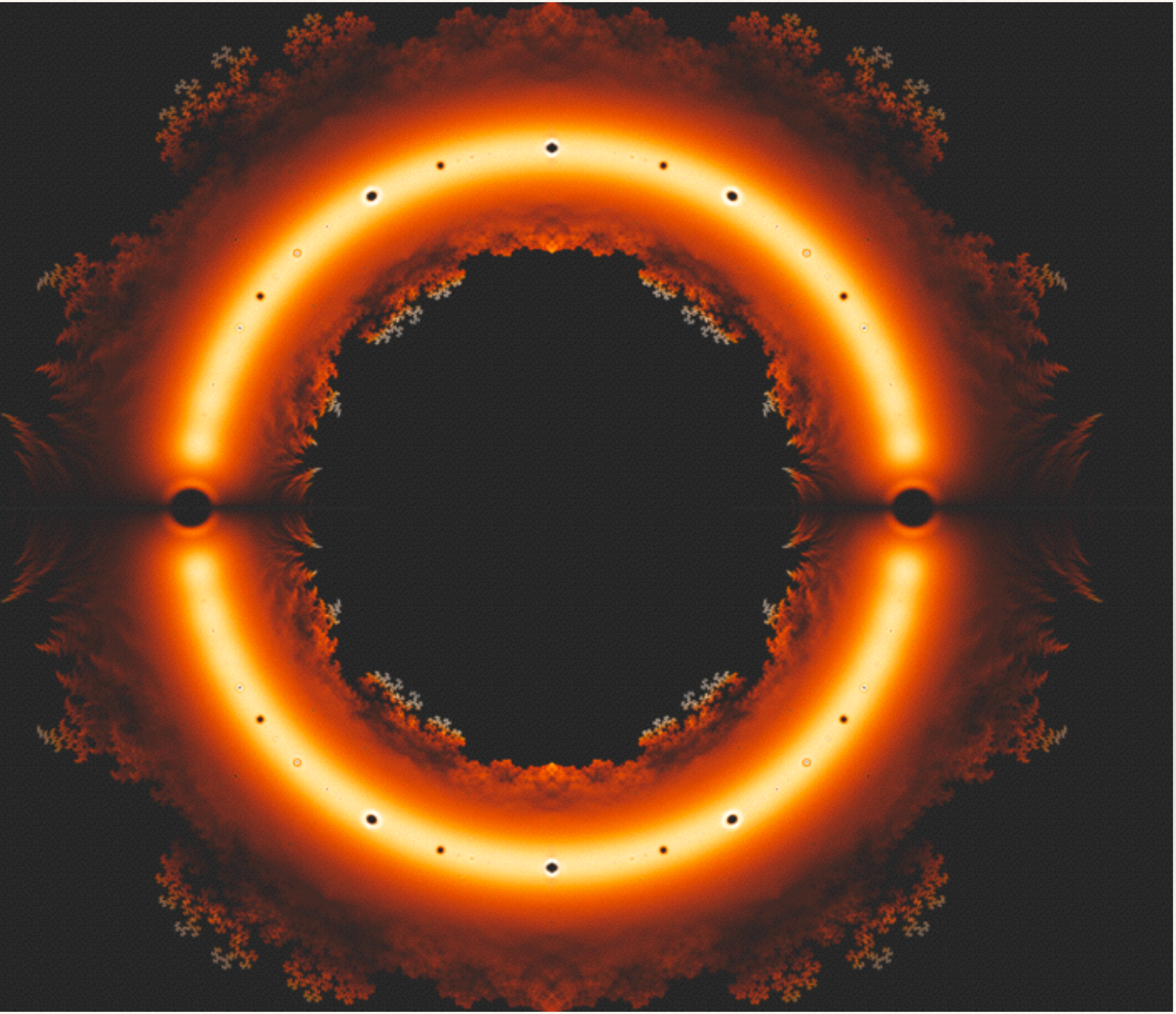
$$w = l_0 l_1 \cdots l_{n-1}, \quad \text{where } l_i \in \{f_c, g_c\}$$

$$w : z \mapsto a_0 + a_1 c + \cdots + a_{n-1} c^{n-1} + c^n z, \quad a_i \in \{1, -1\}$$

$$\Lambda_c = \{\text{values at } c \text{ of power series with coefficients in } \{-1, 1\}\}$$

$$\begin{aligned} M &= \{c \in \mathbb{D}^* \mid f_c(\Lambda_c) \cap g_c(\Lambda_c) \neq \emptyset\} \\ &= \{\text{zeros of power series with coefficients in } \{-1, 0, 1\}\} \\ &= \text{closure of } \{\text{roots of polynomials with coefficients in } \{-1, 0, 1\}\} \end{aligned}$$

