Contributions of D. Gromoll to Riemannian Geometry

Classical Sphere Theorem

(1951–1961 Rauch, Klingenberg, Berger):
M complete, $\pi_1(M^n) = 0$, $\frac{1}{4} < K_M \leq 1 \Rightarrow$
$M \approx_{\mathrm{homeo}} \mathbb{S}^n$.

For a compact manifold M with $K_M > 0$

$\delta_M \colon = \frac{\min K_M}{\max K_M}$ is scale-invariant.

M is said to be δ-pinched if $\delta_M > \delta$.
M is said to be weakly δ-pinched if $\delta_M \geq \delta$.

Lecture Notes

Riemannsche Geometrie im Grossen

1962 mimeographed notes, Bonn

1968 Springer Lecture Notes Series Vol. 55

1964 Detlef's Thesis:
Gromoll filtration

$$0 = \Gamma^n_{n-2} \subset \cdots \subset \Gamma^n_k \subset \cdots \subset \Gamma^n_1 \subset \Gamma^n \ .$$

of the Kervaire–Milnor group Γ^n of twisted spheres.

Recursive construction of a sequence δ_ν, $\delta_1 = \frac{1}{4}$, $\delta_\nu < \delta_{\nu+1}$ with $\lim_{\nu\to\infty} \delta_\nu = 1$.

Theorem 2.1 *If M^n is complete, simply connected and δ_k-pinched, then $M \in \Gamma^n_k$.*
In particular M is diffeomorphic to $\mathbb{S}^n$ when $k = n - 2$.

Example: $\delta_5 \leq 0.819$, hence $M^7 \approx_{\text{diffeom}} \mathbb{S}^7$ if $\delta_M > 0.819$.

Improvements by H. Karcher, E. Ruh, K. Shiohama, M. Sugimoto, and Y. Suyama: Differentiable sphere theorems with a pinching constant independent of n.
Suyama (1995): If M is 0.654-pinched then M is diffeomorphic to $\mathbb{S}^n$.

Most recently S. Brendle and R. Schoen solved the problem completely:

Theorem 3.1 (S. Brendle, R. Schoen 2008) *Manifolds with pointwise* 1/4*-pinched sectional curvature are diffeomorphic to space forms.*

In a follow up paper they give a full classification of all pointwise weakly 1/4-pinched manifolds.
The progress was possible due to results by **Böhm and Wilking (2006)**. They found a general set of "curvature conditions" that are invariant under the Ricci flow.

Antonelli, Burghelea and Kahn [1970] investigated the Γ_k^n further and introduced the name Gromoll groups.

N. Hitchin (1972): Used Gromoll groups to construct examples of spin–manifolds M^n with a spin cobordism invariant $\alpha(M^n) \neq 0$. In particular these manifolds do not admit metrics with positive scalar curvature.

M. Weiss (1993): Used the Gromoll groups to show that certain exotic spheres cannot be 1/4-pinched. In particular Milnor's generator Σ^7 of Γ^7 does not admit any $\frac{1}{4}$–pinched metric.

Theorem 4.1 (K. Grove, F. Wilhelm [1995]) *$n \geq 2$, $2 \leq q \leq n$ M closed manifold, $K_M \geq 1$ and $\operatorname{pack}_q M > \frac{\pi}{4}$. Then $M \in \Gamma^n_{q-1}$. If $n \geq 4$ and $q = n-2$, M is diffeomorphic to $\mathbb{S}^n$.*

$\operatorname{pack}_q(M)$ is the largest number r for which M contains q disjoint balls of radius r.

Exotic spheres with nonnegative sectional curvature

Open question:
Is there any exotic sphere with $K > 0$?
D. Gromoll, W. M. (1974):
Σ^7 is a submersion of $\mathsf{Sp}(2)$ via the two–sided action of $\mathsf{Sp}(1)$ given by $(q, Q) \mapsto \left(\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix}\right) Q \left(\begin{smallmatrix} \bar{q} & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
Hence $K_{\Sigma^7} \geq 0$.
$K_{\Sigma^7} > 0$ on an open set of points. But there is also an open set of points with planes of zero curvature as pointed out by F. Wilhelm.

