Wiggles in the anti-holomorphic quadratic family

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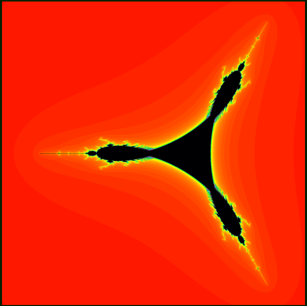
Tricorn $\mathcal{M}^*$

Consider the anti-holomorphic quadratic family

$$f_c(z) = \bar{z}^2 + c.$$

Observe that $f_c^2(z) = (z^2 + \bar{c})^2 + c$ is holomorphic.

- $K_c = \{z \in \mathbb{C};\ \{f_c^n(z)\}_{n \geq 0}\text{: bdd}\}$: filled Julia set,
- $J_c = \partial K_c$: Julia set,
- $\mathcal{M}^* = \{c \in \mathbb{C};\ K_c\text{: connected}\}$: the tricorn.



Tricorn is connected (Nakane)

- φ_c: the Böttcher coordinate, i.e., a holomorphic germ at ∞ tangent to the identity s.t. $\varphi_c \circ f_c = f_0 \circ \varphi_c$ or

$$\varphi_c(f_c(z)) = (\overline{\varphi_c(z)})^2.$$

- For $c \in \mathcal{M}^*$, $\varphi_c : \mathbb{C} \setminus K_c \to \mathbb{C} \setminus \overline{\Delta}$ is an isomorphism.

Theorem 1 (Nakane)

For $c \notin \mathcal{M}^*$, $\Phi(c) = \varphi_c(c)$ is well-defined and
$\Phi : \mathbb{C} \setminus \mathcal{M}^* \to \mathbb{C} \setminus \overline{\Delta}$ is a real-analytic diffeomorphism.
In particular, $\mathcal{M}^*$ is connected and full.

Remark

Φ is not complex analytic (different from the Riemann map).

External rays

For $\theta \in \mathbb{R}/\mathbb{Z}$, let

- $R_c(\theta) = \{\varphi_c^{-1}(re^{2\pi i\theta});\ r \in (1,\infty)\}$: dynamical ray,
- $\mathcal{R}(\theta) = \{\Phi^{-1}(re^{2\pi i\theta});\ r \in (1,\infty)\}$: parameter ray.

Then we have

$$f_c(R_c(\theta)) = R_c(-2\theta).$$

Proposition 2

For $\theta \in \mathbb{Q}/\mathbb{Z}$,

$$\lim_{r \searrow 1} \varphi_c^{-1}(re^{2\pi i\theta})\ (\in J_c)$$

exists, i.e., $R_c(\theta)$ lands.

Question

For $\theta \in \mathbb{Q}/\mathbb{Z}$, does $\mathcal{R}(\theta)$ land?

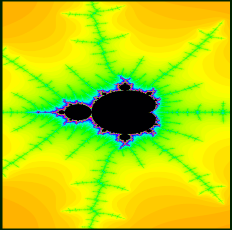
(cf. Every rational parameter ray lands for the Mandelbrot set.)

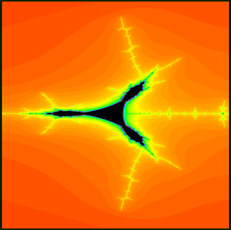
Hyperbolic components

Proposition 3 (Nakane-Schleicher)

Let $\mathcal{H} \subset \operatorname{int} \mathcal{M}^*$ be a hyperbolic component of period p.

- ***p*: even** The return map on an attracting basin is holomorphic. $\mathcal{H}$ is parametrized by the multiplier.
- ***p*: odd** The return map is anti-holomorphic. The attracting cycle (for f_c^2) always has real multiplier. $\partial\mathcal{H}$ consists of 3 **parabolic arcs** with 3 **cusps** as the endpoints.





Root arc and co-root arcs of $\mathcal{H}$ with odd period

- $\mathcal{H}$: hyperbolic component with odd period $p \geq 3$,
- $\gamma \subset \partial\mathcal{H}$: a parabolic arc

Root arc A parabolic periodic point for f_c disconnects K_c for every $c \in \gamma$.
Two parameter rays of period $2p$ accumulate to γ.

Co-root arc No parabolic periodic point disconnects K_c for $c \in \gamma$.
One parameter ray of period p accumulates to γ.

(Nakane-Schleicher)

By definition, every parabolic arc is a co-root arc when $p = 1$ for simplicity.

Landing properties of periodic parameter rays

The landing property of periodic rational rays for the Mandelbrot set depends on the fact that parabolic maps of given period are discrete.

