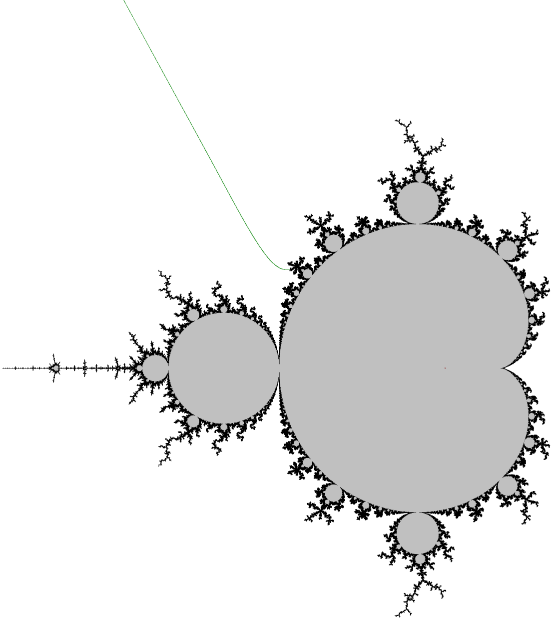
Stretching rays

Pascale Roesch

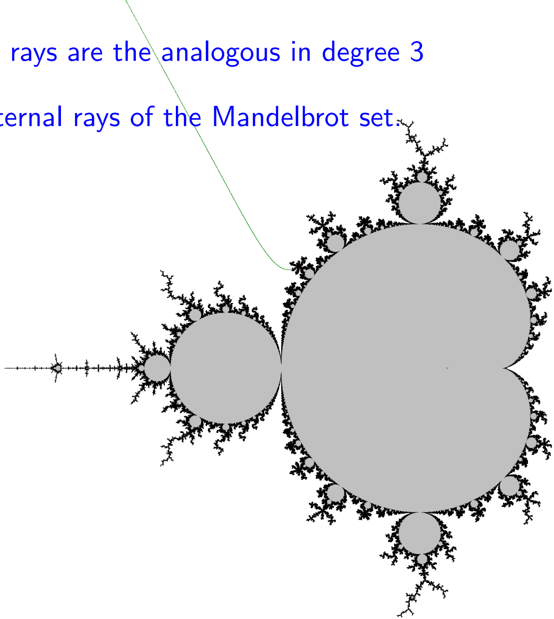
work in progress with Shizuo Nakane

june 2016

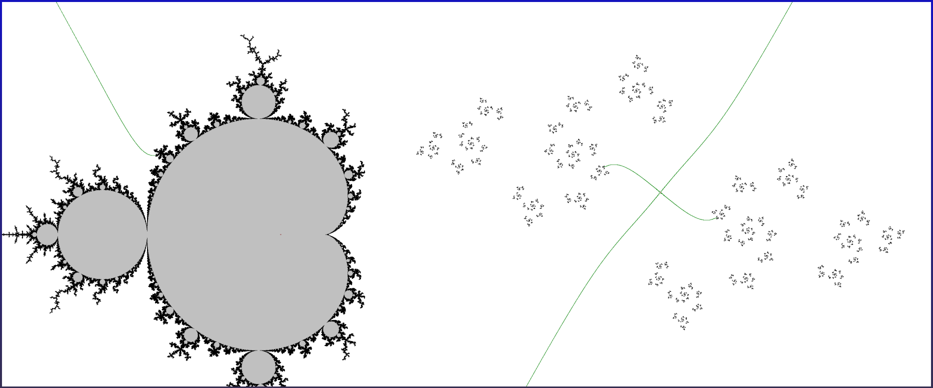


Streching rays are the analogous in degree 3

of the external rays of the Mandelbrot set.



Remember that they allow to switch between parameter plane and dynamical plane.



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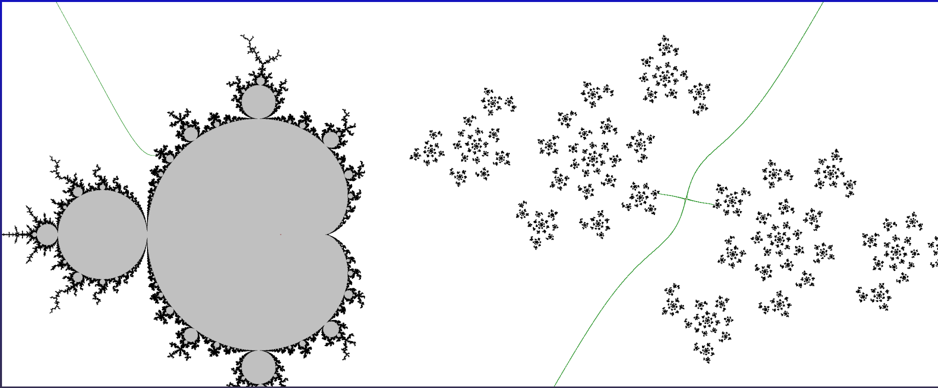

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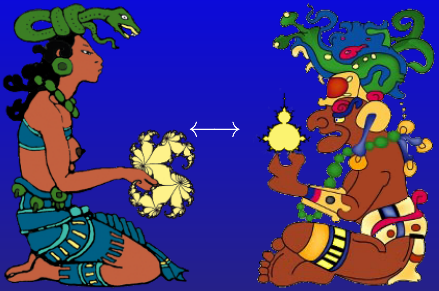

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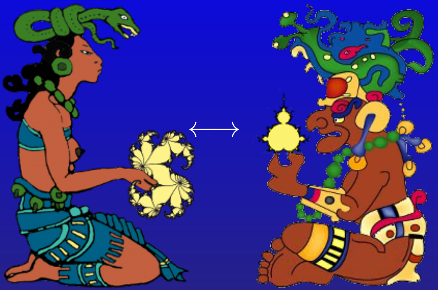

Streching rays were defined by Branner and Hubbard in the paper describing

$\mathcal{P}_3$ the space of cubic polynomials.

Through each $P \in \mathcal{P}_3$ there is a stretching ray.

Roughly speaking it is a curve of polynomials

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P_s is obtained by changing the complex structure near infinity

in the basin of P

stretching **along the external rays** of P.

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Otherwise, one critical point

$$c_1(P_s) \in R_{P_s}(\theta_1)$$

or two

$$c_2(P_s) \in R_{P_s}(\theta_2)$$

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The stretching ray accumulates the connectedness locus

$$\mathcal{C} = \{R \in \mathcal{P}_3 \mid J(R) \text{ is connected}\}.$$

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- Is there a place where no stretching ray accumulate?

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Recall $c \in R_M(\theta)$ means that $R_c(\theta/2)$ and $R_c(\theta/2 + 1/2)$ break on the critical point.

Finite number of parabolic point of a given period.

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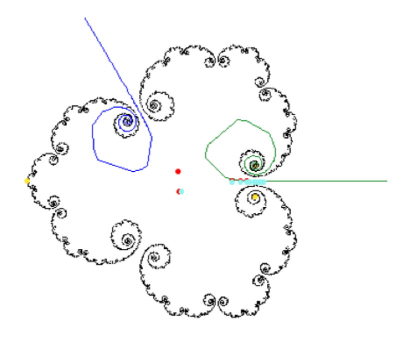
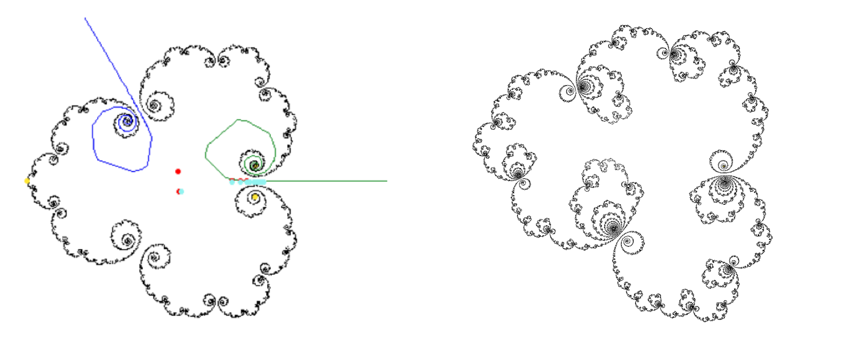
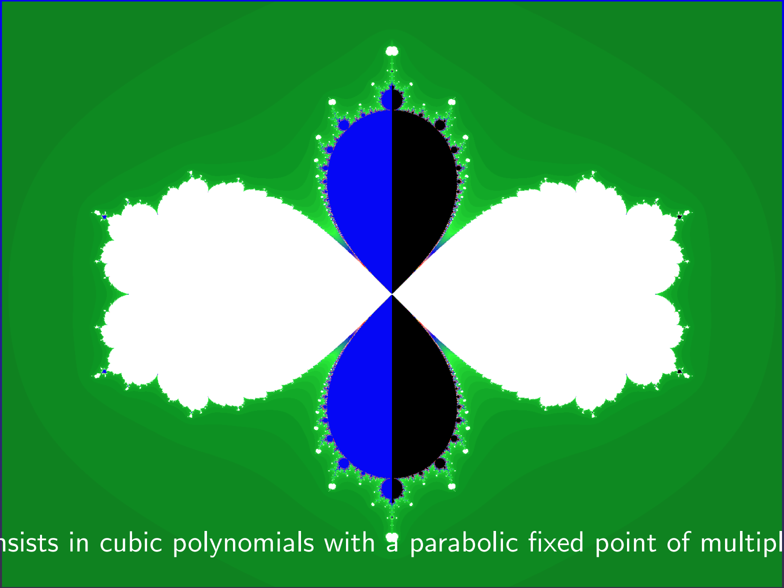
The second critical point allows some flexibility when P is in the shift locus