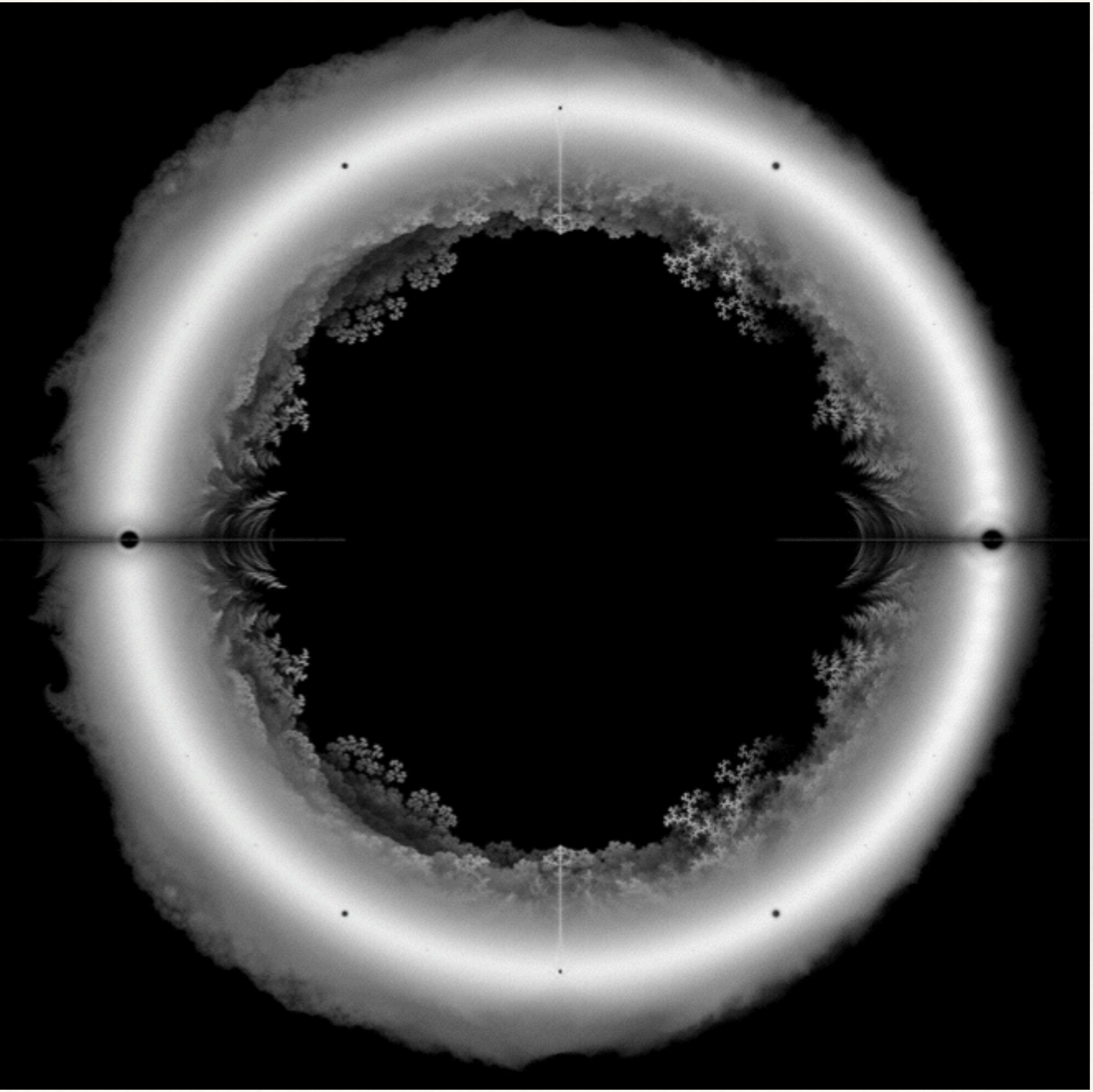
$$\begin{aligned} M' &:= \{c \in \mathbb{D}^* \mid \Lambda_c \ni 0\} \\ &= \{\text{roots of power series with coefficients in } \{-1, 1\}\} \\ &= \text{closure of } \{\text{roots of polynomials with coefficients in } \{-1, 1\}\} \end{aligned}$$



