On cobordism of rational functions.

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Problem

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We try to associate three dimensional objects to rational maps in a way consistent with the conformal structure and, hopefully, with the dynamics.

Two basic examples.

The map $z \mapsto z^n$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto nz} & \mathbb{C} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle exp} \\ \mathbb{C}^* & \xrightarrow{z \mapsto z^n} & \mathbb{C}^*. \end{array}$$

The Lattés family.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{z \mapsto nz} & \mathbb{C} \\ {\scriptstyle \wp}\downarrow & & \downarrow{\scriptstyle \wp} \\ \mathbb{C} & \xrightarrow{R_n} & \mathbb{C}. \end{array}$$

Geometric Extensions

Let S_1 and S_2 be two conformal orbifolds supported on the Riemann sphere such that

$$R : S_1 \to S_2$$

is a holomorphic covering. Assume that there exist two Kleinian groups Γ_1 and Γ_2 with components W_1 and W_2 of the discontinuity sets $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$, respectively, and

$$S_i = W_i / Stab_{W_i}(\Gamma_i).$$

Assume that there exist $\alpha(R) : W_1 \to W_2$ a Möbius map with

$$\begin{array}{ccc} W_1 & \xrightarrow{\alpha(R)} & W_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{R} & S_2 \end{array}$$

which induces a homomorphism from Γ_1 to Γ_2.
If $M_i := B^3 \cup \Omega(\Gamma_i)/\Gamma_i$. Then $\alpha(R)$ induces a unique Möbius morphism

$$\tilde{R} : M_1 \to M_2.$$

Poincaré extensions

Definition

If $\Omega(\Gamma_i)/\Gamma_i \cong S_i$, we call $\tilde{R}$ the Poincaré extension of R.

Note that the degree is

$$deg(\tilde{R}) = [\Gamma_2 : \alpha(R)\Gamma_1\alpha(R)^{-1}].$$

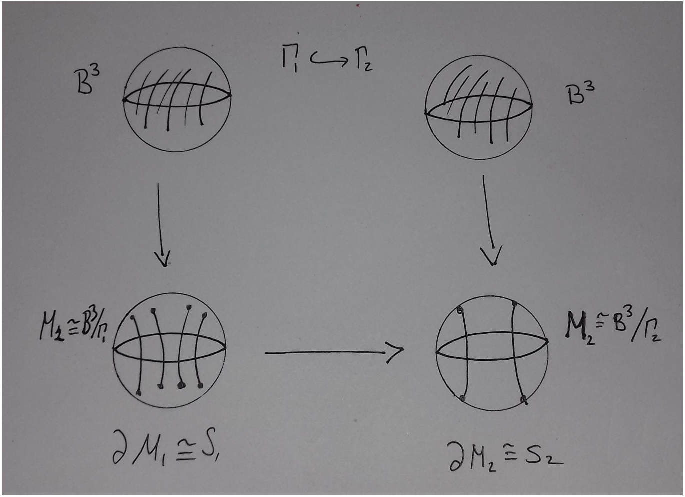
In fact, $deg(R) \leq deg(\tilde{R})$ with equality when

$$Stab_{W_i}(\Gamma_i) = \Gamma_i.$$

Geometric extension

Let $\phi_i : \partial M_i \to S_i$ be identification maps. Assume that there is a homeomorphic extension $\Phi_i : M_i \to \bar{B}^3$. Then the map $\Phi_2 \circ \tilde{R} \circ \Phi_1^{-1}$ is called *geometric* if and only if satisfies the following conditions.

1. The sets $\Phi_i(M_i \cup \partial M_i)$ are of the form $\bar{B}^3 \setminus \{\bigcup \gamma_j\}$ where each γ_j is either a geodesic or a family of finitely many geodesic rays with common starting point. There are no more than countably many curves γ_j.
2. There exist a continuous extension, on all B^3, which maps complementary geodesics to complementary geodesics.