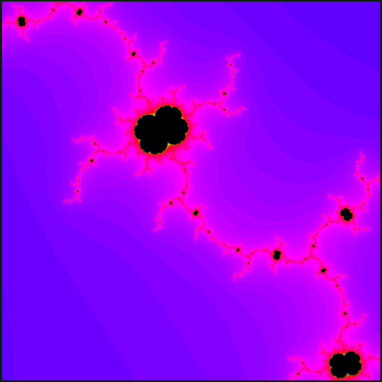
Proposition 4 (Mukherjee-Nakane-Schleicher)

Let $\theta \in \mathbb{Q}/\mathbb{Z}$ periodic of period p.

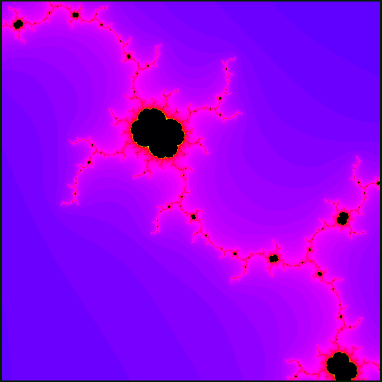
- If ***p* is odd**, then $\mathcal{R}(\theta)$ accumulates to a co-root arc of an odd period hyperbolic component;
- If $\boldsymbol{p = 2k}$ **for** $\boldsymbol{k}$ **odd**, then $\mathcal{R}(\theta)$ either
  - accumulates to the root arc of a hyperbolic component of period k, or
  - lands at the root of a hyperbolic component of period $2k$.
- If $\boldsymbol{p = 4k}$ **(**$\boldsymbol{k \in \mathbb{N}}$**)**, then $\mathcal{R}(\theta)$ lands at the root of a hyperbolic component of period p.

Let c be the landing point or in the accumulating parabolic arc. The dynamical ray $R_c(\theta)$ lands at the parabolic periodic point whose immediate basin contains the critical value.

Julia sets in parabolic arcs



root arc



co-root arc

Umbilical cords

It seems there is an "**umbilical cord**" connecting the origin to $\mathcal{H}$ in $\mathcal{M}^*$.

Conjecture

The Mandelbrot set $\mathcal{M}$ is locally connected.
In particular, for every hyperbolic component $\mathcal{H} \subset \operatorname{int} \mathcal{M}$, there is an arc connecting the origin to the root of $\mathcal{H}$ in $\mathcal{M}$.[1]

Remark

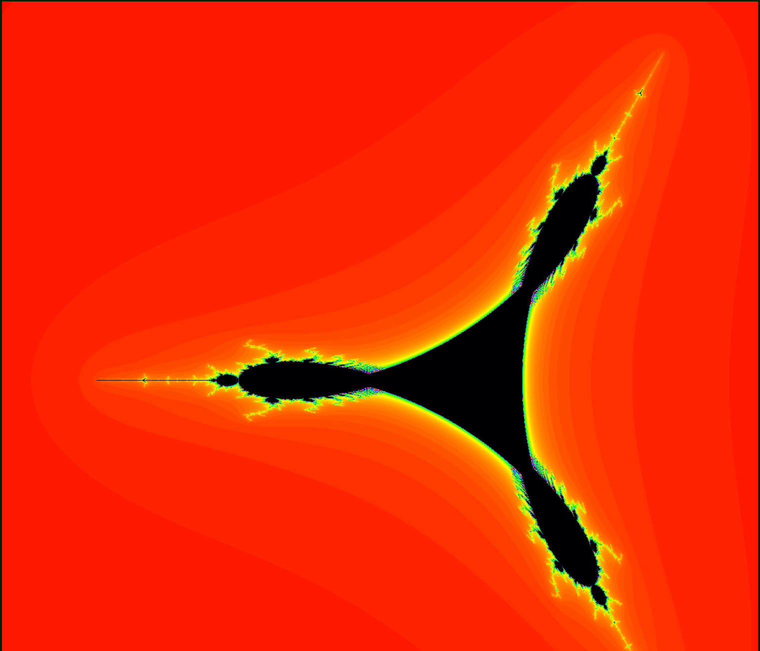
There is no definition of umbilical cord (for now).

[1] After the talk, Petersen and Roesch pointed out that they have already proved the existence of such arcs.