$\mathcal{S}_3 := \{P \in \mathcal{P}_3 \mid \text{ both critical points escape to } \infty\}$





The cubic parabolic slice $Per_1(1)$



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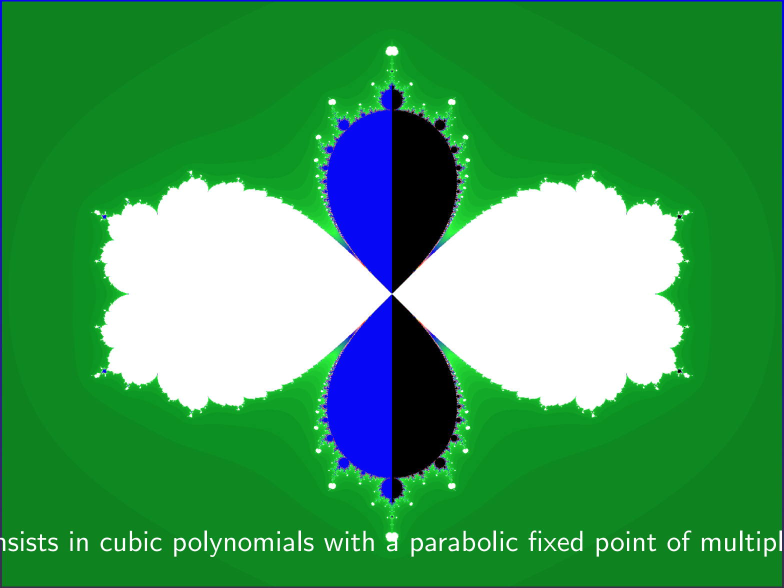

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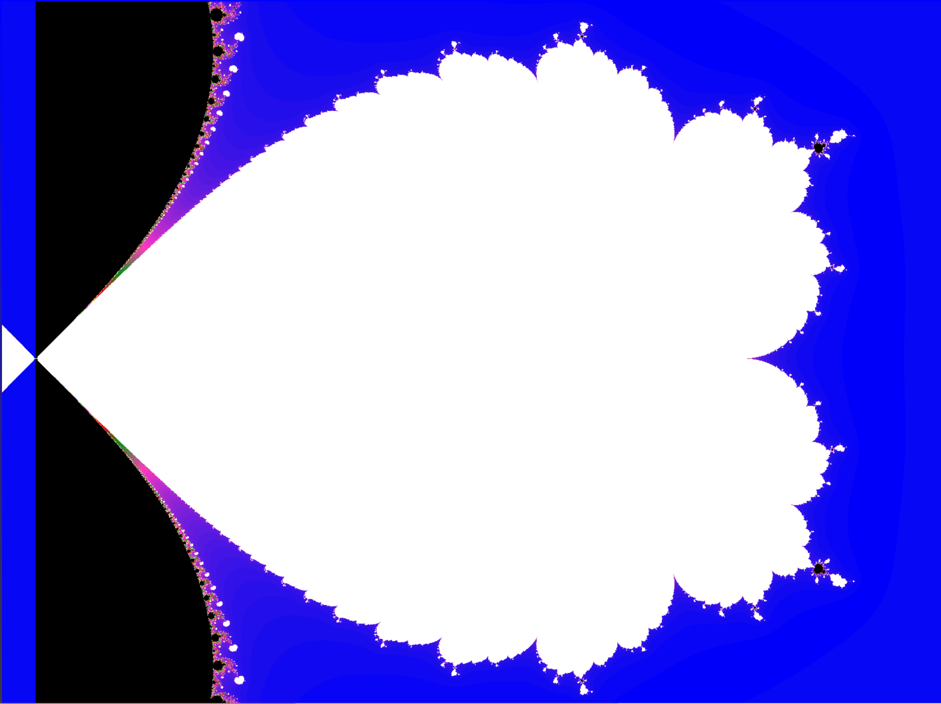
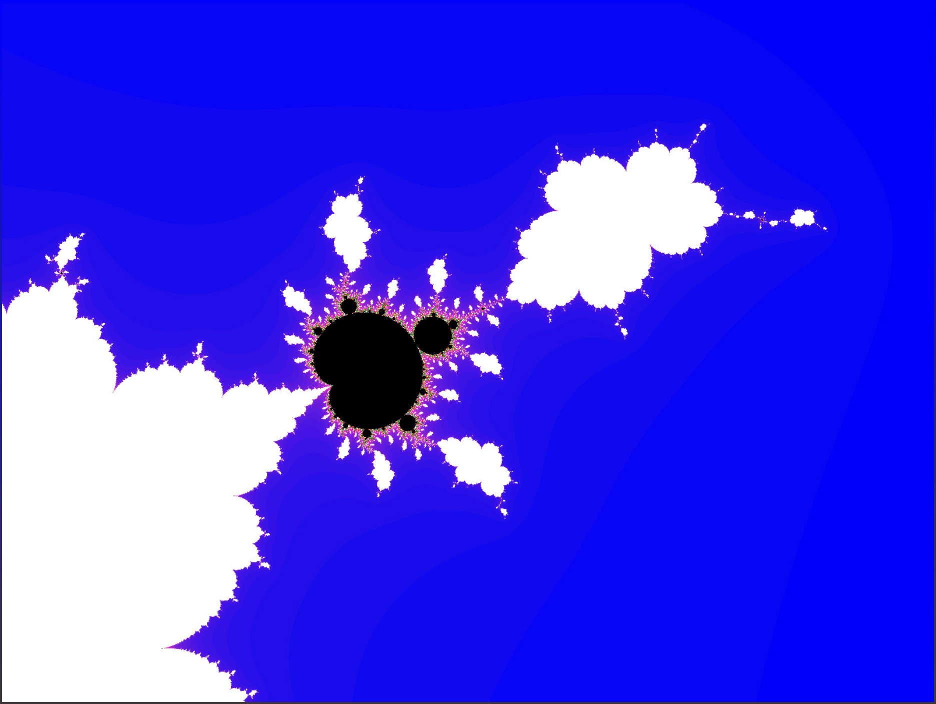
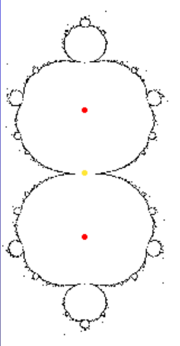
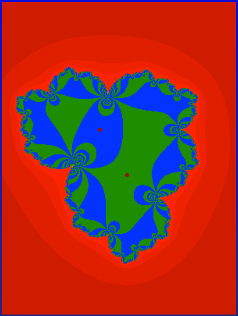
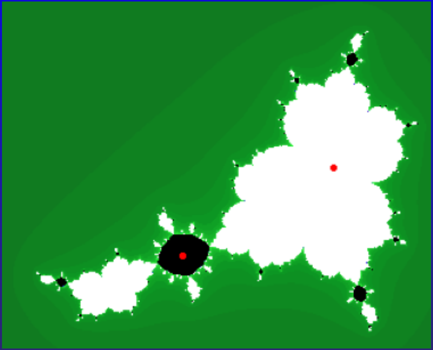
$$Per_1(1) = \{P_a(z) = z^3 + az^2 + z \mid a \in \mathbb{C}\}$$

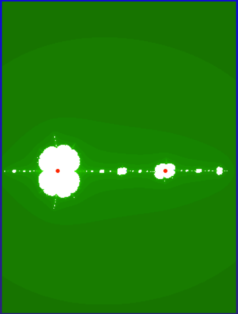










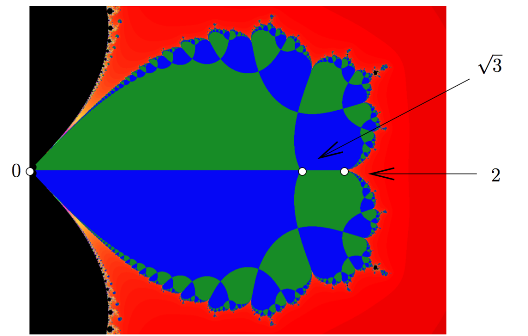


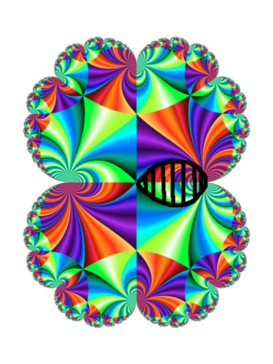
Note that P_a and $P_{a'}$ are analytically conjugate iff $a' = -a$.
We can follow outside $[-\sqrt{3}, \sqrt{3}]$ the critical points $c_+(a)$ and $c_-(a)$.

