Tropical Complex Dynamics

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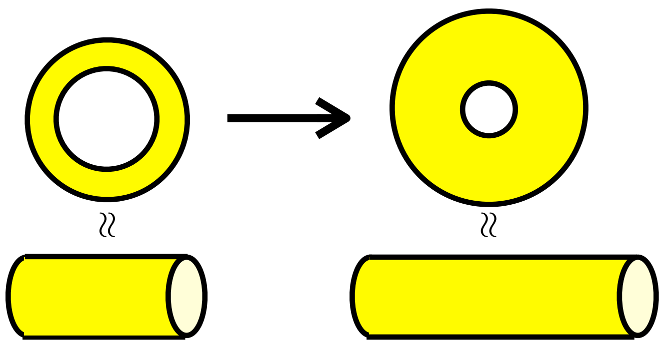
North-American Workshop in Holomorphic Dynamics
JackFest 2016
Cancún, Mexico
June 2, 2016

Why Tropical?

- CANCÚN!

- Tropical Geometry ⟵ Algebraic Geometry
  piecewise linear eqns polynomials eqns
  Degeneration, or the boundary of Moduli space

- Tropical Complex Dynamics ⟵ Complex Dynamics
  piecewise linear maps on trees
  Degeneration, or the boundary of Moduli space



Tropical Geometry (non-expert's view)

Tropical semi-ring

For $x, y \in \mathbb{R}$, define new operations:

$$x \text{ “}+\text{” } y := \max\{x, y\}, \quad x \text{ “}\times\text{” } y := x + y$$

Interpretation: let $x = \log_T \tilde{x}$, or $\tilde{x} = T^x$, etc. with $T > 1$.

Then when $T \to \infty$ or $t = \frac{1}{T} \to 0$,

$$x \text{ “}+\text{” } y = \lim_{T\to\infty} \frac{\log(T^x + T^y)}{\log T} = \lim_{T\to\infty} \log_T(\tilde{x} + \tilde{y}),$$

$$x \text{ “}\times\text{” } y = \log_T(\tilde{x}\tilde{y}).$$

“Tropical varieties” defined by piecewise linear equations in terms of “$+$” and “$\times$” can be interpreted as a limit of algebraic varieties X_t as $t \to 0$.

Objects appearing in the boundary of the moduli space.

Formulation in terms of the field of Puiseux series

Counting objects (enumerative geometry)

Tropical Complex Dynamics

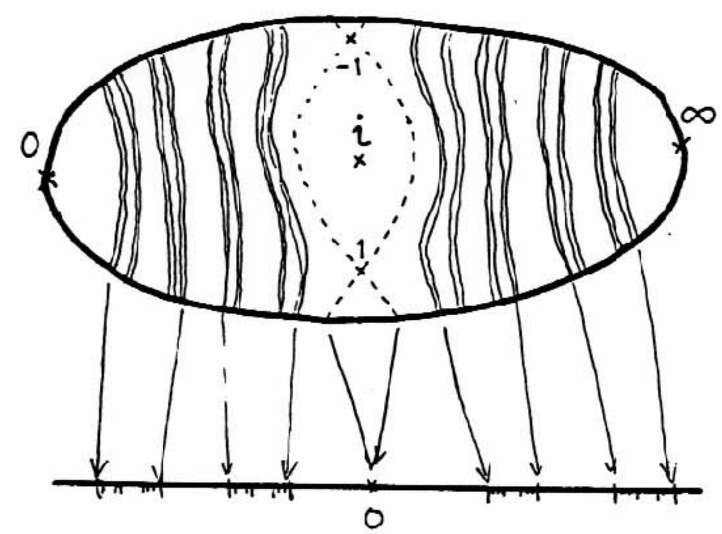
Rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with (multiply connected) Fatou components

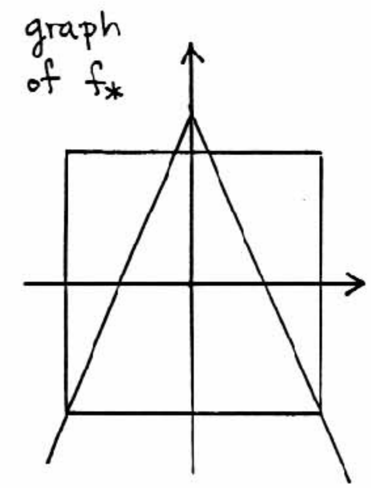
$\longrightarrow$ Tree $T = T_f$ with a metric d
Piecewise linear map $F = F_f : T \to T$
they encode the dynamics of f on Fatou components

The tree is also related to the limit of quasiconformal deformation

S. 1989 (Tree for Config of HRs), cf. McMullen-DeMarco-Pilgrim, Cui-Tan Lei,
J. Rivera-Letelier (arithmetic dynamics), J. Kiwi (arithmetic dynamics of Puiseux series field), M. Arfeux

