Pseudo-Automorphims with an invariant curve

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Problem

Let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ *be a birational map.*
When is there a blowup $\pi : X \to \mathbb{P}^k$ *such that*

$$f_X := \pi^{-1} \circ f \circ \pi \quad \textit{is a automorphism ?}$$

Dimension 2

Theorem (Noether)

Every birational map of $\mathbb{P}^2$ is a composition of J's and linear automorphisms on $\mathbb{P}^2$ where J is the cremona involution

$$J : [x_0 : x_1 : x_2] \mapsto [1/x_0 : 1/x_1 : 1/x_2].$$

Question

What are linear automorphisms $S_i \in PGL(3, \mathbb{C})$ $i = 1, \ldots, n$ such that $f := S_1 \circ J \circ S_2 \circ J \circ \cdots \circ J \circ S_n$ is equivalent to an automorphism?

The Cremona involution $J : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

$$J : [x_0 : x_1 : x_2] \mapsto [1/x_0 : 1/x_1 : 1/x_2].$$

1. The indeterminacy locus

$$\mathrm{Ind}(J) = \cup_i e_i \quad \text{where } e_i = \{x_j = 0, j \neq i\}$$

2. The exceptional locus

$$\mathrm{Exc}(J) = \cup_i \Sigma_i \quad \text{where } \Sigma_i = \{x_i = 0\}$$

3. Let X be a blowup of $\mathbb{P}^2$ along $\{e_0, e_1, e_2\}$.
   The induced map J_X is an automorphism.

$f = S \circ J \circ T^{-1}$

1. $\mathrm{Ind}(f) = \{T(e_i), i = 0, 1, 2\}$
2. $\mathrm{Exc}(f) = \{T(\Sigma_i), i = 0, 1, 2\}$
3. Suppose there are positive integers n_0, n_1, n_2 and a permutation σ on $\{0, 1, 2\}$ such that

$$f : T(\Sigma_i) \mapsto S(e_i) \mapsto * \mapsto \cdots \mapsto * \mapsto f^{n_i} T(\Sigma_i) = e_{\sigma(j)}$$

$$f^k T(\Sigma_i) \notin \mathrm{Ind}(f) \qquad 1 \leq k \leq n_i - 1$$

4. Let X be a blowup of $\mathbb{P}^2$ along a set of points $\{f^j T(\Sigma_i), 0 \leq i \leq 2, 1 \leq j \leq n_i\}$.
   The induced map f_X is an automorphism.

- By requiring the existence of an invariant elliptic curve, McMullen showed how one can construct $L \circ J : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ with $n_0 = n_1 = 1, n_2 \geq 7$ and a cyclic permutation.
- Diller constructed all possible rational surface automorphisms with invariant elliptic curves that are obtained as lifts of quadratic birational maps.
- For each possible entropy, Uehara showed one can always construct a rational surface automorphism with an invariant elliptic curves whose entropy is the correct value.
- Assuming the existence of an invariant elliptic normal curve, Perroni and Zhang showed that one can construct pseudoautomorphisms with the dynamical degree > 1

Remark

There exist rational surface automorphism which doesn't have an invariant curve. (Bedford-K)

Dimension 3 or higher

Problem

Let $f : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ be a birational map.
When is there a blowup $\pi : X \to \mathbb{P}^k$ such that

$$f_X := \pi^{-1} \circ f \circ \pi \quad \textit{is a automorphism ?}$$

Theorem (Truong, Bayracktar and Cantat)

If X is the iterated blowup of $\mathbb{P}^3$ along a finite sequence of points, then every automorphism on X has entropy zero.

If X is the iterated blowup of $\mathbb{P}^k$ along a finite sequence of smooth varieties of dimension $< (k-2)/2$, then every automorphism on X has entropy zero.

