Limiting Dynamics of Conformal dynamical systems

Jackfest, Cancun 2016

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I dedicate these lectures to the memory of

Tan Lei

who died April 1, 2016 of pancreatic cancer.

I beg Hans Henrik, Charlotte and Paul
to accept my most sincere sympathy

I imagine everyone the complex dynamics world knew and loved Tan Lei. She was one reason why the group is so friendly, and mathematically so productive.

Of course, understanding limits of dynamical systems is of the greatest interest.

Think of Perelman's proof of the geometrization conjecture!

In any generality the project appears unreasonable.

I will focus on limits of polynomials as dynamical systems, and on limits of Kleinian groups.

Of course, the interesting case occurs when
the system is unstable,
when the system is bifurcating.

It is much easier to say just what the
limiting dynamical system
is in the case of Kleinian groups,
because of the Chabauty topology

The Chabauty topology is a topology on the
space of closed subgroups of an arbitrary
locally compact group.

Let G be a locally compact group, and for convenience, suppose that the 1-point compactification

$$\overline{G} = G \sqcup \{\infty\} \quad \text{is metrizable.}$$

Give the space of closed $\mathrm{Cl}(\overline{G})$ of subsets of $\overline{G}$ the Hausdorff metric.

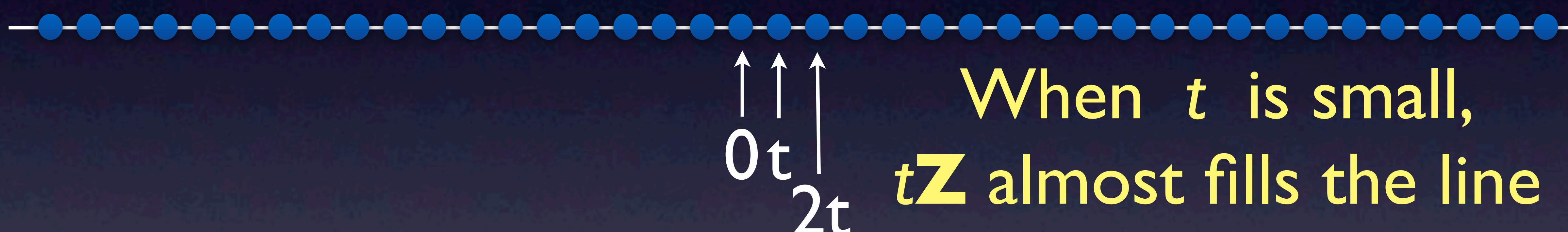
For every closed subgroup $H \subset G$, let $\overline{H} = H \cup \{\infty\}$.

The map $H \mapsto \overline{H}$ makes the space of closed subgroups of G into a compact subset of $\mathrm{Cl}(\overline{G})$.

The easiest example: $G = \mathbb{R}$

The closed subgroups are $t\mathbb{Z},\ t > 0$,
$\{0\}$ and $\mathbb{R}$.

$$\lim_{t \to 0} t\mathbb{Z} = \mathbb{R}, \qquad \lim_{t \to \infty} t\mathbb{Z} = \{0\}.$$



So the space of closed subgroups is homeomorphic to the closed interval $[0, \infty]$

The space of closed subgroups of $\mathbb{R}^2$ is a 4-sphere containing a knotted 2-sphere.

Nobody understands the set of closed subgroups of $\mathbb{R}^3$.

As these examples show, Chabauty limits of discrete groups may be non-discrete.

This doesn't happen nearly so much for non-elementary groups.

Vicky Chuckrow's theorem

Let Γ be a non-elementary group, and let $\rho_n : \Gamma \to G$ be a sequence of representations with discrete images. If the ρ_n converge pointwise, then all Chabauty limits of the $\rho_n(\Gamma)$ are discrete.

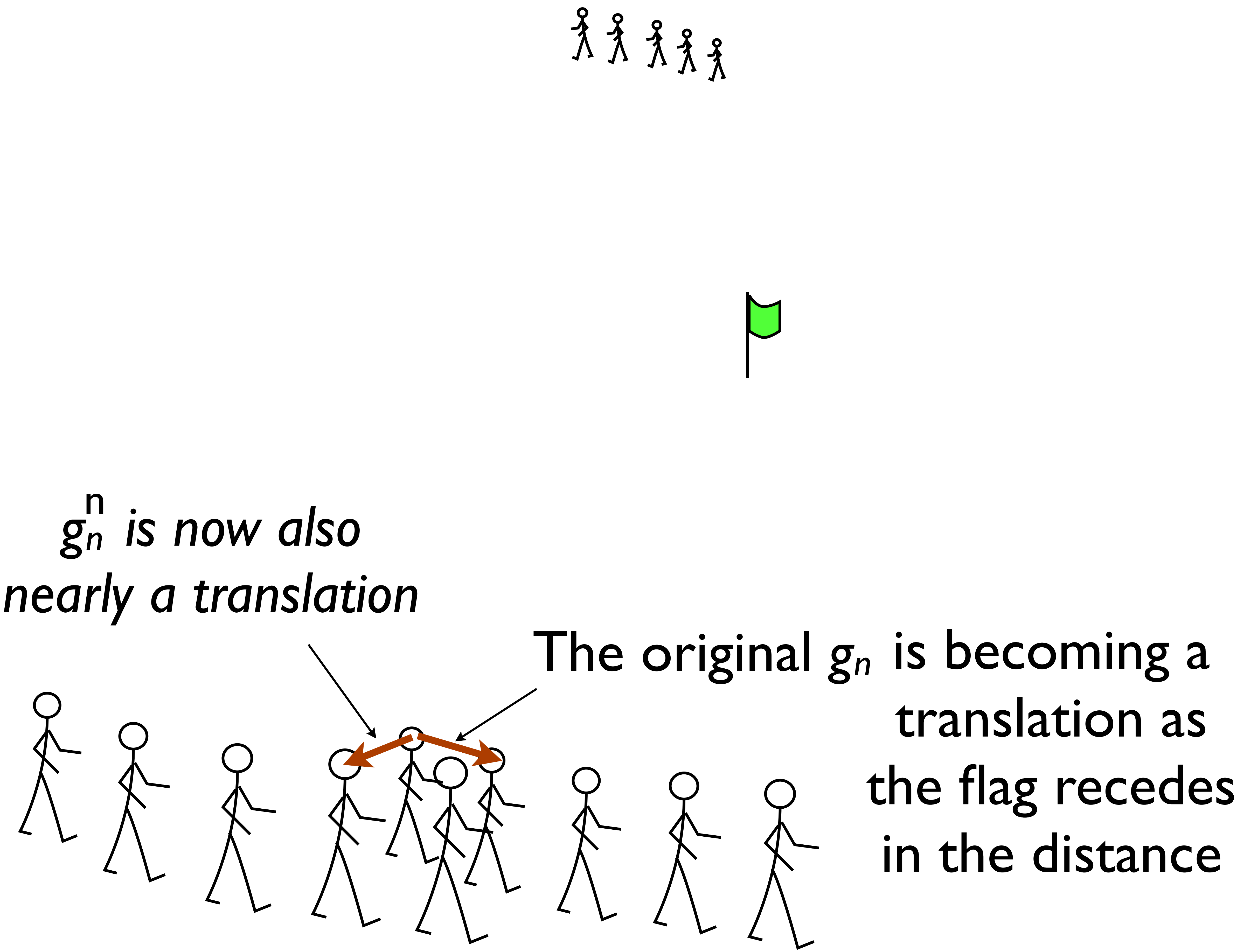
Discrete these limits may be, but they are not necessarily isomorphic to the algebraic limiting groups.

They may be enriched, meaning that they have acquired extra generators.

Text

This process of "enriching" exists for both Kleinian groups (Thurston, Kerchoff) and for polynomials (and rational functions, ...) (Douady, Lavaurs, Epstein)

Enriching is the key to this lecture.
Let us see it at work!


g_n^n is now also nearly a translation
The original g_n is becoming a translation as the flag recedes in the distance

The same in formulas

$$\text{Let } f_n(z) = \left(1 - \frac{a}{n^2}\right) e^{2\pi i/n}\,(\text{z-n})+\text{n}$$

$$Then \qquad \lim_{n\to\infty} f_n(z) = z + 2\pi i$$

$$\text{but} \qquad \lim_{n\to\infty} f_n^{\circ n}(z) = z + a$$

Note that $\mathrm{Tr}\, f_n \to \pm 2$, but not any old way.

it approaches tangentially to $[-2, 2]$

Let us see how these ideas apply to non-elementary Kleinian groups.

There is a beautiful program called Opti written by Masaaki Wada.

It examines 2-generator subgroups of $PSL_2 \mathbb{C}$ whose commutator $[a, b]$ is parabolic, in fact $[a, b] : z \mapsto z + 2$.

The space of such groups has 3 complex dimensions.

