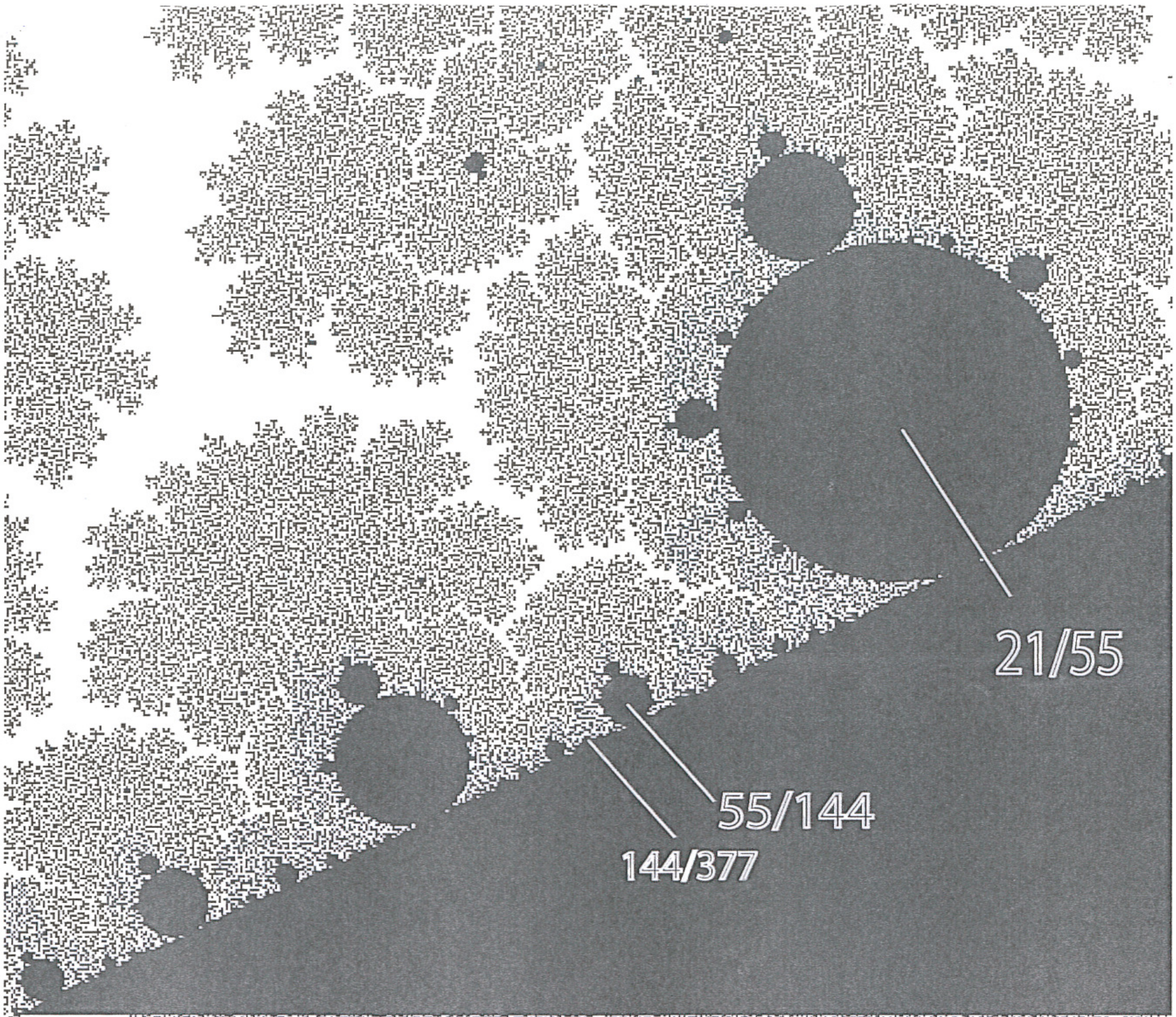
Pacman Renormalization

and

self-similarity of M near Siegel parameters

joint w. Dima Dudko
& Nikita Selinger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