Limiting Dynamics of Conformal dynamical systems

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with help from Harry Baik, Adam Epstein, Xavier Buff, Sarah Koch and Bill Thurston

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Pictures due to Papadantonakis (Fractalasm) and Masaaki Wada (Opti)

The basic theory of parabolic implosion
is due to
DOUADY and LAVAURS

The essential constructions for
parabolic blowups

Maps of finite type
Conformal dynamical systems
Parabolic towers
The Epstein rigidity theorem

Are all due to
ADAM EPSTEIN

Definitions for dynamical systems

$$p_c(z) = z^2 + c$$

K_c is the filled-in Julia set

$$K_c = \{z \mid \text{the sequence } z, p_c(z), p_c(p_c(z)), \cdots \not\to \infty\}$$

The set $\mathcal{C}(\mathbb{C})$ is the set of compact subsets of $\mathbb{C}$

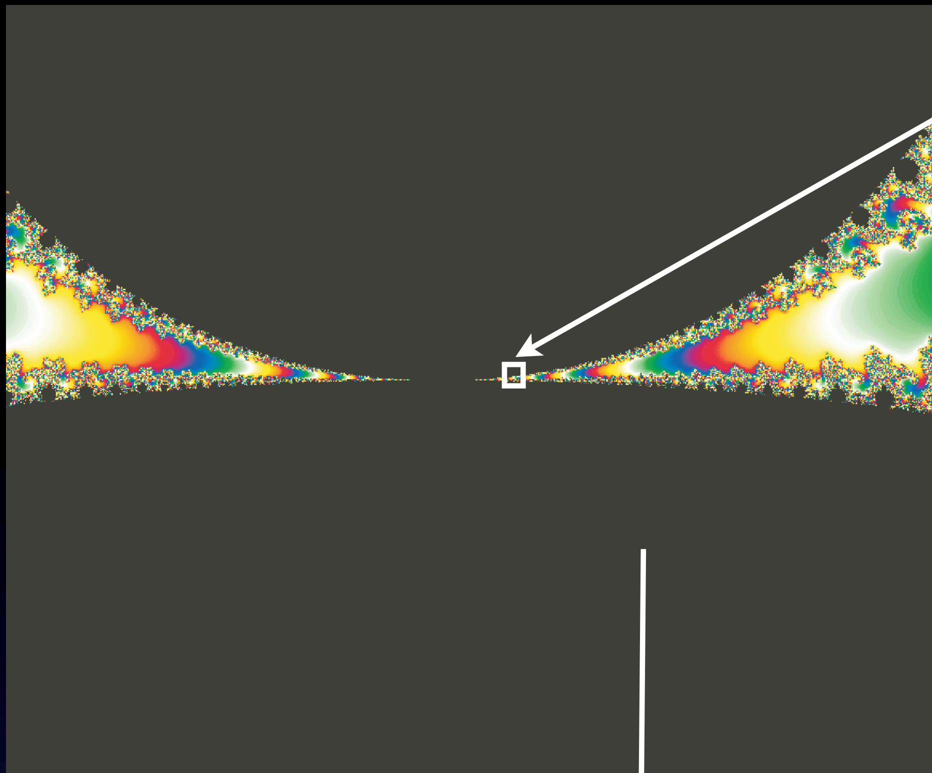
Give $\mathcal{C}(\mathbb{C})$ the Hausdorff metric.

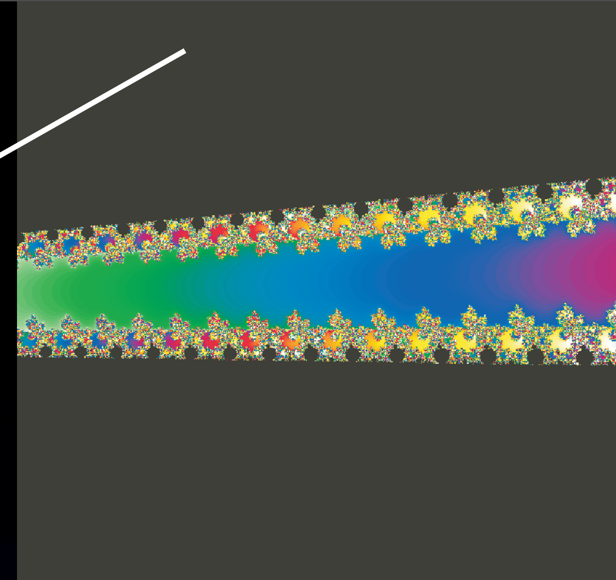
There is a dichotomy $\begin{cases} 0 \in K_c \iff K \text{ connected} \\ 0 \notin K_c \iff K \text{ Cantor set} \end{cases}$

The Mandelbrot set M is

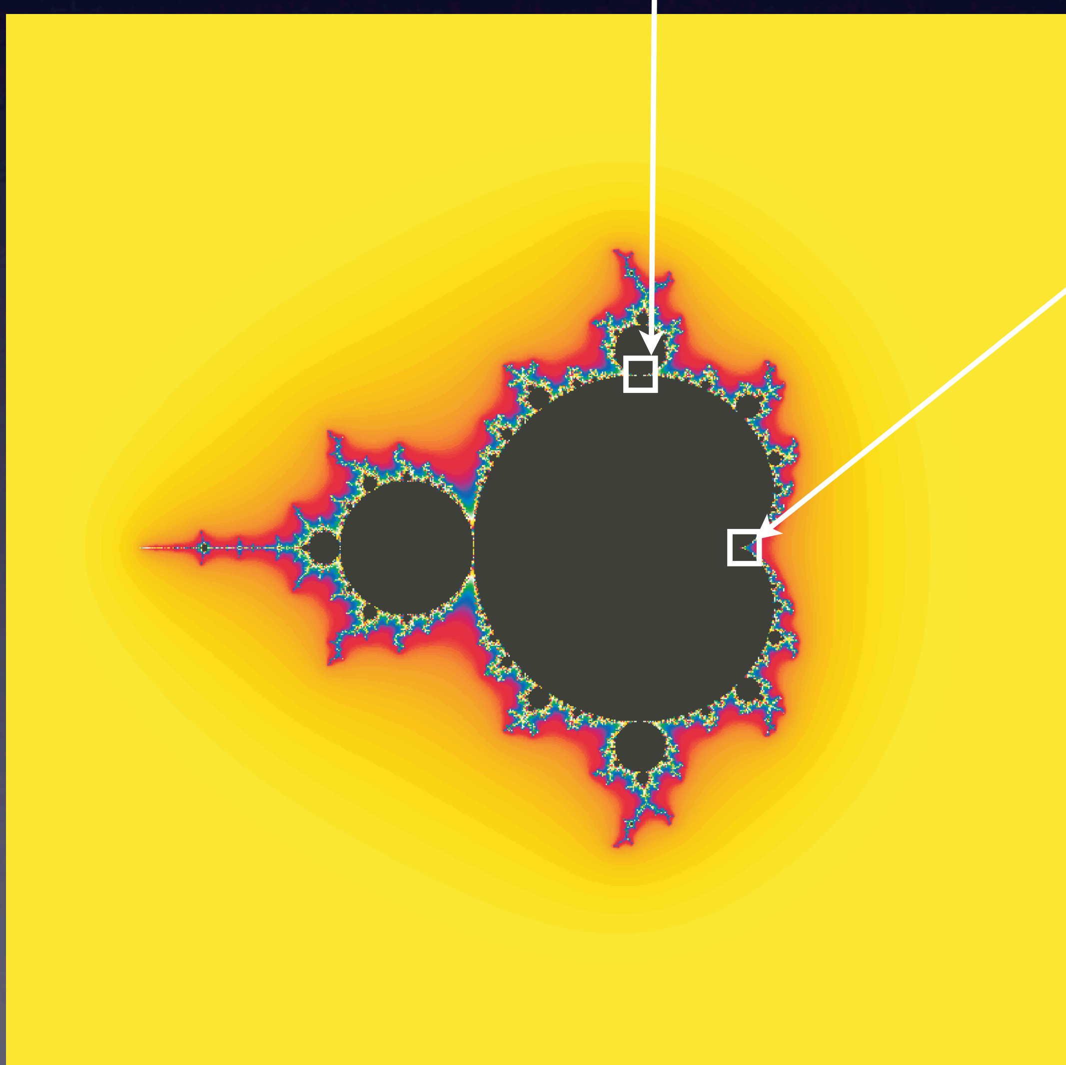
$$M = \{c \in \mathbb{C} \mid K_c \text{ connected}\}$$

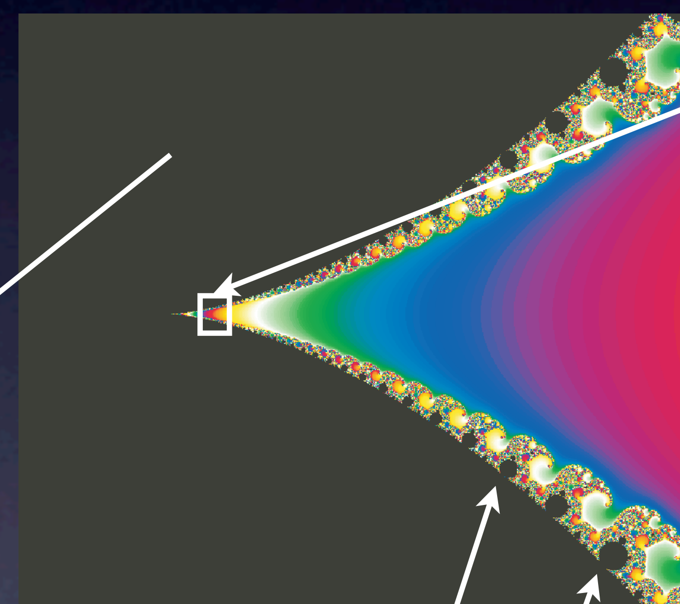
M is the important object in parameter space

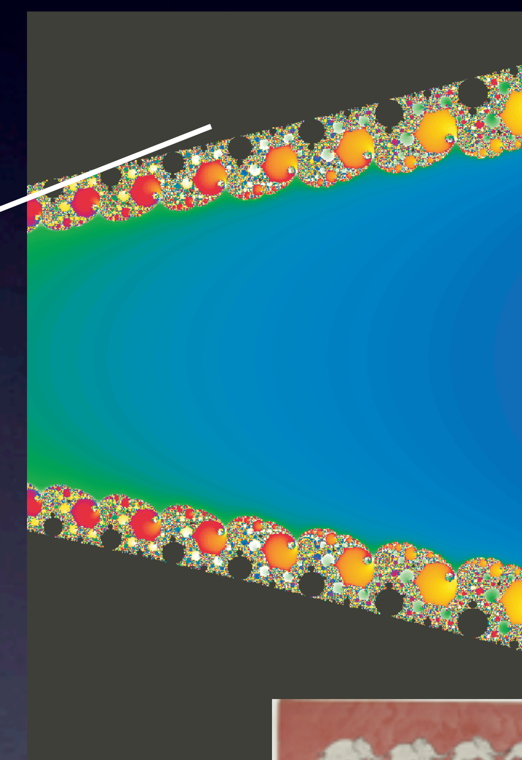




The set M
and various blow-ups
that will come up
during the lecture







do you see elephants?

Basic observation

The map

$$c \mapsto K_c$$

is not continuous

The goal is to describe the closure of its image

According to Douady:

The map $c \mapsto K_c$

is continuous if and only if

p_c has no parabolic cycles

So we need to understand the possible limits of K_c as p_c approaches a polynomial with a parabolic cycle

Definitions for Kleinian groups

Let G be a Lie group,
and $\overline{G} = G \cup \{\infty\}$ be its 1-point compactification.