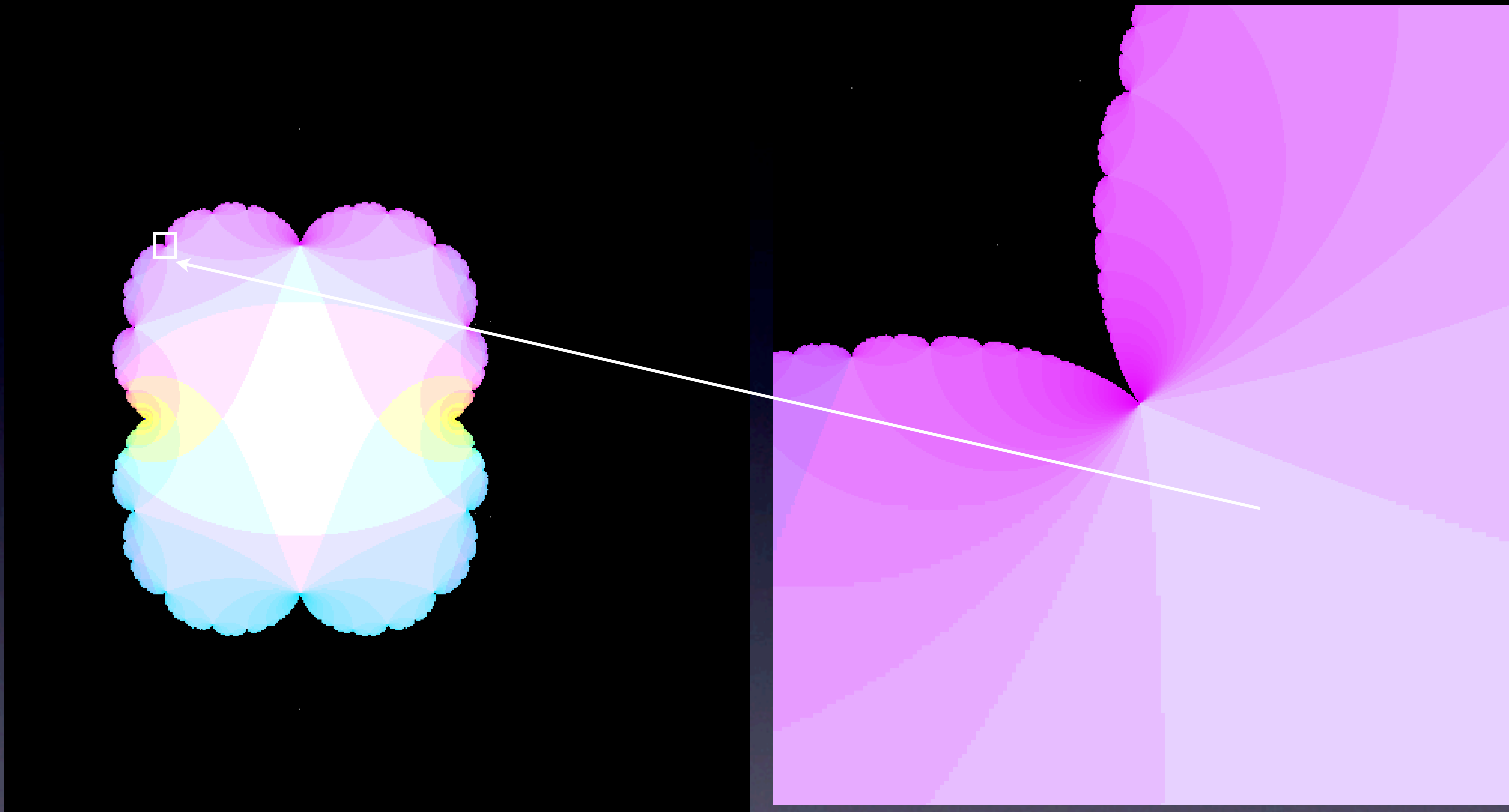
You can still make one more normalization, requiring that 0 is a fixed point of a.

The program makes fascinating pictures.

In the dynamical plane, it draws fundamental domains and limit sets.

In the parameter space, it draws complex slices through the 2-dimensional space of groups.

It isn't obvious how it does this.



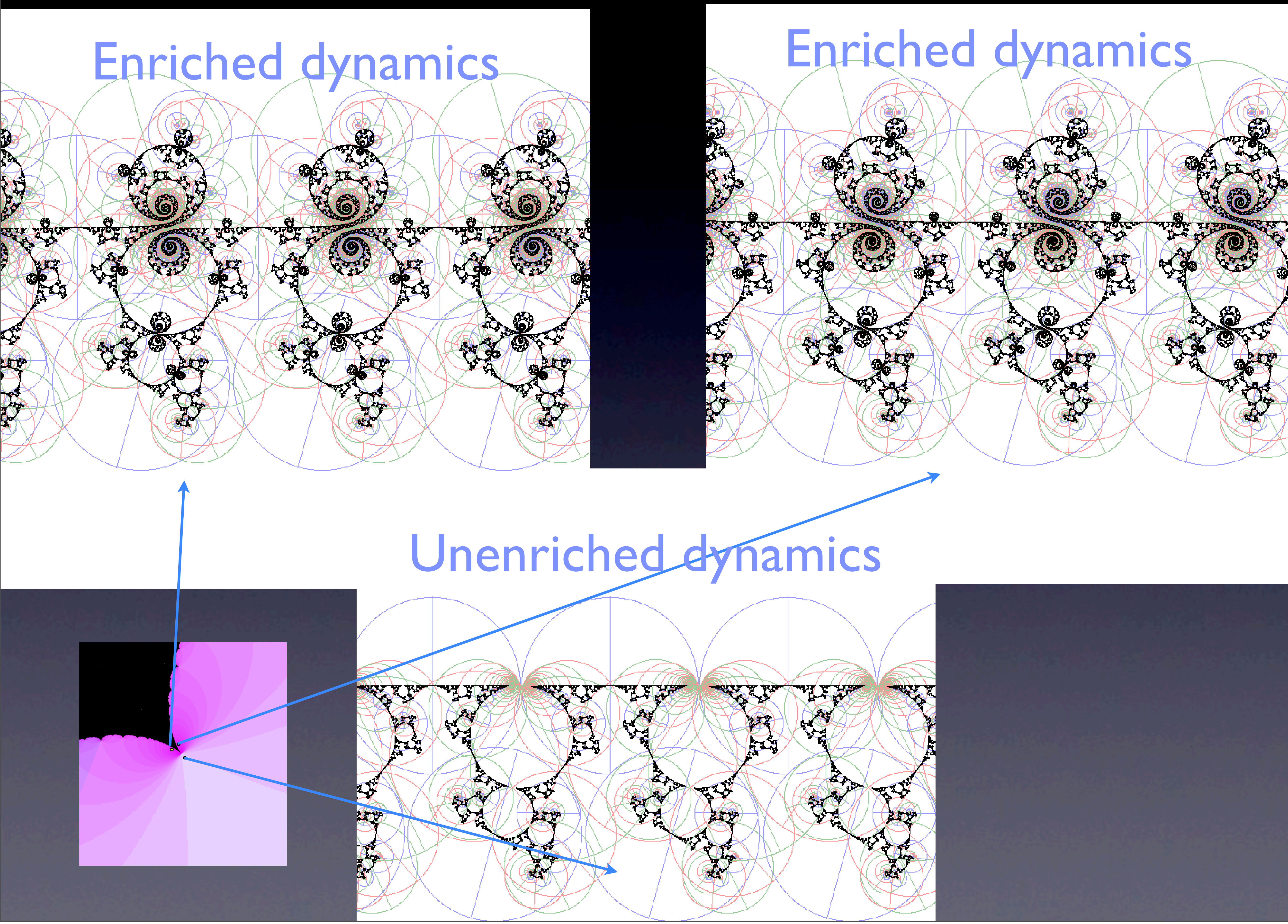
In color are the groups in the slice
that correspond to discrete faithful representations.

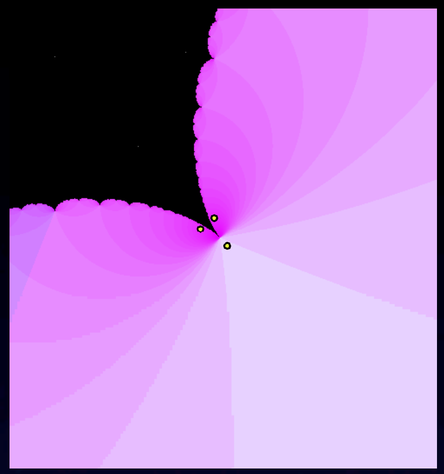
Jorgensen's Inequality

Suppose $A, B \in \mathrm{PSL}_2\,\mathbb{C}$ generate a non-elementary discrete group. Then

$$|(\mathrm{Tr}\, A)^2 - 4| + |\,\mathrm{Tr}([A, B]) - 2| \geq 1$$

Applying this inequality to any word in a and b eliminates a disk of possible discrete groups.


