The Secret Combinatorial Garden of Siegel

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(C. L. Siegel, 1942): Quite often!

Hailed by a member of this audience as

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$$\widehat{c_k} := \varepsilon_{k-1} \cdot \left(\sum_{r=2}^{k} \sum_{\ell_1+\ldots+\ell_r=k} \widehat{c_{\ell_1}} \cdot \ldots \cdot \widehat{c_{\ell_r}} \right)$$

$$\begin{aligned}
\widehat{c_1} &= 1 \\
\widehat{c_2} &= \varepsilon_1\big([\widehat{c_1}\widehat{c_1}]\big) = \varepsilon_1\,, \\
\widehat{c_3} &= \varepsilon_2\big([\widehat{c_1}\widehat{c_2} + \widehat{c_2}\widehat{c_1}] + [\widehat{c_1}\widehat{c_1}\widehat{c_1}]\big) = 2\varepsilon_2\varepsilon_1 + \varepsilon_2\,, \\
\widehat{c_4} &= \varepsilon_3\big([\widehat{c_1}\widehat{c_3} + \widehat{c_2}\widehat{c_2} + \widehat{c_3}\widehat{c_1}] + [\widehat{c_1}\widehat{c_1}\widehat{c_2} + \widehat{c_1}\widehat{c_2}\widehat{c_1} + \widehat{c_2}\widehat{c_1}\widehat{c_1}] + [\widehat{c_1}\widehat{c_1}\widehat{c_1}\widehat{c_1}]\big) \\
&= \underbrace{4\varepsilon_3\varepsilon_2\varepsilon_1 + 2\varepsilon_3\varepsilon_2 + \varepsilon_3\varepsilon_1\varepsilon_1 + 3\varepsilon_3\varepsilon_1 + \varepsilon_3}\,.
\end{aligned}$$

$\widehat{c_k}$ is the sum of several products of SD-terms. Which one is largest? It depends on λ

$$\delta_k := \max\{\text{product of SD-terms in } \widehat{c_k}\}$$

$$\tau_k := \#\{\text{SD-products counted with multiplicity}\} \quad (\text{e.g., } \tau_4 = 4+2+1+3+1 = 11)$$

$\widehat{c_k} \leq \delta_k \cdot \tau_k$ $\longleftarrow$ Find exponential bounds for both δ_k and τ_k!

Diophantine vs Combinatorial

$$\widehat{c_k} := \varepsilon_{k-1} \cdot \left(\sum_{r=2}^{k} \sum_{\ell_1 + \ldots + \ell_r = k} \widehat{c_{\ell_1}} \cdot \ldots \cdot \widehat{c_{\ell_r}} \right)$$

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$$\delta_1 = 1$$

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(or how to deal with the τ_k)

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(or how to deal with the δ_k)

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Lemma

Given $r+1$ indices $k_0 > \ldots > k_r \geq 1$, the following holds:

$$\prod_{p=0}^{r} \varepsilon_{k_p} < \left(2^{2\nu+1}\right)^{r+1} \cdot k_0^\nu \prod_{p=1}^{r} \left(k_{p-1}-k_p\right)^\nu .$$

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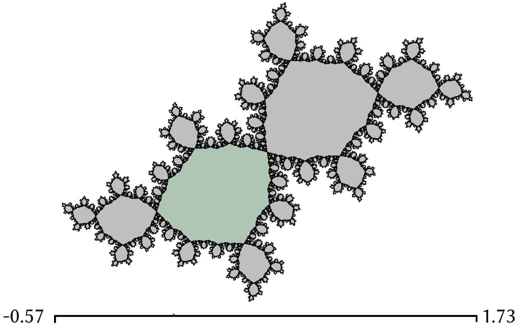

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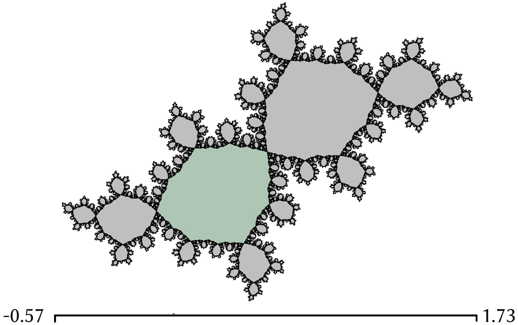

$$|c_k| \leq \widehat{c}_k \leq \delta_k \cdot \tau_k$$

Back to Schröder

$$\tau_1 = 1\,, \quad \tau_k = \left(\sum_{r=2}^{k} \sum_{\ell_1+\ldots+\ell_r=k} \tau_{\ell_1} \cdot \ldots \cdot \tau_{\ell_r}\right)$$

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When $k = 4$:

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Unravel the combinatorics

(joint with M. Aspenberg) The coefficients of the inverse linearization map φ^{-1} are given by the recursion

$$a_n = \varepsilon_{n-1} \cdot \sum_{r=\lceil \frac{n}{2} \rceil}^{n-1} \binom{r}{n-r} a_r$$

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Given n, every sequence $b_{s+1}, b_s, \ldots, b_1$ satisfying

$$n = b_{s+1} \succ b_s \succ \ldots \succ b_1 = 1$$

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Q: What do the binomials count?