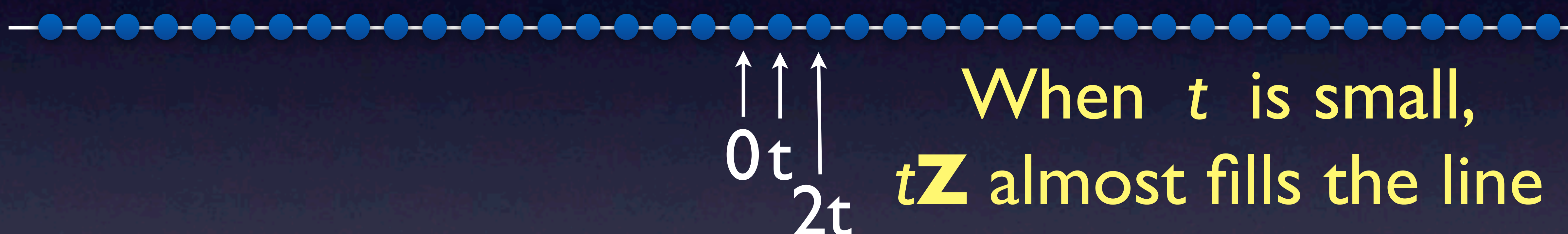
Then $\Gamma \mapsto \Gamma \cup \{\infty\}$ maps the set
of closed subgroups of G to $\mathcal{C}(\mathcal{G})$

This topology on the space of closed subgroups of G
is called the Chabauty topology

The easiest example: $G = \mathbf{R}$

The closed subgroups are $t\mathbb{Z},\ t > 0$, $\{0\}$ and $\mathbb{R}$.

$$\lim_{t \to 0} t\mathbb{Z} = \mathbb{R}, \qquad \lim_{t \to \infty} t\mathbb{Z} = \{0\}.$$



When t is small,
$t\mathbf{Z}$ almost fills the line

So the space of closed subgroups is homeomorphic to the closed interval $[0, \infty]$

The space of closed subgroups of $\mathbb{R}^2$ is a 4-sphere containing a knotted 2-sphere.

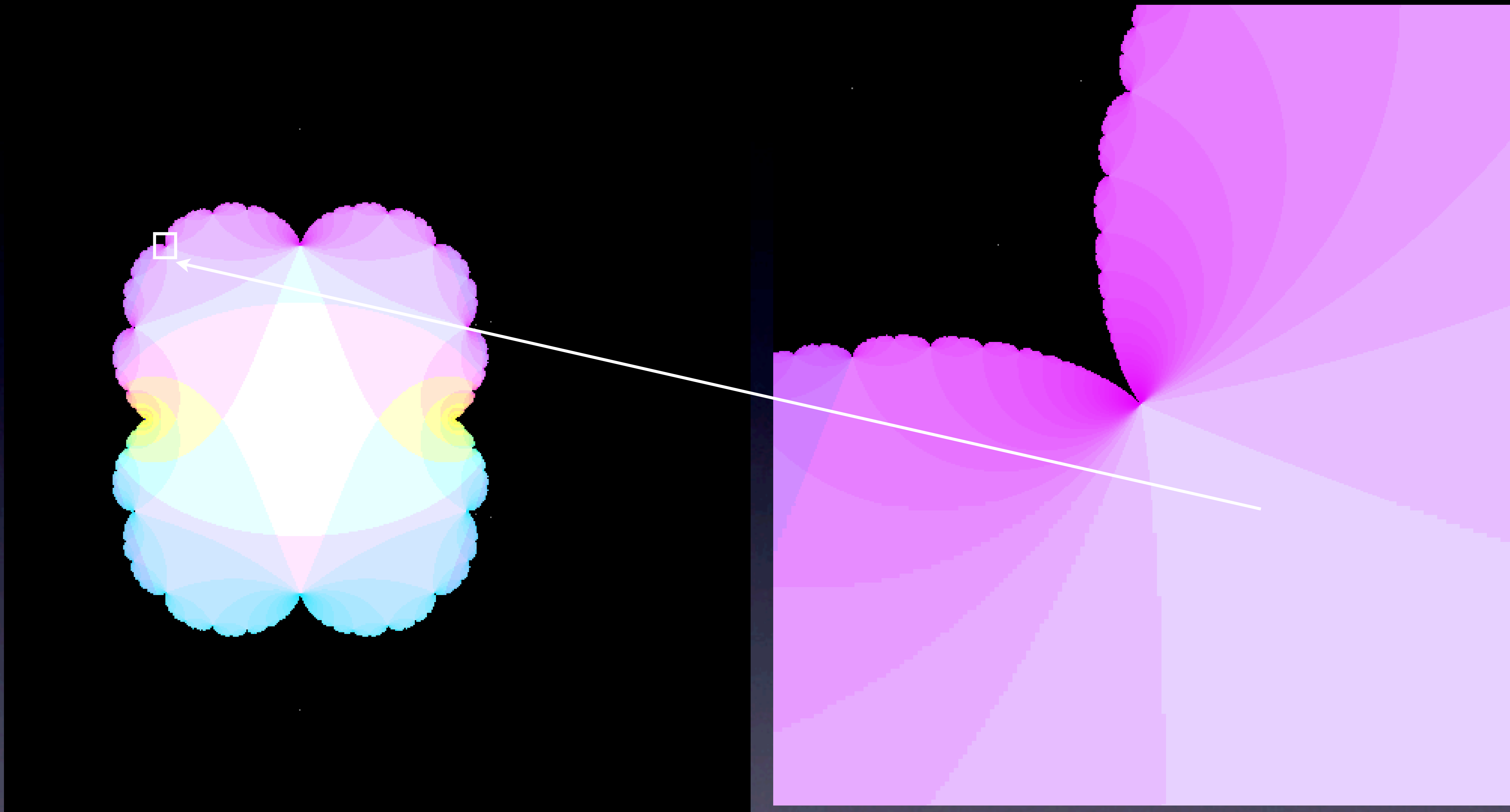
Nobody understands the set of closed subgroups of $\mathbb{R}^3$.

A Kleinian group is a discrete subgroup of $\mathrm{PSL}_2\mathbb{C}$.

The space of closed subgroups of $\mathrm{PSL}_2\mathbb{C}$
is presumably incomprehensibly complicated.

We will try to understand the closure of the set
of free subgoups on 2 generators
with parabolic commutator.

After appropriate normalization, the set of such groups
is two dimensional: the following pictures
represent sections of this space



In color are the groups in the slice
that correspond to discrete faithful representations.

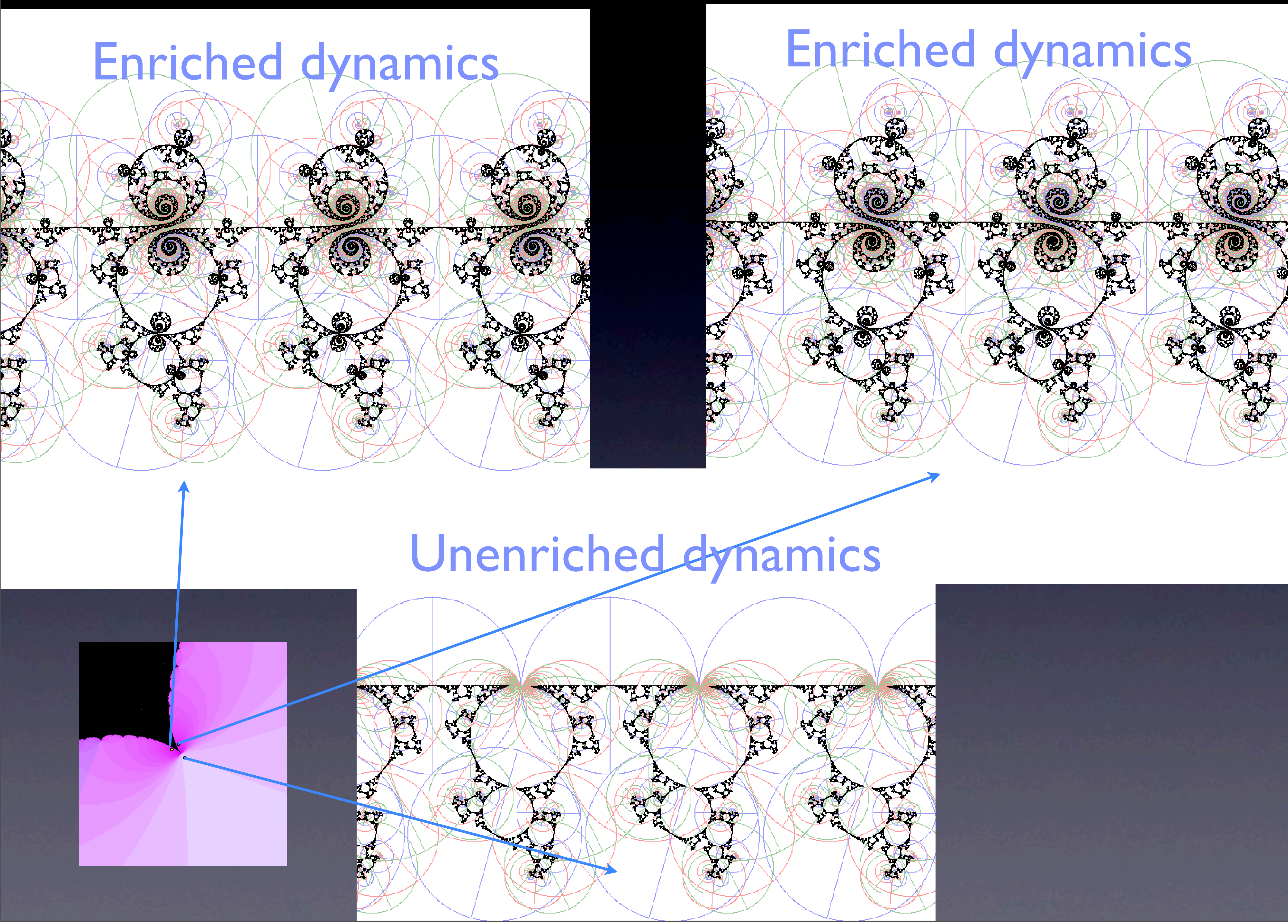
A Kleinian group Γ has a limit set Λ_Γ:
The closure of the set of fixed points of all its elements.

It is the analog of the Julia set of a polynomial.