Enriched dynamics
Enriched dynamics
Unenriched dynamics

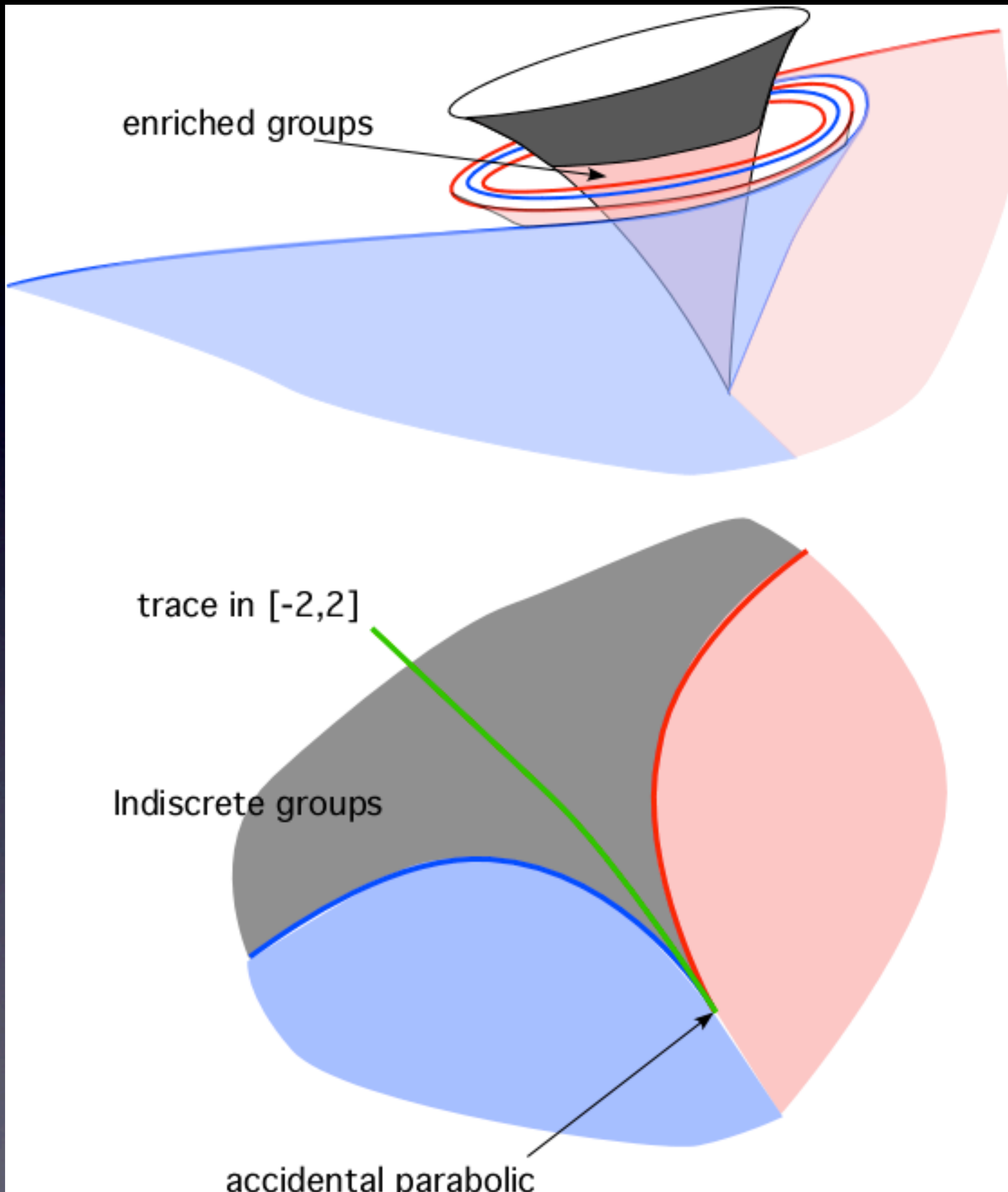


This picture is parametrized by $\mathrm{Tr}\, w$ for some particular word w in the two generators a and b.

In this picture, $w = ab$

The next picture attempts to locate the enriched groups in the Hausdorff topology.

It is a first attempt at drawing the space of closed subgroups of $\mathrm{PSL}_2\,\mathbb{C}$.



The accidental parabolic γ has a fixed point $p \in \overline{\mathbb{C}}$.

The Riemann surface $\widetilde{C}_p = \overline{\mathbb{C}} - \{p\}$ is isomorphic to $\mathbb{C}$.

$C_p = C_p/\langle\gamma\rangle$ is isomorphic to $\mathbb{C}/\mathbb{Z}$.

The enrichment is an element of $\mathrm{Hom}(C_p, C_p)/\pm 1$

The parabolic maps $z \mapsto z + 2\pi i$ and

$z \mapsto z + a$,

have a unique fixed point $\{\infty\}$, and

$z \mapsto z + 2\pi i$ is really

an endomorphism of $\widetilde{C}_p = \overline{\mathbb{C}} - \{\infty\}$.

The enriching map $z \mapsto z + a$ is also

an element of $\mathrm{Hom}(\widetilde{C}_p, \widetilde{C}_p)$.

But this isn't quite the right description:

$\lim_{n \to \infty} f_n^{\circ(n+1)}(z) = z + a + 2\pi i$,

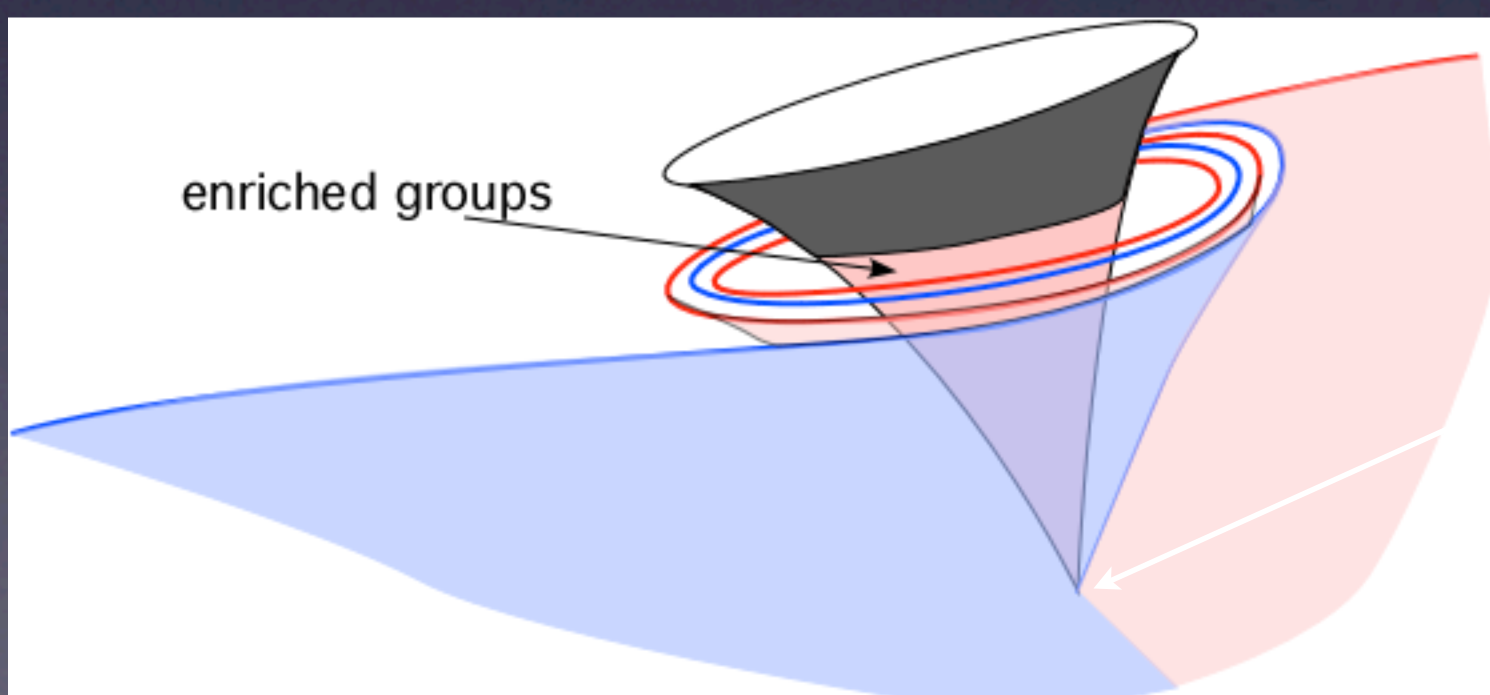
is also an element of the enriched group.

So the enrichment is really an element of $\mathrm{Hom}(C_p, C_p)$, where $C_p = \widetilde{C}_p/\langle\gamma\rangle \approx \mathbb{C}/\mathbb{Z}$ and the (once) enriched groups are parametrized by $\mathrm{Hom}(C_p, C_p)/\pm 1$ since enriching by $z \mapsto z + a$ or by $z \mapsto z - a$ gives rise to the same enriched group.

I say once-enriched, because enrichments exist for every possible accidental parabolic.

The enriched group may (and will) have new accidental parabolics, which lead to new (twice, or 3-times, . . . , ∞-times) endriched groups.

Note that in the Chabauty topology, as the enriching map tends to infinity, the enriched group tends to the unenriched group. So the point $\pm\infty \in \overline{C_p/\pm 1}$ corresponds to a perfectly good group, the unenriched group.



The point at infinity corresponds to the unenriched limit group

The situation for polynomials

The Chabauty topology is the correct topology on the space of closed subgroups, because enriched groups are still groups.

We will now see that there is a very similar construction of enriched polynomials, but the dynamical systems including enriched polynomials are not polynomials.

Some terminology

We will view polynomials as dynamical systems. This means we will try to understand the behavior of orbits: sequences

$$z,\ p(z),\ p(p(z)), \ldots,\ p^{\circ n}(z), \ldots$$

A point with a finite orbit, i.e. $p^{\circ k}(z) = z$ is a periodic point. and its orbit is called a k-cycle.

If $|(p^{\circ k})'(z)| < 1$ the cycle is attracting;
If $|(p^{\circ k})'(z)| = 1$ the cycle is indifferent;
If $|(p^{\circ k})'(z)| > 1$ the cycle is repelling.

Attracting cycles do attract nearby points.
Repelling cycles do repel nearby points.
But indifferent cycles are much more complicated

The derivative $(f^{\circ k})'(z)$
a.k.a the multiplier of the cycle
can be written $e^{2\pi i t}$, $t \in \mathbb{R}$.

The cycle is parabolic if $t \in \mathbb{Q}$,
i.e., if the multiplier is a root of unity.

If $t \notin \mathbb{Q}$, there is a whole
zoology of possible behaviors,
starting with linearizable or non-linearizable.