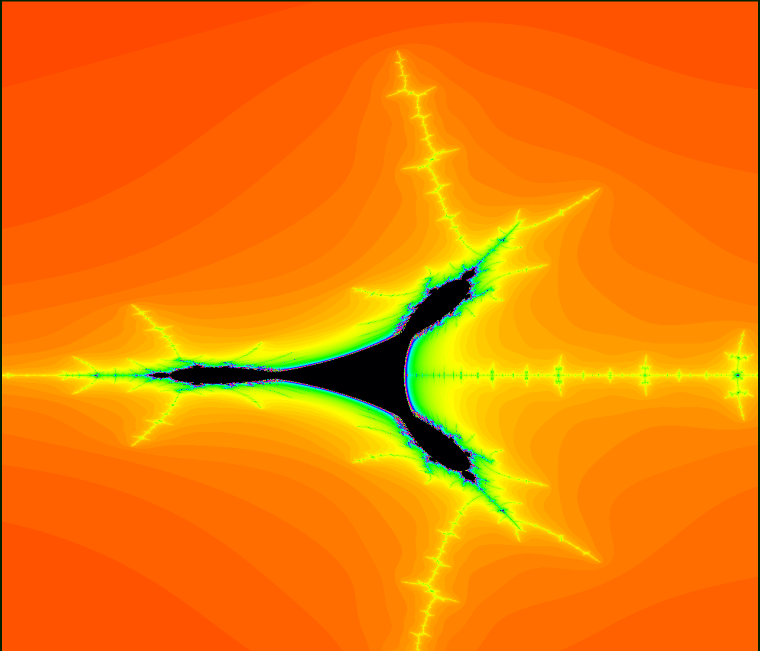
Numerical experiments

It seems parameter rays accumulating parabolic arcs and "umbilical cords" oscillates and do not converge to a point except when the period is one.