B^3
$\Gamma_1 \hookrightarrow \Gamma_2$
B^3
$M_1 \cong B^3/\Gamma_1$
$M_2 \cong B^3/\Gamma_2$
$\partial M_1 \cong S_1$
$\partial M_2 \cong S_2$

Equivariance under Möbius actions

Let $A \subset Rat_d(\mathbb{C})$. Assume that there exist a map

$$Ext : A \to End(\bar{B}^3)$$

such that $Ext(R)$ is an extension of R for every $R \in A$. Then for every pair of maps h, g in Mob we define

$$\widetilde{Ext}(g \circ R \circ h) = \hat{g} \circ Ext(R) \circ \hat{h}$$

where $\hat{g}$ and $\hat{h}$ are the classical Poincaré extensions of the maps g and h, respectively.
If $\widetilde{Ext}$ is a map from the Möbius bi-orbit of A to $End(\bar{B}^3)$, then we call Ext a *conformally natural extension* of A.

A list of desirable conditions

1. **Geometric.**
2. **Same degree.**
3. **Dynamical.** These are extensions Ext such that $Ext(R^n) = Ext(R)^n$ for $n = 1, 2, ...$
4. **Semigroup Homomorphisms.** A stronger version of the previous property is to find semigroups $\mathcal{S}$, of rational maps, for which there is an extension Ext defined in all $\mathcal{S}$ such that

$$Ext(R \circ Q) = Ext(R) \circ Ext(Q).$$

5. **Equivariance under Möbius actions.** When defined on saturated sets under the left and right actions of $PSL(2, \mathbb{C})$.

Blaschke maps

A *Blaschke map* $B : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a rational map that leaves the unit disk Δ invariant. If d is the degree of B, then there exist $\theta \in [0, 2\pi]$ and d points $\{a_1, ..., a_d\}$ in Δ such that

$$B(z) = e^{i\theta} \left(\frac{z - a_1}{1 - \bar{a}_1 z} \right) ... \left(\frac{z - a_d}{1 - \bar{a}_d z} \right).$$

For the semigroup of Blaschke maps we have the following theorem:

Theorem

Let B be the semigroup of all Blaschke maps, then there exist an extension defined on B that satisfies conditions 1 to 4. This extension is conformally natural with respect to $PSL(2, \mathbb{R})$.

Theorem

Let S be a subsemigroup of Blaschke maps, then the extension above restricted on S is conformally natural with respect to all Möbius transformations if and only if S does not intersect the Möbius bi orbit of maps of the form $z \mapsto z^n$.

Motivation

Given Fuchsian uniformizations Γ_1 and Γ_2 for $R : S_1 \to S_2$, then we get another rational map Q^* induced by the action on the complement of the unit disk such that we have the following diagram commutes:

$$\begin{array}{ccc}
\bar{\mathbb{C}} & \xrightarrow{R} & \bar{\mathbb{C}} \\
\phi \downarrow & & \downarrow \psi \\
\bar{\mathbb{C}} & \xrightarrow{Q^*} & \bar{\mathbb{C}} \\
\bar{z} \downarrow & & \downarrow \bar{z} \\
\bar{\mathbb{C}} & \xrightarrow{Q} & \bar{\mathbb{C}}.
\end{array}$$

This construction motivates the following definition.

Definition

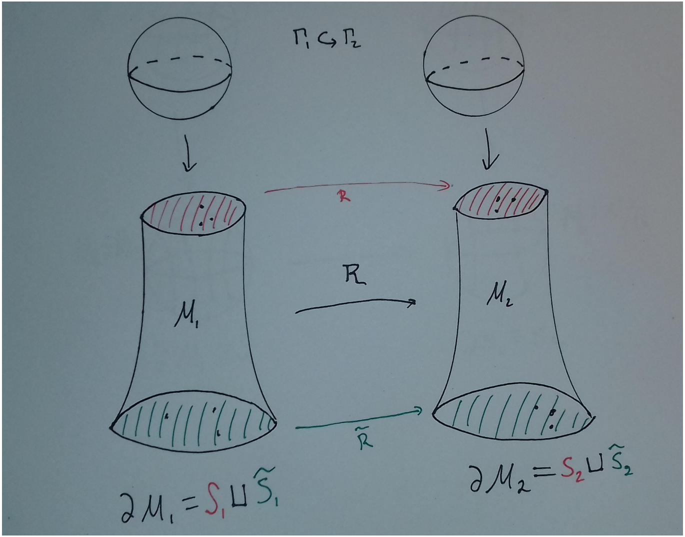
Two rational maps $R : S_1 \to S_2$ and $\tilde{R} : \tilde{S}_1 \to \tilde{S}_2$ are *cobordant* if:

- There are geometrically finite Kleinian groups Γ_1 and Γ_2 such that $B^3 \bigcup \Omega(\Gamma_1)/\Gamma_1 = M_1$ and $B^3 \bigcup \Omega(\Gamma_2)/\Gamma_2 = M_2$ so the following diagram commutes:

$$\begin{array}{ccc} B^3 \bigcup \Omega(\Gamma_1) & \longrightarrow & B^3 \bigcup \Omega(\Gamma_2) \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{\quad \Re \quad} & M_2 \end{array}$$

- $\partial M_1 = S_1 \sqcup \tilde{S}_1$, $\partial M_2 = S_2 \sqcup \tilde{S}_2$.
- The restriction of $\Re$ to the boundaries S_1 and $\tilde{S}_1$ belong to the same conformal class of R and $\tilde{R}$, respectively.

Cobordism is an equivalence relation.


$\Gamma_1 \hookrightarrow \Gamma_2$
R
R
$\mathcal{M}_1$
$\mathcal{M}_2$
$\tilde{R}$
$\partial\mathcal{M}_1 = S_1 \sqcup \tilde{S}_1$
$\partial\mathcal{M}_2 = S_2 \sqcup \tilde{S}_2$

We say that two branched coverings R and Q, of the Riemann sphere onto itself, are *Hurwitz equivalent* if there are quasiconformal homeomorphisms ϕ and ψ, making the following diagram commutative

$$\begin{array}{ccc} \bar{\mathbb{C}} & \xrightarrow{R} & \bar{\mathbb{C}} \\ \phi \downarrow & & \downarrow \psi \\ \bar{\mathbb{C}} & \xrightarrow{Q} & \bar{\mathbb{C}}. \end{array}$$

Given a rational map R, the Hurwitz space $H(R)$ is the set of all rational maps Q that are Hurwitz equivalent to R. The topology we are considering on $H(R)$ is the compact-open topology.

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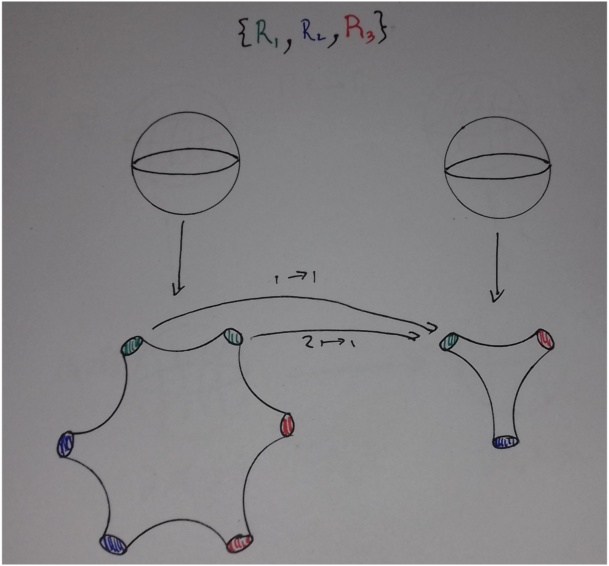
Theorem

If R_1 and R_2 are Hurwitz equivalent, then $R_1(z) \sim_{cob} \bar{R}_2(\bar{z})$.

Cobordisms of families of rational functions

Once we have considered cobordisms of two maps, we can define cobordisms of finite family of rational maps as shown in the following image:


{R1, R2, R3}
1 ↦ 1
2 ↦ 1

Given a finite family of Riemann surfaces of finite type $\{S_1, S_2, ..., S_n\}$, then there exist Riemann surface S_0 and a Kleinian group Γ such that

$$\Omega(\Gamma)/\Gamma = S_0 \sqcup S_1 \sqcup ... \sqcup S_n.$$

In fact, it is possible to find a group Γ without extra components.
What can we say about rational maps?

Theorem

Given a finite family of rational maps $\{R_1, ..., R_n\}$, then there exist a finite collection of rational maps $\{Q_1, ..., Q_m\}$ such that the extended collection $\{Q_1, ..., Q_m, R_1, ..., R_n\}$ forms a family of cobordant rational maps.