$$f(z) = c\left(\frac{z}{1+z^2}\right)^3$$

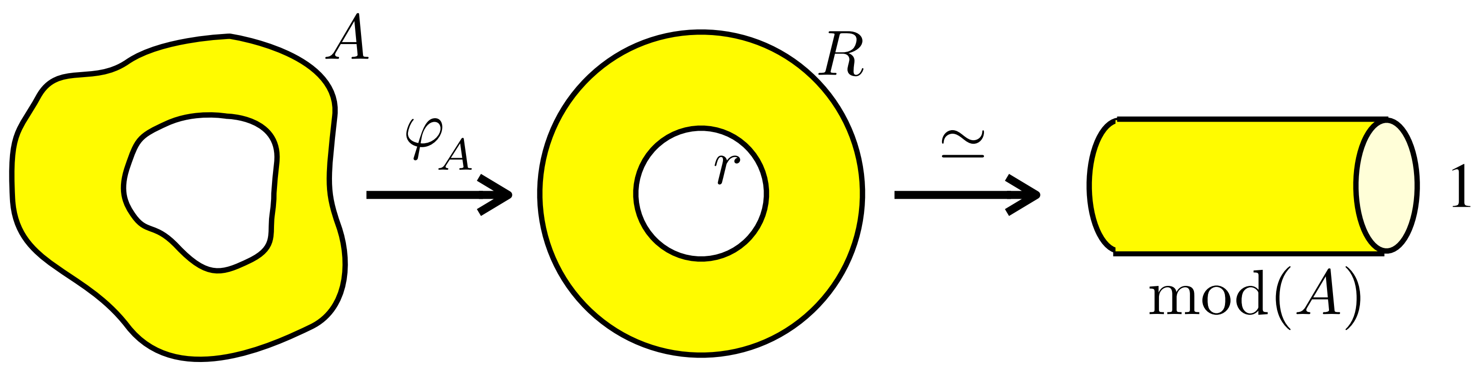




Application of the tree map:
Surgery construction (How to design), and limit of degeneration.
Connectivity of Julia set for Newton's method of a polynomial.
Determination of configuration of Herman rings.
Wandering Julia components.

Annuli

An annulus A is a doubly connected domain in $\widehat{\mathbb{C}}$.



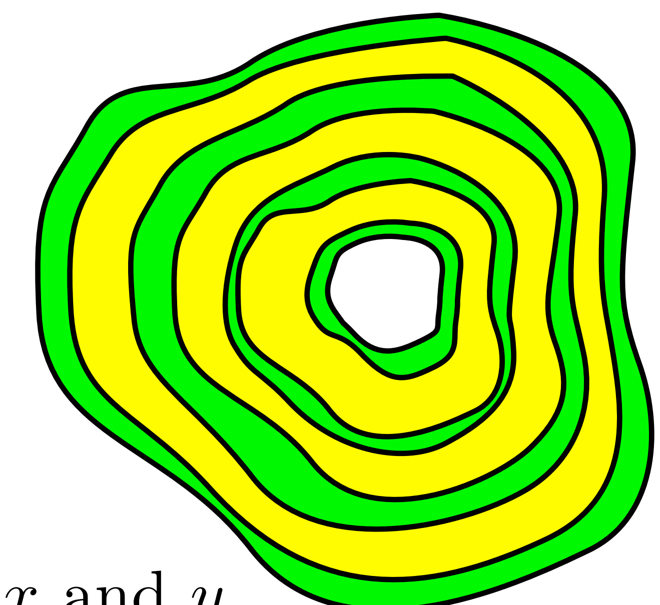
There is a conformal map $\varphi_A : A \to \{z \in \mathbb{C} | r < |z| < R\}$, and its *modulus* is defined by

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{R}{r}.$$

$\operatorname{mod}(A)$ is conformal invariant; if $f : A_1 \to A_2$ is a conformal covering of degree k then $\operatorname{mod}(A_2) = k \operatorname{mod}(A_1)$. Moreover,

Grötzsch inequality: Let $A_1, A_2, \ldots$ be disjoint annuli contained in an annulus A. If A_i's are essential (π_1 injects), then

$$\sum_i \operatorname{mod}(A_i) \leq \operatorname{mod}(A).$$

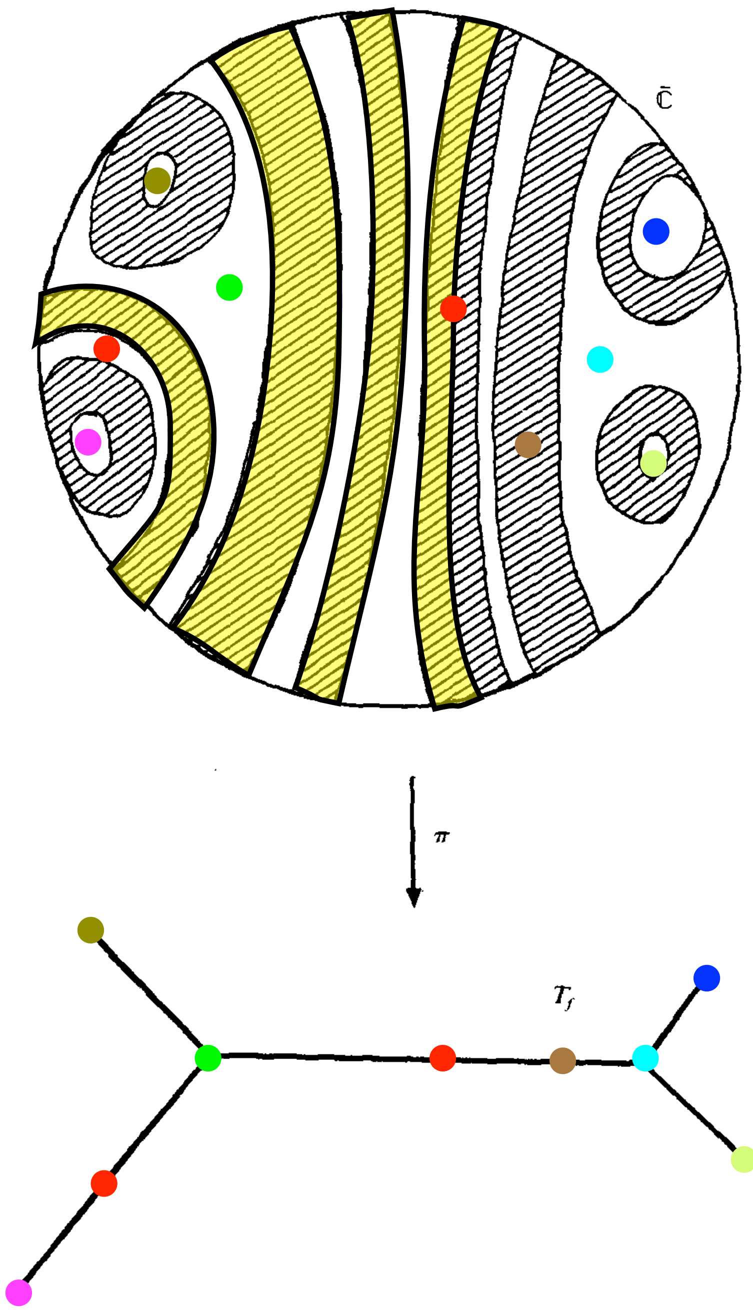


An annulus A is foliated by "circles" $\varphi_A^{-1}(r)$.

