Quasisymmetries of the Feigenbaum Julia set and transcendental dynamics

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Abstract of the Dissertation

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We give an explicit description of the group of topologically extendable quasisymmetric self-maps of the Feigenbaum quadratic Julia set. We also describe the transcendental dynamics of the Feigenbaum renormalization fixed point whose structure plays an important role in restricting what maps can belong to the group of quasisymmetries.

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Chapter 1

Introduction

In the last decade there has been a substantial increase in interest in the field of quasisymmetric geometry. Perhaps the most natural invariant for this setting is the group of quasisymmetries of a given space. One can coarsely classify sets as being either *little quasisymmetric* or *highly quasisymmetric* depending on whether its group of quasisymmetries is finite or infinite, respectively. In the world of Julia sets of rational maps, quasisymmetric groups have been computed for Sierpínski carpets in [BLM14] and for the basilica Julia set in [LM18]. The Sierpínski carpets from [BLM14] are examples of sets that are little quasisymmetric while in [LM18], Lyubich and Merenkov show the basilica is highly quasisymmetric. The methods from [LM18] also extend to every hyperbolic Julia set in the main molecule of the Mandelbrot

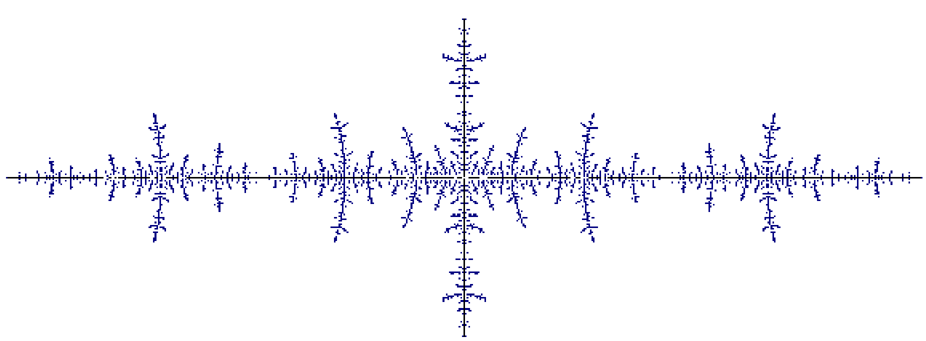

Figure 1.1: Julia set of the Feigenbaum quadratic, $\mathcal{J}_c$

set (i.e. maps reachable by a finite series of bifurcations from z^2). The goal of this thesis is to describe the group of quasisymmetries of the Julia set of the Feigenbaum quadratic, $\mathcal{J}_c$, (see Figure 1.1), thus giving the first example of the group of quasisymmetries of the Julia set of an infinitely renormalizable map and, more generally, a non-hyperbolic quadratic.

The Feigenbaum quadratic's lack of hyperbolicity prohibits us from applying the methods of [LM18] in our setting. Instead, we rely heavily on the topology and geometry of the transcendental dynamics of the quadratic-like renormalization fixed point from which we are able to fully determine the group of extendable quasisymmetries of $\mathcal{J}_c$. We give now some quick definitions in order to state the main results and give better intuition as to the ideas of the proofs.

Given a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ and metric spaces $(X, d_X), (Y, d_Y)$, a homeomorphism $\phi : X \to Y$ is said to be a *quasisymmetry with distortion function* η (η-*quasisymmetry* or η-qs) if for all distinct triples $x_1, x_2, x_3 \in X$,

$$\frac{d_Y(\phi(x_1), \phi(x_2))}{d_Y(\phi(x_1), \phi(x_3))} \leq \eta\left(\frac{d_X(x_1, x_2)}{d_X(x_1, x_3)}\right).$$

When $X = Y$, we call ϕ a *quasisymmetry* (or η-*quasisymmetry* if η is specified). As is done for the basilica in [LM18] we restrict our scope to quasisymmetries of $\mathcal{J}$ that are *topologically extendable* (to an orientation-preserving homeomorphism of $\mathbb{C}$).

A *quadratic-like map* (ql map) is a degree 2 branched covering between topological disks $U \Subset V$. Let $0 \in U_0 \Subset V_0$ be suitable open topological disks and let $f : U_0 \to V_0$ be the period-doubling renormalization fixed point in the space of quadratic-like maps – for more on this map and its existence, see [Buf97] or [McM96]. As such, f has the desirable property of satisfying the functional equation

$$f(z) = -\mu f^2(\mu^{-1} z) \tag{1.0.1}$$

where $\mu > 1$. Define the Julia set of f, $\mathcal{J} \equiv \mathcal{J}(f)$, to be the set of non-escaping points; that is,

$$\mathcal{J} := \{z \in U_0 \mid \text{for all } n \in \mathbb{N}, f^n(z) \in U_0\}. \tag{1.0.2}$$

By (1.0.1) and (1.0.2), for all $k \in \mathbb{N}$,

$$\mu^{-k}\mathcal{J} \subset \mathcal{J} \tag{1.0.3}$$

and $f^{2^k} : \mu^{-k}\mathcal{J} \to \mu^{-k}\mathcal{J}$ is a degree two mapping. We say a subset $J \subset \mathcal{J}$ is a *little Julia set of depth* k if J is a pre-image of $\mu^{-k}\mathcal{J}$ under f^n for some $n \in \mathbb{N}_0$. For each little Julia set, J, we define a "shift" map $\sigma_J : \mathcal{J} \to \mathcal{J}$ (under which J is invariant) which acts as a shift on the little Julia sets contained in J and of depth one more than that of J. Given the top-level shift map on $\mathcal{J}$, every other shift map is then defined canonically in terms of the dynamics and scaling. See Section 5.1 for formal definitions and Figures 5.2 and 5.3 for drawings of these maps. Let $\rho : \mathcal{J} \to \mathcal{J}$ be the symmetry given by $\rho(z) = -z$.

Let X_k be the collection of little Julia sets of depth less than k. From this, define $Y_k := \{\rho, \sigma_J \mid J \in X_k\}$ and G_k to be the group of maps generated by elements of Y_k.

The following statement is the main result of this dissertation and the sentiment is that any quasisymmetry of $\mathcal{J}$ is dynamical in nature for all sufficiently deep little Julia sets and the action of a quasisymmetry on $\mathcal{J}$ can be fully described in terms of shift maps on little Julia sets of shallow depth (where what is considered shallow depends on the distortion of the quasisymmetry).

Theorem 1.0.1. *For any distortion function η there exists a $k \in \mathbb{N}$ and distortion function η' such that for any topologically extendable η-quasisymmetry $\phi : \mathcal{J} \to \mathcal{J}$, there exists a sequence of η'-quasisymmetries $(g_n)_{n\in\mathbb{N}} \subset G_k$ such that $g_n \to \phi$ uniformly.*

As mentioned earlier, the proof of Theorem 1.0.1 relies heavily on the topology and geometry of the transcendental dynamics of the renormalization fixed point, f. The following set, $\mathcal{J}_\infty$, is of particular importance. By (1.0.3),

$$\mathcal{J}_\infty := \bigcup_{n\in\mathbb{N}} \mu^n \mathcal{J} \tag{1.0.4}$$

is an increasing union and therefore well defined. By Proposition 4.1.1 it is dense in $\mathbb{C}$. We use $\mathcal{J}_\infty$ to obtain rigidity in the quasisymmetric group of $\mathcal{J}$. A primary example of how this is done is via Lemma 5.2.4 which lets us pass from a quasisymmetry of $\mathcal{J}$ that fixes 0 to a limiting map which preserves "limb structures" in $\mathcal{J}_\infty$ and via Propositions 4.8.8, 4.8.9, and 4.8.10 which essentially say that if the closure of these "limb structures" in $\mathcal{J}_\infty$ don't intersect then they are bounded away from each other relative to their diameters. Since the "deep limb structures" in $\mathcal{J}$ approximate the "limb structures" of $\mathcal{J}_\infty$ the quasisymmetric property implies these deep "virtually touching" limbs must go to "virtually touching" limbs under qs maps which gives rigidity to the quasisymmetric group of $\mathcal{J}$.

In proving Theorem 1.0.1, we also obtain the following theorem:

Theorem 1.0.2. *The uniform closure of the group of topologically extendable quasisymmetries of $\mathcal{J}$ is properly contained in the group of topologically extendable homeomorphisms of $\mathcal{J}$.*

The fact that it is a proper subset highlights a key difference between the quasisymmetric group of $\mathcal{J}$ versus that of the quasisymmetric groups for the basilica and other hyperbolic Julia sets coming from the main molecule, for which the uniform closure is the entire group of extendable homeomorphisms. In this way, the Feigenbaum Julia set is in some sense a middle case between that of the Sierpínski carpet Julia sets and the hyperbolic Julia sets of the main molecule. Theorem 1.0.2 essentially follows from the fact that there exists topological "rotations by 90 degrees" that rotate by 1 the 4 connected components of $\mathcal{J} \setminus \{0\}$ but that any such map cannot be qs.

1.1 Organization of the chapters

Chapter 2 gives formal definitions, states some theorems from the literature, and proves some lemmas that rely only on the general theory and tools from their respective areas.

Chapter 3 gives definitions and proves lemmas regarding geometry of $\mathcal{J}$. For example, that it is tetravalent and uniquely arc-wise connected (Lemmas 3.2.7 and 3.2.3.) This chapter serves primarily as a preliminary to both chapters 4 and 5: defining objects and proving lemmas that will have analogs in the transcendental setting or will be of use for exploring quasisymmetries of $\mathcal{J}$.

Chapter 4 is the main chapter detailing transcendental dynamics of the renormalization fixed point. Much of this chapter mirrors Section 5 of [DL18] by Dudko and Lyubich as the structures are very similar. The recorded lecture [Dud20] by Dudko was a great help in the writing of this chapter and it informed both the style and substance of much of what is presented here. Detailed here are the concepts of limbs, lakes, and β-points (an analog of the standard definition in the quadratic setting.) We show here that the escaping set consists of quasi-arcs that are continuously parametrized by escaping time and that closed limbs are uniform quasidisks. A key observation that is formally proved here is also the discrepancy between *real* and *imaginary* limbs which, as we'll see, prohibits the existence of qs "rotations by 1/4". (See Proposition 4.9.6.)

Chapter 5 is the main chapter that goes through the proof of Theorem 1.0.1. There are 5 steps to the proof, each with their own section. Section 5.2 shows that while rotations by 1/4 *do* exist topologically, by Proposition 4.9.6 they do not exist as qs maps. For this, a limiting argument is used where we pass from the quadratic-like setting to the transcendental setting. Section 5.3 greatly restricts what is allowable under a qs map and shows that every qs map is k dynamical for some k, meaning that the action of a qs map on a deep Julia set is given by the dynamics: as a unit, it maps forwards to the center via the dynamics, it maybe rotates by 180 degrees, and then it maps backwards under the dynamics to an image little Julia set of the same depth. Section 5.4 shows that the lift of any qs map of $\mathcal{J}$ to $\mathbb{T}$ via the Böttcher coordinate is again a qs map – something that does not come for free. The argument for this is by explicitly showing that the lift satisfies the qs property. This is done by starting in simple cases and gradually working our way to the general setting. Section 5.5 shows that any topologically extendable qs map of $\mathcal{J}$ is qc extendable (to $\mathbb{C}$). While this is possible via cutting and pasting together qc maps and taking advantage of the result that the Lebesgue measure of $\mathcal{J}$ is 0 (by [DS20]), we instead opt to show this via qc extension of tunings of the Basilica that limit on $\mathcal{J}$. The last step is done in Section 5.6 in which, for any quasisymmetry of $\mathcal{J}$, ϕ, we construct an explicit sequence of shift maps with the property that the n-th map of the sequence and ϕ have the same action on the spine children of the n largest (extended) little Julia sets of depth less than k. This sequence converges uniformly to ϕ because the diameters of the (extended) little Julia sets go to 0 and because, from Section 5.3, we know ϕ moves little Julia sets of depth k dynamically. That is, we are able to show that on "large" (extended) little Julia sets of depth k the maps are equal, while on "small" (extended) Julia sets of depth k, the maps differ by at most ϵ.

Chapter 2

Background

2.1 qs and qc maps

Definition 2.1.1. We say a map $\phi : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open is *conformal* if ϕ is holomorphic and injective.

Definition 2.1.2. We say an orientation-preserving homeomorphism $\phi : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open is *K-quasiconformal (K-qc)* if ϕ has locally square-integrable distributional derivatives satisfying

$$|\partial_{\overline{z}}\phi(z)| \leq k|\partial_z\phi(z)|$$

for a.e. $z \in U$ where $K = \dfrac{1+k}{1-k}$. A map ϕ is said to be *quasiconformal (qc)* if it is K-qc for some $K \geq 1$.

The following is a special case of the Koebe Distortion theorem (see e.g. [Lyu]).

Theorem 2.1.1 (Koebe Distortion)**.** *Let $\phi : \mathbb{D} \to \mathbb{C}$ be conformal. Then for any $r < 1$, $\phi|_{r\mathbb{D}}$ is η-qs where η depends on r.*

The following is Theorem 3.4 in the paper by Tukia and Väisälä [TV80].

Theorem 2.1.2. *Let X, Y be metric spaces, let $M > 0$, and let $a, b \in X$ be distinct points. If F is a family of η-qs maps from X to Y such that for each $f \in F$, $d_Y(f(a), f(b)) \leq M$, then F is equicontinuous.*

We define an η*-quasi-arc* to be an η-qs image of an interval $I \subset R$. Similarly, we define an η*-quasi-circle* to be an η-qs image of $\mathbb{T}$ and we say a topological disk, D is an η*-quasi-disk* if D is the image of $\mathbb{D}$ under an η-qs map. One remark is that if D is a topological disk and ∂D is an η-quasi-circle then D is an η'-quasi-disk where η' depends only on η. Hence, to show something is a quasi-disk, it suffices to show that its boundary is a quasi-circle. We say something is a quasi-arc (quasi-circle, quasi-disk) if it is an η-quasi-arc (η-quasi-circle, η-quasi-disk) for some distortion function η. We say a family of quasi-arcs (quasi-circles, quasi-disks) is *uniform* if there exists a distortion function η such that each is an η-quasi-arc (η-quasi-circle, η-quasi-disk).

We say an arc or topological circle $A \subset \mathbb{C}$ has *c-bounded turning* if for any $a, b \in A$, $\text{diam}([a, b]) \leq c|a - b|$ where $c \geq 1$ and $[a, b] \subset A$ is a sub-arc connecting a and b whose diameter is less than or equal to the complementary arc. An arc has *bounded turning* if it has c-bounded turning for some $c \geq 1$.

In Theorem 4.9 of the same paper by Tukia and Väisälä [TV80], they prove a generalization of the following:

Theorem 2.1.3. *Let $\gamma \subset \mathbb{C}$ be homeomorphic to $[0, 1]$ or $\mathbb{T}$. Then γ is qs if and only if γ has bounded turning.*

Lemma 2.1.4. *Let $\lambda \in \mathbb{D}$, and A, B disjoint closed quasi-arcs such that*

- $X = \{0\} \cup \bigcup_{n \in \mathbb{N}_0} \lambda^n (A \cup B)$ *is an arc*
- $\text{diam}(A \cap \lambda A), \text{diam}(B \cap \lambda B) > 0.$

Then X is a quasi-arc. (See Figure 2.1.)

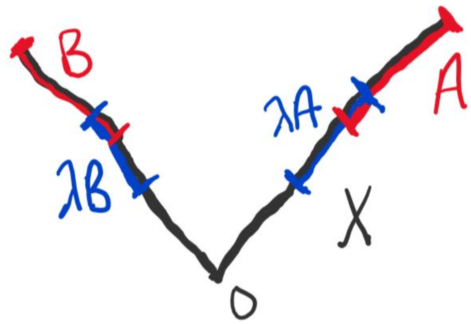

Figure 2.1: The set X and the subsets $A, B, \lambda A, \lambda B \subset X$ from Lemma 2.1.4

Proof. By Theorem 2.1.3 it suffices to show that X is BT. Also by Theorem 2.1.3 there exists c_1 such that A, B are c_1-BT. Let $a, b \in X$. Since scaling by λ has the same effect on $\operatorname{diam}([a,b])$ and $|a-b|$, we may assume WLOG that $a \in X \setminus \lambda X$. With this assumption, a, b satisfy one of the following cases.

(1) If $a, b \in A$ or $a, b \in B$ then $|a-b| \geq \operatorname{diam}([a,b])/c_1$ since A, B are c_1-BT.

(2) If we're not in the first case, then either $a \in A \setminus \lambda X$ and $b \in X \setminus A$ or $a \in B \setminus \lambda X$ and $b \in X \setminus B$. Suppose that $a \in A \setminus \lambda X$ and $b \in X \setminus A$. Since X is an arc and $\operatorname{diam}(A \cap \lambda A) > 0$, $A \cap \lambda A \subset X$ is a (non-singleton) closed sub-arc. Hence, $\overline{A \setminus \lambda X} \cap \overline{\lambda X \setminus A} = \emptyset$ as they are the closure of the connected components of X after removing the sub-arc $A \cap \lambda A$. Therefore, $d(A \setminus \lambda X, X \setminus A) > 0$ and so for $c_2 := \operatorname{diam}(X)/d(A \setminus \lambda X, X \setminus A)$ we have that

$$\operatorname{diam}([a,b]) \leq \operatorname{diam}(X) = c_2 d(A \setminus \lambda X, X \setminus A) \leq c_2|a-b|.$$

(3) If the first two are not satisfied, then $a \in B \setminus \lambda B$ and $b \in X \setminus B$ (or vice-versa.) This case is exactly analogous to the previous and so there exists c_3 such that $\operatorname{diam}([a,b]) \leq c_3|a-b|$.

Hence, X is c-BT where $c = \max\{c_1, c_2, c_3\}$. ∎

Lemma 2.1.5. *Let X be a topological circle, A, B quasi-arcs such that*

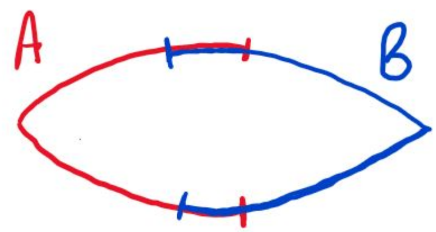

Figure 2.2: Drawing to accompany Lemma 2.1.5.

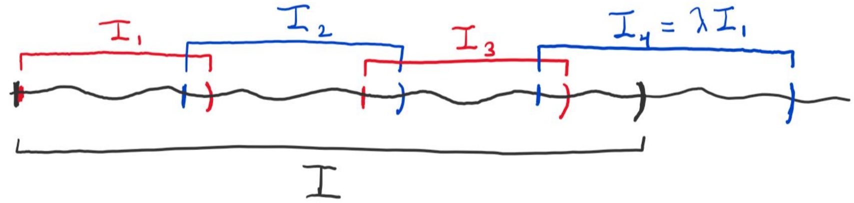

Figure 2.3: Drawing to accompany Lemma 2.1.6 for the case $n = 3$.

1. $A \cup B = X$,

2. $d(A \setminus B, B \setminus A) > 0$.

Then X is a quasi-circle. (See Figure 2.2.)

Proof. By Theorem 2.1.3, let c_1 such that A, B are c_1-BT. Let $c_2 = \mathrm{diam}(X)/d(A \setminus B, B \setminus A)$. Let $a, b \in X$. If $a, b \in A$ or $a, b \in B$ then $\mathrm{diam}([a, b]) \leq c_1|a - b|$ since A, B are c_1-BT. If not, then $a \in A \setminus B$ and $b \in B \setminus A$ (or vice-versa). Therefore,

$$\mathrm{diam}([a, b]) \leq \mathrm{diam}(X) = c_2 d(A \setminus B, B \setminus A) \leq c_2|a - b|.$$

Let $c = \max\{c_1, c_2\}$. Then X is c-BT. $\square$

We say a map $\chi : U \to V$ where $U \subset V$ is *expanding* if there exists a $\lambda > 1$ and $\epsilon > 0$[1] such that if $x, y \in U$ and $d(x, y) < \epsilon$ then $d(\chi(x), \chi(y)) > \lambda d(x, y)$.

Lemma 2.1.6. *Let $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, and $I, I_1, I_2, ..., I_{n+1}$ be arcs such that the following hold:*

1. $I_{n+1} = \lambda I_1$ *and for* $i = 1, 2, ..., n$, $I_i \subset I$.

[1]If U is not compact, then ϵ may depend on x.

2. *$I \subset I_1 \cup I_2 \cup \cdots \cup I_{n+1}$.*

3. *There exists an open set $U \supset I$ such that for each $i = 1, 2, ..., n$, there exists an open set $U_i \supset I_i$ and an expanding, conformal or anti-conformal map $\chi_i : U_i \to U$ such that $\chi_i(I_i) = I$.*

4. *For $i = 1, 2, ..., n$, $I_i \cap I_{i+1}$ is an arc and for $|i - j| \geq 2$, $I_i \cap I_j = \emptyset$.*

Then I is a quasiarc. (See Figure 2.3.)

Proof. For each $i \in \{1, 2, ..., n+1\}$, define

$$J_i := I_i \setminus \bigcup_{j \neq i} I_j.$$

By the fourth assumption, the closures of the J_i are all disjoint. Hence, we have that

$$C := \min_{i \neq j} \{d(J_i, J_j)\} > 0.$$

Therefore, if $z_1, z_2 \in I$ and $d(z_1, z_2) < C$ then there exists an $i \in \{1, 2, ..., n+1\}$ such that $I_i \ni z_1, z_2$.

Let $\chi : I \to I$ be defined as

$$\chi(z) = \begin{cases} \chi_i(z), & z \in I_i \setminus I_{i+1}, i \in 1, 2, ..., n, \\ \chi_1(\lambda^{-1} z), & z \in I_{n+1}. \end{cases}$$

By Schwarz Lemma each χ_i is expanding. Therefore χ is also expanding. Let $0 < \epsilon < C$ such that for any $a, b \in I$ there exists $N \in \mathbb{N}$ such that $d(\chi^N(a), \chi^N(b)) > \epsilon$.

Let $c = \text{diam}(I)/\epsilon$. Let $a, b \in I$ and let $N \in \mathbb{N}_0$ be the smallest non-negative integer such that $d(\chi^N(a), \chi^N(b)) > \epsilon$. Observe that for $0 \leq i < N$, $d(\chi^i(a), \chi^i(b)) < \epsilon$ and so for each i, $\chi^i(a), \chi^i(b)$ belong to some common I_j, $j \in 1, ..., n+1$. In this way,

$$\text{diam}([\chi^N(a), \chi^N(b)]) \leq \text{diam}(I) = c\epsilon < cd(\chi^N(a), \chi^N(b)).$$

Let χ^{-N} represent the branch sending $\chi^N(a), \chi^N(b)$ to a, b. Since this map is conformal or anti-conformal on U, it is η-qs on I, where η is independent of a, b or the choice of branch. Since

$$\operatorname{diam}([\chi^N(a), \chi^N(b)]) \leq cd(\chi^N(a), \chi^N(b)),$$

then for any $z \in [a, b]$,

$$d(\chi^N(z), \chi^N(b)) \leq cd(\chi^N(a), \chi^N(b)).$$

Therefore,

$$d(z, b) \leq \eta(c)|a - b|.$$

And so

$$\operatorname{diam}([a, b]) \leq 2\eta(c)|a - b|.$$

Hence, I is BT and so by Theorem 2.1.3, I is a quasi-arc. □

2.2 Quadratic-like maps

Recall from the introduction that a quadratic-like (ql) map is a degree 2 branched covering $g : U \to V$ where $U \Subset V$ are (open) topological disks. We define the *filled Julia set* of g,

$$\mathcal{K}(g) := \{z \in U \mid f^n(z) \in U \text{ for all } n \in \mathbb{N}\},$$

to be the set of non-escaping points and we define the Julia set, $\mathcal{J}(g) := \partial\mathcal{K}(g)$.

Definition 2.2.1. Two ql maps $g : U \to V$, $g' : U' \to V'$ are *hybrid equivalent* if they are conjugate under a qc map ϕ on neighborhoods of their filled Julia sets such that $\phi|_{\mathcal{K}(g)}$ is conformal.

The theory of quadratic-like maps, along with the following theorem were introduced and proved in [DH85], (see also [Lyu] or [McM96].)

Theorem 2.2.1 (The Straightening Theorem)**.** *Any ql map g is hybrid equivalent to a quadratic map $f_c : z \mapsto z^2 + c$. Moreover, if $\mathcal{K}(g)$ is connected then f_c is unique.*

2.3 Tuning basilicas

It is well known that there exists a unique decreasing sequence $(c_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$ such that the critical orbit of f_{c_n} is periodic of period 2^n and for $c \in [c_n, 0]$, if $z \mapsto z^2 + c$ has an attracting cycle then its period is at most 2^n. For each $n \in \mathbb{N}$, we define $B_n := \mathcal{J}(f_{c_n})$ to be the *n-th tuning of the basilica* where B_1 is the basilica itself. This sequence $(c_n)_{n\in\mathbb{N}_0}$ is convergent, see [McM96]. Define c_* as the limiting value. Then the *Feigenbaum quadratic* is defined as the map $z \mapsto z^2 + c_*$ and we denote it merely by f_c. As the next proposition from [Dou94] shows, as $n \to \infty$, $B_n \to \mathcal{J}_c$.

Proposition 2.3.1. *If $\mathcal{J}(f_c) = \mathcal{K}(f_c)$ then for any sequence $c_n \to c$, $\mathcal{J}(f_{c_n}) \to \mathcal{J}(f_c)$.*

f_c may be alternatively defined as the straightening of f to a quadratic. Therefore, by The Straightening Theorem, f and f_c are hybrid-equivalent and so $\mathcal{J}(f)$ and $\mathcal{J}(f_c)$ are qs-equivalent. Hence, they have canonically isomorphic quasisymmetric groups. Since f satisfies (1.0.1), we prove most of our results using f instead of f_{c_*} as its scaling properties make it easier to use.

2.4 Böttcher coordinate

Since f_c is a quadratic, we have the following theorem which uniformizes $\mathbb{C} \setminus \mathcal{J}_c$, the basin of infinity of f_c, by conjugating it with the squaring map on the exterior of the closed disk. This uniformization of the basin of ∞ by the exterior of the closed disk is known as the *Böttcher coordinate*. For a proof, see [Lyu] or [Mil06].

Theorem 2.4.1. *There exists a conformal map $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathcal{J}_c$ such that*

$$f_c(\psi_c(w)) = \psi_c(w^2). \tag{2.4.1}$$

Since $\mathcal{J}_c$ is locally connected by Theorem 3.2.2, the map ψ_c extends continuously to a continuous map $\psi_c : \mathbb{T} \to \mathcal{J}_c$ satisfying (2.4.1). Since f and f_c are hybrid conjugate, we can also define the Böttcher coordinate for $\mathcal{J}$ as the lift to $\mathbb{T}$ under $\psi := h \circ \psi_c$ where h is the qc map conjugating f and f_c.

Chapter 3

Structures within $\mathcal{J}$

3.1 Notation

- $v := f(0)$ which is also known as the critical value.

- α refers to the fixed point which is the landing point of the $1/3, 2/3$ external rays. β refers to the fixed point which is the landing point of the 0 ray. By (1.0.1), for $k \in \mathbb{N}$, $\alpha_k := (-\mu)^{-k}\alpha$ is a periodic point of period 2^k.

- An α-point, β-point, or α_k-point is a pre-image of α, β, or α_k, respectively, under f^n for some $n \in \mathbb{N}_0$.

- For $A, B, C \in \mathbb{R}_{>0}$ we say $A \cong_C B$ if $C^{-1}B \leq A \leq CB$.

- The set of pre-critical points is denoted

$$PCP := \{c \in \mathcal{J} \mid f^n(c) = 0, \text{ some } n \in \mathbb{N}_0\}.$$

  Note that for convenience, we include the critical point, 0, in the set of *pre*-critical points.

- Throughout the paper, $\mathbb{T} \subset \mathbb{C}$ denotes the unit circle and instead of writing $e^{2\pi i t}$ where $t \in \mathbb{R}$ for points in $\mathbb{T}$, we adopt the more concise notation $t \in [0, 1]/\sim$.

3.2 First properties

For simplicity, we previously defined $\mathcal{J}$ as the set of non-escaping points of $f : U \to V$. This is, however, the traditional way of defining the *filled* Julia set, denoted $\mathcal{K}(f)$. In this way, the Julia set would then be defined as $\mathcal{J}(f) := \partial\mathcal{K}(f)$. We can get away with this simpler definition of $\mathcal{J}$ because of the following proposition which is proved in [McM94].

Proposition 3.2.1. *Since f is infinitely renormalizable, $\mathcal{J}(f) = \mathcal{K}(f)$.*

The following theorem is due to Jiang and Hu [JH93].

Theorem 3.2.2. *$\mathcal{J}$ is locally connected.*

A *path* between points z_1, z_2 is understood to be the image of a continuous mapping $p : [0, 1] \to \mathbb{C}$ such that $p(0) = z_1$ and $p(1) = z_2$. We say a path between z_1 and z_2 is an *arc* if the path is injective. We say a path connected set is *uniquely arc-wise connected* if for any two points z_1, z_2 in the set, there is a unique arc connecting them.

Proposition 3.2.3. *$\mathcal{J}$ is uniquely arc-wise connected.*

Proof. $\mathcal{J}$ is arc-wise connected by Theorem 31.2 of [Wil70] which says that a compact, connected, locally connected metric space is arc-wise connected. Since $\mathcal{J} = \mathcal{K}$, by the maximum modulus principle $\mathcal{J}$ is simply connected. Therefore, since $\mathcal{J}$ has no interior, $\mathcal{J}$ is uniquely arc-wise connected. □

The following theorem is proven in [Mil06].

Theorem 3.2.4. *Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree at least 2. For any $z \in \mathcal{J}(p) \subset \mathbb{C}$ and any neighborhood $N \ni z$, the union of forward images of N under p contains all but at most 1 point (in $\mathbb{C}$).*

A point $b \in X$ is said to be a *branch point* if $X \setminus \{b\}$ has at least 3 connected components. We say a branch point, b, is *of degree d* if $X \setminus \{b\}$ has d connected components. A set X is said to be *tetravalent* if for any branch point b, the degree of b is 4.

Lemma 3.2.5. *$\mathcal{J} \setminus \{v\}$ has exactly 2 connected components.*

Proof. By Lemma 3.2.3, since $v \in [-\beta, \beta] \subset \mathbb{R}$, $\mathcal{J} \setminus \{v\}$ has $d \geq 2$ connected components. We know there exists 2 distinct components: one containing $[-\beta, v)$ and one containing $(v, \beta]$. Suppose v is a branch point of degree $d \geq 3$ and let X be one of the components that does not intersect $\mathbb{R}$. Since 0 is the unique critical point and is not periodic, for all $n \geq 0$, the degree of $f^n(v)$ is equal to the degree of v. Since $\mathcal{J} \cap \mathbb{R}$ is forward invariant, by taking a sufficiently small neighborhood of v, we find that the two connected components of $\mathcal{J} \setminus \{f^n(v)\}$ containing $\mathcal{J} \cap \mathbb{R} \setminus \{f^n(v)\}$ must correspond to the two components of $\mathcal{J} \setminus \{v\}$ that intersected $\mathbb{R}$. Hence, $f^n(X) \cap \mathbb{R} = \emptyset$ for all $n \geq 0$. Since X is a connected component of $\mathcal{J} \setminus \{v\}$, there exists an open set $U \supset X$ such that $U \cap \mathcal{J} \setminus X = \emptyset$. By the invariance of $\mathcal{J}$ under f, U also satisfies the property that $f^n(U) \cap (\mathcal{J} \cap \mathbb{R}) = \emptyset$ for all $n \in \mathbb{N}$. This is a contradiction by Theorem 3.2.4. □

Corollary 3.2.6. *$\mathcal{J} \setminus \{0\}$ has 4 connected components.*

Proof. Since $\mathcal{J} \setminus \{v\}$ has 2 connected components, there are exactly 2 external rays that land at v. Since v is the critical point, this means there are 4 rays that land at 0. These rays cut the plane into 4 sectors. Hence, $\mathcal{J} \setminus \P\{0\}$ has 4 connected components. □

Lemma 3.2.7. *$\mathcal{J}$ is tetravalent and the only branch points are pre-critical points.*

Proof. Let $b \in \mathcal{J}$ be a branch point. By Corollary 3.2.6 if b is a pre-image of 0 then degree$(b) = 4$. Suppose b is not a pre-image of 0. Then degree$(b) =$ degree$(f^n(b))$ for all $n \geq 0$. By Theorem 3.2.4 combined with invariance of $\mathcal{J}$ under f, every component of $\mathcal{J} \setminus \{b\}$ must eventually intersect $\mathbb{R}$. Since $f(\mathbb{R}) \subset \mathbb{R}$ this means that there exists an $N \in \mathbb{N}$ such that for every component X of $\mathcal{J} \setminus \{b\}$, $f^N(X) \cap \mathbb{R} \neq \emptyset$. Since for all n, $\mathcal{J} \setminus \{f^n(b)\}$ can have at most two components that intersect $\mathbb{R}$ we have a contradiction. Hence, the only branch points of $\mathcal{J}$ are 0 and pre-images of 0. □

Lemma 3.2.8. *Every $x \in (-\beta, \beta)$ is a cut-point of $\mathcal{J}$.*

Proof. We already know that if x is a pre-critical point then it is a degree 4 branch point, and hence a cut-point.