Sam Derbyshire

roots

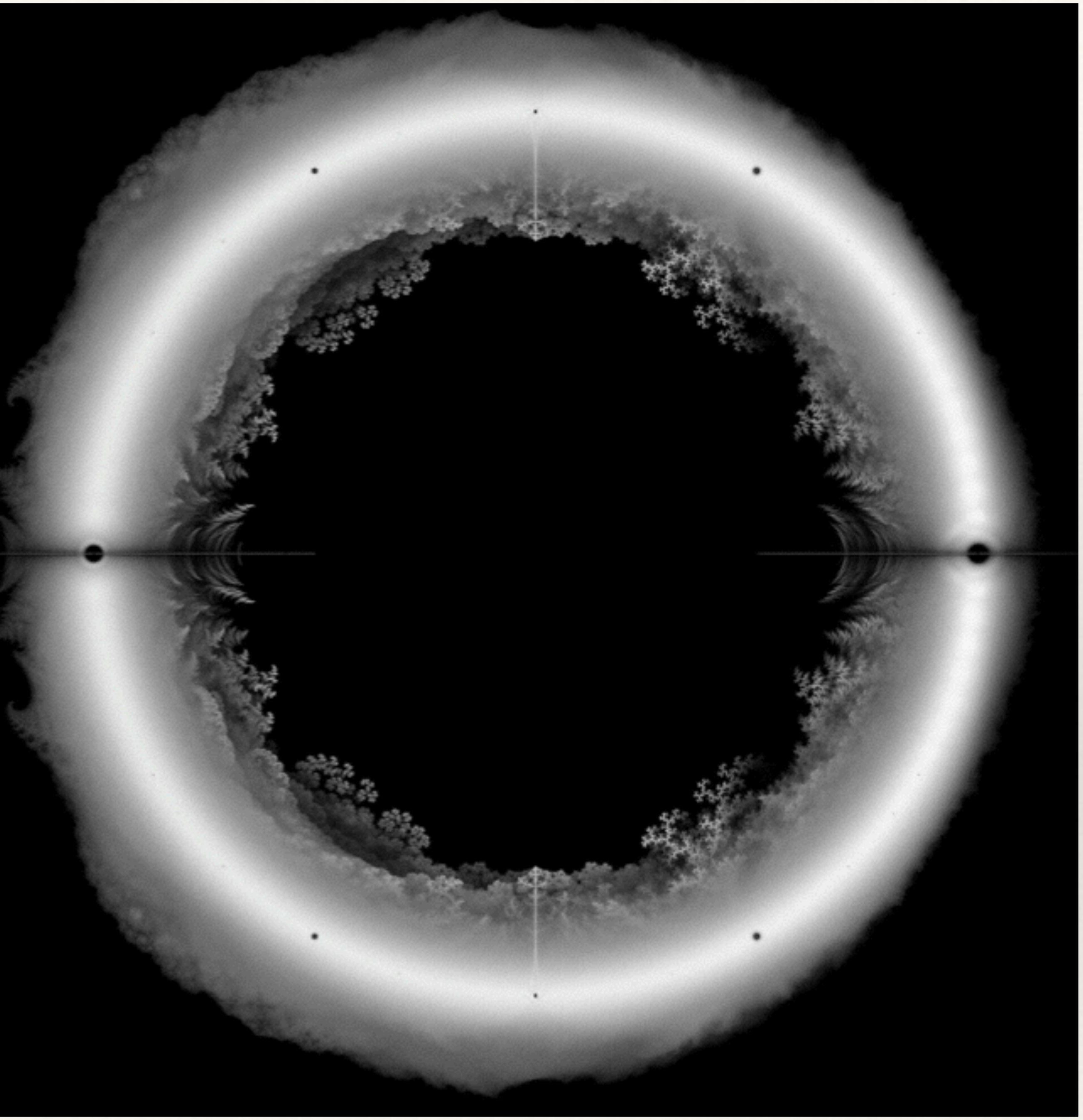
$0 < |c| < 1$



