Perturbations of maps tangent to $z \mapsto \bar{z}$

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Local anti-holomorphic dynamics

- $f : (X, x) \to (X, x)$ is a anti-holomorphic germ fixing $x \in X$.
- $\mathrm{D}_x f : \mathrm{T}_x X \to \mathrm{T}_x X$ is an anti-$\mathbb{C}$-linear map; it has two eigenvalues $\rho \geq 0$ and $-\rho \leq 0$.
- $f^{\circ 2}$ is holomorphic; it fixes x with multiplier $\rho^2 \geq 0$.

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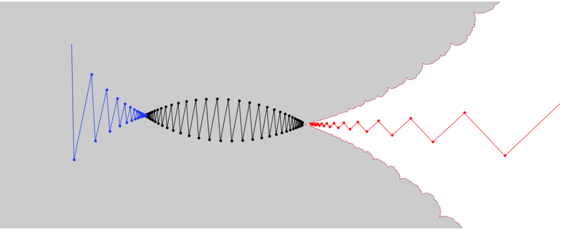
An example: the tricorn family.

- $P_c(z) = \bar{z}^2 + c$.
- x is a fixed point of P_c.



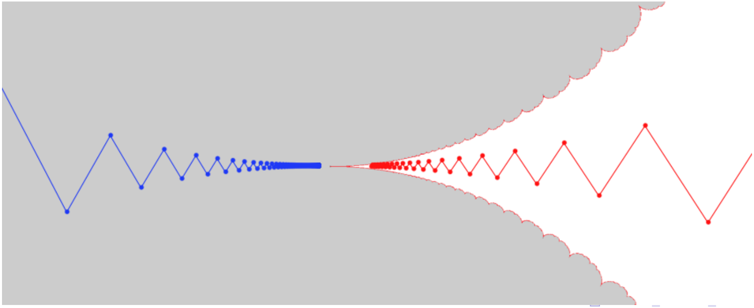
Germs tangent to $z \mapsto \bar{z}$

We are interested in the case $\rho = 1$.

- $\mathrm{D}_x f : \mathrm{T}_x X \to \mathrm{T}_x X$ is conjugate to $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$.
- $\mathrm{D}_x f$ fixes a line $\Delta_x \subset \mathrm{T}_x X$.
- $f^{\circ 2}$ has m attracting axes and m repelling axes.

Lemma

Δ_x is a union of attracting and/or repelling axes for $f^{\circ 2}$.



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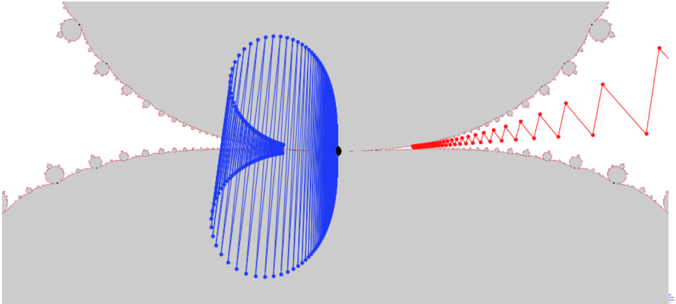

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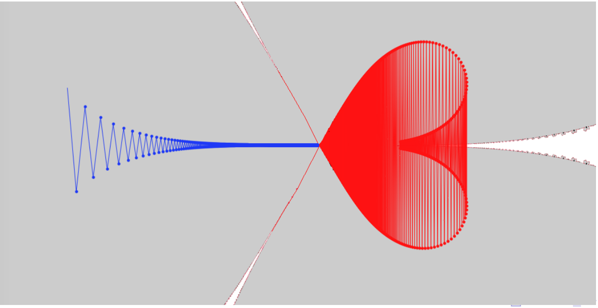

The tricorn family

- $P_c(z) = \bar{z}^2 + c$.
- x is periodic of odd period p for P_c and $f := P_c^{\circ p}$.
- The number of attracting petals is either $m = 1$ or $m = 2$.
- If $m = 1$, then Δ_x is the union of the attracting direction and the repelling direction.