Dimension 3

Problem

Let $f : (\mathbb{P}^k)^m \dashrightarrow (\mathbb{P}^k)^m$ *be a birational map.*
When is there a blowup $\pi : X \to (\mathbb{P}^k)^m$ *such that*

$$f_X := \pi^{-1} \circ f \circ \pi \quad \textit{is a pseudo-automorphism ?}$$

Definition

A birational map $f : X \dashrightarrow X$ is a *pseudo-automorphism* if neither f nor f^{-1} contracts hypersurfaces,
i.e. there are sets $S_1, S_2 \subset X$ of codimension ≥ 2 such that

$$f : X \setminus S_1 \to X \setminus S_2 \text{ is biregular.}$$

For Dimension $k \geq 3$,

The standard Cremona involution $J : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$:

$$J : [x_0 : x_1 : \cdots : x_k] \mapsto [1/x_0 : 1/x_1 : \cdots : 1/x_k]$$

Question

What are linear automorphisms $S_i \in PGL(k+1, \mathbb{C})$ $i = 1, \ldots, n$ such that $f := S_1 \circ J \circ S_2 \circ J \circ \cdots \circ J \circ S_n$ is equivalent to a pseudo-automorphism?

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Question

What are linear automorphisms $S, T \in PGL(k+1, \mathbb{C})$ such that $f := S \circ J \circ T^{-1}$ is equivalent to a pseudo-automorphism?

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Question

What are linear automorphisms $S, T \in PGL(k+1, \mathbb{C})$ such that $f := S \circ J \circ T^{-1}$ is equivalent to a pseudo-automorphism on a blowup of $\mathbb{P}^k$ along a finite set of points?

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Question

What are linear automorphisms $S, T \in PGL(k+1, \mathbb{C})$ such that $f := S \circ J \circ T^{-1}$ is equivalent to a pseudo-automorphism on a blowup of $\mathbb{P}^k$ along a finite set of distinct points?

For Dimension $k \geq 3$,

The standard Cremona involution $J : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$:

$$J : [x_0 : x_1 : \cdots : x_k] \mapsto [1/x_0 : 1/x_1 : \cdots : 1/x_k]$$

Question

Find linear automorphisms $S, T \in PGL(k+1, \mathbb{C})$ such that

- $f := S \circ J \circ T^{-1}$ *is equivalent to a pseudo-automorphism on a blowup of $\mathbb{P}^k$ along a finite set of distinct points*
- f *has an invariant curve*

The standard Cremona involution $J : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$:

$$J : [x_0 : x_1 : \cdots : x_k] \mapsto [1/x_0 : 1/x_1 : \cdots : 1/x_k]$$

$$\mathrm{Ind}(J) = \cup_{i \neq j}\{x_i = x_j = 0\}, \quad \mathrm{Exc}(J) = \cup_i \{x_i = 0\}$$

- Each coordinate plane $\{x_i = 0\}$ maps to a point

$$J : \{x_i = 0\} \mapsto e_i = \cap_{j \neq i}\{x_j = 0\}$$

- Suppose $I \subset \{0, 1, \ldots, k\}$,
  each point in $\{x_i = 0, x_j \neq 0 : i \in I, j \notin I\}$ blows up to $\{x_i \neq 0, x_j = 0 : i \in I, j \notin I\}$.

We say a birational map $F : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ is a *basic cremona map* if

$$F = S \circ J \circ T^{-1}, \qquad S, T \in PGL(k+1, \mathbb{C})$$

- Exceptional hypersurfaces : $T(\Sigma_j),\ j = 0, 1, \ldots, k$

$$F : T(\Sigma_j) \mapsto S(e_j)$$

- Points of indeterminacy which blows up to hyper surfaces : $T(e_j),\ j = 0, 1, \ldots, k$

Observation

Let $\pi : X \to \mathbb{P}^k$ be a blowup of $\mathbb{P}^k$ along a set of $k+1$ points $e_0, \ldots, e_k$. Then the induced map $J_X : X \dashrightarrow X$ is a pseudo-automorphism.

Observation

Let F be a basic cremona map. Suppose for each $0 \le j \le k$ there is a positive integer n_j such that

1. $F^{n_j - 1}(S(e_j)) = T(e_\ell)$ *for some* $0 \le \ell \le k$
2. $F^i(T(e_j)) \notin Ind(F)$ *for all* $0 \le i \le n_j - 2$.

