On the work of Igor Frenkel

Introduction

by Pavel Etingof

Igor Frenkel is one of the leading representation theorists and mathematical physicists of our time. Inspired by the mathematical philosophy of Herman Weyl, who recognized the central role of representation theory in mathematics and its relevance to quantum physics, Frenkel made a number of foundational contributions at the juncture of these fields. A quintessential mathematical visionary and romantic, he has rarely followed the present day fashion. Instead, he has striven to get ahead of time and get a glimpse into the mathematics of the future – at least a decade, no less. In this, he has followed the example of I. M. Gelfand, whose approach to mathematics has always inspired him. He would often write several foundational papers in a subject, and then leave it for the future generations to be developed further. His ideas have sometimes been so bold and ambitious and so much ahead of their time that they would not be fully appreciated even by his students at the time of their formulation, and would produce a storm of activity only a few years later. And, of course, as a result, many of his ideas are still waiting for their time to go off.

This text is a modest attempt by Igor's students and colleagues of various generations to review his work, and to highlight how it has influenced in each case the development of the corresponding field in subsequent years.

1 Contributions of Igor Frenkel to the representation theory of affine Lie algebras

by Alistair Savage and Anthony Licata

Among infinite-dimensional Lie algebras, it is the theory of affine Lie algebras that is the richest and most well understood. Igor Frenkel's contributions to this subject are both numerous and diverse, and his are among the deepest and most fundamental developments in the subject.

These contributions began in his 1980 Yale University thesis, the core of which was later published in the paper [Fth]. In his thesis, Frenkel adapts the orbital theory of A. A. Kirillov to the setting of affine Lie algebras, giving, in particular, a formula for the characters of irreducible highest weight representations in terms of orbital integrals. The technical tools required for Frenkel's orbital theory include a tremendous amount of interesting mathematics, including "the Floquet theory of linear differential equations with periodic coefficients, the theory of the heat equation on Lie groups, the theories of Gaussian and Wiener measures, and of Brownian motion." (Quote from the MathSciNet review of [Fth]). Thus Frenkel's thesis gives one of the early examples of a central theme in the theory of affine Lie algebras, namely, the rich interaction between their representation theory and the rest of mathematics.

A fundamental contribution of Frenkel to infinite-dimensional representation theory came in his joint paper with Kac [FK80]. In this paper, the authors formally introduced vertex operators into mathematics, and used them to give an explicit construction of the basic level one irreducible representation of a simply-laced affine Lie algebra. (A very similar construction was given independently around the same time by Segal [S]). In important earlier work, Lepowsky-Wilson [LW] gave a twisted construction of the basic representation for $\widehat{\mathfrak{sl}_2}$, and this twisted construction was then generalized to other types by Kac-Kazhdan-Lepowsky-Wilson [KKLW]. Vertex operators themselves had also been used earlier in the dual resonance models of elementary particle physics. But it was the ground-breaking paper of Frenkel and Kac that developed their rigorous mathematical foundation, and established a direct link between vertex operators and affine Lie algebras. Thus began the mathematical subject of vertex operator algebras, a subject which has had profound influence on areas ranging from mathematical physics to the study of finite simple groups. Frenkel also gave closely related spinor constructions of fundamental representations of affine Lie algebras of other types in [FPro].

Another important example of Frenkel's work at the interface of affine Lie algebras and mathematical physics is his work on the boson-fermion correspondence [Fre81]. In the course of establishing an isomorphism between two different realizations of simply-laced affine Lie algebras, he realized that his result could be reformulated in the language of quantum field theory, implying an equivalence of physical models known to physicists as the *boson-fermion correspondence*. This paper was the first on the connection between infinite dimensional Lie algebras and 2d conformal field theory. Also, in [FF85], Feingold and Frenkel obtained bosonic and fermionic constructions of all classical affine Lie algebras. Further related but independently important developments appeared in [Fre85] and in [Flr], where Frenkel established what is now known as *level-rank duality* for representations of affine Lie algebras of type A, and obtained upper bounds for root multiplicities for hyperbolic Kac-Moody algebras applying the no-ghost theorem from physics. In another paper with Feingold, [FF83], Frenkel suggested a relation between hyperbolic Kac-Moody algebras and Siegal modular forms, which was further studied in the works of Borcherds and Gritsenko-Nikulin.

The relevance of affine Lie algebras and their representation theory was highlighted by Frenkel in his invited address, entitled “Beyond affine Lie algebras”, at the 1986 ICM in Berkeley ([FBa]). Since then, his foundational work in and around the subject of affine Lie algebras has been extremely influential in other areas, perhaps most notably in vertex algebra theory, in the representation theory of quantum groups, and in geometric representation theory and categorification. Frenkel’s work on affine Lie algebras comprises his first major contributions to mathematics, and the fundamental nature of this work has been repeatedly confirmed by the relevance of affine Lie algebras and their representation theory in both mathematics and mathematical physics.

2 Igor Frenkel’s work on the quantum Knizhnik-Zamolodchikov equations

by Pavel Etingof

In 1984 Knizhnik and Zamolodchikov studied the correlation functions of the Wess-Zumino-Witten (WZW) conformal field theory, and showed that they satisfy a remarkable holonomic system of differential equations, now called the Knizhnik-Zamolodchikov (KZ) equations. Soon afterwards Drinfeld and Kohno proved that the monodromy representation of the braid group arising from the KZ equations is given by the R-matrices of the corresponding quantum group, and Schechtman and Varchenko found integral formulas for solutions of the KZ equations. At about the same time, Tsuchiya and Kanie proposed a mathematically rigorous approach to the WZW correlation functions, by using intertwining operators between a Verma module over an affine Lie algebra and a (completed) tensor product of a Verma module with an evaluation module:

$$\Phi(z) : M_{\lambda,k} \longrightarrow M_{\mu,k}\hat{\otimes}V(z).$$

Namely, they proved that highest matrix elements of products of such operators (which are the holomorphic parts of the correlation functions of the WZW model) satisfy the KZ equations. This construction can be used to derive the Drinfeld-Kohno theorem, as it interprets the monodromy of the KZ equations in terms of the exchange matrices for intertwining operators $\Phi(z)$, which are twist equivalent (in an appropriate sense) to the R-matrices of the quantum group.