Since $\psi : \mathbb{T} \to \mathcal{J}$ is surjective, there exists an external ray R_θ that lands at x. Since $\mathcal{J}$ is symmetric about $\mathbb{R}$, $R_{-\theta}$ also lands at x. Since the arc $R_\theta \cup \{x\} \cup R_{-\theta}$ separates the plane into two pieces and only intersects $\mathcal{J}$ at x, x is a cut-point. □

Corollary 3.2.9. *For any $k > 0$, any iterated pre-image of $(-\mu)^{-k}\beta$ is a non-branching cut-point of $\mathcal{J}$.*

Proof. By Lemma 3.2.8, $(-\mu)^{-k}\beta$ is a cut-point. Since $(-\mu)^{-k}\beta$ is not in the post-critical set, all pre-images of $(-\mu)^{-k}\beta$ are still cut-points.

□

3.3 Limbs

Let X be the component of $\mathcal{J} \setminus \{0\}$ that intersects $i\mathbb{R}_+$. Define $L_\uparrow := X \cup \{0\}$. We say a subset $L \subset \mathcal{J}$ is a *(quadratic) limb* if L is an iterated pre-image of $\pm L_\uparrow$ under f. The *spine* of $L_\uparrow$ is

$$\mathrm{spine}(L_\uparrow) := i\mathbb{R} \cap L_\uparrow$$

and more generally if L is any limb and $n \in \mathbb{N}_0$ is such that $f^n(L) = \pm L_\uparrow$ then

$$\mathrm{spine}(L) := \{z \in L \mid f^n(z) \in i\mathbb{R}\}.$$

For both pre-critical points and limbs we use the term *generation* to describe their dynamical distance to 0 or $\pm L_\uparrow$. That is, for a limb L, $\mathrm{Gen}(L) = n$ if $f^n(L) = \pm L_\uparrow$ and for a pre-critical point c, $\mathrm{Gen}(c) = n$ if $f^n(c) = 0$.

The following lemma follows directly from the definition of a limb.

Corollary 3.3.1. *For every pre-critical point, $c \in PCP$, c is the root of exactly two limbs: L_c, L'_c.*

Proof. This follows immediately from Lemma 3.2.7 and the definition of a limb. $\square$

Since $\mathcal{J}$ is tetravalent and uniquely arc-wise connected, for every point $z \in \mathcal{J}$ we may define the *level of* z in the following inductive manner:

- If $z \in \mathcal{J} \cap \mathbb{R}$, then $\text{level}(z) = 0$.
- If $z \notin \mathbb{R}$ and $z \in \text{spine}(L)$ for some limb L that is rooted in $\mathbb{R}$, then $\text{level}(z) = 1$.
- More generally, if $z \in \text{spine}(L_c) \setminus \{c\}$ for some limb rooted at c and $\text{level}(c) = n$ then $\text{level}(z) = n + 1$.
- If z does not belong to any spine in $\mathcal{J}$ then $\text{level}(z) = \infty$.

We say that an arc p has a *turn* at a point c if:

1. $c \in \text{Int}(p)$ is a pre-critical point, and
2. The two components of $\mathcal{J} \setminus \{c\}$ that p intersects are not opposite of each other. That is, under a rotational ordering of the components of $\mathcal{J} \setminus \{c\}$, the two components that intersect p are adjacent.

This definition of levels coincides with the intuitive notion of the number of turns in the unique path from 0 to z in $\mathcal{J}$.

For $L \subset \mathcal{J}$ a limb rooted at c, define $|L| := t_2 - t_1$ where $0 < t_1 < t_2 < 1 \in \mathbb{T}$ such that $\psi(t_1) = \psi(t_2) = c$. In other words, $|L|$ is the size of L in the Böttcher coordinate.

Lemma 3.3.2. *Given two distinct limbs L_1, L_2 rooted at c_1, c_2 respectively such that $c_1 \neq c_2$. If $L_1 \cap L_2 \neq \emptyset$, then either $L_1 \subset L_2$ or $L_2 \subset L_1$.*

Proof. If $c_2 \in L_1$ then $c_1 \in [0, c_2]_{\mathcal{J}}$. Since only one of the 4 components of $\mathcal{J} \setminus \{c_2\}$ can contain c_1, the others are contained in L_1. Since the component that contains c_1 also intersects $\mathbb{R}$, it is not a limb. Hence, the limbs rooted at c_2 are contained in L_1.

If instead $c_2 \notin L_1$, then for $z \in L_1 \cap L_2$, $[c_2, z] \ni c_1$ as any path going from outside the limb to inside the limb must pass through the root. Hence, $c_1 \in L_2$. By the same argument as in the previous case, this means that $L_1 \subset L_2$. □

Lemma 3.3.3. *There is a unique pre-critical point $c_1 \in L_\uparrow \setminus \{0\}$ of generation 3. Every other pre-critical point in $L_\uparrow \setminus \{0\}$ is of greater generation.*

Proof. Let $x_1, -x_1$ be the two pre-images of 0 under f in $\mathcal{J}$ where $x_1 > 0$. By Lemma 4.3.2, it suffices to show that this is true along $(0, \beta_\uparrow] := \text{spine}(L_\uparrow) \setminus \{0\}$. That is, $\beta_\uparrow \in i\mathbb{R}_+$ is defined to be the tip of $\text{spine}(L_\uparrow)$.

We can do this easily by observing the dynamics. $f((0, \beta_\uparrow]) = [-\beta, v) \not\ni 0$. Since $f|_{[0,\beta]}$ and $f|_{[-\beta,0]}$ are both injective, $f([-\beta, v)) = (f(v), \beta]$. Since $v < -x_1$, $0 < f(v) < x_1$. Hence, $f^2((0, \beta_\uparrow]) = (f(v), \beta] \not\ni 0$. Therefore, $f^3((0, \beta_\uparrow]) = f((f(v), \beta]) = (f^2(v), \beta]$. Since $0 < f(v) < x_1$ and $f(x_1) = 0$, by injectivity $f^2(v) < 0$. Hence, $f^3((0, \beta_\uparrow]) = (f^2(v), \beta] \ni 0$ while for $n = 0, 1, 2$, $f^n((0, \beta_\uparrow]) \not\ni 0$. □

Lemma 3.3.4. *For any $\epsilon > 0$ there are only finitely many limbs whose diameter is at least ϵ.*

Proof. Let $(L_n)_{n\in\mathbb{N}}$ be any sequence of limbs. Since $\mathcal{J}$ is compact, the limbs L_n converge on a subsequence. Since there are only finitely many limbs

We prove this by contradiction. Suppose there exists an $\epsilon > 0$ and infinitely many limbs with diameter at least ϵ. Thus, there exists a sequence of limbs L_n such that $\text{diam}(L_n) > \epsilon$ and the dynamical height of L_n is at least n. Since $\mathcal{J}$ is compact, there exists a subsequence n_k such that $L_{n_k} \to Y$ such that Y is connected and $\text{diam}(Y) \geq \epsilon$. However, $\psi_{-1}(L_n) \subset \mathbb{T}$ is an interval and $\text{diam}(\psi_{-1}(L_n)) \leq C \cdot 2^{-n}$ for some $C > 0$ that doesn't depend on n. Hence, $\psi^{-1}(L_{n_k})$ goes to a singleton as $n_k \to \infty$. But since $\psi : \mathbb{T} \to \mathcal{J}$ is well-defined, this implies Y is a singleton, a contradiction. □

Corollary 3.3.5. *If $(L_n)_{n\in\mathbb{N}}$ is a sequence of limbs such that $L_n \supsetneq L_{n+1}$ then $\bigcap_{n\in\mathbb{N}}$ is a singleton.*

Proof. By Lemma 3.3.4, $\mathrm{diam}(L_n) \to 0$ as $n \to \infty$. Hence, $\mathrm{diam}(\bigcap_{n\in\mathbb{N}} L_n) = 0$. Furthermore, since each L_n is compact and non-empty, $\bigcap_{n\in\mathbb{N}} L_n$ is non-empty. Therefore, it is a singleton. □

Let

$$X_s = (\mathcal{J} \cap \mathbb{R}) \cup \bigcup_{c\in PCP} \mathrm{spine}(L_c) \cup \mathrm{spine}(L'_c)$$

where L_c, L'_c denote the two limbs rooted at the pre-critical point c.

Proposition 3.3.6. *$\mathcal{J} \setminus X_s$ is totally disconnected.*

Proof. Let $\{L_n\}_{n\in\mathbb{N}}$ be a sequence of limbs such that $L_n \supsetneq L_{n+1}$. Then $\bigcap_{n\in\mathbb{N}} L_n$ is compact and non-empty. Let $z \in \bigcap_{n\in\mathbb{N}} L_n$. Then $z \notin \mathrm{spine}(L)$ for any limb L because otherwise z could only belong to finitely many limbs. Hence, $z \in \mathcal{J} \setminus X_s$ and so $\mathcal{J} \setminus X_s$ is non-empty.

Let $z \in \mathcal{J} \setminus X_s$ and, by Proposition 3.2.3, let p be the unique arc from 0 to z in $\mathcal{J}$. Since $z \notin X_s$, p must have infinitely many turns. Otherwise z would lie in $\mathbb{R}$ or $\mathrm{spine}(L)$ for some limb L. Hence, z lies in a decreasing nested sequence of limbs.

Let $z_1 \neq z_2 \in \mathcal{J} \setminus X_s$. Since z_1, z_2 belong to different decreasing nested sequences of limbs, there exists a limb $L_1 \ni z_1$ such that $L_1 \not\ni z_2$ and let c_1 denote the root of L_1. Let $\theta_1, \theta'_1 \in \mathbb{T}$ such that $\psi^{-1}(L_1) = [\theta_1, \theta'_1]$. Let $R_{\theta_1}, R_{\theta'_1}$ be the corresponding external rays. Then since $R_{\theta_1} \cup \{c_1\} \cup R_{\theta'_1} \cap \mathcal{J} = \{c_1\}$, it separates z_1 from z_2 in X_s, hence they lie in different connected components □

Corollary 3.3.7. $\mathcal{J} = \overline{X_s}$.

Proof. If $w \in \mathcal{J} \setminus X_s$ then for all n, $f^n(w) \in \mathcal{J} \setminus X_s$ since spines of limbs map either to spines of limbs or to $\mathcal{J} \cap \mathbb{R}$. Therefore if $z \in X_s$ and $w \in \mathcal{J}$ such that $f^n(w) = z$ then $w \in X_s$. Since the set of all iterated pre-images of z is dense in $\mathcal{J}$, X_s is dense in $\mathcal{J}$. □

3.4 Important Points

There are some points in $\mathcal{J}$ (or in $\mu^2\mathcal{J}$) that will be of use in later sections which we will define here. To see these points drawn on $\mathcal{J}_\infty$, see Figure 4.3.

- x_1 is defined to be the unique positive pre-image of 0 under f and will be particularly relevant throughout Chapters 4 and 5.

- By Lemma 3.3.3 there is a unique point, c_1, of generation 3 in $L_\uparrow$. For convenience, define $y_1 := \mu^2 c_1$.

- By an analogous argument to that of Lemma 3.3.3 there is a unique pre-critical point of generation 5 separating 0 and c_1. Define $y_2 := \mu^2 c_2$.

3.5 Little Julia sets

Within $\mathcal{J}$ are little copies of itself. Recall from the introducation that a *little Julia set of depth* $k \geq 0$, $J \subset \mathcal{J}$, is a pre-image of $\mu^{-k}\mathcal{J}$ under f^n for some $n \geq 0$. As we will see, these little Julia sets – and the variations of them which we are about to define – play an important role in understanding the group of quasisymmetries of $\mathcal{J}$.

Let J be a little Julia set of depth k and let $n \in \mathbb{N}_0$ such that $f^n(J) = \mu^{-k}\mathcal{J}$. The *spine of* J, denoted spine(J), is the set $I \subset J$ such that $f^n(I) = \mu^{-k}[-\beta, \beta]$. We say a set $I \subset \mathcal{J}$ is a *1-dimensional (1d) little Julia set of depth* k if $I = \text{spine}(J)$ for some little Julia set, J, of depth k. A 1d little Julia set is called a *patriarch* if it is not contained in any other 1d little Julia set. The term patriarch would have little meaning for regular little Julia sets because the only little Julia set not contained in any other is $\mathcal{J}$ itself.

Given J, the *extended little Julia set* containing J is

$$\widehat{J} := \text{spine}(J) \cup \bigcup_{c \in PCP \cap \text{spine}(J)} (L_c \cup L'_c)$$

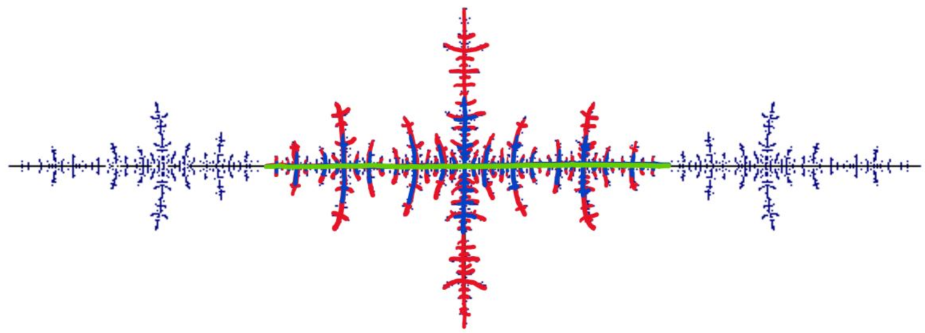

Figure 3.1: For the little Julia set, $J := \mu^{-1}\mathcal{J}$. Depicted is $\widehat{J} \supset J \supset \mathrm{spine}(J)$ in red, blue, and green, respectively.

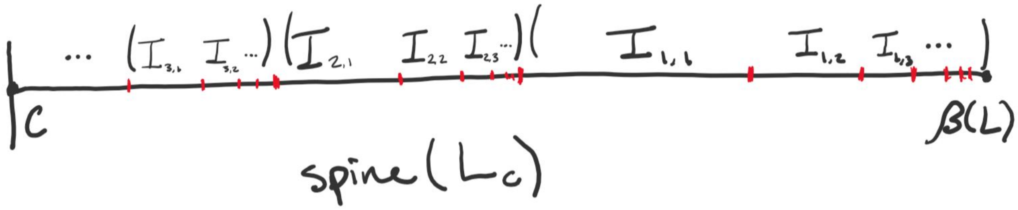

Figure 3.2: The spine of every limb consists of the root, the tip, and a countable union of patriarchal 1d little Julia sets. $I_{i,j}$ is the j-th patriarchal 1d little Julia set of depth i from the root.

(i.e. $\widehat{J}$ extends J by taking the full limbs rooted in spine(J), as opposed to trimming off the "decorations" as would be done to achieve J.)

Given a little Julia set, J, of depth k. A *child* of J is a little Julia set, $J_c \subset J$ of depth $k+1$. Similarly, a *spine child* of J, is a child of $\mathcal{J}$, J_c, such that $\mathrm{spine}(J_c) \subset \mathrm{spine}(J)$. Analogously, a *child* of a 1d little Julia set, I, of depth k is a 1d little Julia set, I_c of depth $k+1$, such that $I_c \subset I$. It is worth noting that there is a natural indexing by $\mathbb{Z}$ of the spine children of any little Julia set. This indexing corresponds to the ordering of the real bounded Fatou components of the Basilica from left to right, such that the 0-th component contains 0.

Lemma 3.5.1. *For any limb L_c rooted at c,*

$$\mathrm{spine}(L_c) = \{c\} \cup \bigcup_{i,j \in \mathbb{N}} I_{i,j}$$

where $I_{i,j}$ is the j-th closest 1d little Julia set of depth i to the root of L. Furthermore, $I_{i,j}$ is a patriarch 1d little Julia set. (See Figure 3.2.)

Proof. Since every limb is a pre-image of $\pm L_\uparrow$, it suffices to show that this is the case for $L_\uparrow$.

$\mu^{-1}\beta(L_\uparrow)$ is the α-point in $\mathrm{spine}(L_\uparrow)$ closest to 0. Since two consecutive α-points in $\mathrm{spine}(L_\uparrow)$ form the endpoints of a 1d little Julia set of depth 1, the α-point in $\mathrm{spine}(L_\uparrow)$ closest to 0 is also the closer to 0 of the two endpoints of $I_{1,1}$. Since α-points accumulate on $\beta(L_\uparrow)$ in $\mathrm{spine}(L_\uparrow)$, it follows that

$$\bigcup_{j\in\mathbb{N}} I_{1,j} = [\mu^{-1}\beta(L_\uparrow), \beta(L_\uparrow)).$$

Since $\mu^{-i+1}I_{1,j}$ is the j-th closest 1d little Julia set of depth i to $\mu^{-i}\beta(L_\uparrow)$ in $\mathrm{spine}(L_\uparrow)$, it follows that it is also the closest one to 0 as there can be no 1d little Julia sets of depth i in $\mathrm{spine}(L_\uparrow) \cap \mu^{-i}\mathcal{J} = (0, \mu^{-i}\beta(L_\uparrow)]_{\mathcal{J}}$. Hence, $I_{i,j} = \mu^{-i+1}I_{1,j}$. As such,

$$\bigcup_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} I_{i,j} = \bigcup_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} \mu^{-i+1}I_{1,j} \tag{3.5.1}$$

$$= \bigcup_{i\in\mathbb{N}} \mu^{-i+1}[\mu^{-1}\beta(L_\uparrow), \beta(L_\uparrow)) \tag{3.5.2}$$

$$= (0, \beta(L_\uparrow)). \tag{3.5.3}$$

Since $[-\beta, \beta]$ is the unique 1d little Julia set of depth 0, any 1d little Julia set not contained in $\mathbb{R}$ is a patriarch 1d little Julia set. Hence, $I_{1,j}$ is a patriarch 1d little Julia set for all j. Similarly, $\mu^{-i}[-\beta, \beta]$ is the unique 1d little Julia set of depth i in $\mu^{-i}\mathcal{J}$. Since $I_{i,j}$ lies closer to 0 than any 1d little Julia set of depth $i-1$, for any $i > 1$, $I_{i,j}$ is also a patriarch 1d little Julia set. □

Corollary 3.5.2. *If I is a patriarch 1d little Julia set then $I = [-\beta, \beta]$ or $I \subset \mathrm{spine}(L)$ for some limb L in which case it can be characterized as the j-th closest 1d little Julia set of depth i to the root of L among those in* $\mathrm{spine}(L)$*.*

Proof. By Proposition 3.3.6, there are no 1d little Julia sets outside of $\mathcal{J} \cap \mathbb{R}$ and the spines of limbs. If $I \neq [-\beta, \beta]$ then by Lemma 3.5.1, $I \subset \mathrm{spine}(L) = \{c\} \cup \bigcup_{i,j\in\mathbb{N}} I_{i,j}$ where $I_{i,j}$ is the j-th closest 1d little Julia set of depth i to the root of L. Since I is a patriarch 1d little Julia set, $I = I_{i,j} \subset \text{spine}(L)$ for some $i, j \in \mathbb{N}$. $\square$

Corollary 3.5.3. *If c is a pre-critical point and $I \ni c$ is a 1d little Julia set then for L_c a limb rooted at c, $I \cap L_c = \{c\}$.*

Proof. Suppose $I \ni c$ is a 1d little Julia set. Let $I_p \supset I$ be the patriarch 1d little Julia set containing I. Let $X \ni c$ be a spine. By Corollary 3.5.2, $I_p \subset X$. Hence, $I_p \cap L_c = \{c\}$ and so $I \cap L_c = \{c\}$.

$\square$

Lemma 3.5.4. *I is a 1d little Julia set if and only if it is the closure of the union of a unique 2-sided sequence of distinct 1d little Julia sets such that consecutive elements overlap at an end-point.*

Proof. We begin by proving the forward direction. Consider first the depth 0 1d little Julia set, $I_0 = [-\beta, \beta]$. For $j \geq 0$, define $I_{j,0} := f^{-j}([\alpha, -\alpha])$ where the branch of f^{-j} is chosen such that it fixes β. For $j < 0$, define $I_{j,0} := f^{-j}([\alpha, -\alpha])$ where the branch of f^{-j} is chosen such that it sends β to $-\beta$. In this setting $\bigcup_{j \in \mathbb{Z}} I_{j,0} = (-\beta, \beta)$, I_0 is the closure of the union and consecutive elements overlap at a point.

Now consider the general case where I is a 1d little Julia set of depth k. Let n such that $f^n(I) = \mu^{-k}[-\beta, \beta]$. Let f^{-n} be the branch such that $f^{-n}(\mu^{-k} I_0) = I$. Define $I_j := f^{-n}(\mu^{-k} I_{j,0})$. This sequence satisfies both desired properties: its union is a dense subset of I and consecutive elements overlap at a point.

For the backwards direction, suppose $(I'_j)_{j \in \mathbb{Z}}$ is a 2-sided sequence of distinct 1d little Julia sets such that consecutive elements overlap at a point. Since the end-points of a 1d little Julia set of depth k are pre-images of $(-\mu)^{-k}\beta$, the assumption that consecutive elements overlap at an end-point implies that each 1d little Julia set is of the same depth. Let $k := \text{depth}(I'_j)$ for all $j \in \mathbb{Z}$. By Corollary 3.5.2, the elements of this sequence are not patriarch 1d little Julia sets. Therefore, let $I_p \supsetneq I'_0$ be a 1d little Julia set of depth 1 less than I'_0. Let n be the dynamical height of I_p. Then $f^n(I'_0) = \mu^{-k} I_{i,0}$ for some $i \in \mathbb{Z}$. Therefore, $f^n(I'_j) = \mu^{-k} I_{i+j,0}$ for all $j \in \mathbb{Z}$ or $f^n(I'_j) = \mu^{-k} I_{i-j,0}$ for all $j \in \mathbb{Z}$. Hence, $\overline{\bigcup_{j\in\mathbb{Z}} I'_j} = I_p$.

□

Corollary 3.5.5. *Every 1d little Julia set is uniquely determined by the patriarch 1d little Julia set it is contained in as well as an element of $\mathbb{Z}^n$ for some $n \in \mathbb{N}_0$ describing where amongst the descendants of the patriarch it is located.*

Proof. By Lemma 3.5.4, the children of a 1d little Julia set are indexed by $\mathbb{Z}$. Since each of their children are also indexed by $\mathbb{Z}$, and so on, an element of $\mathbb{Z}^n$ for some $n \in \mathbb{N}_0$ can specify where amongst a patriarch 1d little Julia set's descendants a given 1d little Julia set lies. □

3.6 More properties of $\mathcal{J}$

Note that the term *spine* is used to refer to subsets of both little Julia sets and limbs. We say a spine is *maximal* if it is not contained in any other spine.

Lemma 3.6.1. *If X is a maximal spine then $X = \mathcal{J} \cap \mathbb{R}$ or $X = \mathrm{spine}(L)$ for some limb L.*

Proof. If $X = \mathcal{J} \cap \mathbb{R}$ or $X = \mathrm{spine}(L)$ for some limb L then X is maximal. If X is any other spine then X is the spine of a little Julia set of depth at least 1 and so is a 1d little Julia set. If X is not a patriarchal 1d little Julia set then X is not a maximal spine. If it is a patriarch 1d little Julia set then by Corollary 3.5.2 $X \subset \mathrm{spine}(L)$ for some limb L. □

Lemma 3.6.2. *Let X be a maximal spine. Then the set of pre-critical points in X is dense in X.*

Proof. We prove this by contradiction in the context of $(f_c, \mathcal{J}_c)$ using the fact that pre-critical points are dense in $\mathcal{J}_c$, (see Corollary 4.13 of [Mil06]). By the conjugacy of $(f, \mathcal{J})$ and $(f_c, \mathcal{J}_c)$ the result also holds for $\mathcal{J}$.

Suppose there is some open interval $I \subset X$ such that I contains no pre-critical points. Since we may further restrict I if necessary, we assume WLOG that the end-points of I are also not pre-critical points and that neither are endpoints of X. By Lemma 3.2.8, every point in X other than its end-points are cut-points of $\mathcal{J}_c$. By Lemma 3.2.7, the only branch points of $\mathcal{J}_c$ are pre-critical points. Therefore, every point in I is a non-branching cut-point. Since I is an open interval and since its end-points are non-branching cut-points, $\mathcal{J}_c \setminus I$ has 2 connected components, both of which are closed in $\mathbb{C}$. Hence, $\mathbb{C} \setminus (\mathcal{J}_c \setminus I)$ is an open neighborhood of I that contains no pre-critical points: a contradiction. $\square$

Chapter 4

Transcendental dynamics of the renormalization fixed point

4.1 Definition of f as a transcendental map onto $\mathbb{C}$

Recall $f : U_0 \to V_0$ is the period-doubling renormalization fixed point which satisfies both (1.0.1) and $\mu^{-1}\mathcal{J} \subset \mathcal{J}$.

By (1.0.1), we may define $f^{1/2}(z) := \mu f(\mu^{-1}z)$ where $f^{1/2} : \mu U_0 \to \mu V_0$ is a rescaled quadratic-like map. (See Figure 4.2.) Letting $U_1 := f^{-2\cdot 1/2}(\mu V_0)$ we get a degree 4 branched covering $f : U_1 \to \mu V_0$. Similarly, for $n \in \mathbb{N}$, we may define the ql map $f^{1/2^n} : \mu^n U_0 \to \mu^n V_0$ by $f^{1/2^n}(z) := \mu^n f(\mu^{-n}z)$. Letting $U_n := f^{-2^n \cdot 1/2^n}(\mu^n V_0)$, we get that $f : U_n \to \mu^n V_0$ is a degree 2^n branched covering. Defining $\Omega_1 := \bigcup_{n\in\mathbb{N}} U_n$, we get a ($\sigma$-proper) transcendental map $f : \Omega_1 \to \mathbb{C}$. This construction can be extended to find the maximal domain of f^s for any dyadic $s > 0$, which will be denoted Ω_s.

Two useful objects of study are

$$\mathcal{J}_\infty := \bigcup_{n\in\mathbb{N}} \mu^n \mathcal{J}$$

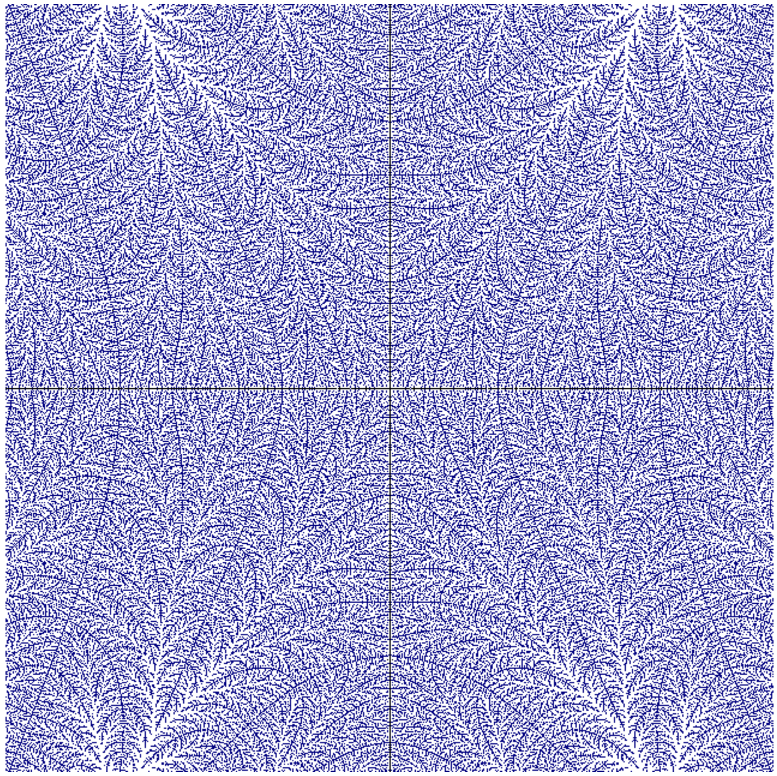

Figure 4.1: Drawn is a zoomed in picture of $\mathcal{J}_c$, which gives a close approximation of the structure of $\mathcal{J}_\infty$

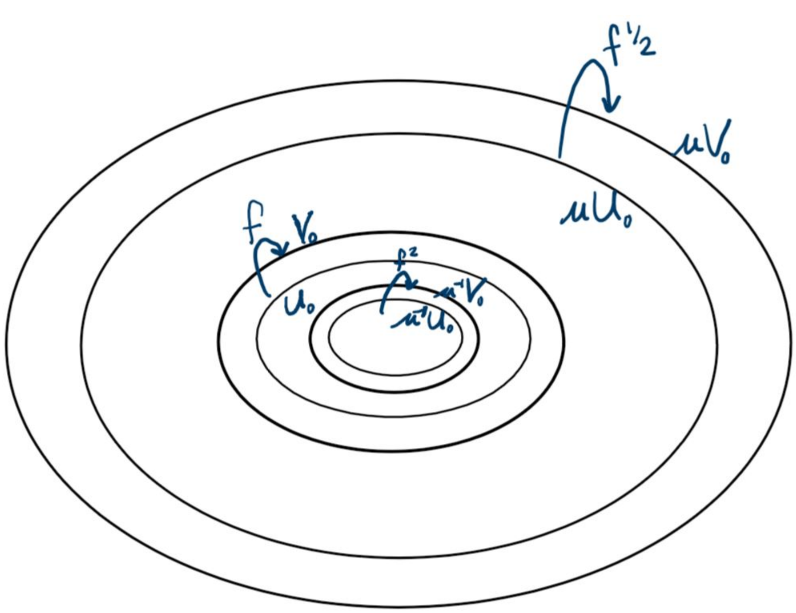

Figure 4.2: Re-scalings of the quadratic-like map f used to build the transcendental dynamics of $f : \Omega_1 \to \mathbb{C}$

and

$$\mathrm{Esc} = \bigcup_{t>0} \mathrm{Esc}_t$$

where t is dyadic and $\mathrm{Esc}_t := \mathbb{C} \setminus \mathrm{Dom}(f^t)$.

A proof of the following proposition can be found in [McM96]:

Proposition 4.1.1. *$\mathcal{J}_\infty$ is dense in $\mathbb{C}$.*

Corollary 4.1.2. *Each component of* Esc *is simply connected and has empty interior. Furthermore, for any two points $z_1, z_2 \in \mathrm{Esc}$, there exists at most 1 arc connecting them.*

Proof. Since $\mathcal{J}_\infty$ is dense by Proposition 4.1.1 and since $\mathrm{Esc} \subset \mathbb{C} \setminus \mathcal{J}_\infty$, Esc has empty interior. Since $\mathcal{J}_\infty$ is also connected, Esc cannot contain any non-contractible loops. That is, if $p_1, p_2 \subset \mathrm{Esc}$ are two arcs with the same starting and ending point, then $p_1 = p_2$ since otherwise they would bound some open region that would be disjoint from $\mathcal{J}_\infty$. □

Since f is an even function, we can define F to be the function satisfying $f(z) = F(z^2)$. The following proposition is due to Henri Epstein, see [Eps92] or [Buf97].