Let p be a polynomial of degree d. The **filled in Julia set** is

$$K_p = \{z \in \mathbb{C} \mid z, p(z), p(p(z)), \ldots \text{ is bounded.}$$

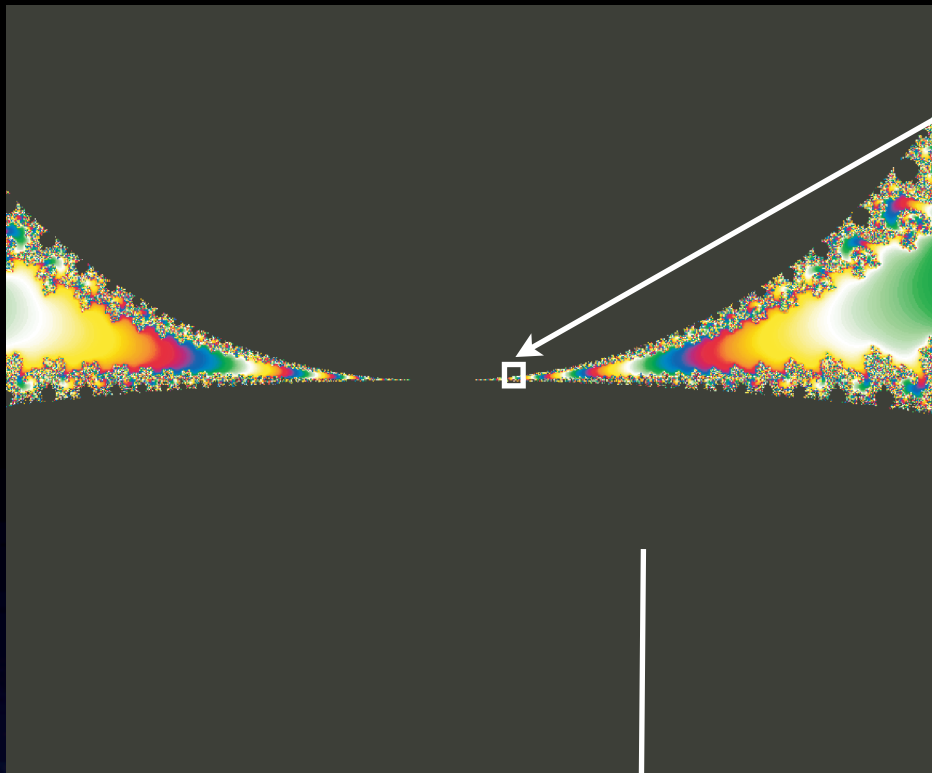
The **Julia set** is the topological boundary ∂K_p.

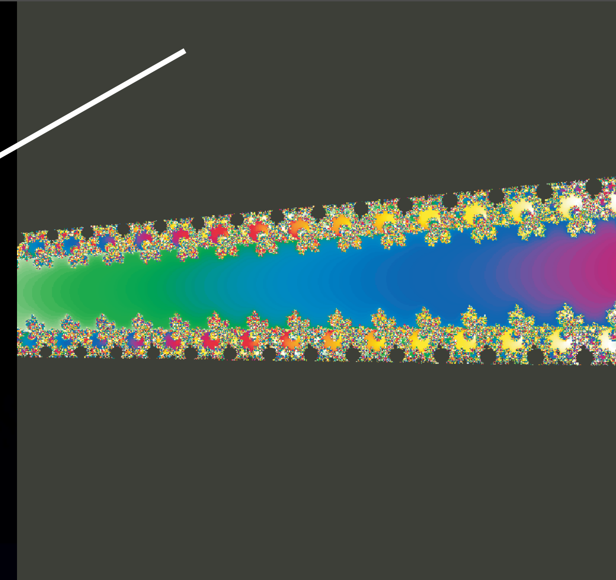
Write quadratic polynomials $p_c : z \mapsto z^2 + c$. with critical point 0 and critical value c.

$$0 \in K_c \iff K_c \text{ is connected}$$

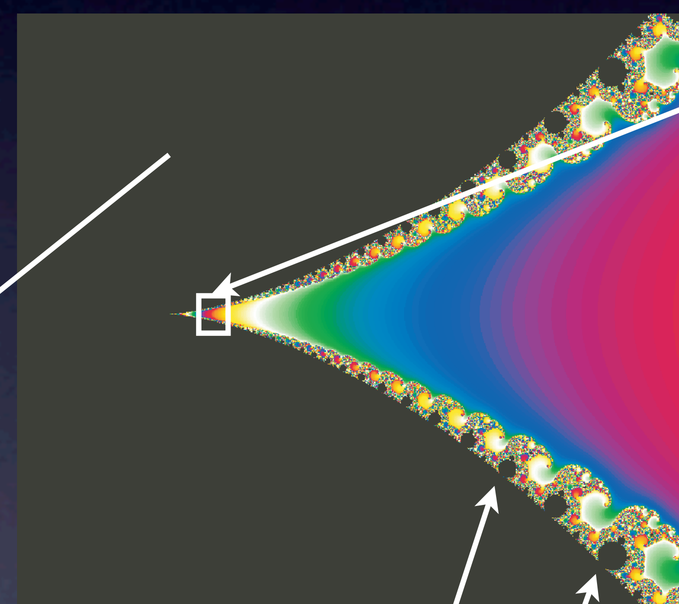
$$0 \notin K_c \iff K_c \text{ is a Cantor set}$$

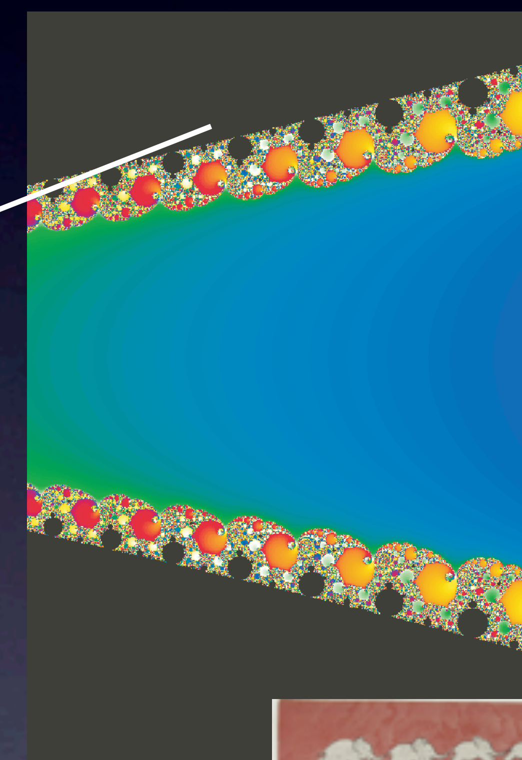
$$M = \{c \mid 0 \in K_c\} = \{c \mid K_c \text{ is connected}\}$$

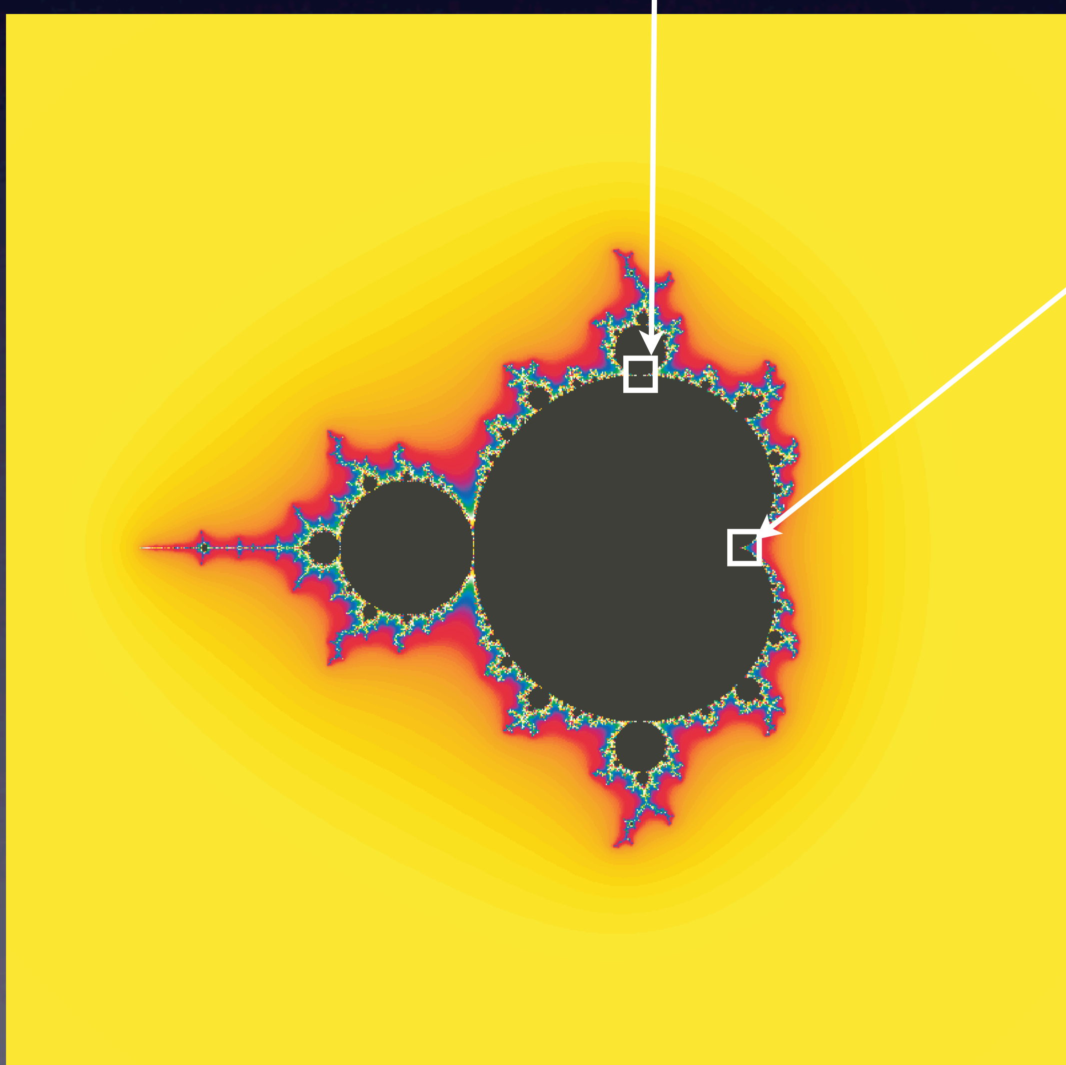




The set M
and various blow-ups
that will come up
during the lecture







do you see elephants?