This set the stage for the pioneering paper by I. Frenkel and N. Reshetikhin [FR], which was written in 1991 (see also the book [EFK] based on lectures by I. Frenkel, which contains a detailed exposition of this work). In this groundbreaking work, Frenkel and Reshetikhin proposed a q-deformation of the theory of WZW correlation functions, KZ equations, and their monodromy, and, in effect, started the subject of q-deformed conformal field theory, which remains hot up to this day [1]. Namely, they considered the intertwining operators $\Phi(z)$ for quantum affine algebras, and showed that highest matrix elements of their products, $\langle \Phi_1(z_1)....\Phi_n(z_n)\rangle$, satisfy a system of difference equations, which deform the KZ equations; these equations are now called the quantum KZ equations. They also showed that the monodromy of the quantum KZ equations is given by the exchange matrices for the quantum intertwining operators, which are elliptic functions of z, and suggested that such matrices should give rise to "elliptic quantum groups".

This work had a strong influence on the development of representration theory in the last 20 years, in several directions.

First of all, the quantum KZ equations arose in several physical contexts (e.g., form factors of F. Smirnov, or solvable lattice models considered by Jimbo, Miwa, and their collaborators).

Also, Felder, Tarasov, and Varchenko, building on the work of Matsuo, generalized the Schectman-Varchenko work to the q-case, and found integral formulas for solutions of the quantum KZ equations.

At about the same time, G. Felder proposed the notion of elliptic quantum groups based on the dynamical Yang-Baxter equation, which is satisfied by the exchange matrices. This theory was further developed by Felder, Tarasov, and Varchenko, and also by Etingof-Varchenko, who proposed a theory of dynamical quantum groups and dynamical Weyl groups (generalizing to the q-case the theory of Casimir connections).

Another generalization of the quantum KZ equation, corresponding to Weyl groups, was considered by Cherednik, and this generalization led to his proof of Macdonald's conjectures and to the discovery of double affine Hecke algebras, also called Cherednik algebras, which are in the center of attention of representation theorists in the past 15 years.

Yet another generalization is the theory of elliptic quantum KZ equations (or quantum Knizhnik-Zamolodchikov-Bernard equations), which was developed in the works of Etingof, Felder, Schiffmann, Tarasov, and Varchenko.

[1] We note that q-deformation of some structures of conformal field theory, namely the vertex operator construction of [FK80], was already considered in an earlier paper, [FJ88].

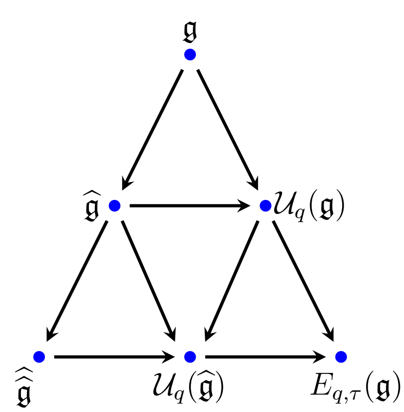
The paper [FR] also served as a motivation for Etingof, Schedler, and Schiffmann in their consruction of explicit quantization of all non-triangular Lie bialgebra structures on simple Lie algebras (classified by Belavin and Drinfeld) and to Etingof and Kazhdan in their work on quantization of Lie bialgebras associated to curves with punctures.

Finally, the ideas of this paper played an important role in the work of Etingof and Kirillov Jr. on the connection between Macdonald polynomials and quantum groups, and their definition of affine Macdonald polynomials, and in a generalization of this work by Etingof and Varchenko (the theory of traces of intertwining operators for quantum groups). These structures and functions are now arising in algebraic geometry (e.g. the work of A. Negut on integrals over affine Laumon spaces). Also, quantum KZ equations and q-Casimir connections are expected to arise in the study of quantum K-theory of quiver varieties.

3 Igor Frenkel's ideas and work on double loop groups

by Pavel Etingof

Around 1990, when the loop algebra/quantum group revolution of the 1980s and early 1990s had reached its culmination, Igor Frenkel suggested that the next important problem was to develop a theory of double loop algebras. More specifically, he proposed a philosophy of three levels in Lie theory (and thereby in mathematics in general), illustrated by the following diagram:



In this diagram, the left downward arrows stand for affinization (taking loops), and the right downward arrows stand for quantization (q-deformation). The first level represents "classical" Lie theory, i.e., the structure and representation theory of complex semisimple Lie groups and Lie algebras. The second level represents affine Lie algebras and quantum groups, i.e., structures arising in 2-dimensional conformal and 3-dimensional topological field theory. The connection between them, depicted by the horizontal arrow, is the Drinfeld-Kohno theorem on the monodromy of the KZ equations, which is a part of the Kazhdan-Lusztig equivalence of categories. Finally, the third level is supposed to represent double affine Lie algebras, quantum affine algebras, and double (or elliptic) quantum groups. These three levels are supposed to correspond to discrete subgroups of the complex plane of ranks 0,1,2, respectively, and higher levels are not supposed to exist in the same sense because there are no discrete subgroups of $\mathbf{C}$ of rank > 2. [2]

At the time this philosophy was formulated, there wasn't much known about the third level of the diagram. Specifically, while quantum affine algebras were being actively studied, and Igor Frenkel's work with Reshetikhin on quantum KZ equations (subsequently developed by Felder, Tarasov, Varchenko, and others) shed a lot of light on what elliptic quantum groups and the quantum Drinfeld-Kohno theorem should be, the left lower corner of the diagram – the double loop algebras – remained mysterious. Yet, Igor insisted that this corner is the most important one, and that the study of double loops holds a key to the future of representation theory.