F. Wilhelm [2001] and also J. Eschenburg-M. Kerin [2008] : The metric of Σ^7 can be deformed to a metric with positive curvature almost everywhere.

P. Petersen and F. Wilhelm [arXiv 2009]: Σ^7 admits a metric of positive curvature. (third version, over 90 pages.)

K. Grove, W. Ziller [2000]: Metrics with $K \geq 0$ on all the 3–sphere bundles over $\mathbb{S}^4$ with structure group SO(4), including 14 of the 28 Milnor spheres.

Diameter Rigidity, metric fibrations of spheres

The Classical Rigidity Theorem:

Theorem 7.1 (M. Berger [1960])
M^n closed, $\pi_1(M) = 0$, $\frac{1}{4} \leq K_M \leq 1$.
Then M is homeomorphic to $\mathbb{S}^n$ or isometric to a symmetric space of rank 1.

The proof depends on Klingenberg's estimate inj$(M) \geq \pi$ for the injectivity radius of M. A correct proof for this estimate in odd dimensions was given in:

Theorem 7.2 (Cheeger, Gromoll [1980])
M closed, $\pi_1(M) = 0$, $\frac{1}{4} \leq K_M \leq 1$.
Then inj$(M) =$ conj$(M) \geq \pi$.

Brendle and Schoen (2008):
homeomorphic can be replaced by *diffeomorphic* in Berger's rigidity Theorem.

The Diameter Rigidity Theorem:

By rescaling the metric in Berger's rigidity theorem, the curvature assumptions can be transformed to $1 \leq K_M \leq 4$. These conditions imply via the injectivity radius estimate that $\operatorname{diam} M \geq \pi/2$.

Theorem 8.1 (D. Gromoll, K. Grove [1987]) M^n *connected, complete,* $n \geq 2$, *and* $K_M \geq 1$, $\operatorname{diam} M^n \geq \frac{\pi}{2}$. *Then*
(i) M^n *is homeomorphic to* $\mathbb{S}^n$, *or*
(ii) *the universal covering* $\tilde{M}^n$ *of* M^n *is isometric to a rank one symmetric space, except possibly when* $H^*(m) \cong H^*(\mathrm{Ca}\mathbb{P}^2)$.

This generalizes the Grove-Shiohama Diameter Sphere Theorem (1977).

Due to **B. Wilking [2001]** the exception in (ii) can be can be deleted.

Basic idea: construct a "dual" pair of convex sets A and A' in M at maximal distance $\frac{\pi}{2}$ and analyze its properties:

A' and A are simply connected manifolds without boundary if M is not a sphere. Let $p \in A$, S_p fiber of the normal sphere bundle of A. Then there is a submersion $S_p \to A'$. This essentially reduces the proof to the

Classification of metric fibrations of spheres by Gromoll–Grove (1985) and (1987), Wilking (2001):
Fibers are spheres of dimension 1, 3, or 7. The fibration is metrically equivalent to a Hopf fibration. The case when the fiber dimension is 7 and $n = 15$ was solved by Wilking.

For example if $M = \mathbb{CP}^m$, then:
$A = \mathbb{CP}^k$, $A' = \mathbb{CP}^{m-k-1}$, and A is the cut locus of A' and vice versa. If $A = \{pt\}$, M is the Thom space of the normal bundle of A'.

The Soul Theorem of Cheeger and Gromoll

1966–1968 D. Gromoll:
Miller Fellow at Berkeley

1966-1967 W. Meyer: Visiting Berkeley.
Joined work of Gromoll-M. on periodic geodesics and also on complete non compact manifolds with positive sectional curvature.
1967-1968 J. Cheeger: Visiting Berkeley.
Joined work of Gromoll-Cheeger on the structure of complete manifolds with nonnegative curvature.

