Sheaves and Cohomology

I want to claim that I understand something about the space $\widehat{Quad}$

What does that mean?

Certainly one possible meaning is that I can compute its cohomology.

The authorities (Araceli) say that I need to define Čech cohomology.

The leitmotiv

- Sheaves are local data
- cohomology is a tool to extract global information from local data

Let me spell out what this means.

Sheaves are local data

A *sheaf* $\mathcal{F}$ on a topological space X
with values in a category $\mathcal{C}$
assigns an object $\mathcal{F}(U)$ of $\mathcal{C}$
to every open subset $U \subset X$;
together with a restriction map

$$\rho_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that $\rho_V^W \circ \rho_U^V = \rho_U^W$
when $W \subset V \subset U$ are nested open sets.

A different (better) way of saying this:

Let $\mathrm{Open}(X)$ be the category whose objects are open subsets of X, and whose morphisms are inclusions.

$\mathcal{F}$ is a functor $\mathrm{Open}(X) \to \mathcal{C}$.

For $\mathcal{F}$ to be a sheaf, it must be *local*:

If $U \subset X$ is open and
$U = \bigcup_i U_i$ is an open cover of U,
then the sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact. That means that $\mathcal{F}(\mathcal{U})$ is the equalizer of the pair of arrows on the right.

If you think about, you will see that this is what local ought to mean.

This sounds like gobbledygook without examples

Example 1. If A is any set,

$A_X(U)$ is the set of locally constant maps $U \to A$.

Note that the "set of constant maps $U \to A$" is not a sheaf, it is not local.

Essentially all of algebraic topology studies locally constant sheaves.

If A is a group, abelian group, ring, field, vector space, Banach algebra,. . .
then A_X is a sheaf of groups, . . .
Each of these is an important example.

There are zillions of other important examples.
I will give several used by
Xavier Gomez-Mont in his lecture yesterday

Let S be a complex surface, and F a (singular) analytic foliation of F by Riemann surfaces.
Like Xavier, ignore the singularities.
The sheaves Xavier studied were:
the sheaf of holomorphic vector fields on S tangent to F;
the sheaf of holomorphic vector fields on S;
the same replacing “holomorphic” by “C^∞”;
the sheaf of normal vector fields.
(I felt he was a bit swift about those!)

Cohomology

Comology exists only for sheaves of abelian groups:

sheaves where you can add in $\mathcal{F}(U)$
and addition is commutative.

We will define Čech cohomology.
This only works for paracompact spaces.

Let $\mathcal{U}$ be an open cover of X.
For convenience, order it.
Make a cochain complex

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F}) \to \dots$$

Here

$$C^k(\mathcal{U},\mathcal{F}) = \prod_{i_0<i_1<\cdots<i_k} \mathcal{F}(U_{i_0}\cap\cdots\cap U_{i_k})$$