Proposition 4.1.3. *There exists a real-symmetric bounded domain $W \ni 0$ such that*

- *$F|_W$ is univalent*
- *$F(W) = \mathbb{C} \setminus ((-\infty, v] \cup [-\mu v, +\infty))$.*

4.2 First properties of $\mathcal{J}_\infty$

Lemma 4.2.1. *$\mathcal{J}_\infty$ has empty interior.*

Proof. Since $\mathcal{J}$ has empty interior, this follows as a direct consequence of Baire Category Theorem. □

The following is a corollary of Lemma 3.2.3

Corollary 4.2.2. *$\mathcal{J}_\infty$ is uniquely arc-wise connected.*

Proof. Let $z_1, z_2 \in \mathcal{J}_\infty$. There exists an N such that $z_1, z_2 \in \mu^N \mathcal{J}$. By Lemma 3.2.3, there exists a unique arc $\gamma = [z_1, z_2]_{\mu^N \mathcal{J}} \subset \mu^N \mathcal{J}$ between z_1 and z_2. Let γ' be any arc between z_1 and z_2 in $\mathcal{J}_\infty$. If $\gamma \neq \gamma'$ then these paths will bound a planar region, U, which by maximum modulus principle must be contained in $\mathcal{J}_\infty$. However, by Lemma 4.2.1 this is a contradiction. $\square$

Lemma 4.2.3. *Removing any pre-critical point separates $\mathcal{J}_\infty$ into 4 path connected components. Moreover, any point c such that $\mathcal{J}_\infty \setminus \{c\}$ has at least 3 path connected components is a branch point of $\mu^N \mathcal{J}$ for some $N \in \mathbb{N}$.*

Proof. Let $N \in \mathbb{N}$ such that c is a branch point of $\mu^N \mathcal{J}$. By Lemma 3.2.7, for any $M \geq N$, $\mu^M \mathcal{J} \setminus \{c\}$ has 4 connected components since c is also a branch point of $\mu^M \mathcal{J}$. For $z_1, z_2 \in \mu^M \mathcal{J}$, being in the same component of $\mu^M \mathcal{J} \setminus \{c\}$ is equivalent to the path $[z_1, z_2]_{\mu^M \mathcal{J}} \not\ni c$. Therefore, by Corollary 4.2.2, if $z_1, z_2 \in \mu^M \mathcal{J}$ then they are in different path connected components of $\mu^M \mathcal{J} \setminus \{c\}$ if and only if they are in different path components of $\mathcal{J}_\infty \setminus \{c\}$. Hence, $\mathcal{J}_\infty \setminus \{c\}$ has 4 path connected components.

Let $b \in \mathcal{J}_\infty$ such that $\mathcal{J}_\infty \setminus \{b\}$ has at least 3 path connected components. For sake of contradiction, suppose b is not a branch point for $\mu^n \mathcal{J}$ for any n. Let $z_1, z_2, z_3 \in \mathcal{J}_\infty \setminus \{b\}$ belong to different path connected components. Let M such that $z_1, z_2, z_3, b \in \mu^M \mathcal{J}$. Since b is not a branch point of $\mu^M \mathcal{J}$ we may assume WLOG that z_1, z_2 are in the same connected component of $\mu^M \mathcal{J} \setminus \{b\}$. Hence, $[z_1, z_2]_{\mu^M \mathcal{J}} \not\ni b$. However, since z_1, z_2 are in *different* components of $\mathcal{J}_\infty \setminus \{b\}$ any path between them in $\mathcal{J}_\infty$ must pass through b. This contradicts the fact that $[z_1, z_2]_{\mu^M \mathcal{J}} = [z_1, z_2]_{\mathcal{J}_\infty}$. Hence, each branch point of $\mathcal{J}_\infty$ is a branch point of $\mu^n \mathcal{J}$ for some n.

Hence, as in the case for $\mathcal{J}$, every such point $b \in \mathcal{J}_\infty$ is a pre-image of 0, and so its removal separates $\mathcal{J}_\infty$ into 4 path connected components. $\square$

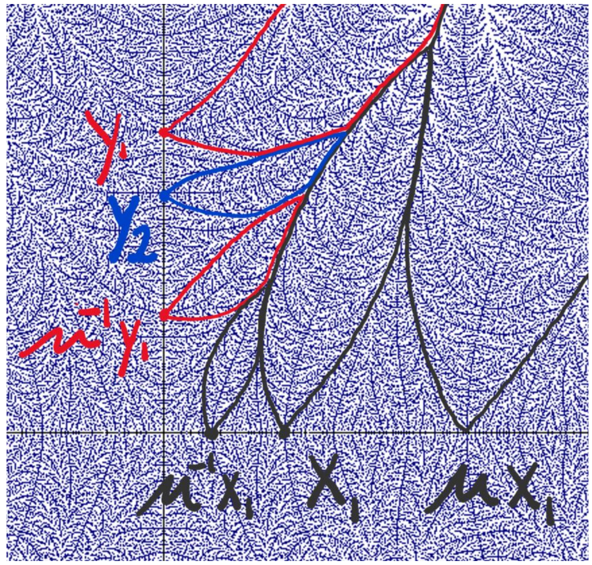

Figure 4.3: Seen is a computer drawing of an approximation of $\mathcal{J}_\infty$ with the boundaries of the closures of limbs rooted at $\mu^{-1}x_1, x_1, \mu x_1, \mu^{-1}y_1, y_2$, and y_1 added by hand for emphasis.

Remark 4.2.4. Actually, Lemma 4.2.3 can be strengthened to the stronger result that $\mathcal{J}_\infty$ is tetravalent. This can be done by identifying certain arcs in Esc landing at a given pre-critical point c such that removing these arcs and c cuts the plane into 4 open sets.

4.3 Limbs

Let X be the connected component of $\mathcal{J}_\infty \setminus \{0\}$ containing $i\mathbb{R}_{>0}$. Define $\mathcal{L}_\uparrow := X \cup \{0\}$. We say $\mathcal{L}$ is a *limb (of generation t)* if it is a pre-image of $\pm\mathcal{L}_\uparrow$ under f^t for some dyadic $t \geq 0$. A limb of generation t, $\mathcal{L}$, is said to be *rooted at c* if $c \in \mathcal{L}$ and $f^t(c) = 0$. Two (disjoint) limbs $\mathcal{L}_1, \mathcal{L}_2$ are said to *touch* if $\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2} \neq \emptyset$. We define the *spine* of $\mathcal{L}_\uparrow$ to be

$$\text{spine}(\mathcal{L}_\uparrow) := i\mathbb{R}_{\geq 0} \subset \mathcal{L}_\uparrow.$$

The spine of a general limb is defined in the natural way in terms of pre-images of spine$(\mathcal{L}_\uparrow)$. More generally, we say a set $X \subset \mathcal{J}_\infty$ is a *spine* if $X = \mathbb{R}$ or if $X = \text{spine}(\mathcal{L})$ for some limb $\mathcal{L}$. The spines in $\mathcal{J}_\infty$ are analogous to the maximal spines of $\mathcal{J}$. For examples of some limbs, see Figure 4.3.

Proposition 4.3.1. $\pm\mathcal{L}_\uparrow$ *are the only unbounded limbs.*

Proof. Let W be the bounded domain from Proposition 4.1.3 such that $F|_W$ is univalent onto $\mathbb{C} \setminus ((-\infty, v] \cup [-\mu v, +\infty))$. Let $c \neq 0$ be a pre-critical point. By rescaling if necessary we may assume WLOG that $c^2 \in W$. Therefore, $f(c) \in \mathbb{C} \setminus ((-\infty, v] \cup [-\mu v, +\infty))$. Since the branch of f^{-1} that sends $f(c)$ to c maps $\mathbb{C} \setminus ((-\infty, v] \cup [-\mu v, +\infty))$ to a bounded set (by Proposition 4.1.3), the branch of f^{-1} maps the (transcendental) limbs at $f(c)$ to the limbs at c. Hence, we get that the limbs rooted at c are bounded. $\square$

Lemma 4.3.2. *Let $c \neq c'$ be pre-critical points such that $c' \in \mathcal{L}_c$ where $\mathcal{L}_c$ is a limb rooted at c. Then* $\mathrm{Gen}(c') > \mathrm{Gen}(c)$.

Proof. Let $s = \mathrm{Gen}(c)$. Then $f^s(c') \in f^s(\mathcal{L}_c) = \pm\mathcal{L}_\uparrow$. Since the post-critical set is contained in $\mathbb{R}$, $f^s(c')$ is not in the post-critical set. Hence, $\mathrm{Gen}(c') > s$. $\square$

Lemma 4.3.3. *Let $\mathcal{L}_1, \mathcal{L}_2$ be limbs of generation at least s. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are touching then $f^s(\mathcal{L}_1), f^s(\mathcal{L}_2)$ are touching.*

Proof. This is a direct consequence of continuity of f^s on its domain. $\square$

Corollary 4.3.4. *If $\mathcal{L}_1, \mathcal{L}_2$ are limbs of the same generation, then $\mathcal{L}_1, \mathcal{L}_2$ do not touch.*

Proof. This follows from Lemma 4.3.3 because $\pm\mathcal{L}_\uparrow$ are the only limbs of generation 0 and they do not touch. $\square$

Lemma 4.3.5. *For any limb $\mathcal{L}$ rooted at c, $\mathcal{J}_\infty \setminus \{c\}$ has 4 connected components, one of which is $\mathcal{L} \setminus \{c\}$.*

Proof. Let $s = \mathrm{Gen}(c)$. Since $\mathcal{J}_\infty$ is connected and c is a pre-image of 0, which is not in the post-critical set, there exists a neighborhood of c, U_c, such that $f^s|_{U_c}$ is conformal. Since 0 is a branch point of degree 4, c is also a branch point of degree 4. $\square$

Corollary 4.3.6. *Let $\mathcal{L}$ be a limb. Any path in $\mathcal{J}_\infty$ between a pair of points $z_1 \in \mathcal{J}_\infty \setminus \mathcal{L}$ and $z_2 \in \mathcal{L}$ must pass through the root of $\mathcal{L}$.*

Proposition 4.3.7. *Given two limbs $\mathcal{L}_1, \mathcal{L}_2$. If $\mathcal{L}_1 \cap \mathcal{L}_2 \neq \emptyset$ then $\mathcal{L}_1 \subset \mathcal{L}_2$, $\mathcal{L}_2 \subset \mathcal{L}_1$, or $\mathcal{L}_1 \cap \mathcal{L}_2 = \{c\}$ where c is their common root.*

Proof. The proof is the same as for Lemma 3.3.2. $\square$

Corollary 4.3.8. *If $X_1 \neq X_2$ are spines, then X_1, X_2 are either disjoint or overlap on one of their endpoints.*

Proof. Since $X_1 \neq X_2$, we may assume WLOG that $X_2 = \mathrm{spine}(\mathcal{L}_2)$ for some limb $\mathcal{L}_2$. If $X_1 = \mathbb{R}$ then $X_1 \cap X_2 = \{c_2\}$ where c_2 is the root of $\mathcal{L}_2$ if and only if $c_2 \in \mathbb{R}$. If $X_1 \neq \mathbb{R}$ then $X_1 = \mathrm{spine}(\mathcal{L}_1)$ for some limb $\mathcal{L}_1$ rooted at c_1. If $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$ then $X_1 \cap X_2 = \emptyset$. If not, then by Proposition 4.3.7, $\mathcal{L}_1 \supset \mathcal{L}_2$ or vice-versa. For concreteness, assume $\mathcal{L}_1 \supset \mathcal{L}_2$. Then $\mathcal{L}_2$ is rooted in X_1 if and only if $X_1 \cap X_2 = \{c_2\}$. If $\mathcal{L}_2$ is *not* rooted in X_1 then $X_1 \cap X_2 = \emptyset$. $\square$

Corollary 4.3.9. *For every pre-critical point c, there is a unique maximal spine X_c such that $c \in \mathrm{Int}(X_c)$.*

Proof. If X is a maximal spine and $c \in \mathrm{Int}(X)$ then by Corollary 4.3.8, X is unique. Let $N \in \mathbb{N}$ such that $c \in \mu^N \mathcal{J}$. Since the number of turns away from $\mathbb{R}$ is bounded by $2^N \,\mathrm{Gen}(c)$, since each turn occurs at a pre-critical point, and since each pre-critical point is the root of a limb, and hence the endpoint of a spine, it follows that c belongs to a spine. $\square$

Lemma 4.3.10. *If X is a spine and $c \in \mathrm{Int}(X)$ is a pre-critical point of generation $s > 0$ then there exists $c_1, c_2 \in X$ such that $\mathrm{Gen}(c_1), \mathrm{Gen}(c_2) < s$ and $[c_1, c_2]_{\mathcal{J}_\infty} \ni c$.*

Proof. If $X = \mathbb{R}$, then we may choose $c_1, c_2 = 0, \mu c$. If $X = i\mathbb{R}_{\geq 0}$ or $i\mathbb{R}_{\leq 0}$ then we may again choose $c_1, c_2 = 0, \mu c$. If not, then X is some pre-image of $i\mathbb{R}_{\geq 0}$ or $i\mathbb{R}_{\leq 0}$ and so the result also holds for general X. $\square$

Lemma 4.3.11. *For any bounded limb, $\mathcal{L}$, there exists s dyadic and $k \in \mathbb{Z}$ such that $f^s(\mathcal{L})$ is a limb rooted at $\pm\mu^k x_1$.*

Proof. Recall that $\pm x_1$ are the only two points in $\mathcal{J}$ that map to 0 under f. Let c be the root of $\mathcal{L}$. Since $c \in \mathcal{J}_\infty$, $c \in \mu^k \mathcal{J}$ for some $k \in \mathbb{Z}$. Hence, for some s, $f^s(c) = \pm\mu^k x_1$. The statement of the lemma follows. □

Recall the points x_1, y_1, and y_2 from Section 3.4.

Lemma 4.3.12. *For any $k \in \mathbb{Z}$,*

1. *$\mu^k x_1$ is the unique point of least generation in $(0, \mu^{k+1}x_1) \subset \mathbb{R}$.*
2. *$\mu^k y_1$ is the unique point of least generation in $(0, \mu^{k+1}y_2) \subset i\mathbb{R}$.*
3. *$\mu^k y_2$ is the unique point of least generation in $(0, \mu^k y_1) \subset i\mathbb{R}$.*

Proof. By (1.0.1) it suffices to prove each for a single choice of k.

1. We want to show that $\mu^{-1}x_1$ is the point of least generation in $(0, x_1)$. By (1.0.1), $\mathrm{Gen}(\mu^{-1}x_1) = 2\,\mathrm{Gen}(x_1) = 2$. Since $(0, x_1) \subset \mathcal{J}$ and there are only 7 points of generation less than or equal to 2 in $\mathcal{J}$ – $0, \pm x_1, \pm\mu^{-1}x_1, \pm f^{-1}(x_1)$ – we see the statement holds by examining where each of them lies.

2. For the case $k = -3$. $\mu^{-3}y_1, \mu^{-2}y_2 \in \mathcal{J}$ and it is simple to verify that there is no point of generation less than or equal to $6 = \mathrm{Gen}(\mu^{-3}y_1)$ in $(0, \mu^{-2}y_2)$. This has been partially done already in Lemma 3.3.3 where it is shown that $\mu^{-2}y_1$ is the point of least generation in $i\mathbb{R}_+ \cap \mathcal{J}$.

3. For this one, it is similarly simple to verify for the case $k = -2$.

□

We have the following immediate corollary.

Corollary 4.3.13. *1. The points $\{\pm\mu^k x_1 \mid k \in \mathbb{Z}\}$ are exactly the points in $\mathbb{R}$ that cannot be separated from 0 by a point of lesser generation.*

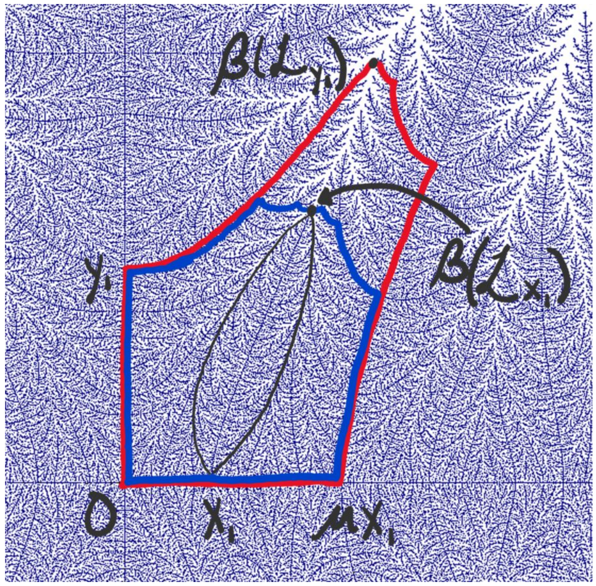

Figure 4.4: We see in blue $\partial^c\mathbb{O}(\mathcal{L}_{x_1})$, the coast of the lake associated to $\mathcal{L}_{x_1}$, and in red $\partial\mathbb{O}_\rho(\mathcal{L}_{y_1})$, the boundary of the right half-lake associated to $\mathcal{L}_{y_1}$

2. *The points $\{\pm\mu^k y_1, \pm\mu^k y_2 \mid k \in \mathbb{Z}\}$ are exactly the points in $i\mathbb{R}$ that cannot be separated from 0 by a point of lesser generation.*

4.4 Lakes

For any s, $\mathrm{Dom}(f^s)$ has a certain "chessboard" type structure, as introduced in [Buf97]. We use here a slightly more general approach to demonstrate this structure via the following lemma, which is taken from [DL18].

Lemma 4.4.1. *Let $g : \mathrm{Dom}\, g \to \mathbb{C}$ be a σ-proper map, where $\mathrm{Dom}\, g$ is either $\mathbb{D}$ or $\mathbb{C}$. Suppose that the set of critical values $CV(g)$ of g is discrete and assume that $l : \mathbb{R} \to \mathbb{C}$ is a simple properly embedded arc such that*

1. *$l(\mathbb{R}) \supset CV(g)$, and*

2. *l splits $\mathbb{C}$ into two open half-planes V and W.*

Then

- $g^{-1}(l)$ *is a tree in* $\operatorname{Dom} g$*; in particular, if* U *is a connected component of* $\operatorname{Dom} g \setminus g^1(l)$*, then* U *has a single access to infinity; and*

- *there is a "chess-board rule": if* U_1 *and* U_2 *are two different components of* $g^1(V)$*, then* $\partial U_1 \cap \partial U_2 \cap \operatorname{Dom} g$ *is either empty or a single critical point of* g*.*

The "squares" of the chess board cut out by $f^{-s}(\mathbb{R})$ are what we call *lakes*. That is, a *lake of generation* s is a pre-image of $\mathbb{H}_+$ or $\mathbb{H}_-$ under f^s. There is a natural correspondence between limbs and lakes. Let $\mathcal{L}$ be a limb of generation s. Then $\mathbb{O}(\mathcal{L})$ is defined to be the unique lake of generation s containing $\mathcal{L}$. Given a lake $\mathbb{O}$ of generation s, its *coast* is

$$\partial^c\mathbb{O} := \partial\mathbb{O} \cap \operatorname{Dom}(f^s).$$

The following proposition is also from [DL18] and follows from Lemma 4.4.1. Using slightly different language, it was also known by Epstein, see [Eps92] or [Buf97].

Proposition 4.4.2. *Let* $\mathbb{O}$ *be a lake of generation* s*.*

- $\mathbb{O}$ *has a single access to* Esc *(i.e. after identifying* $\mathbb{C} \setminus \mathrm{Esc}_s \cong \mathbb{D}$*)*

- $f^s : \mathbb{O} \to \mathbb{H}$ *is conformal*

- $\partial^c\mathbb{O}$ *is an arc in* $f^{-s}(\mathbb{R})$

- $f^s : \partial^c\mathbb{O} \to \mathbb{R}$ *is a homeomorphism*

Corollary 4.4.3. *Let* $\mathbb{O}$ *be a lake of generation* s*. Then there is a unique* $c \in \partial^c\mathbb{O}$ *of generation* s*.*

Proof. By Proposition 4.4.2, $f^s|_{\partial^c\mathbb{O}}$ is a homeomorphism and 0 is the unique point of generation 0 in $\mathcal{J}_\infty$ (and hence, in $\mathbb{R}$.) Therefore c is the unique point in $\partial^c\mathbb{O}$ of generation s. □

Corollary 4.4.4. *Let* $\mathbb{O} = \mathbb{O}(\mathcal{L}_c)$ *be a lake of generation* $s > 0$*.* $\partial^c_\lambda\mathbb{O}, \partial^c_\rho\mathbb{O}$ *are two paths in* $\mathcal{J}_\infty$ *from* c *that limit on* $\beta(\mathcal{L}_c)$ *with infinitely many turns such that each turn is the first encountered pre-critical point* c' *such that* $\operatorname{Gen}(c') < s$*.*

Proof. There are two parts to the above statement: first, that following the coast from c in either direction has infinitely many turns, and second, that these paths limit on a single point.

By 4.3.9, let X_0 be the unique spine containing c in its interior. By Lemma 4.3.10 there exist points c_0', c_1' of generation less than s on either side of c in X_0. Since $[c_0', c_1'] \subset \mu^N \mathcal{J}$ for some N and since points of generation less than s are discrete in $\mu^N \mathcal{J}$, we can define c_0, c_1 to be the *closest* pre-critical points of generation less than s in X_0 that are separated by c. It follows that $\partial^c \mathbb{O}(\mathcal{L}_c) \cap X_0 = [c_0, c_1]_{\mathcal{J}_\infty}$.

Since c_1 is a critical point of f^s, $\partial^c \mathbb{O}$ has a turn at c_1. Let $X_1 \ni c_1$ be the spine that $\partial^c \mathbb{O}$ turns on to when traveling away from c. If $c_1 \in \mathrm{Int}(X_1)$ then by the same argument as for X_0, there exists a closest point of generation less than s in X_1, c_2, for which $\partial^c \mathbb{O} \cap X_1 = [c_1, c_2]_{\mathcal{J}_\infty}$. If instead, c_1 is the root of X_1, then there exist points in X_1 of generation arbitrarily close to $\mathrm{Gen}(c_1)$. Since $\mathrm{Gen}(c_1) < s$, there exist points in X_1 of generation less than s. In particular, there is a closest point (to c_1) of generation less than s, c_2, for which $\partial^c \mathbb{O} \cap X_1 = [c_1, c_2]_{\mathcal{J}_\infty}$.

Continuing in this way, we find that the path from c starting in the direction of c_1 and going along $\partial^c \mathbb{O}$ has infinitely many turns. Since the path in the direction of c_0 is analogous, it also has infinitely many turns.

□

Within a lake $\mathbb{O}$ of generation s, we define the *right and left half-lakes of generation* s, denoted, $\mathbb{O}_\rho, \mathbb{O}_\lambda$, to be the subsets of $\mathbb{O}$ such that $f^s(\mathbb{O}_\rho)$ equals Q_I or Q_{III} and $f^s(\mathbb{O}_\lambda)$ equals Q_{II} or Q_{IV}. That is, while lakes are pre-images of the upper or lower half-planes, *half-lakes* are pre-images of quadrants. The definition of right and left half-lakes is such that if $\mathbb{O} = \mathbb{O}(\mathcal{L})$ then the right half-lake of $\mathbb{O}$ is the portion of $\mathbb{O}$ lying to the right of spine($\mathcal{L}$) – when viewing spine($\mathcal{L}$) as an oriented path from the root of $\mathcal{L}$ to $\beta(\mathcal{L})$ – while the left half-lake lies to the left.

By Proposition 4.4.2, we may define simple paths

$$\partial_\lambda^c \mathbb{O} := \partial^c \mathbb{O} \cap \partial \mathbb{O}_\lambda,$$

$$\partial^c_\rho \mathbb{O} := \partial^c \mathbb{O} \cap \partial \mathbb{O}_\rho.$$

From these definitions, it follows that

$$\partial^c_\lambda \mathbb{O} \cup \partial^c_\rho \mathbb{O} = \partial^c \mathbb{O}$$

and

$$\partial^c_\lambda \mathbb{O} \cap \partial^c_\rho \mathbb{O} = \{c\}.$$

Lemma 4.4.5. *Every lake is contained in either the upper or lower half-plane.*

Proof. Since $\mathbb{R}$ is forward-invariant, $\mathbb{C} \setminus \mathbb{R}$ is backwards invariant. Since every lake is defined as a pre-image of $\mathbb{H}_+$ or $\mathbb{H}_-$, every lake is disjoint from $\mathbb{R}$ and so by connectedness is contained in either $\mathbb{H}_+$ or $\mathbb{H}_-$. □

Lemma 4.4.6. *If $\mathbb{O}_1, \mathbb{O}_2$ are lakes such that $\mathbb{O}_1 \cap \mathbb{O}_2 \neq \emptyset$ then $\mathbb{O}_1 \subset \mathbb{O}_2$ or $\mathbb{O}_2 \subset \mathbb{O}_1$.*

Proof. Let $s_1 = \mathrm{Gen}(\mathbb{O}_1)$ and $s_2 = \mathrm{Gen}(\mathbb{O}_2)$. WLOG, suppose $s_1 \leq s_2$. Then $\mathbb{H} = f^{s_1}(\mathbb{O}_1)$ is the upper or lower half-plane and $f^{s_1}(\mathbb{O}_2)$ is a lake of generation $s_2 - s_1$. Since $f^{s_1}(\mathbb{O}_2) \cap \mathbb{H} \neq \emptyset$, by Lemma 4.4.5, $f^{s_1}(\mathbb{O}_2) \subset \mathbb{H}$. Hence, $\mathbb{O}_2 \subset \mathbb{O}_1$. □

Lemma 4.4.7. *Let $s > t > 0$. For any limb, $\mathcal{L}$, of generation s, there is a unique lake of generation t, $\mathbb{O}_t$ such that $\mathbb{O}_t \supset \mathcal{L}$.*

Proof. Since all the lakes of generation t are disjoint, it suffices to show that there exists one that contains $\mathcal{L}$. Since $s > t$, $f^t(\mathcal{L})$ is a bounded limb contained in $\mathbb{H}_+$ or $\mathbb{H}_-$. Hence, there is a lake of generation t containing $\mathcal{L}$. □

Lemma 4.4.8. *Let $\mathcal{L}$ be a limb rooted at c such that $\mathrm{Gen}(c) = s > 0$. If $c \in \mathbb{R}$, let $X := \mathbb{R}$. Otherwise, let $X := \mathrm{spine}(\mathcal{L}_p)$ such that $\mathrm{Int}(X) \ni c$*[1]*. Then*

$$\partial^c \mathbb{O}(\mathcal{L}) \cap X = [c_1, c_2]_{\mathcal{J}_\infty}$$

where c_1, c_2 are the closest pre-critical points of generation less than s in X.

1

[1] $\mathrm{Int}(X)$ refers to the 1-dimensional interior.

Proof. By Proposition 4.4.2, $f^s : \partial^c\mathbb{O}(\mathcal{L}) \to \mathbb{R}$ is a homeomorphism. Since $f^s|_{(c_1,c_2)_{\mathcal{J}_\infty}}$ has no critical points $\partial^c\mathbb{O}(\mathcal{L})$ cannot leave X before leaving $(c_1, c_2)_{\mathcal{J}_\infty}$. On the other hand, since c_1, c_2 are critical points of f^s, $\partial^c\mathbb{O}(\mathcal{L})$ cannot extend past c_1, c_2 in X because this would break injectivity of $f^s|_{\partial^c\mathbb{O}(\mathcal{L})}$. □

4.5 β-points

In the quadratic setting, β-points are pre-images of the β-fixed point and are also the ends of the spines of quadratic limbs. We now define an analog for the transcendental setting in which β-points here are again the ends of the spines of limbs and can also be viewed as pre-images of ∞.

Lemma 4.5.1. *For any bounded limb, $\mathcal{L}_c$, $\beta(\mathcal{L}_c) := \overline{\mathrm{spine}(\mathcal{L})} \setminus \mathrm{spine}(\mathcal{L})$ is a singleton.*[2]

Proof. By Lemma 4.3.11 and by the self-similarity of $\mathcal{J}_\infty$ under scaling by μ, it suffices to show this is true for $\mathcal{L}_{-x_1} \subset \mathbb{H}_+$. By Lemma 4.4.7, let $\mathbb{O}_{3/4}$ be the lake of generation 3/4 containing $\mathcal{L}_{-x_1}$. We use the following map

$$\chi_{-x_1} = f^3 \circ \mu^{-2} : \mathbb{O}_{3/4} \to \mathbb{H}_+. \tag{4.5.1}$$

Note that by (1.0.1), scaling by μ^{-2} sends the lake of generation 3/4, to a lake of generation 3. It is a simple matter of verifying that $f^3(-\mu^{-2}x_1) = -x_1$ and that the image is $\mathbb{H}_+$ as opposed to $\mathbb{H}_-$. Therefore this map is conformal. Since $\mathbb{O}_{3/4} \subset \mathbb{H}_+$, it is expanding in the hyperbolic metric on $\mathbb{H}_+$. Since $\chi_{-x_1}(\mathcal{L}_{-x_1}) = \mathcal{L}_{-x_1}$ the set $\beta(\mathcal{L}_{-x_1})$ is invariant under χ_{-x_1}, an expanding map. Hence, it is a singleton. □

[2]Without using this language, this lemma follows from work of Epstein, see [Eps92], who knew that as $z \to \infty$ in $\mathbb{H}_+$, $F^{-1}(z)$ tended to a single point.

4.6 Tree order on Esc

Borrowing notation from [DL18], we define the partial order "$\succ$" as the *tree order* on β-points. We write $\beta_v \succ \beta_w$ and $\beta_w = \beta_v \wedge \beta_w$ if $\beta_v \in \mathbb{O}(\beta_w)$. Note that $\beta_v \succ \beta_w$ implies $\mathrm{Gen}(\beta_v) > \mathrm{Gen}(\beta_w)$. Whenever $\beta_v \succ \beta_w$, we may define the *external chain* as

$$[\beta_w, \beta_v] := \mathrm{Esc}_{\mathrm{Gen}(\beta_v)} \cap \overline{\mathbb{O}(\beta_w)} \setminus \bigcup_{\mathbb{O}^\iota(\beta_s) \not\ni \beta_v} \mathbb{O}^\iota(\beta_s)$$

where $\iota \in \{\rho, \lambda\}$ and β_s is any β-point. In other words we remove any pieces of Esc not lying between β_v and β_w. From the definition, it is clear that for $\beta_v \prec \beta_w$, $[\beta_v, \beta_w]$ is connected. We will see in the next section that it is an arc parametrized by escaping time.

If $\beta_v \in \mathbb{O}_+(\beta_x)$ and $\beta_w \in \mathbb{O}_-(\beta_x)$ then we say that β_v and β_w are $\prec$-separated by β_x and write $\beta_x = \beta_v \wedge \beta_w$.

Lemma 4.6.1. *For any $\beta_v \succ \beta_w \succ \beta_s$,*

- $[\beta_s, \beta_w] \cap [\beta_w, \beta_v] = \{\beta_w\}$,
- $[\beta_s, \beta_w] \cup [\beta_w, \beta_v] = [\beta_s, \beta_v]$.

Proof. Since β_w is a branch point of Esc it is a cut-point of $[\beta_s, \beta_v]$. Hence, $[\beta_s, \beta_w] \cap [\beta_w, \beta_v] = \{\beta_w\}$.

For the second part, let $t = \mathrm{Gen}(\beta_v)$. Then

$$[\beta_s, \beta_w] = \mathrm{Esc}_t \cap \overline{\mathbb{O}(\beta_s)} \setminus \bigcup_{\mathbb{O}^\iota(\beta_u) \not\ni \beta_w} \mathbb{O}^\iota(\beta_u)$$

because all the half-lakes of generation at least $\mathrm{Gen}(\beta_w)$ are removed because they do not contain β_w. Hence, $[\beta_s, \beta_w] \cup [\beta_w, \beta_v] \subseteq [\beta_s, \beta_v]$. This argument also shows the other direction because it shows that if $z \in [\beta_s, \beta_v] \setminus [\beta_w, \beta_v]$ then $z \in [\beta_s, \beta_w]$. $\square$

4.7 qs structure in Esc and $\mathcal{J}_\infty$

Define $\beta_0 := \beta(\mathcal{L}_{x_1}) \in Q_I$. For $k \in \mathbb{Z}$, define $\beta_k := \mu^{-k}\beta_0$. This notation is such that $\mathrm{Gen}(\beta_k) = 2^k$.

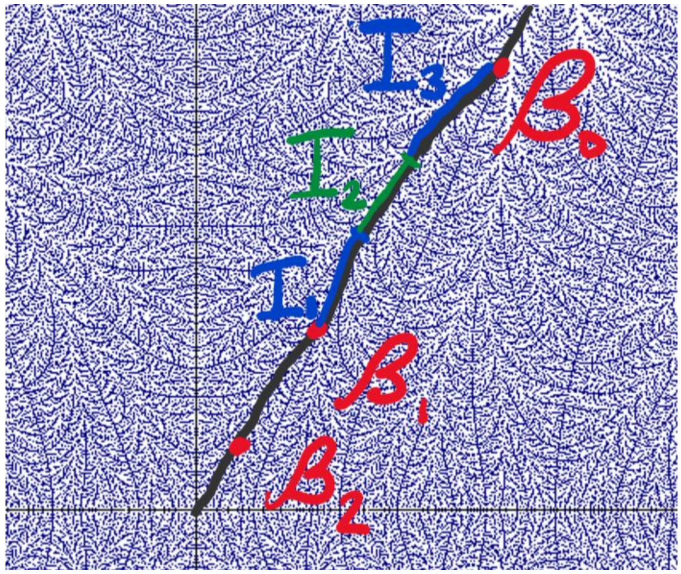

Figure 4.5: Drawn in black is the ray R_I^0, with the points $\beta_0, \beta_1, \beta_2$ labeled in red, and the level 1 tiles I_1, I_2, I_3 in blue and green.