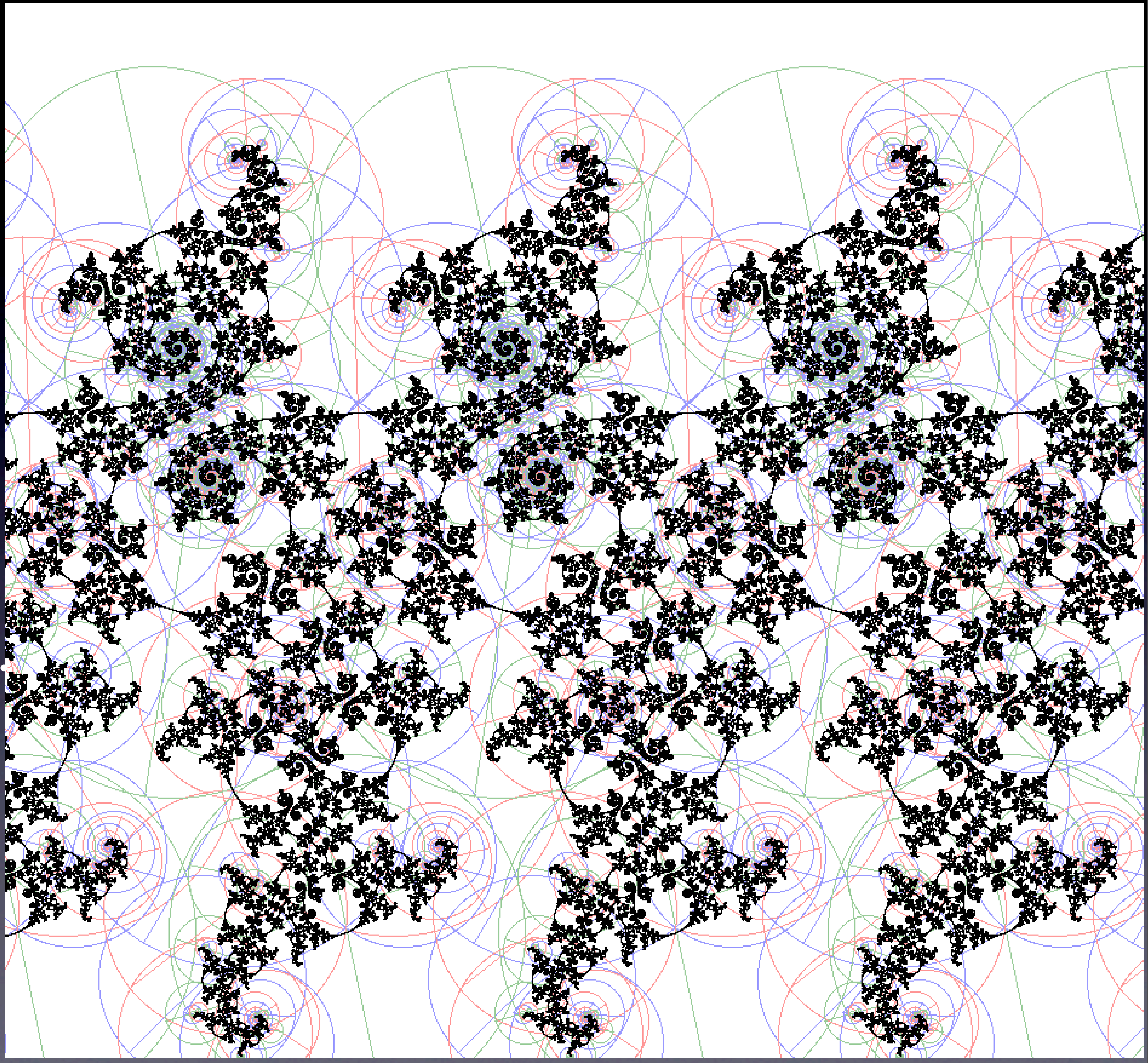
Just as the set K_c does not depend continuously on c

The map $\Gamma \mapsto \Lambda_\Gamma$ is not continuous.

We will show three very close groups whose limit sets are very far apart.


Enriched dynamics
Enriched dynamics
Unenriched dynamics

Such limit sets can be remarkably complicated.

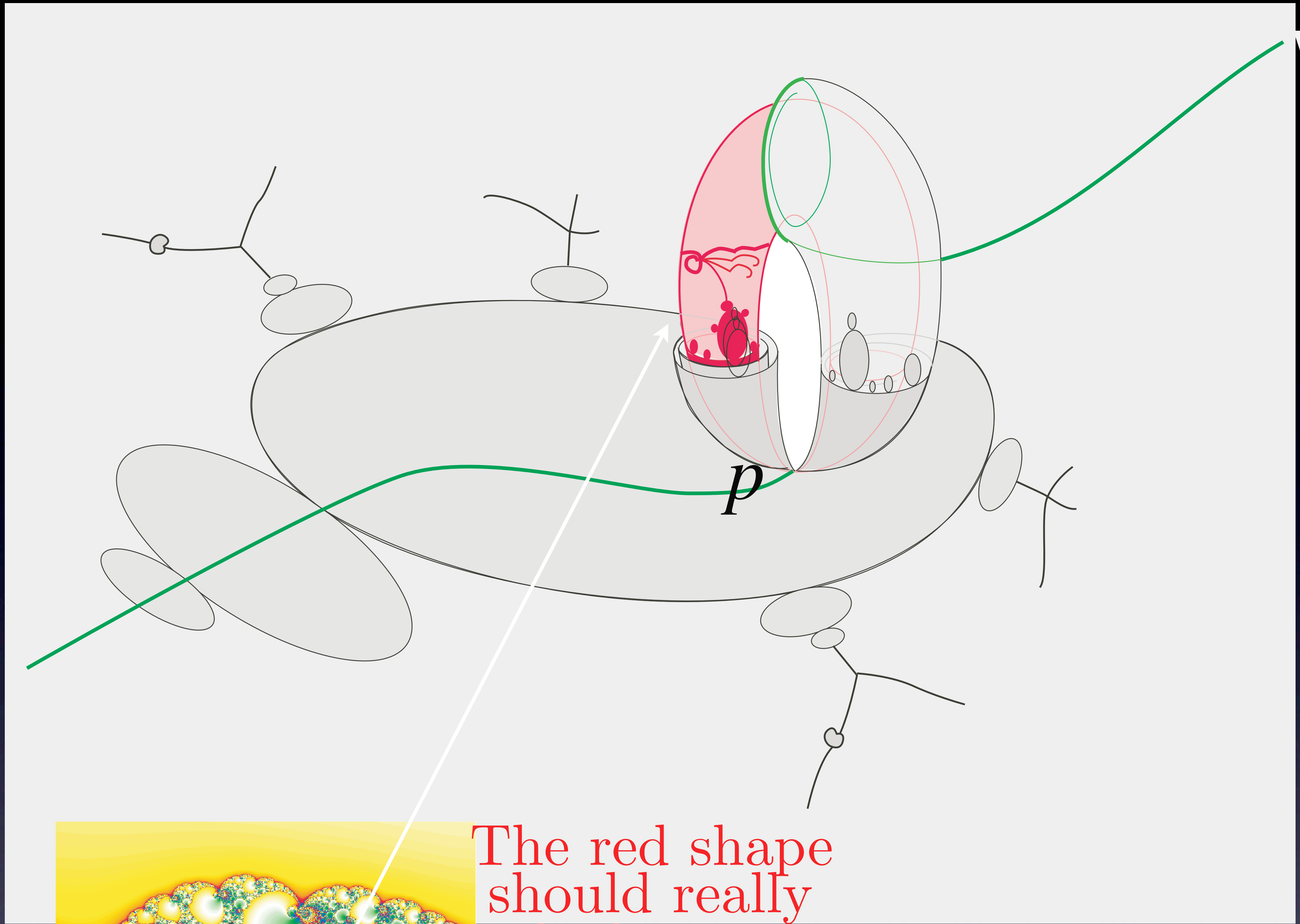


In the case of quadratic polynomials, our (conjectural) answer is:

The closure of $\{K_c, c \in \mathbb{C}\}$ in $\mathcal{C}(\mathbb{C})$ is the projective limit $\widehat{Quad}$ of all systems of finitely many projective blow-ups.

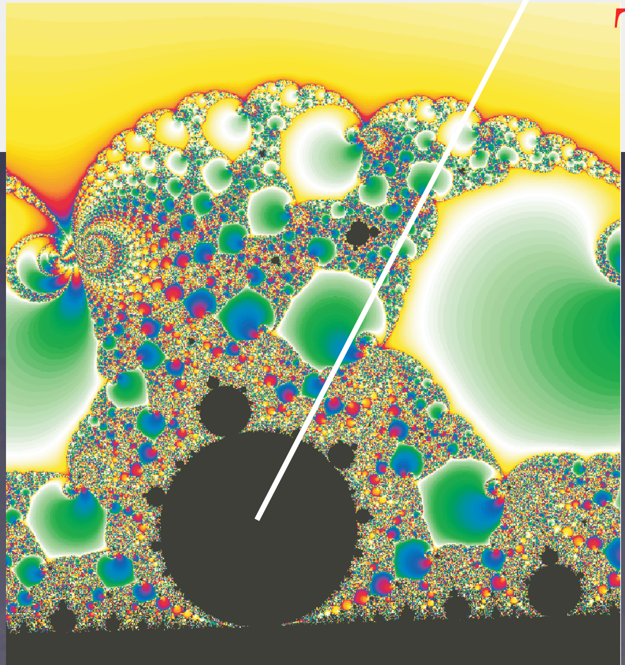
Before giving a precise definition of a parabolic blow-up I will show pictures of two examples

First the parabolic blow-up of $\mathbb{C}$ at $c = \frac{1}{4}$



We replace the cusp of the Mandelbrot set M by a copy of $\overline{\mathbb{C}/\mathbb{Z}}$ with its ends identified at the point p

The part of the real axis $c < \frac{1}{4}$ lands at p, whereas the part of the real axis $c > \frac{1}{4}$ spirals towards $\mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$



The red shape should really look like this

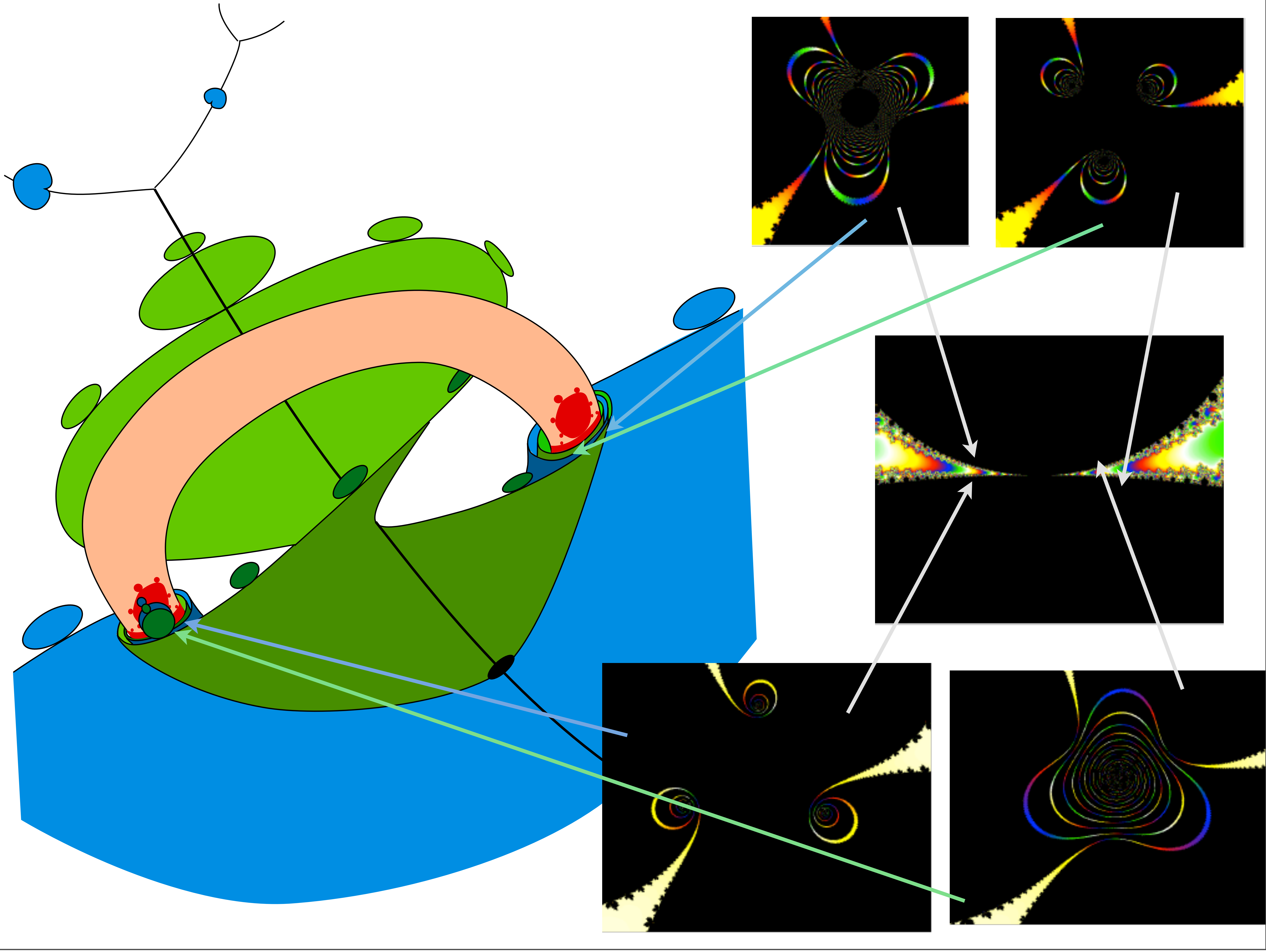
The copy of the cylinder $\mathbb{C}/\mathbb{Z}$ is called the *exceptional divisor* or *universal elephant (Douady)*

Next I will sketch the parabolic blow-up at $c = \frac{\lambda}{2} - \frac{\lambda^2}{4}$, with $\lambda = e^{2\pi i/3}$, the root of the "rabbit component".