For $x, y \in \widehat{\mathbb{C}}$, let $A[x, y]$ be the union of the circles that separate x and y.

Define $\operatorname{mod} \emptyset = 0$.

(Non-dynamical) Tree from disjoint annuli



Suppose that $\mathcal{A}$ is a collection of disjoint annuli in $\widehat{\mathbb{C}}$.

Define $d = d_{\mathcal{A}} : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ by

$$d(x,y) = \sum_{A \in \mathcal{A}} \operatorname{mod} A[x,y].$$

Grötzsch inequality implies that

$$d(x,z) \leq d(x,y) + d(y,z).$$

Let $T = T_{\mathcal{A}} = \widehat{\mathbb{C}}/\sim_{\mathcal{A}}$, where $x \sim_{\mathcal{A}} y \Leftrightarrow d(x,y) = 0$.

"circle" $\longrightarrow$ a point, annulus $A \in \mathcal{A} \longrightarrow$ a segment,
a complementary component of $\cup\mathcal{A} \longrightarrow$ a point.

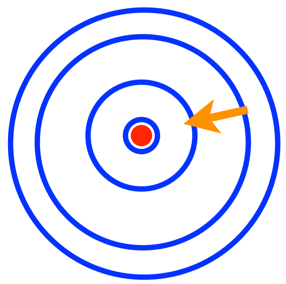
Fact: $T_{\mathcal{A}}$ is a tree and $d(\cdot,\cdot)$ is a (geodesic) metric on $T_{\mathcal{A}}^{finite}$, where $T_{\mathcal{A}} = T_{\mathcal{A}}^{finite} \sqcup T_{\mathcal{A}}^{\infty}$ with $T_{\mathcal{A}}^{\infty} = \{x \in \widehat{\mathbb{C}} : d(x,y) = \infty \text{ for } y \neq x\}$. The image of circles are dense.

pf. Jordan's curve theorem.

Tree associated with a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\deg f \geq 2$

Superattracting basin (SAB)

$z \mapsto z^k \ (k \geq 2)$

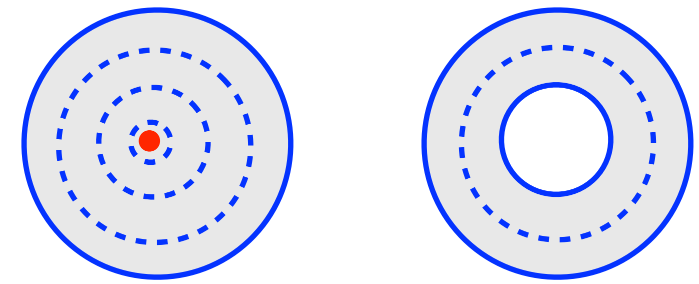


Attracting basin (AB)

$z \mapsto \lambda z \ (0 < |\lambda| < 1)$

Siegel Disk (SD), Herman ring (HR)

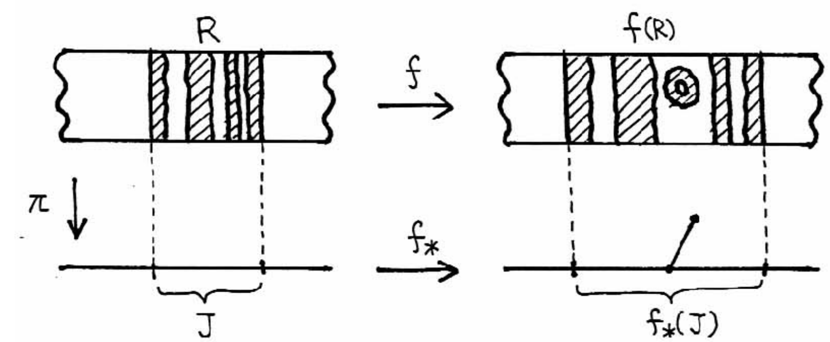
$z \mapsto e^{2\pi i \alpha} z \ (\alpha \in \mathbb{R} \setminus \mathbb{Q})$



The model dynamics provides a foliation by circles. Critical points cause branching of inverse images, so remove all circles intersecting the grand orbits of critical points. Let $\mathcal{A}_f$ be the inverse image of these circles, which is a collection of disjoint annuli.

Define the tree $T = T_f = T_{\mathcal{A}_f}$ as before.

The map f induces a map $F = F_f : T_f \to T_f$.



Fact: $F : T \to T$ is continuous and if $[x, y]$ is an arc in T such that the "full annulus" A corresponding to (x, y) (bounded by $\pi^{-1}(x)$ and $\pi^{-1}(y)$) does not contain critical points, then $A' = f(A)$ is an annulus, $f : A \to A'$ is a covering map and

$$d(F(x), F(y)) = k\, d(x, y),$$

where k is the covering degree of $f|_A$. (We will denote $DF(z) := k$ for an unbranched point $z \in (x, y)$.)

Extracting finite skeleton

Sometimes it is more convenient to extract a tree with finite topological type.

Choose a set $X \subset \widehat{\mathbb{C}}$ such that X is disjoint from $\mathcal{A}_f$, $f(X) \subset X$ and X consists of a finite number of connected components.

Example: $X = \{\text{all non-repelling periodic points}\} \cup (\text{union of all the boundaries of Siegel disks and Herman rings})$. (together with a finite number of inverse images)

Let $\mathcal{A}_{f.X} = \{A \in \mathcal{A}_f | \, A \text{ separates } X\}$.
Then $T = T_{f,X} = T_{\mathcal{A}_{f,X}}$ and $F = F_{f,X} : T \to T$ is defined as before.