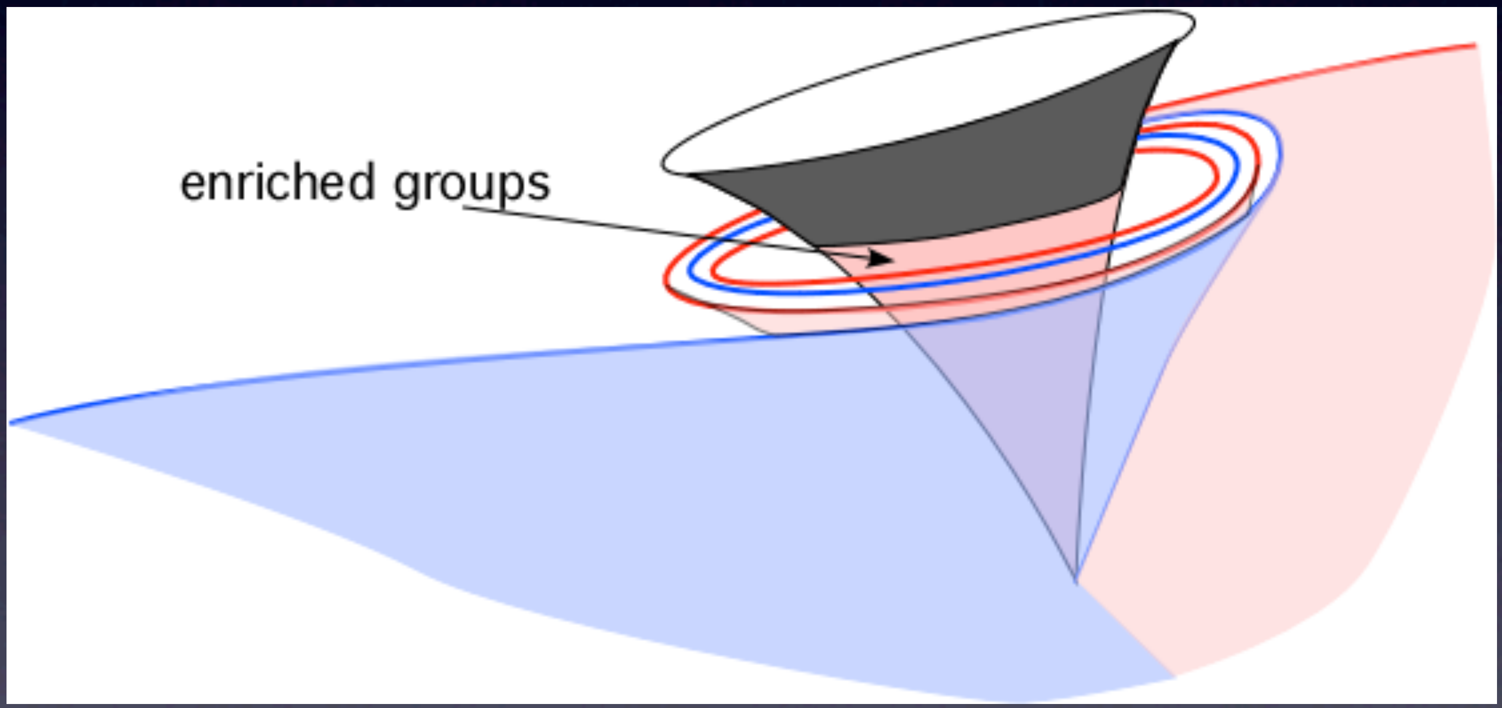
A theorem of Douady says that

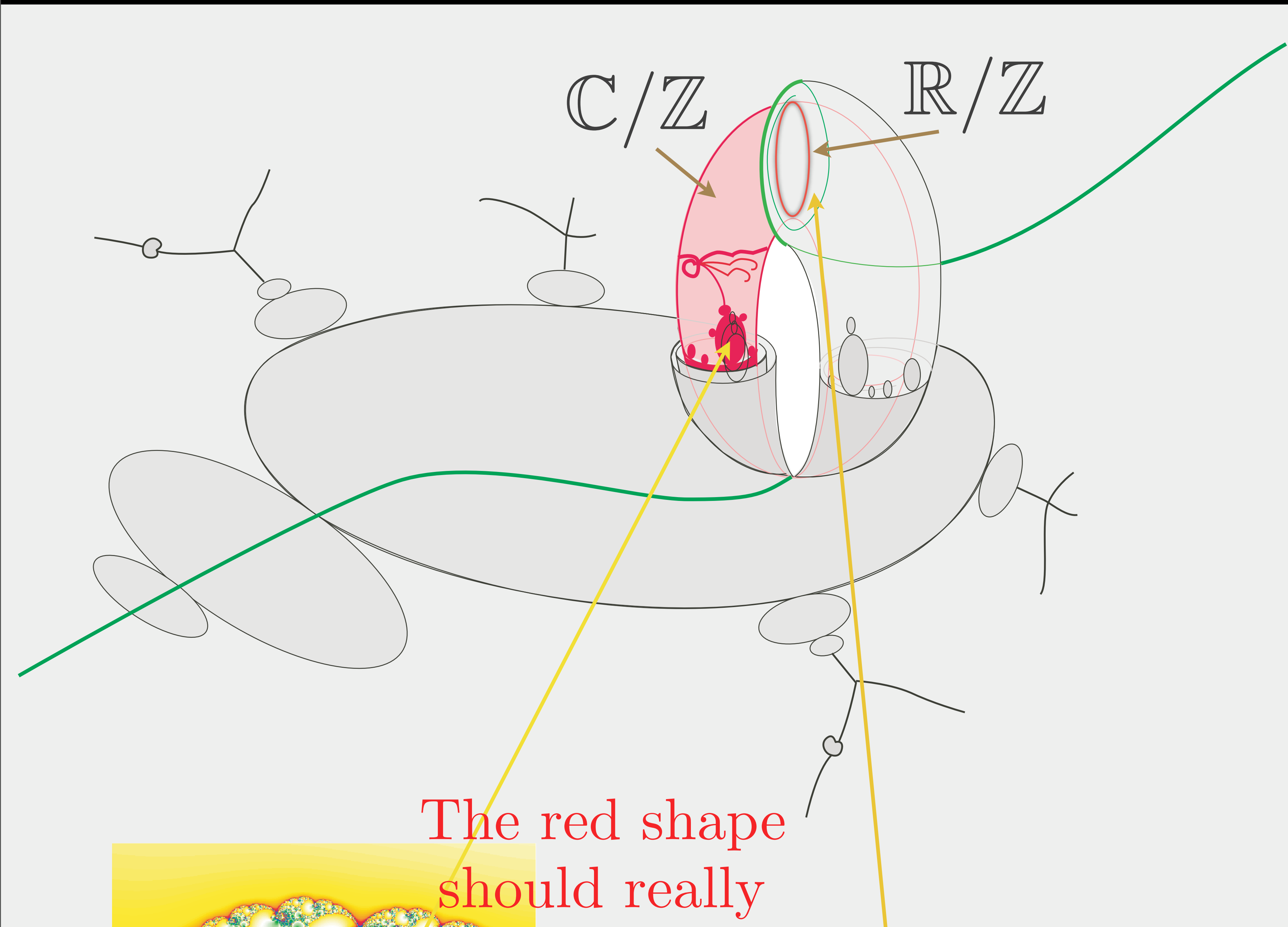
The filled in Julia set depends continuously on p unless p has a parabolic cycle.

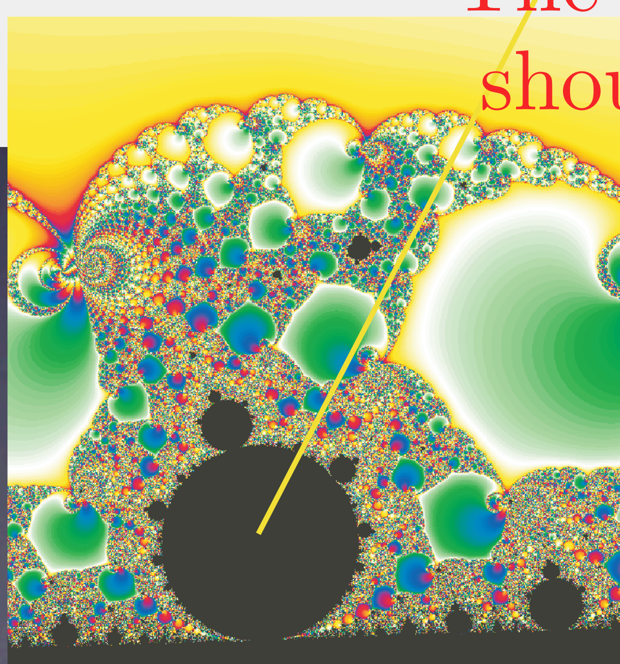
The archetype of a polynomial with a parabolic cycle is the polynomial $z \mapsto z^2 + z$, that is conjugate to $z \mapsto z^2 + \frac{1}{4}$.

Let us try to understand the enrichments of this polynomial.