Then, there is a blowup space X of $\mathbb{P}^k$ along a set of points such that the induced map F_X is a pseudo automorphism.

Remark

Note that $F^{i_1}(T(e_{j_1})) \ne F^{i_2}(T(e_{j_2}))$ for all $(i_1, j_1) \ne (i_2, j_2)$.

Question

what are the basic cremona maps on $\mathbb{P}^k$ satisfying the followings?

1. $F^{n_j-1}(S(e_j)) = T(e_\ell)$ *for some* $0 \leq \ell \leq k$
2. $F^i(T(e_j)) \notin Ind(F)$ *for all* $0 \leq i \leq n_j - 2.$

Suppose $C \subset \mathbb{P}^k$

$$C = \{\gamma(t) = [1 : t : \cdots : t^{k-1} : t^{k+1}], t \in \mathbb{C}\} \cup \{[0 : \cdots : 0 : 1]\}$$
$$= \text{ a degree } k+1 \text{ curve with a cusp}$$

- For hyperplanes $H \subset \mathbb{P}^k$,

$$C \cap H = \{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_{k+1})\}$$

- Let $H = \{\sum a_i x_i = 0\}$ then t_i's are the solution of

$$a_0 + a_1 t + \cdots a_{k-1} t^{k-1} + a_k t^{k+1} = 0$$

  Thus we have $\sum_{i=1}^{k+1} t_i = 0$
- Similarly for any hypersurface $S \subset \mathbb{P}^k$,

$$C \cap S = \{\gamma(t_i) : i = 1, \ldots, (\deg S)(k+1)\}, \quad \sum t_i = 0$$

Suppose $F = S \circ J \circ T^{-1}$ preserve C and

$$C \cap \mathrm{Ind}(F) \subset \{T(e_0), T(e_1), \ldots, T(e_k)\}$$

- Since no hyperplane contains C, each exceptional hypersurface, there is a regular point in $T(\{x_i = 0\}) \cap C$. Thus
$$S(e_i), T(e_i) \in C \quad \text{for all} \quad 0 \leq i \leq k$$
- If F is equivalent to a pseudo-automorphism then $F|_C$ is an automorphism fixing a singular point at $t = \infty$.
$$F|_C : \gamma(t) \mapsto \gamma(\delta t + \tau)$$
- After rescaling parameter, we may choose $\tau = 1 - \delta$.

Constructing a Basic Cremona map $F = S \circ J \circ T^{-1} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that

- F fixes $C = \{\gamma(t) = [1 : t : t^2 : t^4]\}$ and $F(\gamma(\infty)) = (\gamma(\infty))$
- $C \cap \mathrm{Ind}(F) = \{T(e_i), i = 0, 1, 2, 3\}$
- $F : T(\{x_i = 0\}) \mapsto S(e_i) = T(e_{i+1})$ for $i = 0, 1, 2$
- $F : T(\{x_3 = 0\}) \mapsto S(e_3) \to \cdots \to F^{n-1}S(e_3) = T(e_0)$.

Then on the blowup of $\mathbb{P}^3$ along a set of points $\{T(e_i), i = 0, \dots, k$ and $F^j(S(e_3)), j = 0, \dots, n-2\}$, the induced map F_X is a pseudo-automorphism.

Let t_i be a parameter for $T(e_i)$, i.e. $T(e_i) = \gamma(t_i)$ and set

$$F^{-1}|_C : \gamma(t) \mapsto \delta t + 1 - \delta$$

Consider the exceptional hyperplane $\Sigma_1 := T(\{x_1 = 0\})$.

