Complex Dynamics with focus on the real part

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(joint work with Han Peters)

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It is important to ask whether the Mathematical results still describe the situation accurately. In this paper, we will investigate rigorously whether one can recover precise results when one suppresses some variables.
We will do this in the case of one dimensional complex dynamics, where the exact theory is highly developed.

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Theorem

Takens Let M be a compact manifold of dimension m. For pairs (ϕ, y),
$\phi : M \to M$ a $\mathcal{C}^2$ diffeomorphism
and $y : M \to \mathbb{R}$ a $\mathcal{C}^2$ function,
it is a generic property that the map $\Phi_{(\phi,y)} : M \to \mathbb{R}^{2m+1}$, defined by

$$\Phi_{(\phi,y)}(x) = (y(x), y(\phi(x)), \dots, y(\phi^{2m}(x)))$$

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Hence all information about the original dynamical system can be retrieved from the suppressed dynamical system.

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Is it at all possible to say anything? What kind of precise questions can one ask? What kind of theorems can one try to prove? What are the right fruitful concepts?

To approach this problem, we focus on a system which is mathematically well understood and then suppress one variable. We chose complex dynamics in one complex dimension. The dynamical system is that of a complex polynomial $P : \mathbb{C} \to \mathbb{C}$. Let $z_0 = x_0 + iy_0$ be a point, and let $P^{\circ n}(z_0) = z_n = x_n + iy_n$. This is the orbit of x_0 and the study of dynamics is the study of orbits.

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Our goal is to restrict attention to the sequence of x_n. Try to find Theorems about these sequences. But to actually prove these theorems we feel free to use all we know about the complex dynamics.

So we have a real dynamical system where complex analysis is used essentially to prove the theorems. In fact as we found out during this project we need to use the theory of complex analysis in higher dimension as well. This is especially since we need properties of complex varieties.

There are some complex polynomials where the orbits of the real parts does not at all reflect the dynamics of the original system. One such example is $P(z) = -iz^2 + ia$ where a is a large real number. This maps the $y-$axis to itself and the interesting dynamics happens on the $y-$ axis. So all we see is the real map $0 \to 0$. The map on the y axis is $y \to y^2 + a$. For such a map all the chaotic behaviour happens on a Cantor set in the y axis.

Lemma

Let P be a complex polynomial of degree $d \geq 2$. Then there can be at most one vertical line which is mapped to a vertical line. For all other lines, the number of points which is mapped to any other given vertical line is at most d.

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Lemma

There exists an integer N so that if z_0, w_0 are two points with orbits $z_n = x_n + iy_n, w_n = u_n + iv_n$ and $x_n = u_n$ for all $n \leq N - 1$, then $x_n = u_n$ for all n.

Proof.

We consider the polynomials $Q_n =$Re$(P^n(x+iy)-P^n(u+iv))$, $Z_n=\{Q_0=\cdots=Q_n=0\}$. If we complexify, these are holomorphic polynomials and since $Z_n \supset Z_{n+1}$ these must be stationary. $\square$

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Let $\Phi : \mathbb{C} \to \mathbb{R}^N$, $\Phi(z) = (x, x_1, \ldots, x_{N-1})$ Set $S = \Phi(\mathbb{C})$. Then we have a semiconjugacy with the map $Q : S \to S$, $Q(x_0, \ldots, x_{N-1}) = (x_1, \ldots, x_N)$

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Unfortunate difficulty: In the example $P = -iz^2 + ia$ the map $\overline{\Phi}$ is not continuous. Also $S \cup \infty$ is not compact.

Let $P(z) = a_d + \cdots$ be a polynomial of degree d for which $a_d z^d$ is purely imaginary when z is imagninary. Then P is said to be exceptional. Otherwise, P is non-exceptional.

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One useful concept is that of mirrors. Two points z_0, w_0 are mirrors of each other if the orbits $x_n = u_n$ for all n. So these points are indistuingshable. If $P(z) = \sum a_j z^j$ and all the a_j are real, then the points $z, \overline{z}$ are mirrored.

Let $M = \{(z, w) \in \mathbb{C}^2\}$ where z, w are mirrored.

Lemma

If P does not have real coefficients, then M is real analytic of dimension at most 1 (if we add the diagonal we get dimension 2.) If P has real coefficients we get in addition the set $(z, \overline{z})$.

Sketch of proof. Suppose there exists open sets $U, V \subset \mathbb{C}$ and a real analytic diffeomorphism $\Psi : U \to V$ such that U, V are mirrored. Then one complexifies $z = x + iy, w = u + iv$ as complex variables z', w', u', v' in $\mathbb{C}^4$. Then the equation of being mirrored becomes holomorphic. One then can use the complexification of $\text{Re}(P(x + iy))$ to get a holomorphic function on $\mathbb{C}^2$. Mirroring sends (x', y') to $(x', \lambda(x', y'))$. This can be continued by analytic continuation to $|x'|$ large. Restricting back to $\mathbb{R}^4$ one shows that this continuation preserves reals for $|x|$ large. Hence one gets an open set of mirrored points arbitrarily close to ∞ in $\mathbb{C}$. In $\mathbb{C}$ one has Bottcher coordinates so that $P(z) = az^d$. This allows after calculations to conclude that the original P has real coefficients, so the mirroring was just z and $\overline{z}$.

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Theorem

(Brolin, Lyubich, Mane) Let P be a complex polynomial of degree $d \geq 2$. Then there is a unique probability measure μ on $\mathbb{C}$ which is invariant, ergodic and of maximal entropy, $\log d$.

Our main result is that the same is true for the real dynamical system $\overline{Q} : \overline{S} \subset \overline{\mathbb{R}}^N \to \overline{S}$, measure ν. But we need to assume that P is non exceptional.

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Explain words: Let $F : X \to X$. A probability measure λ is **invariant** if $\lambda(F^{-1}(E)) = \lambda(E)$ for all measurable sets E.

An invariant measure is **ergodic**: If $E \subset X$ and $F^{-1}(E) = E$ except for sets of measure 0. Then $\lambda(E)$ equals 0 or 1.

Metric entropy: Let $U \subset X \times X$ be an open neighborhood of the diagonal. For $n \geq 1$, Let

$B(x, U, n) = \{(y \in X; (x,y), (F(x), F(y)), \dots, (F^{n-1}(x), F^{n-1}(y)) \in U.\}$

The entropy at x:

$$h_\lambda(F, x, U) = \liminf_n \frac{-1}{n} \log(\lambda(B(x, U, n)))$$

The smaller the balls, the larger the entropy. Large entropy means the points spread out faster when you iterate.

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Result: The relation between μ and ν. The measure ν is the push-forward of μ : For $E \subset S$, $\nu(E) = \mu(\Phi^{-1}(E))$.

THANK YOU FOR LISTENING