Smooth Metric Measure Spaces

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Namely a Riemannian manifold with a conformal change in the measure

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Recall $(X_i, \mu_i) \stackrel{\text{mGH}}{\longrightarrow} (X_\infty, \mu_\infty)$ (compact) if for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty.$$

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We have, as $\epsilon \to 0$,

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where $g_\epsilon = g_M + (\epsilon e^{-\frac{f}{m}})^2 g_F$.

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By O'Neill's formula, the Ricci curvature of the warped product metric g_ϵ in the M direction is

$$\mathrm{Ric}_M + \mathrm{Hess} f - \frac{1}{m} df \otimes df.$$

m-Bakry-Emery Ricci tensor

Therefore for smooth metric measure spaces $(M^n, g, e^{-f} dvol_g)$, the corresponding Ricci tensor is

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df \quad \text{for } m > 0,$$

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If $m_1 \geq m_2$, then $\mathrm{Ric}_f^{m_1} \geq \mathrm{Ric}_f^{m_2}$.

So $\mathrm{Ric}_f^m \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

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when m is a positive integer, (1) $\Leftrightarrow$ the warped product metric $M \times_{e^{-\frac{f}{m}}} F^m$ is Einstein for some F^m.
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- Corresponding versions for non-smooth metric measure spaces (Lott-Villani, Sturm)
- diffusion processes
- Sobolev inequality
- conformal geometry

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Lott 2003: if M is compact with $\mathrm{Ric}_f^m \geq \lambda$ (m positive integers), then $M \times_{\epsilon e^{-\frac{f}{m}}} \mathbb{S}^m$ has $\mathrm{Ric} \geq \lambda$ when ϵ is small.

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Another way later!

What about $m = \infty$?

Examples

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$\mathbb{H}^n$ the hyperbolic space. Fixed any $p \in \mathbb{H}^n$, let $f(x) = (n-1)d^2(p,x)$, then $Ric_f \geq (n-1)$.

Myers' theorem and Cheeger-Gromoll's isometric splitting theorem do not hold for Ric_f.

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Example

$\mathbb{R}^n$ with Euclidean metric, $f(x_1, \cdots, x_n) = x_1$. $Ric_f = Ric = 0$. $vol_f(B(0,r)) = \int_{B(0,r)} e^{-f} dvol$ is of exponential growth.

Bishop-Gromov's volume comparison doesn't extend.

Need Conditions

Many results do extend
when f or ∇f are bounded!

Notations

With respect to the measure $e^{-f} dvol$:

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- $\mathrm{vol}_f(B(p,r)) = \int_{B(p,r)} e^{-f} dvol_g$
- m_H^k be the mean curvature of the geodesic sphere in the model space M_H^k

Mean Curvature (Laplacian) Comparison for Ric_f

Theorem (Wei-Wylie2007)

Fix $p \in (M^n, g, e^{-f} dvol_g)$. *Assume* $Ric_f(\partial_r, \partial_r) \geq (n-1)H$,
a) if $\partial_r f \geq -a$ *along a minimal geodesic segment from p (when* $H > 0$ *assume* $r \leq \pi/2\sqrt{H}$*) then*

$$m_f(r) - m_H(r) \leq a$$

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b) if $|f| \leq k$ *along a minimal geodesic segment from* p *(when* $H > 0$ *assume* $r \leq \pi/4\sqrt{H}$*) then*

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Volume Comparison for ∞-Bakry-Emery

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In particular, if f *is bounded and* $Ric_f \geq 0$ *then* M *has polynomial* f*-volume growth.*

Some Applications of Laplacian Comparison

Theorem (Myers' Theorem)

If $(M^n, g, e^{-f}dvol_g)$ *has* $Ric_f \geq (n-1)H > 0$ *and* $|f| \leq k$, *then* M *is compact and* $diam_M \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.

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Theorem (Cheeger-Gromoll's SplittingTheorem)

If $(M^n, g, e^{-f}dvol_g)$ has $Ric_f \geq 0$, $|f|$ is bounded, and M contains a line, then $M = N^{n-1} \times \mathbb{R}$ and f is constant.

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Remark 2 It's enough to assume f is bounded from above (then f is linear along the line) (Fang-Li-Zhang) .

Theorem (Abresch-Gromoll's Excess Estimate)

Let $Ric_f \geq 0$, $|f| \leq k$ *and* $h(x) < \min\{d(p,x), d(q,x)\}$ *then*

$$e_{p,q}(x) \leq 2\left(\frac{n+4k-1}{n+4k-2}\right)\left(\frac{1}{2}Ch^{n+4k}\right)^{\frac{1}{n+4k-1}}$$

where

$$C = 2\left(\frac{n+4k-1}{n+4k}\right)\left(\frac{1}{d(p,x)-h(x)} + \frac{1}{d(q,x)-h(x)}\right)$$

Idea of Proof for Mean Curvature Comparison

A main tool for Ricci curvature is the Bochner formula:
For smooth function u on (M^n, g),

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

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This characterizes Ricci curvature lower bound.

Bochner formulas for the m-Bakry-Emery Ricci tensor

With respect to the measure $e^{-f} dvol$, $\Delta_f = \Delta - \nabla f \cdot \nabla$:

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When m is finite

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Therefore, (Bakry-Qian2005) if $\text{Ric}_f^m \geq (n+m-1)H$, then

$$m_f(r) \leq m_H^{n+m}(r).$$

When $m = \infty$, we have

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We start from the usual Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \mathrm{Ric}(\partial r, \partial r).$$

Let $\mathrm{sn}_H(r)$ be the solution to

$$\mathrm{sn}_H'' + H\mathrm{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$.