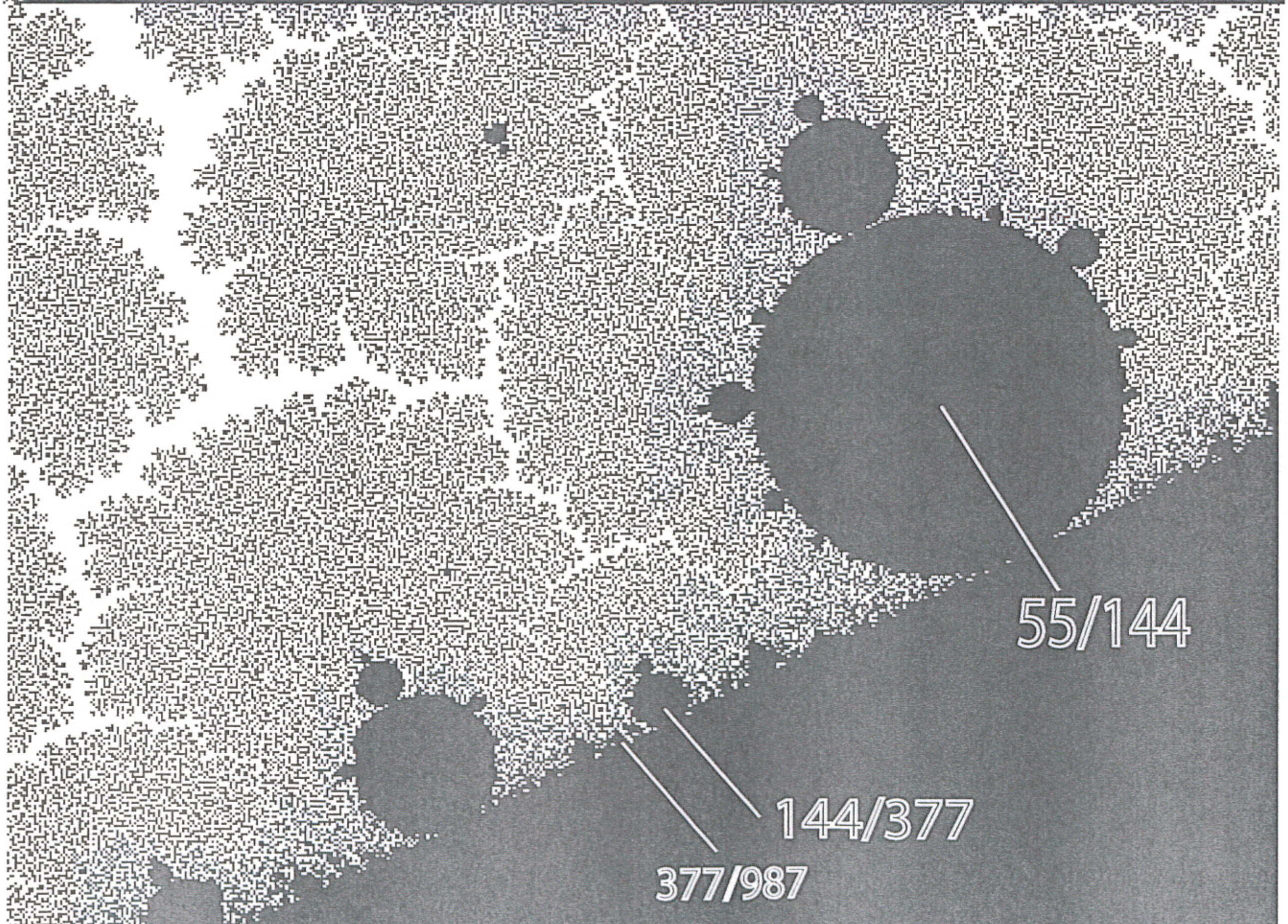
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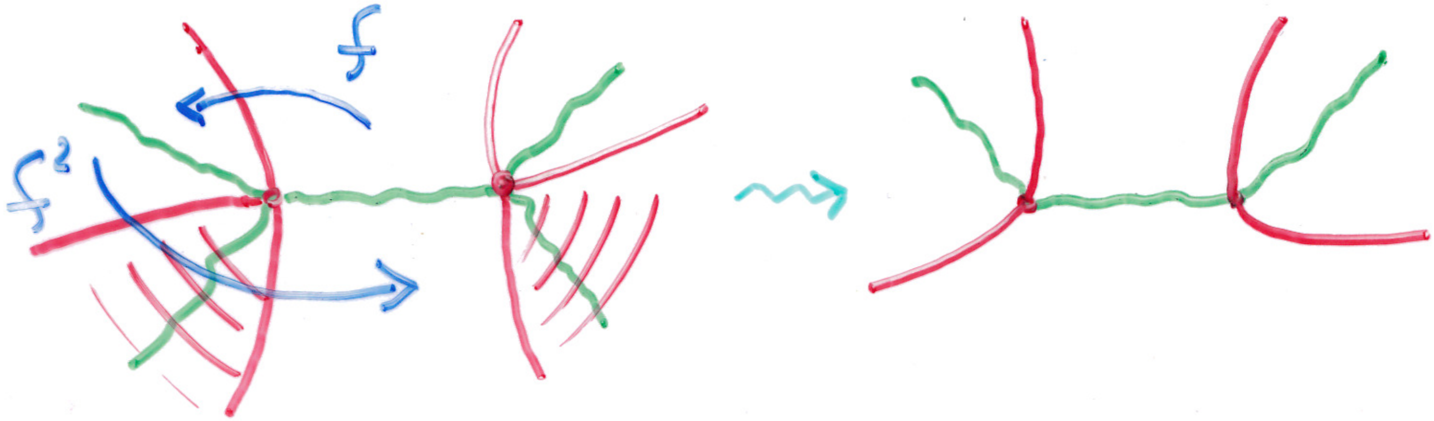
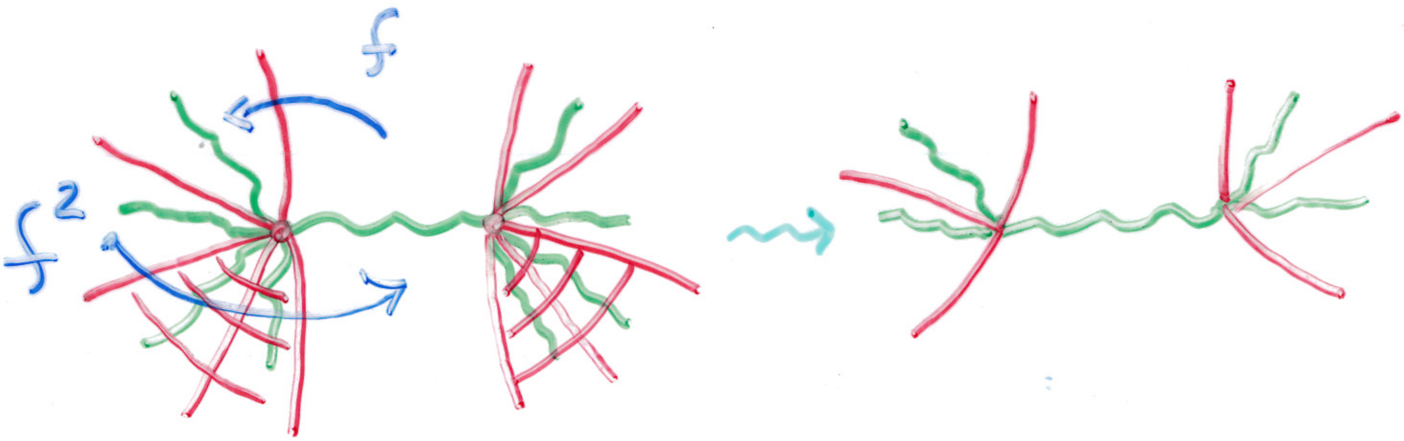
Branner-Douady Surgery