Theorem 10.1 (D. Gromoll, W. M. [1969]) *Let M^n be a complete non compact Riemannian manifold with positive sectional curvature. Then M^n is diffeomorphic to $\mathbb{R}^n$.*

Theorem 11.1 (J. Cheeger, D. Gromoll [1972]) *Let M^n be a complete non compact manifold of nonnegative curvature K. Then there is a compact totally geodesic submanifold S in M such that M is diffeomorphic to the normal bundle $\nu(S)$ of S.*

Cheeger-Gromoll called S a soul of M. The theorem is known as the **Soul Theorem**.

Question at the end of the paper:
Suppose M is complete and non-compact with $K \geq 0$ but $K > 0$ at some point. Is then the soul of M always a point, or equivalently, is M diffeomorphic to euclidian space?

Perelman (1994) gave a positive answer to this Question in a paper *"Proof of the soul conjecture of Cheeger and Gromoll"*.

Main ideas for the proof of the Soul Theorem

1. The basic construction of an expanding family of totally convex sets.

Definition: A nonempty subset C of M is called *totally convex* if for arbitrary points $p, q \in C$ any geodesic with endpoints p and q is contained in C.

Definition: A *ray* in M is a normal geodesic $c : [0, \infty) \to M$ for which any finite segment is minimal. For a ray $c : [0, \infty) \to M$ we define the *open half-space* B_c by

$$B_c = \bigcup_{t>0} B(c(t), t)$$

where $B(c(t), t)$ is the open metric ball of radius t around $c(t)$.

Remark: For any $p \in M$ there exists a ray $c : [0, \infty) \to M$ with initial point $c(0) = p$.

Lemma 13.1 *If M is complete, noncompact of nonnegative sectional curvature, then the closed half-space $M - B_c$ is totally convex for any ray in M.*

Proof: Suppose $M - B_c$ is not totally convex, i.e. there is a geodesic $c_0 : [0, 1] \to M$ with endpoints $c_0(0), c_0(1) \in M - B_c$ but $c_0(s_0) \in B_c$ for some $s_0 \in (0, 1)$. Then $q := c_0(s_0) \in B(c(t_0), t_0)$ for some $t_0 > 0$.
By the triangle inequality $q \in B(c(t), t)$ for any $t \geq t_0$. In fact, setting

$$t_0 - \varepsilon = \text{dist}(q, c(t_0)), \quad \varepsilon > 0$$

we have

$$\begin{aligned} \text{dist}(q, c(t)) &\leq \text{dist}(q, c(t_0)) + \text{dist}(c(t_o), c(t)) \\ &= (t_0 - \varepsilon) + (t - t_0) = t - \varepsilon \end{aligned}$$

for $t \geq t_0$.

Let $t \geq t_0$.

Choose $c_0(s_t)$ on $c_0(0,1)$ closest to $c(t)$,
c_1^t a minimal geodesic from $p = c_0(s_t)$ to $c(t)$.
- Then $|c_1^t| \leq t - \varepsilon$ and
- $\measuredangle(\dot{c}_0(s_t), \dot{c}_1^t(0)) = \frac{\pi}{2}$

Let $c_0^t(\tau) :\, = c_0(s_t - \tau)$ for $\tau \in [0, s_t]$.
- Then $|c_0^t| < |c_0|$

Since $c_0^t(s_t) = c_0(0) \notin B(c(t), t)$, we have
- $t \leq \mathrm{dist}(c_0^t(s_t), c(t))$

Apply Toponogov's comparison theorem to the hinge $(c_1^t, \pi/2, c_0^t)$:

$\mathrm{dist}(c_0^t(s_t), c(t))^2 \leq |c_0^t|^2 + |c_1^t|^2 \leq |c|^2 + (t - \varepsilon)^2$

Therfore $t^2 \leq |c|^2 + (t - \varepsilon)^2$, a contradiction.

Fix a point $p \in M$. For a ray $c : [0, \infty[\to M$ we also consider the restricted ray $c_t(s) := c(t+s)$, $s \in [0, \infty)$. Let

$$C_t := \bigcap_c (M - B_{c_t})$$

where the intersection is taken over all the rays c emanating from p.