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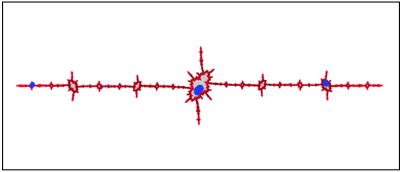
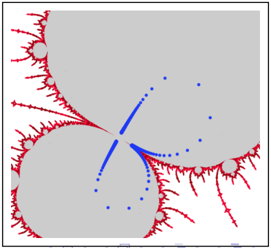
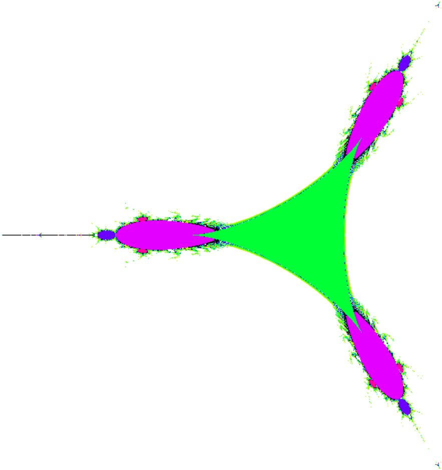
If $m = 2$, then Δ_x is the union of the two repelling directions.





The bifurcation locus for the family $(P_c(z) = \bar{z}^2 + c)_{c \in \mathbb{C}}$



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Question

What does the set of parameters $c \in \mathbb{C}$ for which P_c has a parabolic periodic orbit of odd period p look like?

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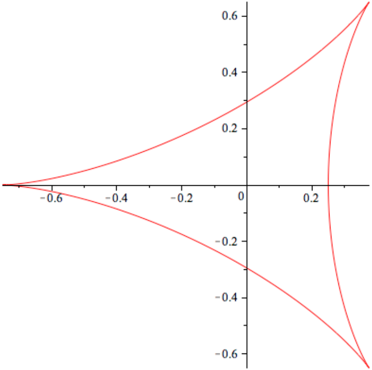
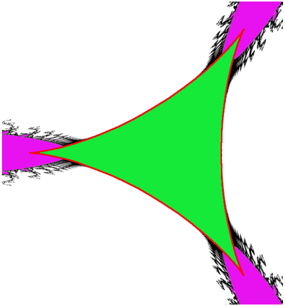


For $p = 1$, the locus is the image of the circle $C(0, 1/2)$ by the map $z \mapsto c = z + \bar{z}^2$.

The parabolic locus

Theorem (Mukherjee, Nakane, Schleicher)

The boundary of every hyperbolic component of odd period is a simple closed curve consisting of exactly 3 *parabolic cusp points as well as* 3 *parabolic arcs, each connecting two parabolic cusps.*



Period $p = 1$

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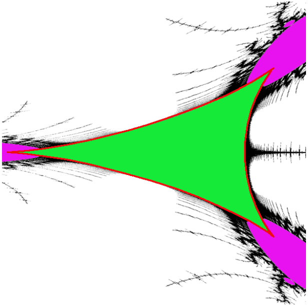

Period $p = 3$

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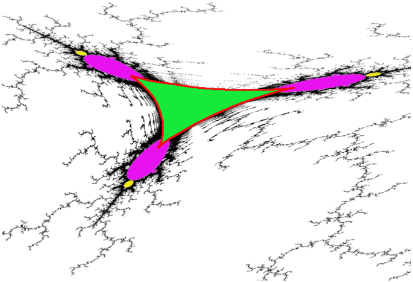

Period $p = 5$

Local picture near a parabolic arc

Proposition (Bonifant-B-Milnor)

Arcs are smooth.

Local picture near a parabolic arc

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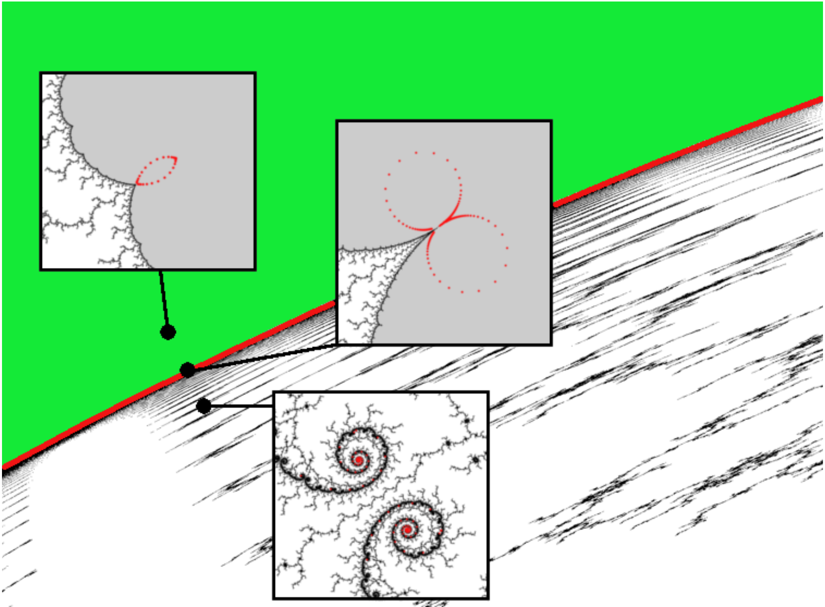
If P_{c_0} has a parabolic point with odd period p and 1 attracting petal, there is a coordinate function $u : (\mathbb{C}, c_0) \to (\mathbb{R}, 0)$ such that