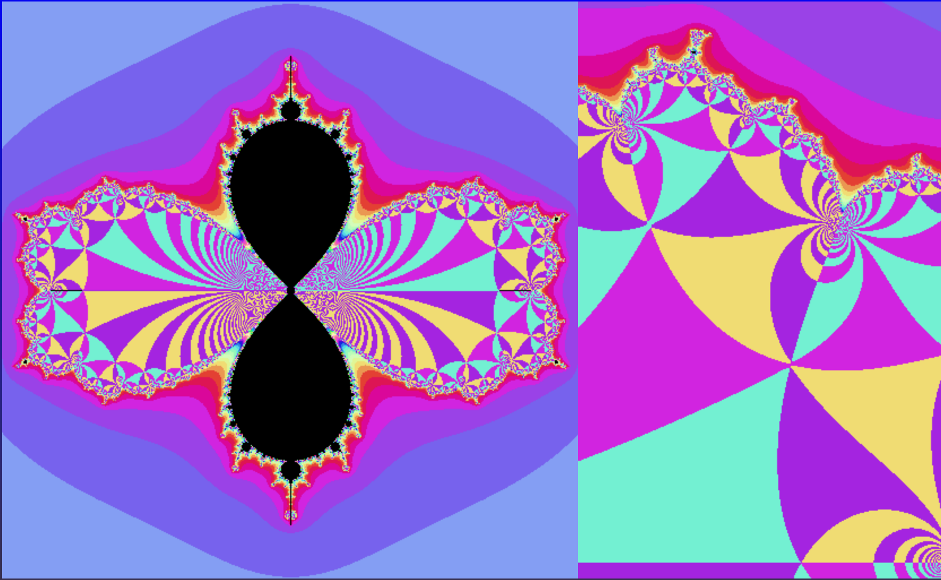
$$Crit(k, n) = \{a \in Per_1(1) \mid Q_a^k(c_-(a)) = Q_a^n(c_+(a))\}$$

Theorem (Nakane, R)

For any $Q \in Crit(k, n) \cap \mathcal{C}$ there exists $P \in \mathcal{S}_3$ such that $S(P)$ lands at Q.





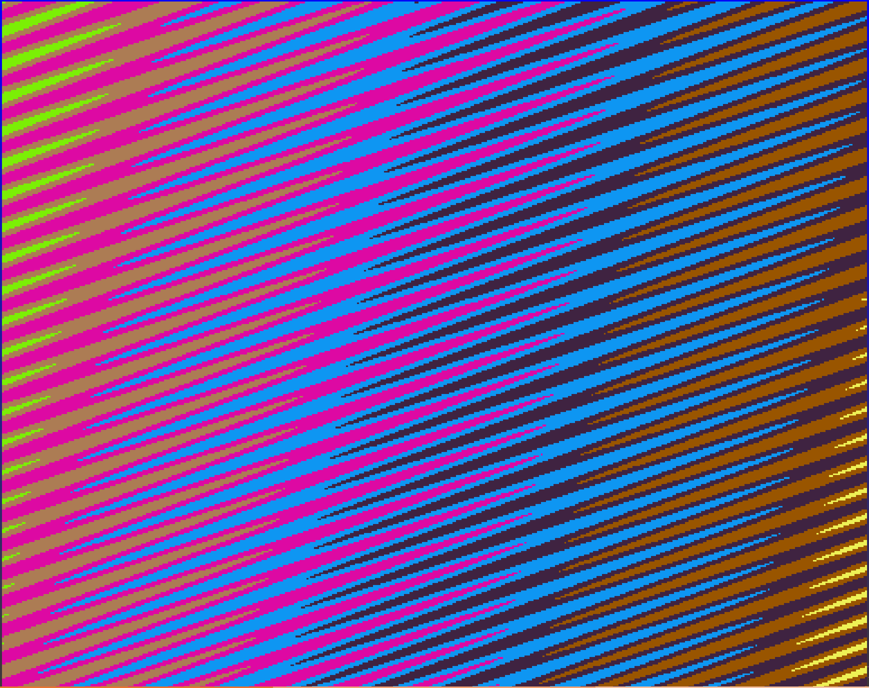


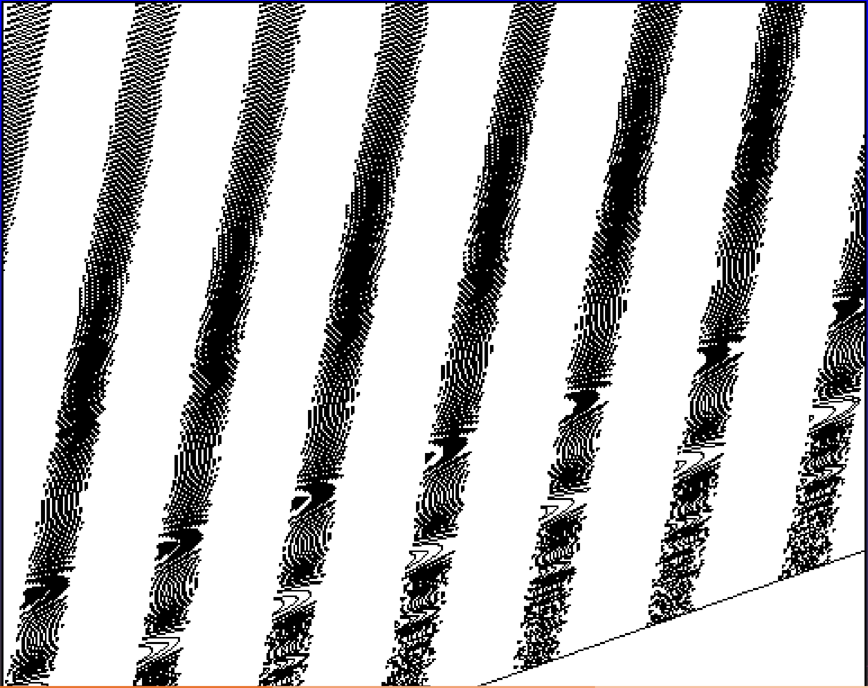
Theorem (Nakane, R)

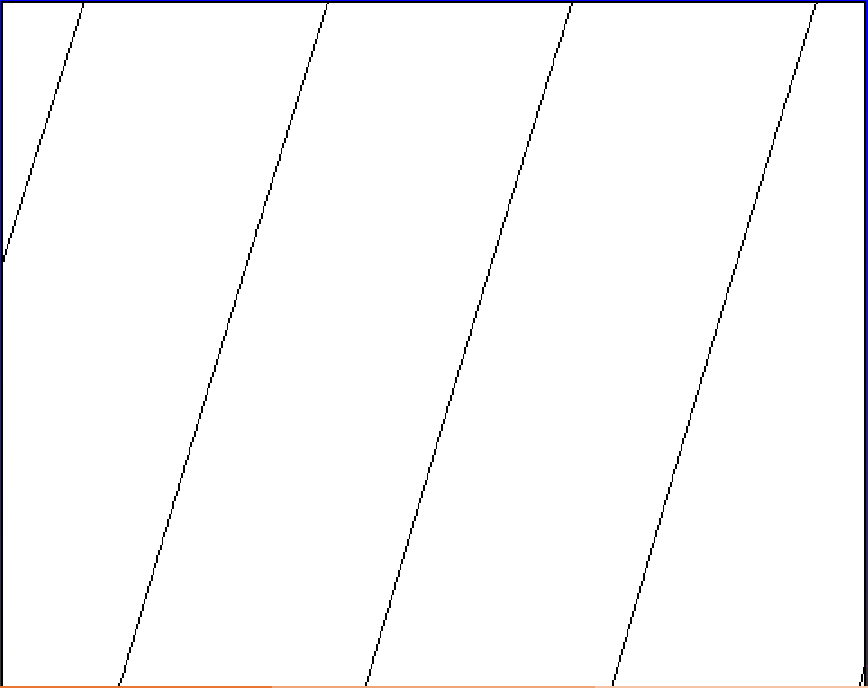
Let $P \in \mathcal{S}_3$ *having critical portrait* $\{0, 1/3\}, \{\theta + 1/3, \theta + 2/3\}$ *with* $3^m\theta = 0 \mod 1$.