Theorem: Let f, X, $F : T \to T$ be as above.
Then T is a tree of finite topological type and F is "piecewise linear" with $DF \in \mathbb{N}$.
For a dense set of points the orbit lands on a periodic segment or half line on which F^p is conjugate to $x \mapsto kx$ (coming from SAB), $x \mapsto x + c$ (coming from AB) or id (coming from SD or HR) on a half line or a finite segment.

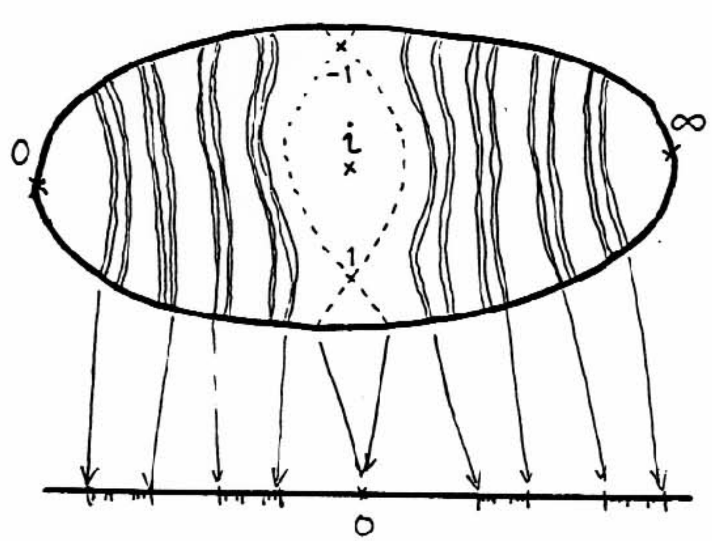
Example

$$f(z) = c\left(\frac{z}{1+z^2}\right)^3$$

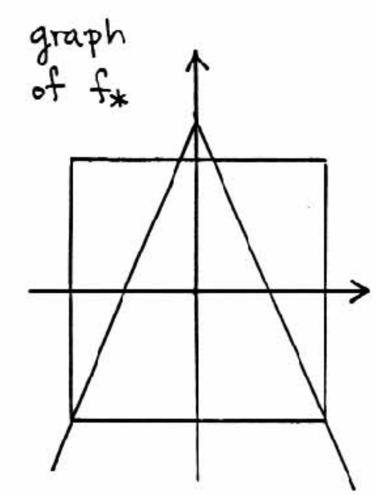
or

$$f(z) = z^m + \frac{\varepsilon}{z^n}$$

with $1/m + 1/n < 1$
and $|\varepsilon| << 1$



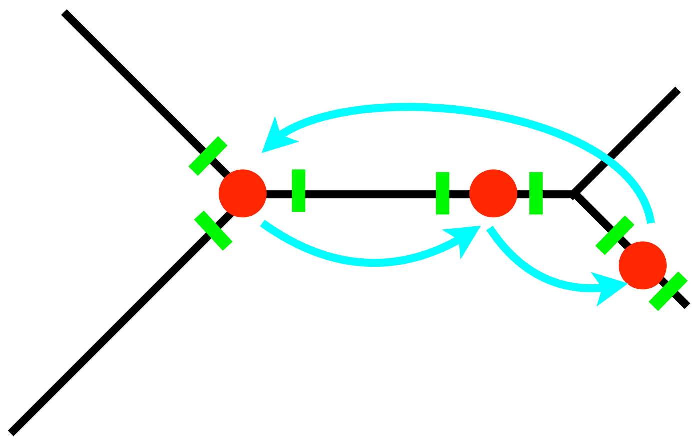
$T = \mathbb{R} \cup \{\pm\infty\}$

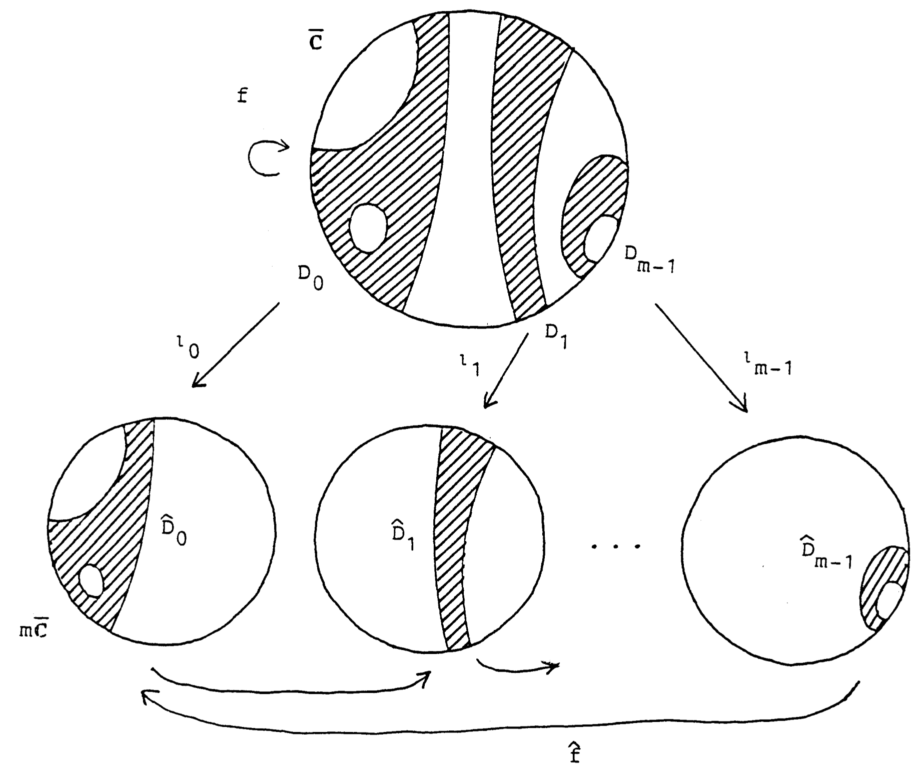


What information is lost by passing to the tree?