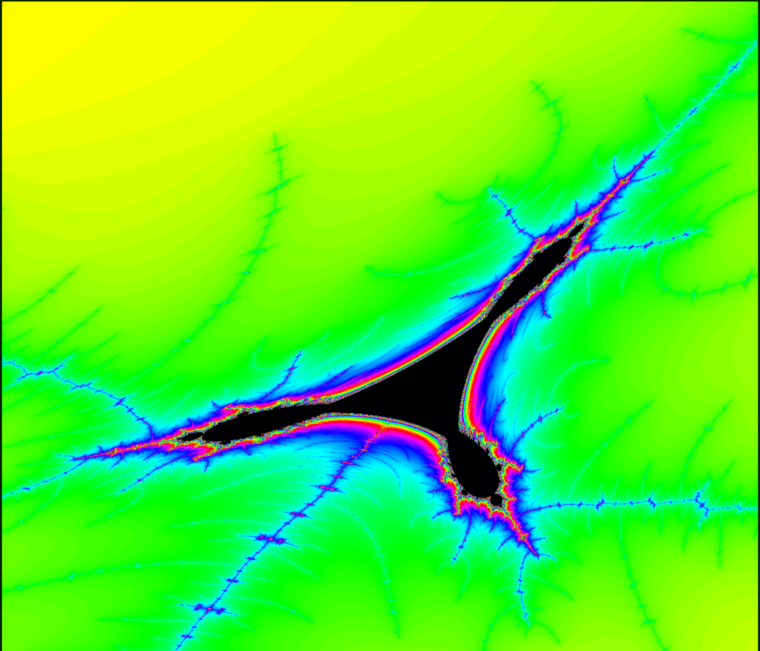
Wiggly umbilical cords



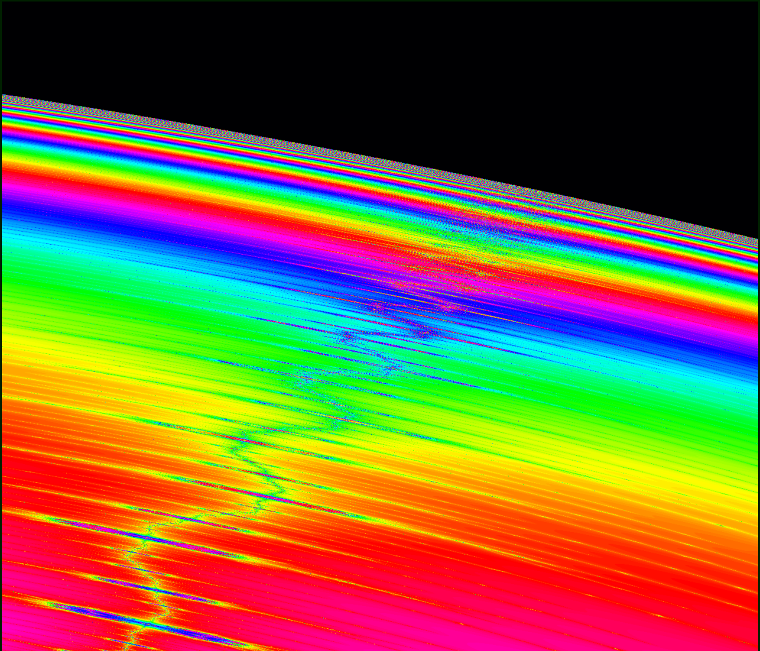
Wiggly umbilical cords



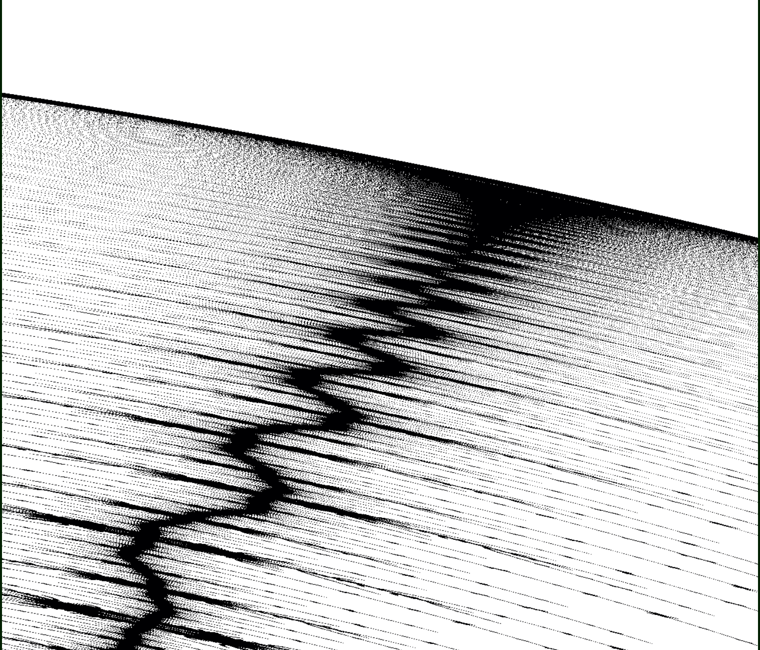
Wiggly umbilical cords



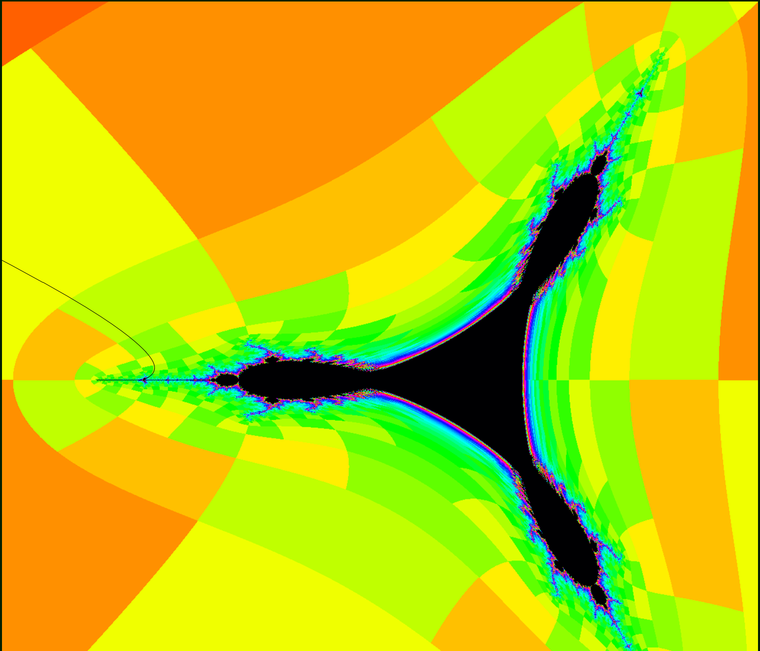
Wiggly umbilical cords



Wiggly umbilical cords



External ray of angle 3/7



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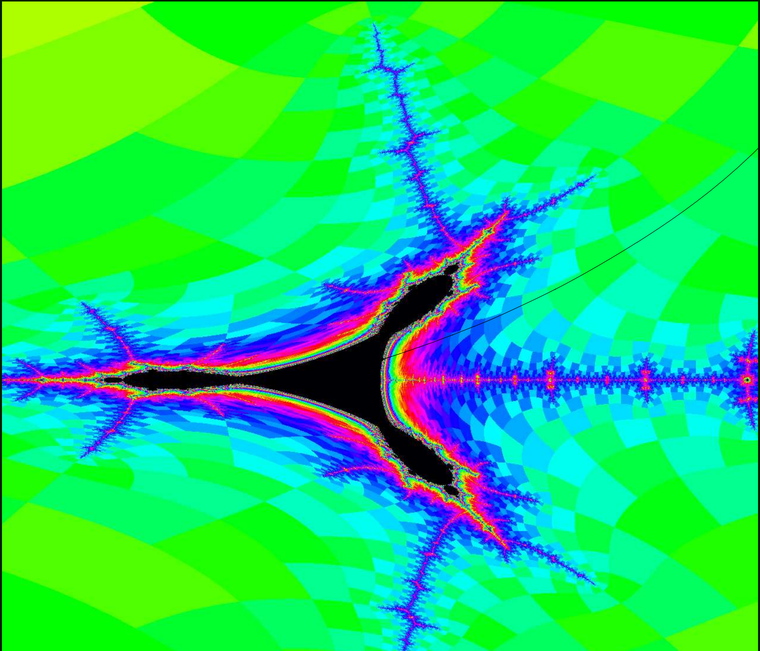