Again we replace the point by a copy of $\overline{\mathbb{C}/\mathbb{Z}}$

This time we show how the boundary of the cardioid and of the rabbit component spiral towards the exceptional divisor.

They "cross": the part from the right of the cardioid spirals towards the same circle as the part from the left of the rabbit component.



Temporarily, let us assume that

1. We know how to define a parabolic blow-up.

2. That each point P of the projective limit $\widehat{Quad}$ of all finite systems of parabolic blowups corresponds to a "conformal dynamical system"

3. That each such conformal dynamical system has a "filled-in Julia set" K_P that is a compact subset of $\mathbb{C}$

Main theorem

The map $\widehat{Quad} \to \mathcal{C}(\mathbb{C})$ given by

$$P \mapsto K_P$$

is continuous.

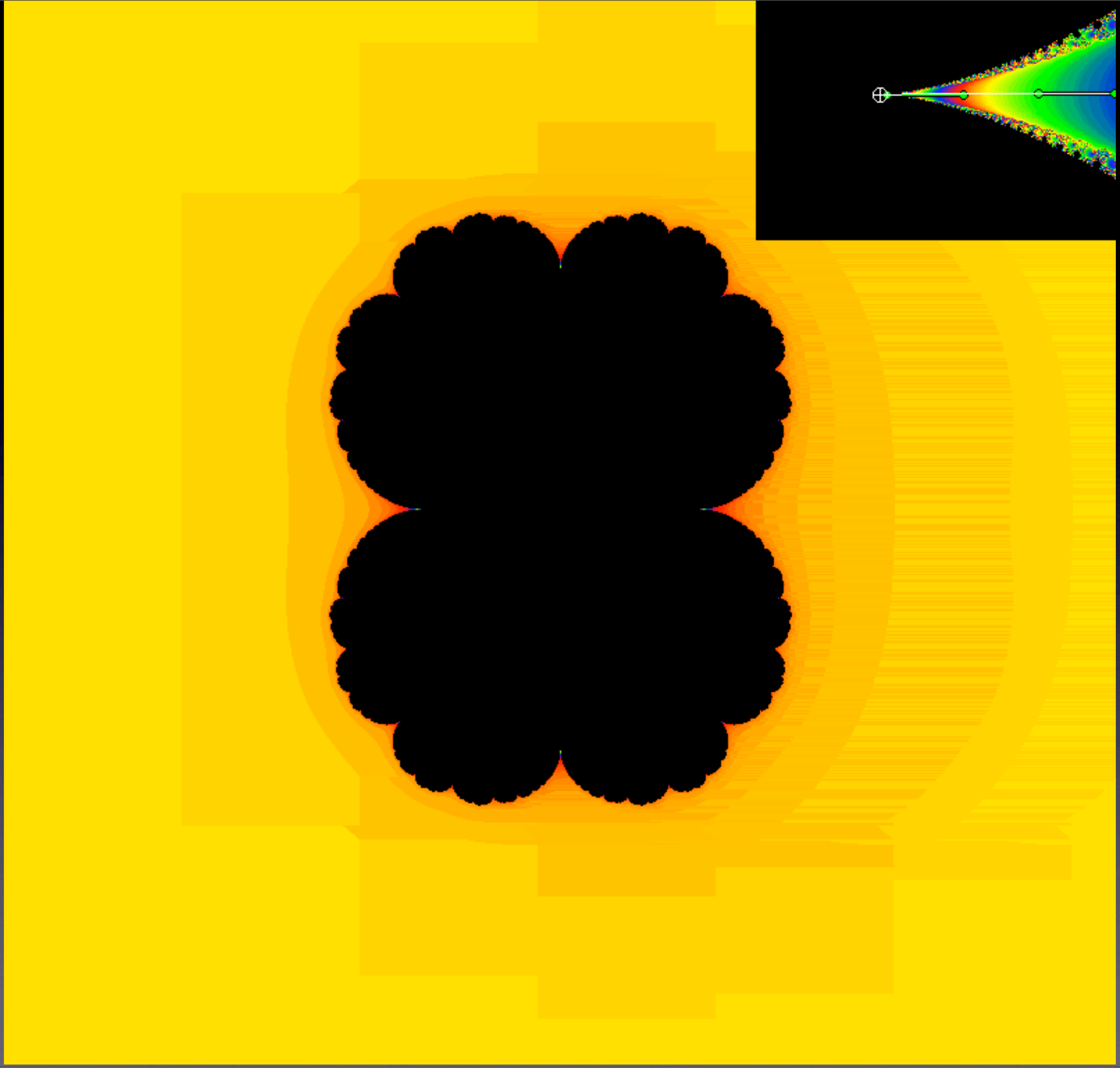
Conjecture:

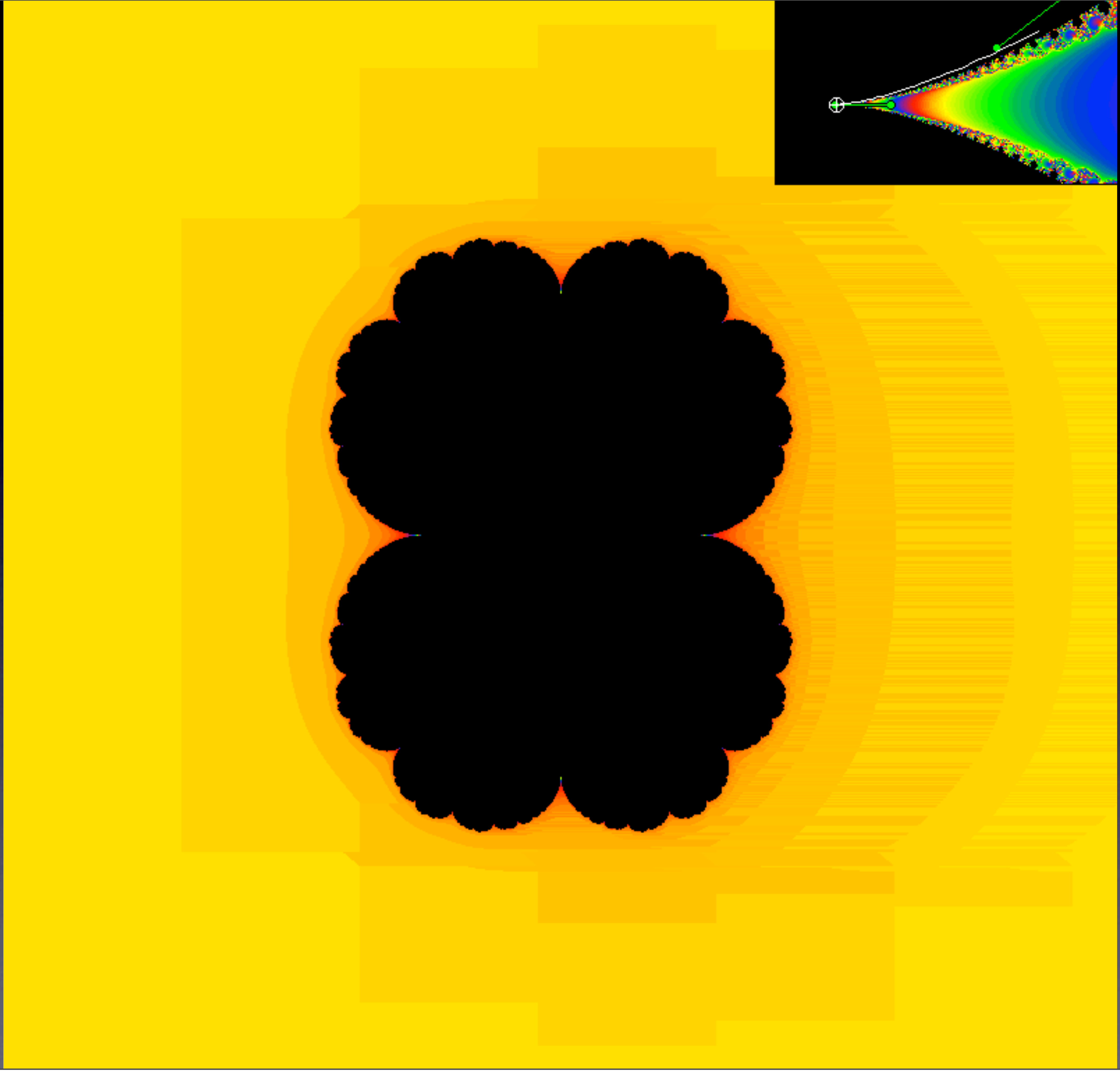
It is also injective, hence a homeomorphism to its image.

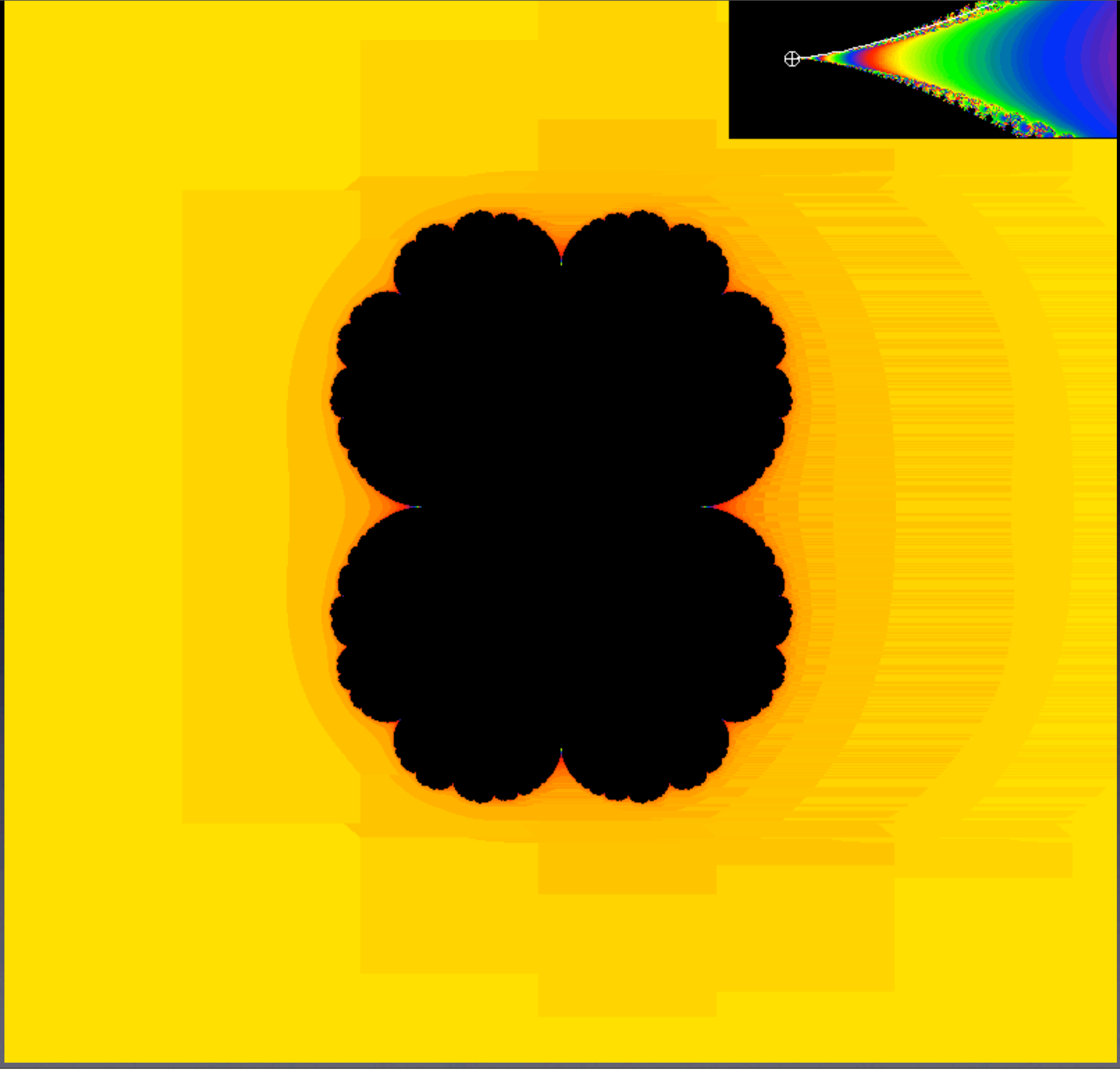
Why the spiraling behavior in parabolic blow-ups

