A disconnected deformation space of rational maps

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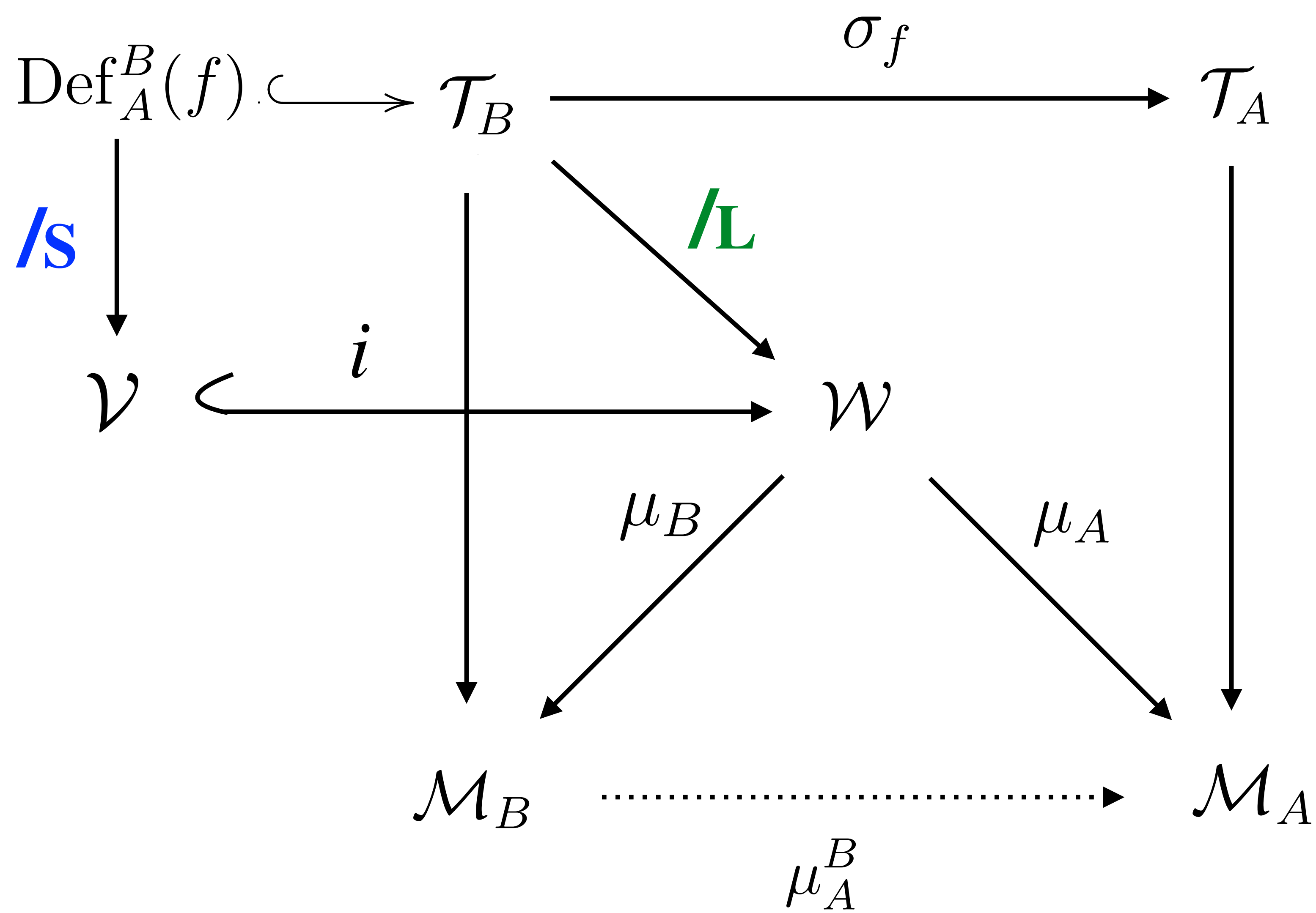
Part II

Fix $\langle f \rangle \in \mathrm{Per}_4(0)^*$

Theorem: $\mathrm{Def}_A^B(f)$ has infinitely many components.

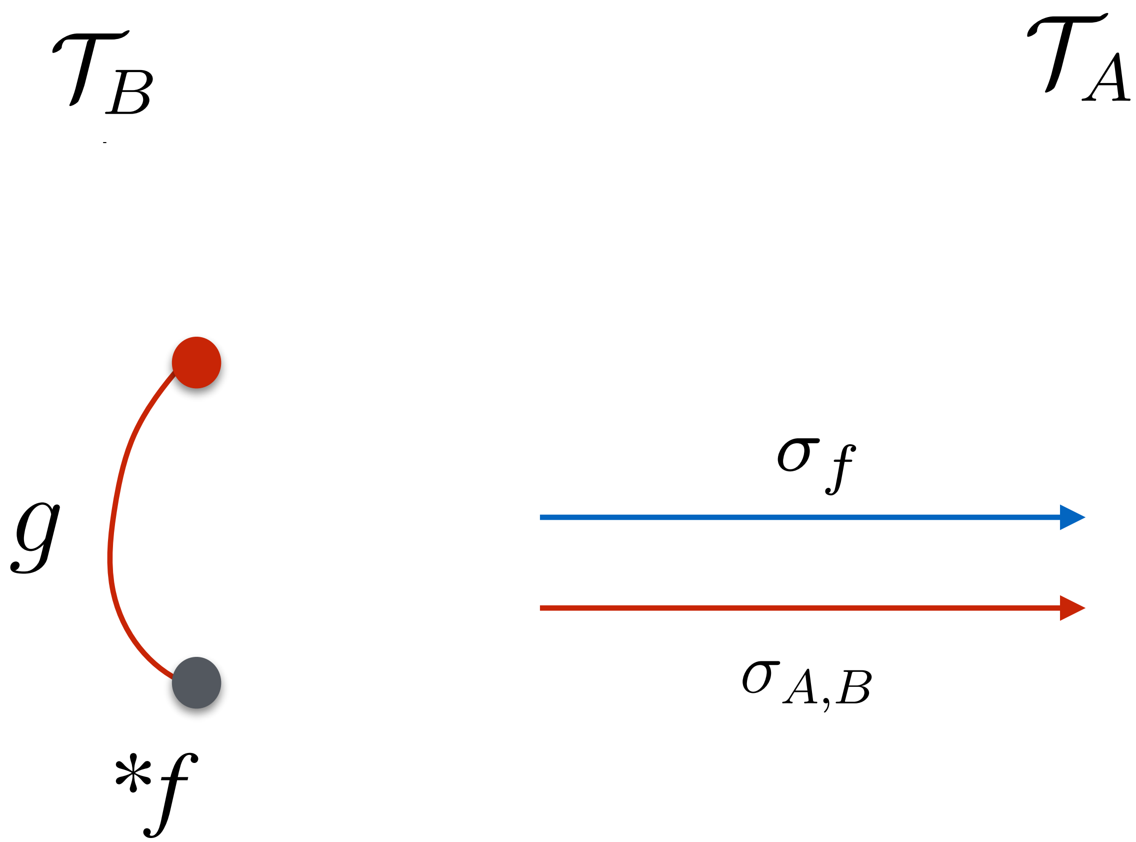
Recall



To prove the Theorem:
Enough to show that the index of E:= $i_*(\pi_1(\mathcal{V}, \circledast_{\mathcal{V}}))$ in S is infinite.