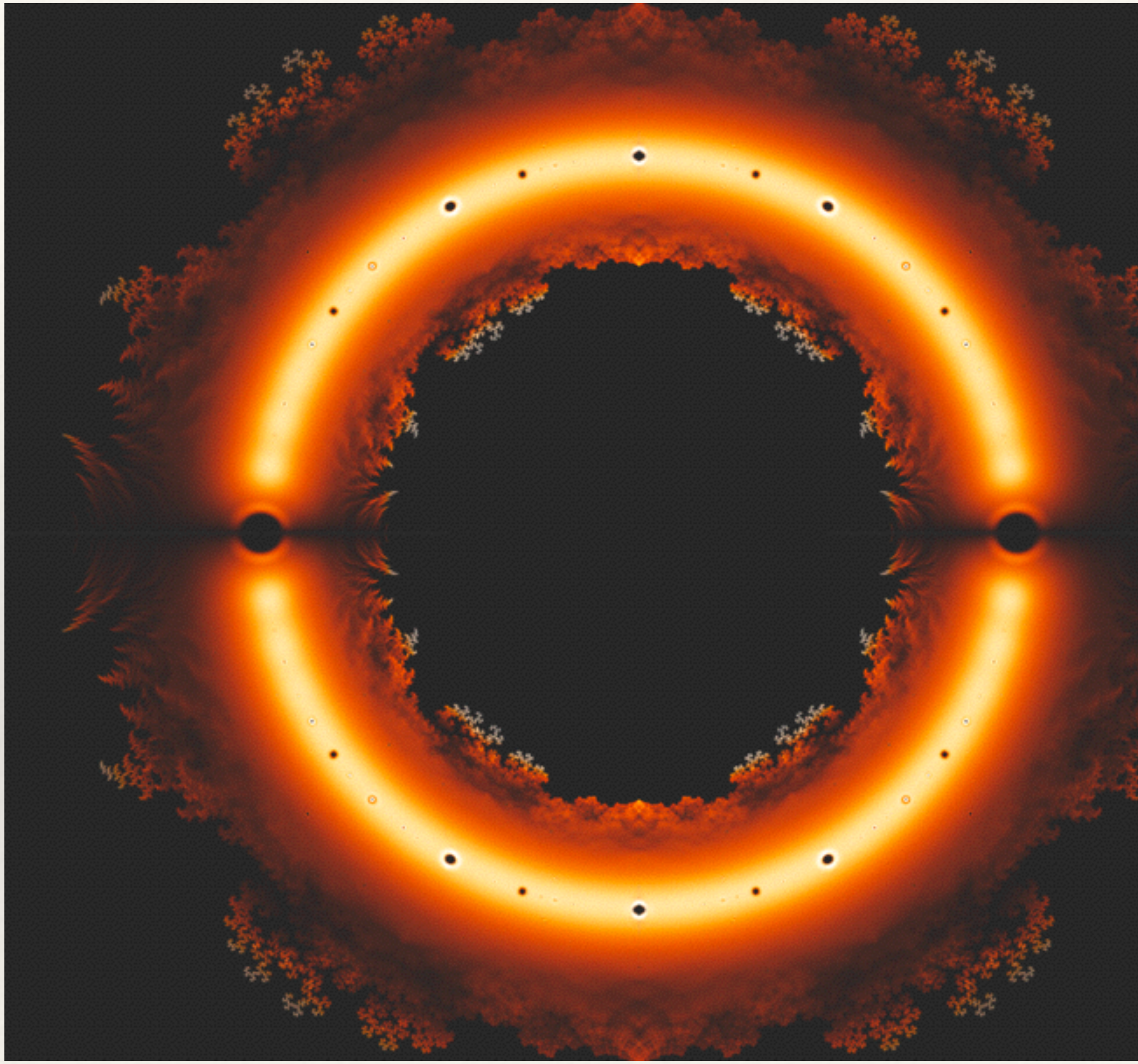
Bill Thurston

entropy