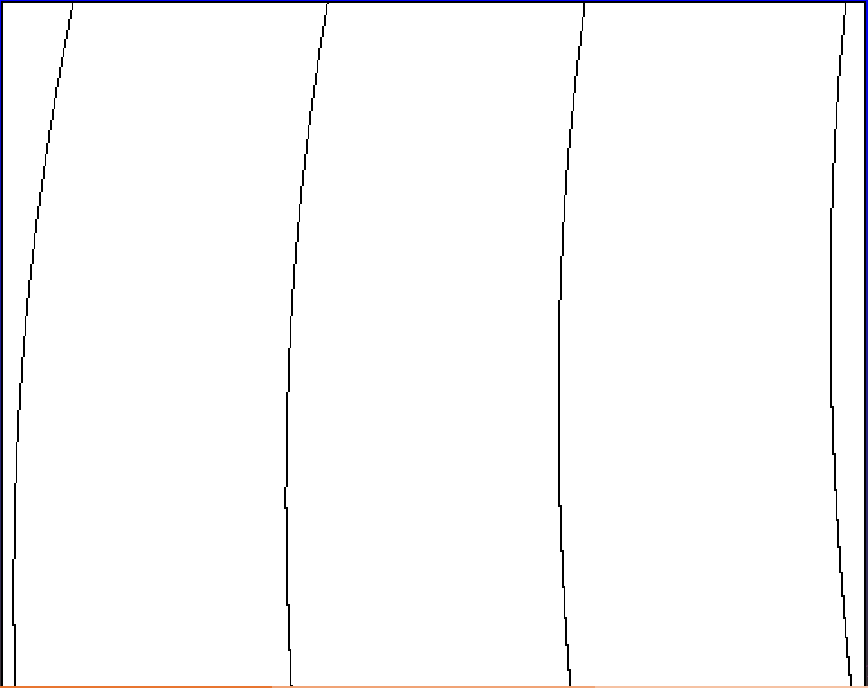
If the stretching ray $S(P)$ *lands*

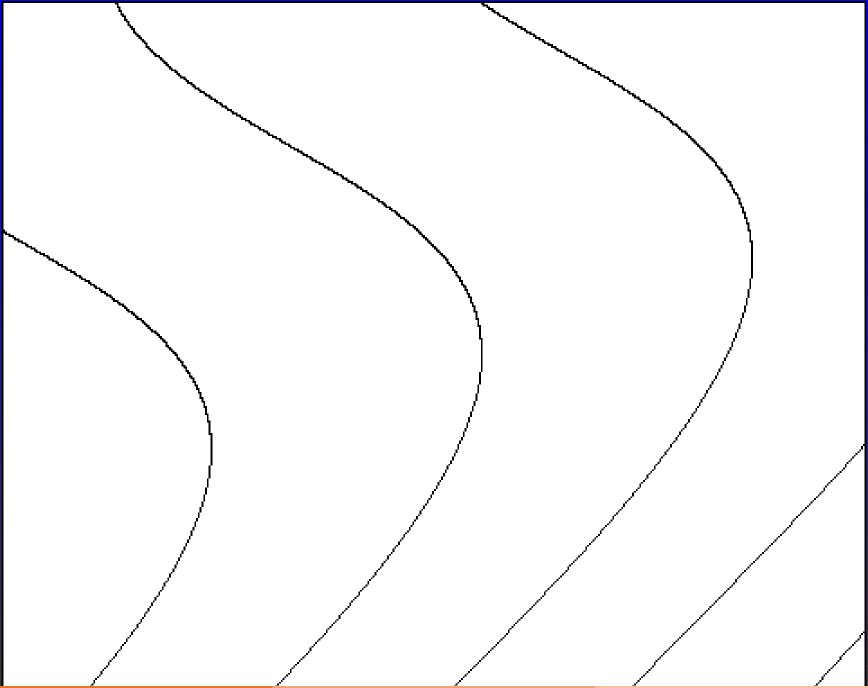
- *either there is a critical orbit relation between* c_1 *and* c_2
- *or it lands at* $Q_0(z) = z + z^3$.

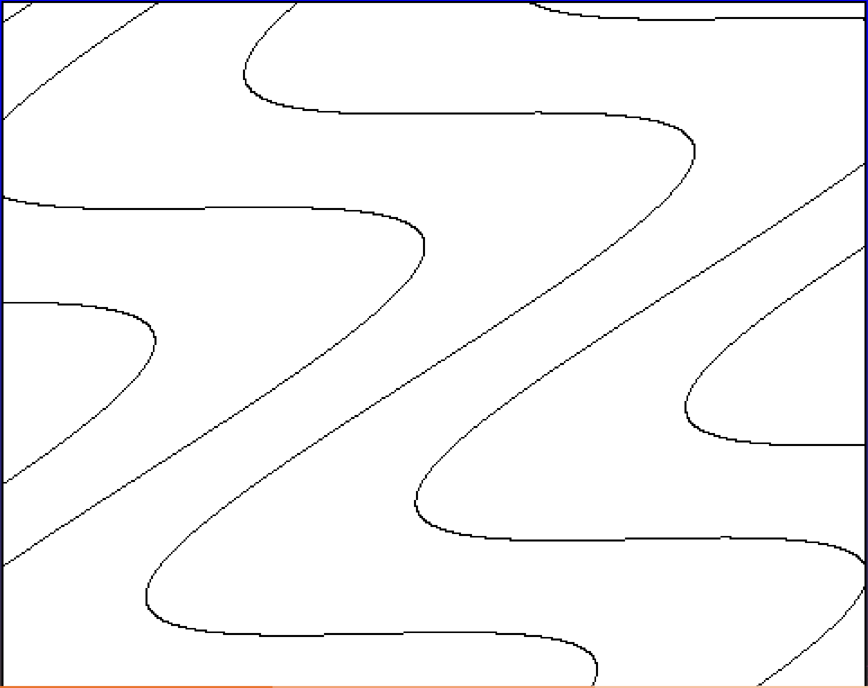


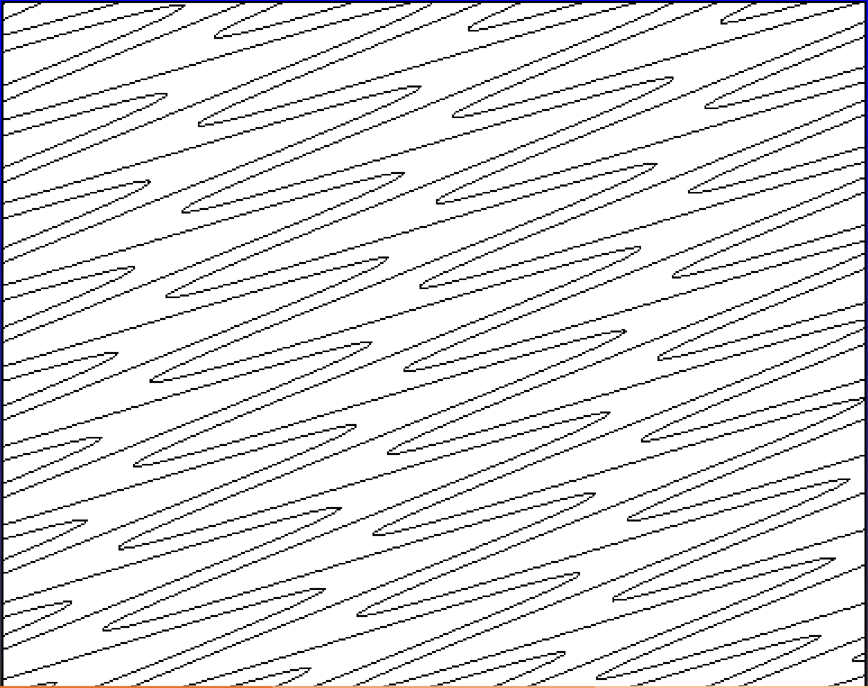


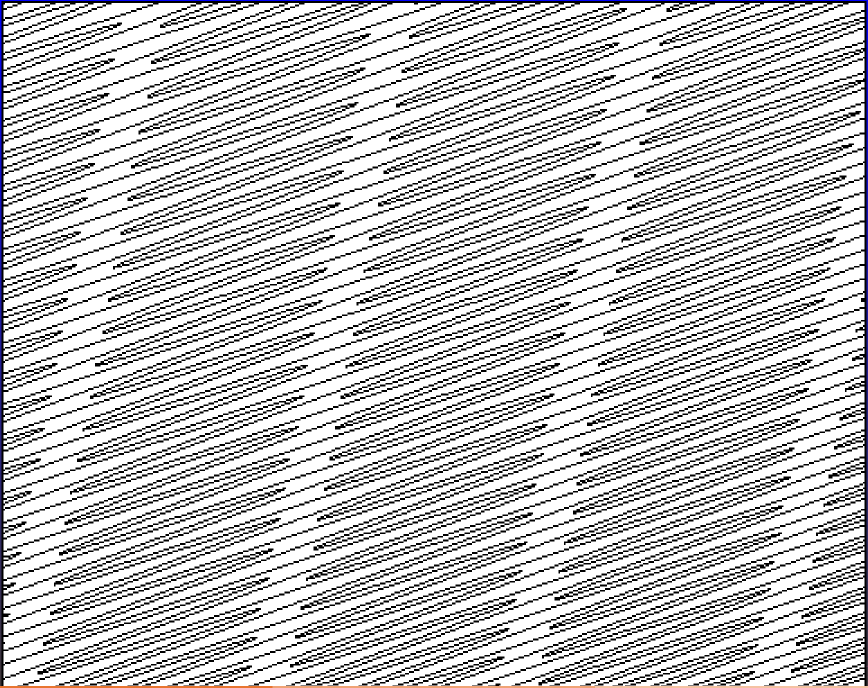












Beltrami forms

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A Beltrami form is P invariant if $(P^*\mu) = \mu$ where

$$(P^*\mu)(z) = \mu(P(z))\frac{\overline{P'(z)}}{P'(z)}$$

Quasi-conformal deformations of P

Let μ be a P-invariant Beltrami form with $\|\mu\|_\infty \leq 1$.