- *if $u(c) = 0$, $P_c^{\circ 2p}$ has a multiple fixed point close to x;*
- *if $u(c) \neq 0$, $P_c^{\circ 2p}$ has two distinct fixed points close to x;*
  - *if $u(c) > 0$, they are fixed by $P_c^{\circ p}$, one is attracting, one is repelling;*
  - *if $u(c) < 0$, they form a repelling cycle of period 2 for $P_c^{\circ p}$.*

Local picture near a parabolic arc



Local picture near a parabolic cusp

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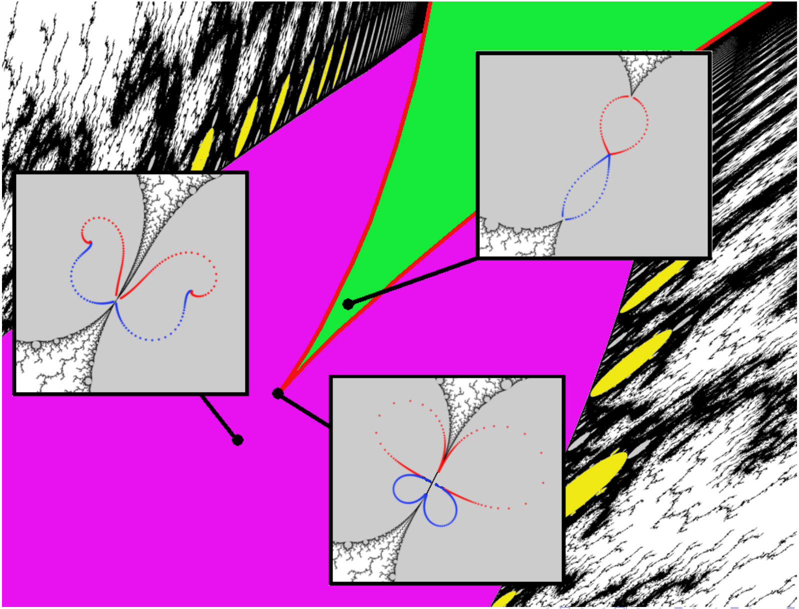
If P_{c_0} has a parabolic point with odd period p and 2 attracting petal, there is coordinate system $(u,v):(\mathbb{C},c_0)\to(\mathbb{R}^2,(0,0))$ such that

- *if $u^3(c)=v^2(c)$, then $P_c^{\circ 2p}$ has two distinct fixed points close to x; one is repelling and the other has multiplier 1; both are fixed by $P_c^{\circ p}$;*
- *if $u^3(c)\neq v^2(c)$, then $P_c^{\circ 2p}$ has three distinct fixed points close to x;*
    - *if $u^3(c)>v^2(c)$, they are fixed by $P_c^{\circ p}$; one is attracting and the other two are repelling;*
    - *if $u^3(c)<v^2(c)$, one is repelling and fixed by $P_c^{\circ p}$; the other two are attracting and form a cycle of period 2 for $P_c^{\circ p}$.*

Local picture near a parabolic cusp



Splitting of a multiple fixed point

- X is a Riemann surface.
- $(f_\lambda : X \to X)_{\lambda \in \Lambda}$ is a holomorphic family of holomorphic maps.
- f_{λ_0} has a multiple fixed point x with multiplicity $m+1$.
- As λ moves away from λ_0, the fixed point splits into $m+1$ fixed points $x_1(\lambda), \ldots, x_{m+1}(\lambda)$, counting multiplicities.

A priori, those fixed points do not depend holomorphically on λ.

Question

How can we study the splitting of those fixed points?

Splitting of a multiple fixed point

- $\zeta : (X, x) \to (\mathbb{C}, 0)$ is a local coordinate such that

$$\zeta \circ f = \zeta + \zeta^{m+1} + \mathrm{O}(\zeta^{2m+1}).$$

- $\beta(\lambda)$ is the barycenter of the points $\zeta\big(x_i(\lambda)\big)$.
- for $k \in [2, m+1]$, $\sigma_k(\lambda)$ are the elementary symmetric functions of the differences $\zeta\big(x_i(\lambda)\big) - \beta(\lambda)$.

