Non-Abelian Hodge Theory, Zeros of Holomorphic One-Forms, and Generic Vanishing

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Abstract of the Dissertation

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This dissertation is a compilation of the results from two related research projects.

In the first, I give a new and significantly easier proof of the theorem that every holomorphic one-form on a smooth complex projective variety of general type must vanish at some point, first proven by Popa and Schnell using generic vanishing theorems for Hodge modules. My proof relies on Simpson's results on the relation between rank one Higgs bundles and local systems of one-dimensional complex vector spaces, and the structure of the cohomology jump loci in their moduli spaces.

In the second, I give a new and technically simpler proof of a theorem by Pareschi, Popa and Schnell that the direct image of the canonical bundle of a smooth projective variety along a morphism to an abelian variety admits a Chen-Jiang decomposition. My argument uses only results on variations of Hodge structures, rather than Hodge modules.

Both projects relate to cohomology jump loci in the moduli space of rank one Higgs bundles, and involve applying Simpson's results on the structure of these loci. I give a largely self-contained, accessible exposition of these results.

To my parents and my brother.

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Chapter 1

Introduction

This dissertation is built from my two previous papers [Vil22b; Vil22a] which study the Hodge theory of abelian varieties and applications in two different directions. In both papers, I studied theorems whose only available proofs previously relied heavily on vanishing theorems and the decomposition theorem for Hodge modules, and instead provide technically much simpler proofs relying on tools from classical Hodge theory.

In particular, a key ingredient in both papers is a theorem of Simpson regarding the Dolbeault cohomology of rank one Higgs bundles. A Higgs bundle on a complex manifold X is a holomorphic vector bundle E and a $\mathcal{O}_X$-linear map $\theta\colon E \to \Omega^1_X \otimes E$, satisfying the integrability condition $\theta \wedge \theta = 0$. The integrability condition implies that we can construct a Dolbeault complex

$$E \xrightarrow{\theta} \Omega^1_X \otimes E \to \cdots \to \Omega^n_X \otimes E,$$

where $n = \dim X$; denote the hypercohomology of this complex by $H^\bullet_{\mathrm{Dol}}(X; E, \theta)$.

When $E = L$ is a line bundle, θ is just a holomorphic one-form $\theta \in H^0(X, \Omega^1_X)$, so the moduli space of rank one Higgs bundles is just $\mathrm{Pic}(X) \times H^0(X, \Omega^1_X)$. Note that the tensor product operations gives this a group structure. Simpson [Sim93] proved the following linearity theorem.

Theorem 1.0.1. *Let X be a projective manifold. Then each irreducible component of the cohomology jump loci*

$$\Sigma^k_m(X) = \{(L, \theta) \in \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X) \mid \dim H^k_{\mathrm{Dol}}(X; L, \theta) \geq m\}$$

is of the form

$$\{(\alpha \otimes L, \omega + \theta) \mid (L, \theta) \in \hat{A} \times H^0(A, \Omega^1_A)\},$$

where $(\alpha, \omega) \in \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X)$ is a point of finite order, $\hat{A} \subseteq \mathrm{Pic}^0(X)$ is an abelian subvariety, and $H^0(A, \Omega^1_A) \subseteq H^0(X, \Omega^1_X)$ is its tangent space.

Except for the arithmetic part of the theorem (about points of finite order), Simpson's proof follows by elementary considerations from the non-abelian Hodge theorem in rank one. While the non-abelian Hodge theorem [Sim92] is in general a difficult result, the rank one case can be proven directly starting from the Kähler identities from classical Hodge theory. I give a complete account of both this case of the non-abelian Hodge theorem, and of Simpson's argument, excluding the arithmetic part, in Chapter 2.

1.1 Zeros of holomorphic one-forms

The results of the paper [Vil22a] are reproduced, with some improvements to exposition, as Chapter 3. In that paper, I gave a much simpler proof of the following theorem, originally proven by Popa and Schnell [PS14].

Theorem 1.1.1. *Let X be a smooth projective variety of general type over the complex numbers. Then every holomorphic one-form on X must vanish at some point of X.*

The first part of my argument proceeds in the same way as that of [PS14], by using results on the birational geometry of maps to abelian varieties due to Kawamata to show that the result follows from the following more precise theorem.

Theorem 1.1.2. *Let X be a smooth complex projective variety and $f\colon X \to A$ a morphism to an abelian variety. If $H^0(X, \omega_X^{\otimes d} \otimes f^*L^{-1}) \neq 0$ for some integer $d \geq 1$ and some ample line bundle L on A, then for every holomorphic one-form ω on A, the pullback $f^*\omega$ vanishes at some point of X.*

From here, Popa and Schnell use vanishing theorems for Hodge modules combined with a cyclic covering trick to prove this more precise theorem. In contrast, the main tool in my proof is Simpson's Theorem 1.0.1 discussed above, on the structure of the cohomology jump loci for Dolbeault cohomology of rank one Higgs bundles. The basic idea behind the use of Higgs bundles is that for a rank one Higgs bundle (L, θ), consisting of a line bundle L and a holomorphic one-form θ, the Dolbeault cohomology groups $H^i_{\mathrm{Dol}}(X; L, \theta)$ vanish if θ has no zeros. From this, one produces a complex of sheaves C_X on $X \times H^0(A, \Omega^1_A)$ related to the Dolbeault cohomology of rank one Higgs bundles, supported on the incidence locus of (x, θ) where $f^*\theta$ vanishes at x. The image of this locus in $H^0(A, \Omega^1_A)$ is the locus of one-forms that acquire zeros somewhere on X. Combining Simpson's results on the cohomology jump loci of Higgs bundles with a base change argument proves that the derived direct images of C_X on $H^0(A, \Omega^1_A)$ are vector bundles, but they are also supported in the locus of one-forms with zeros. A cyclic covering argument finally produces some non-zero torsion-free sheaves supported on the same locus, proving the theorem.