First me show you a picture of the once-enriched dynamics of $z \mapsto z^2 + \frac{1}{4}$. You should think of it as analogous to





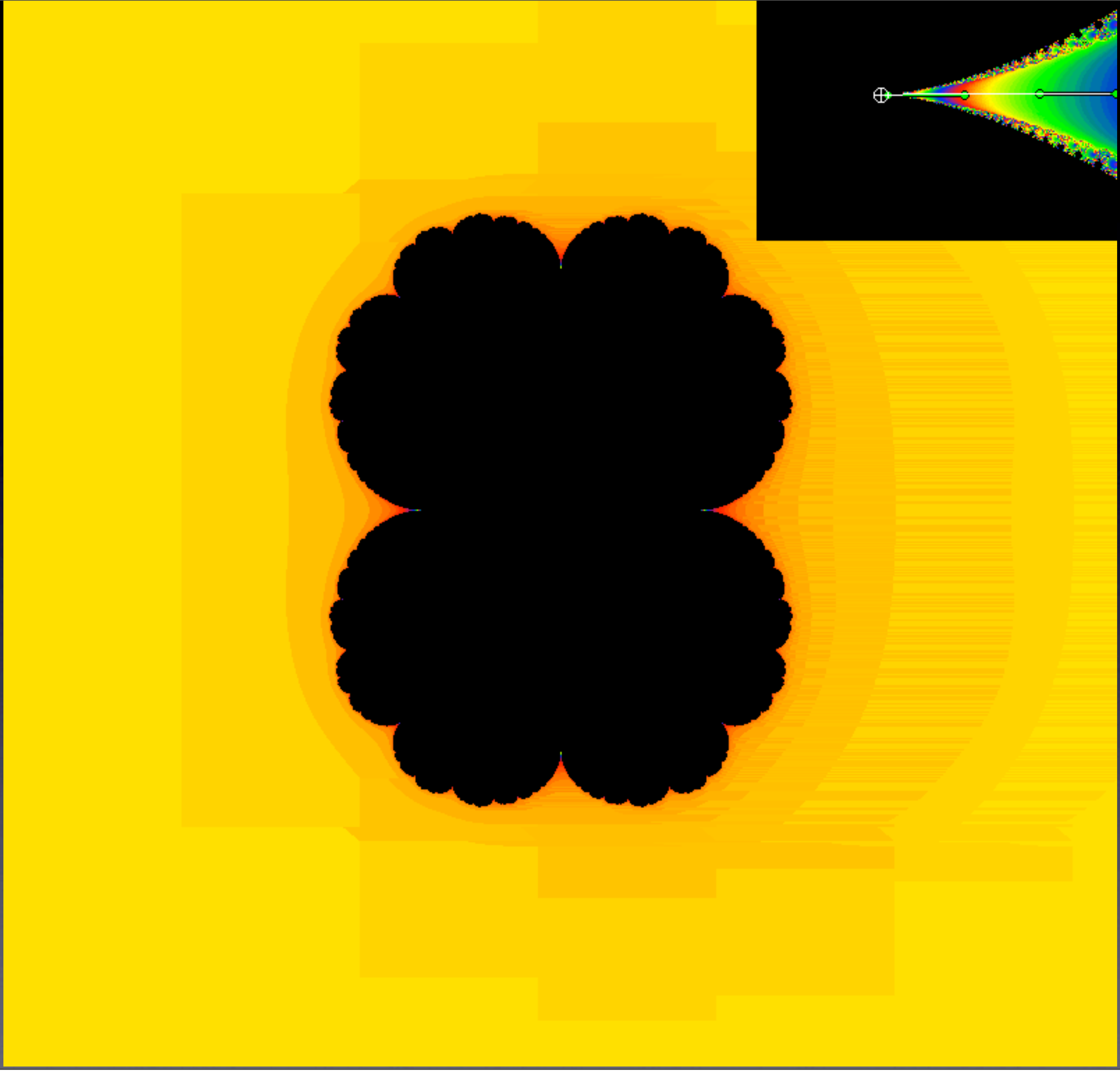


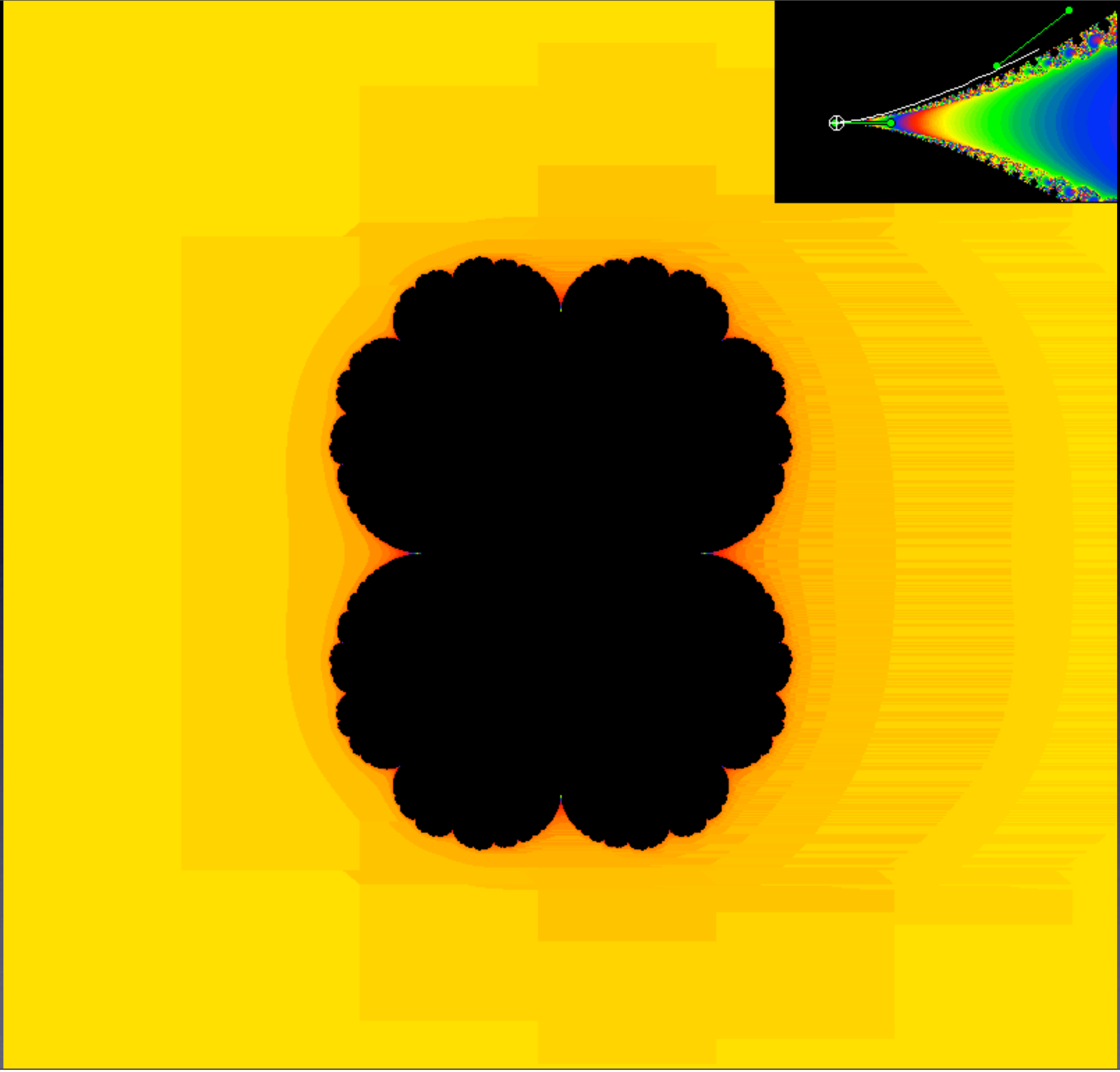
We replace the cusp of the Mandelbrot set M by a copy of $\overline{\mathbb{C}/\mathbb{Z}}$ with its ends identified at the point p

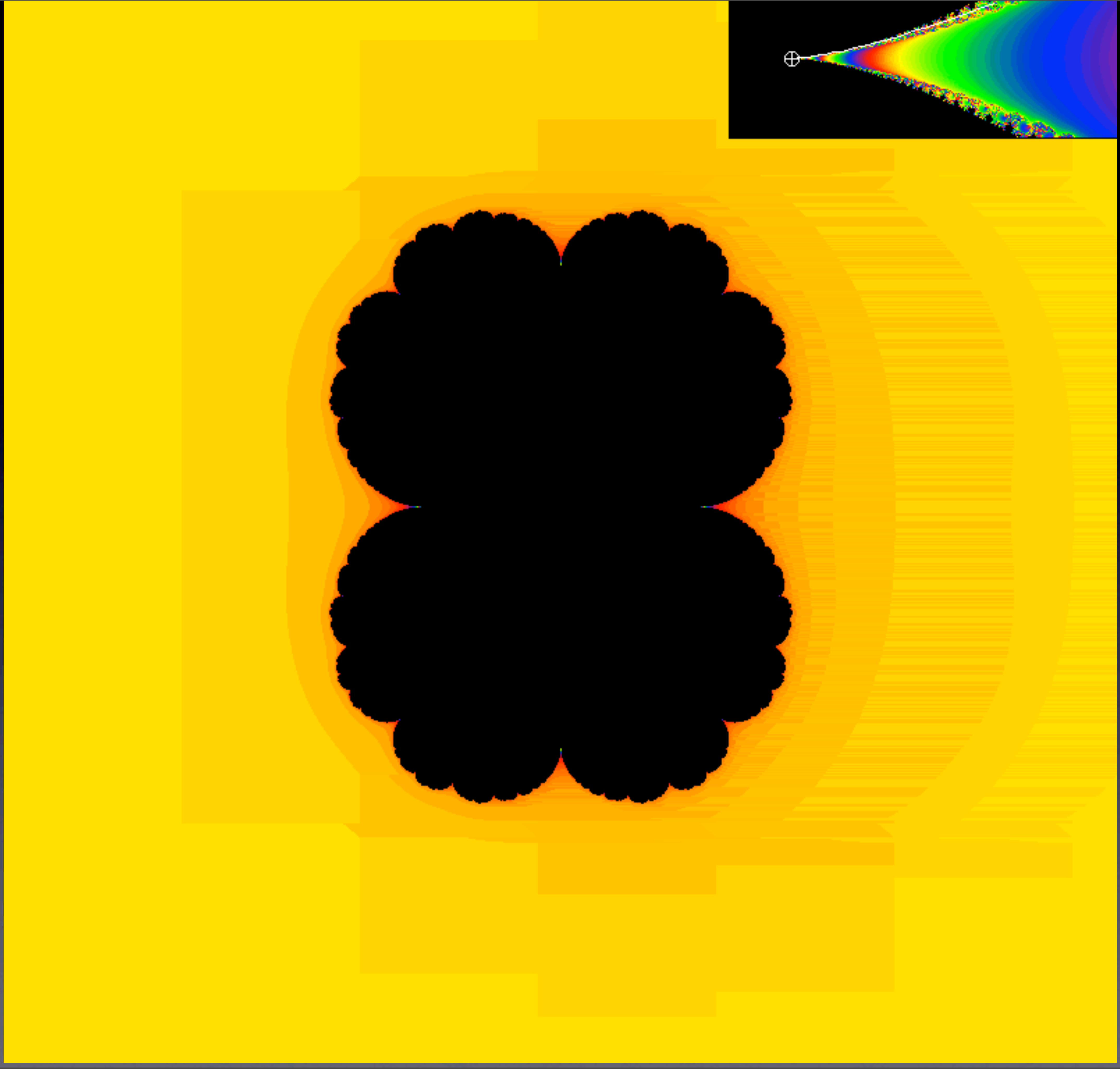
The part of the real axis $c < \frac{1}{4}$ lands at p, whereas the part of the real axis $c > \frac{1}{4}$ spirals towards $\mathbb{R}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$