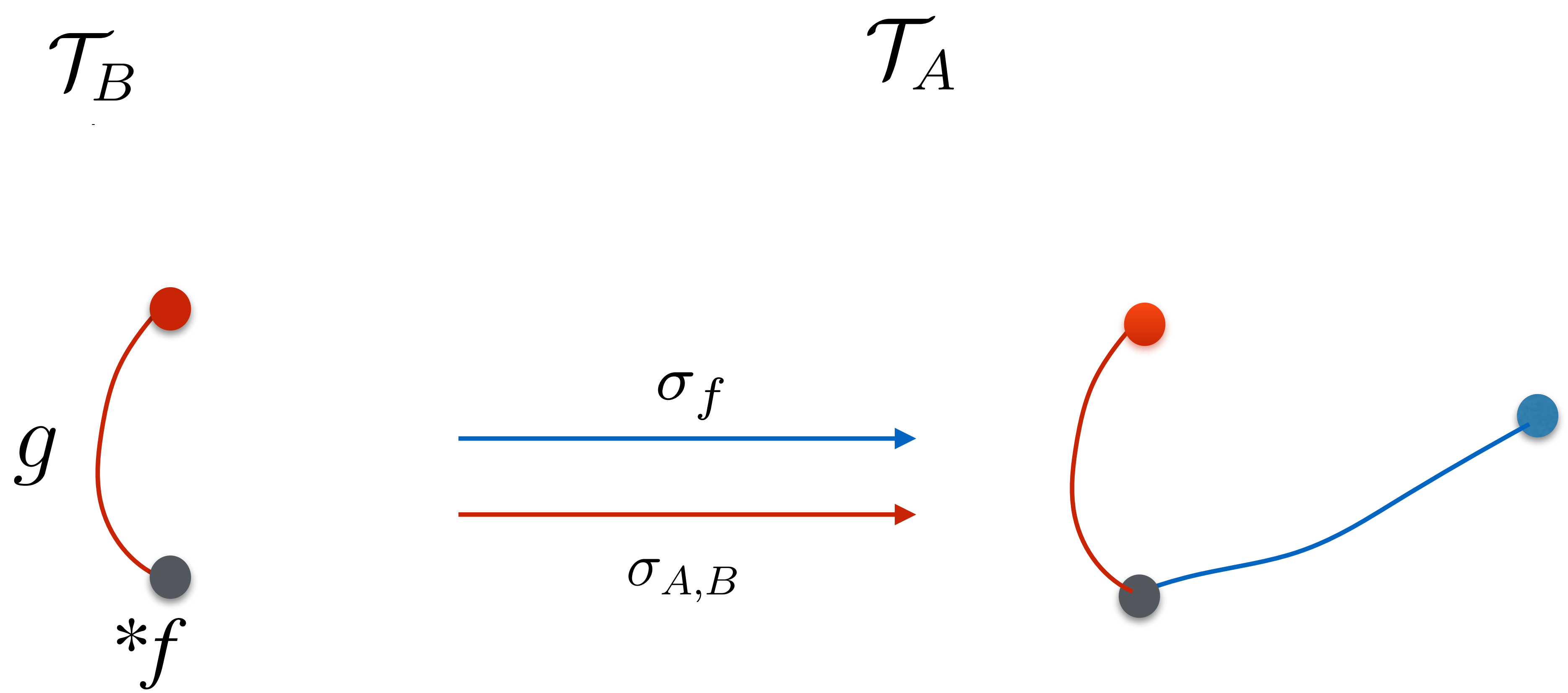
Picturing the subgroups:

$$E \subseteq S \subseteq L \subseteq \mathrm{Mod}_B$$



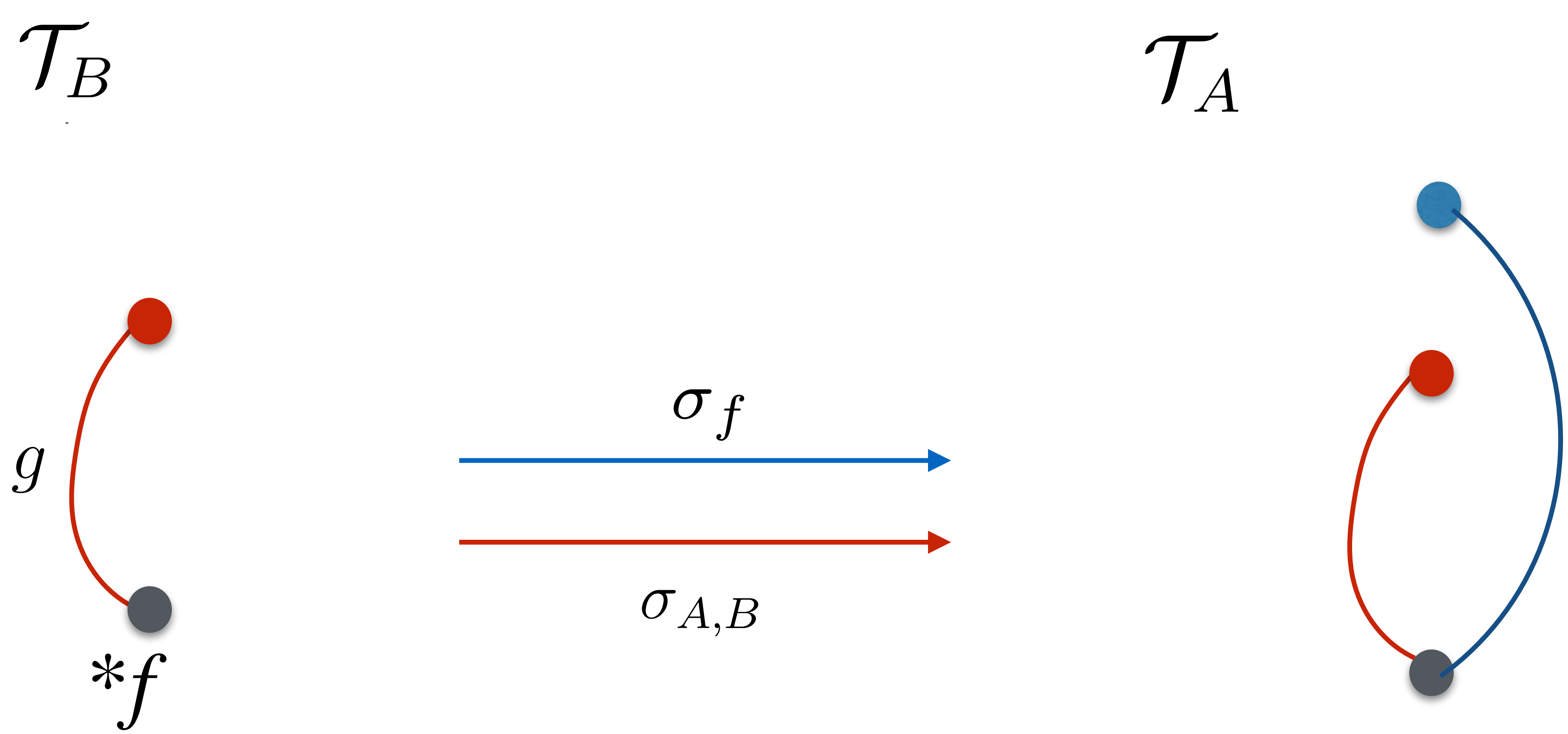
Represent $g \in \mathrm{Mod}_B$ as a path on $\mathcal{T}_B$ emanating from the (canonical) basepoint $*f$ in $\mathrm{Def}_A^B(f)$

General Case: $g \in \mathrm{Mod}_B$



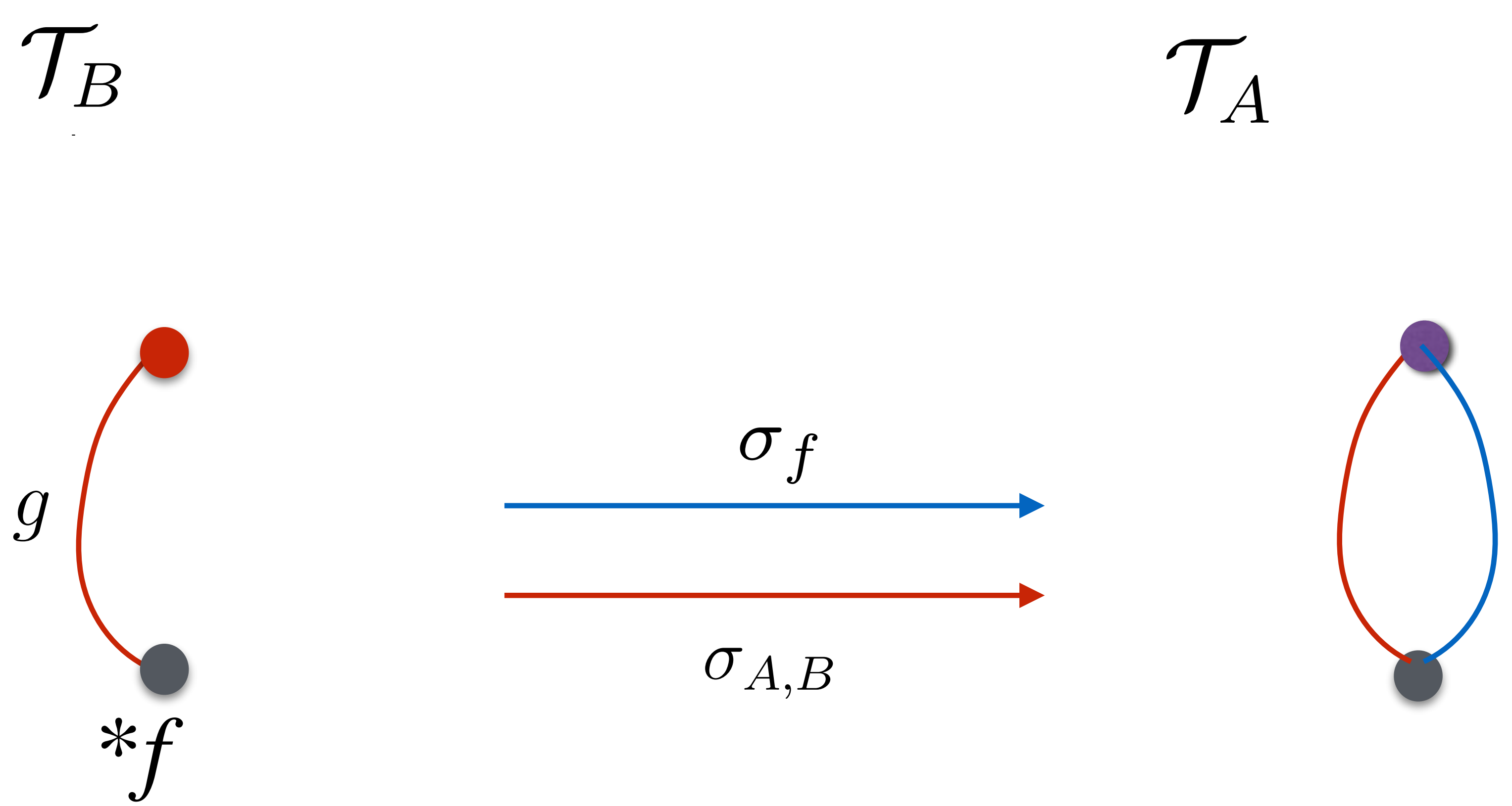
The red endpoint of g need not map to the same point or even in the same fiber over $\mathcal{M}_A$