For this argument, I don't need the arithmetic part of Simpson's theorem. Hence this dissertation provides an essentially self-contained proof of Theorem 1.1.2, starting from the classical Hodge theorem.

1.2 Chen-Jiang decompositions

The paper [Vil22b] is reproduced as Chapter 4 with minor editing. In it, I give a technically simpler proof of the Chen-Jiang decomposition theorem (explained below), Theorem 4.0.1, due in a special case to Chen and Jiang [CJ18] and in full generality to Pareschi-Popa-Schnell [PPS17]. Instead of the Hodge modules in their proof, I give a more direct and constructive proof using only variations of Hodge structure, explaining in particular where the projection maps in the decomposition come from.

To explain this result, consider a morphism $f\colon X \to A$ from a complex smooth projective variety to an abelian variety. The theory of generic vanishing was initiated by Green and

Lazarsfeld [GL87; GL91], and proposes to study the sheaf $f_*\omega_X$ by studying the cohomology support loci

$$V^k(A, f_*\omega_X) = \{\alpha \in \hat{A} \mid H^k(A, f_*\omega_X \otimes \alpha) \neq 0\},$$

where $\hat{A} = \mathrm{Pic}^0(A)$ parametrizes line bundles α on A with $c_1(\alpha) = 0$. Green and Lazarsfeld shows that $f_*\omega_X$ is a *GV-sheaf*, meaning that $\operatorname{codim}_{\hat{A}} V^k(A, f_*\omega_X) \geq k$ for every $k = 0, \dots, \dim A$. Furthermore, they showed that each irreducible component of the cohomology support loci is linear, meaning a translate of an abelian subvariety of $\hat{A}$. Simpson [Sim93] later showed, as a corollary of Theorem 1.0.1, that each component contains a torsion point.

However, it can happen that $\operatorname{codim}_{\hat{A}} V^k(A, f_*\omega_X) = k$ for some $k > 0$, hence $f_*\omega_X$ fails to be an *M-regular* sheaf, a positivity notion later defined by Pareschi and Popa [PP03] which implies various useful global generation properties. Motivated by this, Chen and Jiang [CJ18] proved for generically finite morphisms, and Pareschi-Popa-Schnell [PPS17] in the general case, that while $f_*\omega_X$ is not necessarily itself M-regular, it admits a decomposition as follows.

Theorem 1.2.1. *For any morphism $f\colon X \to A$ from a smooth projective variety to an abelian variety, the sheaf $f_*\omega_X$ admits a Chen-Jiang decomposition*

$$f_*\omega_X \cong \bigoplus_{\substack{k=0,\dots \dim A \\ \tau + \hat{B} \subseteq V^k(A, f_*\omega_X)}} \tau \otimes p_B^* \mathscr{F}_B,$$

where $\tau + \hat{B}$ ranges over a set of representations of the components of $V^k(A, f_*\omega_X)$ of codimension exactly k, $p_B\colon A \to B$ is the projection dual to the inclusion $\hat{B} \subseteq \hat{A}$, and $\mathscr{F}_B$ is an M-regular sheaf on B. Here τ can be taken to be any point in the component; in particular, it can be taken to be a torsion point.

The generalization by Pareschi-Popa-Schnell relies heavily on vanishing and decomposition theorems for Hodge modules, and the decomposition is ultimately obtained rather abstractly from semi-simplicity of the category of polarizable Hodge modules.

My proof in [Vil22b] proceeds by producing, for each codimension k component of each locus $V^k(A, f_*\omega_X)$, the corresponding M-regular sheaf via induction on dimension. This induction relies crucially on Simpson's torsion translate result, hence on Theorem 1.0.1; specifically, Simpson's result implies that after passing to a suitable étale cover of A, all the cohomology support loci are abelian subvarieties, i.e. they pass through the origin.

The M-regular sheaves $\mathscr{F}_B$ are produced very constructively starting from the sheaf $f_*\omega_X$, and the construction comes with a map $f_*\omega_X \to p_B^*\mathscr{F}_B$ for each $\mathscr{F}_B$ as in the theorem (remember that we've killed off all the torsion points τ at this point). This map is essentially a Gysin morphism, hence topological in nature. Following this observation, I show that this morphism lifts to a morphism of certain variations of Hodge structures, at least on a Zariski open locus where f and $p_B \circ f$ are smooth. Using the polarizations, theorems of Kollár on recovering $f_*\omega_X$ globally from a variation of Hodge structures on the smooth locus, and the global invariant cycles theorem, I produce the desired direct sum decomposition from these Gysin morphisms; see Theorem 4.3.4 for the splitting argument.

As part of my exposition on Theorem 1.0.1 in this dissertation, I explain the consequences of Simpson's theorem needed for Chapter 4 in Section 2.2.1. For this argument, the torsion points in Simpson's theorem are quite important. However, since the torsion translate part of Simpson's theorem relies on arithmetic results quite far from the Hodge theory that is otherwise the focus in this dissertation, I do not explain the proof of that part of Simpson's result.

Chapter 2

Non-abelian Hodge theory in rank one

In this chapter, we will explain the basics of rank one Higgs bundles on a compact Kähler manifold, and the structure of their moduli space. Following Simpson [Sim92] in general and notes of Christian Schnell [Sch13] in the rank one case, we will explain the non-abelian Hodge theorem relating Higgs bundles to flat connections and local systems. We will establish this correspondence in rank one using only basic properties of harmonic differential forms on compact Kähler manifolds, as covered e.g. in [Voi02].

Using this correspondence, we will review the proof of Simpson's theorem [Sim93] on the structure of the cohomology jump loci in the moduli space of Higgs bundles.

Throughout this section, let X be a fixed compact Kähler manifold and ω a Kähler form on X. Let $A^{p,q}(X)$ be the space of $\mathcal{C}^\infty$ differential forms of type (p,q) on X, and let $\mathcal{H}^{p,q}(X)$ be the space of harmonic (p,q)-forms with respect to the Kähler metric associated to ω.