The copy of the cylinder $\overline{\mathbb{C}/Z}$ is called the *exceptional divisor* or *universal elephant (Douady)*

I will next show three approaches to $c = 1/4$, which all lead to a circle of enriched dynamics. I hope it is clear that the paths spiral towards that circle of limits.





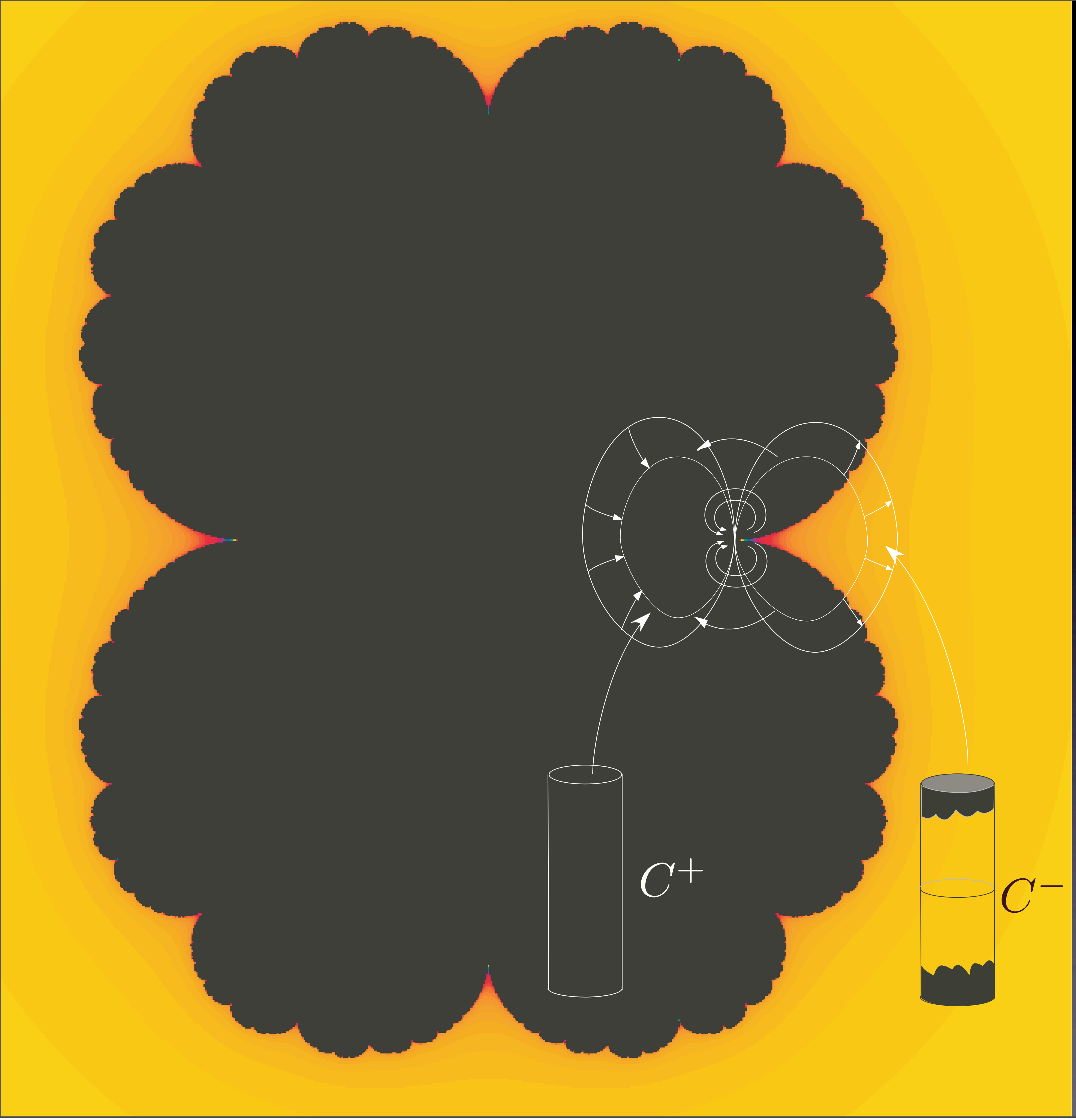


Douady and Lavaurs investigated the limiting dynamics, using

Ecalle Cylinders

and

Horn Maps



The quotient of the filled in Julia set by the dynamics is a cylinder C^+

There is also an outgoing cylinder C^-

Douady and Lavaurs investigated the limiting dynamics, using

Ecalle Cylinders

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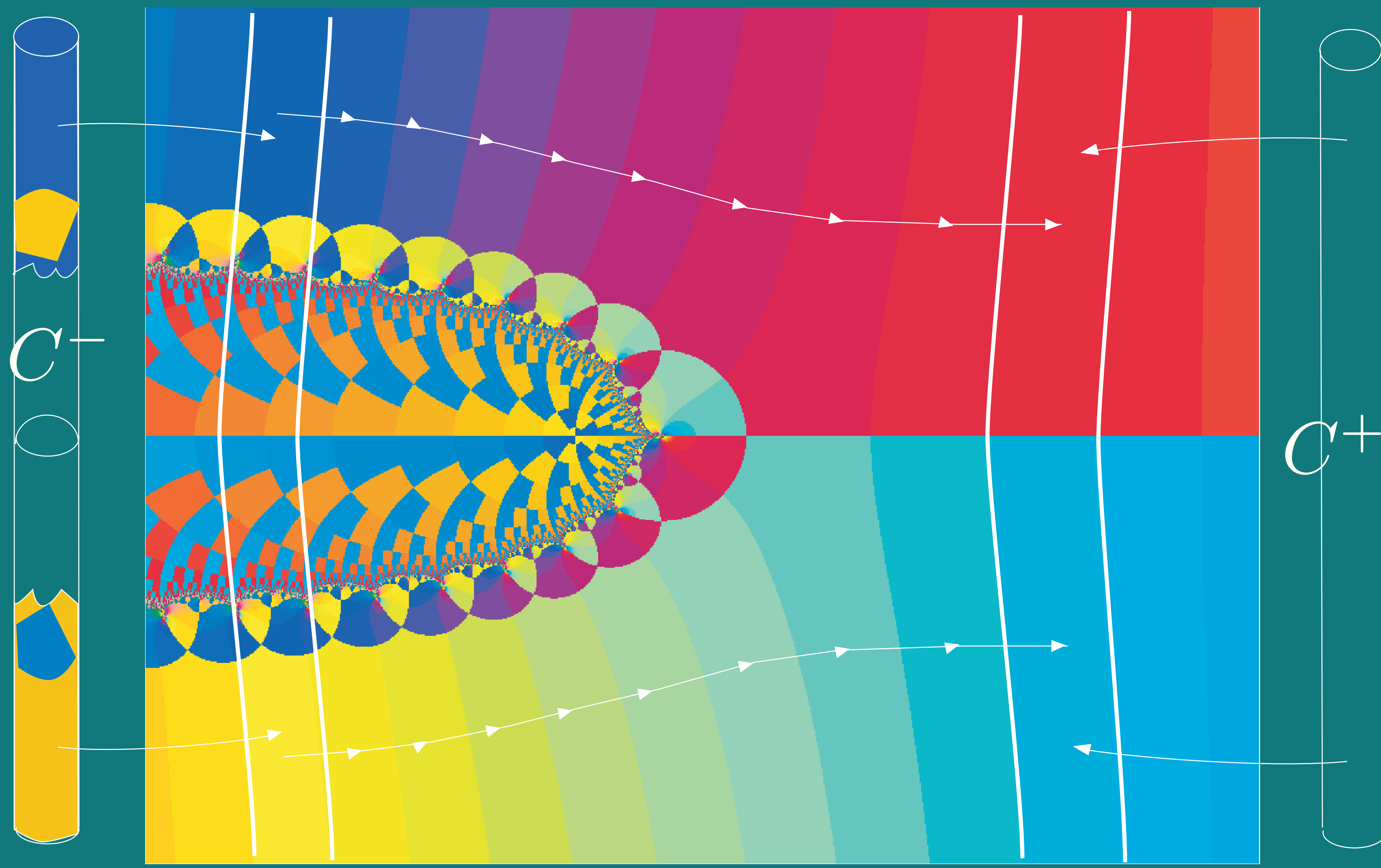
Horn Maps