Case: $g \in$ L



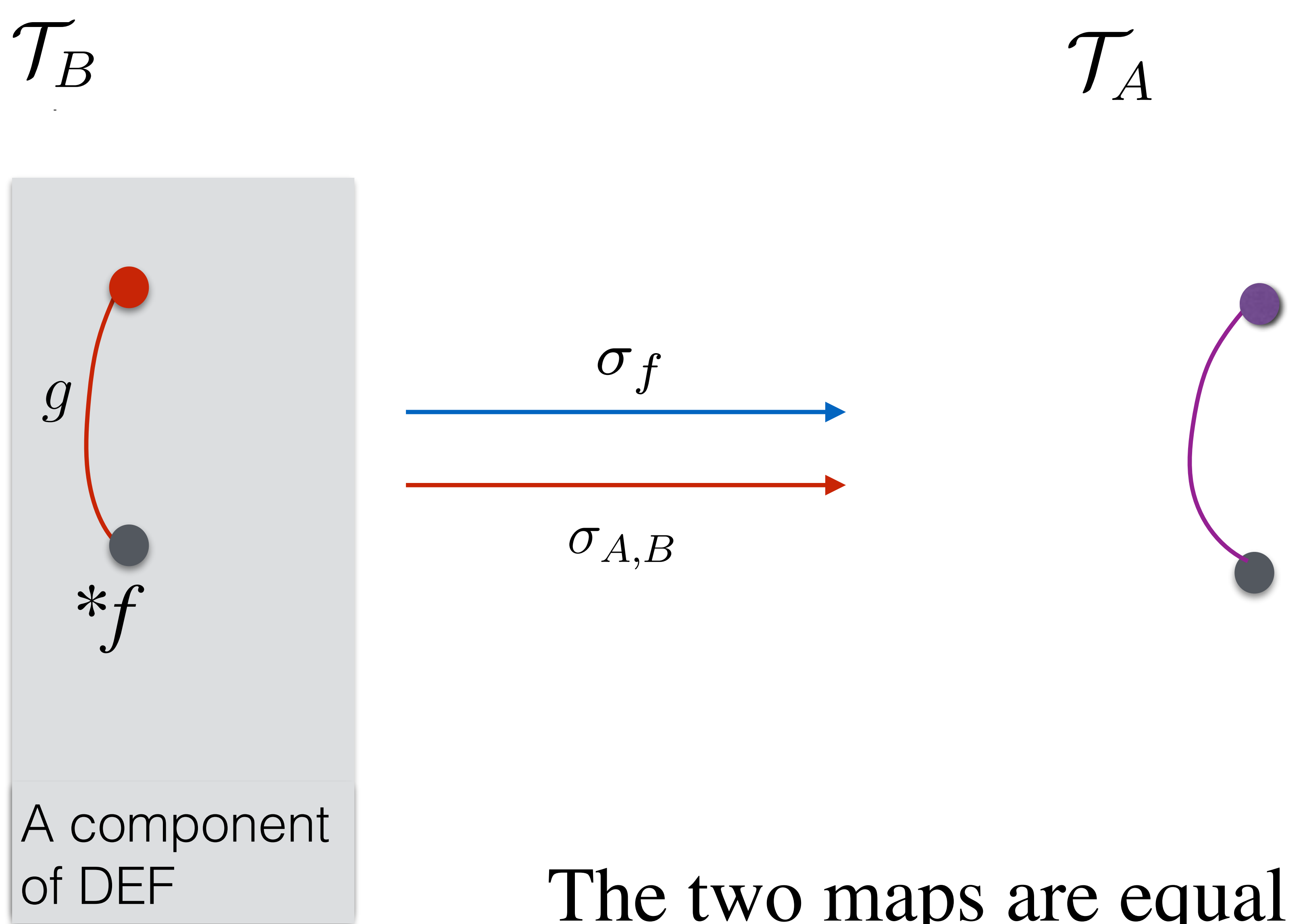
The red endpoint maps under the two maps to points in the same fiber over $\mathcal{M}_A$ as the image of $*f$

Case: $g \in S$



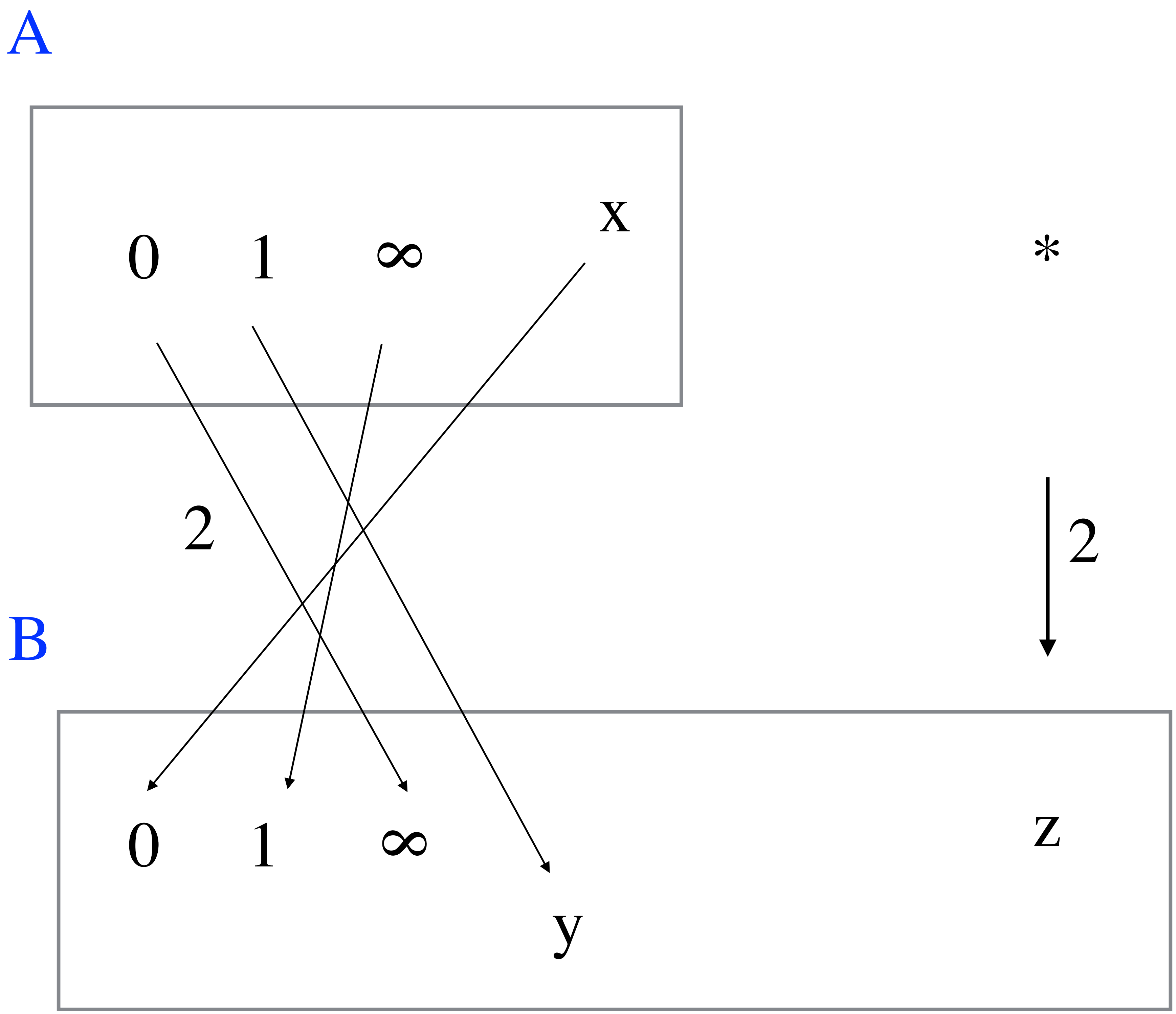
Maps agree on endpoints of the path corresponding to g