To develop the theory of double loop groups following the parallel with ordinary loop groups, one has to start with central extensions. This direction was taken up in our joint paper [EtF], where we constructed the central extension of the group of maps from a Riemann surface to a complex simple Lie group by the Jacobian of this surface (i.e., for genus 1, by an elliptic curve), and showed that the coadjoint orbits of this group correspond to principal G-bundles on the surface. This work was continued in the paper [FrKh], which extends to the double loop case the Mickelsson construction of the loop group extension by realizing the circle as a boundary of a disk, and then realizing a union of two such disks as a boundary of a ball. Namely, the circle is replaced by a complex curve (Riemann surface), the disk by a complex surface, and the ball by a complex threefold; then a similar formula exists, in the context of Leray's residue theory instead of De Rham theory. This work led to subsequent work by Khesin and Rosly on polar homology, as well as to the work of Frenkel and Todorov on a complex version of Chern-Simons theory, [FTo]. In this latter work, they start to develop the complex version of knot theory, in which the role of the 3-sphere is played by a Calabi-Yau threefold, and the role of the circle is played by a complex curve. In particular, these works led to a definition of the holomorphic linking number between two complex curves in a Calabi-Yau threefold, which is a

[2] I must admit that initially I did not take this philosophy too seriously, and we used the diagram in a Holiday party skit. However, with time it acquired quite a few concrete mathematical incarnations, and, ironically, defined much of my own work.

complex analog of the classical Gauss linking number, previously studied by Atiyah in the case of $\mathbb{CP}^1$.

In spite of this progress, however, it is still not clear what the representation theory of central extensions of double loop groups should be like. Perhaps we don't yet have enough imagination to understand what kind of representations (or maybe analogous but more sophisticated objects) we should consider, and this is a problem for future generations of mathematicians.

4 Igor Frenkel's work on Vertex Operator Algebras

by John Duncan

The *(normalised) elliptic modular invariant*, denoted $J(\tau)$, is the unique $SL_2(\mathbb{Z})$-invariant holomorphic function on the upper-half plane $\mathbb{H}$ with the property that $J(\tau) = q^{-1} + O(q)$ for $q = e^{2\pi i\tau}$. In the late 1970's McKay and Thompson made stunning observations relating the coefficients of the Fourier expansion

$$J(\tau) = q^{-1} + \sum_{n>0} c(n)q^n \tag{1}$$

of $J(\tau)$ to the dimensions of the irreducible representations of the (then conjectural) Monster sporadic simple group. This led to the conjecture [Tho79b] that there is a naturally defined infinite-dimensional representation

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots \tag{2}$$

for the Monster group with the property that $\dim V_n = c(n)$. Consideration [Tho79a] of the functions $T_g(\tau)$ obtained by replacing $\dim V_n = \mathrm{tr}|_{V_n} e$ with $\mathrm{tr}|_{V_n} g$ for g in the Monster led to the birth of *monstrous moonshine* and the *monstrous game* of Conway–Norton [CN79]. Thus, for the elucidation of monstrous moonshine, it became an important problem to construct such a representation—a *moonshine module* for the monster—explicitly. Igor Frenkel's pioneering work on vertex representations of affine Lie algebras, such as appears in [FK80, Fre81, Fre85, FF85], furnished important foundations for the work [FLM84, FLM85, FLM88] that would eventually realise this goal.

In [Gri82] Griess constructed the Monster group explicitly as the automorphism group of a certain commutative non-associative finite-dimensional algebra, thereby establishing its existence. The great insight of Frenkel–Lepowsky–Meurman was to recognise this algebra as a natural analogue of a simple finite dimensional complex Lie algebra $\mathfrak{g}$, viewed as a subalgebra of its affinization $\hat{\mathfrak{g}}$. Identifying Griess's algebra with (a quotient of) V_1 they attached vertex operators to the elements of this space and used them in [FLM84, FLM85] (see also [FLM88]) to recover the Griess algebra structure. In this way the non-associativity of the finite-dimensional Griess algebra was replaced with the associativity property of vertex operators.

The Frenkel–Lepowsky–Meurman construction [FLM84, FLM85] of the moonshine module V utilised the Leech lattice in much the same way as the root lattice of a Lie algebra of ADE type had been used to construct its basic representation in [FK80], but an important twisting procedure was needed in order to ensure the vanishing of the subspace V_0 in (2). This procedure (realised in full detail in [FLM88]) turned out to be the first rigorously constructed example of an *orbifold conformal field theory* and thus represented a significant development for mathematical physics.

Building upon the work of [FLM84, FLM85], Borcherds discovered a natural way to attach vertex operators to all elements of V, and several other examples, in [Bor86] and used this to define the notion of *vertex algebra*, which has subsequently met many important applications in mathematics and mathematical physics. The closely related notion of *vertex operator algebra (VOA)* was introduced in [FLM88]. A VOA comes equipped with a representation of the Virasoro algebra, and this hints at the importance of VOAs in conformal field theory. The central charge of the Virasoro representation attached to a VOA is called its *rank*, and a VOA is called *self-dual* if it has no irreducible modules other than itself. According to [FLM88] the Monster group can be characterised (conjecturally) as the automorphism group of the (conjecturally unique) self-dual VOA $V^\natural = \bigoplus_{n\in\mathbb{Z}} V_n^\natural$ of rank 24 satisfying $V_n^\natural = 0$ for $n < -1$, $V_{-1}^\natural \simeq \mathbb{C}$ and $V_0^\natural = 0$.