Lemma 15.1 *C_t is a compact totally convex set for all $t \geq 0$, moreover*

a) $C_{t_2} \supset C_{t_1}$ *for* $t_2 \geq t_1$ *and*
$C_{t_1} = \{q \in C_{t_2} \mid \text{dist}(q, \partial C_{t_2} \geq t_2 - t_1\}$,
in particular
$\partial C_{t_1} = \{q \in C_{t_2} \mid \text{dist}(q, \partial C_{t_2}) = t_2 - t_1\}$

b) $\bigcup_{t \geq 0} C_t = M$

c) $p \in \partial C_0$

Clearly C_t is closed, totally convex and $p \in C_t$. If C_t were not compact, one can construct a ray contained in C_t emanating from p, contradicting the definition.

2. The basic construction of minimal totally convex sets.

Local convexity.

Definition A subset A of M is called *strongly convex* if for any $q, q' \in A$ there is a unique minimal geodesic from q to q' which is contained in A.

Recall that there is a continuous function $r : M \to]0, \infty]$, the *convexity radius* such that for any $p \in M$, any open metric ball B which is contained in $B(p, r(p))$ is strongly convex.

Definition We say that a subset C of M is *convex* if for any $p \in \overline{C}$ there is a number $0 < \varepsilon(p) < r(p)$ such that $C \cap B(p, \varepsilon(p))$ is strongly convex.

Notice that a totally convex set is convex and connected. Also the closure of a convex set is again convex.

Structure theorem for convex sets:

Theorem 17.1 (J. Cheeger, D.Gromoll) *Let C be a connected nonempty convex subset of an arbitrary Riemannian manifold M. Then C carries the structure of an imbedded k-dimensional submanifold of M with smooth totally geodesic interior $N = \operatorname{int} C$ and (possibly nonsmooth) boundary $\partial C = \overline{N} - N$.*

Definition Let C be a convex subset of M. The *tangent cone* to C at a point $p \in C$ is by definition the set $T_pC = \{v \in T_pM \mid \exp(t\frac{v}{\|v\|}) \in \operatorname{int} C$ for some $0 < t < r(p)\} \cup \{0\}$.

Clearly if $p \in \operatorname{int}(C)$, then $T_pC = T_pN$.

The following lemma contains all the technical information about T_pC we need.

Lemma 18.1 (tangent cone lemma)
Let $C \subset M$ be convex and $p \in \partial C$.

a) *Then $T_pC \setminus \{0\}$ is contained in an open half-space of T_pM.*

b) *Suppose that there exists $q \in \operatorname{int} C$ and a minimal normal geodesic $c : [0, d] \to C$ from q to p such that $|c| = \operatorname{dist}(q, \partial C)$. Then*

$$T_pC \setminus \{0\} = \{v \in \hat{T}_pC \mid \measuredangle(v, -\dot{c}(d)) < \tfrac{\pi}{2}\},$$

where $\hat{T}_pC$ is the subspace of T_pM spanned by T_pC.

Lemma 19.1 (contraction lemma) *Suppose M has nonnegative sectional curvature and $C \subset M$ is a closed totally convex subset with $\partial C \neq \emptyset$. We set*

$$C^a = \{p \in C \mid \mathrm{dist}(p, \partial C) \geq a\}\,, \quad C^{max} = \bigcap_{C^a \neq \emptyset} C^a\,.$$

Then

a) C^a is closed and totally convex.

b) $\dim C^{max} < \dim C$*.*

c) If $K > 0$ then C^{max} is a point.