Local models for the periodic points on the tree (+Dehn twists?)

a periodic cycle of $F : T \to T$





via quasiconformal surgery

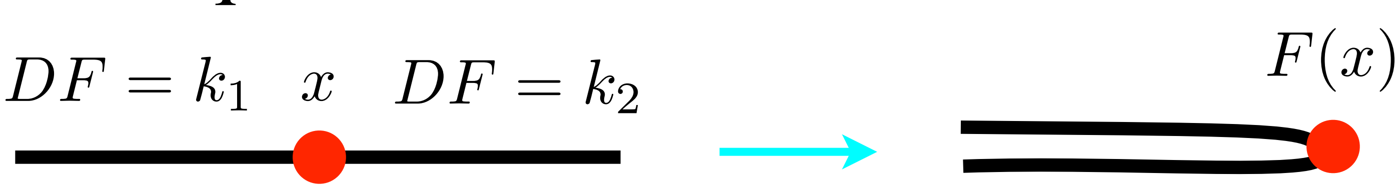
by simplifying the dynamics
in the complementary components

a cycle of rational maps

local model

edge with $DF = k$
$\longrightarrow$ a point with local degree k

Consequences



Then $\pi^{-1}(x)$ contains at least $k_1 + k_2$ critical points.

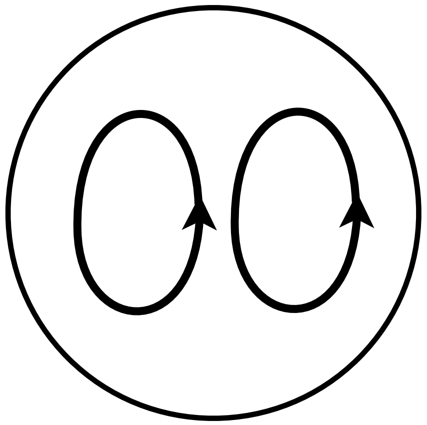
A folding costs two critical points! (bounds in terms of # of crit. pts)

If x_0 is a fixed point of F and not in the interior of periodic segment, then $\pi^{-1}(x_0)$ must contain a “weak-repelling” fixed point (repelling or parabolic with multiplier 1).

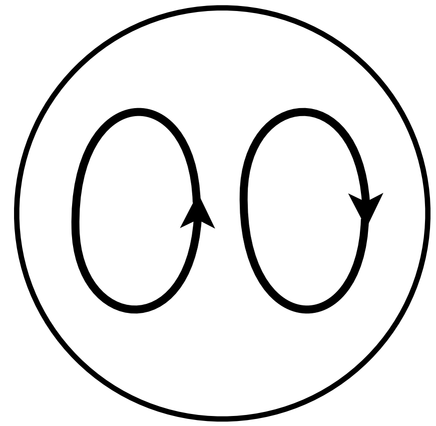
Bounds in terms of # of weakly-repelling fixed pts

Applications

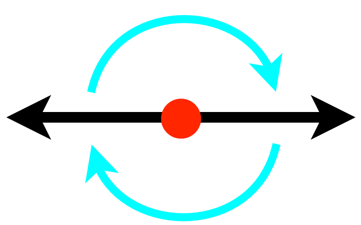
Configuration of Herman rings of period 2 (use the bounds on # of crit pts)