- $\Sigma_1 \cap C$ has 4 distinct points.
- $T(e_0), T(e_2,)T(e_3) \in \Sigma_1 \cap C$
- Since $F : \Sigma_1 \mapsto S(e_1) = T(e_2)$,

$$F^{-1}(T(e_2)) = F^{-1}|_C(\gamma(t_2)) \in \Sigma_1 \cap C.$$

Thus we have

$$t_0 + t_2 + t_3 + \delta t_2 + 1 - \delta = 0 \quad (1)$$

Similarly, with exceptional hyperplanes $T(\{x_0 = 0\})$ and $T(\{x_2 = 0\})$, we get

$$t_1 + t_2 + t_3 + \delta t_1 + 1 - \delta = 0 \tag{0}$$

and

$$t_0 + t_1 + t_3 + \delta t_3 + 1 - \delta = 0 \tag{2}$$

Now for $T(\{x_3 = 0\})$, We have

$$F^n : T(\{x_3 = 0\}) \mapsto T(e_0)$$

Thus

$$T(e_0), T(e_1), T(e_2), \text{ and } F^{-n}(T(e_0)) \in T(\{x_3 = 0\})$$

$$t_0 + t_1 + t_2 + \delta^n t_0 + 1 - \delta^n = 0 \tag{3}$$

We need one more piece of information to determine t_i's and δ.

Let H be a generic hyperplane in $\mathbb{P}^3$.

- $\#H \cap C = 4$. Let $H \cap C = \{\gamma(q_1), \gamma(q_2), \gamma(q_3), \gamma(q_4)\}$
- $q_1 + q_2 + q_3 + q_4 = 0$.
- $\deg F^{-1}H = 3$ and thus $\#(F^{-1}H) \cap C = 12$.
- For $i = 1, \ldots, 4$,
$$F^{-1}|_C(\gamma(q_i)) = \gamma(\delta q_i + 1 - \delta) \in (F^{-1}H) \cap C$$
- For each $i = 1, \ldots, 4$, $H \cap S(\{x_i = 0\})$ = a line, thus with multiplicity 2
$$F^{-1}S(\{x_i = 0\}) = \gamma(t_i) \in (F^{-1}H) \cap C$$

For $(F^{-1}H) \cap C$, we found 12 points

$$\sum_{i=1}^{4}(\delta q_i + 1 - \delta) + 2\sum_{j=0}^{3} t_j = 0$$

Since $q_1 + q_2 + q_3 + q_4 = 0$, we see that

$$t_0 + t_1 + t_2 + t_3 = 2(\delta - 1) \tag{4}$$

$$t_1 + t_2 + t_3 + \delta t_1 + 1 - \delta = 0 \tag{0}$$

$$t_0 + t_2 + t_3 + \delta t_2 + 1 - \delta = 0 \tag{1}$$

$$t_0 + t_1 + t_3 + \delta t_3 + 1 - \delta = 0 \tag{2}$$

$$t_0 + t_1 + t_2 + \delta^n t_0 + 1 - \delta^n = 0 \tag{3}$$

and

$$t_0 + t_1 + t_2 + t_3 = 2(\delta - 1) \tag{4}$$

It is not hard to get the solutions. For example δ is the root of

$$\delta^n(\delta^3 - \delta^2 - \delta) + \delta^3 + \delta - 1 = 0$$

- $\gamma(t) = [1 : t : t^2 : t^4]$
- $\gamma(t_i) = T(e_i)$, i.e. the i-th column of $T = \lambda_i(1, t_i, t_i^2, t_i^4)^t$.
- $S(e_i) = T(e_{i+1})$ for $i = 0, 1, 2$ and $S(e_3) = F^{-(n-1)}|_C T(e_0)$

Thus we determine two automorphisms in $\mathbb{P}^3$, S, T "almost" . Since F fixes the singular point, $\gamma(\infty)$, we can determine the λ_i by setting

$$T[1 : 1 : 1 : 1] = \gamma(\infty), \text{ and } S[1 : 1 : 1 : 1] = \gamma(\infty)$$

$F|_C$ determines F:

Suppose $F|_C : \gamma(t) \mapsto \gamma(\delta t + 1 - \delta)$.
For each $p \in \mathbb{P}^3 \setminus C$, we have