2.1 Higgs bundles and flat connections

In this section we will define Higgs bundles and bundles with flat holomorphic connections. We will explain how to represent line bundles with vanishing integral first Chern class, as well as Higgs fields and flat connections on them, using harmonic one-forms, and prove the non-abelian Hodge theorem in the rank one case.

2.1.1 Hodge theory of line bundles

Consider then a line bundle L. The complex structure of L is a differential operator $\bar\partial_L \colon A^0(X,L) \to A^{0,1}(X,L)$, where in general $A^{p,q}(X,L)$ is the space of global $\mathcal{C}^\infty$-differential forms of type (p,q) with coefficients in L. If $c_1(L) \in H^2(X,\mathbb{Z})$ vanishes, then L is trivial as a $\mathcal{C}^\infty$-line bundle, so $\bar\partial_L \colon A^0(X) \to A^{0,1}(X)$. Given a global, nowhere vanishing $\mathcal{C}^\infty$ section s of L, we get $\bar\partial_L s = \tau \otimes s$ for some $(0,1)$-form τ, and since $\bar\partial_L^2 = 0$, τ is $\bar\partial$-closed. Any other section (local or global) can be represented as fs for a function f, and $\bar\partial_L(fs) = (\bar\partial f + \tau) \otimes s$, so using s to trivialize L, $\bar\partial_L$ is identified with $\bar\partial + \tau$.

By changing the choice of smooth section s to $e^f s$ for a smooth function f, we can replace τ by $\tau + \bar\partial f$, so the isomorphism class of the holomorphic line bundle determined by τ in this way depends only on the class of τ in $H^{0,1}(X)$.

In particular, we can represent any line bundle L with $c_1(L) = 0$ as the trivial bundle with complex structure $\bar{\partial} + \tau$ for $\tau \in \mathcal{H}^{0,1}(X)$ a harmonic form. Equivalently, $d\tau = 0$; indeed if $d\tau = 0$, then $\overline{\partial\tau} = 0$, so τ is anti-holomorphic, hence harmonic.

As in classical Hodge theory (which is the case where $L = \mathcal{O}_X$), there is a type $(1,0)$-operator $\partial_L = \partial - \bar{\tau}$. Though we will not use this fact, let us mention as motivation for the definition of ∂_L that it is the $(1,0)$-part of the Chern connection on L with respect to the hermitian metric which is constant on a frame s as above.

Similarly, we can take adjoints ∂_L^* and $\bar{\partial}_L^*$ with respect to the given Kähler metric. Then the Kähler identities hold for these operators.

Lemma 2.1.1. *Let τ be a harmonic $(0,1)$-form and L the associated holomorphic line bundle. Then*

$$[\Lambda_\omega, \bar{\partial}_L] = -i\partial_L^*$$
$$[\Lambda_\omega, \partial_L] = i\bar{\partial}_L^*$$

where Λ_ω is the adjoint operator of the Lefschetz operator $L_\omega \colon H^k(X) \to H^{k+2}(X)$ given by taking the wedge product with the Kähler form ω.

Proof. By the classical Kähler identities, we know that $[\Lambda_\omega, \partial] = i\bar{\partial}^*$, so it only remains to show that $[\Lambda_\omega, \bar{\tau}] = -i\tau^*$. As in the proof of the classical Kähler identities, this follows by reducing to the case of the Euclidean metric on $\mathbb{C}^n$ and computing locally. $\square$

With the Kähler identities established, the Hodge theorem follows as in the classical case.

Theorem 2.1.2. *Every class in $H^q(X, \Omega_X^p \otimes L)$ is uniquely represented by a smooth (p,q)-form α on X satisfying*

$$\bar{\partial}_L\alpha = \bar{\partial}_L^*\alpha = \partial_L\alpha = \partial_L^*\alpha = 0.$$

Equivalently, α is both Δ_{∂_L}-harmonic and $\Delta_{\bar{\partial}_L}$-harmonic, i.e. $\Delta_{\partial_L}\alpha = \Delta_{\bar{\partial}_L}\alpha = 0$.

Let again L be a line bundle with $c_1(L) = 0$. Now, L is trivial as a holomorphic line bundle if and only if there exists a global nowhere vanishing holomorphic section, i.e. a nowhere vanishing smooth function f on X such that $\bar{\partial}_L f = \bar{\partial} f + f\tau = 0$. By the Hodge theorem, f is a harmonic section of L, so also $\partial_L f = \partial f - f\bar{\tau} = 0$. It follows that $\bar{\tau} - \tau = \frac{df}{f}$, which is equivalent to asking that the periods of $\bar{\tau} - \tau$ lie in $\mathbb{Z}(1) = 2\pi i\mathbb{Z}$; indeed f is necessarily given by

$$f(x) = \exp\left(\int_{x_0}^{x} \bar{\tau} - \tau\right)$$

for some basepoint $x_0 \in X$.

In conclusion, the moduli space of line bundles with vanishing first Chern class is naturally identified with

$$\frac{\mathcal{H}^{0,1}(X)}{H^1(X, \mathbb{Z}(1))}$$

Under the identification $\mathcal{H}^{0,1}(X) = H^{0,1}(X)$, this recovers the usual description of the Picard variety $\mathrm{Pic}^0(X)$.

2.1.2 Line bundles with flat connection

Definition 2.1.1. A holomorphic connection on a vector bundle E is a $\mathbb{C}$-linear operator

$$\nabla\colon E \to \Omega^1_X \otimes E$$

satisfying the Leibniz rule

$$\nabla(fs) = \partial f \otimes s + f\nabla s$$

for holomorphic functions f. The connection is flat if $\nabla \circ \nabla = 0$, where ∇ is extended to maps $\Omega^p_X \otimes E \to \Omega^{p+1}_X \otimes E$ by applying ∇ to the first factor and taking wedge products. Note that this extension satisfies the Leibniz rule

$$\nabla(\alpha \wedge s) = \partial\alpha \otimes E + (-1)^p \alpha \wedge \nabla s$$

Let $M^0_{\mathrm{dR}}(X)$ be the set of isomorphism classes of pairs (L, ∇) where $L \in \mathrm{Pic}^0(X)$ and ∇ is a flat connection on L.