Then for t in the disk $|t| < 1$ the Beltrami form

$\mu_t = t\mu$ is also P-invariant.

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and an analytic family of polynomials :

$$P_t = \chi_t \circ P \circ \chi_t^{-1}$$

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The extension is still P invariant

$$\mu_P = \begin{cases} (\log \circ \varphi_P \circ P^n)^* \frac{d\overline{z}}{dz} & \text{on } B_\infty \text{ the basin of } \infty \\ 0 & \text{outside } B(\infty) \end{cases}$$

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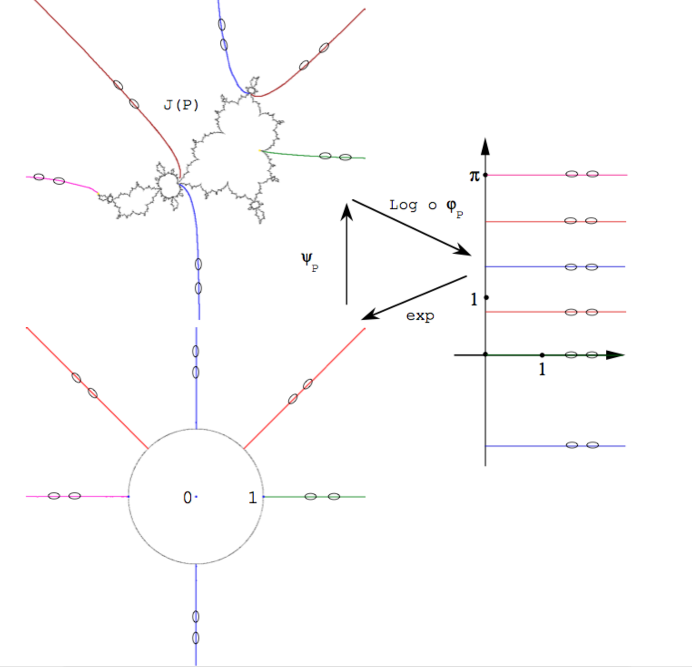
χ_t defines a holomorphic motion (Branner-Hubbard motion) .

The stretching ray is $S(P) = \{P_t \mid t \in [-1, 1]\}$.


J(P)
Log o φ_P
ψ_P
exp
π
1
1
0
1

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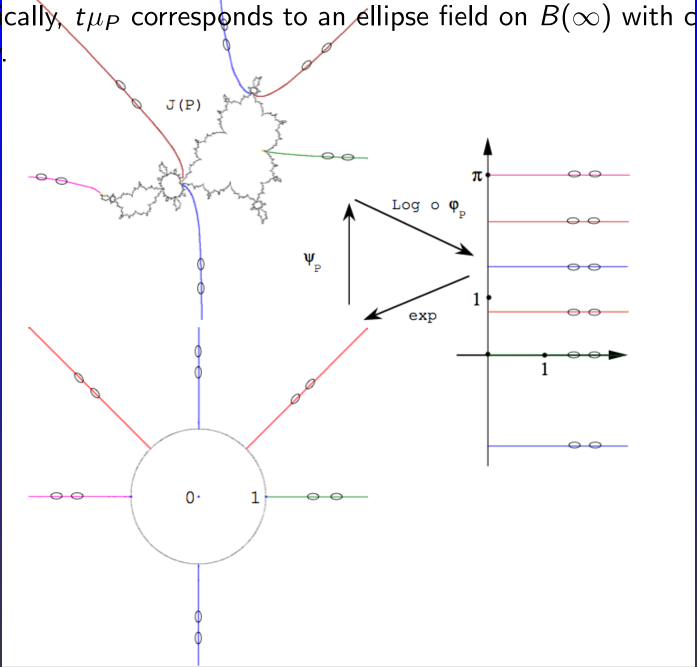

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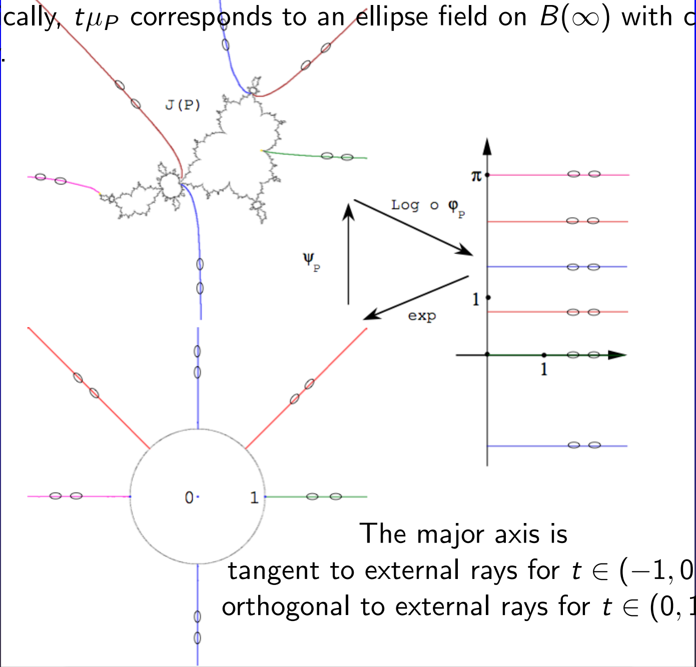

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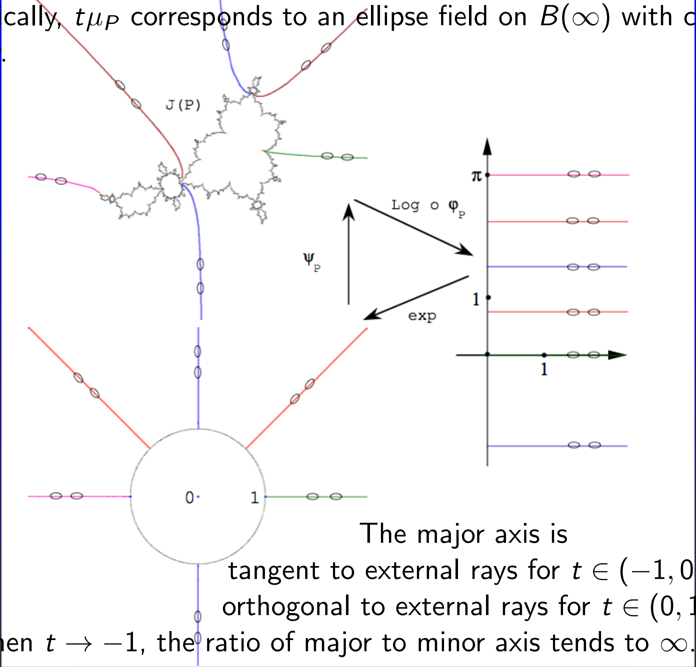

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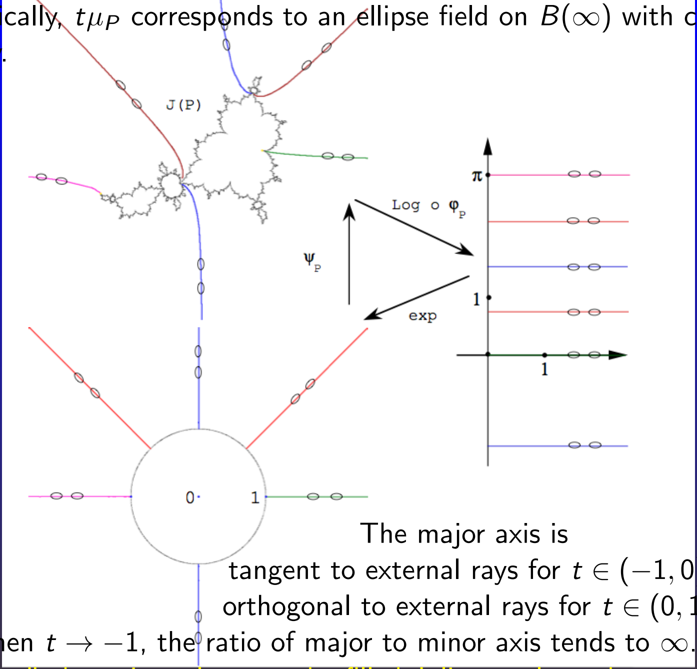

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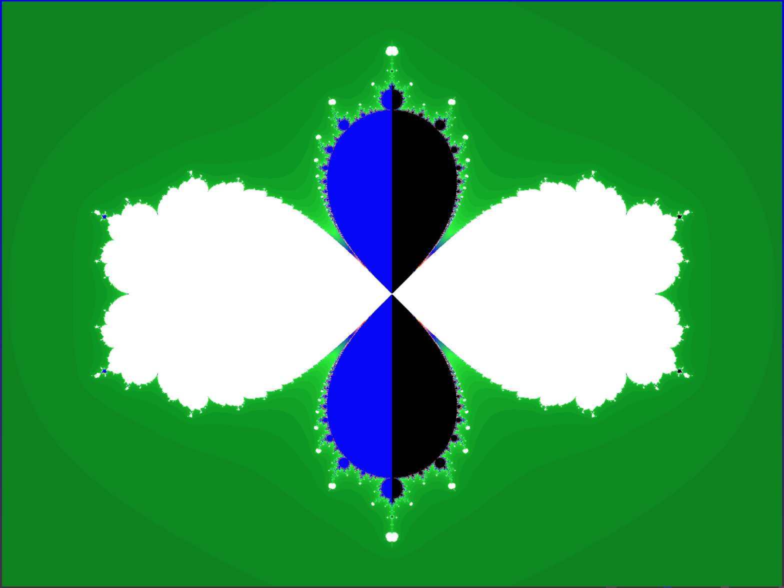
χ_t "pushes" the points closer to the filled Julia set along the rays.