This is a corollary of the following more general lemma:

Lemma 20.1 *With the hypothesis of lemma 19.1, let* $\psi(x) \colon = \operatorname{dist}(x, \partial C)$*. Then for any normal geodesic segment* c *which is contained in* C *the function* $\psi \circ c$ *is (weakly) concave, i.e. for* $\lambda \in [0,1]$

$$\psi(c(\lambda t_1 + (1-\lambda)t_2)) \geq \lambda\psi(c(t_1)) + (1-\lambda)\psi(c(t_2))$$

If the sectional curvature satisfies $K > 0$ *then the strict inequality holds.*

Proof. It is sufficient to show:
For $s_0 \in (0,1)$ and some $\delta > 0$ there is a linear function $h(s)$ on $(s_0 - \delta, s_0 + \delta)$ satisfying

$$h(s_0) = \psi(c(s_0)) =: d \text{ and } h(s) \geq \psi(c(s))$$

Let c_{s_0} be a distance minimizing normal geodesic of length d from $c(s_0)$ to ∂C and $\alpha := \sphericalangle(\dot{c}_{s_0}(0), \dot{c}(s_0))$.
Then we can take

$$h(s) = d - (s - s_0)\cos\alpha\ .$$

To show $h(s) \geq \psi(c(s))$ we consider the three cases $\alpha = \frac{\pi}{2}$, $\alpha > \frac{\pi}{2}$, $\alpha < \frac{\pi}{2}$. Note that we only have to consider points $s \geq s_0$.

Case $\alpha = \frac{\pi}{2}$:
E parallel along c_{s_0} with $E(0) = \dot{c}(s_0)$.
$c_s(t) \colon = \exp(s - s_0)E(t)$, $0 \leq t \leq d$
Rauch II: $|c_s| \leq d = |c_{s_0}|$ for $0 \leq s - s_0$ small.
Let $\bar{c}(\tau) \colon = \exp(\tau E(d))$, $0 \leq \tau \leq (s - s_0)$, then
$q := \bar{c}(0) = c_{s_0}(d) \in \partial C$ and $\dot{\bar{c}}(0) \perp \dot{c}_{s_0}(d)$.

By the tanget cone lemma $\dot{\bar{c}}(0) \notin T_q C$ so that $c_s(d) = \bar{c}(s - s_0) \notin \operatorname{int} C$ for $s - s_0$ small.

Therefore $\psi(c(s)) \leq |c_s| \leq d = d - (s - s_0) \cos \frac{\pi}{2}$.

Case $\alpha > \frac{\pi}{2}$: Let $E(0) \perp \dot{c}_{s_0}(0)$ be the unique unit vector in the convex cone spanned by $\dot{c}(s_0)$ and $\dot{c}_{s_0}(0)$ and extend it to the parallel vector field E along c_{s_0}. Define c_s as in the first case to obtain

$$|c_s| \leq d \ . \qquad (1)$$

Applying Rauch I to the hinge with geodesics $t \mapsto \exp tE(0)$, $0 \leq t \leq (s - s_0)\cos(\alpha - \frac{\pi}{2})$ and $t \mapsto c(s_0 + t)$, $0 \leq t \leq (s - s_0)$ with angle $\alpha - \frac{\pi}{2}$, one obtains

$$\begin{aligned} &\mathrm{dist}\Big(c(s), \exp((s - s_0)\cos(\alpha - \tfrac{\pi}{2})E(0)\Big) \\ &\qquad \leq -(s - s_0)\cos\alpha \ . \qquad (2) \end{aligned}$$

Combining (1) and (2), the inequality $\psi(c(s)) \leq d - (s - s_0)\cos\alpha$ follows.

Case $\alpha < \frac{\pi}{2}$ **:** Choose $c_{s_0}(t_s)$ on c_{s_0} with dist$(c(s), c_{s_0}([0,d])) =$ dist$(c(s), c_{s_0}(t_s))$.
a_s normal minimal geodesic from $c_{s_0}(t_s)$ to $c(s)$. Then $\measuredangle(\dot{a}_s(0), \dot{c}_{s_0}(t_s)) = \frac{\pi}{2}$. Let E be parallel along $c_{s_0}|_{[t_s,d]}$ with $E(t_s) = \dot{a}_s(0)$.
$c_s(t) :=$ exp$(|a_s|\, E(t))$, $t_s \le t \le d$, is of length $|c_s| \le (d - t_s)$ by Rauch II.
As before dist$(c(s), \partial C) \le |c_s|$, thus