Our discussion can be extended to the moduli space $M_{\mathrm{dR}}(X)$ of all line bundles with flat connections; we are restricting to the connected component of $(\mathcal{O}_X, d)$. Note however that L having a flat connection implies that $c_1(L)$ is torsion (in fact the converse holds as well, i.e. any line bundle with torsion c_1 admits a flat connection).

There is a natural map $M^0_{\mathrm{dR}}(X) \to \mathrm{Pic}^0(X)$ that forgets the connection. Given a holomorphic one-form α and a point $(L, \nabla) \in M^0_{\mathrm{dR}}(X)$, $\nabla + \alpha$ is again a flat connection on L, and conversely if $(L, \nabla_1), (L, \nabla_2) \in M^0_{\mathrm{dR}}(X)$, a direct computation shows that ∇_1 and ∇_2 differ by a unique holomorphic one-form. We have proven:

Proposition 2.1.3. *The map $M^0_{\mathrm{dR}}(X) \to \mathrm{Pic}^0(X)\colon (L, \nabla) \to L$ is a torsor under $H^0(X, \Omega^1_X)$ acting affine-linearly on the fibres. In particular,* $\dim M^0_{\mathrm{dR}}(X) = h^{0,1}(X) + h^{1,0}(X) = b_1(X)$.

The word "torsor" here means that the group $H^0(X, \Omega^1_X)$ acts on each fibre of $M^0_{\mathrm{dR}}(X) \to \mathrm{Pic}^0(X)$, that the actions vary holomorphically, and that the action on each fibre is free and transitive.

In fact this proposition determines the natural structure of $M^0_{\mathrm{dR}}(X)$ as a complex manifold. We will later (see Lemma 2.2.2) construct explicitly the Poincaré line bundle P on $\mathrm{Pic}^0(X) \times X$; this is the universal family of line bundles on X with $c_1 = 0$, and shows that $\mathrm{Pic}^0(X)$ is a fine moduli space. After pulling back P to $M^0_{\mathrm{dR}}(X) \times X$ by the map $M^0_{\mathrm{dR}}(X) \to \mathrm{Pic}^0(X)$, we will construct a (relative over X) connection on P, turning it into a universal family of flat line bundles on $M^0_{\mathrm{dR}}(X) \times X$, hence $M^0_{\mathrm{dR}}(X)$ is also a fine moduli space.

Let us describe this moduli space in terms of harmonic forms. Suppose given a harmonic form $\epsilon \in \mathcal{H}^1(X)$ (equivalently, a one-form such that both $\epsilon^{1,0}$ and $\epsilon^{0,1}$ are d-closed). We've seen that $\bar\partial + \epsilon^{0,1}$ gives the complex structure on a line bundle $L \in \mathrm{Pic}^0(X)$. The connection will be given by $\nabla = \partial + \epsilon^{1,0}$. It's clear that this squares to 0 and satisfies the Leibniz rule, so we just need to check that it maps holomorphic section of L to holomorphic sections of $\Omega^1_X \otimes L$.

For this, suppose $\bar{\partial}_L s = (\bar{\partial} + \epsilon^{0,1})s = 0$. By direct computation, we get

$$\begin{aligned}
\bar{\partial}_L \nabla(s) &= \bar{\partial}\partial(s) + \bar{\partial}(\epsilon^{1,0}s) + \epsilon^{0,1} \wedge \partial(s) + \epsilon^{0,1} \wedge \epsilon^{1,0}s \\
&= -\partial\bar{\partial}(s) - \epsilon^{1,0} \wedge \bar{\partial}(s) - \partial(\epsilon^{0,1}s) - \epsilon^{1,0} \wedge \epsilon^{0,1}s \\
&= -\nabla\bar{\partial}(s) \\
&= 0,
\end{aligned}$$

where we use the fact that ϵ is harmonic, hence both ∂-closed and $\bar{\partial}$-closed, to anti-commute $\epsilon^{1,0}$ and $\epsilon^{0,1}$ with $\bar{\partial}$ and ∂.

Note that this computation also implies that $\nabla + \bar{\partial}_L$ is a flat connection on the trivial complex $\mathcal{C}^\infty$-line bundle.

We saw in the previous section that L is trivial if and only if $\overline{\epsilon^{0,1}} - \epsilon^{0,1}$ has periods in $\mathbb{Z}(1)$. In the same way, one sees that (L, ∇) is the trivial flat bundle $(\mathcal{O}_X, d)$ if and only if (L, ∇) admits a nowhere vanishing holomorphic flat section, if and only if the periods of ϵ land in $\mathbb{Z}(1)$. More precisely, f being flat and holomorphic is equivalent to $df + \epsilon f = 0$ so $-\epsilon = \frac{df}{f}$, hence up to normalization,

$$f(x) = \exp\left(\int_{x_0}^{x} -\epsilon\right)$$

Note that in this case ϵ is purely imaginary, so $\bar{\epsilon} = -\epsilon$, hence $\overline{\epsilon^{1,0}} = -\epsilon^{0,1}$ and $-\epsilon = \overline{\epsilon^{0,1}} - \epsilon^{0,1}$, so this is compatible with the previous construction just for line bundles.

The torsor structure from the previous proposition implies that we get every flat line bundle from this construction.