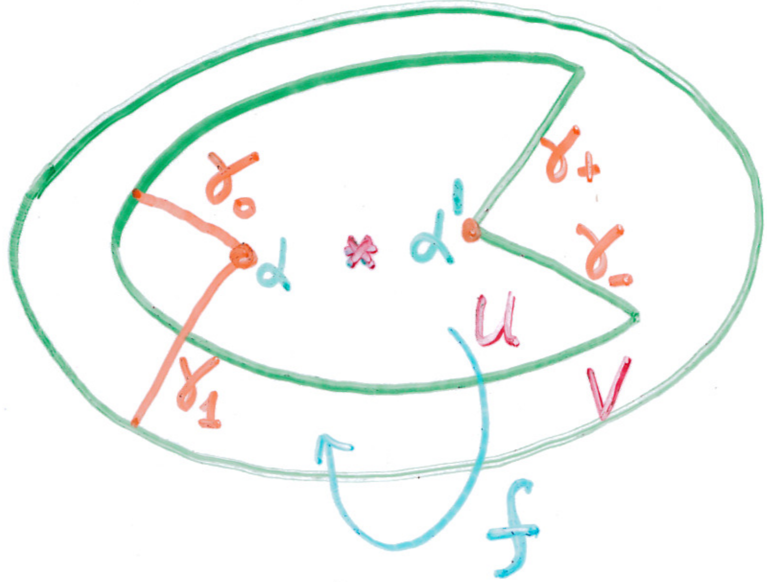
Part of $\mathcal{L}_{1/3} \xrightarrow{\sim} \mathcal{L}_{1/2}$