Let us illustrate this spiraling behavior
with a few approaches to $c = \frac{1}{4}$

We approach different circles
on the exceptional divisor $\mathbb{C}/\mathbb{Z}$
if the multiplier m of the fixed point in $\operatorname{Im} z > 0$
(or the fixed point in $\operatorname{Im} z < 0$)
approach 1 on a circle
tangent to the line $\operatorname{Re} m = 1$





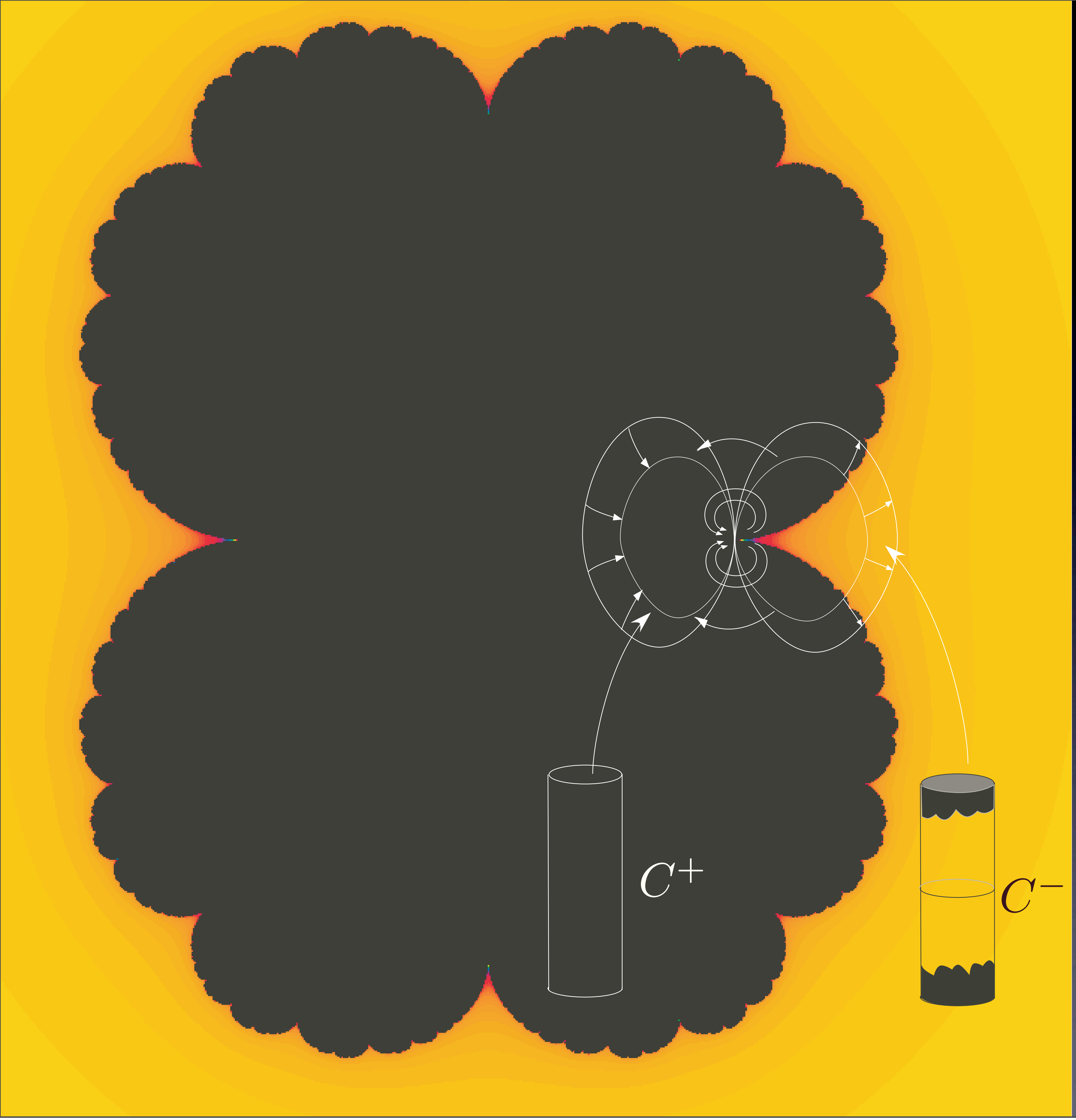


Douady and Lavaurs
investigated the limiting dynamics, using

Ecalle Cylinders

and

Horn Maps



The quotient of the filled in Julia set by the dynamics is a cylinder C^+

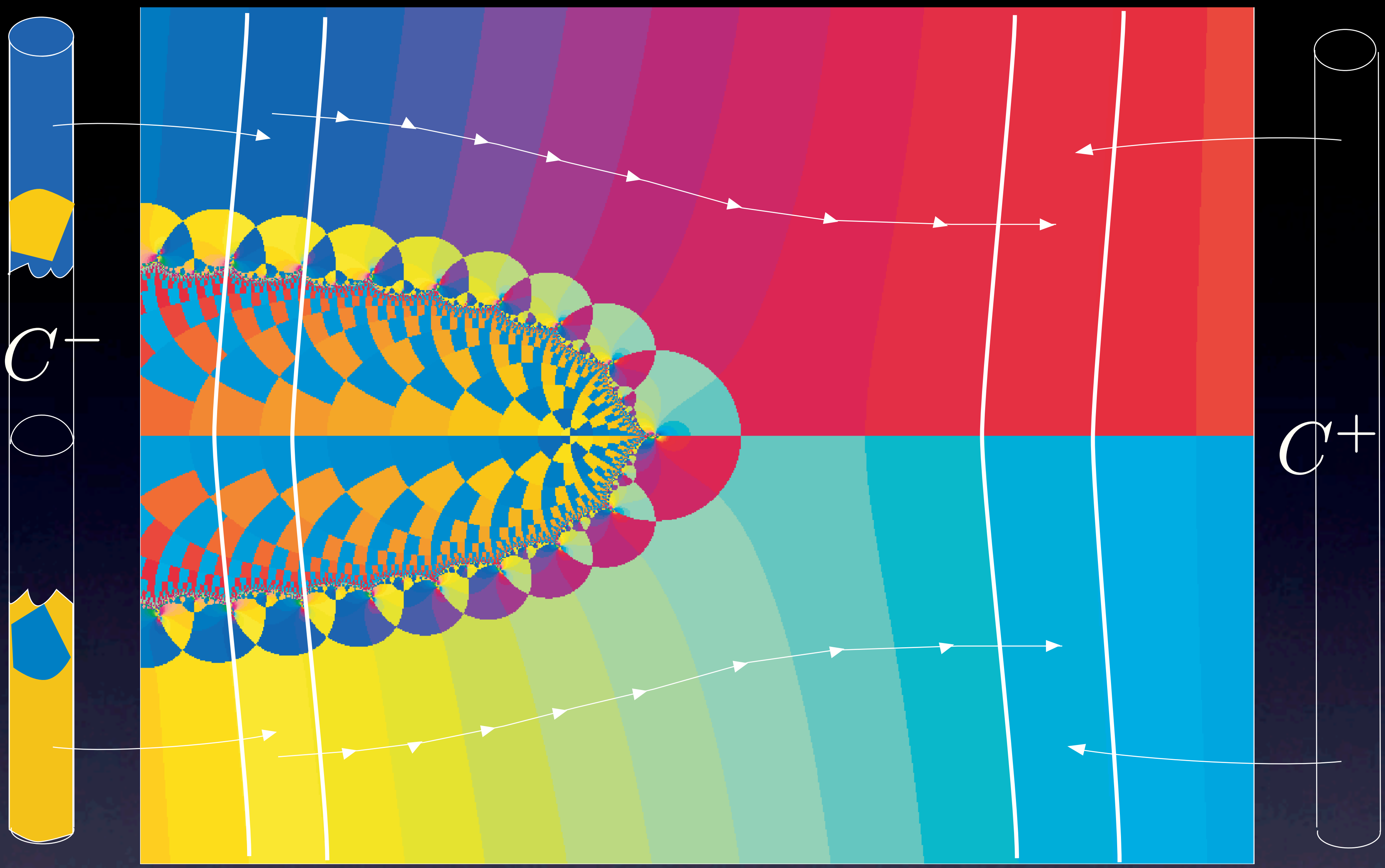
There is also an outgoing cylinder C^-

It is easier to visualize these cylinders

if the parabolic fixed point is placed at ∞

The map being iterated is now

$$z \mapsto z + 1 + \frac{1}{z-1}$$



The map

$$z \mapsto z + 1 + \frac{1}{z-1}$$

is conjugate to

$$z \mapsto z + 1$$

in a neighborhood of ∞

The quotient of

$\{z \mid \operatorname{Re} z < -R\}$ and $\{z \mid \operatorname{Re} z > R\}$

are both isomorphic to $\mathbb{C}/\mathbb{Z}$

Call these cylinders C^- and C^+

The dynamics induces *horn maps* from a neighborhood of the ends of C^- to C^+

To summarise

if p_c has a parabolic cycle
then there are two quotients C^+ and C^-
by the dynamics, and a horn map
$h : U \to \overline{C}^+$ defined in a
neighborhood U of the ends of C^-.

Adam Epstein has proved that horn maps are
analytic maps of *finite type:*
$h : U \to \overline{C}^+$ is a covering map
of all but finitely many points of C^+.