$$\text{dist}(c(s), \partial C) \le (d - t_s) \qquad (3)$$

Applying Rauch I to the hinges
$(c|_{[s_0,s]}, c_{s_0}|_{[0,t_s]}, \alpha)$ and $(c_{s_0}^{-1}|_{[0,t_s]}, a_s, \frac{\pi}{2})$,
we obtain

$$|a_s|^2 \le (s - s_0)^2 + t_s^2 - 2t_s(s - s_0)\cos\alpha \text{ and}$$
$$(s - s_0)^2 \le |a_s|^2 + t_s^2, \text{ hence}$$

$$-t_s \le -(s - s_0)\cos\alpha. \qquad (4)$$

(3) and (4) imply $\psi(c(s)) \le h(s)$.

Proof of the soul theorem: Let $p \in M$ and consider the compact totally convex sets C_t
If $\partial C_0 = \emptyset$ let $S = C_0$.
If $\partial C_0 \neq \emptyset$, consider C_0^{max}, $\dim C_0^{max} < \dim C_0$.
If $\partial C_0^{max} = \emptyset$ let $S = C_0^{max}$.
If $\partial C_0^{max} \neq \emptyset$, consider$(C_0^{max})^{max}$ etc.
Repeating this procedure leads us in a finite number ($\leq n$) of steps to a compact totally convex set $S \subset C_0$ with $\dim S < n$ and $\partial S = \emptyset$.
In particular S is a compact totally geodesic submanifold of M.
A diffeomorphism from the normal bundle to M can be constructed by means of the flow of a gradient like vector field of the function $f(x) = \text{dist}(x, S)$ A gradientlike vector field can be constructed using the tangent cone lemma. This was pointed out by Gromoll and Grove (1978) in their rigidity theorem paper.

The splitting theorem for manifolds with nonnegative Ricci curvature

Theorem 25.1 (J. Cheeger, D. Gromoll [1971])
Let M be a complete manifold of nonnegative Ricci curvature. Then M is the isometric product $\overline{M} \times \mathbb{R}^k$ where $\overline{M}$ contains no lines and $\mathbb{R}^k$ has the standard flat metric.

Recall that a line in M is a normal geodesic γ: $(-\infty, \infty) \rightarrow) M$ each segment of which is minimal.

Basic idea for the proof: It suffices to show that if M contains a line, them M splits isometrically as $M' \times \mathbb{R}$.

Consider a ray γ: $[0, \infty) \rightarrow) M$.
For $t \geq 0$ let g_t be given by

$$g_t(x) = \text{dist}(x, \gamma(t)) - t\,.$$

Fact: for $t \to \infty$, g_t converges uniformly on compact sets to a continuous function g_γ.

Basic observation of Cheeger and Gromoll: g_γ is superharmonic.

It is actually simpler to show: g_γ is superharmonic in the sense of support functions: At $x \in M$ an upper support function is constructed by means of an asymptotic ray: Choose t_i, $0 < t_i < t_{i+1}$, with $\lim_{i\to\infty} t_i = \infty$ and minimal normal geodesics σ_i from x to $\gamma(t_i)$ so that $\dot\sigma_i(0)$ converges to the tangent vector $\dot\sigma(0)$ of a ray σ. The ray σ is called asymptotic to γ.

A support function for g_γ at x is defined by

$$g_{x,t}(y) = \mathrm{dist}(\sigma(t), y) - t + g_\gamma(x)\,.$$

Notice that $g_{x,t}$ is differentiable in a neighborhood of x since $\sigma(t)$ is not on the cut locus of x and hence x can't be on the cut locus of $\sigma(t)$. Obviously $g_{x,t}(x) = g_\gamma(x)$ and it is easy to show

- $g_{x,t}(y) \geq g_\gamma(y)$.

Furthermore, a standard calculation gives $\Delta g_{x,t}(x) \leq \frac{n-1}{\mathrm{dist}(\sigma(t),x)} = \frac{n-1}{t}$, for $\varepsilon > 0$ we have

- $\Delta g_{x,t}(x) \leq \varepsilon$ for t sufficiently large.