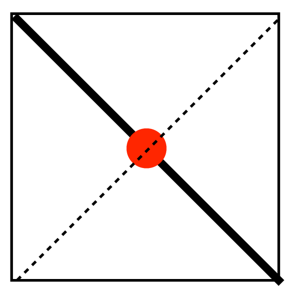


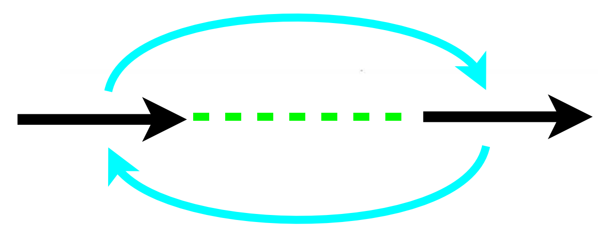
Can be realized by a deg 3 map

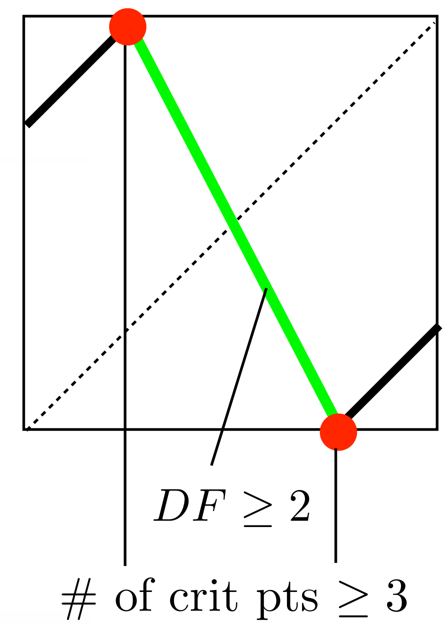


Not realized by a deg 3 map
need deg 4 or more

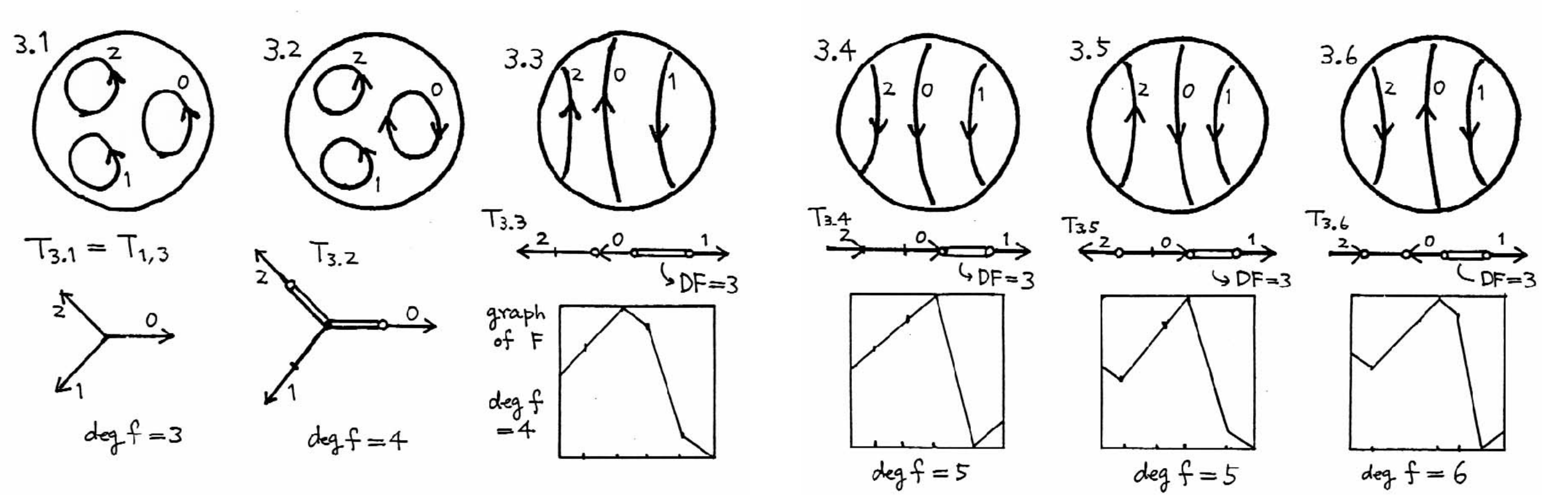








period 3



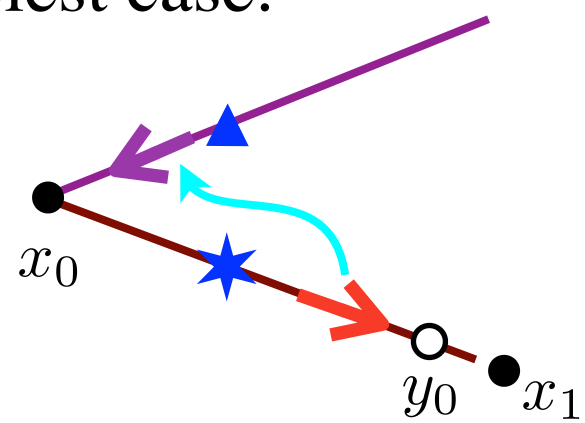
Applications

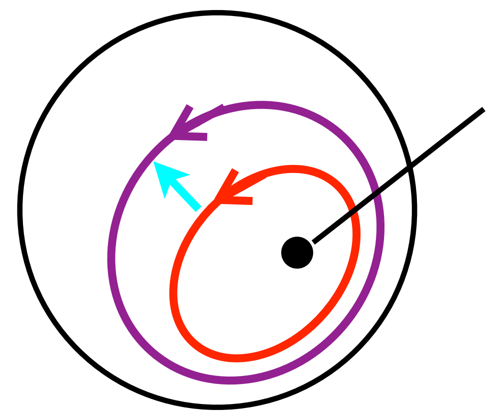
Disconnected Julia set $\Longrightarrow$ <u># of weakly repelling fixed pt</u> is greater than 1

$\exists\, x_0 \in T$ such that $\pi^{-1}(x_0)$ contains a weakly repelling fixed pt

J_f disconnected $\Longrightarrow \exists\, y_0 \in T$ such that $y_0 \neq x_0$ and $F(y_0) = x_0$

The simplest case:





There must be at least one "weakly repelling fixed point" here by Fatou (after a surgery).

Can find another weakly rep. fixed pt in other cases

When J_f is disconnected because of a parabolic basin, perturb it to mult. conn attracting basin.

Theorem: If P is a polynomial of degree ≥ 2, the its Newton's method

$$N_P(z) = z - \frac{P(z)}{P'(z)}$$

has connected Julia set.

Published proof does not use the tree, but underlying case studies were based on the tree.

Surgery construction

Piecewise linear map on a tree (with properties listed before) + local model $\longrightarrow$ rational map

Given a piecewise linear map $F : T \to T$ on a tree and local models for periodic "singular points" with certain conditions, one can do a quasiconformal surgery to construct a rational map.
Idea is to replace edges of the tree by cylinders and "singular points" by small spheres (Plumbing construction). If the constructed map is holomorphic outside regions which are transient, by a surgery principle, one can change the conformal structure to obtain a rational map.