More generally:



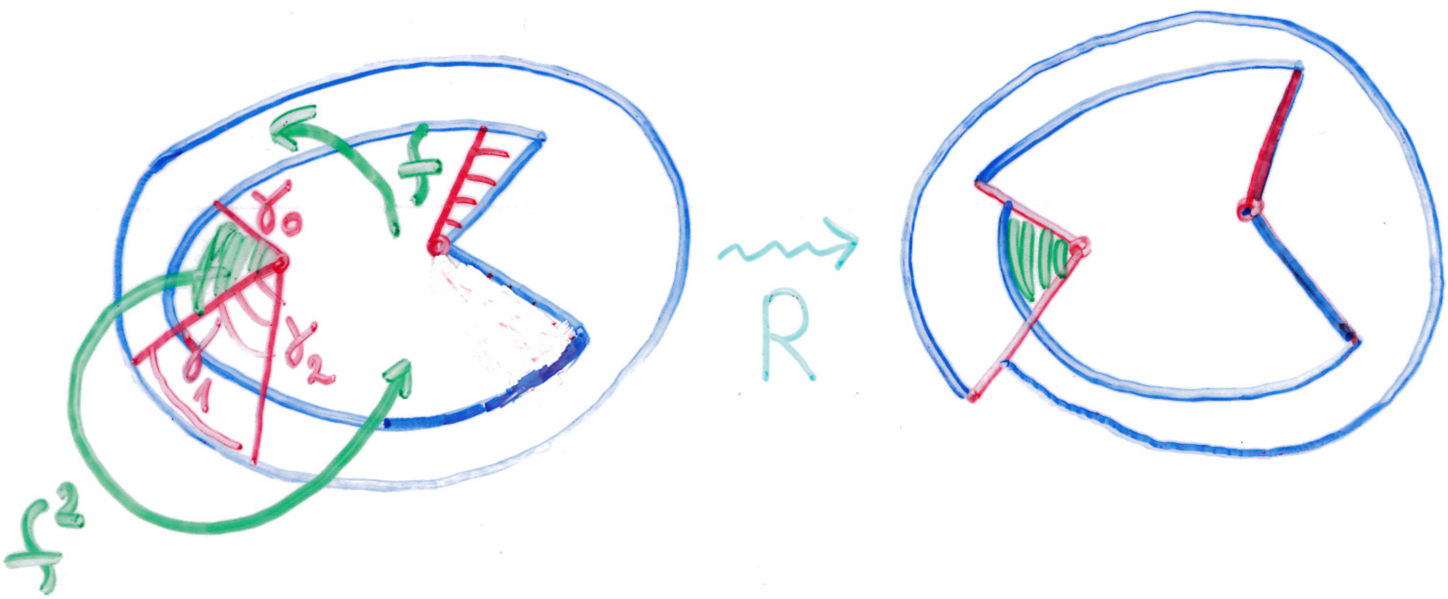
Part of $\mathcal{L}_{p/q} \xrightarrow{\sim} \mathcal{L}_{p/(q-p)}$ $\left(0 < p/q < \frac{1}{2}\right)$

Pacmen



$f : U \setminus \gamma_0 \to V \setminus \gamma_1$ is 2-to-1 branched covering

Pacman Renormalization



Hyperbolicity Theorem

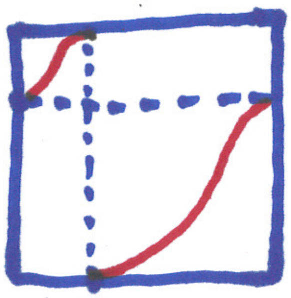
(Dudko – L – Selinger)

- The Pacman Renormalization can be locally realized as a holomorph op-r.
- For any periodic quadratic irrational θ, $\exists$ a renormalization per pt f_* which is a Siegel pacman with rotation number θ.
- f_* is hyperbolic.
- The unstable man-d W^u is 1D and is parametrized by rotation numbers near θ.
- The stable man-d W^s consists of Siegel pacmen with rot # θ; all of them are hybrid equivalent.