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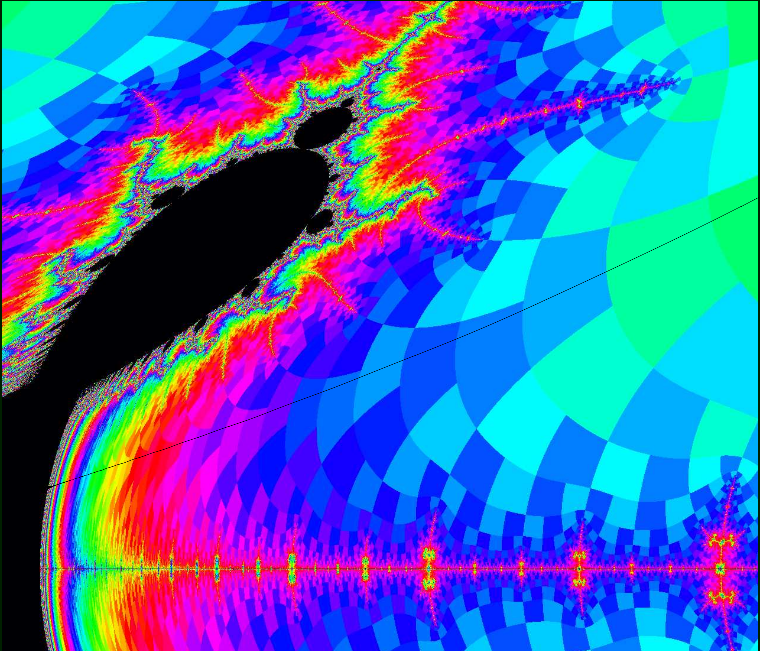

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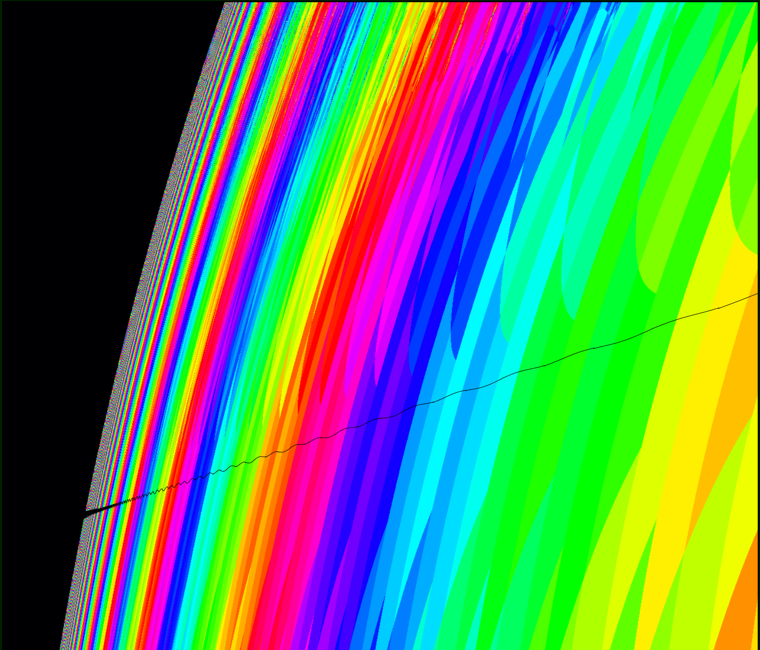
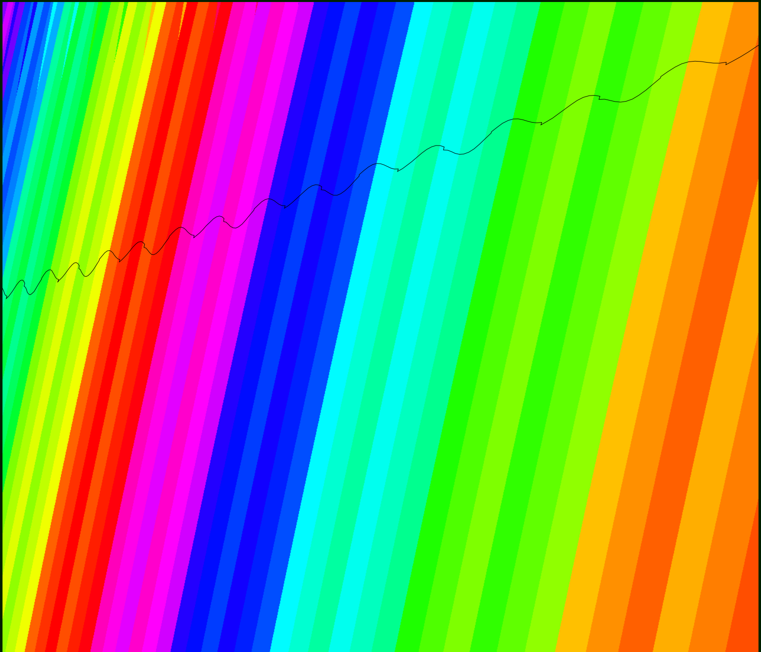
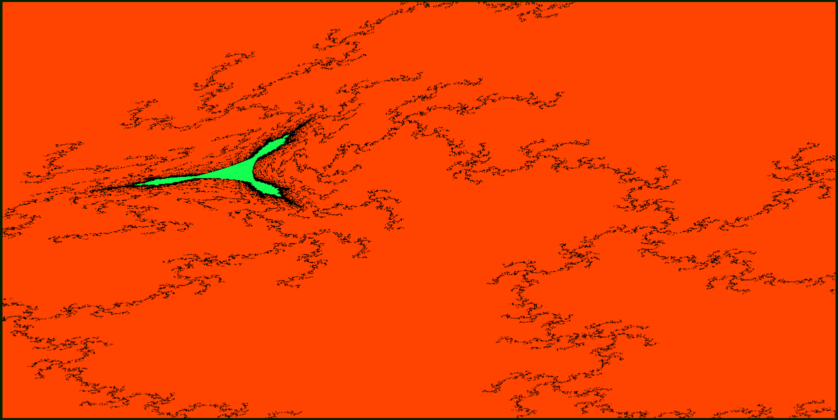