Lemma 4.7.1. *For $k < l$, $\beta_k \prec \beta_l$.*

Proof. By Lemma 4.4.8,

$$\partial^c \mathbb{O}(\beta_k) \cap \mathbb{R} = [0, \mu^{-k+1} x_1] \ni \mu^{-l} x_1.$$

Hence, $\mathcal{L}_{\mu^{-l} x_1} \subset \mathbb{O}(\beta_k)$ and so

$$\beta(\mathcal{L}_{\mu^{-l} x_1}) = \beta_l \in \mathbb{O}(\beta_k).$$

□

By Lemma 4.7.1, we may define R_I^0 in the following way:

$$R_I^0 := \cdots \cup [\beta_{-1}, \beta_0] \cup [\beta_0, \beta_1] \cup [\beta_1, \beta_2] \cup \cdots .$$

For $j \in \{II, III, IV\}$ define $R_j^0 \subset Q_j^0$ accordingly by reflections of R_I^0 across $\mathbb{R}$ and $i\mathbb{R}$.

Lemma 4.7.2. *For any $\beta_v \prec \beta_w$ with $\mathrm{Gen}(\beta_v) = s > 0$. Then the following hold:*

1. $\mathrm{Gen}(\mu^k \beta_w) - \mathrm{Gen}(\mu^k \beta_v) = 2^{-k}(\mathrm{Gen}(\beta_w) - \mathrm{Gen}(\beta_v))$

2. *For $t < s$,* $\mathrm{Gen}(f^t(\beta_w)) - \mathrm{Gen}(f^t(\beta_v)) = \mathrm{Gen}(\beta_w) - \mathrm{Gen}(\beta_v)$

Proof. The first property is a direct consequence of (1.0.1). The second property is also straightforward since for any point z with escaping time s, then for $t < s$, $f^t(z)$ has escaping time $s - t$. $\square$

Lemma 4.7.3. *For $j \in \{I, II, III, IV\}$, R_j^0 is an arc, continuously parametrized by escaping time. That is, for each j, there is a homeomorphism $h_j : (0, \infty) \to R_j^0$ such that* $\mathrm{Gen}(h_j(t)) =$ *t.*

Proof. By (1.0.1) and Lemma 4.7.2, it suffices to show that $I := [\beta_0, \beta_1]$ is continuously parametrized by escaping time. To achieve this, we will define a Markov partitioning $I = I_1 \cup I_2 \cup I_3$ and a map $\chi : I \to I$ such that $\chi(I_i) = I$ for $i = 1, 2, 3$ such that for any two points $a, b \in I$ and any branch of the three branches of χ^{-1},

$$|\mathrm{Gen}(\chi^{-1}(a)) - \mathrm{Gen}(\chi^{-1}(b))| \leq 2^{-1}|\mathrm{Gen}(a) - \mathrm{Gen}(b)|.$$

The maps we'll use are as follows:

$$\chi_1 := \mu^{-1} \circ -\overline{f} : \mathbb{O}(\mathcal{L}_{x_1}) \to \mathbb{H}_+ \tag{4.7.1}$$

$$\chi_2 := \mu^{-2} \circ -\overline{f} : \mathbb{O}(\mathcal{L}_{x_1}) \to \mathbb{H}_+ \tag{4.7.2}$$

$$\chi_3 := -f^3 \circ \mu^{-2} : \mathbb{O}(\mathcal{L}_{\mu^2 y_1}) \to \mathbb{H}_+ \tag{4.7.3}$$

We now explain how for each $i = 1, 2, 3$, the maps χ_i can be joined together to define the map $\chi : I \to I$ with the aforementioned desired properties.

It is easy to verify that $\chi_1(\mu^{-1}x_1) = \mu^{-1}x_1$. Therefore, since χ_1 maps into $\mathbb{H}_+$, $\chi_1(\beta_1) = \beta_1$. Let $\mathcal{L}' := \chi_1^{-1}(\mathcal{L}_{x_1})$. Since $\beta_1 \in \mathbb{O}(\mathcal{L}_{x_1})$ and $\chi_1^{-1}(\beta_1) = \beta_1$, $\beta_1 \in \mathbb{O}(\mathcal{L}')$. Therefore, for $\beta' := \beta(\mathcal{L}')$, $\beta' \prec \beta_1$. By Lemma 4.7.2, $\mathrm{Gen}(\beta_1) - \mathrm{Gen}(\beta') = 2^{-1}(\mathrm{Gen}(\beta_0) - \mathrm{Gen}(\beta_1))$. Hence, $\mathrm{Gen}(\beta') = 1.5$. Since $\beta' \prec \beta_1$, $\beta_0 \prec \beta_1$ and $\mathrm{Gen}(\beta') > \mathrm{Gen}(\beta_0)$, $\beta_0 \prec \beta' \prec \beta_1$.

Since $\chi_1(\beta') = \beta_0$ and $\chi_2 = \mu^{-1} \circ \chi_1$, we have that $\chi_2(\beta') = \beta_1$. Let $\mathcal{L}'' := \chi_2^{-1}(\mathcal{L}_{x_1})$ and let $\beta'' := \beta(\mathcal{L}'') = \chi_2^{-1}(\beta_0)$. Since $\beta_0 \prec \beta_1$, $\chi(\beta'') = \beta_0$, and $\chi(\beta') = \beta_1$, it follows that $\beta'' \prec \beta'$. Using the same argument for finding $\mathrm{Gen}(\beta')$, we can conclude $\mathrm{Gen}(\beta'') = 1.25$.

It is easy to verify that $\chi_3(x_1) = x_1$. This, combined with the fact that χ_3 maps onto $\mathbb{H}_+$ implies that $\chi_3(\beta_0) = \beta_0$. Therefore $\chi_3(\mathbb{O}(\mathcal{L}_{x_1})) = \mathbb{O}(\mathcal{L}_{x_1})$. By Lemma 4.7.2, $\mathrm{Gen}(\chi_3^{-1}(\beta_1)) = 1.25$. One can verify by the real dynamics of f that

$$\chi_3([0, x_1]) = [-f^3(0), x_1] \subset (\mu^{-1}x_1, x_1].$$

By Lemma 4.4.4, $[0, x_1] = \partial^c\mathbb{O}(\mathcal{L}_{\mu^{-1}x_1})$. Hence, $\chi_3(\beta_1) \in \mathbb{O}(\mathcal{L}_{\mu^{-1}x_1})$. Since χ_3 sends lakes to lakes, this in turn implies that $\beta_1 \in \mathbb{O}(\chi_3^{-1}(\mathcal{L}_{\mu^{-1}x_1}))$ and so $\chi_3^{-1}(\beta_1) \prec \beta_1$. Since $\mathrm{Gen}(\beta'') = \mathrm{Gen}(\chi_3^{-1}(\beta_1))$ and since their lakes intersect – as they both contain β_1 – they must be the same point by Lemma 4.4.7.

In summary, $\beta_0 \prec \beta'' \prec \beta' \prec \beta_1$, so we have the following tiling:

$$I_1 := [\beta', \beta_1], \tag{4.7.4}$$

$$I_2 := [\beta'', \beta'], \tag{4.7.5}$$

$$I_3 := [\beta_0, \beta''], \tag{4.7.6}$$

satisfying $\chi_i(I_i) = I$. Let $\chi : I \to I$ be defined piece-wise by the maps χ_i. By Lemma 4.7.2, for any branch of χ^{-n}, $\mathrm{Gen}(\chi^{-n}(\beta_1)) - \mathrm{Gen}(\chi^{-n}(\beta_0)) \leq 2^{-n}$. Since each χ_i is expanding by Schwarz Lemma, we also have that the pre-images of I under χ^n shrink to points as n goes to ∞. Hence, I is an arc which is continuously parametrized by escaping time. $\square$

Proposition 4.7.4. *$R_I^0 \cup \{0\} \cup R_{II}^0$ and $R_{III}^0 \cup \{0\} \cup R_{IV}^0$ are quasiarcs.*

Proof. By symmetry and self-similarity under scaling by μ, it suffices to show that

$$J = R_I^0 \cup \{0\} \cup R_{II}^0 \cap (\overline{\mathbb{O}(\mathcal{L}_{x_1})} \cup \overline{\mathbb{O}(\mathcal{L}_{-x_1})})$$

is a quasiarc.

Let $\beta' \in R_I^0$ be the β-point of generation 1.5. (This corresponds with the definition of β' from the proof of Lemma 4.7.3.) Let $I' := [\mu\beta', \beta_1] \supset [\beta_0, \beta_1]$. For $i = 1, 2, 3$, let $I_i' := \chi_i^{-1}(I')$. By Lemma 2.1.6 applied to χ_i and I_i' for $i = 1, 2, 3$ and $I_4' := \mu I_1'$, I' is a quasiarc.

Let $A := I'$ and let B be the reflection of A about the imaginary axis. Then by Lemma 2.1.4, J is a quasi-arc.

$\square$

Corollary 4.7.5. $R_I^0 \cup \{0\} \cup R_{II}^0 = \partial\overline{\mathcal{L}_\uparrow}$.

Proof. By Lemma 4.7.3, $B = R_I^0 \cup \{0\} \cup R_{II}^0$ is an arc. Since R_I^0 tends to ∞ in quadrant I and R_{II}^0 is the reflection of R_I^0 about the imaginary axis, B separates $\mathbb{C}$ into two regions. Let U be the connected component of $\mathbb{C} \setminus B$ containing $i\mathbb{R}_+$. Since $B \cap \mathcal{J}_\infty = \{0\}$ and $U \supseteq i\mathbb{R}_+$, $\mathcal{L}_\uparrow \subseteq U$. Since $U \cap (\mathbb{R} \cup i\mathbb{R}_-) = \emptyset$, U does not contain any of the 3 other connected components of $\mathcal{J}_\infty \setminus \{0\}$. Hence, $(\mathcal{J}_\infty \setminus \mathcal{L}_\uparrow) \cap U = \emptyset$. Since $\mathcal{J}_\infty$ is dense (by Proposition 4.1.1), $\overline{\mathcal{L}_\uparrow} = \overline{U}$. $\square$

Corollary 4.7.6. *For any limb $\mathcal{L}_c$ rooted at c, $\partial\overline{\mathcal{L}_c} \setminus \{c\} \subset$ Esc.*

Proof. Let $s \geq 0$ such that $f^s(\mathcal{L}_c) = \pm\mathcal{L}_\uparrow$. To ease notation, assume $f^s(\mathcal{L}_c) = \mathcal{L}_\uparrow$. Since $f^s(\partial\overline{\mathcal{L}_c} \setminus \{\beta(\mathcal{L}_c)\}) = \partial\mathcal{L}_\uparrow$, the statement follows from Corollary 4.7.5. $\square$

Lemma 4.7.7. *For any two disjoint limbs $\mathcal{L}_1, \mathcal{L}_2$,*

$$\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2} = \partial\overline{\mathcal{L}_1} \cap \partial\overline{\mathcal{L}_2}.$$

Proof. It suffices to show that $\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2} \subseteq \partial\overline{\mathcal{L}_1} \cap \partial\overline{\mathcal{L}_2}$ since the other direction is immediate.

If $z \in \mathrm{Int}(\overline{\mathcal{L}_1})$ then there exists a neighborhood $N(z) \subseteq \mathrm{Int}(\overline{\mathcal{L}_1})$. By Corollary 4.7.6, $N(z) \cap \mathcal{L}_2 = \emptyset$. Hence, $\mathrm{Int}(\overline{\mathcal{L}_1}) \cap \overline{\mathcal{L}_2} = \emptyset$. Since the same argument works to show $\mathrm{Int}(\overline{\mathcal{L}_2}) \cap \overline{\mathcal{L}_1} = \emptyset$, it follows that

$$\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2} = (\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2}) \setminus (\mathrm{Int}(\overline{\mathcal{L}_1}) \cup \mathrm{Int}(\overline{\mathcal{L}_2})) = \partial\overline{\mathcal{L}_1} \cap \partial\overline{\mathcal{L}_2}.$$

$\square$

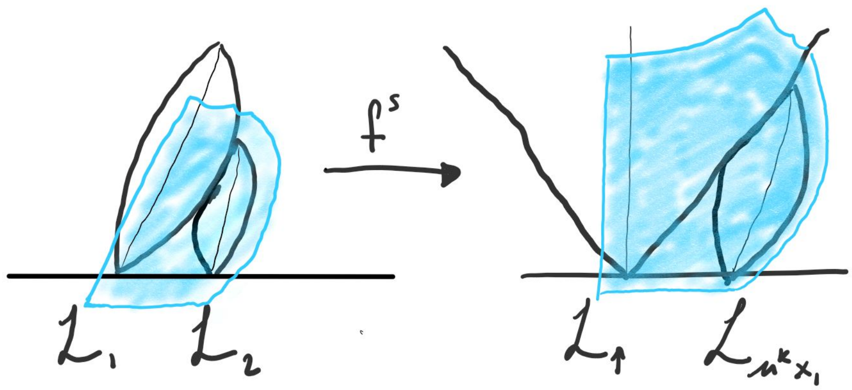

Figure 4.6: Depicted are two limbs, $\mathcal{L}_1, \mathcal{L}_2$, satisfying $A(\mathcal{L}_1, \mathcal{L}_2)$ and mapping to $\mathcal{L}_\uparrow, \mathcal{L}_{\mu^k x_1}$ for some $k \in \mathbb{Z}$ under f^s. Shaded in blue is the region from Corollary 4.7.10 on which f^s has qs bounds.

Definition 4.7.1. For two limbs, $\mathcal{L}_1, \mathcal{L}_2$, say $A(\mathcal{L}_1, \mathcal{L}_2)$ is satisfied if

1. $\mathcal{L}_1, \mathcal{L}_2$ are rooted in the same spine or are both rooted in $\mathbb{R}$

2. $\mathcal{L}_1, \mathcal{L}_2$ touch

3. $\mathrm{Gen}(\mathcal{L}_1) < \mathrm{Gen}(\mathcal{L}_2)$.

Lemma 4.7.8. *For any limb $\mathcal{L}_1$, there exists a limb $\mathcal{L}_0$ such that $A(\mathcal{L}_0, \mathcal{L}_1)$ is satisfied.*

Proof. By Corollary 4.3.9 let X be the unique spine such that $c \in \mathrm{Int}(X)$. By Lemma 4.4.8, $\partial^c\mathbb{O}(\mathcal{L}_1) \cap X = [c_a, c_b]$ where $\mathrm{Gen}(c_a), \mathrm{Gen}(c_b) < \mathrm{Gen}(c_1)$ and for any $c \in (c_a, c_b)_{\mathcal{J}_\infty}$, $\mathrm{Gen}(c) > \mathrm{Gen}(c_1)$. If $X = \mathbb{R}$ then both $c_a, c_b \in \mathrm{Int}(X)$. If instead X is the spine of a limb, then X contains only one endpoint (as the other end is a β-point. Hence, at least one of c_a, c_b belongs to $\mathrm{Int}(X)$. Therefore, we can define c_0 to be one of c_a, c_b such that $c_0 \in \mathrm{Int}(X)$ and we can define $\mathcal{L}_0$ as the limb rooted at c_0 and lying on the same side of X as $\mathcal{L}_1$. Since one of the two connected components of $\partial^c\mathbb{O}(\mathcal{L}_1) \setminus X$ is contained in $\mathcal{L}_0$, $\beta(\mathcal{L}_1) \in \overline{\mathcal{L}_0}$. Therefore, $\mathcal{L}_0$ and $\mathcal{L}_1$ touch and so $A(\mathcal{L}_0, \mathcal{L}_1)$ is satisfied. $\square$

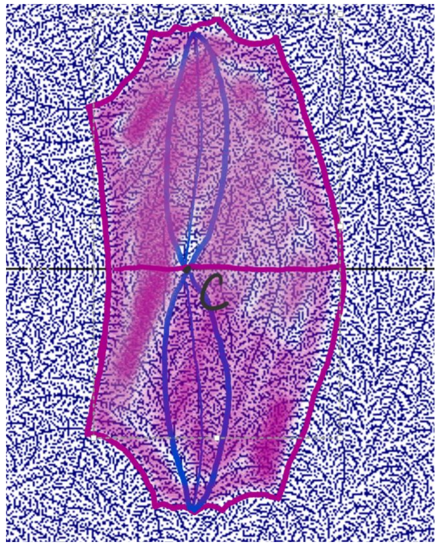

Figure 4.7: A double lake at some $c \neq 0$.

For any pre-critical point c of generation s, by Corollary 4.3.9, let X be the unique spine such that $c \in \mathrm{Int}(X)$. Let $\mathcal{L}_c, \mathcal{L}'_c$ be the two limbs rooted at c. By Corollary 4.4.4,

$$\partial^c \mathbb{O}(\mathcal{L}_c) \cap X = \partial^c \mathbb{O}(\mathcal{L}'_c) \cap X = [c_1, c_2],$$

where $c_1, c_2 \in X$ are the closest pre-critical points on opposite sides of c of generation less than s. Therefore, we may define the *double lake at c* to be

$$\mathbb{O}(\mathcal{L}_c) \cup \mathbb{O}(\mathcal{L}'_c) \cup (c_1, c_2).$$

In this way, a double lake is the union of two lakes and the interior of their shared boundary and is a topological disk (when $c \neq 0$). (See Figure 4.7.)

Lemma 4.7.9. *For any pre-critical point c, let V_c denote the double lake at c. Then V_c is open and $f^s(V_c) = \mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty))$ where a, b belong to a finite set of points up to scaling by μ.*

Proof. Let c be pre-critical point of generation $s > 0$ and let $\mathcal{L}, \mathcal{L}'$ be the two limbs rooted at c. By Corollary 4.3.9, let X be the unique spine such that $\mathrm{Int}(X) \ni c$. By Lemma 4.4.8, $\partial^c \mathbb{O}(\mathcal{L}) \cap X = \partial^c \mathbb{O}(\mathcal{L}') \cap X = [c_1, c_2]$ where c_1, c_2 are the closest pre-critical points to c in X of smaller generation. Since $f^s(\mathbb{O}(\mathcal{L})) \cup f^s(\mathbb{O}(\mathcal{L}') = \mathbb{H}_+ \cup \mathbb{H}_-$, $f^s(V_c) = \mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty))$ for some $a < 0 < b \in \mathbb{R}$. Since $f^s(V_c)$ is open, V_c is also open.

We now show that there exists a finite set of points, $P \subset (x_1, \mu x_1)$, such that for some $k_a, k_b \in \mathbb{Z}$, $\mu^k|a|, \mu^k b \in P$. Let s_1 be the minimum of the generations of c_1, c_2. Let $I_0 := [c_1, c_2]$. Then $I_1 := f^{s_1}(I_0)$ has 0 as an endpoint, is contained in $\mathbb{R}$ or $i\mathbb{R}$, and by the definition of I_0, the other endpoint of I_1 is such that there is no point of smaller generation between it and 0. Therefore, by Corollary 4.3.13 I_1 is one of the three possible types of intervals: $\pm[0, \mu^k x_1], \pm[0, \mu^k y_1], \pm[0, \mu^k y_2]$ for some $k \in \mathbb{Z}$. Since f is even, and since scaling by μ does not affect distortion, we may assume WLOG that I_1 is one of $[0, \mu x_1], [0, y_1]$, or $[0, \mu y_2]$.

Case 1: $I_1 = [0, \mu x_1]$. By Lemma 4.3.12, $f^{s_1}(c) = x_1$. Since $\mathrm{Gen}(x_1) = 1$, applying f sends x_1 to 0, and the double lake at x_1 to $\mathbb{C} \setminus ((-\infty, v] \cup [-\mu v, +\infty))$.

Case 2: $I_1 = [0, y_1]$. By Lemma 4.3.12, $f^{s_1}(c) = y_2$. Since $\mathrm{Gen}(y_2) = 5$, and $f^5([0, y_1]) = [f^5(0), f^2(0)] = [f^5(0), -\mu^{-1}v]$, applying f^5 sends the double lake at y_2 to $\mathbb{C}\setminus((-\infty, f^5(0)] \cup [-\mu^{-1}v, +\infty))$.

Case 3: $I_1 = [0, \mu y_2]$. By Lemma 4.3.12, $f^{s_1}(c) = y_1$. Since $\mathrm{Gen}(y_1) = 3$, and $f^3([0, \mu y_2]) = [f^3(0), f^{1/2}(0)] = [f^3(0), -\mu v]$.

Combining these three cases, rescaling, and taking the absolute value, we find our set of finite set of points is $P = \{|v|, \mu|f^3(0)|, |f^5(0)|\}$. □

Corollary 4.7.10. *Every bounded limb is uniformly qs-equivalent to $\mathcal{L}_{x_1}$. More generally, there exists a distortion function η_1, such that for any pair of bounded limbs $\mathcal{L}_1, \mathcal{L}_2$ of generations s_1, s_2 satisfying $A(\mathcal{L}_1, \mathcal{L}_2)$, $f^{s_1}|_U$ is η_1-qs where U is an open set containing $\overline{\mathcal{L}_2}$ and the half-lake of generation s_2 intersecting both $\mathcal{L}_2$ and $\mathcal{L}_1$. (See Figure 4.6.)*

Proof. Let $\mathcal{L}_1, \mathcal{L}_2$ be bounded limbs rooted at c_1, c_2, as in the statement of the corollary. By Lemma 4.7.9, the double lake at c_1 maps conformally under f^{s_1} to $\mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty))$ where for some $k_a, k_b \in \mathbb{Z}$, $\mu^{k_a}|a|, \mu^{k_b} b \in P \subset (x_1, \mu x_1)$ by Lemma 4.7.9 where P is a finite set. Furthermore, $f^{s_1}(c_1) = 0$ and $f^{s_1}(c_2) = \pm\mu^k x_1 \in (a, b)$ for some $k \in \mathbb{Z}$. By rotation 180 degrees if necessary and rescaling by μ^{-k}, we may assume WLOG that $f^{s_1}(c_2) = x_1$. Let $p \in P$ be the point of minimal distance to x_1. Since U is the smallest open set containing $\mathcal{L}_2$ and the half-lake of generation s_2 intersecting both $\mathcal{L}_2$ and $\mathcal{L}_1$, $U' := f^{s_1}(U)$ is the smallest open set containing $\mathcal{L}_{x_1} := f^{s_1}(\mathcal{L}_2)$ and the half-lake of generation 1 intersecting $\mathcal{L}_{x_1}$ and $\pm\mathcal{L}_\uparrow$. Since $\overline{U}$ is compact and $\overline{U} \cap \mathbb{R} = [0, x_1]$ the modulus of $\mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty)) \setminus U'$ is bounded. Hence, by Koebe distortion, the result follows.

□

Proposition 4.7.11. *The closure of every bounded limb is a uniform quasidisk.*

Proof. By Corollary 4.7.10, it suffices to show that $X = \partial\overline{\mathcal{L}}_{-x_1}$ is a quasicircle. Recall the map χ_{-1} (see (4.5.1).) Since $\chi_{-x_1}(\mathcal{L}_{-x_1}) = \mathcal{L}_{-x_1}$, fixes $\beta(\mathcal{L}_{-x_1})$, and $|\chi'_{-x_1}(\beta(\mathcal{L}_{-x_1}))| > 1$, $\chi^{-1}_{-x_1}(X) = X$ and is locally linearizable in a neighborhood of $\beta(\mathcal{L}_{-x_1})$ to the map $z \mapsto \lambda z$ where $\lambda = (\chi^{-1}_{-x_1})'(\beta(\mathcal{L}_{-x_1})) \in \mathbb{D}$.

Let N be a neighborhood of $\beta(\mathcal{L}_{-x_1})$ on which $\chi^{-1}_{-x_1}$ is linearizable. Let $N' = \chi^{-2}_{-x_1}(N) \subset N$. Let U be the double lake at $-x_1$. $f(\mathcal{L}_{-x_1}) = -\mathcal{L}_\uparrow$, $f(U) = V = \mathbb{C} \setminus ((-\infty, a] \cup [b, +\infty))$ for some $a < 0 < b$, and $f|_U$ is conformal. Since $f(X \setminus N') \subset K \Subset V$ where K is compact, $f^{-1}|_K$ is qs by Koebe distortion. Therefore, since $(R^0_{III} \cup R^0_{IV})$ is a quasiarc by Proposition 4.7.4, $X \setminus N'$ is a quasiarc.

This means that, in particular, the two connected components of $X \cap (N \setminus N_2)$ are quasiarcs. By Lemma 2.1.4 applied to $X \cap (N \setminus N_2)$ and $\chi^{-1}_{-x_1}$, we have that $X \cap N$ is a quasiarc. Finally, applying Lemma 2.1.5 to $X \setminus N_2$ and $X \cap N$, we get that X is a quasicircle. □

4.8 More properties of limbs

Lemma 4.8.1. $[\beta_0, \beta_1] = \overline{\mathcal{L}_\uparrow} \cap \overline{\mathcal{L}_{x_1}}$.

Proof. By Corollary 4.7.5, $[\beta_0, \beta_1] \subseteq \overline{\mathcal{L}_\uparrow}$. Therefore, it suffices to show $[\beta_0, \beta_1] \subseteq \overline{\mathcal{L}_{x_1}}$. Since we already know $\beta_0 := \beta(\mathcal{L}_{x_1}) \in \overline{\mathcal{L}_{x_1}}$ and since, by Corollary 4.1.2, there is at most 1 arc between any two points in Esc, it suffices to show $\beta_1 \in \overline{\mathcal{L}_{x_1}}$.

By Lemma 4.7.1, $\beta_0 \prec \beta_1$. Since $f(\beta_1) = \beta(\mathcal{L}_{-x_1}) \in R^0_{II}$ and since the branch $f^{-1} : \mathbb{H}_+ \to \mathbb{O}(\mathcal{L}_{x_1})$ satisfies $f^{-1}(R^0_{II}) \subset \partial\overline{\mathcal{L}_{x_1}}$, it follows that $\beta_1 \in \partial\overline{\mathcal{L}_{x_1}}$. Since $\partial\overline{\mathcal{L}_{x_1}}$ is a quasi-circle (Proposition 4.7.11), and $\partial\overline{\mathcal{L}_{x_1}} \setminus \{x_1\} \subset \mathrm{Esc}$, there is an arc between β_1 and β_0 contained in $\partial\overline{\mathcal{L}_{x_1}} \cap \mathrm{Esc}$. By Corollary 4.1.2 this is the unique arc from β_1 to β_0. Hence, $[\beta_0, \beta_1] \subseteq \partial\overline{\mathcal{L}_{x_1}}$.

We now show the other direction, namely, that $[\beta_0, \beta_1] \supseteq \overline{\mathcal{L}_\uparrow} \cap \overline{\mathcal{L}_{x_1}}$. By Lemma 4.7.7,

$$\overline{\mathcal{L}_\uparrow} \cap \overline{\mathcal{L}_{x_1}} = \partial\overline{\mathcal{L}_\uparrow} \cap \partial\overline{\mathcal{L}_{x_1}} = R^0_I \cap \partial\overline{\mathcal{L}_{x_1}}.$$

By Lemma 4.7.3 and since $\beta_0 \prec \beta_1 \in \mathbb{O}_\lambda(\mathcal{L}_{x_1})$,

$$R^0_I \cap \mathbb{O}(\mathcal{L}_{x_1}) = R^0_I \cap \mathbb{O}_\lambda(\mathcal{L}_{x_1}).$$

Therefore, by scaling,

$$R^0_I \cap \mathbb{O}(\mathcal{L}_{\mu^{-1}x_1}) = R^0_I \cap \mathbb{O}_\lambda(\mathcal{L}_{\mu^{-1}x_1}).$$

Therefore,

$$\begin{aligned} z \in R^0_I \cap \partial\overline{\mathcal{L}_{x_1}} &\implies z \in \{\beta_0\} \cup \partial_\lambda\overline{\mathcal{L}_{x_1}} \setminus \mathbb{O}_\rho(\mathcal{L}_{\mu^{-1}x_1}) \\ &\implies \mathrm{Gen}(z) \in [1, 2] \\ &\implies z \in [\beta_0, \beta_1]. \end{aligned}$$

□

Lemma 4.8.2. *For $x \in (\mu^{-1}x_1, x_1)$ a pre-critical point, $\mathcal{L}_x, \mathcal{L}_\uparrow$ do not touch.*

Proof. By Lemma 4.8.1 and symmetry across $i\mathbb{R}$, $\overline{\mathcal{L}_\uparrow}$ and $\overline{\mathcal{L}_{-x_1}}$ intersect on an interval. Pulling back by f^{-1}, we find that $\overline{\mathcal{L}_{\mu^{-1}x_1}} \cap \overline{\mathcal{L}_{x_1}} = [\beta_1, \beta']$ where $\beta_1 \prec \beta'$. This interval blocks any limb $\mathcal{L}_x$, $x \in (\mu^{-1}x_1, x_1)$ from reaching $\mathcal{L}_\uparrow$. For a picture, see Figure 4.3. □

From the previous lemma, we get the following immediate corollary:

Corollary 4.8.3. *A real limb touches $\mathcal{L}_\uparrow$ if and only if it is rooted at $\pm\mu^k x_1$ for some $k \in \mathbb{Z}$ and contained in the closed upper half-plane.*

Lemma 4.8.4. *If c_1, c_2 belong to the interior of a common spine and there is no pre-critical point of order less than $\max\{\mathrm{Gen}(c_1), \mathrm{Gen}(c_2)\}$ between them, then the limbs rooted at c_1, c_2 touch.*

Proof. WLOG, suppose $s_1 = \mathrm{Gen}(c_1) \leq \mathrm{Gen}(c_2) = s_2$. If there is no pre-critical point of generation less than s_2 in $(c_1, c_2)_{\mathcal{J}}$ then by Lemma 4.4.8, $c_2 \in \partial^c \mathbb{O}(\mathcal{L}_{c_1})$. Since there is no pre-critical point of lesser generation between $f^{s_1}(c_1) = 0$ and $f^{s_1}(c_2) \in \mathbb{R}$, then by Corollary 4.3.13, $f^{s_1}(c_2) = \pm\mu^{-k} x_1$ for some $k \in \mathbb{N}_0$. Since the limbs at $f^{s_1}(c_1), f^{s_1}(c_2)$ map backwards to the limbs at c_1 and c_2 along the same branch of f^{-s_1}, the limbs at c_1 and c_2 must also touch. □

Lemma 4.8.5. *If $c_1 \neq c_2$ belong to the interior of a common spine then there exists a unique pre-critical point $c_3 \in (c_1, c_2)_{\mathcal{J}_\infty}$ of minimal generation. Furthermore, $\mathrm{Gen}(c_3) \neq \mathrm{Gen}(c_1)$ and $\mathrm{Gen}(c_3) \neq \mathrm{Gen}(c_2)$.*

Proof. WLOG, assume that $\mathrm{Gen}(c_1) \leq \mathrm{Gen}(c_2)$. Since $c_1, c_2 \in \mu^n \mathcal{J}$ for some $n \in \mathbb{N}$, $[c_1, c_2]_{\mathcal{J}_\infty} \subset \mu^n \mathcal{J}$. Therefore, since $\mu^n \mathcal{J}$ has only finitely many pre-critical points of generation less than s, for any $s > 0$, there exists a point of minimal generation in $(c_1, c_2)_{\mathcal{J}_\infty}$. By Corollary 4.3.4 any two limbs of the same generation cannot touch. Therefore, by Lemma 4.8.4, the pre-critical point of minimal generation must be unique and cannot be the same generation as that of either c_1 or c_2. □

Corollary 4.8.6. *Let c_1, c_2 be pre-critical points belonging to the interior of a common spine with $\mathrm{Gen}(c_1) \leq \mathrm{Gen}(c_2)$ and let c_3 be the unique pre-critical point of least generation strictly between them. Let $\mathcal{L}_1, \mathcal{L}_2$, and $\mathcal{L}_3$ be limbs rooted at c_1, c_2, and c_3, respectively such that all are lying on the same side of their common spine. If $\mathcal{L}_1$ and $\mathcal{L}_2$ do not touch, then*

- $\beta(\mathcal{L}_1) \prec \beta(\mathcal{L}_3) \prec \beta(\mathcal{L}_2)$, *or*

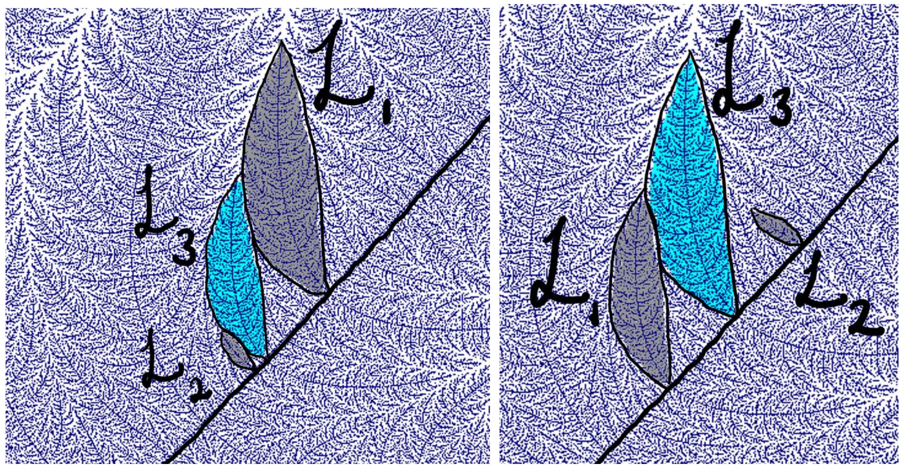

Figure 4.8: This figure is meant to accompany Corollary 4.8.6 and shows the unique limb $\mathcal{L}_3$ of minimal generation that lies between any two non-touching limbs $\mathcal{L}_1$ and $\mathcal{L}_2$ that are rooted in a common spine and lie on the same side of said spine.