More generally, if X is a compact Riemann surface U is a Riemann surface and $f : U \to X$ is analytic, then f is of *finite type* if there is a finite set $Z \subset X$ such that

$$f : U - f^{-1}(Z) \to X - Z \text{ is a covering map.}$$

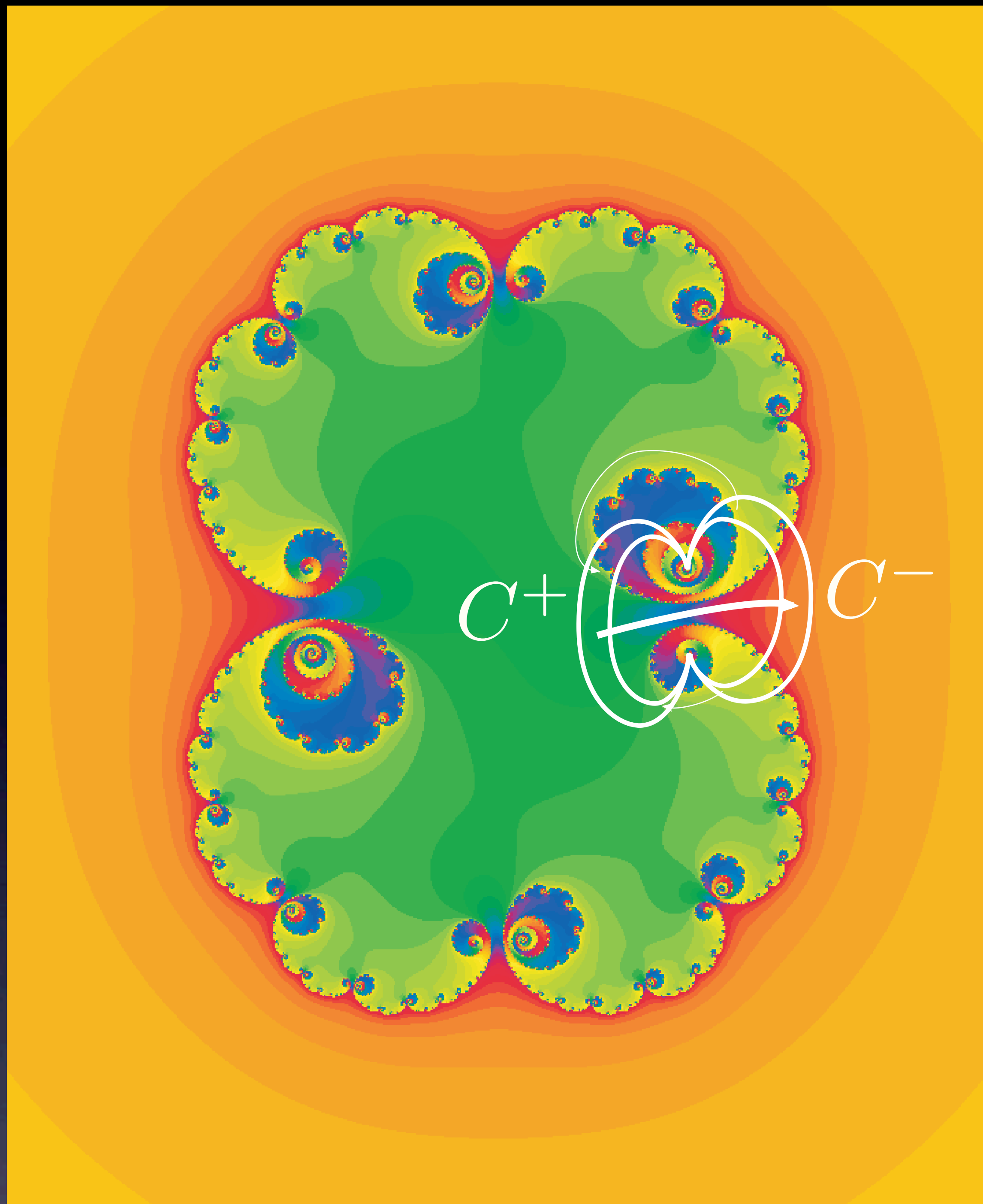
The map f is a *dynamical map of finite type* if $U \subset X$.

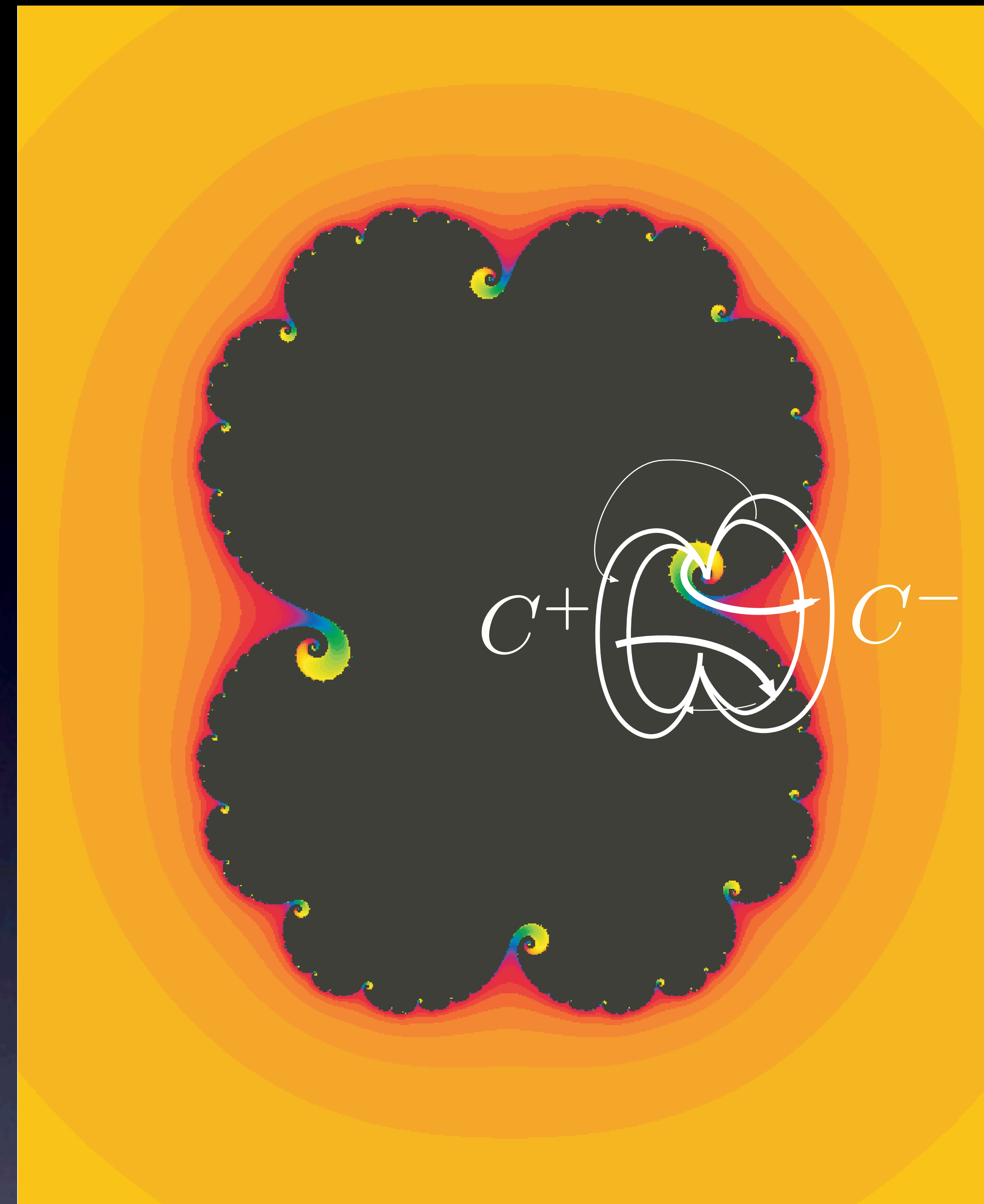
Adam also proves that if a dynamical map of finite type has only one critical value, then it has at most one parabolic cycle, and that parabolic cycle has an ingoing cylinder C^+, an outgoing cylinder C^-, a neighborhood $U \subset C^-$ of the ends of C^-, and a horn map $h : U \to \overline{C}^+$ of finite type.

These cylinders still exist for c in a neighborhood of the parameter value c_0 for which p_{c_0} has a parabolic cycle

The cylinders exist for all values of the parameter with a bit of ambiguity when the cycles emanating from the parabolic cycle are attracting with real derivatives

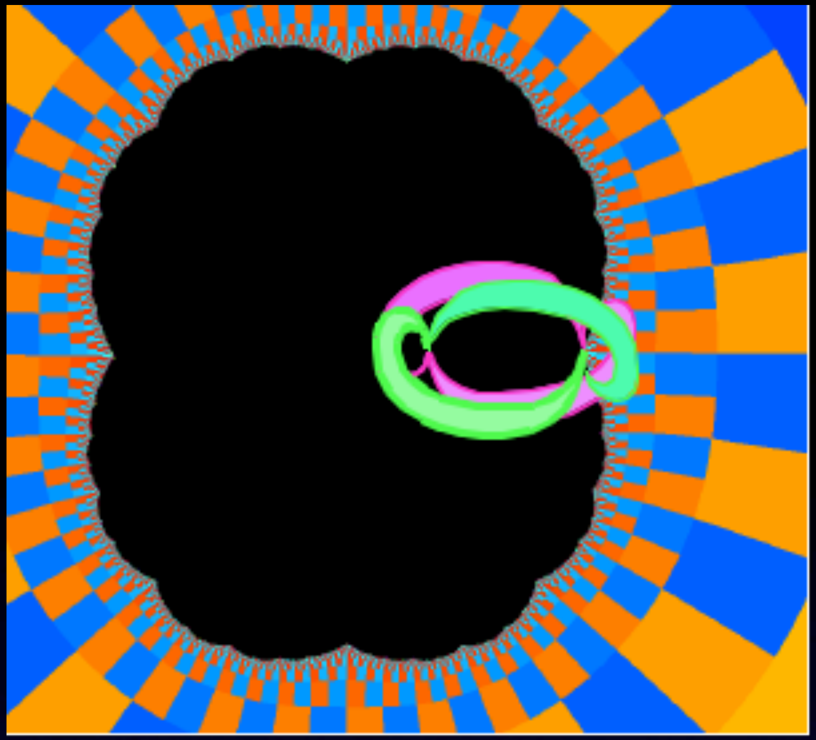
We illustrate this when $c_0 = \frac{1}{4}$.



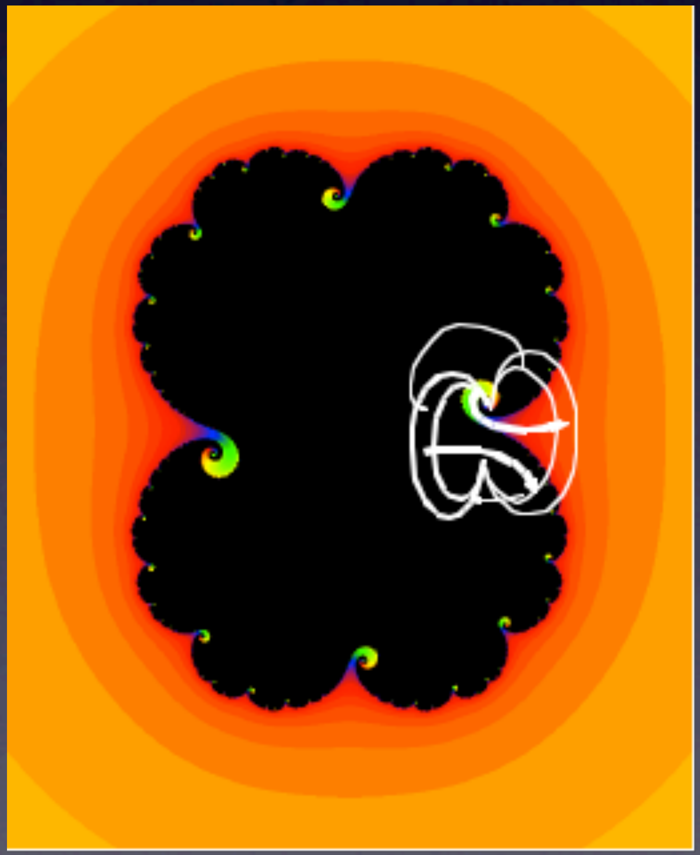


In these two picture of Julia sets K_c
with c close to $c_0 = 1/4$,
we see cylinders C^+ and C^-,
with horn maps defined near
the ends of C^-,

and isomorphisms
$C^+ \to C^-$
referred to as
as *Lavaurs maps*, or
going through the egg beater



In the case where the multiplier of one cycle emating from the parabolic cycle there are two possible sets of cylinders C^+ and C^-, each of which comes with its own horn map.