Hence $\Delta g_\gamma \leq 0$ in the support sense.

Proof of the splitting theorem:
If γ is a line, we consider the two rays

$$\gamma_+ = \gamma|_{[0,\infty]}$$

and γ_- with

$$\gamma_-(t) = \gamma_+(-t)\,.$$

Let $g_+ = g_{\gamma_+}$ and $g_- = g_{\gamma_-}$. By the triangle inequality it follows that

$$\text{dist}(x, \gamma(t)) - t + \text{dist}(x, \gamma(-s)) - s \geq 0$$

with equality for $x \in \gamma([-s, t])$. Hence

$$g_+ + g_- \geq 0$$

and $(g_+ + g_-)(x) = 0$ for $x \in \gamma(\mathbb{R})$.
$\Delta g_+ \leq 0$, $\Delta g_- \leq 0 \Rightarrow \Delta(g_+ + g_-) \leq 0$. It folows that $g_+ + g_- \equiv 0$ by the Hopf-Calabi maximum principle. Now $g_+ = -g_-$ and therefore $\Delta g_+ = 0$ in the barrier sense. It follows that g_+ is smooth harmonic. Now it is straight forward to show that M splits isometrically as $M' \times \mathbb{R}$ by the level-surfaces and gradient lines of g_+.

Examples of manifolds with positive Ricci curvature

Students of D. Gromoll, who have been working on this: **H. Hernandez–Andrade, P. Ingram, J. Nash, W.A. Poor.**
They obtained metrics with $Ric > 0$ on certain bundles and also on submanifolds of $\mathbb{R}^n$ that are defined via equations, notably on Brieskorn varieties.

Does any manifold with Ric > 0 **admit a metric with** $K_M \geq 0$**?**

1983 Humboldt Prize for D. Gromoll, visit to Münster.
D. Gromoll, —— (1985): Examples of non–compact manifolds with Ric > 0, which cannot carry any metric with $K_M \geq 0$.

Sha and Yang [1989],[1991]

(1) Metrics with Ric > 0 on connected sums

$$\mathop{\#}_{i=1}^{k} S^n \times S^m$$

with k arbitrary large.

Gromov: For k large, there is no metric with $K \geq 0$ on these manifolds.

(2) Metrics with Ric > 0 on the manifiold M arising from $\mathbb{R}^4 \times \mathbb{S}^3$ by attaching infinitely many copies of $\mathbb{S}^3 \times \mathbb{CP}^2$ to it by surgery. certainly M is not of finite topological type.

Diameter growth and topological finiteness

By the Soul Theorem a complete noncompact manifold with $K_M \geq 0$ is of finite topological type.

A similar result for manifolds of nonnegative Ricci curvature does not hold.

However, Abresch and Gromoll have obtained a finiteness result with some reasonable assumptions.

Definition of diameter growth. Given $r > 0$, and $p \in M$, let $C_r(p)$ be the union of the unbounded connected components of $M - \overline{B(p,r)}$. Let $\partial C_r(p) = \bigcup \Sigma_{r,k}$ where $\Sigma_{r,k}$ are the components of $\partial C_r(p)$. The diameter growth function with respect to p is defined as $\mathrm{diam}_p(r) = \sup_k \mathrm{diam}\, \Sigma_k$. Let $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function. M is said to have diameter growth of order $o(f(r))$, if $\mathrm{diam}_p(r)/f(r)$ converges to zero as $r \to \infty$.

Theorem 32.1 (Abresch, Gromoll [1990]) *Let M^n be a complete open manifold with nonnegative Ricci curvature. Suppose that M^n has diameter growth of order $o(r^{1/n})$ and suppose the sectional curvature is bounded away from $-\infty$. Then M^n is homotopy equivalent to the interior of a compact manifold with boundary.*

Main Ideas for the proof:

Show that for $p \in M$ the distance function d_p, $d_p(x) = d(p, x)$, has no critical points outside a compact ball $\overline{B(p, r)}$ of large radius r.