Critical circle maps

$f : \mathbb{R}/\mathbb{Z} \circlearrowleft$

$f'(0) = 0, \sim x^3$



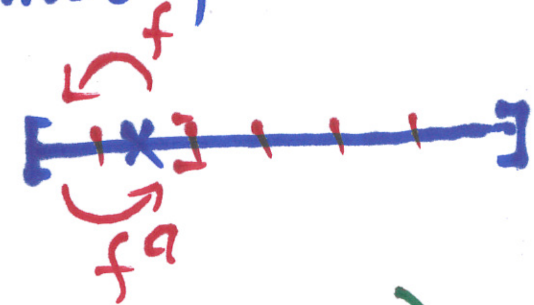
0

commuting pairs

periodic

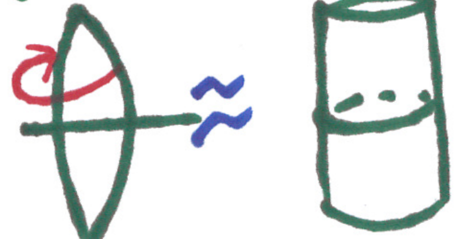
Standing Assumption: Θ is quadr irr-l

- Renormalization of commut pairs (Feigenbaum-Kadanoff...)



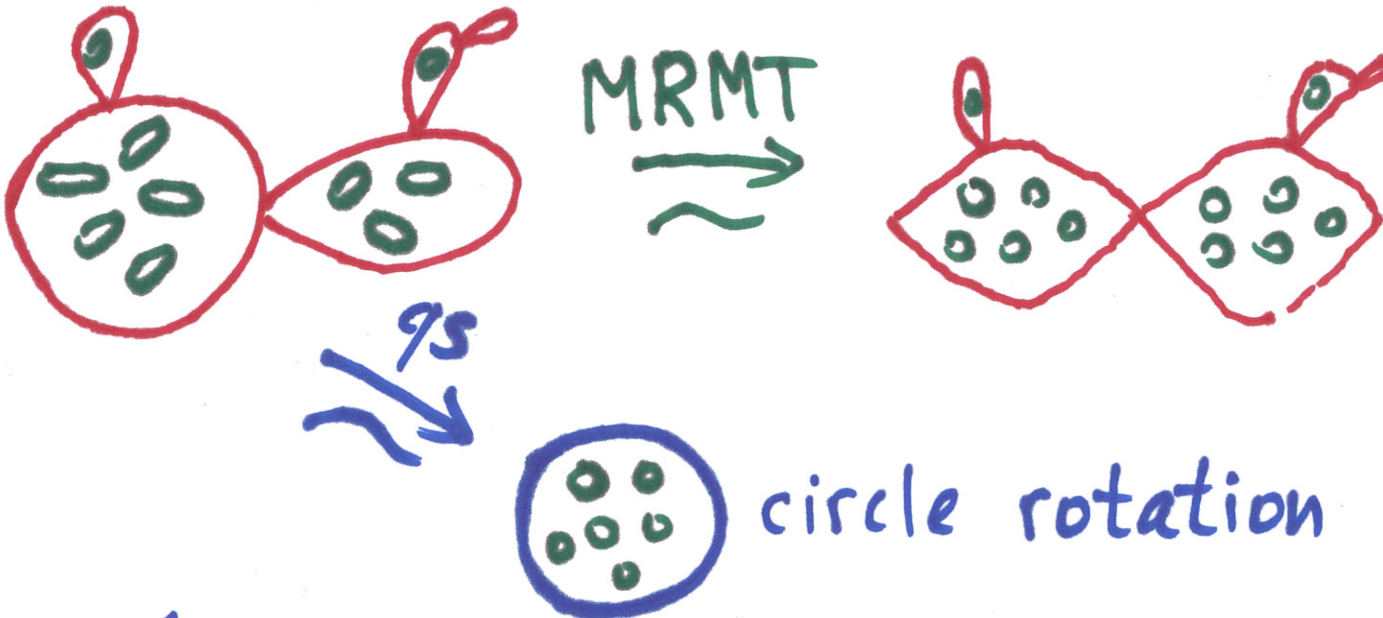
- Real a priori bounds & qs conjugacy (Swiatek, Herman)
- Butterfly Renormalization & Complex Bounds (de Faria)