Definition

The splitting of the fixed points is generic in the family $(f_\lambda)_{\lambda\in\Lambda}$ if the map $\lambda \mapsto \big(\sigma_2(\lambda), \ldots, \sigma_{m+1}(\lambda)\big)$ is a local submersion at λ_0.

Complexification of the tricorn family

- $X = \mathbb{C}_1 \sqcup \mathbb{C}_2$.
- $\Lambda := \mathbb{C}^2$ and for $\lambda := (c_1, c_2) \in \Lambda$, $f_\lambda : X \to X$ is defined by

$$f_\lambda : \mathbb{C}_1 \ni z_1 \mapsto z_1^2 + c_2 \in \mathbb{C}_2 \quad \text{and} \quad f_\lambda : \mathbb{C}_2 \ni z_2 \mapsto z_2^2 + c_1 \in \mathbb{C}_1.$$

- The tricorn family corresponds to the slice $c_2 = \bar{c}_1$.

Proposition

Assume $f_{\lambda_0}^{\circ p}$ has a multiple fixed point. Then the splitting of the fixed point is generic in the family $(f_\lambda^{\circ p})_{\lambda \in \Lambda}$.

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- From now on, we assume $m = 2$. We need to show that $\mathrm{D}_{\lambda_0}\sigma_2$ and $\mathrm{D}_{\lambda_0}\sigma_3$ are linearly independent.

The tangent space to the family $(f_\lambda^{\circ p})_{\lambda \in \Lambda}$

- $f := f_{\lambda_0}$.
- $t \mapsto \lambda_t$ is a complex curve.
- $\xi := \frac{\mathrm{d} f_{\lambda_t}}{\mathrm{d} t}\Big|_{t=0}$.
- η is the meromorphic vector field on X defined by

$$\mathrm{D}f \circ \eta = \xi.$$

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Lemma

$$\eta_p = \eta + f^*\eta + \ldots + f^{\circ(p-1)^*}\eta.$$

The derivative $\mathrm{D}_{\lambda_0}\sigma_k$

Proposition

Writing $\eta_p = (h_0 + h_1\zeta + \cdots)\frac{\mathrm{d}}{\mathrm{d}\zeta}$*, we have*

$$\mathrm{D}_{\lambda_0}\sigma_2(\xi) = h_1 \quad \textit{and} \quad \mathrm{D}_{\lambda_0}\sigma_3(\xi) = -h_0.$$

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Proof.

- Set $Q_t(\zeta) = -\sigma_3(\lambda_t) + \sigma_2(\lambda_t)\cdot\zeta + \zeta^3$, so that

$$\zeta \circ f_{\lambda_t}^{\circ p} - \zeta = u_t(\zeta)\cdot Q_t(\zeta - \beta_t).$$

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- f is tangent to to the identity to order 2, so that

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Quadratic differentials

Definition

A quadratic differential $\boldsymbol{q}$ on X is a field of symmetric and bilinear forms.

If η and θ are two vector fields on X, then

- $\boldsymbol{q}(\eta, \theta) : X \to \mathbb{C}$ is a function,
- $\boldsymbol{q}(\eta, \theta) = \boldsymbol{q}(\theta, \eta)$ and
- $\boldsymbol{q} \cdot \eta := \boldsymbol{q}(\eta, \cdot)$ is a 1-form on X.

In particular, we can consider the residue

$$\mathrm{res}(\boldsymbol{q} \cdot \eta, x).$$

The derivative $\mathrm{D}_{\lambda_0}\sigma_k$

For $j \in [0, p]$, set

$$x_j := f^{\circ j}(x) \quad \text{and} \quad \zeta_j := \zeta \circ f^{\circ(p-j)}.$$

For $k \in \{1, 2\}$, let $\boldsymbol{q}_k$ be the meromorphic quadratic differential on X:

- which is holomorphic outside the cycle,
- whose polar part at x_j is that of $\mathrm{d}\zeta_j^2/\zeta_j^k$ and
- which has at most triple poles at infinity.

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Proposition

$$\mathrm{D}_{\lambda_0}\sigma_2(\boldsymbol{\xi}) = \sum_{j=1}^{p} \mathrm{res}(\boldsymbol{q}_2 \cdot \boldsymbol{\eta}, x_j).$$

$$\mathrm{D}_{\lambda_0}\sigma_3(\boldsymbol{\xi}) = -\sum_{j=1}^{p} \mathrm{res}(\boldsymbol{q}_1 \cdot \boldsymbol{\eta}, x_j).$$

Pushing-forward

- $f : X \setminus \{0_1, 0_2\} \to X \setminus \{c_1, c_2\}$ is a covering of degree 2.
- the push-forward $f_* \boldsymbol{q}$ is defined by

$$f_* \boldsymbol{q} := \sum_g g^* \boldsymbol{q}$$

where g ranges among the inverse branches of f.