Case: $g \in$ E

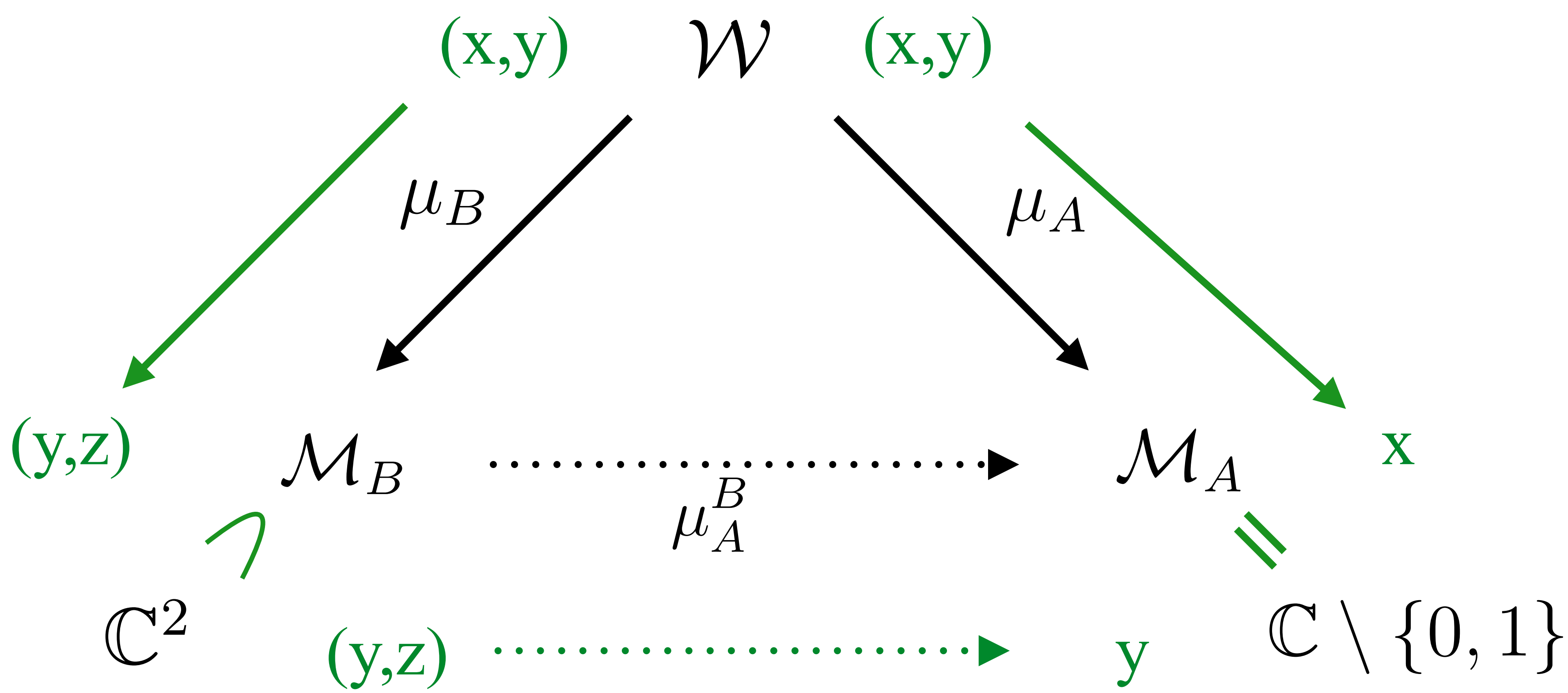


The two maps are equal on the path corresponding to g

To compute E and S, we fix coordinates for $\mathcal{W}$



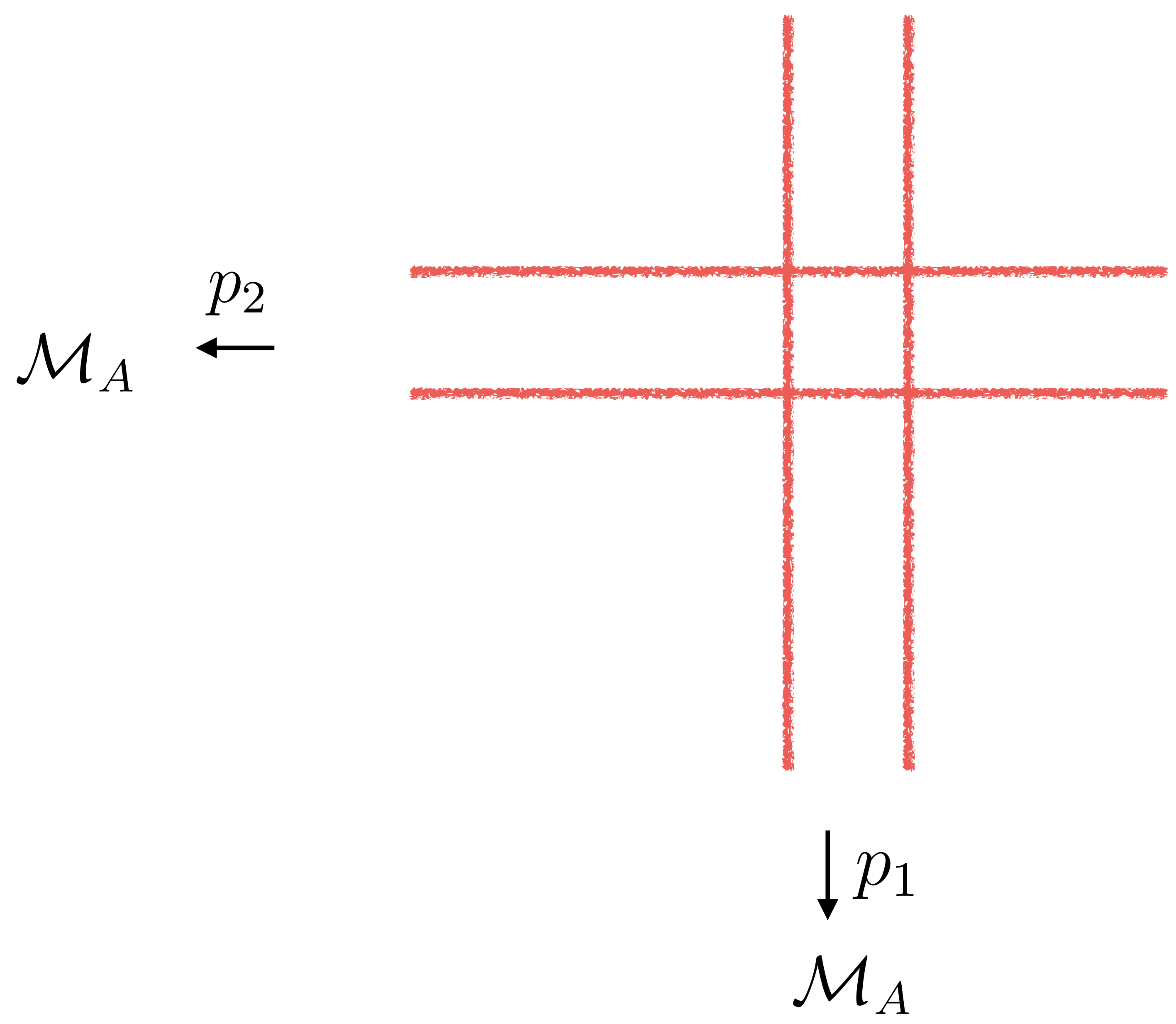
Coordinates for $\mathcal{W}$: Embed $\mathcal{W}$ in $\mathcal{M}_A \times \mathcal{M}_A = \mathbb{C}^2 \setminus \mathcal{L}$



$$\mathcal{W} \subset \mathcal{M}_A \times \mathcal{M}_A = (\mathbb{C} \setminus \{0,1\})^2$$

Coordinates for $\mathcal{W}$: Embed in $\mathcal{M}_A \times \mathcal{M}_A = \mathbb{C}^2 \setminus \mathcal{L}$