$\mathcal{A}$ is the set of polynomial Q in $Per_1(1)$ such that both critical points belong to the same Fatou component (immediate basin of 0).



Accumulation to $\mathcal{A}$

Let $P \neq Q$ be such that $Q \in Acc(S(P)) \cap \mathcal{A}$ then,

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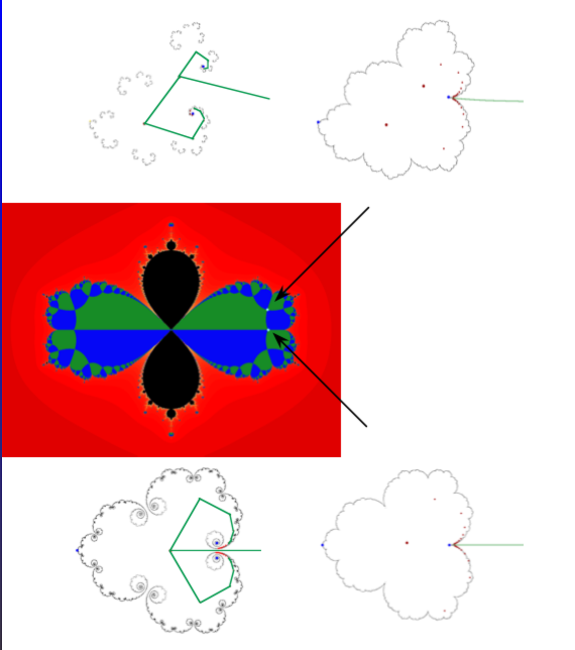
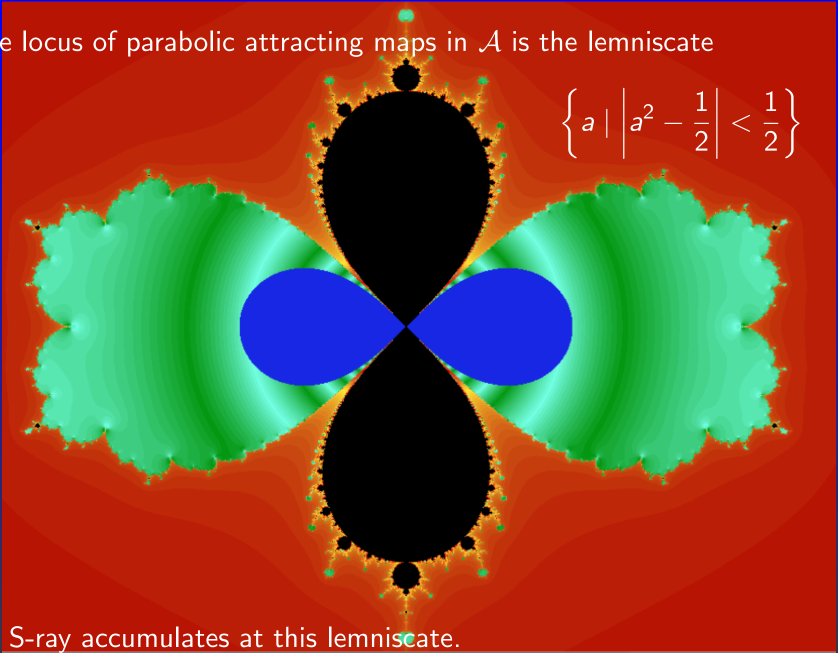
  - *it is the landing or branching point of a θ-external ray with*

$$0 \notin \{3^k\theta, k \in \mathbf{N}\} \quad \textit{but} \quad 0 \in \overline{\{3^k\theta, k \in \mathbf{N}\}}$$



The locus of parabolic attracting maps in $\mathcal{A}$ is the lemniscate

$$\left\{ a \mid \left| a^2 - \frac{1}{2} \right| < \frac{1}{2} \right\}$$



No S-ray accumulates at this lemniscate.

The parabolic point 0 is parabolic attracting if

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Here $Q_a(z) = z + az^2 + z^3$ we get $i(Q,0) = \frac{1}{a^2}$

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The fixed point $-a$ is attracting iff its multiplier $a^2 + 1$ is in $\mathbf{D}$.

Sketch of proof for the first Theorem

Let $P(z) = \lambda z + az^2 + z^3$ with critical points $c_-(a,\lambda)$, $c_+(a,\lambda)$.

$$c_\pm(a,\lambda) = \frac{-a \pm \sqrt{a^2 - 3\lambda}}{3}$$

Let $Crit(k,0) = \{P \mid P^k(c_-(a,\lambda) = c_+(a,\lambda)\}$ to simplify $n = 0$.

- For any $P \in Crit(k,0)$ with critical point on the 0-external ray, every polynomial in $S(P)$ is in $Crit(k,0)$ with critical point on the 0-external ray.
- $Crit(k,0) \cap Per_1(1)$ is finite.
- Then $S(P)$ lands in $Crit(k,0) \cap Per_1(1)$.

Conversely let $Q \in Crit(k, 0) \cap Per_1(1)$

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  For n large the equations

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A subsequence P_{n_j} belongs to the same stretching ray. Hence this S-ray accumulates to Q.

Sketch of proof of the second Theorem

Theorem (Nakane, R)

Let $P \in \mathcal{S}_3$ *having critical portrait* $\{0, 1/3\}, \{\theta + 1/3, \theta + 2/3\}$ *with* $3^m\theta = 0 \mod 1$.

If the stretching ray $S(P)$ *lands*

- *either there is a critical orbit relation between* c_1 *and* c_2
- *or it lands at* $Q_0(z) = z + z^3$.

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$$\eta(P) = \frac{\log(g_P(c_2)) - \log(g_P(c_1))}{\log 3}, \quad g_P \text{ is the Green map}$$