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  where g ranges among the inverse branches of f.
- The polar part of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ along the cycle are invariant, so that

$$\nabla_f \boldsymbol{q}_1 := \boldsymbol{q}_1 - f_* \boldsymbol{q}_1 \quad \text{and} \quad \nabla_f \boldsymbol{q}_2 := \boldsymbol{q}_2 - f_* \boldsymbol{q}_2$$

  belong to

$$\mathrm{Vect}\left(\frac{\mathrm{d}z_1^2}{z_1 - c_1}, \frac{\mathrm{d}z_2^2}{z_2 - c_2}\right).$$

The derivative $\mathrm{D}_{\lambda_0}\sigma_k$

Proposition

$$\mathrm{D}_{\lambda_0}\sigma_2(\xi) = \mathrm{res}\big(\nabla_f \boldsymbol{q}_2 \cdot \xi(0_1), f(0_1)\big) + \mathrm{res}\big(\nabla_f \boldsymbol{q}_2 \cdot \xi(0_2), f(0_2)\big).$$

and

$$\mathrm{D}_{\lambda_0}\sigma_2(\xi) = -\mathrm{res}\big(\nabla_f \boldsymbol{q}_1 \cdot \xi(0_1), f(0_1)\big) - \mathrm{res}\big(\nabla_f \boldsymbol{q}_1 \cdot \xi(0_2), f(0_2)\big).$$

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Proof.

$$\begin{aligned} -\mathrm{res}\big(\nabla_f \boldsymbol{q}_2 \cdot \xi(0_1), f(0_1)\big) &= -\mathrm{res}\big(f_* \boldsymbol{q}_2 \cdot \xi(0_1), f(0_1)\big) \\ &= -\mathrm{res}(\boldsymbol{q}_2 \cdot \eta, 0_1) \\ &= \sum_{j=1}^{p} \mathrm{res}(\boldsymbol{q}_2 \cdot \eta, x_j). \end{aligned}$$

Injectivity of ∇_f

To prove that $\mathrm{D}_{\lambda_0}\sigma_2$ and $\mathrm{D}_{\lambda_0}\sigma_3$ are linearly independent, it is enough to prove that $\nabla_f \boldsymbol{q}_1$ and $\nabla_f \boldsymbol{q}_2$ are linearly independent.

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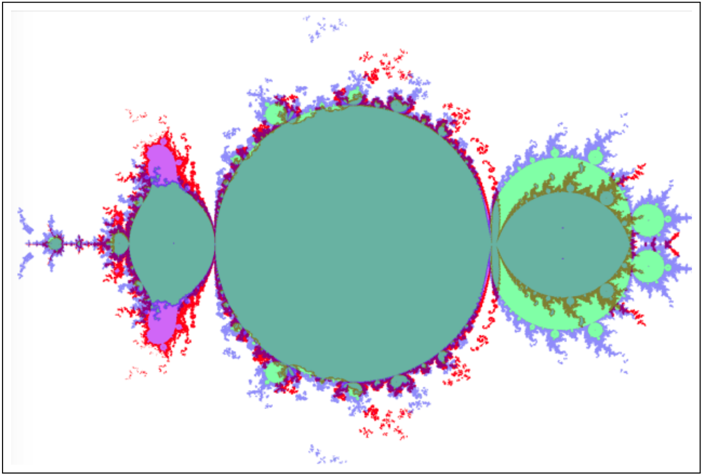
Lemma (Epstein)

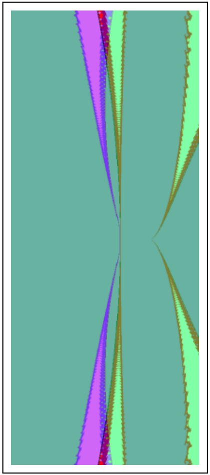
∇_f is injective on $\mathrm{Vect}(\boldsymbol{q}_1, \boldsymbol{q}_2)$.

Proof. The proof relies on the Contraction Principle: if V is compactly contained in $\mathbb{C} \setminus \langle x \rangle$, then

$$\int_V |f_* \boldsymbol{q}| = \int_V \left| \sum_g g^* \boldsymbol{q} \right| \leq \int_V \sum_g |g^* \boldsymbol{q}| = \sum_g \int_V g^* |\boldsymbol{q}| = \int_{f^{-1}(V)} |\boldsymbol{q}|.$$

The bifurcation locus for the family $(\lambda z + z^2 + 10z^3)_{\lambda \in \mathbb{C}}$





Happy Birthday Jack