Important axiomatic foundations for the study of VOAs appeared in [FHL93] and in [FZ92] Frenkel–Zhu established the importance of VOAs in the representation theory of affine Lie algebras and the Virasoro algebra, and the Wess–Zumino–Witten model of mathematical physics. The notion of VOA was generalised and applied—simultaneously—to the affine E_8 Lie algebra and Chevalley's exceptional 24-dimensional algebra (arising from triality for D_4) in [FFR91].

The moonshine module $V^\natural$ exemplifies a close connection between VOAs and modular forms. Frenkel conjectured (cf. [Zhu96]) that the graded dimensions of irreducible modules over a *rational* VOA (being a VOA having finitely many irreducible modules up to isomorphism) should span a representation of the modular group $SL_2(\mathbb{Z})$. Y. Zhu added an important co-finiteness condition and subsequently proved the modularity conjecture for VOAs in his Ph.D. thesis [Zhu90] which was written under the supervision of Igor Frenkel. The subsequent article [Zhu96] remains one of the most influential works in the VOA literature.

Frenkel–Jing–Wang gave a completely new VOA construction of the affine Lie algebras of ADE type via the McKay correspondence in [FJW00b] and they derived a quantum version of this construction in [FJW00a]. These works also furnish vertex operator representations for classical and quantum toroidal algebras; related work appears in [FJ88, FW01, FJW02].

The semi-infinite cohomology of infinite dimensional Lie algebras is an area of research with important applications in string theory. In [FGZ] Frenkel–Garland–Zuckerman established a profound connection between the semi-infinite cohomology of the Virasoro algebra, introduced by Feigin [Fei84], and free bosonic string theories. Later, Frenkel–Styrkas established VOA structures on the modified regular representations of the Virasoro algebra and the affine Lie algebra of type $\hat{A}_1$ and computed their semi-infinite cohomology. Their work was extended to arbitrary affine Lie algebras by M. Zhu in her Ph.D. thesis (see [Zhu08]) using the Knizhnik–Zamolodchikov equations, and she also related this to earlier work [GMS01, AG02] on chiral differential operators. VOA structures on modified regular representations of the Virasoro algebra have been studied further using the Belavin–Polyakov–Zamolodchikov equations in [FZ12]. Braided VOA structures were used to recover the full quantum group $SL_q(2)$ from the semi-infinite cohomology of the Virasoro algebra with values in a suitably constructed module in [FZ10]. Beyond further demonstration of the importance of vertex operators in mathematics and mathematical physics this work promises deep consequences for the geometric and string theoretic understanding of quantum groups.

5 Igor Frenkel's work on Three Dimensional Quantum Gravity

by John Duncan

The most powerful feature of monstrous moonshine is the fact that each *McKay–Thompson series* $T_g(\tau) = \sum_n (\mathrm{tr}|_{V_n^\natural} g) q^n$ for g in the Monster group (where $V^\natural = \bigoplus_n V_n^\natural$ is the *moonshine module* VOA) has the following *genus zero property*: that if $\Gamma_g < PSL_2(\mathbb{R})$ is the invariance group of T_g then T_g is a generator for the field of Γ_g-invariant holomorphic functions on the upper-half plane $\mathbb{H}$. This result was proven by Borcherds in [Bor92] but a conceptual explanation of the phenomenon is yet to be fully elucidated.

In [Wit07] Witten considered the genus one partition function of pure quantum gravity in three dimensions and investigated possible connections with the Monster group, and related work in [MW10] suggested that the genus one partition function of such a theory might be expressible as a sum over certain solid torus geometries (which had appeared earlier in a string-theoretic setting [MS98, DMMV00]). Frenkel observed that the partition function of a chiral version of this (conjectural) quantum gravity theory (such as was considered in [LSS08]) should coincide with a Poincaré series-like expression—called a *Rademacher sum*—for the elliptic modular invariant $J(\tau)$ which was derived by Rademacher [Rad39] in 1939. He saw the potential for this as a mechanism for explaining the genus zero property of monstrous moonshine, and substantial progress towards this goal was achieved in [DF11] where it was established that the Rademacher sum $R_\Gamma(\tau)$ attached to a discrete group $\Gamma < PSL_2(\mathbb{R})$ has the genus zero property—i.e. is a generator for the field of Γ-invariant functions on $\mathbb{H}$—if and only if it is itself Γ-invariant. This result indicates a strong connection between 3-dimensional quantum gravity and monstrous moonshine, for it demonstrates than in the context of quantum gravity, the modular invariance of (twisted) partition functions, necessary for physical consistency, implies that they have the fundamental genus zero property.

The reformulation of the genus zero property obtained in [DF11] has already found important applications in related areas. In particular, it was applied in [CD11] to the moonshine-like phenomena observed by Eguchi–Oogui–Tachikawa [EOT] relating the largest Mathieu group M_{24} to the elliptic genus of a $K3$ surface and used there to obtain a uniform construction of the McKay–Thompson series attached to M_{24} in terms of Rademacher sums. The very fact of this construction elucidates the correct formulation of the genus zero property in the M_{24} case: The graded trace functions arising from the conjectural M_{24}-module underlying the $M_{24}/K3$ observation should coincide with the Rademacher sums attached to their variance groups. In addition to this the result of [CD11] indicates an important rôle for quantum gravity in the $M_{24}/K3$ story. More applications of the approach developed in [DF11] to moonshine both monstrous and otherwise can be expected.