They are limits of cylinders where the multiplier has small imaginary part

In the limit the Lavaurs map maps all C^+ to one of the ends of C^-.

Enriching the dynamics of a polynomial

A sequence of polynomials p_n may converge to a polynomial p_∞,

While a sequence of interates

$$p_n^{\circ m_n}$$

may converge (somewhere) as $n \to \infty$ and $m_n \to \infty$ to a map f not originally part of the dynamics.

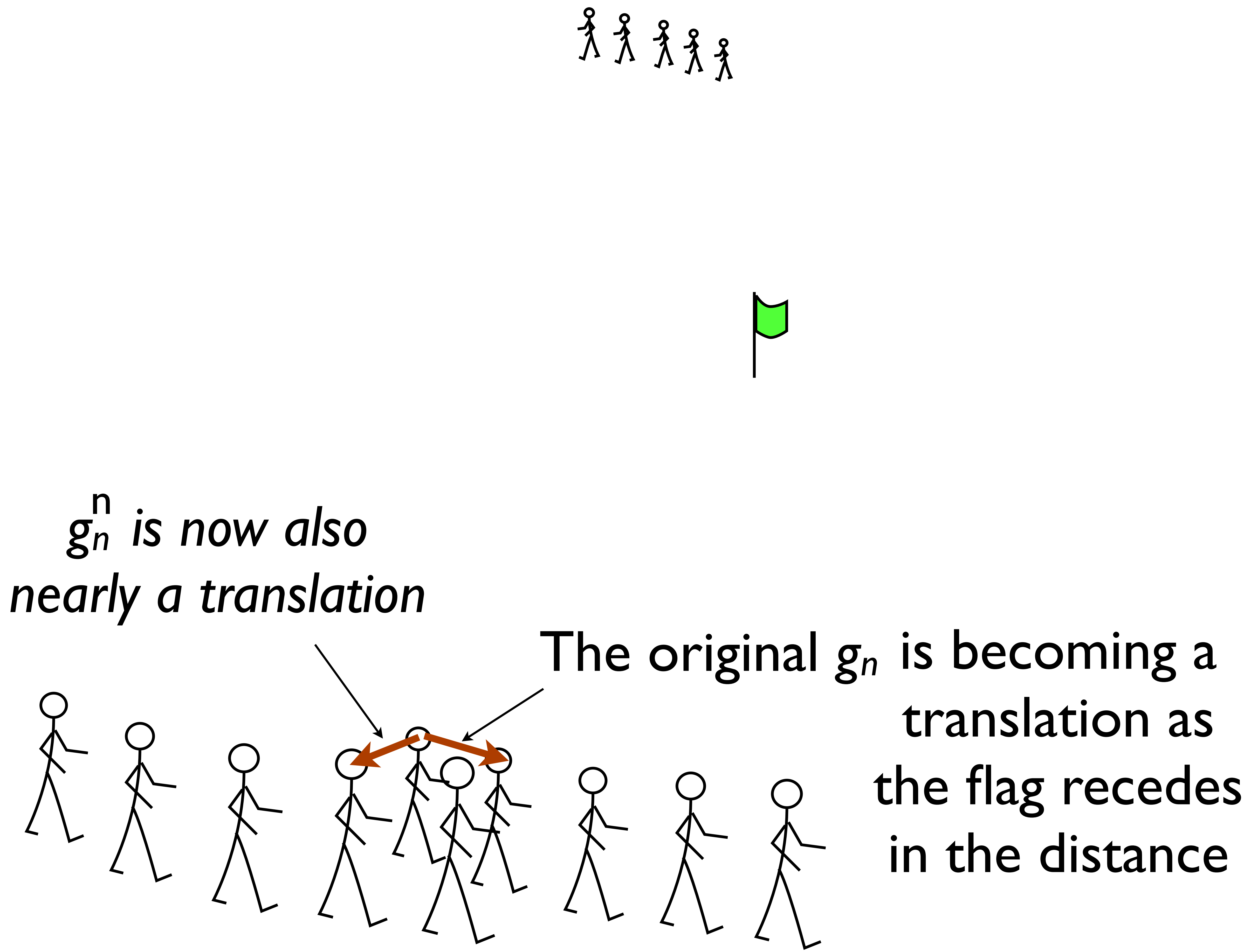
Enriching the dynamics of a Kleinian group

Something similar may happen to a Kleinian group.
Suppose a sequence of Kleinian groups G_n
are generated by $g_{1,n}, \ldots, g_{k,n}$.

Suppose further that each sequence $n \mapsto g_{i,n}$
converges in $\mathrm{PSL}_2\mathbb{C}$ to some $g_{i,\infty}$.
Then $g_{1,\infty}, \ldots, g_{k,\infty}$ generate a group G_∞.

But it may happen that for some sequence $m_n \to \infty$
the sequence $g_{1,n}^{m_n}$ converges to some $g \in \mathrm{PSL}_2\mathbb{C}$

Let us see how that might happen


g_n^n is now also nearly a translation
The original g_n is becoming a translation as the flag recedes in the distance

The same in formulas

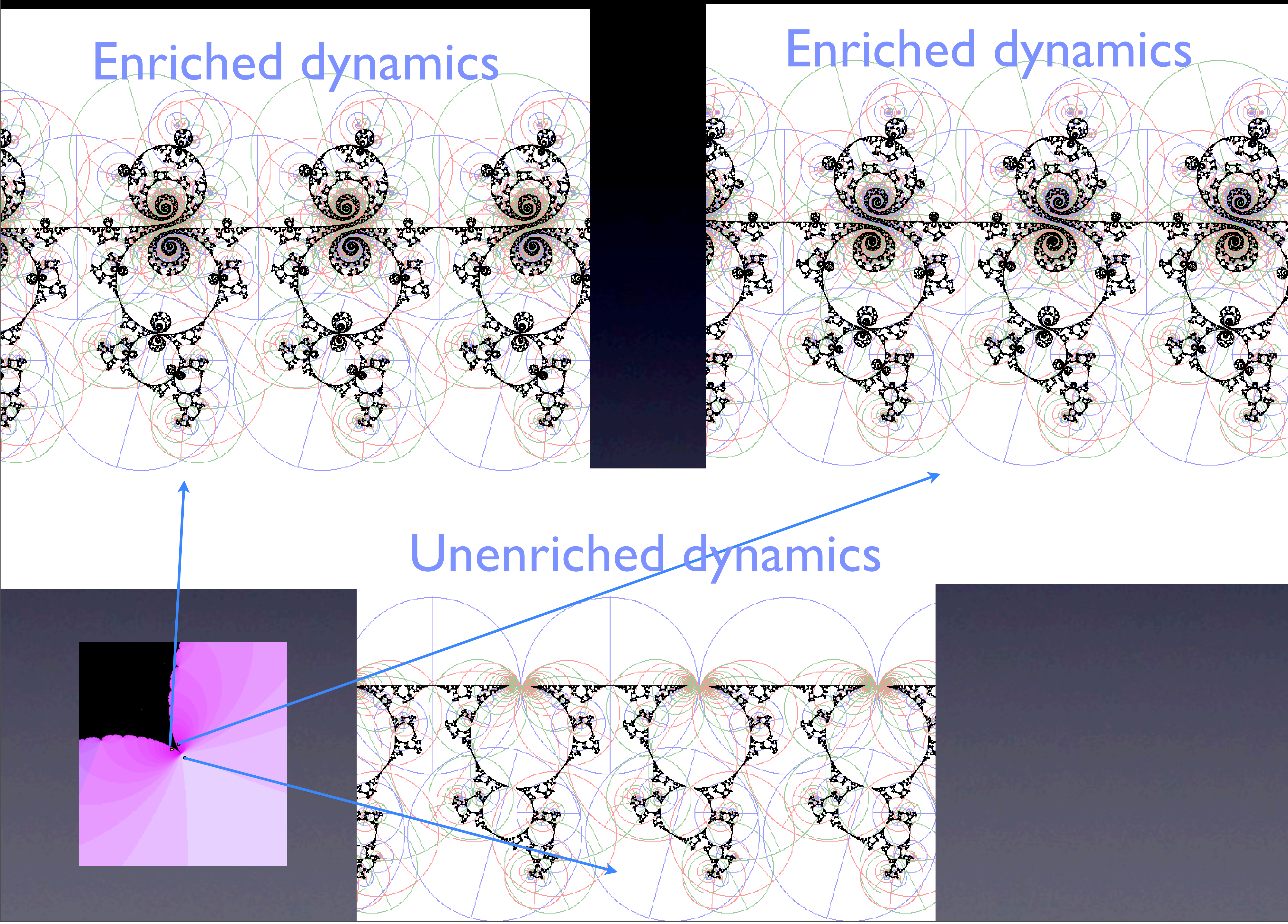
Let $f_n(z) = \left(1 - \frac{a}{n^2}\right)e^{2\pi i/n}$ (z-n)+n

$$Then$$

$$\lim_{n\to\infty} f_n(z) = z + 2\pi i$$

but

$$\lim_{n\to\infty} f_n^{\circ n}(z) = z + a$$


Enriched dynamics
Enriched dynamics
Unenriched dynamics

Defining the parabolic blow-up

The ordinary blow-up of $0 \in \mathbb{C}^2$ is the set

$$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2, l \in \mathbb{P}^1 \right) \mid \begin{pmatrix} x \\ y \end{pmatrix} \in l \right\}$$

We want an analogous definition

of the parabolic blow-up

Suppose that p_{c_0} has a parabolic cycle.
Let V be a neighborhood of c_0 sufficiently small that the cycles emanating from the cycle are well defined, and let $V^* \subset V$ be the subset where no such cycle is attracting with real multiplier

For each $c \in V^*$ we have cylinders C_c^+ and C_c^- which form a trivial principal bundle under $\mathbb{C}/\mathbb{Z}$