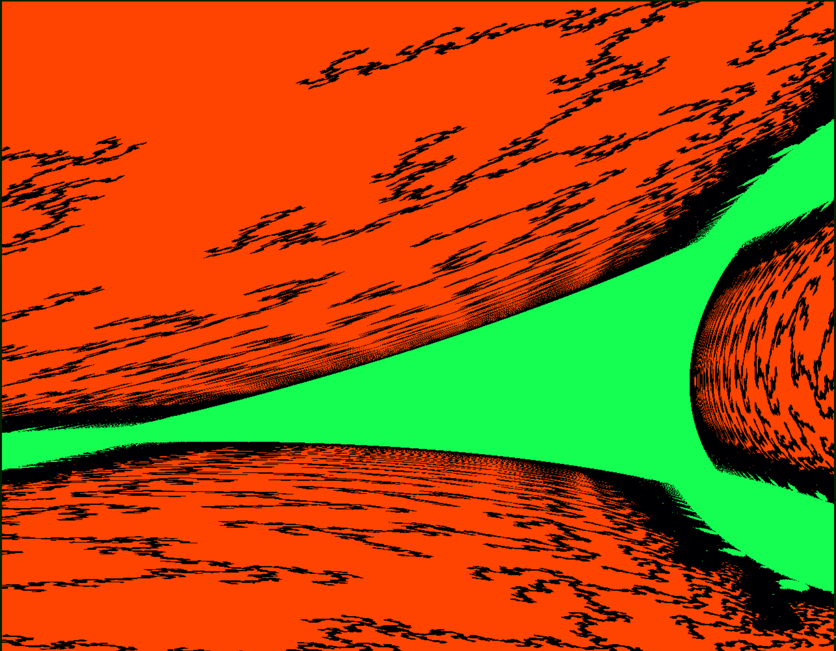
External ray of angle 3/7



Inaccessible hyperbolic component



Inaccessible hyperbolic component



(Non-)existence of baby tricorns

Wiggle of "umbilical cords" is related to (non-)existence of baby tricorns.

- $\mathcal{M}^* \cap \mathbb{R} = [-2, 1/4] (= \mathcal{M} \cap \mathbb{R})$.
- Hence the umbilical cord for every hyperbolic component on the real line exists and is just a real line segment, hence converges to a point.
- By symmetry,
  $\mathcal{M}^* \cap \omega\mathbb{R} = \omega[-2, 1/4], \quad \mathcal{M}^* \cap \omega^2\mathbb{R} = \omega^2[-2, 1/4]$,
  where $\omega = \frac{1+\sqrt{-3}}{2}$.
- If a "baby tricorn-like set" is homeomorphic to the tricorn, then umbilical cords in those segments must land.
- However, at most one of those segments can lie on $\mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$.
- Hence it is a contradiction.