- $\beta(\mathcal{L}_3) = \beta(\mathcal{L}_1) \wedge \beta(\mathcal{L}_2)$.

Proof. By Lemma 4.8.5, let c_3 be the unique point of minimal generation lying strictly between c_1 and c_2. If $\mathcal{L}_1$ and $\mathcal{L}_2$ do not touch then by Lemma 4.8.4,

$$\mathrm{Gen}(c_3) < \max\{\mathrm{Gen}(c_1), \mathrm{Gen}(c_2)\}.$$

Hence, either

$$\mathrm{Gen}(c_1) < \mathrm{Gen}(c_3) < \mathrm{Gen}(c_2) \quad \text{or} \quad \mathrm{Gen}(c_3) < \mathrm{Gen}(c_1) \leq \mathrm{Gen}(c_2).$$

Case 1: Suppose $\mathrm{Gen}(c_1) < \mathrm{Gen}(c_3) < \mathrm{Gen}(c_2)$. Since c_3 is the point of minimal generation in (c_1, c_2), there are no points of lesser generation in $[c_1, c_2]$. Therefore, by Lemma 4.4.8, $c_2, c_3 \in \partial^c\mathbb{O}(\mathcal{L}_1)$ and $\beta(\mathcal{L}_2), \beta(\mathcal{L}_3) \in \mathbb{O}(\mathcal{L}_1)$. Similarly, since there is no point of generation less than $\mathrm{Gen}(c_3)$ in $[c_3, c_2]_{\mathcal{J}_\infty}$, $c_2 \in \partial^c\mathbb{O}(\mathcal{L}_3)$ and $\beta(\mathcal{L}_2) \in \mathbb{O}(\mathcal{L}_3)$.

Case 2: Suppose instead that $\mathrm{Gen}(c_3) < \mathrm{Gen}(\mathcal{L}_1) \leq \mathrm{Gen}(\mathcal{L}_2)$. Since there are no points of lesser generation between c_3 and c_1 or between c_3 and c_2, they each belong to $\partial^c\mathbb{O}(\mathcal{L}_3)$. Since c_3 lies between them, their limbs belong to different half-lakes of $\mathbb{O}(\mathcal{L}_3)$. Hence, $\beta(\mathcal{L}_3) = \beta(\mathcal{L}_1) \wedge \beta(\mathcal{L}_2)$.

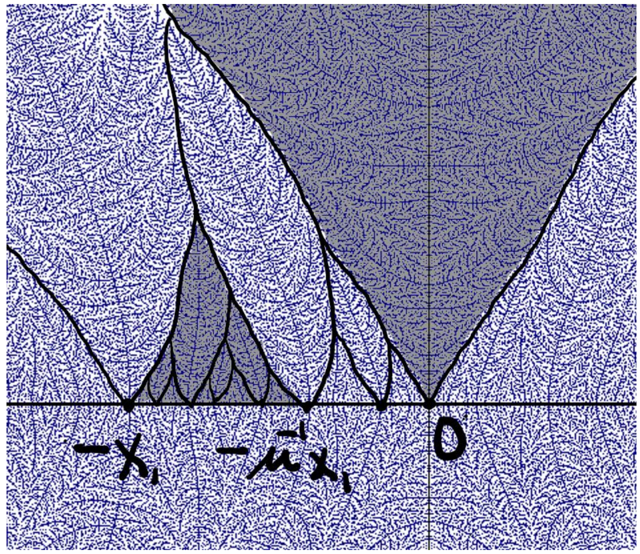

Figure 4.9: Depicted here is the definite gap between the collection of limbs rooted in $(-x_1, -\mu^{-1}x_1)$ and $\mathcal{L}_\uparrow$. By invariance of $\mathcal{J}_\infty$ under scaling by μ, all bounded limbs rooted in $\mathbb{R}$ that don't touch $\mathcal{L}_\uparrow$ cannot be too close to $\mathcal{L}_\uparrow$ relative to their diameter.

□

Let $\mathcal{L}_a, \mathcal{L}_b$ be limbs rooted at a, b such that $A(\mathcal{L}_a, \mathcal{L}_b)$ is satisfied and let $\mathcal{L}_c$ be the unique limb lying between them that touches both of them. Then define $R_{a,b}$ as the closed region bounded by $[a,b]_{\mathcal{J}_\infty}$, $\partial\overline{\mathcal{L}_a} \cap \overline{\mathbb{O}(\mathcal{L}_c)}$, and $\partial\overline{\mathcal{L}_b} \cap \overline{\mathbb{O}(\mathcal{L}_c)}$.

By density of pre-critical points in $[a,b]_{\mathcal{J}_\infty}$ and density of $\mathcal{J}_\infty$ in $\mathbb{C}$, it follows that $R_{a,b}$ is equal to the closure of all the limbs rooted in $(a,b)_{\mathcal{J}}$ and lying on the same side as $\mathcal{L}_a$ and $\mathcal{L}_b$.

Lemma 4.8.7. *There exists a constant $C > 0$ such that if $\mathcal{L}_c \subset \overline{\mathbb{H}_+}$ and does not touch $\mathcal{L}_\uparrow$ then*

$$\frac{d(\mathcal{L}_c, \mathcal{L}_\uparrow)}{\operatorname{diam}(\mathcal{L}_c)} \geq C.$$

Proof. By Lemma 4.8.2, $\mathcal{L}_c$ is not rooted at $\pm\mu^k x_1$ for any $k \in \mathbb{Z}$. By scaling invariance under μ and symmetry of $\mathcal{J}_\infty$ about $i\mathbb{R}$, we may assume WLOG that $\mathcal{L}_c$ is rooted at $c \in (\mu^{-1}x_1, x_1)$. Let $R_{\mu^{-1}x_1,x_1}$ be the closure of all such limbs in that interval. Since $R_{\mu^{-1}x_1,x_1}$ and $\overline{\mathcal{L}_\uparrow}$ are disjoint, we have that

$$\frac{d(\mathcal{L}_c, \mathcal{L}_\uparrow)}{\operatorname{diam}(\mathcal{L}_c)} \geq \frac{d(R_{\mu^{-1}x_1,x_1}, \mathcal{L}_\uparrow)}{R_{\mu^{-1}x_1,x_1}} > 0.$$

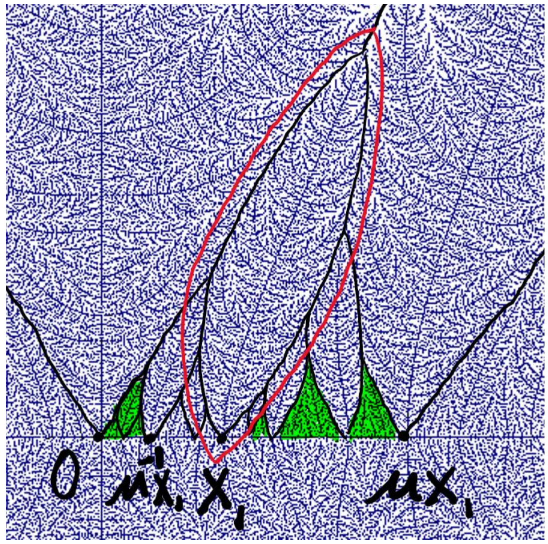

Figure 4.10: Drawn is a δ-neighborhood of $\mathcal{L}_{x_1}$ which is made to be disjoint from the limbs shaded in green where we cannot guarantee good distortion bounds under the dynamics being applied in the proof of Proposition 4.8.8.

$\square$

Proposition 4.8.8. *If $\mathcal{L}_1, \mathcal{L}_2$ are rooted in, and laying on the same side of, a common spine and do not touch, then*

$$\frac{d(\mathcal{L}_1, \mathcal{L}_2)}{\min\{\operatorname{diam}(\mathcal{L}_1), \operatorname{diam}(\mathcal{L}_2)\}} \geq C$$

for some $C > 0$ that does not depend on $\mathcal{L}_1$ or $\mathcal{L}_2$.

Proof. Let $\mathcal{L}_1, \mathcal{L}_2$ be rooted at c_1, c_2, respectively. By Lemma 4.8.4, if $\mathcal{L}_1$ and $\mathcal{L}_2$ don't touch then there exists a limb $\mathcal{L}_3$ rooted at $c_3 \in (c_1, c_2)_{\mathcal{J}_\infty}$ such that $\operatorname{Gen}(c_3) < \max\{\operatorname{Gen}(c_1), \operatorname{Gen}(c_2)\}$. By Corollary 4.8.6, there are two cases to consider corresponding to whether $\beta(\mathcal{L}_1) \prec \beta(\mathcal{L}_3)$ or vice-versa. In each case, we show the desired property by mapping the larger limb – $\mathcal{L}_1$ or $\mathcal{L}_3$ – up to $\mathcal{L}_{x_1}$ or $\mathcal{L}_\uparrow$ with bounded distortion and showing that the property holds there. The main strategy of this proof is to argue the existence of a δ-neighborhood of either $\mathcal{L}_{x_1}$ or of a compact set containing $f^s(\mathcal{L}_1)$ or $f^s(\mathcal{L}_2)$ that is disjoint from the other limb which we can then pull back under a map with good distortion to bound $\mathcal{L}_1$ away from $\mathcal{L}_2$ relative to its diameter or vice-versa.

Case 1: Suppose that $\mathrm{Gen}(\mathcal{L}_1) < \mathrm{Gen}(\mathcal{L}_3)$. Then by Corollary 4.8.6, $\beta(\mathcal{L}_1) \prec \beta(\mathcal{L}_2)$. By Lemma 4.7.8, let $\mathcal{L}_0$ be a limb such that $A(\mathcal{L}_0, \mathcal{L}_1)$ is satisfied. Let $s = \mathrm{Gen}(\mathcal{L}_0)$. Assume WLOG that $f^s(\mathcal{L}_1) = \mathcal{L}_{x_1} \subset \overline{Q_I}$ since otherwise we may post-compose f^s with a symmetry of $\mathbb{C}$ and scaling by μ^n for some $n \in \mathbb{Z}$ so that this is the case. Then by Corollary 4.7.10, f^{-s} is η'-qs on an open set $U \supset \overline{\mathcal{L}_{x_1}}$. Also by Corollary 4.7.10, there are only finitely many points in $(x_1, \mu x_1)$ that can be critical values of the map $f^s : \partial^c \mathbb{O}(\mathcal{L}_0) \to \mathbb{R}$. Let x_b be the greatest pre-critical point in $(x_1, \mu x_1)$ such that $\mathcal{L}_{x_b}$ touches $\mathcal{L}_{x_1}$ and (x_1, x_b) is guaranteed to be without critical values under our map and every limb rooted in (x_1, x_b) is contained in U. Let B be the closure of the collection of limbs that are lying in $\overline{Q_I}$, are rooted in $(0, \mu^{-1}x_1) \cup (x_b, \mu x_1)$, and do not touch $\mathcal{L}_{x_1}$. Since $B \cap \overline{\mathcal{L}_{x_1}} = \emptyset$, there exists a constant $\delta > 0$ such that N_δ, the δ-neighborhood of $\mathcal{L}_{x_1}$ satisfies $N_\delta \subset U$ and $N_\delta \cap B = \emptyset$.

If $f^s(\mathcal{L}_2) \subset B$ then $d(\mathcal{L}_1, \mathcal{L}_2)/\operatorname{diam}(\mathcal{L}_1) \geq C_1$ since $f^s(\mathcal{L}_2) \cap N_\delta = \emptyset$ and $f^{-s}|_{N_\delta}$ has good distortion, where f^{-s} is an appropriately chosen branch. If instead, $f^s(\mathcal{L}_2) \not\subset B$, then $f^s(\mathcal{L}_2)$ is rooted in $(\mu^{-1}x_1, x_b)$. Since we may map $f^s(\mathcal{L}_2)$ and $\mathcal{L}_{x_1}$ by f to $f^{s+1}(\mathcal{L}_2)$ and $\mathcal{L}_\uparrow$ with good distortion, we get that $d(\mathcal{L}_1, \mathcal{L}_2)/\operatorname{diam}(\mathcal{L}_2) \geq C_2$ by Lemma 4.8.7.

Case 2: Suppose instead that $\mathrm{Gen}(\mathcal{L}_3) < \mathrm{Gen}(\mathcal{L}_1)$. Then by Corollary 4.8.6, $\beta(\mathcal{L}_3) = \beta(\mathcal{L}_1) \wedge \beta(\mathcal{L}_2)$. Similarly to Case 1, let $\mathcal{L}_0$ be a limb such that $A(\mathcal{L}_0, \mathcal{L}_3)$ is satisfied. Let $s = \mathrm{Gen}(\mathcal{L}_0)$ and WLOG, assume as before that $f^s(\mathcal{L}_3) = \mathcal{L}_{x_1}$. In this instance, define B_L as the closure of the limbs rooted in $(0, \mu^{-1}x_1]$ and define B_R as the closure of the limbs rooted in $[x_b, \mu x_1)$ where x_b is as defined in Case 1. If $f^s(\mathcal{L}_1)$ and $f^s(\mathcal{L}_2)$ are both in $B_L \cup B_R$ then at least one of them is rooted in $(0, \mu^{-1}x_1]$. Observe that by Corollary 4.7.10, $B_L \subset U$ where $U \supset \mathcal{L}_{x_1}$ is an open set on which f^{-s} has good distortion bounds. Since $B_L \cap B_R = \emptyset$, and $U \supset B_L$ is open, there exists a $\delta > 0$ such that the δ-neighborhood of B_L, is both contained in U and disjoint from B_R. WLOG, suppose $f^s(\mathcal{L}_1) \subset B_L$ (as opposed to $f^s(\mathcal{L}_2)$.) Pulling this δ-neighborhood back by f^s, we find that $d(\mathcal{L}_1, \mathcal{L}_2)/\operatorname{diam}(\mathcal{L}_1) \geq C_3$.

If $f^s(\mathcal{L}_1), f^s(\mathcal{L}_2)$ are not both contained in $B_L \cup B_R$ then at least one of them is rooted in $(\mu^{-1}x_1, x_b)$. Mapping this forward by f sends it to $(-x_1, f(x_b))$ where $f(x_b) \in (0, x_1)$. WLOG, suppose $f^{s+1}(\mathcal{L}_1)$ is rooted in $(-x_1, 0)$. By Corollary 4.7.10, any suitably chosen branch f^{-s-1} sending $f^{s+1}(\mathcal{L}_1)$ to $\mathcal{L}_1$ has good distortion on a neighborhood $U \supset \mathbb{O}^+(\mathcal{L}_{-x_1})$. Let G be the closure of all the limbs rooted in $(-x_1, -\mu^{-1}x_1]$. Since G is contained in the (open) left half-plane and $G \subset U$, there exists a δ-neighborhood $G \subset N_\delta \subset U$ such that N_δ is still contained in the left half-plane. Scaling down G by μ^{-k} so that it contains $f^{s+1}(\mathcal{L}_1)$ and then pulling $\mu^{-k}N_\delta$ back by f^{s+1}, we find that $d(\mathcal{L}_1, \mathcal{L}_2)/\operatorname{diam}(\mathcal{L}_1) \geq C_4$.

Taking $C := \min\{C_1, C_2, C_3, C_4\}$, we obtain the desired result for any such limbs $\mathcal{L}_1, \mathcal{L}_2$. $\square$

Proposition 4.8.9. *If $\mathcal{L}_1, \mathcal{L}_2$ are limbs such that $A(\mathcal{L}_2, \mathcal{L}_1)$ is satisfied, then*

$$\frac{d(\beta(\mathcal{L}_2), \mathcal{L}_1)}{\min\{\operatorname{diam}(\mathcal{L}_1), \operatorname{diam}(\mathcal{L}_2)\}} \geq C$$

for some $C > 0$ that does not depend on $\mathcal{L}_1$ or $\mathcal{L}_2$.

Proof. If $\mathcal{L}_2 = \pm\mathcal{L}_\uparrow$ then this is immediate as $d(\beta(\mathcal{L}_2), \mathcal{L}_1) = \infty$. If $\mathcal{L}_2 = \mathcal{L}_{x_1}$ then the inequality also holds for some $C > 0$ since all the real limbs of greater generation that touch $\mathcal{L}_{x_1}$ are bounded away from $\beta(\mathcal{L}_{x_1})$. All other cases follow from Koebe distortion, symmetry, and scaling since we may map $\mathcal{L}_2, \mathcal{L}_1$ up to $\mathcal{L}_{x_1}, \mathcal{L}_1'$ with bounded distortion where $A(\mathcal{L}_{x_1}, \mathcal{L}_1')$ is satisfied. $\square$

Proposition 4.8.10. *If $\mathcal{L}_1, \mathcal{L}_2$ are limbs satisfying $A(\mathcal{L}_1, \mathcal{L}_2)$ and $\mathcal{L}_{1,s} \subset \mathcal{L}_1$ is a limb rooted in* $\operatorname{spine}(\mathcal{L}_1)$ *that does not touch $\mathcal{L}_2$, then*

$$\frac{d(\mathcal{L}_{1,s}, \mathcal{L}_2)}{\min\{\operatorname{diam}(\mathcal{L}_{1,s}), \operatorname{diam}(\mathcal{L}_2)\}} \geq C$$

for some $C > 0$ that does not depend on $\mathcal{L}_1, \mathcal{L}_2$, or $\mathcal{L}_{1,s}$.

Proof. The proof is similar to that of Proposition 4.8.8. We may map $\mathcal{L}_{1,s}, \mathcal{L}_2$ with bounded distortion by Koebe under the dynamics and scaling to $\mathcal{L}_{1,s}', \mathcal{L}_2'$ where $\mathcal{L}_{1,s}'$ is an imaginary limb and $\mathcal{L}_2'$ is rooted at $\pm x_1$. From there, the problem can be reduced to finally many cases such that for each, the distance between the limbs relative to their diameters is bounded from below. $\square$

4.9 Contrast between layouts of real limbs and imaginary limbs

A limb rooted at c is said to be *real* if $c \in \mathbb{R}$. Similarly, a limb rooted at c is said to be *imaginary* if $c \in i\mathbb{R} \setminus \{0\}$.

Corollary 4.9.1. *Every pair of touching limbs touch on an interval and there is a unique limb in between them that touches both. That is, let $\mathcal{L}_1, \mathcal{L}_2$ be touching limbs rooted at c_1, c_2 such that* $\mathrm{Gen}(\mathcal{L}_1) < \mathrm{Gen}(\mathcal{L}_2)$*, then*

$$\overline{\mathcal{L}_1} \cap \overline{\mathcal{L}_2} = [\beta_v, \beta_w] \subseteq \mathrm{Esc}$$

where $\beta_v = \beta(\mathcal{L}_2)$ and $\beta_w = \beta(\mathcal{L}_3)$ where $\mathcal{L}_3$ is the unique limb rooted in $[c_1, c_2]_{\mathcal{J}_\infty}$ such that $\mathcal{L}_3$ touches both $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. We first observe that this statement holds for $\mathcal{L}_\uparrow$ and $\mathcal{L}_{x_1}$. By Lemma 4.8.1 we already know their closures intersect on an interval, namely the quasi-arc $[\beta_0, \beta_1]$. By scaling, $\overline{\mathcal{L}_\uparrow} \cap \overline{\mathcal{L}_{\mu^{-1}x_1}} = [\beta_1, \beta_2]$. Since f maps $\mathcal{L}_{\mu^{-1}x_1}, \mathcal{L}_{x_1}$ to $\mathcal{L}_{-x_1}, \mathcal{L}_\uparrow$ whose closures intersect on a quasi-arc, namely the reflection of $[\beta_0, \beta_1]$ about $i\mathbb{R}$, it follows that $\mathcal{L}_{\mu^{-1}x_1}, \mathcal{L}_{x_1}$ must also intersect on a quasi-arc. Since $\mathcal{L}_{\mu^{-1}x_1}$ touches both $\mathcal{L}_\uparrow$ on an interval every limb rooted in $(0, \mu^{-1}x_1)$ is bounded away from $\mathcal{L}_{x_1}$ and so cannot touch it. Similarly, since $\mathcal{L}_{\mu^{-1}x_1}$ touches $\mathcal{L}_{x_1}$ on an arc, any limb rooted in $(\mu^{-1}x_1, x_1)$ is bounded away from $\mathcal{L}_\uparrow$ and so cannot touch it. Hence, $\mathcal{L}_{\mu^{-1}x_1}$ is the *unique* limb rooted in $[0, x_1]$ that touches both $\mathcal{L}_\uparrow$ and $\mathcal{L}_{x_1}$. By invariance of $\mathcal{J}_\infty$ under scaling and symmetry this immediately generalizes to the case when $\mathcal{L}_1 = \pm\mathcal{L}_\uparrow$ and $\mathcal{L}_2$ a limb rooted at $\pm\mu^k x_1$ for any $k \in \mathbb{Z}$.

For the general case, suppose $\mathcal{L}_1, \mathcal{L}_2$ are touching limbs such that $\mathrm{Gen}(\mathcal{L}_1) < \mathrm{Gen}(\mathcal{L}_2)$. Let z be in the intersection of the closed limbs. By Lemma 4.7.7, z belongs to the shared boundaries of the closed limbs and so by Corollary 4.7.6, $z \in \mathrm{Esc}$ and $\mathrm{Gen}(z) \geq \mathrm{Gen}(\beta(\mathcal{L}_2))$. Hence $\beta_1 \prec z$ and so $z \in \mathbb{O}(\mathcal{L}_1)$. Furthermore, $z = \beta_2$ or $\beta_2 \prec z$. Therefore, by Lemma 4.4.6, $\mathbb{O}(\mathcal{L}_1) \supset \mathbb{O}(\mathcal{L}_2)$. We can then reduce to the case when $\mathcal{L}_1 = \pm\mathcal{L}_\uparrow$ by mapping $\mathbb{O}(\mathcal{L}_1)$ to the upper or lower half-plane under the dynamics. □

Corollary 4.9.2. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are touching limbs such that $\mathrm{Gen}(\mathcal{L}_1) < \mathrm{Gen}(\mathcal{L}_2)$ then $\mathbb{O}(\mathcal{L}_1) \supset \mathbb{O}(\mathcal{L}_2)$.*

Proof. By Corollary 4.9.1, $\beta(\mathcal{L}_1) \prec \beta(\mathcal{L}_2)$ since $\beta(\mathcal{L}_2) \in \partial\overline{\mathcal{L}_1}$. Hence, if $\beta(\mathcal{L}_2) \prec z$ then $\beta(\mathcal{L}_1) \prec z$. This means that if $z \in \mathbb{O}(\mathcal{L}_2) \cap \mathrm{Esc}$ then $z \in \mathbb{O}(\mathcal{L}_1) \cap \mathrm{Esc}$. Therefore, by Lemma 4.4.6 $\mathbb{O}(\mathcal{L}_1) \supset \mathbb{O}(\mathcal{L}_2)$. □

Lemma 4.9.3. *Consider $\mathcal{L}_{\mu^{-1}y_1}, \mathcal{L}_{y_2}, \mathcal{L}_{y_1} \subset Q_I$. $\mathcal{L}_\mu^{-1}y_1, \mathcal{L}_{y_2}$ touch and $\mathcal{L}_{y_2}, \mathcal{L}_{y_1}$ touch.*

Proof. This follows from Lemma 4.3.12. Since y_2 is the point of smallest generation in $(0, y_1)$, mapping forward by $f^{3/4}$ sends y_2 to a point of smallest generation in $(f^{3/4}(0), 0)$ and so $f^{3/4}(y_2) = -\mu^k x_1$ for some $k \in \mathbb{Z}$. Since $f^{3/4}(\mathcal{L}_{y_2}), f^{3/4}(\mathcal{L}_{y_1})$ touch, so do $\mathcal{L}_{y_2}, \mathcal{L}_{y_1}$. The argument is analogous for $\mathcal{L}_{\mu^{-1}y_1}, \mathcal{L}_{y_2}$. □

Lemma 4.9.4. *$\mathcal{L}_{\mu^{-1}y_1}, \mathcal{L}_{y_2}, \mathcal{L}_{y_1} \subset Q_I$ all touch $\mathcal{L}_{x_1}$.*

Proof. Recall β', β'' from (4.7.5). By Lemma 4.8.1, Lemma 4.9.3 and Corollary 4.9.1, it suffices to show $\beta(\mathcal{L}_{\mu^{-1}y_1}) = \beta'$ and $\beta(\mathcal{L}_{y_2}) = \beta''$.

By Corollary 4.4.4 and Lemma 4.3.12 $\beta(\mathcal{L}_{\mu^{-1}y_1}) \prec \beta_1$ and $\beta(\mathcal{L}_{y_2}) \prec \beta_1$. Since $\mathrm{Gen}(\beta(\mathcal{L}_{\mu^{-1}y_1}) = \mathrm{Gen}(\beta') = 3/2$ and $\mathrm{Gen}(\beta(\mathcal{L}_{y_2})) = \mathrm{Gen}(\beta'') = 5/4$ and since the lake of any given generation containing β_1 is unique, it follows that $\beta(\mathcal{L}_{\mu^{-1}y_1}) = \beta'$ and $\beta(\mathcal{L}_{y_2}) = \beta''$. □

Corollary 4.9.5. *For any pre-critical point $y \in i\mathbb{R}_+ \setminus \{\mu^k y_1, \mu^k y_2 \mid k \in \mathbb{Z}\}$, $\mathcal{L}_y$ does not touch any real limbs.*

Proof. This follows immediately from Corollary 4.9.1 and Lemma 4.9.3. The idea is the same as for Lemma 4.8.2. □

Proposition 4.9.6. *Real limbs touch 0 or 3 imaginary limbs. Imaginary limbs touch 0, 1, or 2 real limbs. All of the above cases are realized.*

Proof. This follows immediately from Corollary 4.8.3 and Lemmas 4.9.4 and 4.9.5. □

See Figure 4.3 for a picture demonstrating Proposition 4.9.6.

Chapter 5

Quasisymmetries of $\mathcal{J}$

5.1 Defining the maps of the generating set

The most important homeomorphisms of $\mathcal{J}$ to itself are a collection of "shift" maps. For any little Julia set, J, there is a shift map and they are named as such because they shift the little Julia sets centered along the spine of J. In order to define these maps, we first define the involution $\iota : \mathcal{J}_\infty \to \mathcal{J}_\infty$ as follows:

$$\iota(z) = \begin{cases} f(z), & z \in [-\beta, \alpha] \cup \bigcup_{c \in PCP \cap (-\beta,\alpha)} \mathcal{L}_c \cup \mathcal{L}'_c \\ f^{-1}(z), & z \in (\alpha, \beta] \cup \bigcup_{c \in PCP \cap (\alpha,\beta)} \mathcal{L}_c \cup \mathcal{L}'_c \\ -z, & \text{otherwise.} \end{cases}$$

The branch of f^{-1} is chosen such that it fixes α. See Figure 5.1 for a picture of the different regions of the map corresponding to each case in the definition of ι. The following proposition gives an important remark about ι.

Proposition 5.1.1. *ι does not extend continuously to $\mathbb{C} = \overline{\mathcal{J}_\infty}$.*

Proof. It suffices to show that ι does not extend continuously to $\beta(\mathcal{L}_{-x_1})$. Indeed, as $z \to \beta(\mathcal{L}_{-x_1})$ along spine$(\mathcal{L}_{-x_1})$, $\iota(z) \to \infty$ since $\mathrm{Gen}(\beta(\mathcal{L}_{-x_1})) = 1$. However, as $z \to \beta(\mathcal{L}_{-x_1})$ in $\mathcal{L}_\uparrow$, $\iota(z) \to f^{-1}(\beta(\mathcal{L}_{-x_1})) \in \mathbb{C}$ where the branch of f^{-1} is the one from the definition of ι. $\square$

Recall from the introduction, the map $\rho : \mathcal{J} \to \mathcal{J}$ is defined as $\rho(z) = -z$. Abusing notation, we also will use ρ as the negation map for any point $z \in \mathbb{C}$. Before we define shift maps for general little Julia sets, we start by defining one for the central little Julia set of depth k, $\mu^{-k}\mathcal{J}$ for each $k \in \mathbb{N}_0$. For $k \in \mathbb{N}_0$, define $\sigma_k : \mathcal{J} \to \mathcal{J}$ by

$$\sigma_k = \mu^{-k} \circ \rho \circ \iota \circ \mu^k|_{\mathcal{J}}. \tag{5.1.1}$$

See Figure 5.2 for a drawing of the map σ_0. By looking at the definition of σ_0 one can see that to the right of α, σ_0 acts as f^{-1} where the branch is chosen such that it fixes β. In this way, σ_0 shifts all the little Julia sets of depth one that are to the right of α and centered on $\mathbb{R}$ to the right by one. To the left of α, σ_0 acts as $-f$ and in so doing shifts all of the little Julia sets on that side to the right by one. While not necessary, it may be easier to first picture this map acting on the Basilica Julia set by replacing f with $z \mapsto z^2 - 1$ in the definition of σ_0.

Proposition 5.1.2. *σ_k acts as identity outside of the extended little Julia set $\widehat{\mu^{-k}\mathcal{J}}$.*

Proof. This follows from the definitions of σ_k and ι. For $z \notin \widehat{\mu^{-k}\mathcal{J}}$, $\iota(\mu^k z) = -\mu^k z$. Hence, $\mu^{-k}\rho\iota\mu^k(z) = z$. $\square$

Proposition 5.1.3. *For each k, σ_k is η_k-qs where η_k depends only on k. However, these distortion bounds deteriorate as $k \to \infty$. Moreover, the maps σ_k converge pointwise to a map that is identity outside of $\pm L_\uparrow$ and maps $\pm L_\uparrow$ to 0.*

Proof. The fact that each σ_k is η_k-qs follows from Proposition 5.5.2 which we prove later in this chapter.

As $k \to \infty$, $\text{spine}(\mu^{-k}\mathcal{J}) = \mu^{-k}[-\beta, \beta] \to \{0\}$. Therefore $\widehat{\mu^{-k}\mathcal{J}} \to L_\uparrow \cup -L_\uparrow$ as $k \to \infty$. Therefore, for $z \in \mathcal{J} \setminus (L_\uparrow \cup -L_\uparrow)$ there exists k_z such that for $k \geq k_z$, $\sigma_k(z) = z$. For $z \in L_\uparrow$, $\sigma_k(z) \in L_{\mu^{-k}x_1} \subset \mu^{-k}\mathcal{L}_{x_1}$. Since $\mu^{-k}\mathcal{L}_{x_1} \to \{0\}$ as $k \to \infty$, the maps σ_∞ converge pointwise to a map that is identity off of $L_\uparrow \cup -L_\uparrow$ and projects $L_\uparrow \cup -L_\uparrow$ to $\{0\}$. Hence, because each σ_k is normalized in that it fixes $0, \beta, -\beta$, there can be no uniform distortion bound η_{unif} such that σ_k is η_{unif} for all k. $\square$

Using the maps σ_k, we can define a shift map for each little Julia set. Let J be a little Julia set of depth k and let n be the smallest number such that $f^n(J) = \mu^{-k}\mathcal{J}$. Define $\sigma_J : \mathcal{J} \to \mathcal{J}$ by $\sigma_J(z) = \begin{cases} f^{-n} \circ \sigma_k \circ f^n(z), & z \in \widehat{J} \\ z, & \text{otherwise.} \end{cases}$ where f^{-n} is the well-defined branch that sends $\mu^{-k}\mathcal{J}$ to J. Since σ_k fixes $\pm\mu^{-k}\beta$, the above map is continuous.

Proposition 5.1.4. *For $J = \mu^{-k}\mathcal{J}$, $\sigma_J = \sigma_k$.*

Proof. This follows immediately from Proposition 5.1.2 and the definitions of σ_J and σ_k. $\square$

We call the maps σ_J, where J is a little Julia set, shift maps because it shifts the little Julia sets centered along the spine of J of depth one greater than depth(J). Figure 5.3 depicts σ_J for a seemingly random choice of J.