Proposition 2.1.4. *Let X be a compact Kähler manifold. Then*

$$M^0_{\mathrm{dR}}(X) \cong \frac{\mathcal{H}^1(X)}{H^1(X, \mathbb{Z}(1))}$$

We recall finally that a flat bundle (L, ∇) has a natural de Rham complex (starting in degree 0)

$$\mathrm{DR}(L, \nabla) = \left(L \xrightarrow{\nabla} \Omega^1_X \otimes L \xrightarrow{\nabla} \Omega^1_X \otimes L \to \cdots \to \Omega^n_X \otimes L\right)$$

associated to it, where $n = \dim_{\mathbb{C}} X$. The de Rham cohomology $H^i_{\mathrm{dR}}(X; L, \nabla)$ of the bundle is defined as the hypercohomology of this de Rham complex. As is standard, the holomorphic Poincaré lemma shows that the holomorphic de Rham complex is quasi-isomorphic to the $\mathcal{C}^\infty$-de Rham complex $(\mathscr{A}^\bullet(X), \nabla + \bar{\partial}_L)$ of L, with differential $\nabla + \bar{\partial}_L = d + \epsilon$. Since the sheaves of $\mathcal{C}^\infty$-forms are soft, this implies that de Rham cohomology of (L, ∇) is the cohomology of the complex

$$A^0(X) \xrightarrow{d+\epsilon} A^1(X) \to \cdots \to A^{2n}(X)$$

More precisely, each sheaf $\Omega^p_X \otimes L$ is resolved by the complex

$$\mathscr{A}^{p,0}(X) \xrightarrow{\bar{\partial}_L} \mathscr{A}^{p,1}(X) \to \cdots \to \mathscr{A}^{p,n}(X),$$

where $\mathscr{A}^{p,q}(X)$ is now the sheaf of (p,q)-forms; putting these together for various p gives a double complex $(\mathscr{A}^{\bullet,\bullet}(X), \nabla, \bar{\partial}_L)$ whose associated simple complex is the $\mathcal{C}^\infty$-de Rham complex above.

2.1.3 Higgs bundles

Definition 2.1.2. A *Higgs bundle* on X is a pair (E, θ) where

1. E is a vector bundle and
2. $\theta \colon E \to \Omega^1_X \otimes E$ is an $\mathcal{O}_X$-linear map such that $\theta \wedge \theta \colon E \to \Omega^2_X \otimes E$ vanishes.

In other words, θ is a section of the sheaf $\Omega^1_X \otimes \underline{\mathrm{End}}\, E$. If $E = L$ is a line bundle, we in particular have $\underline{\mathrm{End}}\, L = \mathcal{O}_X$, so the Higgs field θ is equivalently just a holomorphic one-form.

In general, the non-abelian Hodge theorem [Sim92] gives an equivalence of categories, and corresponding homeomorphism of moduli spaces, between polystable Higgs bundles with vanishing first and second complex Chern character on one hand, and semisimple flat bundles on the other hand. In rank one, this gives an equivalence between Higgs bundles whose underlying line bundle has torsion first Chern class in integral cohomology, and semisimple rank one flat bundles. We will treat the case where the first Chern class on the Higgs side vanishes, so the underlying $\mathcal{C}^\infty$-bundle is trivial.

A rank one Higgs bundle (L, θ) with $c_1(L)$ can now be represented by the complex structure $\bar\partial + \tau$ with $\tau \in \mathcal{H}^{0,1}(X)$, and a holomorphic one-form $\theta \in \mathcal{H}^{1,0}(X)$.

Definition 2.1.3. Let $M^0_{\mathrm{Dol}}(X) = \mathrm{Pic}^0(X) \times H^0(X, \Omega^1_X)$ be the moduli space of rank one Higgs bundles with vanishing first Chern class.

The condition that $\theta \wedge \theta = 0$ implies that we get a natural Dolbeault complex (starting in degree 0)

$$\mathrm{Dol}(L, \theta) = \left(L \xrightarrow{\theta} \Omega^1_X \otimes L \xrightarrow{\theta} \Omega^2_X \otimes L \to \cdots \to \Omega^n_X \otimes L \right)$$

The hypercohomology of this complex is the Dolbeault cohomology

$$H^i_{\mathrm{Dol}}(X; L, \theta) = \mathbb{H}^i(X, \mathrm{Dol}(L, \theta)).$$

As with de Rham cohomology, we can compute this using $\mathcal{C}^\infty$ forms. Resolving $\Omega^p_X \otimes L$ by $(\mathscr{A}^{p,\bullet}(X), \bar\partial_L)$ as before, we get the double complex $(\mathscr{A}^{\bullet,\bullet}(X), \theta, \bar\partial_L)$, with associated simple complex $(\mathscr{A}^\bullet(X), \bar\partial_L + \theta) = (\mathscr{A}^\bullet(X), \bar\partial + \tau + \theta)$; the cohomology of this complex gives the Dolbeault cohomology of (L, θ).

2.1.4 The non-abelian Hodge theorem

Given a flat line bundle (L, ∇), described by a harmonic one-form ϵ as above, Simpson realized that while the Kähler identities do not hold for the obvious decomposition $d + \epsilon = (\partial + \epsilon^{1,0}) + (\bar\partial + \epsilon^{0,1})$, it does hold for a different decomposition. Let

$$\theta = \frac{\epsilon^{1,0} + \overline{\epsilon^{0,1}}}{2} \in \mathcal{H}^{1,0}(X)$$
$$\tau = \frac{\epsilon^{0,1} - \overline{\epsilon^{1,0}}}{2} \in \mathcal{H}^{0,1}(X)$$

such that $d+\epsilon = (\partial - \bar{\tau} + \bar{\theta}) + (\bar{\partial} + \tau + \theta)$. Note that the pair (τ, θ) defines a Higgs bundle (K, θ), where $\bar{\partial}_K = \bar{\partial} + \tau$ (note that L and K are different holomorphic line bundles in general, unless $\tau = \epsilon^{0,1}$, or equivalently $\theta = 0$).

Note that Lemma 2.1.1 gives the Kähler identities for the pair of operators $\bar{\partial}_K$ and $\partial_K = \partial - \bar{\tau}$. By the same method, we can extend the Kähler identities to account for the Higgs field θ as well.