- either $p \in T(\{x_i = 0\})$ for some i : thus we have

$$F : p \mapsto S(e_i)$$

- or $p \in (\cup_i T(\{x_i = 0\}))^c$:
  - H_i : the unique hyperplane containing $\{e_j, j \neq i\}$ and p.
    $\Rightarrow$ there is $\omega_i \in C$ such that $H_i \cap C = \{\omega_i, e_j, j \neq i\}$.
  - Since H_i contains three points of indeterminacy, $F(H_i)$ will be again a hyperplane determined by the $F|_C(e_j), j \neq i$ and $F|_C(\omega_i)$.
  - $F(p) = \cap F(H_i)$

Theorem

Suppose $\delta \in \mathbb{C}^*$ *and* $t_j \in \mathbb{C}$, $0 \leq j \leq k$, *are distinct parameters satisfying* $\sum t_j \neq 0$. *Then there exists a unique basic cremona map* $F = S \circ J \circ T^{-1}$ *and* $\tau \in \mathbb{C}$ *such that*

- F *properly fixes* C *with* $F|_C$ *given by* $F(\gamma(t)) = \gamma(\delta t + \tau)$.
- $\gamma(t_j) = T(e_j)$ *for each* $0 \leq j \leq k$.

Specifically,

- $\tau = \frac{k-1}{k+1}\delta \sum t_j$; *and*
- $S(e_j) = \gamma\left(\delta t_j - \frac{2\tau}{k-1}\right)$.

Theorem (Bedford-Diller-K)

Suppose F is a basic cremona map on $\mathbb{P}^k$ such that

- F *fixes* $C = \{\gamma(t) = [1 : t : t^2 : \cdots : t^{k-1} : t^{k+1}]\}$ *and* $F(\gamma(\infty)) = (\gamma(\infty))$
- $C \cap Ind(F) = \{T(e_i), i = 0, 1, \ldots, k\}$
- $F : T(\{x_i = 0\}) \mapsto S(e_i) = T(e_{i+1})$ *for* $i = 0, 1, k-1$
- $F : T(\{x_k = 0\}) \mapsto S(e_k) \to \cdots \to F^{n-1}S(e_k) = T(e_0)$.

Then F is linearly conjugate to $L \circ J$ where $L = T^{-1} \circ S$ and

Theorem

$$L = \begin{pmatrix} 0 & 0 & & & 0 & 1 \\ \beta_1 & 0 & & & 0 & 1-\beta_1 \\ 0 & \beta_2 & 0 & & 0 & 1-\beta_2 \\ & & \ddots & \ddots & & \vdots \\ 0 & & 0 & \beta_{k-1} & 0 & 1-\beta_{k-1} \\ 0 & & & 0 & \beta_k & 1-\beta_k \end{pmatrix}$$

and $\beta_i = (\delta^i - 1)/(\delta(\delta^{k+1} - \delta^i))$ *for* $i = 1, \ldots, k$ *and* δ *is a valois conjugate of the largest real root of* $\delta^n(\delta^{k+2} - \delta^{k+1} - \delta^k + 1) + \delta^{k+2} - \delta^2 - \delta + 1 = 0$

The same procedure works with other invariant curves:

- A rational normal curve with its tangent line.

$$\gamma_1(t) = [1 : t : t^2 : \cdots : t^k], \quad \gamma_2(t) = [0 : \cdots : 0 : 1 : -t]$$

  Every point of indeterminacy is in the rational normal curve $\{\gamma_1(t)\}$

- $k+1$ concurrent lines in general position.

$$\gamma_0(t) = [-t : 1 : \cdots : 1], \gamma_i(t) = [t : 0 : \cdots : 1 : \cdots : 0]$$

  Each line contains exactly one point of indeterminacy.