The map

$$z \mapsto z + 1 + \frac{1}{z-1}$$

is conjugate to

$$z \mapsto z + 1$$

in a neighborhood of ∞

The quotient of

$\{z \mid \operatorname{Re} z < -R\}$ and $\{z \mid \operatorname{Re} z > R\}$

are both isomorphic to $\mathbb{C}/\mathbb{Z}$

Call these cylinders C^- and C^+

The dynamics induces *horn maps* from a neighborhood of the ends of C^- to C^+

To summarise

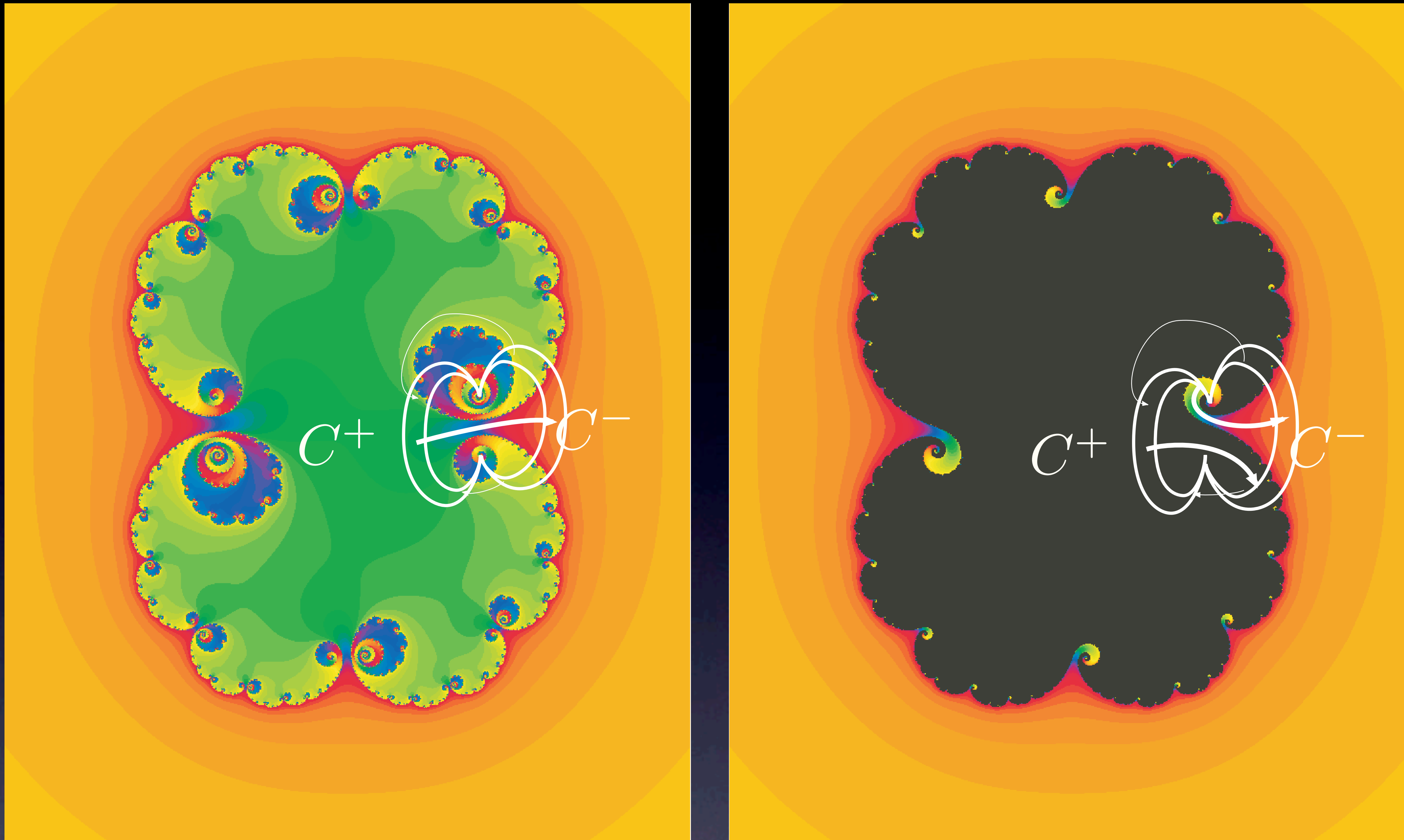
if p_c has a parabolic cycle
then there are two quotients C^+ and C^-
by the dynamics, and a horn map
$h : U \to \overline{C}^+$ defined in a
neighborhood U of the ends of C^-.

Adam Epstein has proved that horn maps are
analytic maps of *finite type:*
$h : U \to \overline{C}^+$ is a covering map
of all but finitely many points of C^+.

These cylinders still exist for c in a neighborhood of the parameter value c_0 for which p_{c_0} has a parabolic cycle

The cylinders exist for all values of the parameter with a bit of ambiguity when the cycles emanating from the parabolic cycle are attracting with real derivatives

We illustrate this when $c_0 = \frac{1}{4}$.



In these two picture of Julia sets K_c
with c close to $c_0 = 1/4$,
we see cylinders C^+ and C^-,
with horn maps defined near
the ends of C^-,

and isomorphisms
$C^+ \to C^-$
referred to as
as *Lavaurs maps*, or
going through the egg beater

Defining the parabolic blow-up

The ordinary blow-up of $0 \in \mathbb{C}^2$ is the set

$$\left\{ \left(\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2, l \in \mathbb{P}^1 \right) \mid \begin{pmatrix} x \\ y \end{pmatrix} \in l \right\}$$

We want an analogous definition

of the parabolic blow-up

Suppose that p_{c_0} has a parabolic cycle.

Let V be a neighborhood of c_0 sufficiently small that the cycles emanating from the cycle are well defined, and let $V^* \subset V$ be the subset where no such cycle is attracting with real multiplier

For each $c \in V^*$ we have cylinders C_c^+ and C_c^- which form a trivial principal bundle under $\mathbb{C}/\mathbb{Z}$

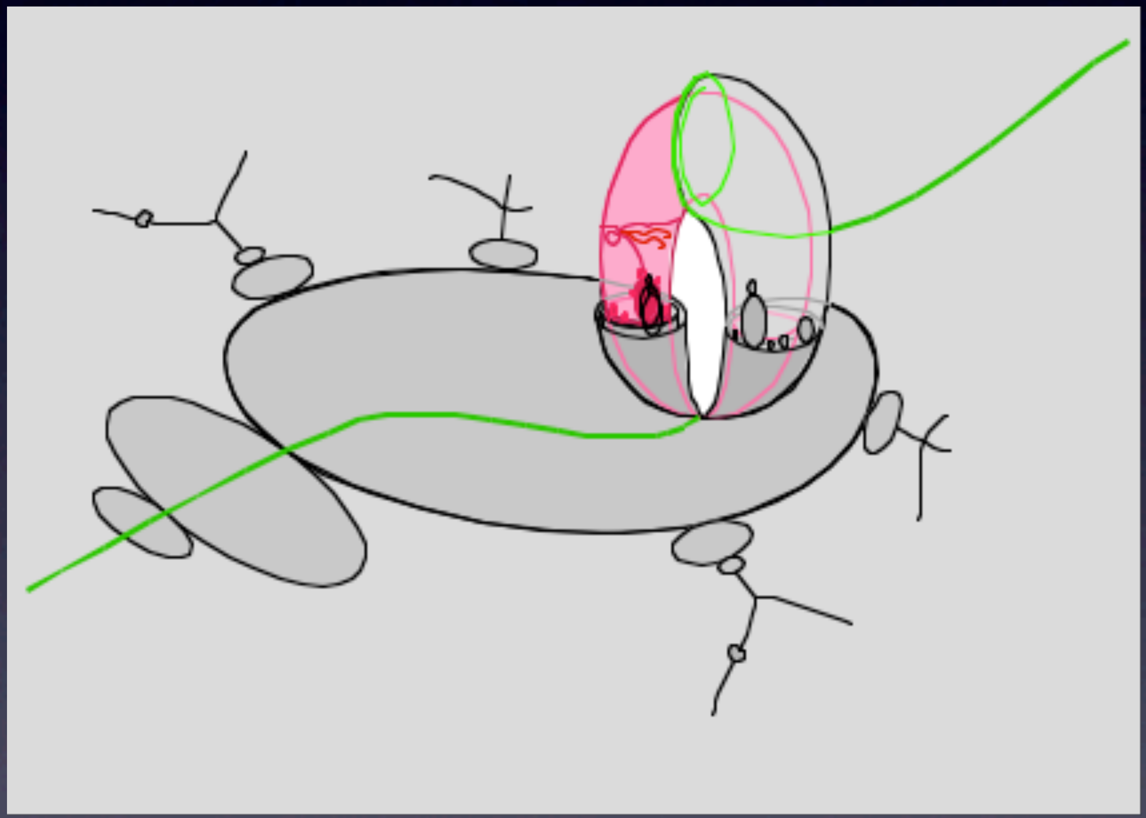
Moreover for all $c \in V^*$, $c \neq c_0$, there is a natural isomorphism $L_c : C^+ \to C^-$

We define the parabolic blowup of $\mathbb{C}$ at c_0

to be the closure in $V \times \mathrm{Isom}(\mathcal{C}^+, \mathcal{C}^-)$

of all pairs (c, L_c).

Thus in the picture the pink "croissant" is $\mathrm{Isom}\,(C^+, C^-)$ and a sequence $i \mapsto c_i$ converges to a point $\phi \in \mathrm{Isom}\,(C^+, C^-)$



if the Lavaurs maps L_{c_i} converge to ϕ.

If $c \uparrow 1/4$, you converge to the identified ends of $\mathrm{Isom}\,(C^+, C^-)$.