- Hybrid equivalence, renormalization fixed pt, exp convegence (dF - de Melo)
- Cylinder Renorm-n & Hyperbolicity (Yampolsky)



Douady-Ghys surgery

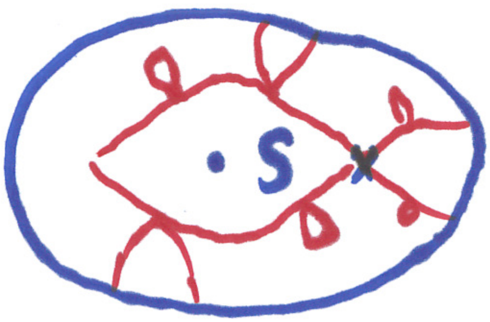
Blaschke product $B_\alpha(z) = e^{2\pi i \alpha} z^2 \frac{z-3}{1-3z} \rightsquigarrow$ Siegel pol-l $f_\theta(z) = e^{2\pi i \theta} z + z^2$



circle rotation

<u>Thm</u> (Petersen) $J(f_\theta)$ is loc conn [bubble rays]

Siegel maps



$f : U \to V$, $f'(c) = 0$

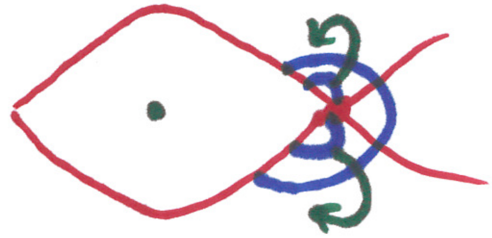
$S \Subset U$, ∂S is a q-circle, $\partial S \ni c$

quasi-critical circle map

<u>Corollary</u> (Avila-L) Controlled geometry, hybrid equivalence.

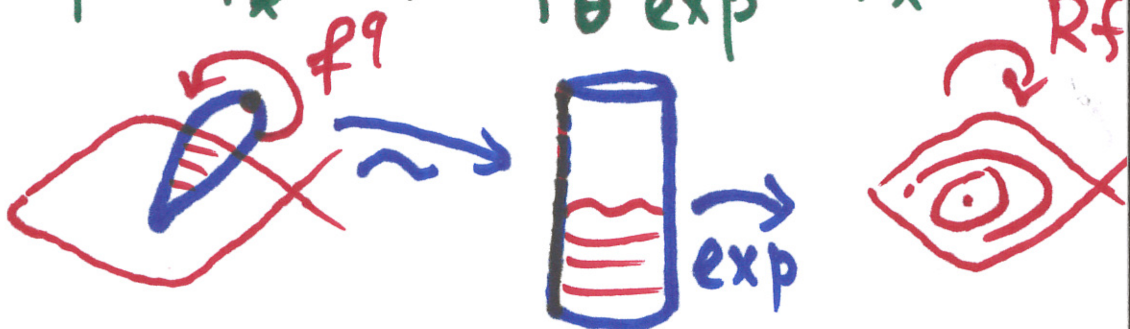
Siegel Renormalization

Butterfly:



<u>Thm</u> (McMullen) $\exists$ a Siegel butterfly renorm fixed pt f_*; $R^n f_{\theta\,exp} \longrightarrow f_*$.

Cylinder:



<u>Prop</u> (Y). The cylinder renormalization has a stable man-d of codim $< \infty$ and an unstable man-d of dim ≥ 1.