Finer properties of the Rademacher sums $R_\Gamma(\tau)$ were used to give a quantum gravity partition function based characterisation (reformulated from [CMS04]) of the functions of monstrous moonshine in [DF11], and observations were also made relating the Rademacher sum construction to certain generalised Kac–Moody Lie algebras closely related to those utilised by Borcherds in his proof of the moonshine conjectures in [Bor92]. Motivated by this conjectures were formulated in [DF11] which identify the Monster as the symmetry group of a certain distinguished chiral 3-dimensional quantum gravity and specify the rôle of Rademacher sums in recovering the twisted partition functions of this theory and its second quantisation. Beyond monstrous moonshine, the further elucidation of this conjectural quantum gravity theory remains a fertile area for research that promises deep applications in algebra, geometry and mathematical physics.

6 Igor Frenkel's Contribution to Quaternionic Analysis

by Matvei Libine

The history of quaternionic analysis began on 16 October 1843 when an Irish physicist and mathematician William Rowan Hamilton (1805-1865) discovered the algebra of quaternions $\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$. He was so excited by that discovery that he carved the defining relations on a stone of Dublin's Brougham Bridge. After that W. R. Hamilton devoted the remaining years of his life developing the new theory which he believed would have profound applications in physics. But one had to wait another 90 years before von Rudolf Fueter [F1, F2] produced a key result of quaternionic analysis, an exact quaternionic counterpart of the Cauchy integral formula

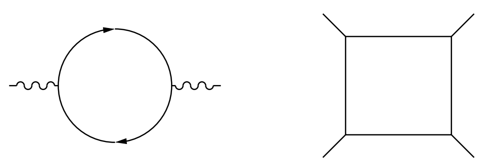
$$f(w) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z-w}.$$

Since then quaternionic analysis has generated a lot of interest among mathematicians and physicists, many results were extended from complex analysis to quaternionic analysis. For example, there is a quaternionic analogue of the Poisson formula for harmonic functions on $\mathbb{H}$. The spaces of harmonic and (left or right) regular functions are invariant under the conformal (fractional linear) action of the group $SL(2,\mathbb{H})$. There is a notion of the quaternionic cross-ratio which is very similar to the complex cross-ratio. See, for example, [Su] for a contemporary review of quaternionic analysis. There were also found many applications to physics (see, for instance, [GT]).

Unfortunately, this promising parallel between complex and quaternionic analysis essentially ends here. The difficulty seems to be in the non-commutative nature of quaternions. As a consequence, unlike the complex analytic case, the product or composition of two quaternionic regular functions is almost never regular. Such difficulties have discouraged mathematicians from working with quaternionic regular functions and developing a satisfactory theory of functions of quaternionic variable.

Igor Frenkel's groundbreaking idea was to approach quaternionic analysis from the point of view of representation theory of the conformal group $SL(2,\mathbb{H})$ and its Lie algebra $\mathfrak{sl}(2,\mathbb{H})$. While some aspects of representation theory of compact groups were used in quaternionic analysis before, using representation theory of non-compact reductive Lie groups is entirely new. This approach has been proven very fruitful and resulted in a series of fundamental papers [FL1, FL2, FL3] pushing further the parallel with complex analysis. In the course of developing this rich and beautiful theory Igor Frenkel found some very striking connections between quaternionic analysis and some of the most fundamental objects of the four dimensional classical and quantum field theories.

To give an example of such a connection between quaternionic analysis and physics, let us recall that Feynman diagrams are a pictorial way of describing integrals predicting possible outcomes of interactions of subatomic particles in the context of quantum field physics. As the number of variables which are being integrated out increases, the integrals become more and more difficult to compute. But in the cases when the integrals can be computed, the accuracy of their prediction is amazing. Many of these diagrams corresponding to real-world scenarios result in integrals that are divergent in the mathematical sense. Physicists have a collection of competing techniques called "renormalization" of Feynman integrals which "cancel out the infinities" coming from different parts of the diagrams. After renormalization, calculations using Feynman diagrams match experimental results with very high accuracy. However, these regularization techniques appear very suspicious to mathematicians (do you get the same result if you apply a different technique?) and attract criticism from physicists as well. Thus it is highly desirable to find an intrinsic mathematical meaning of Feynman diagrams, most likely in the context of representation theory.



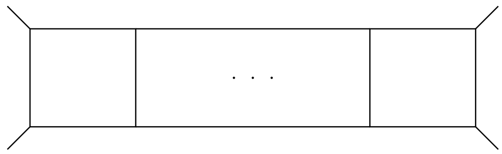
Feynman diagrams

In [FL1] Igor Frenkel found surprising representation-theoretic interpretations of some of the two most fundamental Feynman diagrams. The left figure shows the Feynman diagram for vacuum polarization which is responsible for the electric charge renormalization. This diagram appears in the quaternionic analogue of the Cauchy formula for the second order pole, which in turn can be related to the Maxwell equations for the gauge potential:

$$\begin{aligned} &\overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 & &\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \\ &\overrightarrow{\nabla} \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} & &\overrightarrow{\nabla} \times \overrightarrow{E} = -\frac{\partial \overrightarrow{B}}{\partial t}, \end{aligned}$$

where $\overrightarrow{B}$ and $\overrightarrow{E}$ are three-dimensional vector functions on $\mathbb{R}^4$ (called respectively the magnetic and electric fields) and $\overrightarrow{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$, as usual.

The right figure shows the one-loop Feynman diagram which expresses the hyperbolic volume of an ideal tetrahedron, and is given by the dilogarithm function. This diagram was identified with a projector onto the first irreducible component of a certain representation $\mathcal{H} \otimes \mathcal{H}$ of $SU(2,2)$.



Ladder diagrams

Furthermore, Igor Frenkel has made a conjecture (which is still open) that the so-called "ladder diagrams" correspond to projectors onto the other irreducible components of that representation $\mathcal{H} \otimes \mathcal{H}$ of $SU(2,2)$ (see [FL1] for details). Finding the relationship between ladder diagrams and representations would indicate how the rest of Feynman diagrams relate to representation theory and be a significant progress in four dimensional quantum field theory.