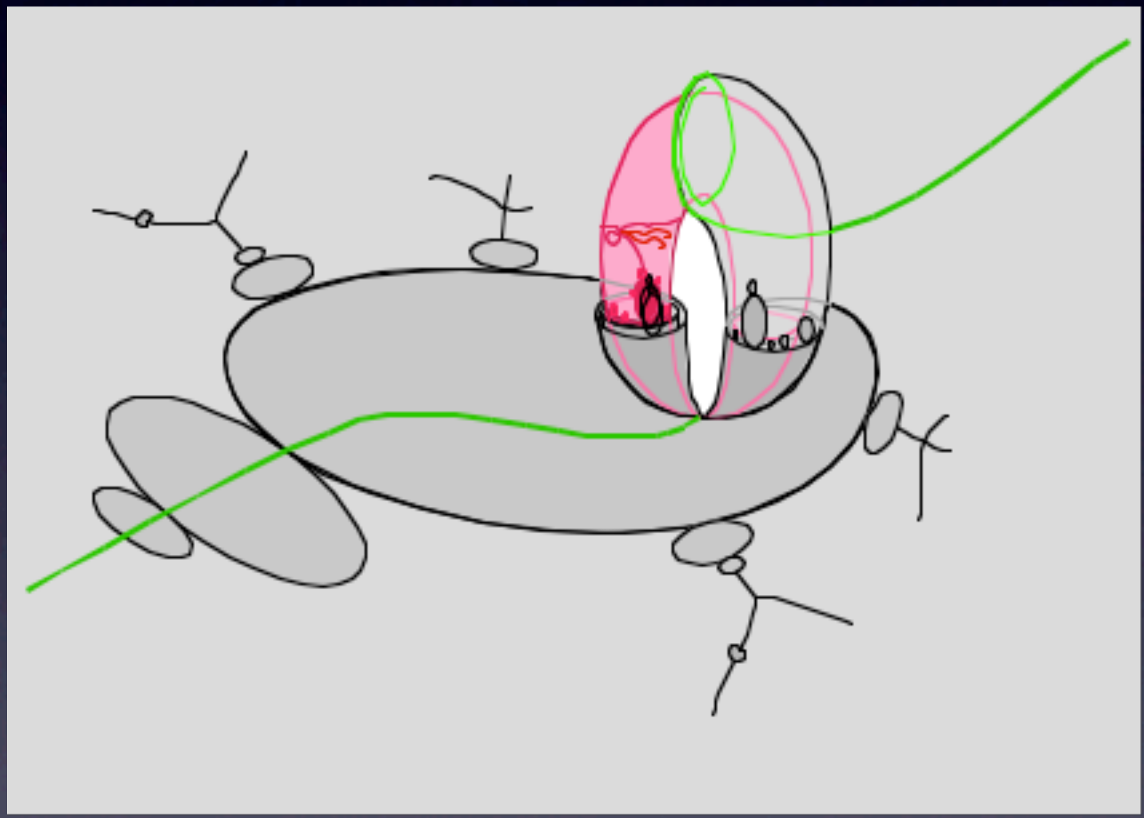
Moreover for all $c \in V^*$, $c \neq c_0$, there is a natural isomorphism $L_c : C^+ \to C^-$

We define the parabolic blowup of $\mathbb{C}$ at c_0

to be the closure in $V \times \mathrm{Isom}\,(C^+, \mathbb{C}^-)$

of all pairs (c, L_c).

Thus in the picture
the pink "croissant"
is $\mathrm{Isom}\,(C^+, C^-)$
and a sequence $i \mapsto c_i$
converges to a point
$\phi \in \mathrm{Isom}\,(C^+, C^-)$



if the Lavaurs maps L_{c_i} converge to ϕ.

If $c \uparrow 1/4$, you converge to the identified ends
of $\mathrm{Isom}\,(C^+, C^-)$.

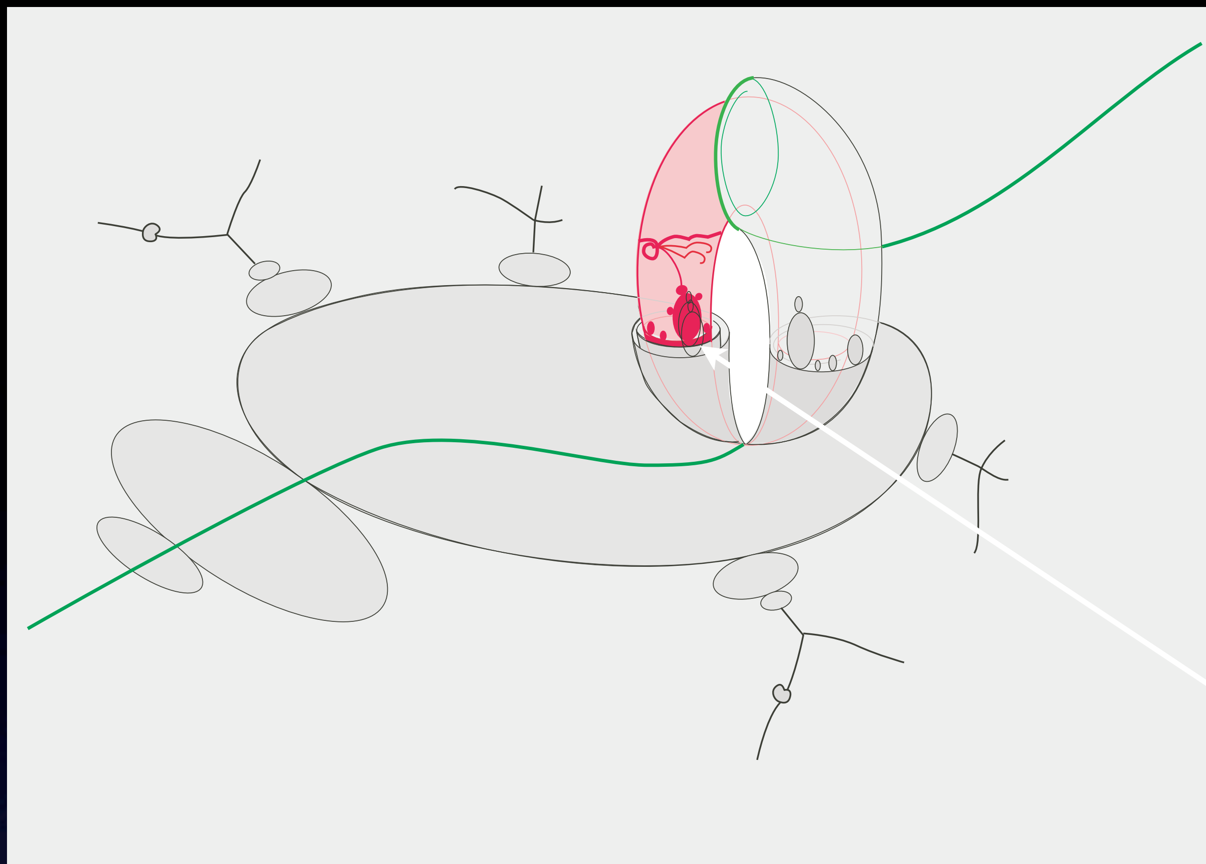
This is just the beginning of the story

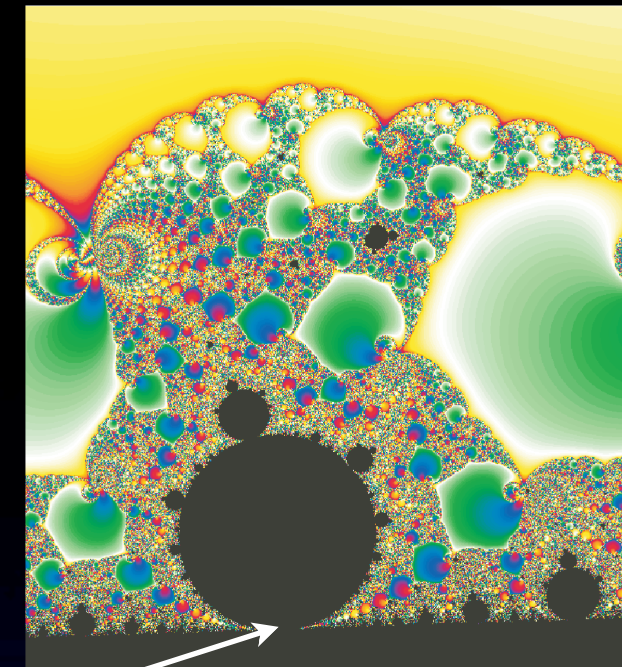
We may have a first dynamical system p_{c_0},
with a parabolic cycle. Then for each
$L \in \mathrm{Isom}\,(C^+, C^-)$ we can define another

$$L \circ h : U \to C^-$$

where $U \subset C^-$ is the domain of the horn map

This composition $L \circ h$ may itself
have parabolic cycles,
and we can iterate the process.





The second order parabolic is an accumulation of first order parabolics, and infinitely many of these must be blown up at the same time.

This leads to the definition of a parabolic tower.

A parabolic tower is a sequence (finite or infinite) of dynamical maps of finite type $f_i : U_i \to X_i$.

Each f_i is of the form $L \circ h$ where h is the horn map associated to a parabolic cycle of f_{i-1} and L is an associated Lavaurs isomorphism.

In our case f_0 is required to be a quadratic polynomial.

The set of parabolic towers
above quadratic polynomials
is exactly the projective limit of all finite
systems of parabolic blow-ups
starting with a quadratic polynomial.

This projective limit $\widehat{Quad}$ comes with a topology.
It can also be understood in terms of parabolic towers.
Adam has shown how to associate
a "conformal groupoid" to each parabolic tower
and how to give the set of such groupoids
the *Fell topology*, the appropriate
variant of uniform convergence on compact sets.

These groupoids
(Adam calls them conformal dynamical systems)
have Julia sets and filled in Julia sets
that have the same semicontinuity properties
as ordinary Julia sets and filled in Julia sets.

Adam proves (in his thesis, 1994) that for
infinite towers the Julia sets and
the filled in Julia set coincide.
Since one is upper semi continuous
and the other lower semi-continuous
at infinite towers both are continuous.

It might seem that this is so complicated as to be useless!

But we do gain some insight from the "projective limit of parabolic blow-ups" approach.

For instance:

We can compute the Čech cohomology

$$H^*\left(\widehat{Quad};\mathbb{Z}\right).$$

Let P be the set of quadratic polynomials with a parabolic cycle

We denote by $\mathbb{Z}^{(P)}$ the sum of copies of $\mathbb{Z}$
(All but finitely many entries 0)

We denote by $\mathbb{Z}^{P}$ the product of copies of $\mathbb{Z}$
(arbitrary entries)

$$H^k(\widehat{Quad};\mathbb{Z}) = 0 \text{ if } k > 2$$

$$\mathbb{Z}^{(P)} \subset \; H^2(\widehat{Quad};\mathbb{Z}) \subset \mathbb{Z}^P$$

$$\mathbb{Z}^{(P)} \subset \; H^1(\widehat{Quad};\mathbb{Z}) \subset \mathbb{Z}^P$$

Note that P is an understandable set:

It is the quotient of the rational angles with odd denominator
by the "companion" equivalence relation.

Another application

The proper transform of the boundary of the cardioid is homeomorphic to the set of finite or infinite sequences of the symbols $1, 2, \ldots, \infty$.

We make the standard identification of continued fractions

If $1 < a_n \leq \infty$, then $[a_1, \ldots, a_n] = [a_1, \ldots, a_n - 1, 1]$.

An N-neighborhood of a sequence $A = [a_1, a_2, \ldots]$ is the set of sequences at most as long as A and whose first N entries coincide with those of A except that any entries ∞ can be replaced by entries $> N$.

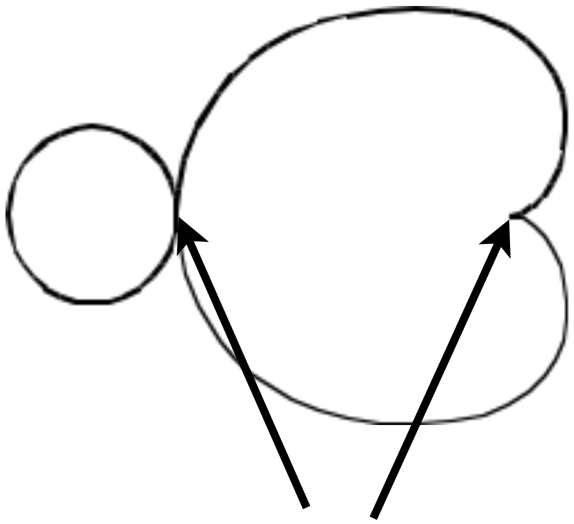
These sequences should be thought of as continued fractions:

$$[a_1, a_2, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

The number of symbols ∞ is the height of the corresponding parabolic tower.

We allow the empty sequence $[\ ]$ to stand for the angle $0 \in \mathbb{Q}/\mathbb{Z}$

Some pictures should illustrate the construction



We will blow up these two points