On cobordism of rational functions.

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Problem

We try to associate three dimensional objects to rational maps in a way consistent with the conformal structure and, hopefully, with the dynamics.
Two basic examples.

The map $z \mapsto z^n$.

The Lattés family.
Let $S_1$ and $S_2$ be two conformal orbifolds supported on the Riemann sphere such that

$$R : S_1 \rightarrow S_2$$

is a holomorphic covering. Assume that there exist two Kleinian groups $\Gamma_1$ and $\Gamma_2$ with components $W_1$ and $W_2$ of the discontinuity sets $\Omega(\Gamma_1)$ and $\Omega(\Gamma_2)$, respectively, and

$$S_i = W_i/\text{Stab}_{W_i}(\Gamma_i).$$
Assume that there exist $\alpha(R) : W_1 \to W_2$ a Möbius map with

\[
\begin{array}{ccc}
W_1 & \xrightarrow{\alpha(R)} & W_2 \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{R} & S_2
\end{array}
\]

which induces a homomorphism from $\Gamma_1$ to $\Gamma_2$.

If $M_i := B^3 \cup \Omega(\Gamma_i)/\Gamma_i$. Then $\alpha(R)$ induces a unique Möbius morphism

\[\tilde{R} : M_1 \to M_2.\]
Poincaré extensions

Definition

If $\Omega(\Gamma_i)/\Gamma_i \cong S_i$, we call $\tilde{R}$ the Poincaré extension of $R$.

Note that the degree is

$$\deg(\tilde{R}) = [\Gamma_2 : \alpha(R)\Gamma_1\alpha(R)^{-1}].$$

In fact, $\deg(R) \leq \deg(\tilde{R})$ with equality when $\text{Stab}_{W_i}(\Gamma_i) = \Gamma_i$. 
Let $\phi_i : \partial M_i \to S_i$ be identification maps. Assume that there is a homeomorphic extension $\Phi_i : M_i \to \bar{B}^3$. Then the map $\Phi_2 \circ \tilde{R} \circ \Phi_1^{-1}$ is called geometric if and only if satisfies the following conditions.

1. The sets $\Phi_i(M_i \cup \partial M_i)$ are of the form $\bar{B}^3 \setminus \bigcup \gamma_j$ where each $\gamma_j$ is either a geodesic or a family of finitely many geodesic rays with common starting point. There are no more than countably many curves $\gamma_j$.

2. There exist a continuous extension, on all $B^3$, which maps complementary geodesics to complementary geodesics.
Problem
Poincaré extensions
Blaschke maps
Cobordisms
Let $A \subset \text{Rat}_d(\mathbb{C})$. Assume that there exist a map

$$\text{Ext} : A \to \text{End}(\tilde{B}^3)$$

such that $\text{Ext}(R)$ is an extension of $R$ for every $R \in A$. Then for every pair of maps $h, g$ in $\text{Mob}$ we define

$$\tilde{\text{Ext}}(g \circ R \circ h) = \hat{g} \circ \text{Ext}(R) \circ \hat{h}$$

where $\hat{g}$ and $\hat{h}$ are the classical Poincaré extensions of the maps $g$ and $h$, respectively.

If $\tilde{\text{Ext}}$ is a map from the Möbius bi-orbit of $A$ to $\text{End}(\tilde{B}^3)$, then we call $\text{Ext}$ a conformally natural extension of $A$. 
A list of desirable conditions


2. Same degree.

3. Dynamical. These are extensions $Ext$ such that $Ext(R^n) = Ext(R)^n$ for $n = 1, 2, ...$

4. Semigroup Homomorphisms. A stronger version of the previous property is to find semigroups $S$, of rational maps, for which there is an extension $Ext$ defined in all $S$ such that

$$Ext(R \circ Q) = Ext(R) \circ Ext(Q).$$

5. Equivariance under Möbius actions. When defined on saturated sets under the left and right actions of $PSL(2, \mathbb{C})$. 
A Blaschke map $B : \mathbb{C} \to \mathbb{C}$ is a rational map that leaves the unit disk $\Delta$ invariant. If $d$ is the degree of $B$, then there exist $\theta \in [0, 2\pi]$ and $d$ points $\{a_1, ..., a_d\}$ in $\Delta$ such that

$$B(z) = e^{i\theta} \left( \frac{z - a_1}{1 - \bar{a}_1 z} \right) \ldots \left( \frac{z - a_d}{1 - \bar{a}_d z} \right).$$
For the semigroup of Blaschke maps we have the following theorem:

**Theorem**

Let $B$ be the semigroup of all Blaschke maps, then there exist an extension defined on $B$ that satisfies conditions 1 to 4. This extension is conformally natural with respect to $\text{PSL}(2, \mathbb{R})$. 
Theorem

Let $S$ be a subsemigroup of Blaschke maps, then the extension above restricted on $S$ is conformally natural with respect to all Möbius transformations if and only if $S$ does not intersect the Möbius bi orbit of maps of the form $z \mapsto z^n$. 
Motivation

Given Fuchsian uniformizations $\Gamma_1$ and $\Gamma_2$ for $R : S_1 \to S_2$, then we get another rational map $Q^*$ induced by the action on the complement of the unit disk such that we have the following diagram commutes:
This construction motivates the following definition.
Two rational maps $R : S_1 \to S_2$ and $\tilde{R} : \tilde{S}_1 \to \tilde{S}_2$ are cobordant if:

- There are geometrically finite Kleinian groups $\Gamma_1$ and $\Gamma_2$ such that $B^3 \cup \Omega(\Gamma_1) / \Gamma_1 = M_1$ and $B^3 \cup \Omega(\Gamma_2) / \Gamma_2 = M_2$ so the following diagram commutes:

$$
\begin{array}{ccc}
B^3 \cup \Omega(\Gamma_1) & \xrightarrow{\mathbb{R}} & B^3 \cup \Omega(\Gamma_2) \\
\downarrow & & \downarrow \\
M_1 & \xrightarrow{\mathbb{R}} & M_2
\end{array}
$$

- $\partial M_1 = S_1 \sqcup \tilde{S}_1$, $\partial M_2 = S_2 \sqcup \tilde{S}_2$.

The restriction of $\mathbb{R}$ to the boundaries $S_1$ and $\tilde{S}_1$ belong to the same conformal class of $R$ and $\tilde{R}$, respectively.

Cobordism is an equivalence relation.
Problem
Poincaré extensions
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Cobordisms
We say that two branched coverings $R$ and $Q$, of the Riemann sphere onto itself, are *Hurwitz equivalent* if there are quasiconformal homeomorphisms $\phi$ and $\psi$, making the following diagram commutative

$$
\begin{array}{ccc}
\overset{\phi}{\mathbb{C}} & \xrightarrow{R} & \overset{\psi}{\mathbb{C}} \\
\downarrow & & \downarrow \\
\overset{\phi}{\mathbb{C}} & \xrightarrow{Q} & \overset{\psi}{\mathbb{C}}
\end{array}
$$

Given a rational map $R$, the Hurwitz space $H(R)$ is the set of all rational maps $Q$ that are Hurwitz equivalent to $R$. The topology we are considering on $H(R)$ is the compact-open topology.
We say that two branched coverings $R$ and $Q$, of the Riemann sphere onto itself, are \textit{Hurwitz equivalent} if there are quasiconformal homeomorphisms $\phi$ and $\psi$, making the following diagram commutative

$$
\begin{array}{ccc}
\tilde{\mathbb{C}} & \xrightarrow{R} & \tilde{\mathbb{C}} \\
\phi \downarrow & & \downarrow \psi \\
\tilde{\mathbb{C}} & \xrightarrow{Q} & \tilde{\mathbb{C}}.
\end{array}
$$

Given a rational map $R$, the Hurwitz space $H(R)$ is the set of all rational maps $Q$ that are Hurwitz equivalent to $R$. The topology we are considering on $H(R)$ is the compact-open topology.

\textbf{Theorem}

\textit{If $R_1$ and $R_2$ are Hurwitz equivalent, then $R_1(z) \sim_{\text{cob}} \bar{R}_2(\bar{z})$.}
Cobordisms of families of rational functions

Once we have considered cobordisms of two maps, we can define cobordisms of finite family of rational maps as shown in the following image:
Given a finite family of Riemann surfaces of finite type \( \{ S_1, S_2, \ldots, S_n \} \), then there exist Riemann surface \( S_0 \) and a Kleinian group \( \Gamma \) such that

\[
\Omega(\Gamma)/\Gamma = S_0 \sqcup S_1 \sqcup \ldots \sqcup S_n.
\]

In fact, it is possible to find a group \( \Gamma \) without extra components. What can we say about rational maps?
Theorem

Given a finite family of rational maps \( \{ R_1, ..., R_n \} \), then there exist a finite collection of rational maps \( \{ Q_1, ..., Q_m \} \) such that the extended collection \( \{ Q_1, ..., Q_m, R_1, ..., R_n \} \) forms a family of cobordant rational maps.