Wiggling umbilical cords

- For simplicity, we say f_c is **real** if there is a real line $L \subset \mathbb{C}$ s.t. f_c is symmetric w.r.t. L.
- Equivalently, if $c \in \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$.
- A hyperbolic component or a parabolic arc is real if f_c is real for some c in it.

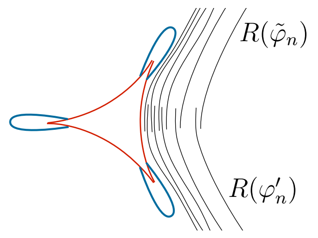
Wiggling and non-path connectivity of $\mathcal{M}^*$

Theorem 5 (Hubbard-Schleicher)

Let $\mathcal{A}$ be a non-real odd period prime principal parabolic (OPPPP) arc. Let $\varphi, \tilde{\varphi} \in \mathbb{Q}/Z$ be such that $\mathcal{R}(\varphi)$ and $\mathcal{R}(\tilde{\varphi})$ accumulate to $\mathcal{A}$.

Then there exists $\tilde{\varphi}_n, \varphi'_n$ such that

- $\varphi'_n \to \varphi$, $\tilde{\varphi}_n \to \tilde{\varphi}$.
- There exists a subarc $\mathcal{A}_\tau \subset \mathcal{A}$ of positive length such that the accumulation sets of $\mathcal{R}(\varphi'_n)$ and $\mathcal{R}(\tilde{\varphi}_n)$ contains $\mathcal{A}_\tau$.



Hubbard, Schleicher. *Multicorns are not path connected.* arXiv:1209.1753

Corollary 6

For such a parabolic arc, **"the umbilical cord does not converge to a point"**.

Remark

- OPPPP implies non-renormalizability, hence the "umbilical cord" for $\mathcal{A}$ (if exists) is not contained in any "baby tricorn-like set".
- Hence we cannot apply this theorem to the above argument for non-existence of baby tricorns.

Conjecture

- Any baby tricorn-like set is not (dynamically) homeomorphic to the tricorn.
- Any two baby tricorn-like sets are not (dynamically) homeomorphic.

Landing umbilical cord

Theorem 5 depends on the following observation:

- $\mathcal{A}$: parabolic root arc, $c \in \mathcal{A}$.
- z_0: parabolic periodic point for f_c,
- $\Psi_c : V_c \to \{\text{Re}\, w > 0\}$: repelling (outgoing) Fatou coordinate at z_0. i.e., solution of

$$\Psi_c(f_c(z)) = \overline{\Psi_c(z)} + \frac{1}{2}.$$

  (Unique up to **real** translation, hence imaginary part makes sense.)
- $V_c/f_c^2 \overset{\Psi_c}{\cong} \mathbb{C}/\mathbb{Z}$: Ecalle cylinder.

Lemma 7

If "the umbilical cord lands at $c \in \mathcal{A}$", then there is a *loose* parabolic Hubbard tree T such that $(T \cap V_c)/f_c^2$ is the equator $\{\text{Im}\, w = 0\}$ $(\subset \mathbb{C}/\mathbb{Z})$.

(Namely, if it is not the equator, then Theorem 5 follows.)

Real analytic Hubbard tree and multiplier

Corollary 8

If there is no loose Hubbard tree T for f_c such that $T \cap V_c$ is not real-analytic for every $c \in \mathcal{A}$, then the umbilical cord does not land.

Corollary 9

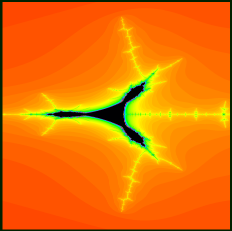
If there is a periodic point x in the Hubbard tree T such that

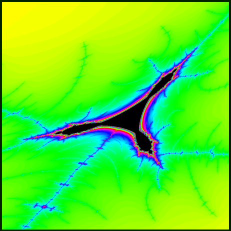
- the multiplier of x is not real,
- the backward orbit (in T) intersects $T \cap V_c$,

then the umbilical cord does not land.

A baby tricorn-like set which is not a baby tricorn

- $c_* = 1.7548...$: airplane (0 is periodic of period 3)
- $\mathcal{M}^*(c_*)$: baby tricorn-like set centered at c_*.
- $c_{**} = c_* \triangleright \omega^2 c_*$: tuning, period 9.
- $\mathcal{A}_{**}$: parabolic root arc for the hyperbolic component centered at c_{**}.
- $\mathcal{A}_{**}$ and its “umbilical cord” is contained in $\mathcal{M}^*(c_*)$.