and

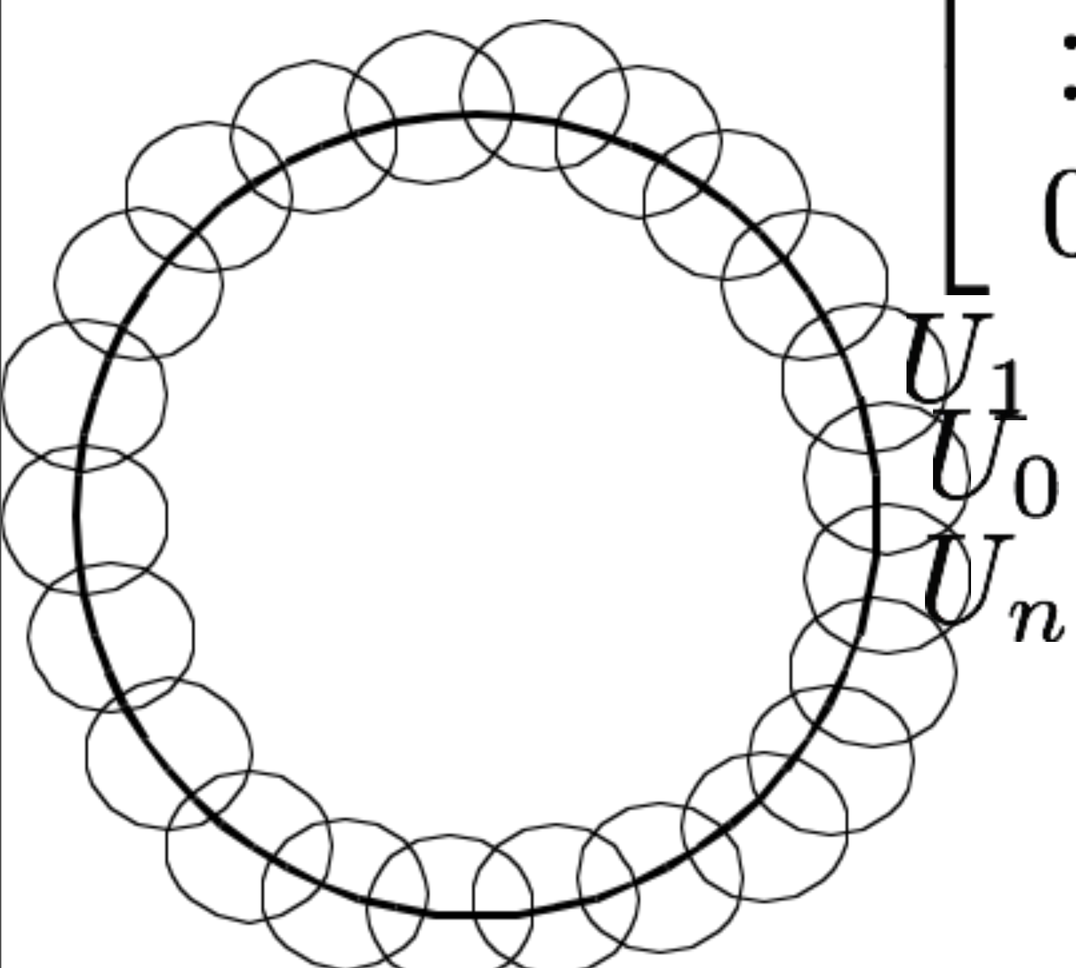
$$d\phi(U_{i_0}\cap\cdots\cap U_{i_{k+1}}) =$$
$$\Sigma_{j=0}^{k+1}(-1)^j\phi(U_{i_0}\cap\cdots\cap\widehat{U_{i_j}}\cap\cdots\cap U_{i_{k+1}})$$

It is easy to check that $d^{k+1}d^k = 0$: this is a cochain complex

Set $H^k(\mathcal{U},\mathcal{F}) = \ker d^k/\,\mathrm{image}\, d^{k-1}$.

Example Cover the circle by $U_i, i = 0, \ldots, n$
Then $C^0(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}^{k+1}$, $C^1(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}^{k+1}$
all others vanish, and d^0 is the matrix

$$\begin{bmatrix} 1 & 0 & \ldots & -1 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ldots & 0 \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$



The kernel is spanned by $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

So the cohomology in dimension 0 and 1 are $\mathbb{Z}$

Getting rid of the cover

We need to define $H^k(X, \mathcal{F})$, not $H^k(\mathcal{U}, \mathcal{F})$.

Define $H^k(X, \mathcal{F}) = \lim\limits_{\rightarrow} H^k(\mathcal{U}, \mathcal{F})$

with the direct limit over all open covers,

ordered by refininement

There is a trap here. Just because $\mathcal{V}$ refines $\mathcal{U}$ doesn't give a map $C^k(\mathcal{U}, \mathcal{F}) \to C^k(\mathcal{V}, \mathcal{F})$.

You need a refining map $\mathcal{V} \to \mathcal{U}$ which tells for each $V \in \mathcal{V}$ a particular element of $U \in \mathcal{U}$ such that $V \subset U$.

But covers with refining maps are not filtering and there is no direct limit on the level of cochains.