$\mathcal{L} = L_{x=0} \cup L_{x=1} \cup L_{y=0} \cup L_{y=1}$



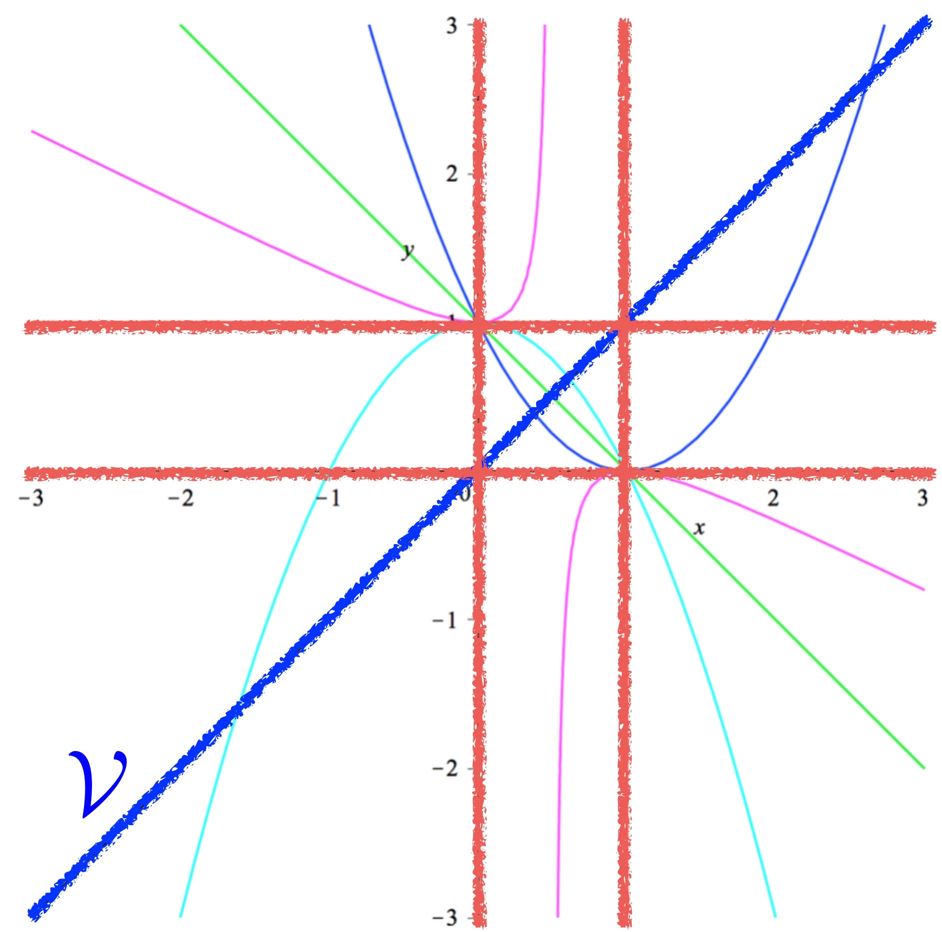
Coordinates for $\mathcal{W}$: embed in $\mathcal{M}_A \times \mathcal{M}_A$ = $\mathbb{C}^2 \setminus \mathcal{L}$

$\mathcal{L} = L_{x=0} \cup L_{x=1} \cup L_{y=0} \cup L_{y=1}$

$\mathcal{W} = \mathbb{C}^2 \setminus (\mathcal{L} \cup \mathcal{C})$

$\mathcal{V} = \{(x, y) \in \mathcal{W} \mid x = y\}$

$\mathcal{M}_A \xleftarrow{p_2}$



$\downarrow p_1$

$\mathcal{M}_A$

What is S?

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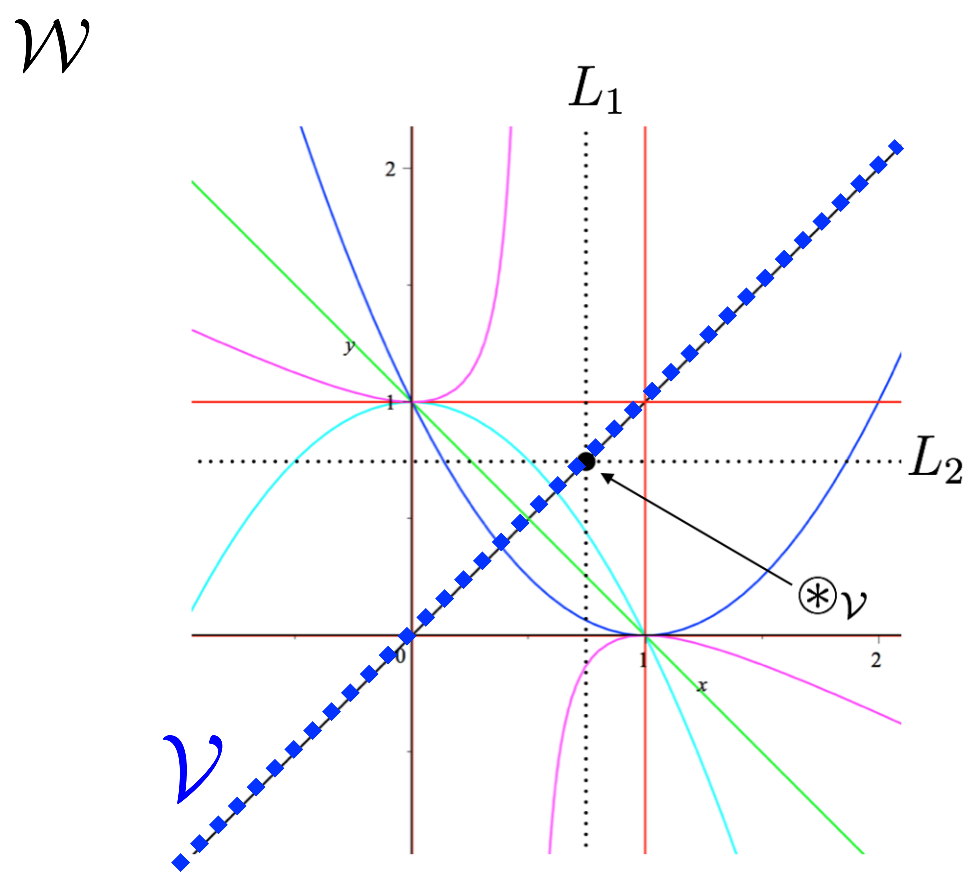

Lemma For $\gamma \in$ L,

$\gamma \in$ S $\quad \Leftrightarrow$

$\gamma = \xi\eta$, where

$\xi \in \mathrm{Im}(\pi_1(L_1)), \eta \in \mathrm{Im}(\pi_1(L_2))$

and

$p_1(\eta) = p_2(\xi)$.

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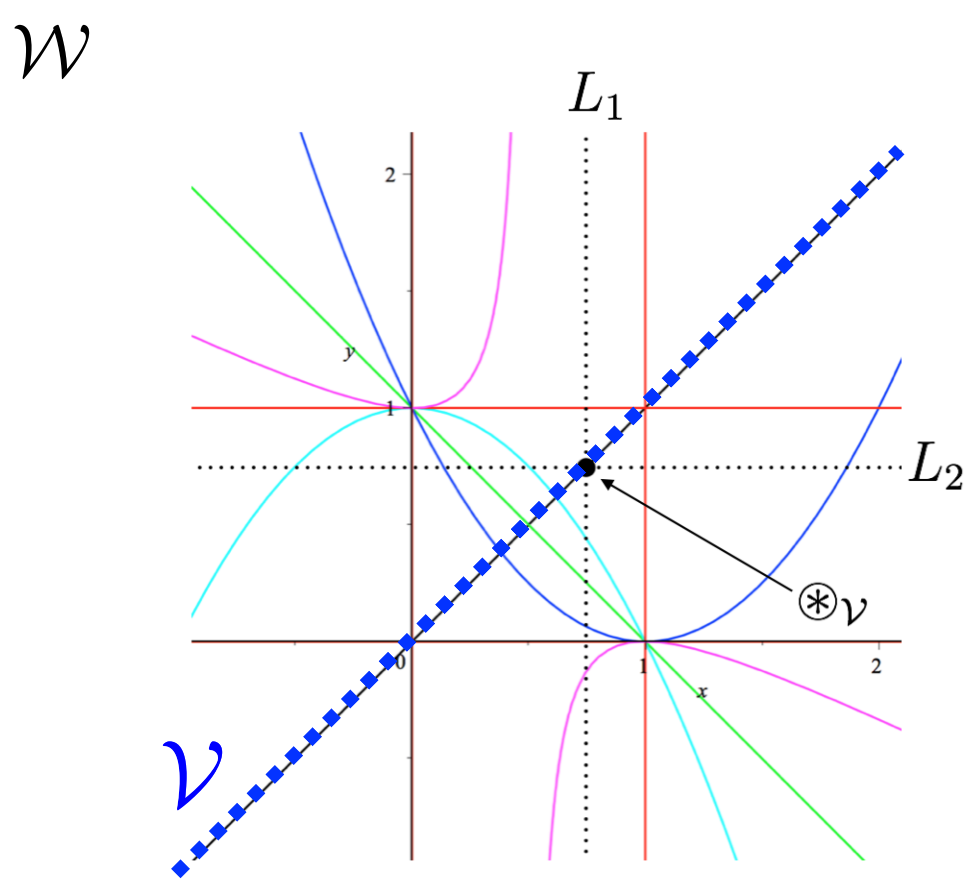

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Remark: it follows that

The subgroup S is not normal in L.