Lemma 2.1.5. *With notation as above, we have*

$$\begin{aligned} [\Lambda_\omega, \bar{\partial} + \tau + \theta] &= -i(\partial - \bar{\tau} + \bar{\theta})^* \\ [\Lambda_\omega, \partial - \bar{\tau} + \bar{\theta}] &= i(\bar{\partial} + \tau + \theta)^* \end{aligned}$$

If ϵ defines the trivial flat bundle $(\mathcal{O}_X, d)$, then ϵ has periods in $\mathbb{Z}(1)$ and is purely imaginary, so $\epsilon^{0,1} = \overline{\epsilon^{1,0}}$, so $\theta = 0$ and $\epsilon = \tau - \bar{\tau}$. But then the associated Higgs bundle (L, θ) is trivial, and we get a bijection

$$M^0_{\mathrm{dR}}(X) \to M^0_{\mathrm{Dol}}(X) \colon [\epsilon] \mapsto ([\tau], \theta).$$

The map involves conjugation so is not holomorphic, but it is still real analytic. Furthermore, $M^0_{\mathrm{dR}}(X)$ and $M^0_{\mathrm{Dol}}(X)$ are complex Lie groups under the natural tensor products, and the map between them is an isomorphism of underlying real Lie groups.

We are almost ready for the non-abelian Hodge theorem in this case. Before stating it, let us finally define $H^{p,q}(X; K, \theta)$ as the cohomology of the complex

$$H^q(X, K) \xrightarrow{\theta} H^q(X, \Omega^1_X \otimes L) \to \cdots \to H^q(X, \Omega^n_X \otimes L)$$

in degree p.

Theorem 2.1.6 (Non-abelian Hodge theorem in rank one)**.** *Given a compact Kähler manifold X, there is an isomorphism of real Lie groups*

$$M^0_{\mathrm{dR}}(X) \to M^0_{\mathrm{Dol}}(X) \colon [\epsilon] \mapsto ([\tau], \theta)$$

as described above. If (L, ∇) is the flat connection induced by ε, and (K, θ) the associated Higgs bundle, then

$$H^i_{\mathrm{dR}}(X; L, \nabla) = H^i_{\mathrm{Dol}}(X; K, \theta)$$

for all i. Finally, the Hodge-de Rham spectral sequence

$$E_0^{p,q} = A^{p,q}(X) \implies H^{p+q}_{\mathrm{Dol}}(X; K, \theta)$$

associated to the double complex $(A^{\bullet,\bullet}(X), \theta, \bar{\partial}+\tau)$ degenerates at E_2, with $E_2^{p,q} \cong H^{p,q}(X; K, \theta)$, so

$$H^i_{\mathrm{Dol}}(X; K, \theta) \cong \bigoplus_{p+q=i} H^{p,q}(X; K, \theta).$$

For $(L, \nabla) = (\mathcal{O}_X, d)$ (so $(K, \theta) = (\mathcal{O}_X, 0)$), this recovers the isomorphism

$$H^i(X, \mathbb{C}) \cong \bigoplus_{p+q=i} H^q(X, \Omega_X^p)$$

from classical Hodge theory. Note however that in the decomposition of $H^i_{\mathrm{Dol}}(X; K, \theta)$ given here, there is not in general any conjugation symmetry between $H^{p,q}(X; K, \theta)$ and $H^{q,p}(X; K, \theta)$ as in the classical case. This is essentially because (L, ∇) underlies a complex variation of Hodge structures, but with no real structure in general (i.e. the monodromy representation $\pi_1(X) \to \mathbb{C}^*$ determined by flat sections of (L, ∇) is not generally induced by a real representation).

Proof. As for the classical Hodge theorem, the Kähler identities (Lemma 2.1.5) imply the equality of Laplacians $\Delta_{d+\epsilon} = 2\Delta_{\bar{\partial}+\tau+\theta} = 2\Delta_{\partial-\bar{\tau}+\bar{\theta}}$. We get further that classes in the de Rham cohomology $H^i_{\mathrm{dR}}(X; L, \nabla)$ can be uniquely represented by $\Delta_{d+\epsilon}$-harmonic forms, and that classes in the Dolbeault cohomology $H^i_{\mathrm{Dol}}(X; K, \Theta)$ can be uniquely represented by $\Delta_{\bar{\partial}_K+\theta}$-harmonic forms. As $\Delta_{\bar{\partial}_K+\theta} = \nabla_{\bar{\partial}+\tau+\theta}$, we get an isomorphism

$$H^i_{\mathrm{dR}}(X; L, \nabla) \cong H^i_{\mathrm{Dol}}(X; K, \theta)$$

The right hand side can be computed by the double complex $(A^{\bullet,\bullet}(X), \theta, \bar{\partial} + \tau)$, as we have already seen. The differential on E_0 of the associated spectral sequence is induced by $\bar{\partial} + \tau$, so $E_1^{p,q} \cong H^q(X, \Omega_X^p \otimes K)$. Then the differential on E_1 is induced by θ, giving $E_2^{p,q} = H^{p,q}(X; K, \theta)$.

Let us show that the differential $d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$ vanishes. Any class in $E_2^{p,q}$ can be represented by a class $[\alpha] \in H^q(X, \Omega_X^p \otimes K)$ such that $\theta \wedge [\alpha] = 0$. Taking a harmonic representative α, equivalently $\theta \wedge \alpha$ is $\bar{\partial}_K$-exact. Recall that the Kähler identities, hence the $\partial\bar{\partial}$-lemma, holds for $\bar{\partial}_K$ and $\partial_K = \partial - \bar{\tau}$ (Lemma 2.1.1). By harmonicity,

$$\partial_K(\theta \wedge \alpha) = \partial\theta \wedge \alpha - \theta \wedge \partial_K \alpha = 0,$$

so $\theta \wedge \alpha = \bar{\partial}_K \partial_K \beta$ for some $\beta \in A^{p-1,q-1}(X)$; then $d_2[\alpha] = [\theta \wedge \partial_K \beta]$. Now $\theta \wedge \partial_K \beta$ is $\bar{\partial}_K$-closed, and ∂_K-exact since $\theta \wedge \partial_K \beta = -\partial_K(\theta \wedge \beta)$. Again by the $\partial\bar{\partial}$-lemma, $\theta \wedge \partial_K \beta = \bar{\partial}\partial\gamma$ for some $\gamma \in A^{p+1,q-2}(X)$, so the class of $\theta \wedge \partial_K \beta$ in $H^{q-1}(X, \Omega_X^{p+2} \otimes L)$ is 0 as desired.