Filling seq.	# of descents	contribution
11111	0	X^6
11115	0	X^6
11121	1	$X^5(1+X)$
11321	2	$X^4(1+X)^2$
12142	2	$x^4(1+X)^2$
12214	1	$X^5(1+X)$
12345	0	X^6

For $n = 6$ there are 5! sequences: 8 with two descents, 70 with one descent, and 42 with none:

$$\begin{aligned} a_6 =& 8X^4 + 86X^5 + 120X^6 \\ =& (8X^4 + 16X^5 + 8X^6) + 70X^5 + 112X^6 \\ =& 8X^4(1+X)^2 + 70X^5(1+X) + 42X^6(1+X)^0 \end{aligned}$$

- $\underline{x > 0}$: Highest degree coefficient is factorial, and therefore a_n grows super-exponentially
- $\underline{x < -1}$: All terms have same sign and will not cancel. Therefore a_n grows super-exponentially
- $\underline{x = -1}$: Only non-zero term comes from sequences without descents. These are classically counted by Catalan numbers

$x \in (-1, 0)$?

Analysis. . . finally!

Define $S_n(r) =$ sum of monomial contributions from sequences that end in r.

$$a_n = \sum_{j=1}^{n-1} S_n(j)$$

By induction

$$S_{n+1}(r) = X \sum_{j=1}^{r} S_n(j) + (1+X) \sum_{j=r+1}^{n-1} S_n(j) \qquad (1 \leq r \leq n-2)$$

There is no descent at the last position, so the last two terms are given by

$$S_{n+1}(n-1) = S_{n+1}(n) = X \sum_{j=1}^{n-1} S_n(j)$$

To simplify notation, define $Y = (1+X)$

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For every n we have a string of $n-1$ values. Collect them into a vector and rescale:

$$s_n := [S_n(1)/(n-2)!, \ldots, S_n(n-1)/(n-2)!]^{\perp} \in \mathbb{R}^{n-1}$$

Consider the $n \times (n-1)$ matrix A_n whose (i,j)-entry is X if $i \geq j$, and Y otherwise. Then (1) becomes

$$s_{n+1} = (A_n \cdot s_n)/(n-1)$$

Let $E_n : \mathbb{R}_{n-1} \longrightarrow L^2[0,1]$ map the standard basis vector e_j to the characteristic function of the interval $\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right)$

- The vector s_n maps to the function $E_n(s_n)$ such that $E_n(s_n)(u) = \frac{S_n(j)}{(n-2)!}$ whenever $u \in \left[\frac{j-1}{n-1}, \frac{j}{n-1}\right)$
- A_n embeds as a linear operator $A_n : L^2[0,1] \longrightarrow L^2[0,1]$ so that (1) becomes

$$E_n(s_{n+1})(u) = [A_n s_n](u) = \int_0^1 \alpha_n(u,v) \cdot E_n(s_n)(v)\, \mathrm{d}v$$

Restate

$$a_n = (n+1)! \int_0^1 s_n(v)\,\mathrm{d}v$$

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The kernel α_n is a piecewise constant function whose value at

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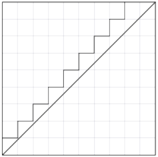

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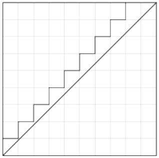

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To prove $\{a_n\}$ grows super-exponentially we need to find a sequence n_k so the exponential rate of decay of $\int s_{n_k}$ is bounded from below

Kernels

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$$(Tf)(u) = \int_0^1 \kappa(u, v) \cdot f(v)\, \mathrm{d}v$$

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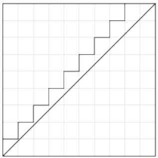

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Lemma T is the limit of $\{A_n\}$ in the operator norm:

$$\|T - A_n\|_2 \leq \frac{1}{\sqrt{n}}$$

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With the correct (weighted) norm

$$\langle f, g\rangle := \int_0^1 \|\tfrac{X}{Y}\|^{-2v} f(v)\overline{g}(v)\,\mathrm{d}v$$

the family of eigenfunctions forms an orthonormal basis for $L^2[0,1]$

Real eigenspace

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Note that for $m \geq 0$ the pair of functions $f_{(m+1)}, f_m$ are complex conjugate and their eigenvalues have the same magnitude. As a consequence, a convenient basis for the subspace $L^2_{\mathbb{R}}[0,1] \subset L^2[0,1]$ of real-valued functions is

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so that $L^2_{\mathbb{R}}[0,1] = E \oplus E^{\perp}$.
By Parseval's theorem we can define the angle θ_n by

$$\sin\theta_: = \frac{\|P^{\perp} s_n\|_2}{\|s_n\|_2}$$

Intuitively, the closer θ_n is to 0, the better s_n resembles a function in E.

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Step 2: The sequence $\{\theta_n\}$ converges to 0, so the functions s_n become progressively sinusoidal.

Step 3: There is a sequence of indices $\{n_k\}$ such that $\{|a_{n_k}|\}$ is comparable to $\{\|s_{n_k}\|_2\}$. Meanwhile, $\|s_n\|_2 \geq (\lambda - \varepsilon)^n$ for arbitrarily small ε, and the result follows

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If done TRULY correctly, the cancellation within a class leaves a polynomially large contribution, and then we can estimate the correct rate of exponential growth of the coefficients a_n of φ^{-1}

Work in progress...

Thank You Jack!!