Simplify $\mathcal{W}$

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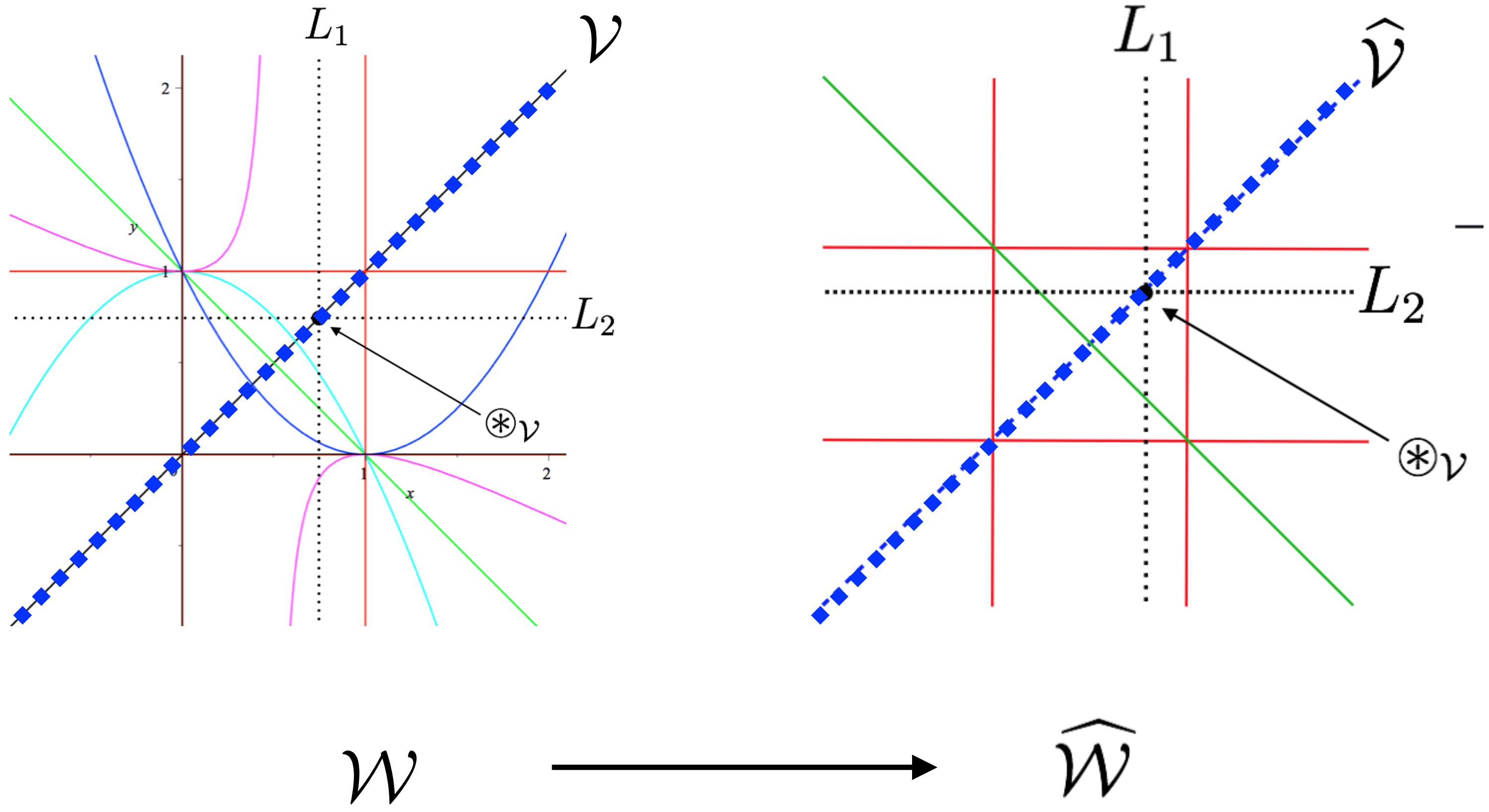
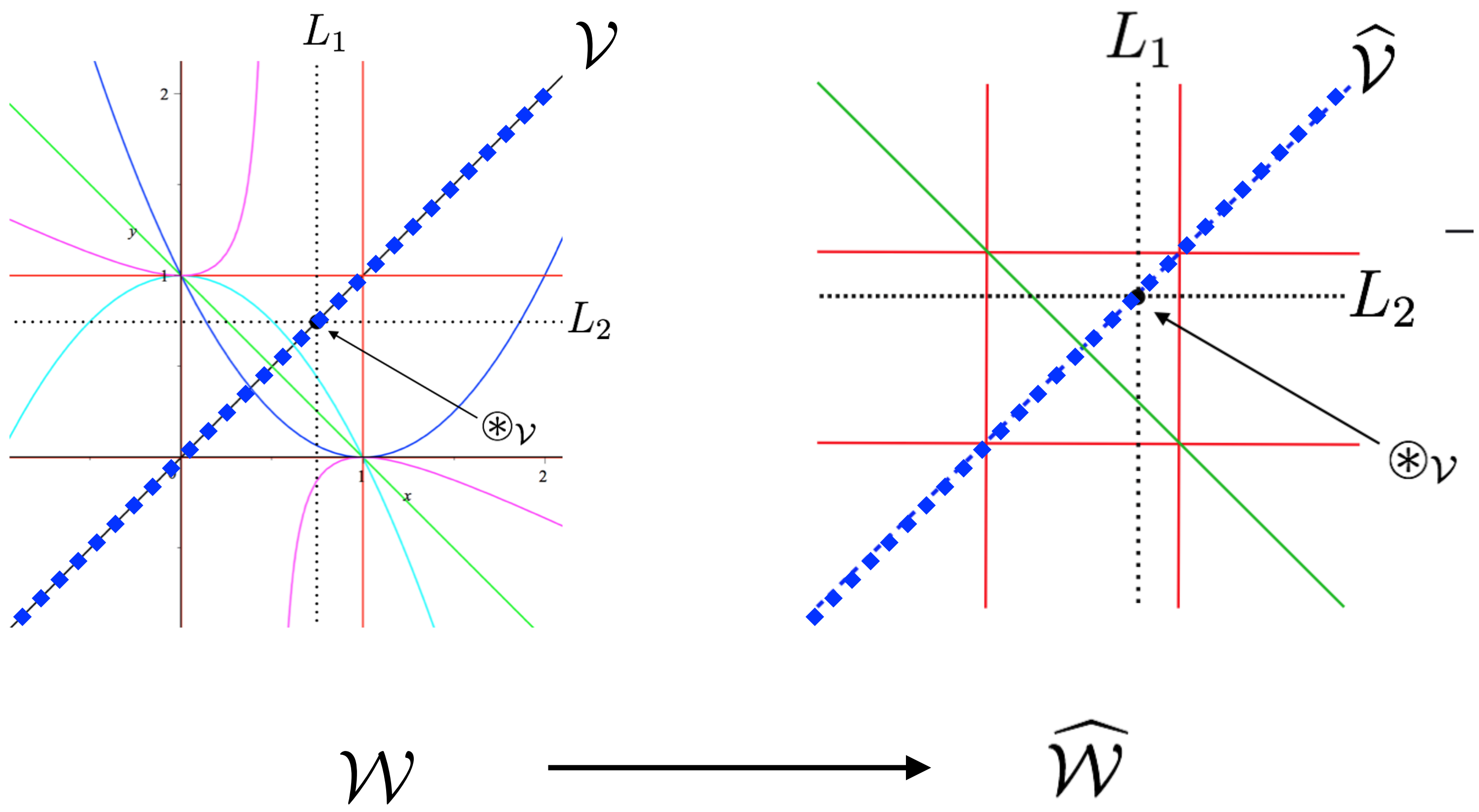