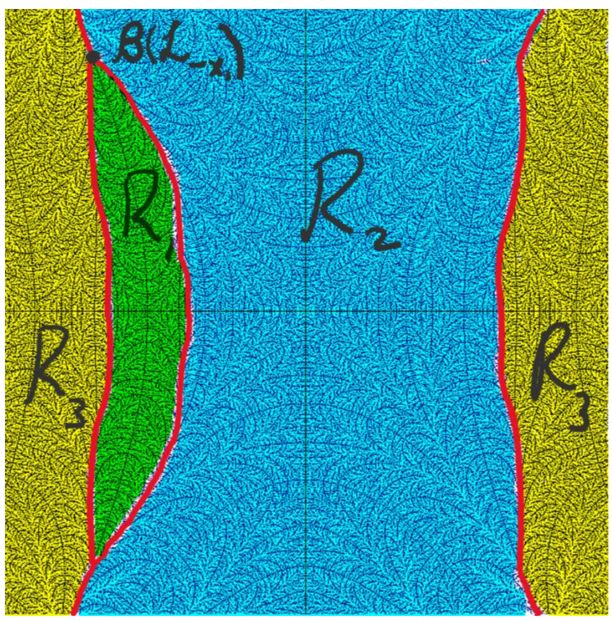

Figure 5.1: Depicted is $\iota : \mathcal{J}_\infty \to \mathcal{J}_\infty$. $\iota|_{R_1} = f : R_1 \to R_2$, $\iota|_{R_2} = f^{-1} : R_2 \to R_1$, $\iota|_{R_3} = \rho$.

5.2 No qs rotations by 1/4

Since f is even, it is immediate that $\rho = z \mapsto -z$, the rotation of $\mathcal{J}$ by $1/2$ is an isometry of $\mathcal{J}$. A natural question is: Is there an analog of ρ that rotates the 4 connected components of

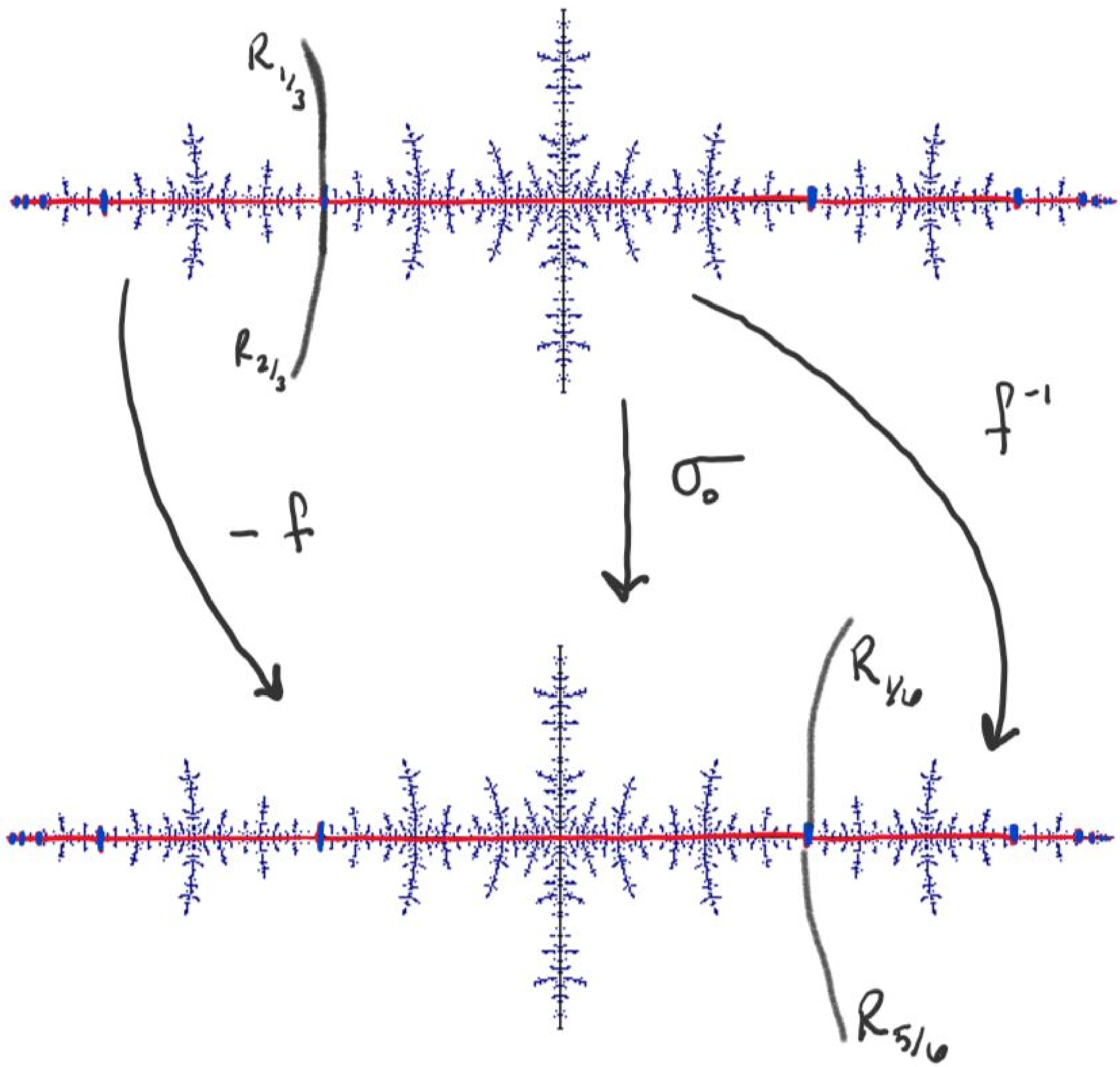

Figure 5.2: Depicted is the map $\sigma_0 : \mathcal{J} \to \mathcal{J}$. Blue dashes mark the real α-points, which each shift one to the right under σ_0.

$\mathcal{J} \setminus \{0\}$ by 1?

Definition 5.2.1. A *rotation by 1/4* of $\mathcal{J}$, $\theta_{1/4} : \mathcal{J} \to \mathcal{J}$, is a topologically extendable homeomorphism (onto its image) such that $\theta_{1/4}(0) = 0$ and $\theta_{1/4}(\mathbb{R}_+ \cap \mathcal{J}) \subset i\mathbb{R}_+ \cap \mathcal{J}$ (i.e. the map "rotates" the 4 connected components of $\mathcal{J} \setminus \{0\}$ sending each to the adjacent one in the counter-clockwise direction). For technical reasons, we allow for rotations by 1/4 to map

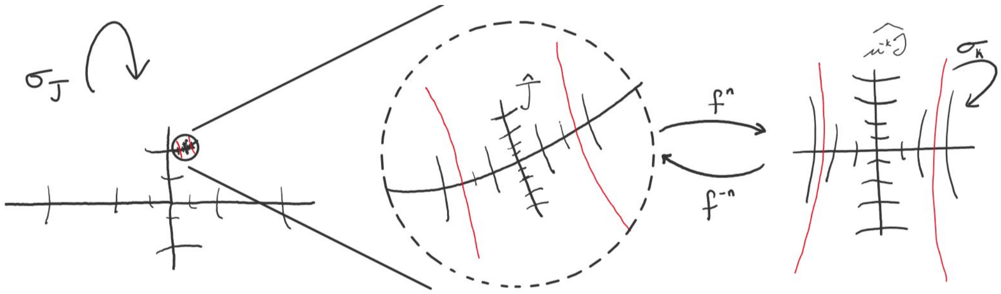

Figure 5.3: Depicted is $\sigma_J : \mathcal{J} \to \mathcal{J}$ where J is a little Julia set of depth k. σ_J acts as identity outside of $\widehat{J}$, which is bounded by the red external rays in the middle and left sketches. $\sigma_J|_{\widehat{J}} = f^{-n} \circ \sigma_k \circ f^n|_{\widehat{J}}$.

strictly inside $\mathcal{J}$.

Proposition 5.2.1. *(Topological) rotations by 1/4 exist.*

Proposition 5.2.1 is shown in three steps:

1. defining an explicit map, $\rho_{1/4}$, that rotates the 4 connected components of $\mathcal{J} \setminus \{0\}$ counter-clockwise,
2. showing $\rho_{1/4}$ is a homeomorphism, and
3. showing $\rho_{1/4}$ is topologically extendable.

Let U_N, U_S, U_E, and U_W correspond to the "north", "south", "east", and "west" components of $\mathcal{J} \setminus \{0\}$. In this way $U_N = L_\uparrow \setminus \{0\}$. Once $\rho_{1/4}$ has been defined on U_N, we can define $\rho_{1/4}$ on the other three components in the following way:

- $\rho_{1/4}|_{U_W} = \rho \circ \rho_{1/4}^{-1}$.
- $\rho_{1/4}|_{U_S} = \rho \circ \rho_{1/4} \circ \rho$.
- $\rho_{1/4}|_{U_E} = \rho \circ \rho_{1/4} \circ \rho$.

In this way, $\rho_{1/4}$ will have the additional desirable property that $\rho_{1/4}^2 = \rho$ and so $\rho_{1/4}^4 = \text{id}$. We also define $\rho_{1/4}(0) := 0$.

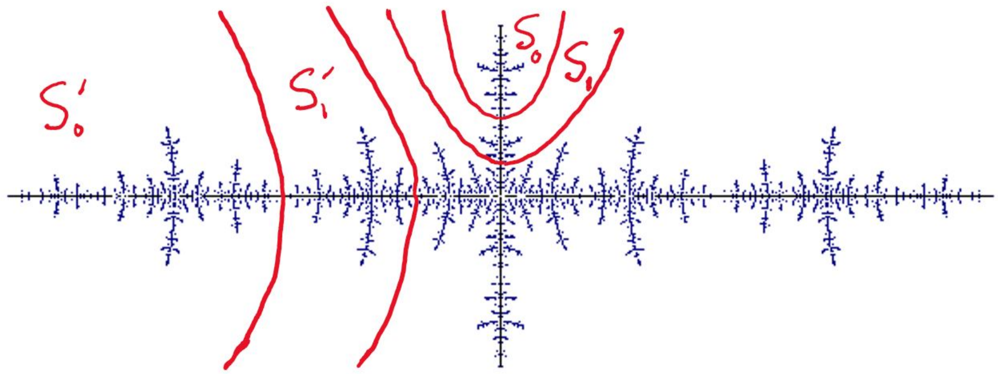

Figure 5.4: Shown here is $\mathcal{J}$ with the strips S_0, S_1, S_0', S_1' marked. $\rho_{1/4}$ rotates the 4 connected components of $\mathcal{J} \setminus \{0\}$ and is defined such that $\rho_{1/4}(S_k) = S_k'$ for $k \in \mathbb{N}_0$.

We now lay the foundation for defining $\rho_{1/4}$ on U_N. Let $\beta_\uparrow := \beta(L_\uparrow)$. For $k \in \mathbb{N}_0$, define

$$I_k := \mu^{-k}(\mu^{-1}\beta_\uparrow, \beta_\uparrow] \subset i\mathbb{R}_+.$$

Using the intervals I_k we define the "strips" of U_N:

$$S_k := I_k \cup \bigcup_{c \in PCP \cap I_k} L_c \cup L_c'.$$

Define

$$\rho_{1/4}|_{S_k} := \mu^{-k} \circ -f^2 \circ \mu^k|_{S_k}. \tag{5.2.1}$$

Since $\bigcup_{k\in\mathbb{N}_0} S_k = U_N$, this fully defines $\rho_{1/4}$ on U_N and therefore is enough to fully define $\rho_{1/4}$ on $\mathcal{J}$. See Figure 5.4.

Lemma 5.2.2. *$\rho_{1/4} : \mathcal{J} \to \mathcal{J}$ is a homeomorphism.*

Proof. We first show that $\rho_{1/4}|_{U_N}$ is a homeomorphism onto U_W.

Observe that $f(\beta_\uparrow) = -\beta$. Therefore, by (1.0.1),

$$-f^2(\mu^{-1}\beta_\uparrow) = \mu^{-1}f(\beta_\uparrow) = \mu^{-1}(-\beta) = \alpha.$$

The last equation is due to the fact that the α fixed point of f is equal to the β fixed point of the pre-renormalization of f. Hence, because f^2 does not have critical points in I_0,

$$-f^2(I_0) = [-\beta, \alpha). \tag{5.2.2}$$

Since every pre-critical point in I_0 is of order at least 3, Lemma 4.3.2 tells us that every pre-critical point in a transcendental limb rooted in I_0 is also of order at least 3. Hence, $-f^2$ moves every transcendental limb rooted in I_0 homeomorphically. Therefore, since the limbs in $\mu^k S_0$ are contained in the transcendental limbs rooted in I_0, $-f^2|_{\mu^k S_k}$ is homeomorphic onto its image for all $k \in \mathbb{N}_0$.

Define $I'_k := \mu^{-k}[-\beta, \alpha)$ and define the strips

$$S'_k := I'_k \cup \bigcup_{c \in PCP \cap I'_k} L_c \cup L'_c.$$

These new strips, which are analogs of $S_k \subset U_N$, similarly satisfy $\bigcup_{k \in \mathbb{N}_0} S'_k = U_W$. Furthermore, by (5.2.2), the definition of $\rho_{1/4}$ is such that $\rho_{1/4}(S_k) = S'_k$. Since the endpoints of I_k, I'_k are related under scaling by μ, the definition of $\rho_{1/4}$ guarantees continuity at these endpoints, which are the points where the closures of consecutive strips intersect.

Hence, $\rho_{1/4}$ is continuous within each connected component of $\mathcal{J} \setminus \{0\}$. Therefore, the only potential for discontinuity in $\rho_{1/4}$ is at 0. However, by the self-similarity of $\mathcal{J}_\infty$ under scaling by μ, the size of transcendental limbs rooted at z goes to 0 as $z \to 0$. Since limbs in $\mathcal{J}$ are contained in transcendental limbs, their diameter, too, must go to 0, as $z \to 0$. Since $I_k, I'_k \to 0$ as $k \to \infty$, it follows that $S_k, S'_k \to 0$ as $k \to \infty$. Hence, as $z \to 0$, $\rho_{1/4}(z) \to 0$ and so we have continuity at 0. This proves $\rho_{1/4} : \mathcal{J} \to \mathcal{J}$ is a homeomorphism. $\square$

Lemma 5.2.3. *$\rho_{1/4}$ is topologically extendable.*

Proof. We do this by showing that there exists an orientation-preserving homeomorphism $\widetilde{\rho}_{1/4}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\widetilde{\rho}_{1/4}} & \mathbb{T} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathcal{J} & \xrightarrow{\rho_{1/4}} & \mathcal{J} \end{array}$$

Recall the strips $S_k \subset U_N$ from the proof of Lemma 5.2.2 and $\beta_\uparrow = \beta(L_\uparrow)$. By Corollary 3.2.9, for all $k \in \mathbb{N}$, $\mu^{-k}\beta_\uparrow$ is a non-branching cut-point. Hence, for each $k \in \mathbb{N}$ there are two points $\theta_k < \theta'_k \in \mathbb{T}$ such that

$$\psi(\theta_k) = \psi(\theta'_k) = \mu^{-k}\beta_{\uparrow}.$$

Since the only branch points of $\mathcal{J}$ are pre-critical points (see Lemma 3.2.7), for each $k \in \mathbb{N}_0$, $S_k \setminus \{\mu^{-k}\beta_{\uparrow}\}$ is the connected component of $\mathcal{J} \setminus \{\mu^{-k}\beta_{\uparrow}, \mu^{-k-1}\beta_{\uparrow}\}$ containing $I_k \setminus \{\mu^{-k}\beta_{\uparrow}\}$. Hence,

$$\psi^{-1}(S_0) = (\theta_1, \theta'_1)$$

and for $k > 0$

$$\psi^{-1}(S_k) = (\theta_{k+1}, \theta_k] \cup [\theta'_k, \theta'_{k+1}).$$

By (1.0.1), $\rho_{1/4}|_{S_k}$ as defined in (5.2.1) can be rewritten as

$$\rho_{1/4}|_{S_k} = (-1)^{k+1} f^{2^{k+1}}. \tag{5.2.3}$$

By symmetry of $\mathcal{J}$ about 0, multiplying by -1 in the dynamical plane corresponds to adding $1/2 \mod 1$ in the Böttcher coordinate of $\mathbb{T}$. Let g be the doubling map on $\mathbb{T}$. Therefore, if we define

$$\widetilde{\rho}_{1/4}|_{A_k} := -\frac{1}{4} + \frac{1}{4}(-1)^{k+1} + g^{2^{k+1}}$$

then by (2.4.1), the diagram commutes.

Since $\rho_{1/4}$ is continuous where the closures of consecutive strips intersect, so is $\widetilde{\rho}_{1/4}$. Piecing together all of the strips, we have that the diagram commutes on all of $\psi^{-1}(U_N)$. We can extend $\widetilde{\rho}_{1/4}$ in the same way to the other components. By continuity, the 4 rays landing at 0 are invariant under $\widetilde{\rho}+_{1/4}$ and rotates them by one counter-clockwise in the same way it does the intervals corresponding to the connected components of $\mathcal{J} \setminus \{0\}$. Hence, $\widetilde{\rho}_{1/4} : \mathbb{T} \to \mathbb{T}$ is a homeomorphism on which the diagram commutes. $\square$

By Proposition 4.9.6 there exists real limbs – limbs rooted in $\mathbb{R}$ – that touch 3 imaginary limbs while an imaginary limb – a limb rooted in $i\mathbb{R}$ – can touch at most 2 real limbs. The following proposition implies that this is an immediate obstruction to any rotations by 1/4 about 0.

Lemma 5.2.4. *Let $\phi : \mathcal{J} \to \mathcal{J}$ be an η-qs map such that $\phi(0) = 0$. Then ϕ induces an η-qs map $\phi_\infty : \mathbb{C} \to \mathbb{C}$ such that for any limb $\mathcal{L}_c$ rooted at c, $\phi_\infty(\mathcal{L}_c) \subseteq \overline{\mathcal{L}_{\phi_\infty(c)}}$.*

Proof. We first begin by defining ϕ_∞ as a limit of scalings of ϕ and showing that it has the desired properties.

Let $m(n) \in \mathbb{Z}$ such that $|\mu^{m(n)}\phi(\mu^{-n}x_1)| \in [x_1, \mu x_1)$. Define $\phi_n : \mu^n\mathcal{J} \to \mu^{m(n)}\mathcal{J}$ by $\phi_n(z) := \mu^{m(n)}\phi(\mu^{-n}z)$. Each ϕ_n is η-qs. For $n \geq k$ define $\phi_{n,k} = \phi_n|_{\mu^k\mathcal{J}}$. The normalization of ϕ_n at x_1 implies the family $\{\phi_n|_{\mu^k\mathcal{J}}\}_{n\geq k}$ is pointwise bounded. Since 0 is fixed and ϕ_n is normalized at x_1, then by Theorem 2.1.2, the sequence $(\phi_n|_{\mu^k\mathcal{J}})_{n\geq k}$ is equicontinuous for each $k \in \mathbb{N}$. Therefore, by Arzela-Ascoli for all $k \in \mathbb{N}$, there is an η-qs map $\phi_{\infty,k} : \mu^k\mathcal{J} \to \mathbb{C}$ such that on a subsequence n_k, which is chosen to be a further subsequence of n_{k-1}, $\phi_{n_k,k} \to \phi_{\infty,k}$ uniformly as $n_k \to \infty$. Since the domains $\mu^k\mathcal{J}$ are increasing with respect to k, by taking further subsequences each time we get the property that for $l > k$, $\phi_{\infty,l}|_{\mu^k\mathcal{J}} = \phi_{\infty,k}$. Therefore, by letting $k \to \infty$ we get an η-qs map $\phi_\infty : \mathcal{J}_\infty \to \mathbb{C}$. Since $\mathcal{J}_\infty$ is dense by Proposition 4.1.1 and since qs maps may always be extended to the closure, ϕ_∞ can be extended to a map $\phi_\infty : \mathbb{C} \to \mathbb{C}$.

We now show that for a limb $\mathcal{L}_c$ rooted at c, $\phi_\infty(\mathcal{L}_c) \subseteq \overline{\mathcal{L}_{\phi_\infty(c)}}$. Let $c \in \mathcal{J}_\infty$ be a branch point, let $\beta(c)$ be the tip of the spine of the (relevant) transcendental limb rooted at c, and let $\beta_n(c)$ be the tip of the spine of the (relevant) limb of $\mu^n\mathcal{J}$ rooted at c. Since these limbs exhaust the transcendental limb rooted at c, then for $n \geq N$ and N sufficiently large, $|\beta_n(c) - c| \geq |\beta(c) - c|/2$. Let n_k be a subsequence such that $\phi_{n_k} \to \phi_\infty$ uniformly on $\mu^N\mathcal{J}$. Since each ϕ_{n_k} is η-qs, fixes 0, and is normalized at x_1, it follows that $\phi_{n_k}(\mathcal{L}_c \cap \mu^N\mathcal{J}) \subset K = K(c, \eta)$, where K is compact. In addition, by the qs property,

$$\left|\frac{\phi_{n_k}(\beta_N(c)) - \phi_{n_k}(c)}{\phi_{n_k}(c) - \phi_{n_k}(0)}\right| \geq \frac{1}{\eta\left(\left|\frac{c-0}{\beta_N(c)-c}\right|\right)}.$$

Hence,

$$\begin{aligned}\operatorname{diam}(\mathcal{L}_{\phi_{n_k}(c)}) &\geq |\phi_{n_k}(\beta_N(c)) - \phi_{n_k}(c)| \\ &\geq \frac{C_0|c|}{\eta\left(\left|\frac{c}{\beta_N(c)-c}\right|\right)} = C_1 = C_1(c,\eta).\end{aligned}$$

Since the closure of transcendental limbs are uniform quasidisks (Proposition 4.7.11,) it follows that for all $n \geq N$, area$(\overline{\mathcal{L}_{\phi_{n_k}(c)}}) \geq C_2$ for some $C_2 > 0$ that depends on C_1. Since a compact set can contain only finitely many limbs whose closure has area greater than C_2, the set $\{\phi_{n_k}(c)\}$ must be finite. Since $\phi_{n_k}(c) \to \phi_\infty(c)$, it follows from discreteness of the roots that for some $N_1 >> 1$ and for all $n_k \geq N_1$, $\phi_{n_k}(c) = \phi_{N_1}(c)$. Hence, $\phi_\infty(\mathcal{L}_c) \subset \overline{\mathcal{L}_{\phi_\infty(c)}}$.

□

Proposition 5.2.5. *If $\theta_{1/4} : \mathcal{J} \to \mathcal{J}$ is a rotation by 1/4 then $\theta_{1/4}$ is not quasisymmetric.*

Proof. Suppose $\theta_{1/4}$ is a qs rotation by 1/4. By Lemma 5.2.4, there is a sequence of normalized scalings of $\theta_{1/4}$ that uniformly converges to a qs map $\theta_{1/4,\infty} : \mathbb{C} \to \mathbb{C}$ which sends limbs to limbs. However, by Proposition 4.9.6 $\theta_{1/4}$ cannot be continuous because of the combinatorial difference between real and imaginary limbs. □

Lemma 5.2.6. *If $\phi : \mathcal{J} \to \mathcal{J}$ is topologically extendable then ϕ preserves the rotational ordering at every branch point.*

Proof. By Lemma 3.2.7, $\mathcal{J}$ is tetravalent and all the branch points are pre-critical points. Let c be a pre-critical point. Since $\mathcal{J} \setminus \{c\}$ has 4 connected components, there are 4 external rays that land at c. Hence, $\psi^{-1}(\mathcal{J} \setminus \{c\})$ consists of 4 open intervals in $\mathbb{T}$. Let $\widehat{\phi} : \mathbb{T} \to \mathbb{T}$ be the lift of ϕ to the Böttcher coordinate. Since $\widehat{\phi}$ is continuous and orientation preserving, $\widehat{\phi}$ sends these 4 open intervals to another 4 open intervals while preserving their rotational ordering. Hence, when ϕ maps the connected components of $\mathcal{J} \setminus \{c\}$ to the connected components of $\mathcal{J} \setminus \{\phi(c)\}$ it preserves their rotational ordering. □

Recall the definition of levels of $\mathcal{J}$ from Section 3.2.

Corollary 5.2.7. *If $\phi : \mathcal{J} \to \mathcal{J}$ is a topologically extendable quasisymmetry, then ϕ preserves the levels of $\mathcal{J}$. In particular, $\phi(\mathcal{J} \cap \mathbb{R}) = \mathcal{J} \cap \mathbb{R}$.*

Proof. It suffices to show that $\phi(\mathcal{J} \cap \mathbb{R}) = \mathcal{J} \cap \mathbb{R}$. Since ϕ is topologically extendable, all the other levels will be preserved.

By Lemma 5.2.6, if $b \in \mathcal{J} \cap \mathbb{R}$, $\phi(b) = 0$, and $\phi(\mathbb{R} \cap \mathcal{J}) \neq \mathbb{R} \cap \mathcal{J}$ then it follows that $\phi(\mathcal{J} \cap \mathbb{R}) = \mathcal{J} \cap i\mathbb{R}$. For n such that $f^n(b) = 0$, we have that $\phi \circ f^{-n}(0) = 0$ where the branch of f^{-n} is chosen such that it is defined on a neighborhood of 0 and sends 0 to b. Hence, if $b \in \mathcal{J} \cap \mathbb{R}$ and $\phi(\mathbb{R} \cap \mathcal{J}) \neq \mathbb{R} \cap \mathcal{J}$ then locally this composition will be a rotation by $1/4$ (or by $-1/4$) which, by Proposition 5.2.5, contradicts the assumption that ϕ is qs.

Suppose instead that $b \in \mathcal{J} \setminus \mathbb{R}$ is a branch point and $\phi(b) = 0$. There are two cases to consider (see Figure 5.5):

Case 1: ϕ maps the limbs of b to $L_\uparrow, L_\downarrow$. This is the same as would happen under locally mapping b up to 0. However, in this case, the path from b to 0 is mapped to a path from 0 to $\phi(0)$ with the same number and order of turns. Since each turn in a path from 0 results in an increase in level, it is then the case that for $z \in \mathcal{J} \cap \mathbb{R} \setminus \{0\}$, $\text{level}(\phi(z)) = \text{level}(\phi(0)) + 1$. What this means is that $\mathcal{J} \cap \mathbb{R}$ is mapped to the union of the spines of the limbs rooted at $\phi(0)$. If we locally map $\phi(0)$ back to 0 under f^n, then we get that in a neighborhood of 0, $f^n \circ \phi$ is locally a rotation by $1/4$. This can be put into a precise form to directly apply Proposition 5.2.5 by taking a large enough k such that $f^n|_{\phi(\mu^{-k}\mathcal{J})}$ is univalent. Thus the map $f^n \circ \phi \circ \mu^{-k} : \mathcal{J} \to \mathcal{J}$ is a rotation by $1/4$. This contradicts the assumption that ϕ is quasisymmetric.

Case 2: The alternative to Case 1 is that ϕ maps the spines of the limbs at b to $\mathcal{J} \cap \mathbb{R} \setminus \{0\}$. This is contrary to what would happen when mapping b to 0 under the dynamics. In a similar spirit to case 1, we would get that for some k, n that the composition $\phi \circ f^{-n} \circ \mu^{-k} : \mathcal{J} \to \mathcal{J}$ is a rotation by $1/4$ mapping into $\mathcal{J}$, a contradiction to the assumption that ϕ is quasisymmetric.

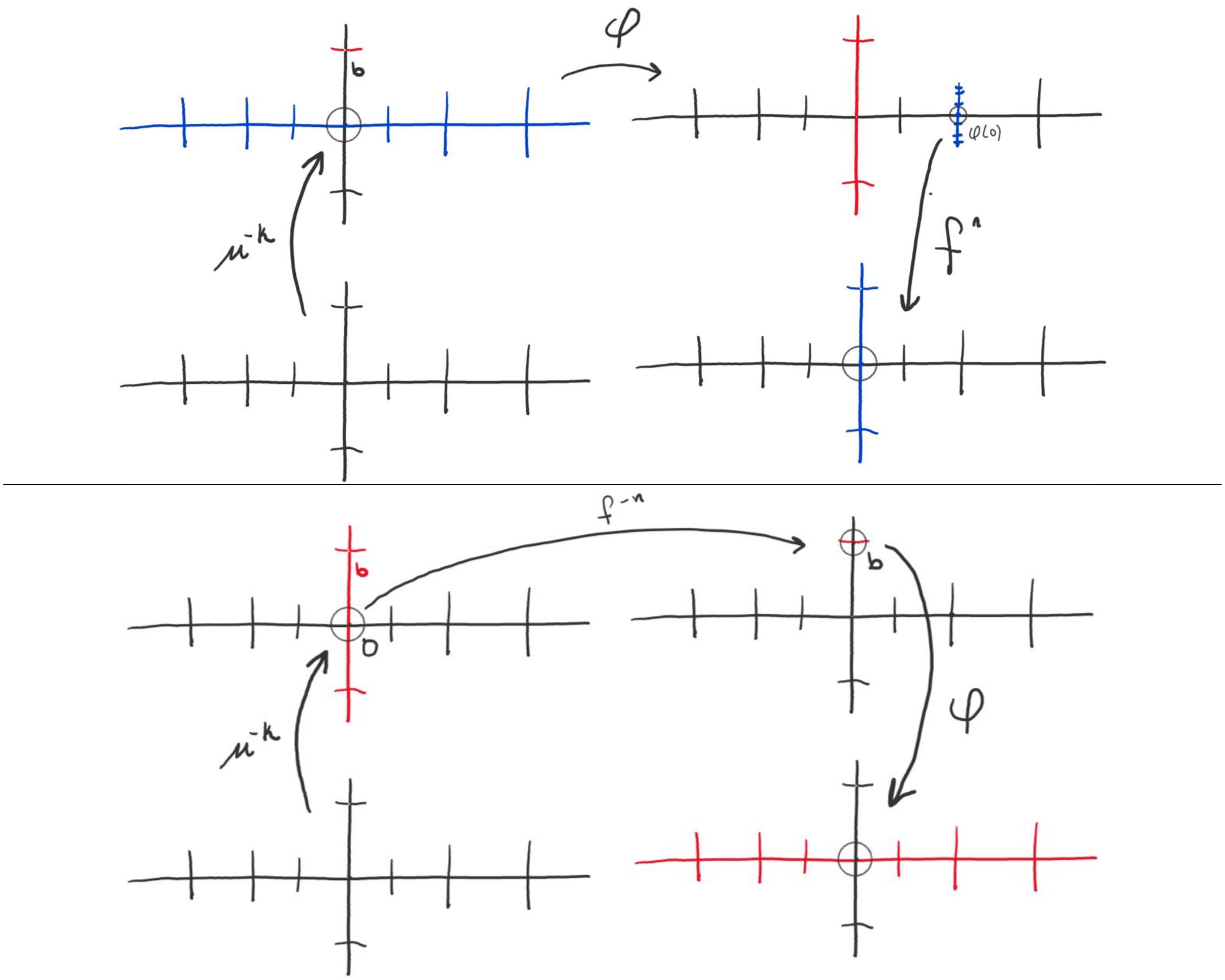

Figure 5.5: Case 1 (upper) and Case 2 (lower) for the proof of Corollary 5.2.7.

□

We are now ready to Prove Theorem 1.0.2.

Proof of Theorem 1.0.2. By Propositions 5.2.1 and 5.2.5, topological rotations by 1/4 exist while qs rotations by 1/4 do not. By Corollary 5.2.7, $\mathbb{R}$ is invariant under extendable quasisymmetries. Therefore, if $\phi : \mathcal{J} \to \mathcal{J}$ is an extendable quasisymmetry such that $\phi(0)$ is close to 0 then $\phi(\beta(L_\uparrow))$ must map to the tip of a limb whose root is close to 0 and so cannot be close to $\pm\beta$. Hence, $\rho_{1/4}$ (and more generally any topological rotation by 1/4) does not belong to the uniform closure of the group of (extendable) quasisymmetries. □

5.3 Deep little Julia sets move dynamically

Definition 5.3.1. We say a little Julia set $J \subset \mathcal{J}$ (of depth $k \in \mathbb{N}$) *moves dynamically with scaling* under a homeomorphism $\phi : \mathcal{J} \to \mathcal{J}$ if

$$\phi|_J = f^{-m} \circ \mu^l \circ \rho^i \circ f^n|_J$$

where n is the number required to map J injectively to the center, (i.e. $f^n(J) = \mu^{-k}\mathcal{J}$), $i \in \{0, 1\}$, $l \in \mathbb{Z}$, and f^{-m} is a well-defined branch. For $I \subset \mathcal{J}$ a *1d* little Julia set, the definition for moving dynamically with scaling is analogous.