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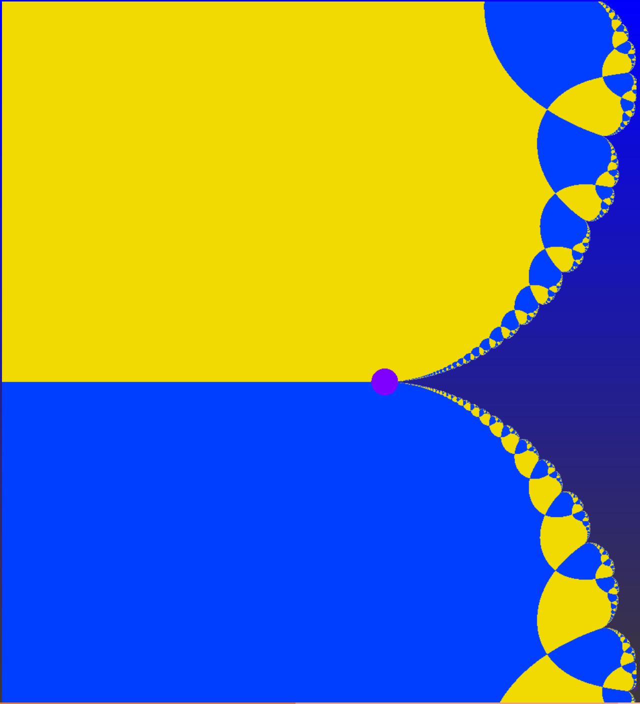
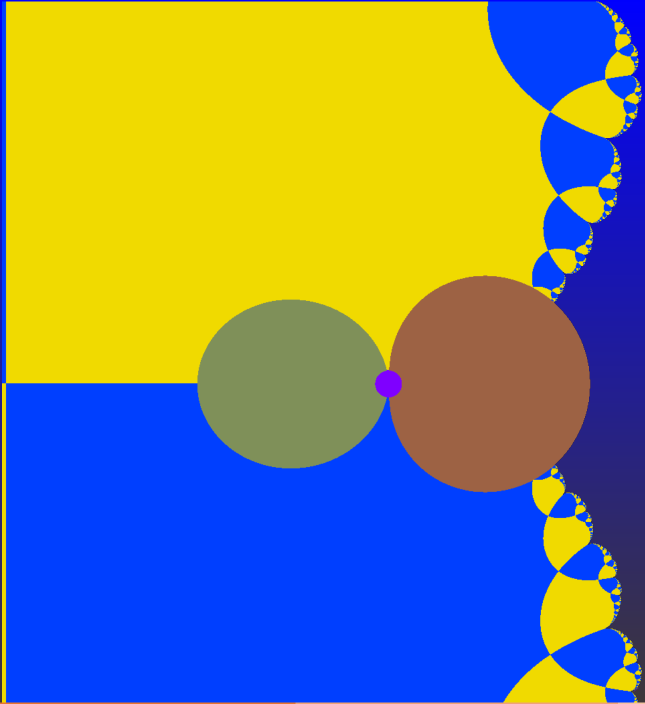
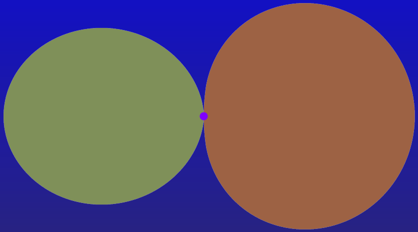
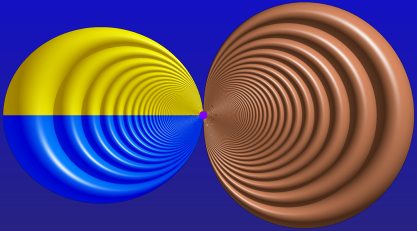
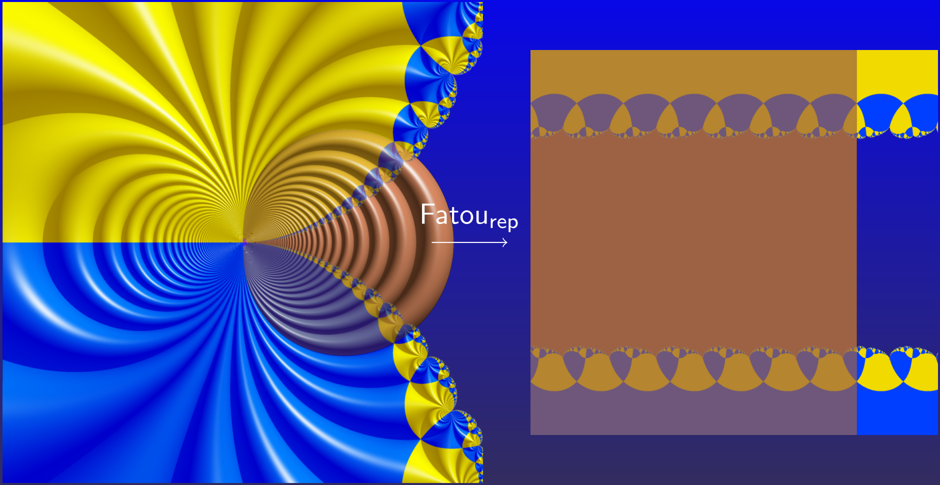
So $1-\lambda_s \sim -(1-\lambda'_s)$ $|Im(\lambda_s - 1)| \geq |Re(\lambda_s - 1)|$ and
$|Im(\lambda_s - 1)| \geq |Re(\lambda_s - 1)|$
otherwise one fixed point is attracting.

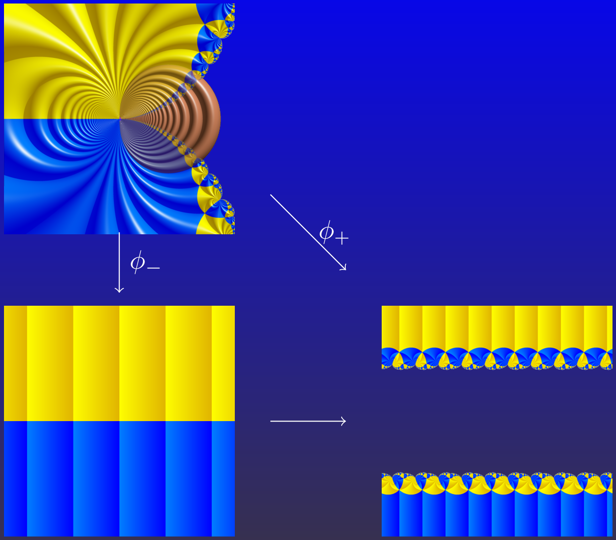




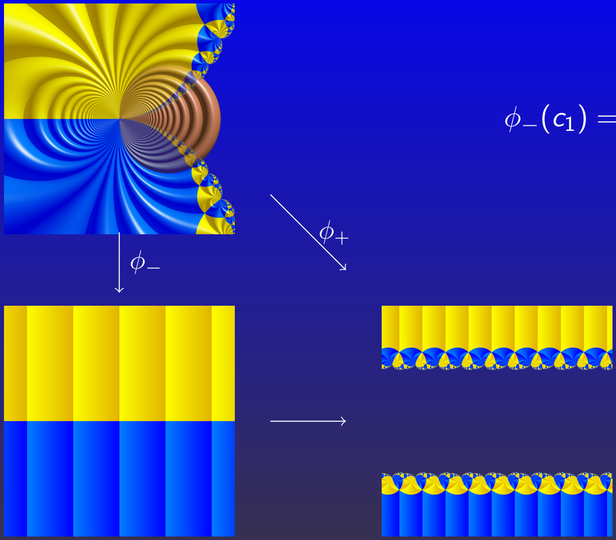


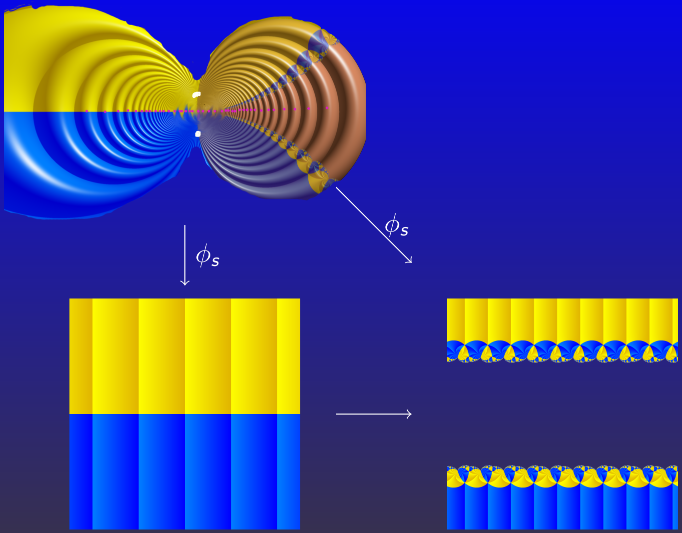



Fatou$_{rep}$


ϕ_+
ϕ_-

$$\phi_-(c_1) = 0$$




ϕ_s
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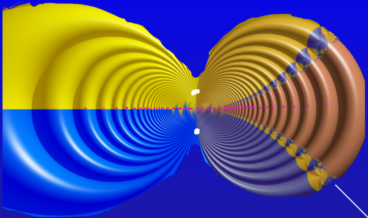
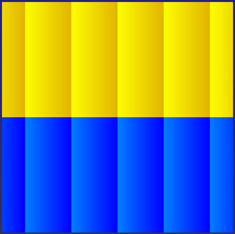
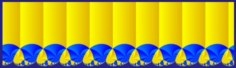
$$\phi_s(c_1) = 0,\ \phi_s \to \phi_-$$

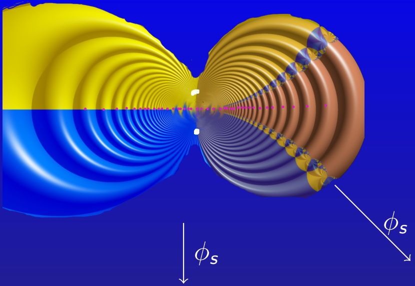


ϕ_s (↓) ϕ_s (↘)