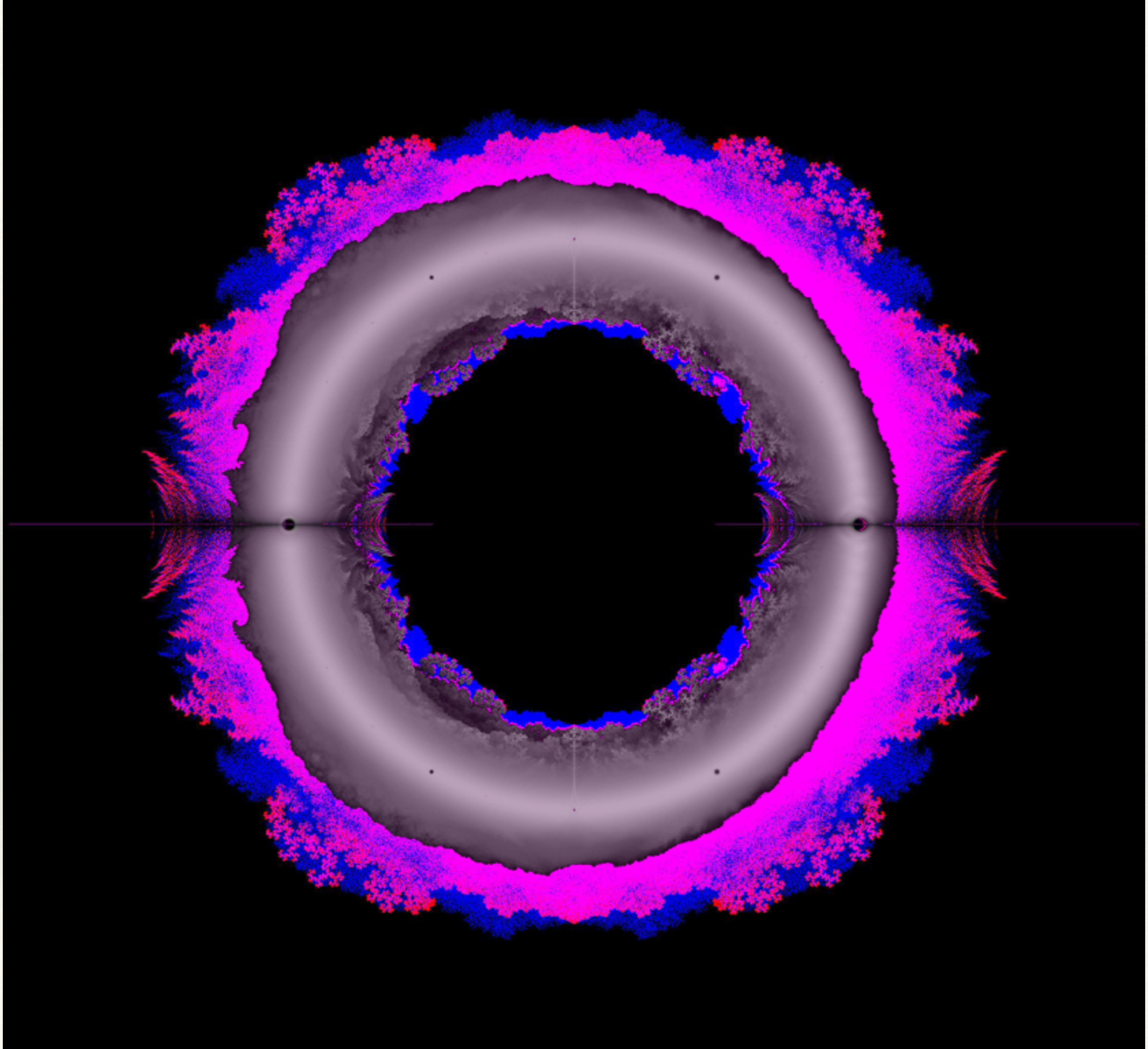
Thurston:
entropy



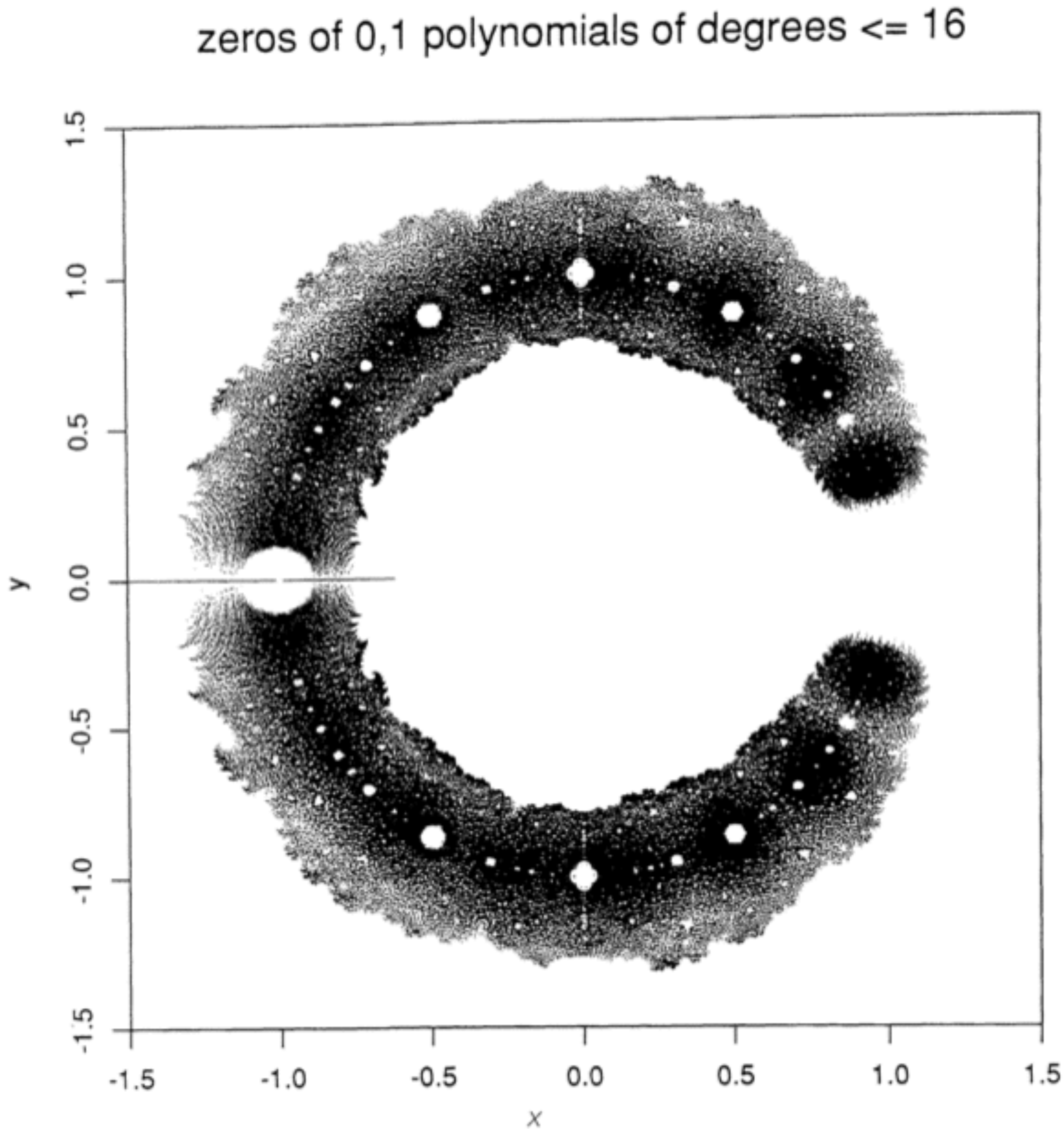
Derbyshire:
roots



Sets have the same closure in $\mathbb{D}$ (Tiozzo).