Then $E_3^{p,q} = E_2^{p,q}$. As above, we can compute $d_3[\alpha]$ as $[\theta \wedge \partial_K \gamma]$, but repeatedly using the $\partial\bar{\partial}$-lemma again shows that this class is $\bar{\partial}_K$-exact. The same argument shows that $d_l = 0$ for all $l \geq 2$, giving the desired degeneration of the spectral sequence. □

We note that there is a natural holomorphic embedding $\mathrm{Pic}^0(X) \subseteq M^0_{\mathrm{Dol}}(X)$, by endowing a line bundle L with the trivial Higgs field $\theta = 0$. Under the non-abelian Hodge correspondence, this gives a real analytic embedding of $\mathrm{Pic}^0(X)$ as the space of unitary flat connections. The unitary connections are those whose associated monodromy representation of $\pi_1(X)$ takes values in the unitary group $U(1)$, or equivalently, the connections which are compatible with a metric, in the sense that $\nabla + \bar{\partial}_L$ is the Chern connection for some hermitian metric.

2.2 Cohomology jump loci

Consider again the moduli spaces $M^0_{\mathrm{dR}}(X), M^0_{\mathrm{Dol}}(X)$ of line bundles with flat connections, and rank one Higgs bundles. Each have natural associated cohomology theories H^k_{dR} and H^k_{Dol} respectively, which define natural cohomology jump loci

$$\Sigma^k_m(X)_{\mathrm{Dol}} = \{(L,\theta) \in M^0_{\mathrm{Dol}}(X) \mid \dim H^k_{\mathrm{Dol}}(X; L, \theta) \geq m\}$$
$$\Sigma^k_m(X)_{\mathrm{dR}} = \{(L,\nabla) \in M^0_{\mathrm{dR}}(X) \mid \dim H^k_{\mathrm{dR}}(X; L, \nabla) \geq m\}$$

By the non-abelian Hodge theorem, these loci are mapped to each other by the correspondence between $M^0_{\mathrm{dR}}(X)$ and $M^0_{\mathrm{Dol}}(X)$, and so we will occasionally just write $\Sigma^k_m(X)$ when we are agnostic about which model to consider.

In general, for a subset $Z_{\mathrm{dR}} \subseteq M^0_{\mathrm{dR}}(X)$, we will write $Z_{\mathrm{Dol}} \subseteq M^0_{\mathrm{Dol}}(X)$ for the associated subset of the Dolbeault moduli space, and just Z if we are agnostic about which we are talking about.

Note that the Dolbeault moduli space admits a natural $\mathbb{C}^\times$-action: If $(L,\theta) \in M^0_{\mathrm{Dol}}(X)$ and $\lambda \in \mathbb{C}^\times$, let $\lambda \cdot (L,\theta) = (L, \lambda\theta)$. Using the complex

$$H^q(X,L) \xrightarrow{\theta} H^q(X, \Omega^1_X \otimes L) \to \cdots \to H^q(X, \Omega^n_X \otimes L)$$

to compute Dolbeault cohomology as $H^k_{\mathrm{Dol}}(X; L, \theta) = \bigoplus_{p+q=k} H^{p,q}(X; L, \theta)$, it is evident that $\Sigma^k_m(X)_{\mathrm{Dol}}$ is $\mathbb{C}^\times$-stable.

Given a map $f\colon X \to Y$ of compact Kähler manifolds, we get natural pullback morphisms

$$f^*\colon M^0_{\mathrm{dR}}(Y) \to M^0_{\mathrm{dR}}(X)$$
$$f^*\colon M^0_{\mathrm{Dol}}(Y) \to M^0_{\mathrm{Dol}}(X)$$

which are compatible with the non-abelian Hodge correspondence.

Note that as $\mathrm{Pic}^0(X)$ and $H^0(X, \Omega^1_X)$ are both obtained, via pullback, from the Albanese variety $\mathrm{Alb}(X)$, we have $M^0_{\mathrm{Dol}}(X) \cong M^0_{\mathrm{Dol}}(\mathrm{Alb}(X))$, and similarly $M^0_{\mathrm{dR}}(X) \cong M^0_{\mathrm{dR}}(\mathrm{Alb}(X))$. Using the same identifications for Y, we see that all pullback maps on M^0_{dR} and M^0_{Dol} can be recovered as pullbacks from morphisms to complex tori.

Definition 2.2.1. A linear subvariety of $M^0_{\mathrm{dR}}(X)$ (equivalently of $M^0_{\mathrm{Dol}}(X)$) is a subset Z_{dR} of the form

$$(L,\nabla) \otimes \mathrm{im}(f^*\colon M^0_{\mathrm{dR}}(T) \to M^0_{\mathrm{dR}}(X)),$$

or equivalently $Z_{\mathrm{Dol}} \subseteq M^0_{\mathrm{Dol}}(X)$ of the form

$$(K,\theta) \otimes \mathrm{im}(f^*\colon M^0_{\mathrm{Dol}}(T) \to M^0_{\mathrm{Dol}}(X))$$

for a morphism $f\colon X \to T$ to a complex torus and $(L,\nabla) \in M^0_{\mathrm{dR}}(X)$ (equivalently $(K,\theta) \in M^0_{\mathrm{Dol}}(X)$ the Higgs bundle associated to (L,∇)).

Linear subvarieties have various nice properties: They are automatically analytic subvarieties in both $M^0_{\mathrm{dR}}(X)$ and $M^0_{\mathrm{Dol}}(X)$, and are $\mathbb{C}^\times$-stable in $M^0_{\mathrm{Dol}}(X)$. We will see that any set which is analytic in $M^0_{\mathrm{dR}}(X)$ and $\mathbb{C}^\times$-stable in $M^0_{\mathrm{Dol}}(X)$ is actually a linear subvariety; in particular, this will apply to the cohomology jump loci defined above, giving the first part of the following theorem due to Simpson [Sim93].