<u>Thm</u> (Gaidashev-Yampolsky) For the golden mean rot #, f_* is hyperbolic. [computer assisted]

<u>Thm</u> (Inou-Shishikura) For θ of high type, f_* is hyperbolic [parabolic perturb]

Strategy

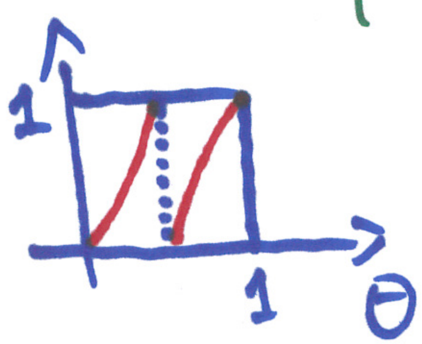
Step 1: Promotion of the Siegel renorm fixed pt to a pacman f_*

← "pre-pacman" built of external rays & equipotentials

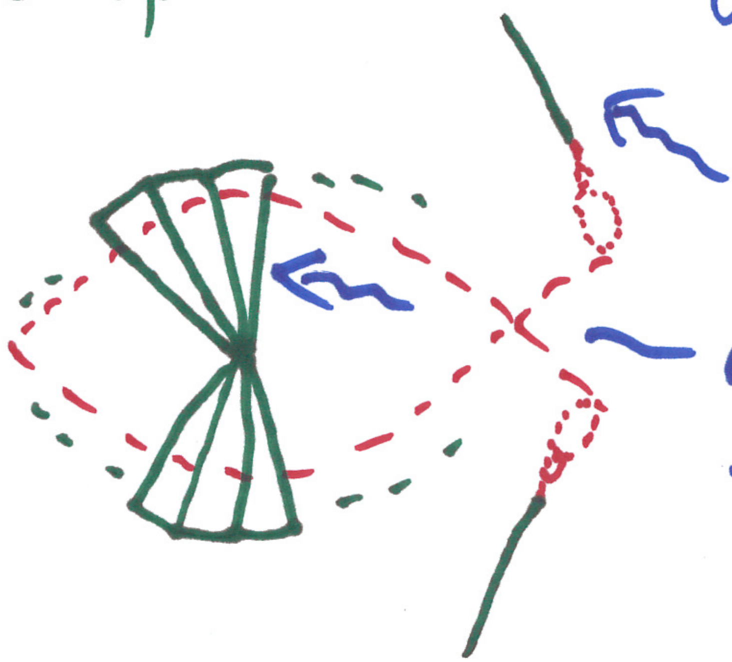
Step 2: $W^s(f_*)$ is a hybrid class
[Pullback Argument]

Step 3: $\dim W^u(f_*) \geq 1$
[expanding action on rot numbers]

$$\theta(Rf) = \begin{cases} \dfrac{\theta}{1-\theta}, & 0 \leq \theta \leq 1/2 \\[2mm] \dfrac{2\theta-1}{\theta}, & \frac{1}{2} \leq \theta \leq 1 \end{cases}$$



Step 4: Robust geometric control of f near f_*.



Step 5: Maximal pre-pacmen.

For $f \in W^u(f_*)$, $\exists$ a σ-proper holomorphic extension $f: \mathbf{W} \to \mathbb{C}$.

Corollary. The critical orbit is captured in $\mathbf{W}$.

Step 6: $\dim W^u(f_*) = 1$

[λ- lemma argument]

Step 7: No neutral directions

[Small Orbits Lemma]

Scaling of the limbs

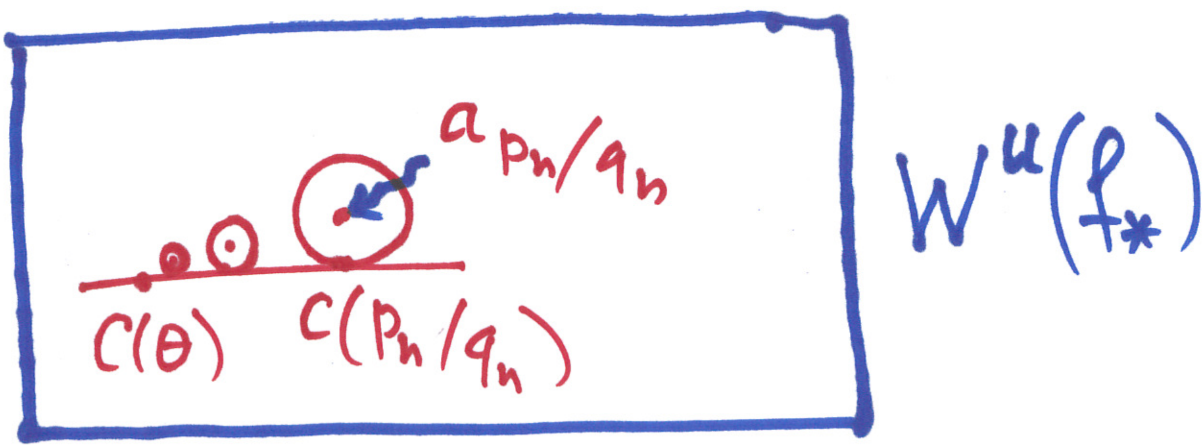
Thm Let Θ be a (periodic) quadratic irrational, and let p_n/q_n be its continued fraction approximands.