Let $\pi : X \to \mathbb{P}^k$ be a blowup of N points $\{\alpha_1, \dots, \alpha_N\}$

1. E_0 : the class of a generic hypersurface in X
2. E_i : the class of the exceptional divisor over $\alpha_j, 1 \leq j \neq N$
3. $\mathrm{Pic}(X) = \langle E_0, E_1, \dots, E_N \rangle$

Let us define a symmetric bilinear form on $\mathrm{Pic}(X)$ as follows

$$\langle \alpha, \beta \rangle := \alpha \cdot \beta \cdot \Phi \qquad \alpha, \beta \in \mathrm{Pic}(X)$$

where

$$\Phi = (k-1)E_0^{k-2} + (-1)^k \sum_j E_j^{k-2} \in H^{k-2,k-2}(X)$$

and $D^n = D \cdot D \cdots D$ is a n-fold intersection product.

$$\langle \alpha, \beta \rangle := \alpha \cdot \beta \cdot \Phi \qquad \alpha, \beta \in \mathrm{Pic}(X)$$

where

$$\Phi = (k-1)E_0^{k-2} + (-1)^k \sum_j E_j^{k-2} \in H^{k-2,k-2}(X)$$

1. $\langle E_0, E_0 \rangle = k - 1$, $\langle E_i, E_i \rangle = -1$ for $i = 1, \ldots, N$
2. $\langle E_i, E_j \rangle = 0$ for $0 \leq i \neq j \leq N$

Let $F : X \dashrightarrow X$ be a pseudo-automorphism that is obtained as lifts of basic cremona map f.

- f has $k+1$ points of indeterminacy, $\alpha_1, \ldots, \alpha_{k+1}$
- Each exceptional hyperplane contains exactly k points of indeterminacy.

$$F^* E_i = \begin{cases} \text{either} & E_j & \text{for some } j \geq k+2 \\ \text{or} & E_0 - \sum_{s=1}^{k} E_{j_s} & \end{cases}$$

- The degree of $f = k$.
- The pre-image of a generic hyperplane contains each point of indeterminacy with multiplicity $k-1$.

$$F^* E_0 = kE_0 - (k-1) \sum_{i=1}^{k+1} E_i$$

$$\langle F^*E_0, F^*E_0\rangle = \langle kE_0 - (k-1)\sum_{i=1}^{k+1} E_i, kE_0 - (k-1)\sum_{i=1}^{k+1} E_i\rangle$$

$$= k^2\langle E_0, E_0\rangle + (k-1)^2 \sum_{i=1}^{k+1}\langle E_i, E_i\rangle = k-1$$

$$\langle E_0 - \sum_{s=1}^{k} E_{j_s}, E_0 - \sum_{s=1}^{k} E_{j_s}\rangle = \langle E_0, E_0\rangle + \sum_{s=1}^{k}\langle E_{j_s}, E_{j_s}\rangle$$

$$= k - 1 - k = -1 = \langle E_i, E_i\rangle$$

Thus

$$\langle F^*E_i, F^*E_i\rangle = \langle E_i, E_i\rangle$$

For $i \neq j$, $\langle F^* E_i, F^* E_j \rangle$ is one of the followings

- $\langle kE_0 - (k-1)\sum_{i=1}^{k+1} E_i, E_0 - \sum_{1 \leq i \leq k+1, i \neq j} E_i \rangle$
  $= k(k-1) - (k-1)k = 0$
- $\langle kE_0 - (k-1)\sum_{i=1}^{k+1} E_i, E_j \rangle$ for some $j \geq k+2$
  $= 0$
- $\langle E_0 - \sum_{1 \leq i \leq k+1, i \neq j} E_i, E_j \rangle$ for some $j \geq k+2$
  $= 0$

Thus

$$\langle F^* E_i, F^* E_j \rangle = \langle E_i, E_j \rangle$$

Theorem (Bedford-Diller-K)

Let $F : X \dashrightarrow X$ be a pseudo-automorphism that is obtained as lifts of basic cremona map on $\mathbb{P}^k$. Then

1. *F^* preserves the bilinear form $\langle \cdot, \cdot \rangle$.*
2. *F^* belongs to the generalized Weyl group $W(2, k+1, N-k-1)$.*

Remark

- F realizes an element in the generalized Weyl group.
- The one we constructed earlier realizes the coxeter element in $W(2, k+1, N-k-1)$. Thus this map has the smallest dynamical degree > 1 among all pseudo-automorphisms on X which preserve the bilinear form we just defined.
- The dynamical degree of F is given by a Salem number.