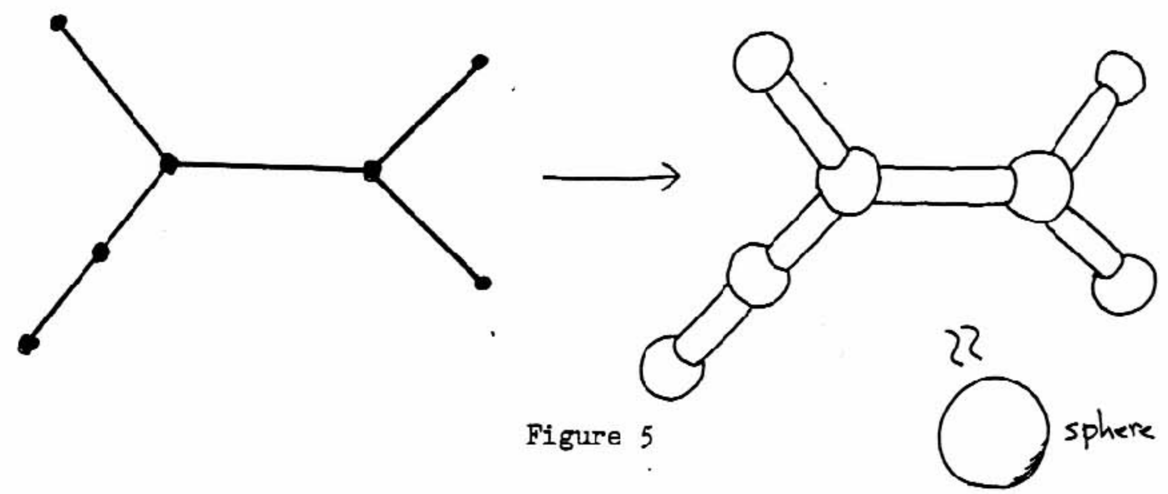
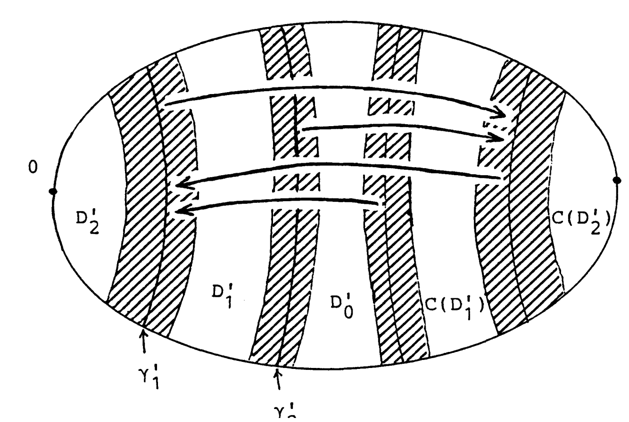
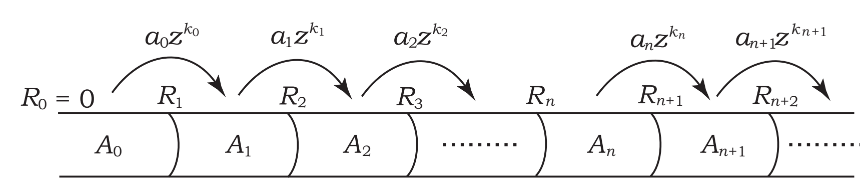


Figure 5

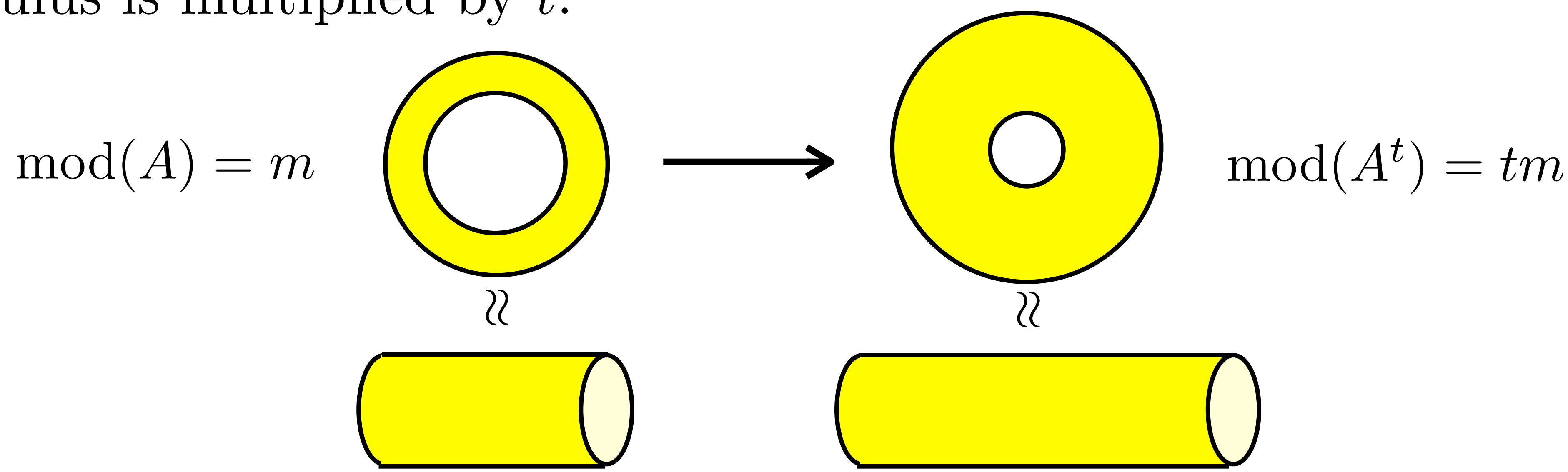
Examples:
Siegel disk $\leftrightarrow$ Herman rings surgery with period > 1;
p-connected Fatou component for $3 \le p < \infty$ (Beardon's book);
Pilgrim-Tan Lei;
Wandering annulus for a transcendental entire function;
P-connected wandering Julia component.





Limit of quasiconformal deformation

t-stretching deformation of an annulus A: Change the conformal structure so that the modulus is multiplied by t.



The family $\mathcal{A}_f$ can be simultaneously t-stretched so that the resulting conformal structure is compatible with a new rational map $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$, where φ_t is a qc-map realizing the deformation. This is the t-stretching deformation of f.

Suppose $x, y \in T$ correspond to pts in the Fatou set. Let $A_{x,y}$ denote the annulus between $\pi^{-1}(x)$ and $\pi^{-1}(y)$. Then by Grötzsch inequality

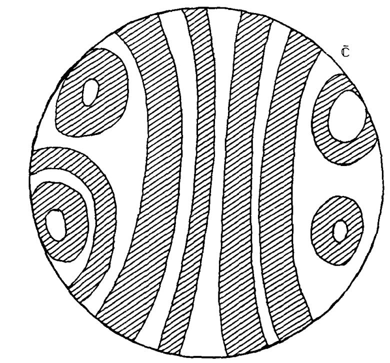
$$mod(\varphi_t(A_{x,y})) \geq t\, d(x,y).$$

Question: $$\lim_{t\to\infty} \frac{mod(\varphi_t(A_{x,y}))}{t} = d(x,y)?$$

If YES, it means that f_t degenerates in the way it is supposed to.
And it helps us to find where these rational maps are in the Rat_d.
Cf. Kiwi's Puiseux series dynamics.