We say a little Julia set (resp. 1d little Julia set) $J \subset \mathcal{J}$ (resp. $I \subset J$) *moves dynamically* under a homeomorphism ϕ if under the above definition, $l = 0$. Note that if J or I moves dynamically, then its image is another little Julia set (resp. 1d little Julia set) of the same depth.

We say a homeomorphism $\phi : \mathcal{J} \to \mathcal{J}$ is *k-dynamical* if every little Julia set of depth k moves dynamically.

Lemma 5.3.1. *If $\phi : \mathcal{J} \to \mathcal{J}$ is k-dynamical then ϕ sends little Julia sets (of all depths) to little Julia sets of the same depth.*

Proof. If J is a little Julia set of depth $k_J \geq k$ then J is contained in a little Julia set of depth k. Hence, J moves dynamically and so its image is a little Julia set of the same depth.

If J is a little Julia set of depth $k - 1$ then J is the closure of a countable union of little Julia sets of depth k. Let $J_0 \subset J$ be the central little Julia set of depth k – that is, for n such that $f^n(J) = \mu^{-k+1}\mathcal{J}$, $f^n(J_0) = \mu^{-k}\mathcal{J}$.

Let $J = \mu^{-k+1}\mathcal{J}$ and let $J_0 = \mu^{-k}\mathcal{J}$. Let $\tilde{J}$ be the little Julia set of depth $k - 1$ such that $\tilde{J} \supset \phi(J_0)$. We will show that $\tilde{J} = \phi(J)$. By Corollary 5.2.7, spine$(\tilde{J}) \subset \mathbb{R}$. Hence, for some $l_1, ..., l_k \in \mathbb{Z}$, $\sigma_{k-1}^{l_k} \circ \cdots \circ \sigma^{l_1}(\tilde{J}) = J$ since we may post-compose ϕ by ρ, we assume WLOG that $\phi(\beta) = \beta$. □

This section is devoted to proving the following proposition:

Proposition 5.3.2. *If $\phi : \mathcal{J} \to \mathcal{J}$ is η-qs then ϕ is k-dynamical for some k that depends only on η.*

Any quadratic limb, L_c, rooted at c has an extension to a unique transcendental limb $\mathcal{L}_c \supset L_c$ that is rooted at c as well. Two limbs, L_1, L_2, are said to *virtually touch* if their extensions to transcendental limbs touch.

Define the property $A(L_1, L_2)$ for quadratic limbs analogously for how it was defined for transcendental limbs, (see Definition 4.7.1.) If $A(L_1, L_2)$ is satisfied, define $\mathrm{depth}(L_1, L_2)$ in the following way: for n satisfying $f^n(L_1) = \pm L_\uparrow$, we have $f^n(L_2)$ is rooted at $\pm\mu^{-k}x_1$ for some $k := \mathrm{depth}(L_1, L_2) \in \mathbb{N}_0$. The following lemma says that when the depth is large, Property A is preserved under qs maps.

Lemma 5.3.3. *Suppose $\phi : \mathcal{J} \to \mathcal{J}$ is η-qs and L_1, L_2 are limbs such that $A(L_1, L_2)$ is satisfied and* $\mathrm{depth}(L_1, L_2) \geq k = k(\eta)$*. Then $A(\phi(L_1), \phi(L_2))$ is satisfied.*

Proof. Since $\mathcal{J}_\infty := \bigcup_{n\in\mathbb{N}} \mu^n \mathcal{J}$ and since $\mathcal{L}_\uparrow$ and $\mathcal{L}_{x_1}$ are touching limbs, then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $n \geq N$,

$$\frac{d(L_\uparrow, L_{\mu^{-n}x_1})}{\mathrm{diam}(L_{\mu^{-n}x_1})} = \frac{d(\mathcal{L}_\uparrow \cap \mu^n\mathcal{J}, \mathcal{L}_{x_1} \cap \mu^n\mathcal{J})}{\mathrm{diam}(\mathcal{L}_{x_1} \cap \mu^n\mathcal{J})} < \epsilon.$$

By Corollary 4.7.10, we can map L_1 and L_2 to $\pm L_\uparrow$ and one of the limbs rooted at $\pm\mu^{-n}x_1$ with good distortion on a neighborhood of L_2. Since the ratio

$$\frac{d(L_1, L_2)}{\min\{\mathrm{diam}(L_1), \mathrm{diam}(L_2)\}} \tag{5.3.1}$$

is quasi-preserved under qs maps, (5.3.1) is small when $\mathrm{depth}(L_1, L_2)$ is large.

Let $\mathcal{L}'_1, \mathcal{L}'_2$ be the extensions of $\phi(L_1), \phi(L_2)$ to transcendental limbs. Since $\phi(L_1) \subset \mathcal{L}'_1$ and $\phi(L_2) \subset \mathcal{L}'_2$, observe that for $i \in \{1, 2\}$

$$\frac{d(\phi(L_1), \phi(L_2))}{\mathrm{diam}(\phi(L_i))} \geq \frac{d(\mathcal{L}'_1, \mathcal{L}'_2)}{\mathrm{diam}(\mathcal{L}'_i)} \tag{5.3.2}$$

Since ϕ is topologically extendable, $\phi(L_1), \phi(L_2)$ must still be rooted in the same spine. Therefore, if $A(\phi(L_1), \phi(L_2))$ is not satisfied then either $\phi(L_1), \phi(L_2)$ don't virtually touch or $\mathrm{Gen}(\phi(L_2)) < \mathrm{Gen}(\phi(L_1))$. If $\phi(L_1), \phi(L_2)$ don't virtually touch then by Proposition 4.8.8 then the right side of (5.3.2) is large, a contradiction. If instead, they do touch, but $\mathrm{Gen}(\phi(L_2)) < \mathrm{Gen}(\phi(L_1))$ then by Proposition 4.8.9, then by a similar argument we again arrive at a contradiction. Hence, $A(\phi(L_1), \phi(L_2))$ must be satisfied when $\mathrm{depth}(L_1, L_2) \geq k = k(\eta)$. $\square$

We also need the following property, which extends Lemma 5.3.3 to secondary limbs as well.

Lemma 5.3.4. *For L_1, L_2 satisfying $A(L_1, L_2)$, there exists a $k_\eta = k_\eta(\eta)$ such that if $\mathrm{depth}(L_1, L_2) \geq k_\eta$ then the secondary limbs of L_1 that virtually touch L_2 must be sent under ϕ to the secondary limbs of $\phi(L_1)$ that virtually touch $\phi(L_2)$. (See Figure 5.6.)*

Proof. By Lemma 5.3.3, there exists $k_\eta \in \mathbb{N}$ such that when $A(L_1, L_2)$ is satisfied and $\mathrm{depth}(L_1, L_2) \geq k_\eta$ then $A(\phi(L_1), \phi(L_2))$ must also be satisfied. Therefore, by Proposition 4.8.10 and by the same logic as in the proof of Lemma 5.3.3, then for k_η sufficiently large, ϕ must send the secondary limbs of L_1 that virtually touch L_2 to the secondary limbs of $\phi(L_1)$ that virtually touch $\phi(L_2)$. $\square$

Corollary 5.3.5. *Let ϕ, L_1, L_2 be as in Lemma 5.3.4. If L_3 is another limb rooted in the same spine as L_1, L_2 such that L_3 virtually touches L_1 and $\mathrm{Gen}(L_3) = \mathrm{Gen}(L_2)$, then $\mathrm{Gen}(\phi(L_3)) = \mathrm{Gen}(\phi(L_2))$. (See Figure 5.6.)*

Proof. Let $n_1 = \mathrm{Gen}(L_1)$. Then $f^{n_1}(L_1) = \pm L_\uparrow$ and $f^{n_1}(L_2), f^{n_1}(L_3)$ are rooted at $\pm\mu^{-k}x_1$ for some common $k \in \mathbb{N}$. By symmetry of $\mathcal{J}$ about $i\mathbb{R}$, the secondary limbs of $\pm L_\uparrow$ that virtually touch $f^{n_1}(L_2)$ share the same roots as the secondary limbs that virtually touch $f^{n_1}(L_3)$. Hence, the secondary limbs of L_1 that virtually touch L_2 have the same roots as the secondary limbs that virtually touch L_3. Therefore, by Lemma 5.3.4, the mapping of the secondary limbs dictates that $\mathrm{Gen}(\phi(L_3)) = \mathrm{Gen}(\phi(L_2))$. $\square$

Given two limbs L_1, L_2 that virtually touch and are rooted at c_1, c_2, respectively, then by Corollary 4.9.1 there is a unique pre-critical point $c_3 \in (c_1, c_2)_{\mathcal{J}}$ such that one of the two limbs at c_3 virtually touches *both* L_1 and L_2. Call c_3 the *step down from* c_1 *and* c_2. We then inductively define a point $c \in (c_1, c_2)$ to be n *steps down from* c_1 *and* c_2 if it is $n-1$ steps down from either c_1 and c_3 or c_2 and c_3. Note that in this way, for each $n \in \mathbb{N}$ there are 2^{n-1} pre-critical points that are n steps down from c_1 and c_2. We say c_1 and c_2 are each *0 steps down from* c_1 *and* c_2. See Figure 4.9 for a picture where one can clearly see from looking at the limbs the pre-critical points that are 1 or 2 steps down from $-x_1$ and $-\mu^{-1}x_1$ (as well as one of the pre-critical points that is 3 steps down.)

Lemma 5.3.6. *Let* $c_1, c_2 \in \mathcal{J}$ *be the roots of virtually touching limbs rooted in the same spine. Then for any pre-critical point* $c \in (c_1, c_2)_{\mathcal{J}}$*,*

$$\mathrm{Gen}(c) > \max\{\mathrm{Gen}(c_1), \mathrm{Gen}(c_2)\}.$$

Furthermore, there is a unique pre-critical point $c \in (c_1, c_2)_{\mathcal{J}}$ *of least generation and this point is the step down from* c_1 *and* c_2*.*

Proof. Suppose WLOG that $\mathrm{Gen}(c_2) > \mathrm{Gen}(c_1) = n_1$. By Lemma 4.3.3, $f^{n_1}(c_1), f^{n_1}(c_2)$ are again the roots of virtually touching limbs. By Corollary 4.8.3, since $f^{n_1}(c_1) = 0$, $c_2' := f^{n_1}(c_2) = \pm\mu^{-k}x_1$ for some $k \in \mathbb{N}_0$. By Lemma 4.3.12, $c_3' := \mu^{-1}f^{n_1}(c_2)$ is the unique point of least generation in $(0, c_2')_{\mathcal{J}}$ and is of greater generation than that of either 0 or c_2'. By Corollary 4.9.1, it is also the step down from 0 and c_2'. Applying the branch of f^{-n_1} that sends $[0, c_2']_{\mathcal{J}}$ to $[c_1, c_2]_{\mathcal{J}}$ we obtain the result with $c_3 := f^{-n_1}(c_3')$ being the unique point of least generation in $(c_1, c_2)_{\mathcal{J}}$ and the step down from c_1 and c_2. □

Lemma 5.3.7. *If* $c_1, c_2 \in \mathcal{J}$ *are the roots of virtually touching limbs rooted in the same spine, then every pre-critical point* $c \in (c_1, c_2)_{\mathcal{J}}$ *is finitely many steps down from* c_1 *and* c_2*.*

Proof. Let $n = \max\{\mathrm{Gen}(c_1), \mathrm{Gen}(c_2)\}$, let $c \in (c_1, c_2)_{\mathcal{J}}$ and let $m = \mathrm{Gen}(c) - n$. By Lemma 5.3.6, $m > 0$.

Let $I_0 = (c_1, c_2)_{\mathcal{J}}$. By Lemma 4.9.1, there exists unique $c_3 \in (c_1, c_2)_{\mathcal{J}}$ such that the limbs rooted at c_3 virtually touch the limbs rooted at c_1 and at c_2. By Lemma 5.3.6 $\text{Gen}(c_3) > n$ and for every $c \in I_1 := (c_1, c_3)_{\mathcal{J}} \cup (c_3, c_2)_{\mathcal{J}}$, $\text{Gen}(c) > n + 1$. In other words, I_1 is equal to I_0 minus the unique pre-critical point that is 1 step down from c_1 and c_2. Define I_j to be I_0 minus the 2^{j-1} pre-critical points that are j steps down from c_1 and c_2.

Since every pre-critical point in I_m has generation greater than $m + n$, c is at most m steps down from c_1 and c_2. $\square$

Lemma 5.3.8. *Let L_1, L_2 be limbs satisfying $A(L_1, L_2)$ and let L_3 be the unique limb of greater generation that virtually touches both L_1 and L_2. Then* $\text{depth}(L_1, L_3)$ *and* $\text{depth}(L_2, L_3)$ *are both greater than or equal to* $\text{depth}(L_1, L_2)$.

Proof. Let $k \in \mathbb{N}_0$ denote $\text{depth}(L_1, L_2)$ and let $n_1 = \text{Gen}(L_1)$. Then, by definition, $f^{n_1}(L_1) = \pm L_\uparrow$ and $f^{n_1}(L_2)$ is rooted at $\pm\mu^{-k}x_1$. By continuity in the setting of $\mathcal{J}_\infty$, $f^{n_1}(L_3)$ is the unique limb of greater generation that virtually touches both $f^{n_1}(L_1)$ and $f^{n_1}(L_2)$. Therefore, $f^{n_1}(L_3)$ is rooted at $\pm\mu^{-k-1}x_1$. Hence, $\text{depth}(L_1, L_3) = k + 1$. Since f^{2^k} sends $\mu^{-k}x_1$ to 0 and $\mu^{-k-1}x_1$ to $(-1)^{k+1}\mu^{-k}x_1$, $\text{depth}(L_2, L_3) = k$. $\square$

Corollary 5.3.9. *Let ϕ, L_1, L_2 as in Lemma 5.3.4. Then the action of ϕ on L_1 and L_2 determines the action on all the limbs in between (rooted in the same spine). That is, for c_1, c_2 the roots of L_1, L_2, respectively,*

$$\phi|_{[c_1,c_2]_{\mathcal{J}}} = f^{-m} \circ (-1)^i \circ \mu^l \circ f^n|_{[c_1,c_2]_{\mathcal{J}}}$$

for some $m, n \in \mathbb{N}_0$, $i \in \{0, 1\}$, and $l \in \mathbb{Z}$ and well-defined branch of f^{-m}.

Proof. Let $n = \text{Gen}(L_1)$. Since $A(L_1, L_2)$ is satisfied, $f^n(L_1) = \pm L_\uparrow$ and $f^n(L_2)$ is rooted at $\pm\mu^{-k}x_1$ for some $k \geq k_\eta$. Since $k \geq k_\eta$, Lemma 5.3.3 tells us that $A(\phi(L_1), \phi(L_2))$ is satisfied. Then for $m = \text{Gen}(\phi(L_1))$, $f^m(\phi(L_1)) = \pm L_\uparrow$ and $f^m(\phi(L_2))$ is rooted at $\pm\mu^{l-k}x_1$ for some $l \leq k$. By topological extendability of ϕ, L_2 lies to the right of L_1 along their spine if and only if $\phi(L_2)$ lies to the right of $\phi(L_1)$ along their spine (oriented with respect to spine(L_1), spine$(\phi(L_1))$, respectively.) Let $X_0 = \{c_1, c_2\}$. Hence, for $i = 0$ or 1 and

$$\tau := f^{-m} \circ (-1)^i \circ \mu^l \circ f^n$$

where f^{-m} is the well-defined branch such that $\tau(c_1) = \phi(c_1)$ and $\tau(c_2) = \phi(c_2)$.

Let X_n (resp. X'_n) denote the set of pre-critical points in $[c_1, c_2]_{\mathcal{J}}$ (resp. $[\phi(c_1), \phi(c_2)]_{\mathcal{J}}$) that are at most n steps down from c_1 and c_2 (resp. $\phi(c_1)$ and $\phi(c_2)$.) We will show using induction that for all $n \in \mathbb{N}$, $\phi|_{X_n} = \tau|_{X_n}$.

The definition of τ requires it to send the point of least generation in a sub-interval of $[c_1, c_2]_{\mathcal{J}}$ to the point of least generation in the image (of the sub-interval.) Therefore, by applying induction to Lemma 5.3.6, we have that $\tau(X_n) = X'_n$ and preserves their relative ordering within their spine.

Suppose that for some n, $\phi(X_n) = X'_n$ and that ϕ preserves the relative ordering of X_n, X'_n within their spines. Let $c \in X_{n+1} \setminus X_n$. By definition, c is a step down from two points $c_a, c_b \in X_n$. Let L_c, L_{c_a}, and L_{c_b} be limbs rooted at their respective points such that they all (pairwise) virtually touch. By Lemma 5.3.8, the depth of each pair of limbs is at least k. WLOG, assume that $\text{Gen}(c_a) < \text{Gen}(c_b)$ and so $A(L_{c_a}, L_{c_b})$ is satisfied. Since $c_a, c_b \in X_n$, we have by assumption that $A(\phi(L_{c_a}), \phi(L_{c_b}))$ is also satisfied. Applying Lemma 5.3.3, we have that $A(\phi(L_{c_a}), \phi(L_c))$ and $A(\phi(L_{c_b}), \phi(L_c))$ must also be satisfied. Hence, $\phi(c)$ is the unique step down from $\phi(c_a)$ and $\phi(c_b)$. Hence, $\phi(X_{n+1}) = X'_{n+1}$ and ϕ preserves their relative order.

Therefore, by induction $\phi|_{\bigcup_{n\in\mathbb{N}} X_n} = \tau|_{\bigcup_{n\in\mathbb{N}} X_n}$. Since $\bigcup_{n\in\mathbb{N}} X_n$ is equal to all pre-critical points in $[c_1, c_2]$ by Lemma 5.3.7 and since the set of pre-critical points in $[c_1, c_2]$ is dense in $[c_1, c_2]$ by Lemma 3.6.2, then by continuity we have that $\phi|_{[c_1,c_2]} = \tau|_{[c_1,c_2]}$. □

Let I be a 1-dimensional little Julia set of depth $k \in \mathbb{N}$, and let $n \in \mathbb{N}_0$ such that $f^n(I) = \mu^{-k}[-\beta, \beta]$ for some $k \in \mathbb{N}$. Given I, define the *center* to be the point in I that maps up to 0 under f^n. Similarly, define the *eldest children* to be the points in I that map up to $\pm\mu^{-k} x_1$ under f^n.

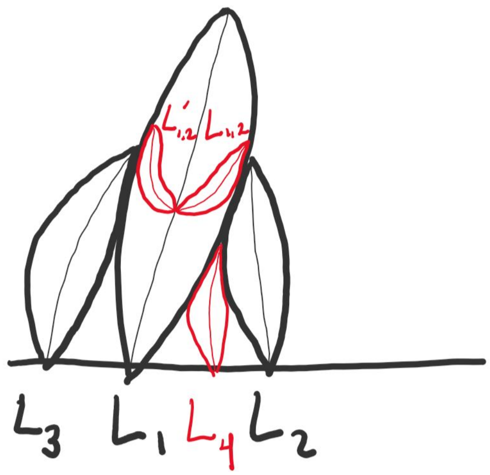

Figure 5.6: For depth$(L_1, L_2) \geq k_\eta$ from Lemma 5.3.4. The overall picture is preserved under ϕ.

Corollary 5.3.10. *1d little Julia sets of depth $k \geq k_\eta$ move dynamically with scaling.*

Proof. Since any little Julia set, J, of depth k can be mapped forward to $\mu^{-k}\mathcal{J}$ on a neighborhood of J with uniformly bounded distortion, it suffices to prove the corollary for the case when $I = \mu^{-k}[-\beta, \beta], \tilde{I} = \mu^{-\tilde{k}}[-\beta, \beta]$.

If $\phi(0) = 0$ and $\phi(\mu^{-k}x_1) = \mu^{-\tilde{k}}x_1$ then by Corollary 5.3.5, $\phi(-\mu^{-k}x_1) = -\mu^{-\tilde{k}}x_1$. Hence, by Corollary 5.3.9, $\phi|_{[-\mu^{-k}x_1, \mu^{-k}x_1]} = x \mapsto \mu^l x$. Define $x_n := f^{-n}(0)$ where the branch of f^{-n} is chosen to fix β. We show that for all n, $\phi|_{\mu^{-k}[-x_n, x_n]} = x \mapsto \mu^l x$.

Since x_2 and $\mu^{-1}x_1$ have the same generation, and their limbs both virtually touch L_{x_1}, then by Corollary 5.3.9 $\phi|_{\mu^{-k}[-x_1, x_2]} = x \mapsto \mu^l x$. By symmetry, this can be extended further to $\mu^{-k}[-x_2, x_2]$. One can continue in this manner to extend to all of I. □

Lemma 5.3.11. *If for some $k \in \mathbb{N}$, $\phi : \mathcal{J} \to \mathcal{J}$ is a qs map that sends 1d little Julia sets of depth at least k to 1d little Julia sets (of any depth) then ϕ sends 1d little Julia sets of all depths to 1d little Julia sets and further, ϕ preserves the depth of any 1d little Julia set.*

Proof. Let $k \in \mathbb{N}$ be such that ϕ sends 1d little Julia sets of depth at least k to 1d little Julia sets. Let I be a little Julia set of depth $k - 1$. By Lemma 3.5.4, there exists sequence $(I_j)_{j \in \mathbb{Z}}$, such that for each $j \in \mathbb{Z}$ $I_j \subset I$ is a 1d little Julia set of depth k and such that for any $j \in \mathbb{Z}$,

$I_j \cap I_{j+1}$ is a point. Furthermore, Lemma 3.5.4 tells us that $I = \overline{\bigcup_{j\in\mathbb{Z}} I_j}$. By assumption, $\phi(I_j), \phi(I_{j+1})$ must be 1d little Julia sets. Since I_j, I_{j+1} share an endpoint, $\phi(I_j), \phi(I_{j+1})$ must also share an endpoint and so must be of the same depth. Hence,

$$\phi(I) = \phi(\overline{\bigcup_{j\in\mathbb{Z}} I_j}) = \overline{\bigcup_{j\in\mathbb{Z}} \phi(I_j)}$$

is a 1d little Julia set. Hence, 1d little Julia sets of depth $k-1$ go to 1d little Julia sets. Continuing in this way, we see that if 1d little Julia sets of depth at least k go to 1d little Julia sets then 1d little Julia sets of *every* depth go to 1d little Julia sets.

We now show that the depth of a 1d little Julia set must be preserved. Since every 1d little Julia set is contained in a patriarchal 1d little Julia set, it suffices to show that patriarchal 1d little Julia sets go to patriarchal 1d little Julia sets of the same depth.

By Corollary 5.2.7 we already know that $[-\beta, \beta]$ maps to itself under any qs map. For I any other patriarch 1d little Julia set, I is uniquely identified by 3 things: a depth k, a limb L such that $I \subset \mathrm{spine}(L)$, and a number $l \in \mathbb{N}$ for which it is the l-th closest 1d little Julia set of depth k to the root of L amongst the 1d little Julia sets in $\mathrm{spine}(L)$. In this way, every patriarch 1d little Julia set other than $[-\beta, \beta]$ is part of a 1-sided sequence of 1d little Julia sets of the same depth with adjacent elements of the sequence sharing an endpoint. Other 1d little Julia sets, however, are part of a 2-sided sequence. Assuming 1d little Julia sets go to 1d little Julia sets, this distinction implies that patriarch 1d little Julia sets go to patriarch 1d little Julia sets. Within the spine of any limb, it is the patriarch 1d little Julia sets of depth 1 that limit on the tip of the spine. Hence, patriarch 1d little Julia sets of depth 1 must map to patriarch 1d little Julia sets of depth 1. One may then use induction to show that the depth of *every* patriarch 1d little Julia set must be preserved as well as its position, l, in the one-sided sequence in which it lies. In summary, if 1d little Julia sets go to 1d little Julia sets, then they must also go to 1d little Julia sets of the same depth. $\square$

The following is an immediate corollary of Corollary 5.3.10 and Lemma 5.3.11.

Corollary 5.3.12. *1d little Julia sets of depth $k \geq k_\eta$ move dynamically under extendable η-quasisymmetries of $\mathcal{J}$.*

Lemma 5.3.13. *If 1d little Julia sets of depth at least k move dynamically, then (full) little Julia sets of depth k move dynamically.*

Proof. Suppose 1d little Julia sets of depth at least k move dynamically under ϕ. Let J be a little Julia set of depth k and let $I = \mathrm{spine}(J)$. By assumption $\phi|_I = f^{-\tilde{n}} \circ \rho^l \circ f^n$. We want to show that $\phi|_J = f^{-\tilde{n}} \circ \rho^l \circ f^n$. Let L be any limb rooted in I. $\mathrm{spine}(L) \cap J$ is a countable union of patriarchal 1d little Julia sets of depth greater than K. Hence, each must move dynamically and go to the corresponding patriarchal 1d little Julia set of the same depth and position in $\mathrm{spine}(\phi(L))$. Hence, $\phi|_{\mathrm{spine}(L)\cap J} = f^{-\tilde{n}} \circ \rho^l \circ f^n$. This argument may be repeated for any limb L' rooted in $\mathrm{spine}(L) \cap J$. Since L, L' were arbitrary, we can induct on the number of turns necessary to reach a point $z \in J$ from $\mathrm{spine}(J)$. This results in a dense subset $X \subset J$ on which $\phi|_X = f^{-\tilde{n}} \circ \rho^l \circ f^n$. Extending to the closure, we have that all of J moves dynamically. □

Corollary 5.3.12 and Lemma 5.3.13 together imply Proposition 5.3.2.

5.4 qs Lift to $\mathbb{T}$

Let $\phi : \mathcal{J} \to \mathcal{J}$ be an η-quasisymmetry. By Proposition 5.3.2, let $k \in \mathbb{N}$ be such that it is k-dynamical. Let J be any little Julia set of depth $i < k$ and let $\tilde{J} = \phi(J)$. Let $n, \tilde{n} \in \mathbb{N}_0$ be the dynamical distances of $J, \tilde{J}$ to $\mu^{-i}\mathcal{J}$. Then we have the following commutative diagram: (by Proposition 5.3.2)

$$\begin{tikzcd} J \arrow[r, "\phi"] \arrow[d, "f^n"] & \tilde{J} \arrow[d, "f^{\tilde{n}}"] \\ \mu^{-i}\mathcal{J} \arrow[r, dashed] & \mu^{-i}\mathcal{J} \end{tikzcd}$$

Since ϕ is η-qs and $f^n|_J, f^{\tilde{n}}|_{\tilde{J}}$ are η'-qs by Koebe, the induced map on $\mu^{-i}\mathcal{J}$ must be η''-qs. By Lemma 5.3.1 and Proposition 5.3.2 the induced map sends little Julia sets in $\mu^{-1}\mathcal{J}$

to little Julia sets in $\mu^{-1}\mathcal{J}$ of the same depth. Therefore, by Corollary 5.2.7, the little Julia sets of depth $i+1$ intersecting $\mu^{-1}\mathcal{J}\cap\mathbb{R}$ form an invariant collection under the induced map. By Lemma 3.5.4, this collection is naturally indexed by $\mathbb{Z}$ such that consecutive elements overlap at a point. Therefore, by continuity, there exists $l\in\mathbb{Z}$ and $\epsilon\in\{0,1\}$ such that $\sigma_i^l\circ\rho^\epsilon$ and the induced map from the diagram have the same action on the spine children of $\mu^{-i}\mathcal{J}$. However, since the induced map is η''-qs, it follows that $|l|\leq N$ for some $N\in\mathbb{N}$ that depends only on η'' which, in turn, depends only on η – not on i or J. We say such a map – one that is k-dynamical and can shift the spine children of any given little Julia set at most N times – is *of type* (k,N). For the rest of this section, $\phi:\mathcal{J}\to\mathcal{J}$ will be any (topologically extendable) homeomorphism of type (k,N).

The following lemma is immediate from the preceding paragraph.

Lemma 5.4.1. *If $\phi:\mathcal{J}\to\mathcal{J}$ is a topologically extendable η-quasisymmetry then it is of type (k,N) where k,N depend only on η.*

The rest of this section is devoted to proving the following proposition:

Proposition 5.4.2. *For $\phi:\mathcal{J}\to\mathcal{J}$ of type (k,N). The lift of ϕ to the Böttcher coordinate $\widehat{\phi}:\mathbb{T}\to\mathbb{T}$ is η'-qs where η' depends only on k,N.*

Lemma 5.4.3. *If J is a little Julia set of depth $l\in\mathbb{N}_0$, then $\widetilde{\sigma}_J$, (the lift of σ_J to $\mathbb{T}$), is 2^{2^l}–bi-Lipschitz.*

Proof. Since σ_J is defined as identity outside of $\widehat{J}$ and by conjugating σ_l under the dynamics so that it acts on $\widehat{J}$, it suffices to show that the lift of σ_l to $\mathbb{T}$, $\widetilde{\sigma}_l$ is 2^{2^l}–bi-Lipschitz. By (1.0.1) and (5.1.1), σ_l is piecewise dynamical and on each segment is identity, f^{2^l}, or a branch of f^{-2^l}. It therefore follows that $\widetilde{\sigma}_l$ is piecewise dynamical under the doubling map on $\mathbb{T}$, g, and on each segment is identity, g^{2^l}, or a branch of g^{-2^l}. Hence, $\widetilde{\sigma}_l$ is piecewise linear and 2^{2^l}–bi-Lipschitz. $\square$

Lemma 5.4.4. *For every $l \in \mathbb{N}$ there exists a correspondence between $\mathbb{Z}^l$ and the real little Julia sets of depth l.*

Proof. We define the correspondence inductively such that $J_{(i_1,i_2,\ldots,i_l)}$ is the i_l-th spine child of $J_{i_1,i_2,\ldots,i_{l-1}}$ where the ordering is from left to right and the 0-th spine child is the central one.

For $l = 1$ the real little Julia sets of depth 1 are naturally indexed by $\mathbb{Z}$ with the integer 0 corresponding to $\mu^{-1}\mathcal{J}$. If $l > 1$ and the correspondence is well-defined for $j < l$, then we can extend it to a correspondence between $\mathbb{Z}^l$ and the little Julia sets of depth l by attaching the correspondence between $\mathbb{Z}$ and the spine children of each real little Julia set of depth $l-1$ onto the existing correspondence between $\mathbb{Z}^{l-1}$ and the real little Julia sets of depth $l-1$. $\square$

Let J be a real little Julia set of depth $l < k$. Since σ_J is defined by bringing J to $\mu^{-l}\mathcal{J}$ under the dynamics, σ_J either shifts the spine children of J to the left or the right. To remove this technicality, we will define a new collection of shift maps that always shift real little Julia sets to the right in the following way.

For $l \in \{1, \ldots, k-1\}$, using the correspondence between real little Julia sets of depth l and $\mathbb{Z}^l$, for each $(i_1, i_2, \ldots, i_l) \in \mathbb{Z}^l$, define $\sigma_{(i_1,i_2,\ldots,i_l)} := \sigma_J^{\epsilon}$ where J and $(i_1, i_2, \ldots, i_l)$ are identified under the correspondence and $\epsilon \in \{1, -1\}$ is such that $\sigma_{(i_1,i_2,\ldots,i_l)}$ shifts the spine children of J to the right.