Fortunately all refining maps induce chain-homotopic maps on the cochains, so they coincide on the cohomology and the direct limit

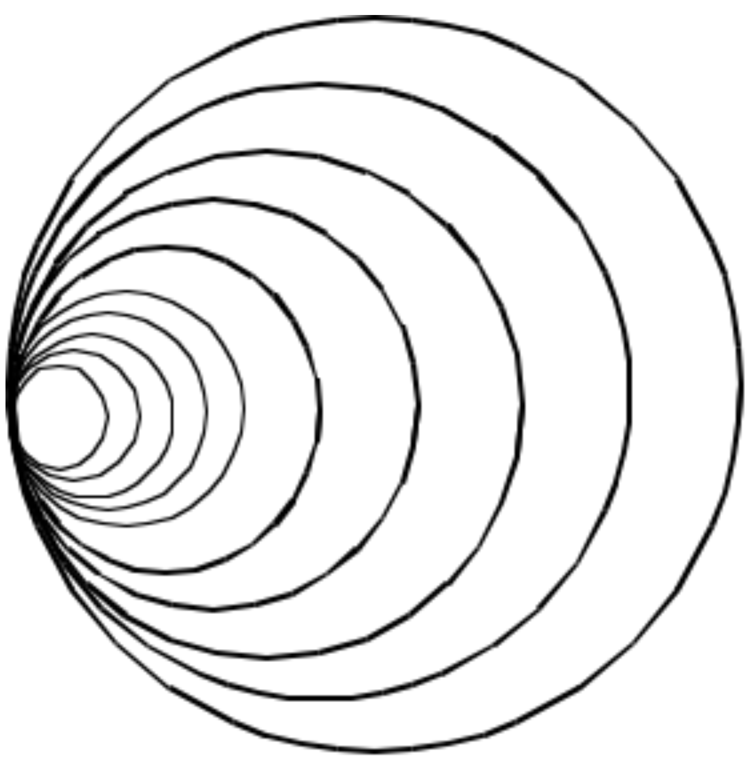
$$\varinjlim H^k(\mathcal{U}, \mathcal{F})$$

is well defined

The meaning of the direct limit

The open covers where one open set covers the bad point are cofinal. For such an open cover $\mathcal{U}$, only some finite number $k(\mathcal{U})$ of the circles are visible,

so $H^1(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}^{k(\mathcal{U})}$.



Exact sequences

The work horse of cohomology theory is the long exact sequence associated to a short exact sequence of sheaves.

A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is what you think: for every open set $U \subset X$ a morphism $\mathcal{F}(U) \to \mathcal{G}(U)$ compatible with restrictions.

It is a natural transformation of functors

For sheaves of groups (or abelian groups...),
there is an obvious kernel sheaf:
if $f : \mathcal{F} \to \mathcal{G}$ is a morphism,
then $(\ker f)(U) = \ker(f_U : \mathcal{F}(U) \to \mathcal{G}(U)$.
you should check that this is indeed a sheaf.

The cokernel is much more delicate to define:
the assignment $U \mapsto \mathrm{coker}(\mathcal{F}(U) \to \mathcal{G}(U))$
is not a sheaf, it doesn't satisfy
the locality condition.

One way to deal with this difficulty is *stalks*.
If $\mathcal{P} : \mathrm{Open}(X) \to \mathcal{C}$ is any functor a.k.a. a presheaf, then the stalk at $x \in X$ is

$$\mathcal{P}_x = \varinjlim \mathcal{P}(\mathcal{U}),$$

where U runs through all neighborhoods of x, ordered by inclusion.

One way to say that a sequence

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact is to say that it is exact on stalks:

Stalks tend to be ridiculously big.
For locally constant sheaves, no problem.
But germs of holomorphic functions are convergent power series (already scary), and germs of smooth functions and continuous functions are hard to imagine.

Another way is to say that for any $U \subset X$, the map $g \circ f : \mathcal{F}(U) \to \mathcal{H}(U)$ is the 0 map, and $\forall U \subset X, \forall \alpha \in \mathcal{G}(U)$ with $g(\alpha) = 0, \forall x \in U$

there exists a neighborhood $V \subset U$ of x and $\beta \in \mathcal{F}(V)$ such that $f(\beta) = \rho_V^U(\alpha)$.