The local models (cycle of rational maps corresponding to a periodic orbit on the tree) can be obtained as the limit of t-stretching deformation after suitable normalizations (moving frame).

Problem: Suppose A an annulus and $\mathcal{A}$ is a collection of disjoint sub-annuli of A. For $A_i \in \mathcal{A}$, let A_i^t denote t-stretching of A_i. Apply t-stretching to all $A_i \subset A$ and obtain a new (measurable) conformal structure for A. Let A^t be A with this conformal structure. Show



$$mod(A^t) \sim \sum_{\substack{A_i \in \mathcal{A} \\ A_i \text{ is essential in A}}} t\; mod(A_i) \quad \text{as } t \to \infty.$$

If one can show this, it is expected that for four points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$, if there is a maximal segment J separating z_1, z_2 from z_3, z_4, then

$$\frac{1}{2\pi} \log CR(z_1^t, z_2^t, z_3^t, z_4^t) \sim t \; \text{length}(J) \text{ as } t \to \infty,$$

where CR denotes a certain cross ratio.

From this, one should be able to deduce how the t-stretching f_t degenerates and one can find the asymptotics of f_t, i.e. where they are in the parameter space.

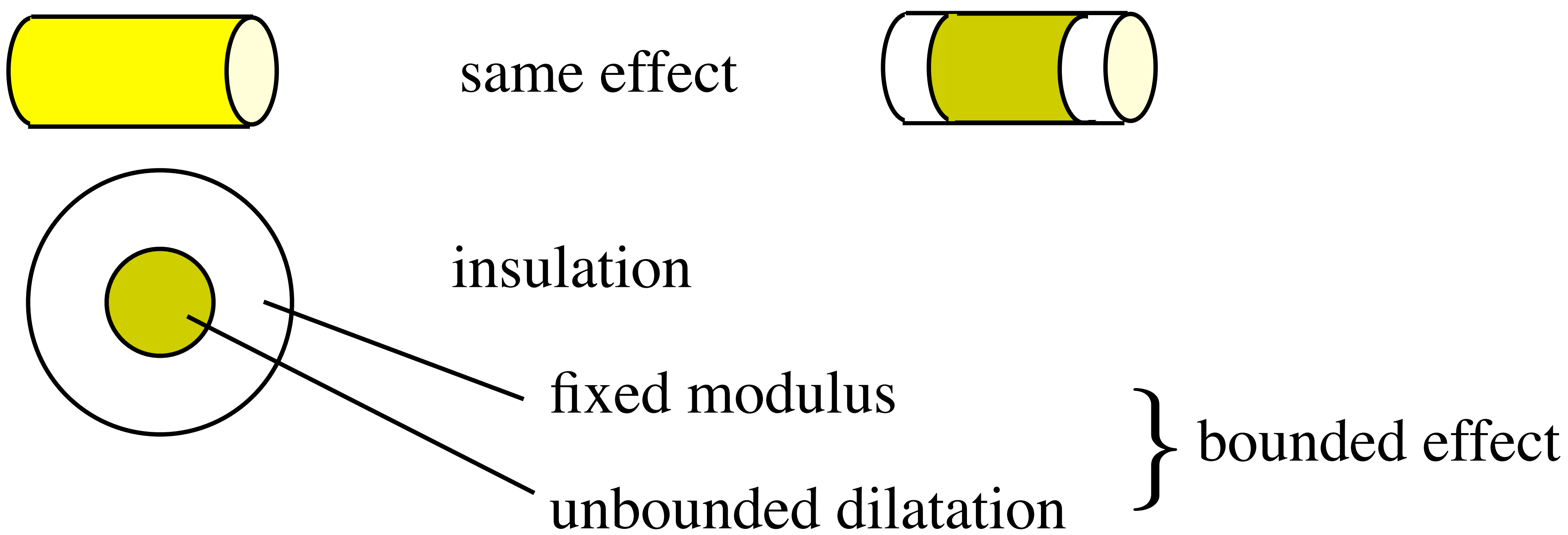
The asymptotics can be deduced from the surgery construction.

Note: The above claim is true when $\mathcal{A}$ is finite.

Partial solutions

Case 1: When A's $\in \mathcal{A}$ have moduli uniformly bounded from below.

How to bound the effect of deformation of non-essential annuli?



Case 2: Inverse surgery construction in hyperbolic case

Quasiconformal surgery construction. Not known if this reproduces the qc-deformation family.

Other directions: arithmetic surgery, realize a family as a rational map on Puiseux series field (non-Archimedean field).

Recent:

Jordi Canela Sanchez: $\exists\, f$ having Fatou components with arbitrarily large connectivity.

Such a question will fit nicely into the frame work of the tree map.

For Fatou components U, V,

$$U \sim V \iff \exists\, n, m \;\; f^n(U) = f^m(V) \text{ and }$$
$$f^n : U \to f^n(U) \text{ and } f^m : V \to f^m(V) \text{ are homeomorphic.}$$

U is p-connected $\iff \widehat{\mathbb{C}} \setminus U$ has p connected components.

Question 1: For any $3 \le p < \infty$, are there only finite number of p-connected Fatou components up to the equivalence?

Question 2: Let N_p be the number of equivalence classes of p-connected Fatou components. What is the growth rate?

$$\lim_{p\to\infty} \frac{1}{p} \log N_p$$

Can it be related to the pressure function for $F : T \to T$ with potential function $\log DF$?

Question 3: When $F : T \to T$ is Markov type, then is the generating function

$$\sum_{p=3}^{\infty} N_p t^p$$

a rational function?