- For any $c \in \mathcal{A}_{**}$, f_c is renormalizable of period 3.
- If there is a periodic point x with non-real multiplier in the Hubbard tree of the period 3 renormalization, then we can apply Corollary 9.
- Hence the umbilical cord does not land and $\mathcal{M}^*(c_*)$ is not homeomorphic to the tricorn.

Theorem 10

Let $\mathcal{H}_{**}$ be the hyperbolic component containing c_{**}. For any $c \in \partial\mathcal{H}_{**}$,

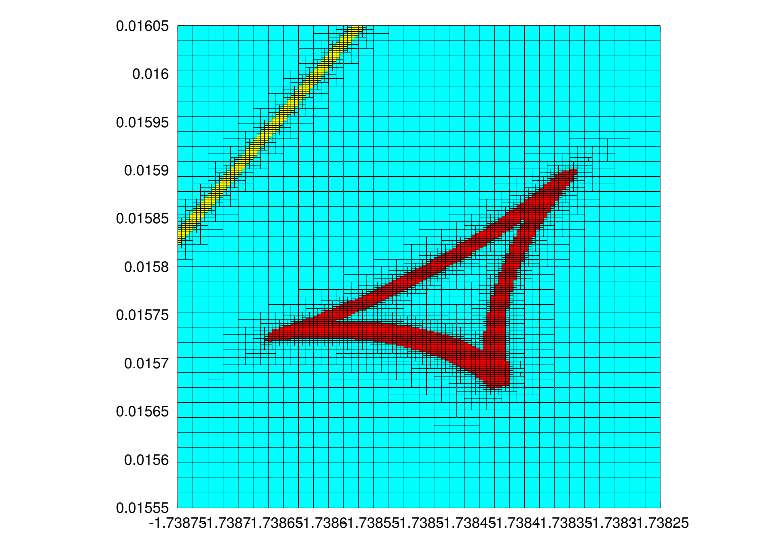
- there is a unique period 6 periodic point x_c of f_c in the period 3 renormalization.
- x_c lies in the Hubbard tree of the renormalization.
- the multiplier of x_c is not real.

(computer-assisted)

Corollary 11

$\mathcal{M}^*(c_*)$ is not homeomorphic to $\mathcal{M}^*$.

Rigorous computations



- Yellow boxes contain the real multiplicity locus of x_c.
- Red boxes contain $\partial\mathcal{H}_{**}$.

Landing parameter rays

Let $\theta \in \mathbb{Q}/\mathbb{Z}$ periodic of period p s.t. $\mathcal{R}(\theta)$ accumulates to a parabolic arc $\mathcal{A}$.
By the same argument as umbilical cords, we have the following:

Lemma 12

If $\mathcal{R}(\theta)$ lands at $c \in \mathcal{A}$, then

$$(R_c(\theta) \cap V_c)/f_c^2 = \{\operatorname{Im} w = k\} \ (\subset \mathbb{C}/\mathbb{Z})$$

for some $k \in \mathbb{R}$.

Two involutions

- $w \mapsto \bar{w} + 2k$ on the Ecalle cylinder $V_c/f_c^2 \cong \mathbb{C}/\mathbb{Z}$:
  - Naturally induces an involution on $\iota_1 : V_c \to V_c$.
  - $R_c(\theta) \cap V_c$ is invariant by ι_1.
- $t \mapsto -t + 2\theta$ in terms of external angle:
  - Defines an involution ι_2 on $\mathbb{C} \setminus K_c$.
  - $R_c(\theta)$ is invariant by ι_2.
- By the Schwarz reflection principle, $\iota_1 = \iota_2$ on $V_c \setminus K_c$.

The involution and rational lamination

- Assume two dynamical rays $R_c(t)$ and $R_c(t')$ land at the same point.
- By taking the involution, $R_c(-t+2\theta)$ and $R_c(-t'+2\theta)$ land at the same point.
- Therefore, the landing relation is invariant by $t \mapsto -t+2\theta$ on $(\theta-\varepsilon, \theta+\varepsilon)$.
- By using the invariance of the landing relation (the **rational lamination**) under $t \mapsto -2t$, it follows that the rational lamination is invariant under $t \mapsto -t+2\theta$.
- However, it is easy to see that this happens only when $\theta = k/3$ for $k = 0, 1, 2$, i.e., θ is of period one and $\mathcal{R}(\theta) = \omega^k(\frac{1}{4}, \infty)$ (it trivially lands).

Theorem 13 (I-Mukherjee)

Parameter ray $\mathcal{R}(\theta)$ accumulating to the boundary of a hyperbolic component $\mathcal{H}$ of odd period p land at a point if and only if $p = 1$, equivalently, if $\mathcal{H} \ni 0$.