The map is surjective on fundamental groups

Simplify $\mathcal{W}$

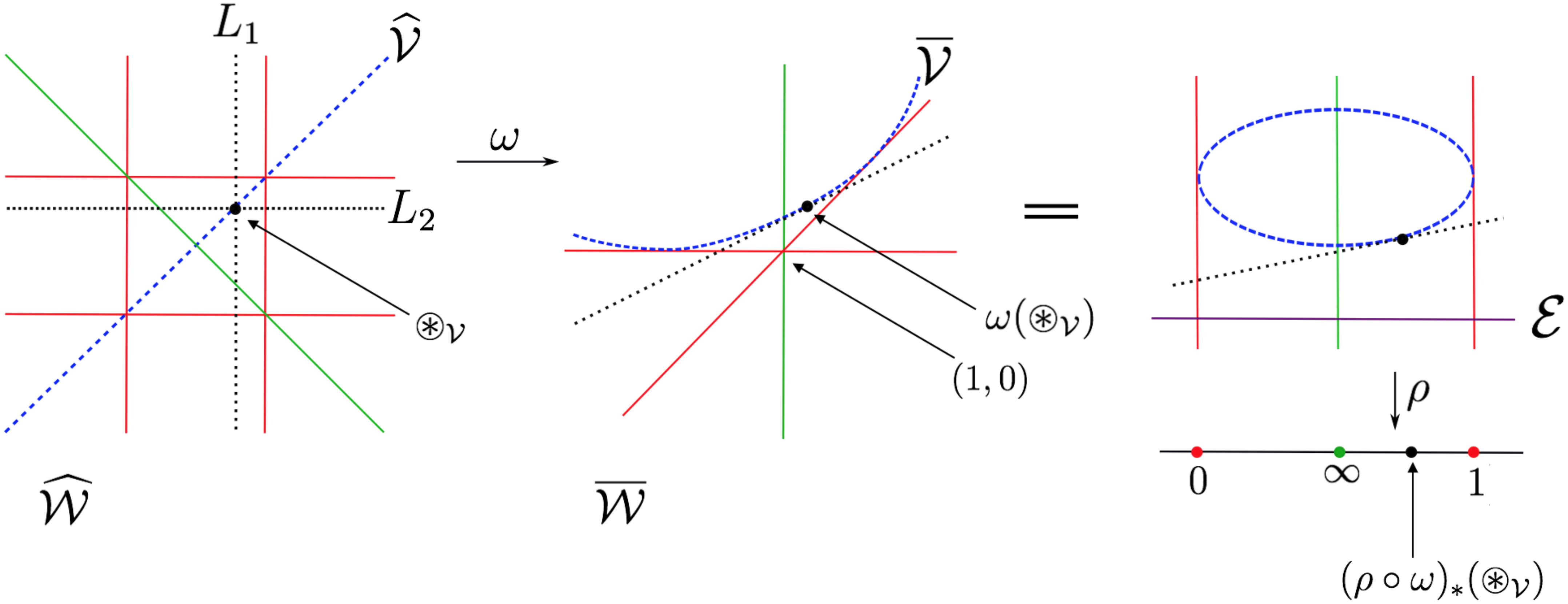


The map is surjective on fundamental groups

Next

Mod out by the diagonal reflection and blow up a point

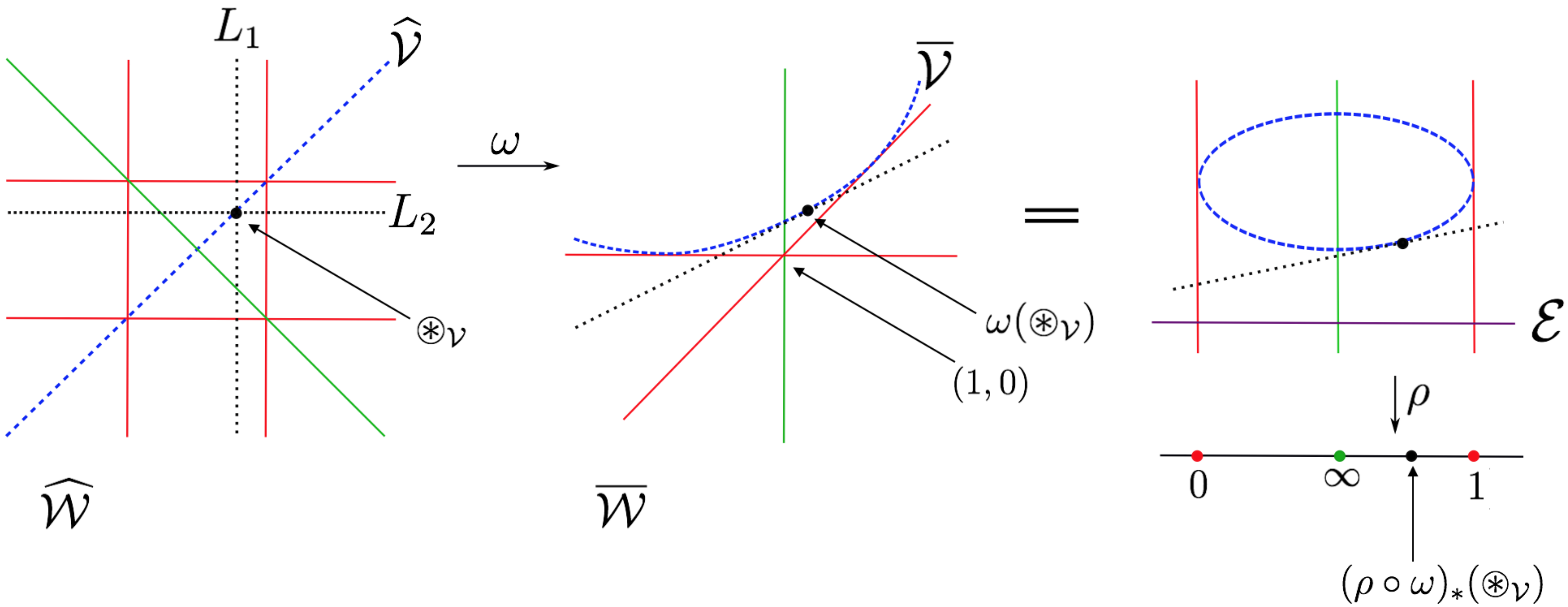
Simplify $\mathcal{W}$



$$\pi_1(\mathcal{W}, \circledast_{\mathcal{V}}) \to \pi_1(\widehat{W}) \xrightarrow{\omega} \pi_1(\overline{W}) \xrightarrow{\rho} \pi_1(\mathbb{C} \setminus \{0, 1\})$$

All these maps are surjective, so index of images of E in S stays the same or decreases.