Odlyzko, Poonen



Zeros of polynomials with 0,1 coefficients

Quadratic polynomials

$$p_c : \mathbb{C} \to \mathbb{C}, \qquad p_c : z \mapsto z^2 + c, \qquad c \in \mathbb{C}$$

distinguished point: the unique critical point $z_0 = 0$

Gleason polynomials: $G_n(c) := p_c^n(0) \in \mathbb{Z}[c]$

$$G_1(c) = c$$

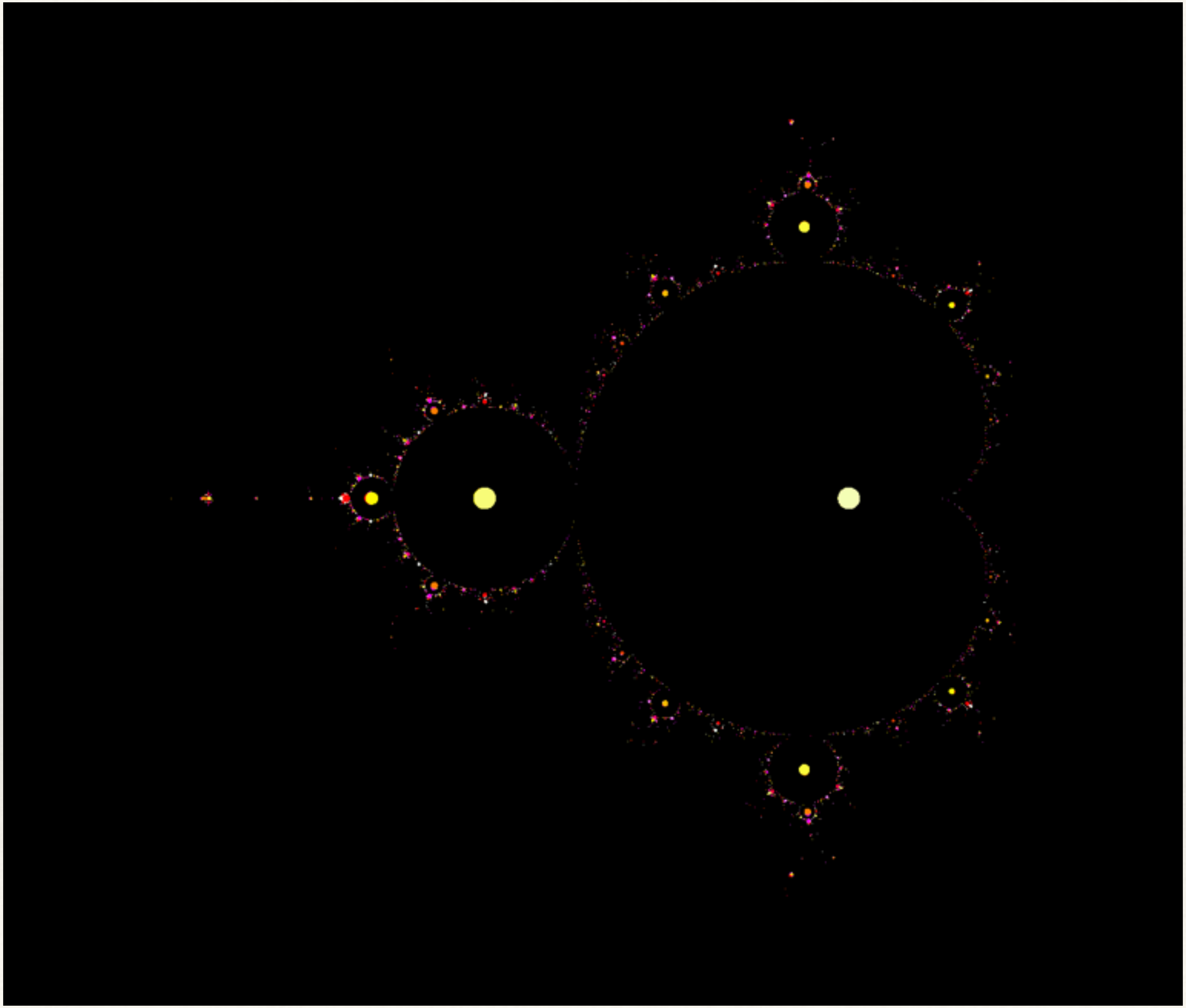
$$G_2(c) = c^2 + c$$

$$G_3(c) = (c^2 + c)^2 + c = c^4 + 2c^3 + c^2 + c$$

$$G_4(c) = c^8 + 4c^7 + 6c^6 + 6c^5 + 5c^4 + 2c^3 + c^2 + c$$

$$\vdots$$

Galois conjugates in parameter space



Thank you for your attention!

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