We compute that

$$\left(\mathrm{sn}_H^2 m - \mathrm{sn}_H^2 m_H\right)' \leq \mathrm{sn}_H^2 \partial_t \partial_t f,$$

which gives

$$\mathrm{sn}_H^2(r)\left(m(r) - m_H(r)\right) \leq \int_0^r \mathrm{sn}_H^2(t) \partial_t \partial_t f(t) dt.$$

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When f is constant (the classical case) this gives the usual mean curvature comparison.

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- when $m = \infty$, this is exactly the gradient Ricci soliton equation ($\lambda > 0$, shrinking soliton)
- when m is positive integer, it corresponds to some warped product Einstein metrics.

Question

What are the properties of quasi-Einstein metrics?
When is it rigid (trivial)?
What are nontrivial examples?

Previous Work

- When $n = 2, 3$ compact Ricci solitons are trivial. (Hamilton, Ivey)
  More generally when Weyl tensor is zero (Eminenti-Nave-Mantegazza,Petersen-Wylie, Ni-Wallach, Cao-Wang-Zhang, Z.H. Zhang)

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- Compact shrinking soliton with positive curvature operators are trivial. (Böhm-Wilking)

Special Examples

- Gaussian soliton: $(\mathbb{R}^n, g_0)$, $f(r) = r^2$. Then $\mathrm{Ric} + \mathrm{Hess}\, f = 2g_0$, a shrinking soliton which is also Einstein.

  The only nontrivial gradient soliton which is Einstein (Petersen-Wylie)

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- $\mathbb{H}^n$ with the warped product metric $g = dt^2 + e^{2t} g_0$. $f(t) = -mt$. Then $\mathrm{Ric}_f^m = -(n+m-1)g$.

  We will see this is essentially the only nontrivial finite m quasi-Einstein metric which is Einstein.

Compact Examples

$m = \infty$, first nontrivial example of shrinking Ricci soliton is $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$ (Koiso, Cao)

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Dancer-Wang(2008), constructed a large class of compact shrinking Ricci solitons.
All known examples are Kälher.

$2 \leq m < \infty$ integers, S^2 bundles over Kähler-Einstein bases (Lu-Page-Pope2004)
These are non-Kälher.

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Our work (joint with J. Case and Y. Shu)

- Extend several properties for Ricci solitons ($m = \infty$) to quasi-Einstein metrics (general m), showing similarity between finite m and infinite m.

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- Extend several properties for Ricci solitons ($m = \infty$) to quasi-Einstein metrics (general m), showing similarity between finite m and infinite m.
- show Kähler quasi-Einstein metrics behave very differently when m is finite and m is infinite.

Results

Proposition

For a quasi-Einstein metric with $m \geq 1$

a) if $\lambda > 0$ and compact, then the scalar curvature

$$R \geq \frac{n(n-1)}{m+n-1}\lambda.$$

Equality if and only if $m = 1$.

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Equality if and only if $m = 1$.
b) if $\lambda = 0$, R *is constant and* $m > 1$, *then it is Ricci flat.*
c) if $\lambda < 0$, R *is constant, then*

$$n\lambda \leq R \leq \frac{n(n-1)}{m+n-1}\lambda,$$

and when $m > 1$, R *equals either of the extreme values iff it is Einstein.*

When $m = \infty$ this is done in Petersen-Wylie.

When $m = 1$, then R is constant and equals $(n-1)\lambda$.

Rigidity

Proposition

A complete finite m quasi-Einstein metric (M^n, g, f) is Einstein if and only if f is constant or M is diffeomorphic to $\mathbb{R}^n$ with the warped product structure $\mathbb{R} \times_{a^{-1}e^{ar}} N^{n-1}$, where N^{n-1} is Ricci flat, a is a constant.

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Theorem

All 2-dimensional (finite m) quasi-Einstein metrics on compact manifolds are trivial.

Kähler quasi-Einstein metrics

Theorem

Let (M^n, g) be an n-dimensional complete simply-connected Riemannian manifold with a Kähler quasi-Einstein metric for finite m. Then $M = M_1^{n-2} \times M_2^2$ is a Riemannian product, and f can be considered as a function of M_2, where M_1 is an Einstein manifold with Einstein constant λ, and M_2 is a 2-dimensional quasi-Einstein manifold.

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Corollary

There are no nontrivial m finite Kähler quasi-Einstein metrics on compact manifolds.

Idea of Proof

When $0 < m < \infty$, consider $u = e^{-\frac{f}{m}}$. Then the quasi-Einstein equation $\mathrm{Ric}_f^m = \lambda g$ becomes

$$\mathrm{Ric} - \frac{m}{u}\,\mathrm{Hess}\, u = \lambda g.$$

Using this and the Kähler structure, show $\mathrm{Hess}\, u(JU, V) = 0$ for all $U, V \perp \nabla u$.

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Using this and the Kähler structure, show $\mathrm{Hess}\, u(JU, V) = 0$ for all $U, V \perp \nabla u$.

Then show $\mathrm{Span}\{\nabla u, J\nabla u\}$ is invariant under parallel transport.

Questions

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If M^n is a compact Riemannian manifold with a measure such that $Ric_f \geq (>)0$, does M^n have a metric on it with $Ric \geq (>)0$?

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Question

Are there examples of non-Kähler compact shrinking solitons?