Simplify $\mathcal{W}$

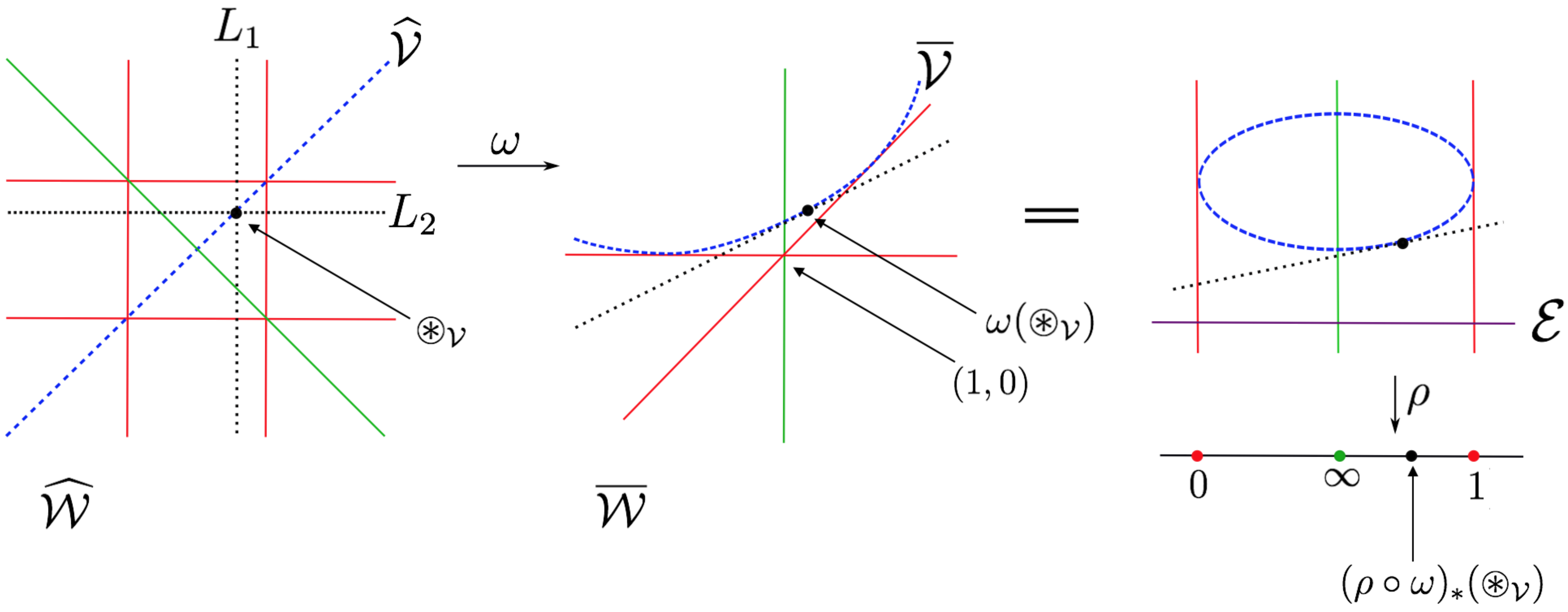


$\omega|_{\widehat{V}} : \widehat{V} \to \overline{V}$ is a homeomorphism

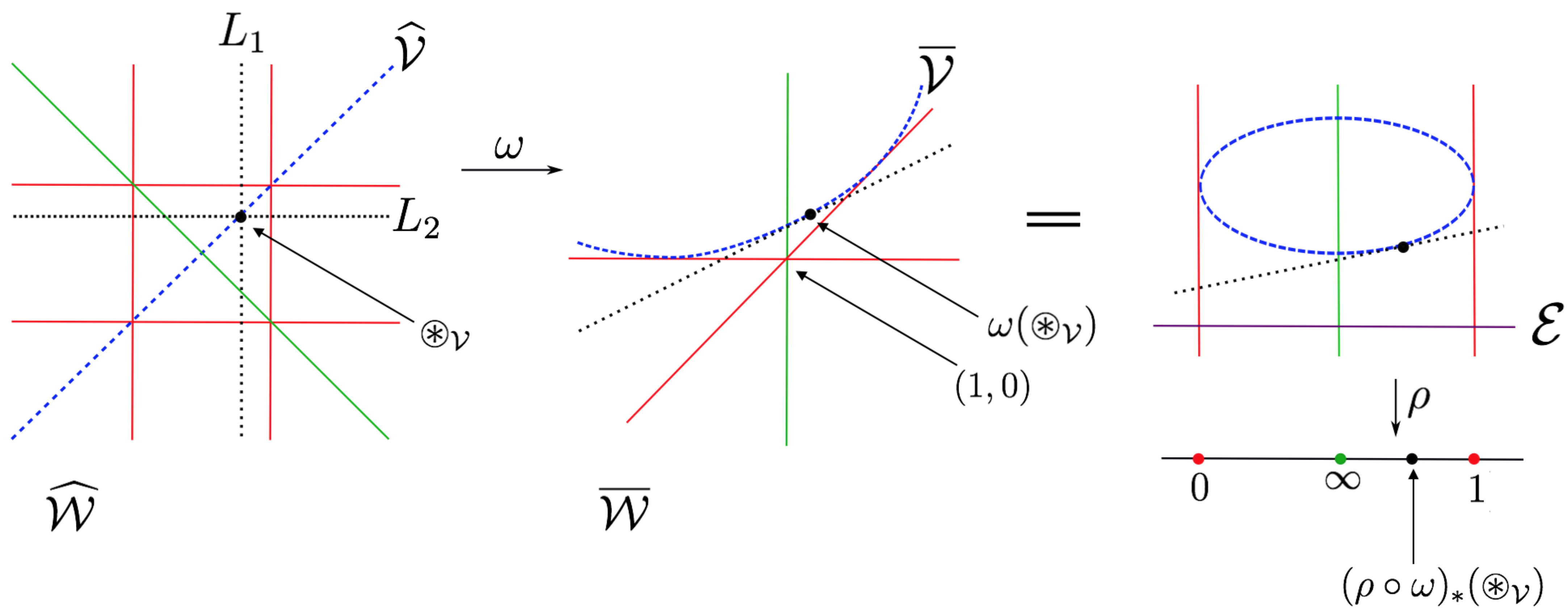
$\overline{V} \to \mathbb{C}^2 \setminus \{0,1\}$ is a (degree 2) covering so injective on fundamental groups

This implies that nontrivial elements of the kernel of $(\rho \circ \omega)_*$ cannot lie in the image of E in $\pi_1(\widehat{\mathcal{W}})$.

Simplify $\mathcal{W}$



Recall that to prove main theorem it is enough to find an infinite set of cosets of E in S.



We can do this explicitly:

Let $\gamma = \xi\eta,$ where

$\xi \in \mathrm{Im}(\pi_1(L_1) \to \pi_1(\mathcal{W}))$, (let ξ a loop on L encircling the intersections of L with the horizontal red lines)

$\omega_*(\xi)$ is nontrivial in $\pi_1(\overline{\mathcal{W}})$ and lies in the kernel of ρ_*,

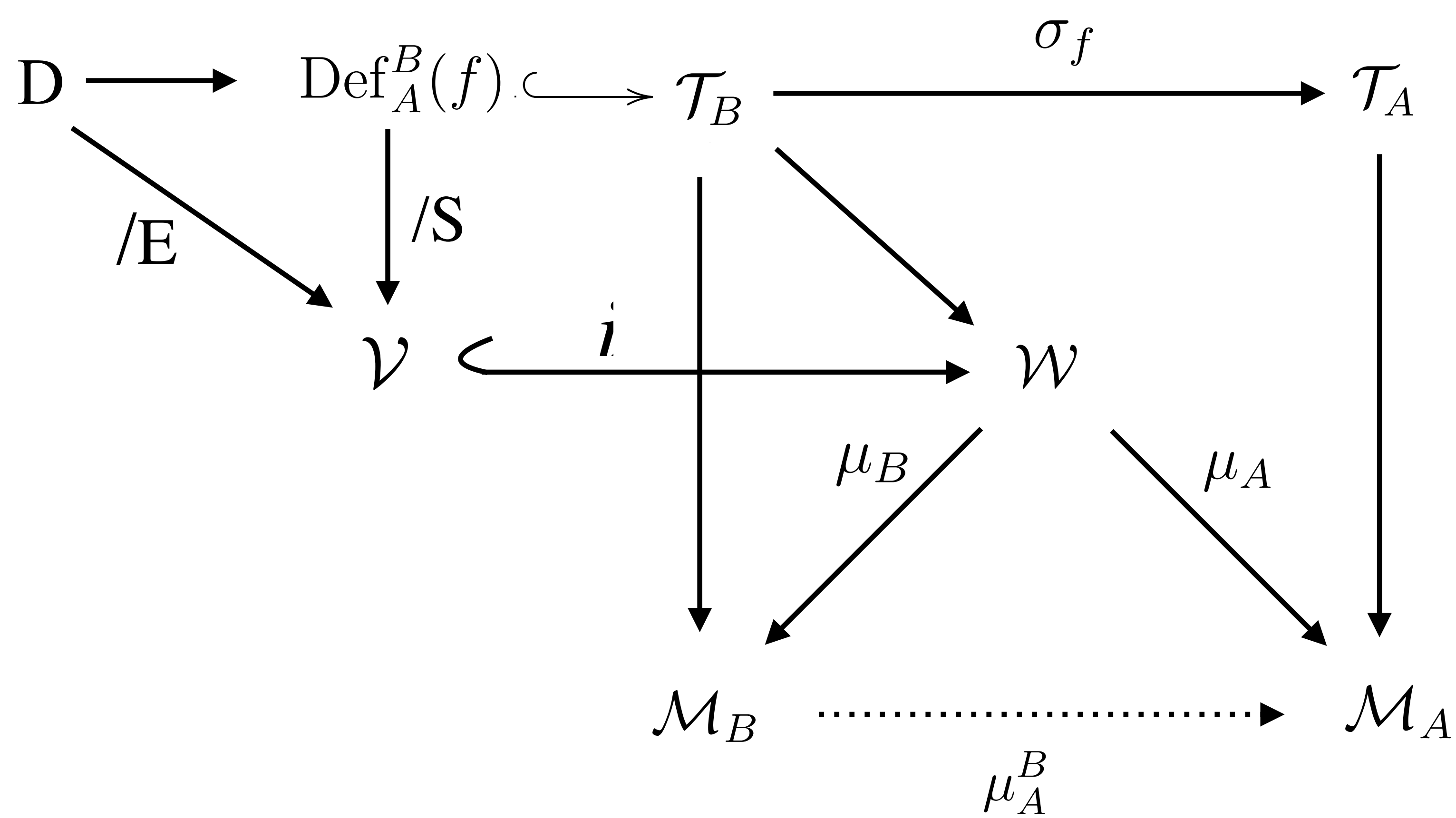
and $\eta = \delta_*(\xi)$ where δ is the symmetry across the diagonal x = y.

Then $\gamma \in \mathrm{S}$ and $\gamma^n \notin \mathrm{E}$ for all n. So $\gamma^n E$ form distinct cosets of E in S.

Summary: Fix $\langle f \rangle \in \mathrm{Per}_4(0)^*$

Theorem: $\mathrm{Def}_A^B(f)$ has infinitely many components.

Proof:



D is a connected component of $\mathrm{Def}_A^B(f)$

We have exhibited an infinite set of cosets of E in S.

It follows that E has infinite index in S and hence

$\mathrm{Def}_A^B(f)$ has infinitely many connected components as claimed.

□

Thank you!