7 Emergence of a new area – elliptic hypergeometric series

by Michael Schlosser

Many special functions which appear in (real-world) applications, such as the trigonometric functions, the logarithm, and Bessel functions, can be expressed in terms of hypergeometric functions. While early occurrences of hypergeometric series already date back to the work of Isaac Newton (who in 1669 discovered the sum for an infinite binomial series), the systematic study of the hypergeometric function (nowadays commonly known as the ${}_2F_1$ series), was commenced by Carl Friedrich Gauß by the end of the 18th century. The theory of "generalized hypergeometric series" thence gradually developed. In 1840 Eduard Heine extended the hypergeometric function to the basic (or q-)hypergeometric function. The latter originally did not receive so much attention, the focus of special functions at that time (and in particular in Germany) being laid on Carl Gustav Jacob Jacobi's theory of *elliptic* functions. Basic hypergeometric series attracted wider interest only in the 20th century, due to important pioneering work done on the British isles (by Rogers, Ramanujan, Jackson, Bailey, and Watson, among others). Basic hypergeometric series have various applications in number theory, combinatorics, statistical and mathematical physics.

The next important step was to extend the basic (or "trigonometric") case to the *modular* or *elliptic* case. Building on (1987) work of the Japanese statistical physicists Date, Jimbo, Kuniba, Miwa and Okado [DJKMO] on the Yang–Baxter equation –elliptic hypergeometric series first appeared there, as elliptic $6j$ symbols, the elliptic solutions of the Yang–Baxter equation– Igor B. Frenkel and his coauthor Vladimir G. Turaev [FT] were in 1997 the first to actually study elliptic $6j$-symbols as elliptic generalizations of q-hypergeometric series and to find transformation and summation formulae satisfied by such series. In particular, by exploiting the tetrahedral symmetry of the elliptic $6j$ symbols, Frenkel and Turaev came across the (now-called) ${}_{12}V_{11}$ transformation (which is an elliptic extension of Bailey's very-well-poised ${}_{10}\phi_9$ transformation) and by specialization obtained the (now-called) ${}_{10}V_9$ summation. These results, involving series satisfying modular invariance, are deep and elegant and, from a higher point of view, lead to a much better understanding (of various phenomena such as "well-poised" and "balanced" series) of the simpler basic case. The new theory beautifully combines the theories of theta (or abelian) functions with the theory of basic hypergeometric series.

The findings of Frenkel and Turaev had big impact and truly initiated an avalanche of further research in the area. Various researchers, first V. Spiridonov and A. Zhedanov, then S.O. Warnaar and others (J.F. van Diejen, H. Rosengren, E. Rains, etc.) joined their forces to build up a yet expanding theory of elliptic hypergeometric series. The importance of this subject is reflected in the fact that already the 2004 second edition of Gasper and Rahman's (already classic) textbook [GR] on basic hypergeometric series devotes a full chapter to elliptic hypergeometric series. Moreover, at several occasions (special functions guru) Richard Askey has suggested that elliptic hypergeometric functions will be *the* special functions of the 21st century.

For further references, see the bibliography of elliptic hypergeometric functions on Hjalmar Rosengren's website `http://www.math.chalmers.ser/~hjalmar/bibliography.html`

8 Igor Frenkel's contributions to the representation theory of split real quantum groups and modular doubles

by Ivan Ip

The quantum group $\mathcal{U}_q(\mathfrak{g})$ defined in 1985 by Drinfeld and Jimbo for a real q can be considered a quantum counterpart of the compact real form $\mathfrak{g}_c \subset \mathfrak{g}$. In particular, its representation theory is in complete parallel with the classical theory. The finite dimensional representations form a braided tensor category which leads for example to certain topological quantum field theories and categorifications.

It is natural to consider other real forms of $\mathfrak{g}$, most notably the split real form $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$, and to address the question about the q-deformation of its representations, which makes sense only when $|q| = 1$. The starting point comes from Faddeev's notion of modular doubles [Fa], which are objects generated by pairs of commuting quantum tori $\{u, v\}, \{\widetilde{u}, \widetilde{v}\}$ acting on $L^2(\mathbb{R})$ which are related by certain transcendental relations. In the case when $|q| = 1$, the representation theory of the quantum plane, a single pair of quantum tori represented by positive self-adjoint operators, is closely related to a remarkable function called the *quantum dilogarithm* $g_b(x)$. Using this function, in the work with Hyun Kyu Kim [FK], Igor Frenkel showed that the quantum Teichmüller space and also its universal version, constructed recently by Kashaev [Ka], and independently by Fock and Chekhov [Fo, CheF], originate from a tensor category of the representations of the modular double of the quantum plane.

In the case for $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2, \mathbb{R})$, J. Teschner et al. [PT] have studied a very special q-deformation of principal series of representations of the quantum group using the modular double, and showed that a class of representations, represented by positive operators, is closed under taking the tensor product. This has profound importance in conformal field theories in physics, and in particular a new kind of topological invariants is expected to be constructed from the tensor category structure. Igor Frenkel has always been emphasizing the analogy between the representation theory of compact and split real quantum group, and the relationship between their classical counterpart. In a recent joint work with Ivan Ip [FI], Igor Frenkel generalized this special class of representations, which we called the *positive principal series representations*, to higher rank $\mathcal{U}_q(\mathfrak{sl}(n, \mathbb{R}))$, and later was further generalized to arbitrary quantum groups of all types [Ip1, Ip2]. This strongly indicates that all the results for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ can be generalized and that one can envision future perspectives for the positive representations of the split real quantum groups comparable to the past developments related to the finite dimensional representations of the quantum group initiated by Drinfeld and Jimbo [F].