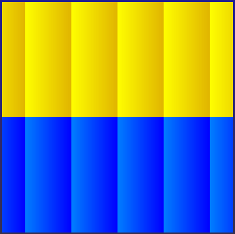
⟶

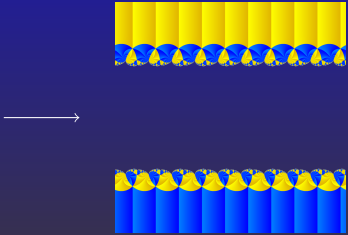


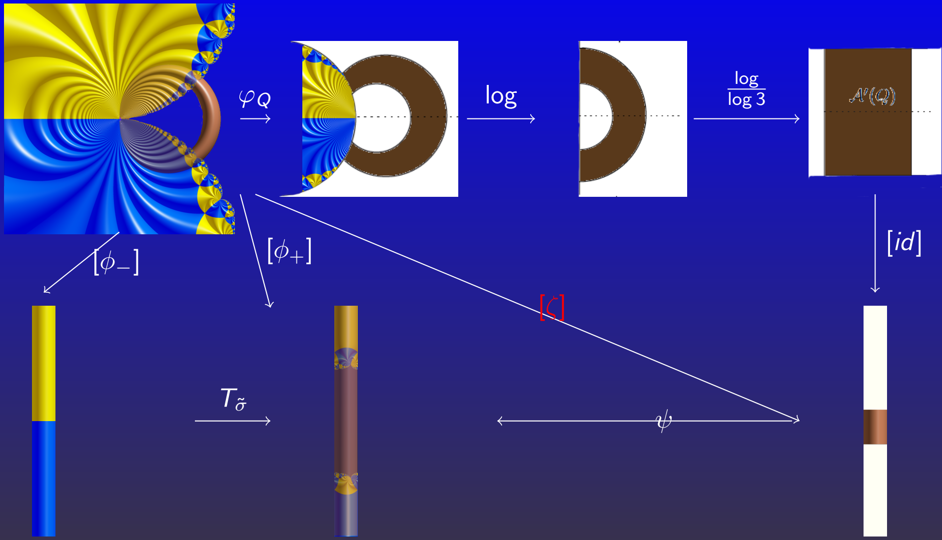


$$\phi_s(c_1) = 0,\ \phi_s \to \phi_-$$

$$\phi_{+,s} = \phi_s + t_s \to \phi_+$$






φ_Q
log
$\frac{\log}{\log 3}$
$\mathcal{A}'(Q)$
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$[\phi_-]$
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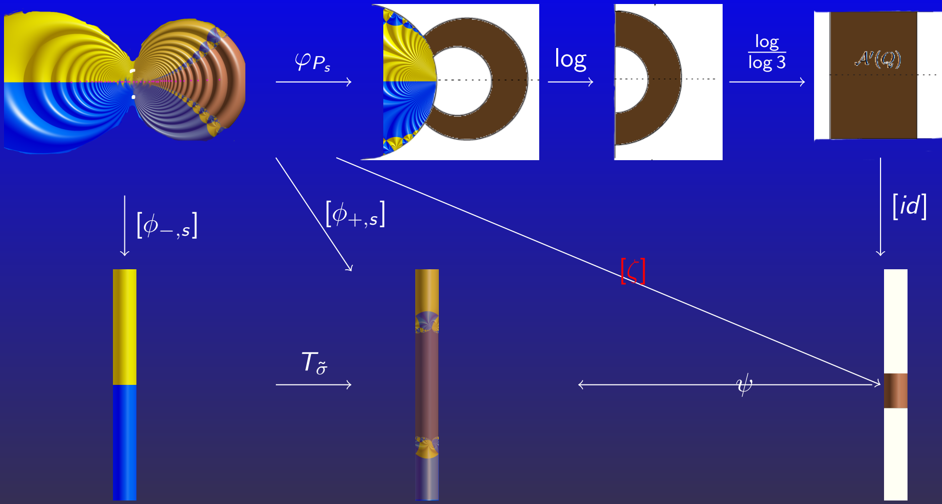
$T_{\tilde{\sigma}}$
ψ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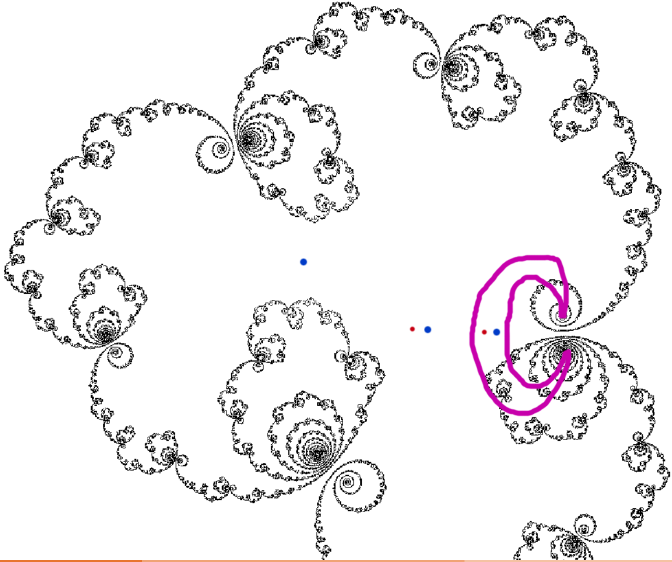
both critical points are in the basin of 0



$$\eta = \eta(P_s) = \frac{\log(g_{P_s}(c_2(P_S))) - \log(g_{P_s}(c_1(P_s)))}{\log 3} = \zeta(c_2) - \zeta(c_1)$$

$$\iff \varphi_{P_s}({P_s}^j(c_2)) \in R_{P_s}(0) \text{ on } [\varphi_{P_s}({P_s}^{j+[\eta]}(c_1)), \varphi_{P_s}({P_s}^{j+[\eta]+1})(c_1)]$$

They all belong to $U_{P_s,-} \to U_{Q,-}$.

$$\tilde{\sigma}(P_s) = \phi_{s,+}(c_1) - \phi_{s,-}(c_1) = \phi_{s,+}(c_1) = \phi_{s,+}(P_s^k(c_1)) - k$$

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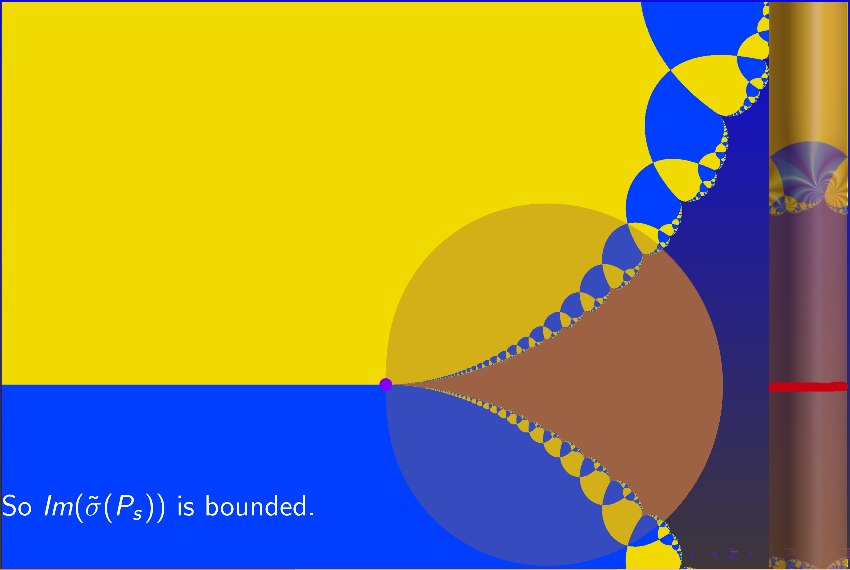

So $Im(\tilde{\sigma}(P_s))$ is bounded.

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For any $\sigma \in \Sigma$, $\exists s_n$ such that $\sigma(P_{s_n}) \to \sigma$.

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$$\Sigma = Acc\{\tilde{\sigma}(P_s)) \mod 1\} \text{ is connected}$$

$Re(\tilde{\sigma}(P_s)) \to -\infty$ ($k_s \to +\infty$) so Σ is a curve turning around $\mathbb{C}/\mathbb{Z}$.

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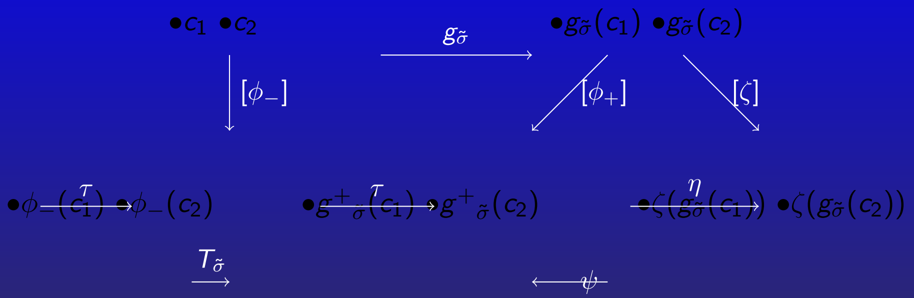
$g_{\tilde{\sigma}}(c_1) \in R_Q(0)$ and $\{\zeta(g_{\tilde{\sigma}}(c_1)) \mid \tilde{\sigma}\text{a lift of } \sigma \in \Sigma\}$ covers $\mathbb{R}/\mathbb{Z}$.



$\bullet c_1 \ \bullet c_2$
$g_{\tilde\sigma}$
$\bullet g_{\tilde\sigma}(c_1) \ \bullet g_{\tilde\sigma}(c_2)$
$[\phi_-]$
$[\phi_+]$
$[\zeta]$
τ
$\bullet\phi_-(c_1) \ \bullet\phi_-(c_2)$
τ
$\bullet g^+{}_{\tilde\sigma}(c_1) \ \bullet g^+{}_{\tilde\sigma}(c_2)$
η
$\bullet\zeta(g_{\tilde\sigma}(c_1)) \ \bullet\zeta(g_{\tilde\sigma}(c_2))$
$T_{\tilde\sigma}$
ψ

So $\psi([\zeta](g_{\tilde{\sigma}}(c_1)) + [\eta]) = \psi([\zeta](g_{\tilde{\sigma}}(c_1))) + \tau \mod 1$ for any $\sigma \in \Sigma$

so for any $x \in \mathbb{R}/\mathbb{Z}$ the magic formula holds

$$\psi(x + [\eta]) = \psi(x) + \tau \mod 1$$

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By analytic extension this is true on the annulus and extends continuously to the boundary so for any z in the annulus

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Thus the Julia set also contradiction since cusps are dense!

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Contradiction. Since the cusps are exactly the pre-images of the parabolic point. (McMullen)

Happy Birthday Jack