An important example

Let X be a topological space, and $\mathcal{C}_X$ be the sheaf of complex-valued continuous functions.
Let $\mathcal{C}_X^*$ be the sheaf of non-vanishing complex valued functions.
The sequence

$$0 \to \mathbb{Z}_X \to \mathcal{C}_X \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{C}_X^* \to \{1\}$$

is a short exact sequence of sheaves.

If X is a complex manifold, we could do the same with analytic functions:

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1.$$

It is amazing how much one can get from the associated long exact sequence

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$

be a short exact sequence of sheaves.

There is then a long exact sequence

$$
\begin{aligned}
0 &\to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to \\
&\to H^1(X,\mathcal{F}) \to H^1(X,\mathcal{G}) \to H^1(X,\mathcal{H}) \to \\
&\to H^2(X,\mathcal{F}) \to H^2(X,\mathcal{G}) \to H^2(X,\mathcal{H}) \to \\
&\to \ldots
\end{aligned}
$$

Of course, the long exact sequence comes from the diagram of complexes

$$
\begin{array}{ccccccccc}
0 & \to & C^{i-1}(\mathcal{U},\mathcal{F}) & \to & C^{i-1}(\mathcal{U},\mathcal{G}) & \to & C^{i-1}(\mathcal{U},\mathcal{H}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{i}(\mathcal{U},\mathcal{F}) & \to & C^{i}(\mathcal{U},\mathcal{G}) & \to & C^{i}(\mathcal{U},\mathcal{H}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{i+1}(\mathcal{U},\mathcal{F}) & \to & C^{i+1}(\mathcal{U},\mathcal{G}) & \to & C^{i+1}(\mathcal{U},\mathcal{H}) & \to & 0
\end{array}
$$

But the lines are not exact on the right!
The connecting homomorphism can only be constructed after refining the cover.
The long exact sequence exists only for cohomology, after taking the direct limit over open covers.

Essentially all the long exact sequences of cohomology are special cases of this one.

For instance the long exact sequence of a pair.

Let $Y \subset X$ be the inclusion of a closed subset.

Let G be an abelian group. Define the sheaves on X

$$G^X_Y(U) = G_X(Y \cap U),$$
$$G_{X/Y}(U) = \{\alpha \in G_X(U) \mid \alpha|_{U \cap Y} = 0.$$

The sequence

$$0 \to G_{X/Y} \to G_X \to G^X_Y \to 0$$

is a short exact sequence of sheves.

It isn't completely obvious that this is true.

It definitely requires that Y be closed.

The associated long exact sequence is well-known:

$$0 \to H^0(X,Y;G) \to H^0(X;G) \to H^0(Y;G) \to$$
$$\to H^1(X,Y;G) \to H^1(X;G) \to H^1(Y;G) \to \dots$$

Another example:
let X be a compact Riemann surface of genus g.

Consider the short exact sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to \{1\}.$$

G_{X/Y}

The associated long exact sequence is

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to$$
$$\to H^1(X;\mathbb{Z}) \to H^1(X;\mathcal{O}_X) \to H^1(X;\mathcal{O}_X^*) \to$$
$$\to H^2(X,\mathbb{Z}_X) \to 0.$$

The top line is exact by itself.
The next two lines contain most of
the theory of compact Riemann surfaces

$H^1(X;\mathbb{Z}_X) \equiv \mathbb{Z}^{2g}$ is embedded as a lattice in $H^1(X;\mathcal{O}_X)$, which is a $\mathbb{C}$-vector space of dimension g.

The map $H^1(X;\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}_X) \approx \mathbb{Z}$ is the first Chern class, and $H^1(X;\mathcal{O}_X^*)$ is the Picard variety, i.e., the set of isomorphism classes of analytic line bundles on X.

The kernel of this map, i.e., the set of analytic line bundles of chern class 0, is (by exactness) the quotient $H^1(X;\mathcal{O}_X)/H^1(X;\mathbb{Z}_X)$. It the the quotient of a vector space by a lattice, i.e., a complex torus called the *Jacobian* of X.

Making sense of all this is what Riemann surface theory is about.