In particular, in the split real case, where the parameters are varying continuously, Igor Frenkel proposed certain notion of "continuous" categorification and geometrization of the quantum groups and their representations. In the past year physicists have observed a remarkable relation between the Chern-Simons-Witten theory for the split real group $SL(2, \mathbb{R})$ and the $N = 2$ super-symmetric gauge theory on a three-dimensional sphere [DGG, TY]. This work can be considered as a first step towards a geometrization of the category of positive representations of the modular double, generalizing the geometric construction of the finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ discovered by Nakajima based on the gauge theory [Na].

One can also discuss the split real version of Kazhdan-Lusztig equivalence between the categories of highest weight representation of affine Lie algebras and quantum groups. In the compact case the explicit construction can be simplified by considering an additional category of representations of W-algebra [Sty]. Although in the split real case it is still an open problem to construct a principal series of representations of the affine Lie algebra $\widehat{\mathfrak{g}_{\mathbb{R}}}$ even for the case $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2,\mathbb{R})$, the work of Igor Frenkel suggests that one can instead discuss the equivalence of categories of representations of the modular double $\mathcal{U}_{q\tilde{q}}(\mathfrak{g}_{\mathbb{R}})$ and the W-algebra associated to $\mathfrak{g}_{\mathbb{R}}$.

9 Igor Frenkel's work on categorification

by Mikhail Khovanov

In 1994 Igor Frenkel and Louis Crane published the paper "Four dimensional topological quantum field theory, Hopf categories, and the canonical bases", advancing the idea that various structures related to quantum invariants of links and 3-manifolds should be just shadows of much richer structures controlling quantum invariants of four-dimensional objects [CF]. They coined the term *categorification* to denote this structural lifting and, in particular, conjectured that the quantum group $U_q(\mathfrak{sl}(2))$ admits such a lifting. After 1994 Igor Frenkel continued to extend these ideas and observed that Grothendieck groups of suitable singular blocks of highest weight categories of the Lie algebra $\mathfrak{sl}(n)$ can be identified with weight spaces of the n-th tensor power of the $\mathfrak{sl}(2)$ fundamental representation, where the generators E and F of $\mathfrak{sl}(2)$ act as translation functors. These insights resulted in the joint paper [BFK], where the authors also constructed a commuting action of the Temperley-Lieb algebra via Zuckerman functors and studied the Koszul dual framework, with the categorification via the direct sum of maximal parabolic blocks and the roles of projective and Zuckerman functors interchanged. Strongly influenced by Igor's revolutionary ideas about categorification and by our joint work, at about the same time I categorified the Jones polynomial into a bigraded link homology theory.

Several years later, Catharina Stroppel [St] proved the conjectures of [BFK], establishing an amazing relation between highest weight categories and low-dimensional topology, via knot homology. Joshua Sussan, Frenkel's graduate student at the time, generalized these constructions from $\mathfrak{sl}(2)$ to $\mathfrak{sl}(k)$, showing that category $\mathcal{O}$ also controls some other link homology theories and proving Lusztig's positivity conjecture for tensor products in the $\mathfrak{sl}(n)$ case.

In the joint paper [FKS], Frenkel, Stroppel, and I extended some of the constructions from [BFK] to arbitrary tensor products of $\mathfrak{sl}(2)$ representations and revisited unpublished ideas of Igor Frenkel on categorification of Lie algebra and quantum group representations via categories of Harish-Chandra modules. More recently, Frenkel, Stroppel, and Sussan [FSS] investigated categorifications of Jones-Wenzl projectors and 3j-symbols in the context of category $\mathcal{O}$, explaining categorification of rational functions in the spin network formulas. The new viewpoint on highest weight categories, originating from Igor Frenkel's ideas and work, has become a fruitful and exciting area of research, with important contributions made by Brundan, Chuang, Kleshchev, Mazorchuk, Rouquier, Webster, Zheng, and many others.

Link homology has also emerged from categories of matrix factorizations (Khovanov-Rozansky), from derived categories of sheaves on quiver varieties and on convolution varieties of affine Grassmannians (Kamnitzer-Cautis), and from Fukaya-Floer categories of quiver varieties (Seidel-Smith, Manolescu). These appearances led to a series of conjectures and results on the existence of equivalences between subcategories of these categories, uncovering a remarkable unity and new connections between various structures of representation theory, algebraic geometry and symplectic topology. Even when the subject was in its infancy, this unity was one of the fundamental goals emphasized by Igor in his conversations with students and colleagues.

Back in 1994, Crane and Frenkel conjectured [CF] that there exists a categorification of quantum $\mathfrak{sl}(2)$ at a root of unity, which should control categorification of the Reshetikhin-Turaev-Witten invariants of 3-manifolds. Frenkel wrote notes (never published) on structural constraints in direct sum decompositions of functors in the desired categorification of quantum $\mathfrak{sl}(2)$. The problem remained open for several years; the major breakthrough came from Chuang and Rouquier, who obtained fundamental results on higher categorical structures of $\mathfrak{sl}(2)$ representations. These were extended by Lauda to a categorification of the Lusztig's idempotented version of quantum $\mathfrak{sl}(2)$ at generic q via a beautiful planar diagrammatical calculus. Categorifications of quantum groups for other simple Lie algebras were developed by Lauda, Khovanov, and Rouquier, while categorification of quantum $\mathfrak{sl}(2)$ at a root of unity remains an open problem. Ben Webster, in spectacular work related to quantum group categorifications, categorified Reshetikhin-Turaev link and tangle invariants for arbitrary simple Lie algebras and their irreducible representations.