More pseudo-automorphisms?

The cremona involution on multi-projective space $(\mathbb{P}^k)^m$

$$J : (x^{(1)}, \dots, x^{(m)}) \mapsto (1/x^{(1)}, x^{(2)}/x^{(1)}, \dots, x^{(m)}/x^{(1)})$$

where $x^{(j)}/x^{(1)} = [x_0^{(j)}/x_0^{(1)} : \dots : x_k^{(j)}/x_k^{(1)}]$

Basic Cremona map $F = S \circ \rho \circ J \circ T^{-1}$ where

- $S = (S_i), T = (T_i) \in (PGL(k+1, (C)))^m$
- $\rho : (x^{(1)}, \dots, x^{(m)}) \mapsto (x^{(p(1))}, \dots, x^{(p(m))})$ where p is a permutation of $\{1, 2, \dots, m\}$.

The Symmetric Bilinear from on $(\mathbb{P}^k)^m$

Let $\pi : X \to (P^k)^m$ be a blowup of N points $\{\alpha_1, \ldots, \alpha_N\}$

1. H_i : the class of $\mathbb{P}^k \times \cdots \times H \times \cdots \times \mathbb{P}^k$ where H is a generic hyperplane in $\mathbb{P}^k$ and H is in the i-th slot.
2. E_i : the class of the exceptional divisor over $\alpha_j, 1 \leq j \neq N$
3. $\text{Pic}(X) = \langle H_1, H_2, \ldots, H_m, E_1, \ldots, E_N \rangle$

Let us define a symmetric bilinear form on $\text{Pic}(X)$ as follows

$$\langle \alpha, \beta \rangle := \alpha \cdot \beta \cdot \Phi \qquad \alpha, \beta \in \text{Pic}(X)$$

$$\Phi = (k-1) \sum_{i=1}^{m} \left(H_i^{k-2} \prod_{j \neq i} H_j^{k-4} \right) + k \sum_{1 \leq i \neq j \leq m} \left(H_i^{k-1} H_j^{k-1} \prod_{\ell \neq i,j} H_\ell^k \right) + (-1)^{km} \sum_j E_j^{mk-2}$$

and $D^n = D \cdot D \cdots D$ is a n-fold intersection product.

Theorem

Let $F : X \dashrightarrow X$ be a pseudo-automorphism that is obtained as lifts of basic cremona map on $(\mathbb{P}^k)^m$. Then

1. *F^* preserves the bilinear form $\langle \cdot, \cdot \rangle$.*
2. *F^* belongs to the generalized Weyl group $W(m+1, k+1, N-k-1)$.*

If $F : X \dashrightarrow X$ be a pseudo-automorphism that is obtained as lifts of basic cremona map $f = S \circ \rho \circ J \circ T^{-1}$ on $(\mathbb{P}^k)^m$, Then we see

$$F^* H_1 = kH_{p(1)} - (k-1) \sum_{i=1}^{k+1} E_i$$

$$F^* H_j = kH_{p(1)} + H_{p(j)} - k \sum \sum_{i=1}^{k+1} E_i, \quad j \neq 1$$

$f = L \circ \rho \circ J$ on $(\mathbb{P}^k)^m$ preserves $k+1$ concurrent lines and f lifts to a pseudo automorphism on a blowup of $(\mathbb{P}^k)^m$.

$$L_j = \begin{pmatrix} 0 & 0 & 0 & 0 & s_j \\ v & 0 & 0 & 0 & s_j - v \\ 0 & v & 0 & 0 & s_j - v \\ 0 & 0 & \ddots & 0 & s_j - v \\ 0 & 0 & 0 & v & s_j - v \end{pmatrix},$$

where

$$v = -\alpha \frac{\alpha^m - 1}{\alpha - 1}, \ s_j = \frac{(\alpha^m - 1)(\alpha^{j+1} - 1)}{\alpha^j (\alpha - 1)(\alpha^{m-j} - 1)}$$

for $j = 0, \ldots, m-1$.