Then
$$|c(\Theta) - a_{p_n/q_n}| \sim \frac{1}{q_n^2}$$

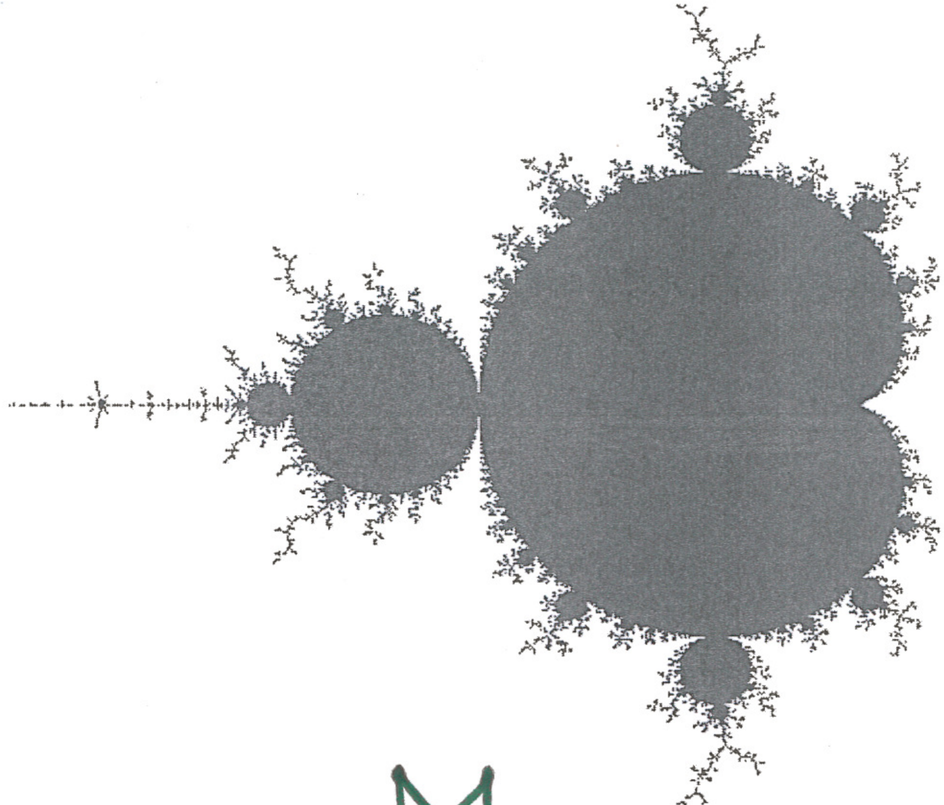
$c(\Theta)$: Siegel pt

a_{p_n/q_n}: centers of satellite comp-s

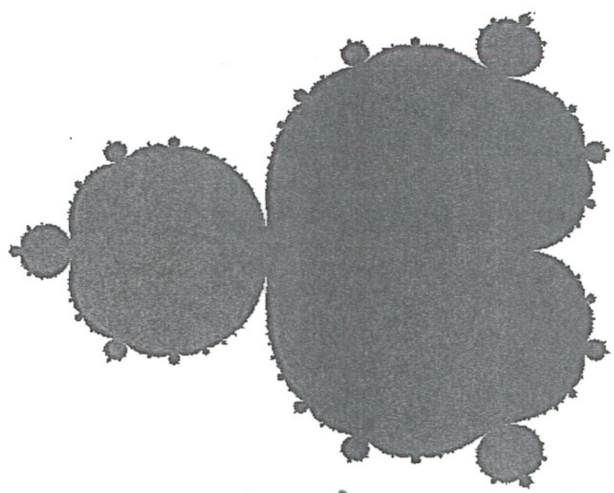
[Qc deformation argument, to realize a_{p_n/q_n} on $W^u(f_*)$]



Conjecture: Full Renormalization Horseshoe



M



Cubic model for Molecule

$z \mapsto z(z+1)^2$