Lemma 5.4.5. *If $J, \tilde{J}$ correspond to $(i_1, i_2, \ldots, i_k), (\tilde{i}_1, \tilde{i}_2, \ldots, \tilde{i}_k)$, respectively then for $l_j := \tilde{i}_j - i_j$, the map $\sigma_0^{l_1}\sigma_{(i_1)}^{l_2}\cdots\sigma_{(i_1,i_2,\ldots,i_{k-1})}^{l_k}$ sends J to $\tilde{J}$.*

Proof. Let $J_{(j_1,j_2,...,j_l)}$ be the real little Julia set that corresponds to $(j_1, j_2, ..., j_l)$.

$$\begin{aligned}
\sigma_0^{l_1}\sigma_{(i_1)}^{l_2}\cdots\sigma_{(i_1,i_2,...,i_{k-2})}^{l_{k-1}}\sigma_{(i_1,i_2,...,i_{k-1})}^{l_k}(J_{(i_1,i_2,...,i_k)}) &= \sigma_0^{l_1}\sigma_{(i_1)}^{l_2}\cdots\sigma_{(i_1,i_2,...,i_{k-2})}^{l_{k-1}}(J_{(i_1,i_2,...,i_k+l_k)}) \\
&= \sigma_0^{l_1}\sigma_{(i_1)}^{l_2}\cdots\sigma_{(i_1,i_2,...,i_{k-2})}^{l_{k-1}}(J_{(i_1,i_2,...,\tilde{i}_k)}) \\
&\vdots \\
&= \sigma_0^{l_1}\sigma_{(i_1)}^{l_2}(J_{(i_1,i_2,\tilde{i}_3,...,\tilde{i}_k)}) \\
&= \sigma_0^{l_1}(J_{(i_1,\tilde{i}_2,\tilde{i}_3,...,\tilde{i}_k)}) \\
&= J_{(\tilde{i}_1,\tilde{i}_2,\tilde{i}_3,...,\tilde{i}_k)}
\end{aligned}$$

□

Lemma 5.4.6. *Let $\Lambda = \psi^{-1}(\mathcal{J} \cap \mathbb{R}) \subset \mathbb{T}$. Then $\widehat{\phi}|_\Lambda$ is M–bi-Lipschitz where*

$$M := 2^{(2^k-1)N}. \tag{5.4.1}$$

Proof. Let J be a real little Julia set of depth k and let $\tilde{J} = \phi(J)$. Since ϕ is k-dynamical then by Lemma 5.3.1 and Corollary 5.2.7 $\tilde{J}$ is a real little Julia set of depth k. Let $J, \tilde{J}$ correspond to $(i_1, i_2, ..., i_k), (\tilde{i}_1, \tilde{i}_2, ..., \tilde{i}_k)$, respectively. By Lemma 5.4.5,

$$\phi|_J = \sigma_0^{l_1}\sigma_{(i_1)}^{l_2}\cdots\sigma_{(i_1,i_2,...,i_{k-1})}^{l_k}|_J$$

where $l_j := \tilde{i}_j - i_j$. However, since ϕ is of type (k, N), $|l_j| \leq N$ for all $j \in \{1, 2, ..., k\}$. By Lemma 5.4.3, $\widetilde{\sigma}_{(i_1,...,i_j)}$ is 2^{2^l}–bi-Lipschitz. Hence, $\widetilde{\sigma}_{(i_1,...,i_j)}^{l_{j+1}}$ is $2^{2^l N}$–bi-Lipschitz. Letting j run over $0, 1, ..., k-1$, we get that for

$$M := 2^{2^0 N} \cdot 2^{2^1 N} \cdots 2^{2^{k-1} N} = 2^{(2^k-1)N}$$

$\widehat{\phi}|_J$ is M–bi-Lipschitz. Since ϕ is k-dynamical it is therefore affine with slope in $[M^{-1}, M]$.

Since the dynamics of the limbs of $\mathcal{J}$ are determined by the dynamics of their roots, we may extend linearly the definition of $\widehat{\phi}|_J$ to all of $\widehat{J}$, the extended little Julia set. Let $\widehat{\phi}_\mathbb{R}$ denote the map that agrees with $\widehat{\phi}$ on (the lift under the Böttcher coordinate of) real little Julia sets of depth k and is defined on the rest of $\mathcal{J}$ by extending linearly each real little

Julia set of depth k to its extended little Julia set. Then $\widehat{\phi}_{\mathbb{R}}$ is a piecewise linear map of $\mathbb{T}$ to itself (with countably many breakpoints) and such that the slope of each segment of linearity lies in $[M^{-1}, M]$. Hence, $\widehat{\phi}_{\mathbb{R}}$ is M–bi-Lipschitz. Therefore, since $\widehat{\phi}_{\mathbb{R}}|_{\Lambda} = \widehat{\phi}|_{\Lambda}$, $\widehat{\phi}|_{\Lambda}$ is M–bi-Lipschitz. □

We now begin gradually expanding the scope of points in $\mathbb{T}$ over which we are able to control the regularity of $\widetilde{\phi}$.

Lemma 5.4.7. *Let $L \subset \mathcal{J}$ be a limb, $t_1, t_2 \in \mathbb{T}$ such that $\psi(t_i) \in \operatorname{spine}(L)$, and M as above. Then*

$$|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)| \cong_M \frac{|\phi(L)|}{|L|} \cdot |t_1 - t_2|.$$

Proof. To begin, consider first a single little Julia set, J, of depth k whose spine is contained in $\operatorname{spine}(L)$. As a corollary of No Rotations, $\phi(\operatorname{spine}(L)) = \operatorname{spine}(\phi(L))$. The location of J within $\operatorname{spine}(L)$ can be effectively described in terms of the maximal little Julia set, $J_{\max}$, which is the largest little Julia set satisfying $\operatorname{spine}(J) \subset \operatorname{spine}(J_{\max}) \subset \operatorname{spine}(L)$. The location of $J_{\max}$ within $\operatorname{spine}(L)$ cannot change under ϕ, hence

$$|\phi(J_{\max})| = \frac{|\phi(L)|}{|L|} \cdot |J_{\max}|.$$

Within $J_{\max}$, the location of J can be given by a finite sequence $(a_{j+1}, a_{j+2}, ..., a_k)$ where j is the depth of $J_{\max}$. Let $\tilde{J}_{\max} = \phi(J_{\max})$ and let $(\tilde{a}_{j+1}, \tilde{a}_{j+2}, ..., \tilde{a}_k)$ give the location of $\tilde{J} = \phi(J)$. Since ϕ is of type (k, N), it follows that $|a_i - \tilde{a}_i| \leq N$ for each $i \in \{j+1, ..., k\}$. From this, we have that $|\tilde{J}|/|\tilde{J}_{\max}| \cong_M |J|/|J_{\max}|$. Combining this with the above equation, we get

$$|\tilde{J}| \cong_M \frac{|\widehat{\phi}(L)|}{|L|} \cdot |J|.$$

Given the last equation, we can now apply the argument used in the proof of the previous lemma for arbitrary, t_1, t_2 satisfying $\psi(t_1), \psi(t_2) \in \operatorname{spine}(L)$. □

Let $t_0, t_k \in \mathbb{T} \cap [0, 1/4]$ such that $\psi(t_0) = 0, \psi(t_k)$ equal to the α_k-point in $\operatorname{spine}(L_\uparrow)$ closest to 0. Let $C_k = |L_\uparrow|/|t_k - t_0|$. It then follows that for any limb $L = \psi([t_a, t_b])$, if $t \in (t_a, t_b)$

such that $\psi(t)$ lies outside the little Julia set of depth k containing the root of L, then

$$|t_a - t|, |t_b - t| \cong_{C_k} |t_b - t_a| = |L|.$$

Lemma 5.4.8. *Let $t_1 < t_2 \in \mathbb{T}$ and define $z_i = \psi(t_i)$ for $i = 1, 2$.*

1. *If there exists a limb L such that $L \ni z_1, z_2$ then*

$$|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)| \cong_{C_k^2 M} \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |t_1 - t_2|$$

   where $L_{1,2}$ is the smallest limb containing both z_1, z_2.

2. *If no such limb L exists, then*

$$|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)| \cong_{C_k^2 M} |t_1 - t_2|.$$

Proof. We prove the first of the two statements. The proof of the second is analogous. The idea of the proof is as follows: divide $[t_1, t_2]$ into three sub-intervals $[t_1, \gamma_{1,1}], [\gamma_{1,1}, \gamma_{2,1}], [\gamma_{2,1}, t_2]$ where the ray $R_{\gamma_{i,1}}$ lands at a pre-critical point, $c_{i,1} \in \text{spine}(L_{1,2})$ such that $\psi(t_i) \in L_{c_{i,1}}$. Since by Lemma 5.4.7, we already have expansion along the spine of a limb controlled, the problem is reduced to showing that expansion in these sub-limbs is controlled. This is done by considering the first turn in $[c_{i,1}, z_i]_{\mathcal{J}}$ that leaves the little Julia set of depth k containing $c_{i,1}$. Before that turn, there can be no added expansion. The

Then $[t_1, t_2]$ is contained in the lift of L to the Böttcher coordinate. For $i = 1, 2$, define $n_i \in \mathbb{N} \cup \{\infty\}$ to be the level of z_i minus the level of $\text{spine}(L_{1,2})$ (i.e. n_i is the number of turns needed to reach z_i from $\text{spine}(L_{1,2})$.) Let $A_i = \{l \in \mathbb{N} \mid l \leq n_i\}$. There is a sequence of pre-critical points $(c_{i,j})_{j \in A_i}$ corresponding to a sequence of nested limbs such that

1. $c_{i,1} \in \text{spine}(L_{1,2})$
2. for $j, j+1 \in A_i$, $c_{i,j+1} \in \text{spine}(L_{c_{i,j}})$.

For every such $c_{i,j}$ (except for possibly when $n_i < \infty$, $j = n_i$, and $c_{i,n_i} = z_i$) there is a unique point $\gamma_{i,j} \in [t_1, t_2]$ such that $\psi(\gamma_{i,j}) = c_{i,j}$.

We have the following two equations.

$$|\gamma_{i,1} - t_i| = \left(\sum_{j=1}^{n_i - 1} |\gamma_{i,j} - \gamma_{i,j+1}| \right) + |\gamma_{i,n_i} - t_i|,$$

$$|\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(t_i)| = \left(\sum_{j=1}^{n_i - 1} |\widehat{\phi}(\gamma_{i,j}) - \widehat{\phi}(\gamma_{i,j+1})| \right) + |\widehat{\phi}(\gamma_{i,n_i}) - \widehat{\phi}(t_i)|.$$

For each $j \in A_i$, define $\alpha_k(\gamma_{i,j}) \in \mathbb{T}$ to be the closer to $\gamma_{i,j}$ of the 2 points in $\mathbb{T}$ that descend to the α_k-point in spine$(L_{c_{i,j}})$ closest to $c_{i,j}$. Suppose for now that there exists an index $j_{\min}$ which is the smallest index such that the arc between $\gamma_{i,j_{\min}}$ and $\gamma_{i,j_{\min}+1}$ contains the arc between $\gamma_{i,j_{\min}}$ and $\alpha_k(\gamma_{i,j_{\min}})$. Then for each $j \in A_i$, $j < j_{\min}$, we have that

$$|\widehat{\phi}(\gamma_{i,j}) - \widehat{\phi}(\gamma_{i,j+1})| = \frac{|\phi(L_{c_{i,j}})|}{|L_{c_{i,j}}|} \cdot |\gamma_{i,j} - \gamma_{i,j+1}|$$

and also that

$$\frac{|\phi(L_{c_{i,j}})|}{|L_{c_{i,j}}|} = \frac{|\phi(L_{c_{i,j+1}})|}{|L_{c_{i,j+1}}|}.$$

Therefore, we have that

$$\begin{aligned}
|\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(\gamma_{i,j_{\min}})| &= \sum_{l=1}^{j_{\min}-1} |\widehat{\phi}(\gamma_{i,l}) - \widehat{\phi}(\gamma_{i,l+1})| \\
&= \sum_{l=1}^{j_{\min}-1} \frac{|\phi(L_{c_{i,l}})|}{|L_{c_{i,l}}|} |\gamma_{i,l} - \gamma_{i,l+1}| \\
&= \frac{|\phi(L_{c_{i,1}})|}{|L_{c_{i,1}}|} \sum_{l=1}^{j_{\min}-1} |\gamma_{i,l} - \gamma_{i,l+1}| \\
&= \frac{|\phi(L_{c_{i,1}})|}{|L_{c_{i,1}}|} \cdot |\gamma_{i,1} - \gamma_{i,j_{\min}}| \\
&\cong_M \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |\gamma_{i,1} - \gamma_{i,j_{\min}}|.
\end{aligned}$$

Beyond $j_{\min}$, we have that

$$|L_{c_{i,j_{\min}}}| \geq |t_i - \gamma_{i,j_{\min}}| \geq |\alpha_k(\gamma_{i,j_{\min}}) - \gamma_{i,j_{\min}}| = C_k^{-1} |L_{c_{i,j_{\min}}}|.$$

Moreover, since $\widehat{\phi}(\alpha_k(\gamma_{i,j_{\min}})) = \alpha_k(\widehat{\phi}(\gamma_{i,j_{\min}}))$, the same thing can be said in the image. That is,

$$|\phi(L_{c_{i,j_{\min}}})| \geq |\widehat{\phi}(t_i) - \widehat{\phi}(\gamma_{i,j_{\min}})| \geq |\widehat{\phi}(\alpha_k(\gamma_{i,j_{\min}})) - \widehat{\phi}(\gamma_{i,j_{\min}})| = C_k^{-1} |\phi(L_{c_{i,j_{\min}}})|.$$

Putting this together, we have the following chain of equations (up to marked constant factors):

$$\begin{aligned}
|\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(t_i)| &= |\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(\gamma_{i,j_{\min}})| + |\widehat{\phi}(\gamma_{i,j_{\min}}) - \widehat{\phi}(t_i)| \\
&\cong_{C_k} |\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(\gamma_{i,j_{\min}})| + |L_{\phi(c_{i,j_{\min}})}| \\
&= \frac{|\phi(L_{c_{i,1}})|}{|L_{c_{i,1}}|} \cdot \Big(|\gamma_{i,1} - \gamma_{i,j_{\min}}| + |L_{c_{i,j_{\min}}}|\Big) \\
&\cong_M \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot \Big(|\gamma_{i,1} - \gamma_{i,j_{\min}}| + |L_{c_{i,j_{\min}}}|\Big) \\
&\cong_{C_k} \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot (|\gamma_{i,1} - \gamma_{i,j_{\min}}| + |\gamma_{i,j_{\min}} - t_i|) \\
&= \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |\gamma_{i,1} - t_i|.
\end{aligned}$$

Hence, $|\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(t_i)| \cong_{C_k^2 M} \dfrac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |\gamma_{i,1} - t_i|$.

If no such index $j_{\min}$ exists, then we obtain the stronger statement:

$$|\widehat{\phi}(\gamma_{i,1}) - \widehat{\phi}(t_i)| \cong_M \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |\gamma_{i,1} - t_i|.$$

By Lemma 5.4.7, we have that

$$|\widehat{\phi}(\gamma_{1,1}) - \widehat{\phi}(\gamma_{2,1})| \cong_M \frac{|\phi(L_{1,2})|}{|L_{1,2}|} \cdot |\gamma_{1,1} - \gamma_{2,1}|.$$

Combining this with the above argument proves the statement.

□

Lemma 5.4.9. *Let $t_1, t_2, t_3 \in \mathbb{T}$ such that $|t_1 - t_2| = |t_1 - t_3| \neq 0$. Suppose also that there is a limb containing 2 of the 3 points. Let L_s be the smallest such limb. There is a number $n \in \mathbb{N}$ depending only on k, N such that the following hold:*

1. *If there is a limb containing all 3 points, let L_b be the smallest such limb. Then*

$$\frac{|\phi(L_s)|}{|L_s|} \cong_{M^N} \frac{|\phi(L_b)|}{|L_b|}.$$

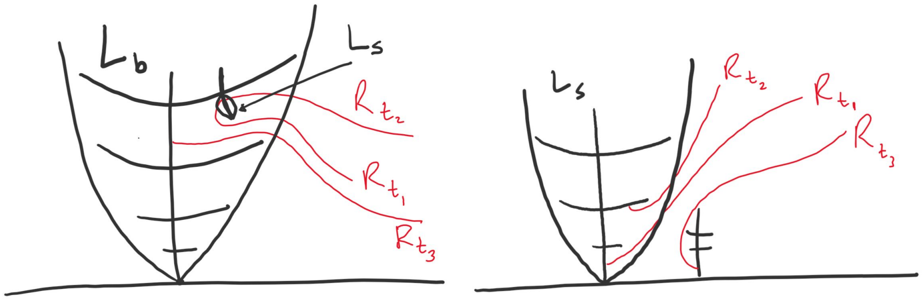

Figure 5.7: Case 1 (left) and Case 2 (right). Shown are the external rays corresponding to $t_1, t_2, t_3 \in \mathbb{T}$. In the drawn example of Case 1, there is only one limb, L such that $L_s \subsetneq L \subsetneq L_b$, however in general there can be arbitrarily many.

2. If not, then

$$\frac{|\phi(L_s)|}{|L_s|} \cong_{M^N} 1.$$

(See Figure 5.7.)

Proof. Let $I_{1,2}, I_{1,3} \subset \mathbb{T}$ be the intervals of equal length between t_1 and t_2, and between t_1 and t_3, respectively. Let $z_i = \psi(t_i)$. Since t_1 lies between t_2 and t_3 it follows that $z_1 \in L_s$. WLOG assume that $z_2 \in L_s$. Hence, $|t_3 - t_1| \leq |L_s|$. If $z_3 \in L_s$ then $L_b = L_s$ and so there is nothing to prove. Let $m \in \mathbb{N}$ be the number of limbs strictly containing L_s and strictly contained in L_b, if it exists. Name them $L_m \supsetneq L_{m-1} \supsetneq \cdots \supsetneq L_1 \supsetneq L_s$. For $i = 1, 2, ..., m$ let c_i be the root of L_i and let $\gamma_i \in \mathbb{T}$ such that $\gamma_i \in I_{1,3}$ and $\psi(\gamma_i) = c_i$. In this way, if L_b exists then $c_m \in \operatorname{spine}(L_b)$; otherwise, $c_m \in \mathbb{R}$. Furthermore, $c_i \in \operatorname{spine}(L_{i+1})$. Under this construction, we have that

$$|t_1 - t_3| = |t_1 - \gamma_1| + |\gamma_1 - \gamma_2| + \cdots + |\gamma_{n-1} - \gamma_n| + |\gamma_n - t_3|.$$

Let n be the smallest positive integer satisfying $2^{3(n-1)} > C_k$. Since C_k depends only on k which depends only on η, n depends only on η. Observe that since every pre-critical point in $i\mathbb{R}_+ \cap \mathcal{J}$ has dynamical distance at least 3, it follows that if $L_s \subset L_t$ then $|L_t| \geq 2^3|L_s|$.

If $1 < l \leq m$ and c_{l-1} lies beyond $\alpha_k(c_l)$ in $\operatorname{spine}(L_{c_l})$ then we have that

$$|t_1 - t_3| \geq |\gamma_l - \gamma_{l-1}| \geq C_k^{-1}|L_l| \geq C_k^{-1}2^{3l}|L_s| \geq C_k^{-1}2^{3l}|t_1 - t_2|.$$

Since $|t_1 - t_3| = |t_1 - t_2|$, it follows that l must be less than n. Hence, for all $n \leq l \leq m$, $|\phi(L_l)|/||L_l| = |\phi(L_n)|/|L_n|$ as they must all lie in the same little Julia set of depth k. Additional expansion/contraction can occur at c_m even when $m > n-1$ since $|t_3 - \gamma_n|$ can be arbitrarily small relative to $|t_3 - t_1|$. Hence, for $R = |\phi(L_b)|/|L_b|$ if L_b exists and $R = 1$ otherwise, we have

$$\begin{aligned} R &\cong_M \frac{|\phi(L_m)|}{|L_m|} = \frac{|\phi(L_{m-1})|}{|L_{m-1}|} = \cdots = \frac{|\phi(L_n)|}{|L_n|} = \frac{|\phi(L_{n-1})|}{|L_{n-1}|} \\ &\cong_M \frac{|\phi(L_{n-2})|}{|L_{n-2}|} \cong_M \cdots \cong_M \frac{|\phi(L_1)|}{|L_1|} \cong_M \frac{|\phi(L_s)|}{|L_s|}. \end{aligned}$$

Since the above chain of similarities and equalities has at most n instances where two sides are equal up to a factor of M, this completes the proof. □

Proof of Proposition 5.4.2. We show that for some $C_1 = C_1(k, N)$, we have that for any $t_1, t_2, t_3 \in \mathbb{T}$ distinct with $|t_1 - t_2| = |t_1 - t_3|$, that

$$\frac{|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)|}{|\widehat{\phi}(t_1) - \widehat{\phi}(t_3)|} \leq C_1.$$

For $i = 1, 2, 3$, define $z_i = \psi(t_i)$. If z_1, z_2 share a common limb, let $L_{1,2}$ be the smallest such limb and define $R_{1,2} = |\phi(L_{1,2})|/|L_{1,2}|$. If not, define $R_{1,2} = 1$. Define $R_{1,3}$ in the same way, depending on whether z_1, z_3 share a limb or not. By Lemma 5.4.8,

$$|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)| \cong_{C^2 M} R_{1,2}|t_1 - t_2|,$$

$$|\widehat{\phi}(t_1) - \widehat{\phi}(t_3)| \cong_{C^2 M} R_{1,3}|t_1 - t_3|.$$

By Lemma 5.4.9, $R_{1,2}/R_{1,3} \cong_{M^n} 1$. Putting those together, we have

$$\frac{|\widehat{\phi}(t_1) - \widehat{\phi}(t_2)|}{|\widehat{\phi}(t_1) - \widehat{\phi}(t_3)|} \leq C^4 M^2 \frac{R_{1,2}|t_1 - t_2|}{R_{1,3}|t_1 - t_3|} \leq C^4 M^{n+2}.$$

Since C, M, n only depend on k, N, this completes the proof.

□

5.5 qc extension to $\mathbb{C}$

Lemma 5.5.1. *If $\phi : \mathcal{J} \to \mathcal{J}$ is of type (k, N), then for all $n \geq k$ the lift of ϕ to the Böttcher coordinate descends to an η'-qs $\phi_n : B_n \to B_n$ that moves dynamically every little copy of B_{n-k}. Furthermore, there exists a K-qc map $\Phi_n : \mathbb{C} \to \mathbb{C}$ such that $\phi_n = \Phi_n|_{B_n}$. K, η' depend only on k, N.*

Proof. By Proposition 5.4.2, $\widehat{\phi} : \mathbb{T} \to \mathbb{T}$ is η''-qs.

Let $\psi_k : \mathbb{T} \to B_k$ be the Böttcher coordinate map for B_k and $\psi : \mathbb{T} \to \mathcal{J}$ be the Böttcher coordinate map for $\mathcal{J}$. The correspondence is such that if J is a little Julia set of depth k then $\psi_k \circ \psi^{-1}(J)$ is a bounded Fatou component of B_k.

Since there is a one-to-one correspondence between little Julia sets of depth k and little copies of B_{n-k} in B_n given by $\psi_n \circ \psi^{-1}$ and since this correspondence agrees with the dynamics of little Julia sets of depth k under the functional equations $\psi \circ g = f \circ \psi$, $\psi_k \circ g = f_k \circ \psi_k$, we get that any k-dynamical homeomorphism of $\mathcal{J}$ corresponds to a homeomorphism of B_n for all $n \geq k$. This correspondence is such that both maps share the same lift to $\mathbb{T}$ under their Böttcher coordinate. For $\phi : \mathcal{J} \to \mathcal{J}$ k-dynamical, we define $\phi_n : B_n \to B_n$ with $n \geq k$ such that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{J} \arrow[r, "\phi"] & \mathcal{J} \\
\mathbb{T} \arrow[u, "\psi"] \arrow[r, dashed, "\widehat{\phi}"] \arrow[d, "\psi_k"'] & \mathbb{T} \arrow[u, "\psi"'] \arrow[d, "\psi_k"] \\
B_n \arrow[r, dashed, "\phi_n"] & B_n
\end{tikzcd}$$

Let K such that the Ahlfors-Beurling extension of $\widehat{\phi}$ to $\mathbb{C} \setminus \mathbb{D}$ is K-qc on $\mathbb{C} \setminus \overline{\mathbb{D}}$. This can then be brought down to an extension of ϕ into the basin of ∞. Since ϕ is k-dynamical, ϕ_k can be extended conformally into every bounded Fatou component. Since B_n is qc-removable, we get that the extension of ϕ_n, $\Phi_n : \mathbb{C} \to \mathbb{C}$ is K-qc. Therefore, Φ_n is η'-qs and so ϕ_n is η'-qs. $\square$

Proposition 5.5.2. *If $\phi : \mathcal{J} \to \mathcal{J}$ is of type (k, N), then ϕ is K-qc extendable to $\mathbb{C}$ and so is η'-qs where η' depends only on k, N.*

Proof. By 2.3.1 $B_n \to \mathcal{J}$ as $n \to \infty$. Since $\Phi_n(\pm\beta(B_n)) = \pm\beta(B_n)$, the maps Φ_n are normalized. Hence, Φ_n converge uniformly on a subsequence to a K-qc map $\Phi : \mathbb{C} \to \mathbb{C}$ such that $\Phi|_{\mathcal{J}} = \phi$. □

Since η-qs maps are of type (k, N) where k, N depend only on η, we have the following immediate consequence.

Proposition 5.5.3. *If $\phi : \mathcal{J} \to \mathcal{J}$ is η-qs, then ϕ is K-qc extendable to $\mathbb{C}$ where K depends only on η.*

5.6 Uniform Approx of ϕ by G_k

Index X_k, the collection of little Julia sets of depth less than k, by $\mathbb{N}$ such that if $i < j$ then

1. $\operatorname{diam}(\widehat{J}_i) \geq \operatorname{diam}(\widehat{J}_j)$
2. $\widehat{J}_i \not\subset \widehat{J}_j$

Furthermore, since $\operatorname{diam}(J_n) \to 0$ as $n \to \infty$ and since there are only finitely many limbs of diameter at least ϵ for any choice of $\epsilon > 0$, $\operatorname{diam}(\widehat{J_n}) \to 0$ as $n \to \infty$.

Lemma 5.6.1. *For any η-quasisymmetry $\phi : \mathcal{J} \to \mathcal{J}$ there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ with $|m_n| \leq L$ for some $L \in \mathbb{N}$ such that for*

$$\tau_n = \sigma_{J_1}^{m_1}\sigma_{J_2}^{m_2}\cdots\sigma_{J_n}^{m_n} \in G_k,$$

satisfying $\tau_n(\widehat{J}_{i,c}) = \phi(\widehat{J}_{i,c})$ for any $i \leq n$ and any spine child $J_{i,c} \subset J_i$,

Proof. By Corollary 5.2.7, $\phi([-\beta, \beta]) = [-\beta, \beta]$. By post-composing with ρ, if necessary, we may assume WLOG that $\phi(\beta) = \beta$.

We prove this by induction. $J_1 = \mathcal{J}$. By Corollary 5.2.7 and Lemma 5.3.1, there exists an m_1 such that $\sigma^{m_1}(\mu^{-1}\mathcal{J}) = \phi(\mu^{-1}\mathcal{J})$. Hence, for any spine child of $\mathcal{J}$, J_c, $\sigma_{J_1}^{m_1}(\widehat{J}_c) = \sigma^{m_1}(\widehat{J}_c) = \phi(\widehat{J}_c)$. Suppose for some $n \in \mathbb{N}$, that $\tau_n = \sigma_{J_1}^{m_1}\sigma_{J_2}^{m_2}\cdots\sigma_{J_n}^{m_n}$ satisfies $\tau_n(J_{i,c}) = \phi(J_{i,c})$ for any $i < n$ and any spine child $J_{i,c} \subset J_i$. We now show that $\tau_n(\widehat{J}_{n+1}) = \phi(\widehat{J}_{n+1})$.

Case 1: If J_{n+1} is the spine child of some little Julia set, J_p, then $J_p = J_i$ for some $i \leq n$. Hence, $\tau_n(\widehat{J}_{n+1}) = \phi(\widehat{J}_{n+1})$.

Case 2: If not, then spine(J_{n+1}) is a patriarchal 1d little Julia set contained in spine(L) for some limb L. Let J be the little Julia set of depth $k-1$ such that L is rooted in spine(J). Since $\widehat{J}_{n+1} \subset \widehat{J}$, $J = J_i$ for some $i \leq n$. Therefore, since ϕ is k-dynamical, for $J_{i,c}$ the spine child of J_i containing the root of L, $\tau_n|_{J_{i,c}} = \phi|_{J_{i,c}}$. Hence, $\tau_n(L) = \phi(L)$. Since patriarchal 1d little Julia sets go to patriarchal 1d little Julia sets, $\tau_n(\text{spine}(J_{n+1})) = \phi(\text{spine}(J_{n+1}))$. Hence, $\tau_n(\widehat{J}_{n+1}) = \phi(\widehat{J}_{n+1})$.

Knowing this, we can move on to mapping the spine children of $\widehat{J}_{n+1}$ to the same place under both ϕ and τ_{n+1}. We now know that τ_n, ϕ both send the spine children of $\widehat{J}_{n+1}$ to the spine children of $\phi(\widehat{J}_{n+1})$. ϕ, however, will shift them by some $m_{n+1} \in \mathbb{Z}$ while τ_n doesn't shift them. (If $m_{n+1} = 0$, this corresponds to no shift.) Define $\tau_{n+1} := \tau_n \circ \sigma_{J_{n+1}}^{m_{n+1}}$. In this way, for every spine child $J_{n+1,c} \subset J_{n+1}$, $\tau_{n+1}(J_{n+1,c}) = \phi(J_{n+1,c})$. Since $\sigma_{J_{n+1}}|_{\mathcal{J}\setminus\widehat{J}_{n+1}} = \text{id}$, τ_{n+1} maintains the same behavior as τ_n on each J_i, $i \leq n$. This completes the proof. $\square$

Lemma 5.6.2. *Let $\phi : \mathcal{J} \to \mathcal{J}$ be η-qs. For all $n \in \mathbb{N}$, let $\tau_n = \sigma_{J_1}^{m_1}\sigma_{J_2}^{m_2}\cdots\sigma_{J_n}^{m_n} \in G_k$ be the maps that satisfy the statement of Lemma 5.6.1. $\tau_n \to \phi$ uniformly as $n \to \infty$.*

Proof. Let $\epsilon > 0$. Let $\delta > 0$ such that if $X \subset \mathcal{J}$ and $\text{diam}(X) < \delta$ then $\text{diam}(\phi(X)) < \epsilon$. Let $n \in \mathbb{N}$ such that if $J_i \in X_k$ satisfies $\text{diam}(\widehat{J}_i) \geq \delta$ then $i \leq n$. The argument is as follows: we show that for every $z \in \mathcal{J}$, either there exists a little Julia set, J, such that $z \in \widehat{J}$, $\text{diam}(\widehat{J}) < \delta$ and $\tau_n(\widehat{J}) = \phi(\widehat{J})$ (Case 1) or, if not, then $\tau_n(z) = \phi(z)$ (Case 2).

Case 1: Suppose there exists an extended little Julia set, $\widehat{J} \ni z$, such that depth$(\widehat{J}) \leq k$ and diam$(\widehat{J}) < \delta$. Given this, we may further suppose that $\widehat{J}$ is the largest such extended little Julia set satisfying these conditions. Let $\widehat{J}_p$ be the smallest extended little Julia set such that depth$(\widehat{J}_p) < k$ and $\widehat{J}_p \supsetneq \widehat{J}$. By our assumption, it follows that diam$(\widehat{J}_p) \geq \delta$. Hence, $\tau_n(\widehat{J}_c) = \phi(\widehat{J}_c)$ for every spine child $\widehat{J}_c$ of $\widehat{J}_p$. If $\widehat{J}$ is not patriarchal, then $\widehat{J}$ is one of these spine children. If instead, $\widehat{J}$ *is* patriarchal, then for L the limb such that spine$(\widehat{J}) \subset$ spine(L), we have that $\widehat{J}_p$ is the unique extended little Julia set of depth $k-1$ such that its spine contains the root of L. Since ϕ is k-dynamical and diam$(\widehat{J}_p) \geq \delta$, it follows that $\tau_n(L) = \phi(L)$. And so $\tau_n(\widehat{J}) = \phi(\widehat{J})$ since patriarchal little Julia sets go to patriarchal little Julia sets.

Case 2: If Case 1 is not satisfied, then every extended little Julia set, $\widehat{J} \ni z$ with depth$(\widehat{J}) \leq k$ satisfies diam$(\widehat{J}) \geq \delta$. Let $\widehat{J} \ni z$ be of depth $k-1$. Observe that $\widehat{J}$ minus the endpoints of its spine is equal to the union of each of its extended spine children. Hence, if z is not in a smaller extended little Julia set of *any* depth, then z is one of the endpoints of spine$(\widehat{J})$ in which case $\tau_n(z) = \phi(z)$. If instead, z belongs to smaller extended little Julia sets but none that have depth less than k, then it must be that depth$(\widehat{J}) = k-1$. Furthermore, we can say that in this case $z \in J$ or z is the β-point of a limb L rooted in J – if z is anything else, then it would belong to a strictly smaller extended little Julia set of depth less than k. Because diam$(\widehat{J}) \geq \delta$ and depth$(\widehat{J}) = k-1$, $\tau_n|_{J_c} = \phi|_{J_c}$ for every spine child of J. This, however, further implies that $\tau_n|_{J_s} = \phi|_{J_s}$ for *every* little Julia set of depth k contained in J. Since J is equal to the closure of a countable union of little Julia sets of depth k we have that $\tau_n|_J = \phi|_J$. Hence, we also have that for any limb L rooted in J, $\tau_n(\beta(L)) = \phi(\beta(L))$.

□

By Proposition 5.5.2, the maps τ_n are all η'-qs. Hence, Lemmas 5.6.1 and 5.6.2 prove Theorem 1.0.1.

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