For this purpose Abresch and Gromoll introduce the excess function e for two given points $p, q \in M$ by

$$e(x) = d(p, x) + d(x, q) - d(p, x)\,,$$

and for a minimal Geodesic γ from p to q the height h given by

$$h(x) = d(x, \gamma)$$

.

Properties of the excess function:

(1) $0 \le e(x) \le 2h(x)$

(2) $e|_\gamma = 0$

(3) If $\mathrm{Ric} \ge 0$,

then on $\{x \,|\, h(x) < \min\{d(p,x), d(q,x)\}\}$

$$\Delta e(x) \le (n-1)\left(\frac{1}{d(x,p)-h(x)} + \frac{1}{d(x,q)-h(x)}\right)$$

in the barrier sense.

Excess estimate for long thin triangles:

Let $s(x) = \min(d(p,x), d(q,x))$.

If $\mathrm{Ric} \ge 0$ and $h(x) \le s(x)/2$, then for $n \ge 3$

$$e(x) \le 2 \cdot \frac{n-1}{n-2} \cdot \left(\frac{n-1}{n} \cdot \frac{h(x)^n}{s(x)}\right)^{1/(n-1)}$$

The lower bound for the sectional curvature gives a lower bound for the excess function if x is a critical point of d_p and q is far away from x: If $K \geq -1$ and $\varepsilon > 0$ then there is an $\delta > 0$ such that for $d(q,x) \geq 1/\delta$

$$e(x) \geq \ln\left(\frac{2}{1+\exp(-2d(p,x))}\right) - \varepsilon\,.$$

The proof for this inequality is based on Toponogov's triangle comparison theorem.

Since $\mathrm{diam}_p(r)$ grows of order $o(r^{1/r})$, the upper estimate for $e(x)$ with $x \in \Sigma_{r,k}$ approaches 0 as $r \to \infty$. This contradicts the fact that the lower bound given above for $e(x)$ is bounded away from 0 if $x \in \Sigma_{r,k}$ is a critical point of d_p.

Periodic Geodesics

Theorem 36.1 (D. Gromoll, — [1969]) *Let M be closed, $\pi_1(M) = 0$, and Ω be the free loop space of M. Suppose the sequence of Betti numbers $b_\nu(\Omega)$ is unbounded.*
Then there are infinitely many geometrically distinct periodic geodesics on M.

Main tools:
1. Generalized Morse Lemma:
H Hilbert space, $f\colon H \to \mathbb{R}$ differentiable, $0 \in H$ a critical point of f, $\operatorname{Hess} f_{|0} = \mathrm{id} + k$, k compact operator. Then $H = E^+ \oplus E^- \oplus N$ and after a coordinate change Φ,

$$f \circ \Phi(x, y, z) = \|x\|^2 - \|y\|^2 + h(z) \ ,$$

where N is the null space of f and $h\colon N \to \mathbb{R}$ is differentiable.
2. Results of Bott on index and nullity of the iterates of a closed geodesic.

Manifolds all of whose geodesics are closed

The metric g of a Riemannian manifold (M, g) is called a C-metric if all geodesics are closed and have the same minimal period. If in addition all the geodesics are without self-intersections g is called an SC-metric. If g is an SC-metric, then the integral cohomology ring is that of a rank one symmetric space.

Theorem 37.1 (D. Gromoll, K. Grove [1981]) *Any Riemannian metric on $\mathbb{S}^2$ all of whose geodesics are closed is an SC-metric, i.e. all the geodesics are simple and have the same minimal period.*

Conjecture: If (M^n, g) is simply connected and all geodesics are closed, then g is a C-metric, i.e. all the geodesics have the same minimal period.

B. Wilking [2009] has shown that the conjecture holds when M is homeomorphic to $\mathbb{S}^n$, $n \geq 4$.The case $n = 3$ remains unsettled.
Methods: Index parity, $\mathbb{S}^1$- equivariant Morse theory and Cohomology of the free loop space.