In the late 90's Igor Frenkel proposed a bold conjecture that the entire conformal field theory and vertex operator algebra theory can be categorified. He suggested to start by categorifying boson-fermion correspondence and related vertex operators. A couple of years later, Frenkel, Anton Malkin, and I spent several months discussing this project, with modest success encapsulated in our unpublished notes. The question of how to categorify vertex operators had a strong background presence in the series of papers by Frenkel, Naihuan Jing, and Weiqiang Wang [FJW00a, FJW00b, FJW02], and in [FW01]. A very recent paper [CL] of Cautis and Licata is a major advancement in this direction, realizing components of vertex operators as functors acting in 2-representations of categorified Heisenberg algebras and giving yet another confirmation of visionary and predictive power of Igor Frenkel's mathematical genius.

Multiple discussions with Igor Frenkel and his remarkable results, ideas, and thoughts on categorification strongly influenced current researchers in the area. Several former graduate students of Frenkel do full-time research in categorification and related fields: Anthony Licata, Alistair Savage, Joshua Sussan, and myself. Categorification has become a dynamic and exciting field, every year boasting more and more connections to various areas and structures in mathematics and mathematical physics. Its practitioners are grateful to Igor Frenkel for his vision which created the subject.

10 Igor Frenkel's work on geometric representation theory

by Anthony Licata

Many of Igor Frenkel's contributions to mathematics have come in the form of foundational ideas introduced at the beginning of a new subject. In contrast, Frenkel's work in geometric representation theory began when the subject was already well developed. As a result, these contributions give some insight into both his ability to understand the deepest parts of others' mathematics in his own terms and also his gift for seeing the implications of this work for future mathematics. Strikingly, some of Frenkel's most important ideas in geometric representation theory remain unpublished by him.

The seminal work of Beilinson-Bernstein and Brylinski-Kashiwara proving the Kazhdan-Lusztig conjectures gave a geometric interpretation of many of the fundamental structures of Lie theory. By the late 1980s and early 1990s, subsequent work of numerous mathematicians produced explicit geometric constructions of representations. In addition to the work of Beilinson-Berstein-Brylinski-Kashiwara, several of these later constructions, including Lusztig's geometric construction of canonical basis and Nakajima's quiver variety constructions, had a profound impact on Frenkel's perspective on representation theory.

When Frenkel began working in the subject in the mid 1990s, he brought a perspective which advocated the geometrization of all important structures in representation theory: the more fundamental the structure, the more beautiful its geometric realization. Moreover, and perhaps more importantly, he proposed that geometric constructions be seen as source for new mathematics, via the principle that these constructions lead to categorifications. In his proposal, vector spaces of geometric origin – like cohomology or K-theory – would be upgraded to categories of sheaves. Once realized geometrically, the symmetries of a vector space should lift to symmetries at the level of categories of sheaves. This idea has had a tremendous influence on the development of categorification in representation theory, breathing new life into the foundational geometric constructions in the subject. Indeed, much of the past decade's work on categorification can be viewed as carrying out the details of this broad vision.

Frenkel's published work made direct contributions to geometric representation theory as well. In collaboration with Kirillov Jr. and Varchenko [FKV], he gave a geometric interpretation of the Lusztig-Kashiwara canonical basis for tensor products of $\mathfrak{sl}(2)$-representations in terms of the homology of local systems on configuration spaces of points in a punctured disk. This construction is a geometric analog of his work with Khovanov [FKh] on a graphical calculus for the representation theory of $\mathfrak{sl}(2)$, work which later influenced Khovanov's categorification of the Jones polynomial. He also wrote several articles on quiver varieties, including joint papers with Malkin and Vybornov [FMV1, FMV2] and with Savage [FS]. He worked with Jardim on quantum instantons [FJ], and with Khovanov and Schiffmann [FKS] on homological realizations of Nakajima quiver varieties. He also contributed to geometric constructions of non-integrable representations in the paper [FFFR].

Quiver varieties of affine type play a prominent role in many of Frenkel's papers from this period. These are distinguished within quiver varieties by their independent appearance as instanton moduli spaces in gauge theory. Frenkel's emphasis on quiver varieties of affine type in geometric representation theory is a legacy of his early foundational work in the representation theory of affine Lie algebras, and the gauge-theoretic origin of affine type quiver varieties was an important motivation for his interest in the subject. More precisely, let $\mathfrak{g}$ be a finite-dimensional simple simply-laced complex Lie algebra and let $\widehat{\mathfrak{g}}$ denote its affinization. Nakajima's construction produces integrable highest weight representations of $\widehat{\mathfrak{g}}$ from moduli spaces of $U(k)$-instantons on the resolution of $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$ related to $\mathfrak{g}$ by the McKay correspondence. In this construction, the algebra is determined by the finite group Γ; the level of the action is determined by the rank of the group $U(k)$, but otherwise this instanton group has no direct connection with the affine Lie algebra $\widehat{\mathfrak{g}}$. Frenkel realized, however, that there should also be a dual picture in which the representations of $\widehat{\mathfrak{g}}$ are realized directly by G-instanton moduli spaces, where G the compact group whose complexification has Lie algebra $\mathfrak{g}$. When both $G = U(k)$ and $\Gamma = \mathbb{Z}_n$ are type A, all the moduli spaces involved in these constructions are Nakajima quiver varieties, and Frenkel's dual picture relates to the Nakajima construction via level-rank duality in affine type A. (In fact, as is mentioned in the affine Lie algebras section of this text, Frenkel discovered level-rank duality in an algebraic setting nearly twenty years earlier.) As with many of his important ideas in geometric representation theory, Frenkel did not publish anything himself about the relationships between representations of $\widehat{\mathfrak{g}}$ and G-instanton moduli spaces. However, his suggestion was both the core of my own thesis [L] and an essential part of the subsequent work of Braverman-Finkelberg on the affine version of the geometric Satake correspondence [BF].

The scope of Frenkel's vision of geometric representation theory, which includes ideas about current algebras and other fantastic mathematical objects, has yet to be fully realized. We sincerely hope that it brings as much to the next stage of the subject as it has to its development thus far.

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