We note that Simpson gives various different proof of this theorem, involving also the Betti moduli space $M_{\mathrm{B}}^0(X)$ of rank one local systems on X. The moral of Simpson's arguments is that any set which is sufficiently nice in two out of three of the models $M_{\mathrm{B}}^0(X), M_{\mathrm{dR}}^0(X)$ and $M_{\mathrm{Dol}}^0(X)$ will in fact be as nice as possible, namely a linear subvariety.

Theorem 2.2.1. *Let X be a compact Kähler manifold. Then the cohomology jump loci $\Sigma_m^k(X)$ are unions of linear subvarieties.*

If furthermore X is projective, then each cohomology jump locus contains a torsion point (i.e. a point of finite order under the group structure on $M_{\mathrm{dR}}^0(X)$ and $M_{\mathrm{Dol}}^0(X)$).

Finally, we will apply this theorem to deduce a result about certain cohomology jump loci in $\mathrm{Pic}^0(X)$ (see Section 2.2.1).

For Chapter 3 about zeros of holomorphic one-forms, only the first part of Simpson's theorem is needed, and for that purpose our treatment here will be essentially self-contained. The reader interested only in this application can skip Section 2.2.1.

For Chapter 4, the second, arithmetic part of the theorem will be crucial. Here, the available proofs require either an input from transcendental number theory (a criterion of Schneider-Lang that implies that we can recognize torsion points by looking at points over number fields) [Sim93]; A reduction to positive characteristic [PR04]; or Hodge modules [Sch15]. These approaches are all quite technical, and would require introducing significant additional material on top of the mostly classical Hodge-theoretic techniques otherwise used in this document, so we will omit the proof of this statement.

It should be noted that Botong Wang [Wan16] extended the Hodge module approach to the compact Kähler case, thus the projectivity assumption in the theorem is actually superfluous.

Theorem 2.2.1 will follow immediately from the following lemma and proposition.

Lemma 2.2.2. *The space $M_{\mathrm{dR}}^0(X)$ is a fine moduli space, in the sense that there exists a universal family (P, ∇_P) of line bundles with flat connections on $M_{\mathrm{dR}}^0(X) \times X$. Here ∇_P is a relative connection*

$$\nabla_P \colon P \to \Omega^1_{M_{\mathrm{dR}}^0(X) \times X / M_{\mathrm{dR}}^0(X)} \otimes P,$$

which is in particular $p_1^* \mathcal{O}_{M_{\mathrm{dR}}^0(X)}$-linear, where $p_1 \colon M_{\mathrm{dR}}^0(X) \times X \to M_{\mathrm{dR}}^0(X)$ is the projection. For each point $(L, \nabla_L) \in M_{\mathrm{dR}}^0(X)$, we have

$$(P, \nabla_P)|_{\{(L, \nabla_L)\} \times X} \cong (L, \nabla_L).$$

If we fix a base point $x_0 \in X$ and require that $P|_{M_{\mathrm{dR}}^0(X) \times \{x_0\}}$ is the trivial bundle, then (P, ∇_P) is unique up to isomorphism.

As a consequence of the existence of this family, the cohomology jump loci

$$\Sigma_m^k(X)_{\mathrm{dR}} \subseteq M_{\mathrm{dR}}^0(X)$$

are analytic subvarieties.

Proof. Write $\mathcal{H}^1 = \mathcal{H}^1(X)$, and similarly for $\mathcal{H}^{0,1}$ and $\mathcal{H}^{1,0}$. Consider the trivial line bundle on $\mathcal{H}^1 \times X$ with differential operators

$$\bar{\partial}_P = \bar{\partial} + \sum_{j=1}^{g} t_j p_2^* \epsilon_j^{0,1}$$

$$\nabla_P = \partial_{\mathcal{H}^1 \times X/\mathcal{H}^1} + \sum_{j=1}^{g} s_j p_2^* \epsilon_j^{1,0},$$

where $\epsilon_j^{0,1} \in \mathcal{H}^{0,1}, \epsilon_j^{1,0} \in \mathcal{H}^{1,0}$ form bases and t_j, s_j are the corresponding holomorphic coordinates on $\mathcal{H}^{0,1}, \mathcal{H}^{1,0}$. Note that $\mathcal{H}^{0,1} = \overline{\mathcal{H}^{1,0}}$ as complex vector spaces, so we can take $s_j = \bar{t}_j$.

For $\epsilon \in \mathcal{H}^1$ and (L, ∇) the corresponding line bundle with connection, the pair $\bar{\partial}_P, \nabla_P$ have the property that they restrict to $\bar{\partial}_L, \nabla$ on $\{\epsilon\} \times X$. Let $\tilde{P}$ be the trivial $\mathcal{C}^\infty$-line bundle on $\mathcal{H}^1 \times X$ with complex structure $\bar{\partial}_p$. Then

$$\nabla_P \colon \tilde{P} \to \Omega_{\mathcal{H}^1 \times X/\mathcal{H}^1} \otimes \tilde{P}$$

is a relative connection, by the same computation we used in Section 2.1.2 to construct flat connections. Note that ∇_P is $p_1^{-1}\mathcal{O}_{\mathcal{H}^1}$-linear since both terms defining it are.

Recall that $M^0_{\mathrm{dR}}(X)$ is the quotient of $\mathcal{H}^1$ by the lattice $\Gamma \subset \mathcal{H}^1$ of one-forms γ such that $\bar{\gamma} - \gamma$ has periods in $\mathbb{Z}(1)$ (note that Γ does not have full rank in this case). We will extend the action of Γ on $\mathcal{H}^1 \times X$ to an action on the total space $\mathcal{H}^1 \times X \times \mathbb{C}$ of the trivial $\mathcal{C}^\infty$-bundle and show that it is compatible with the operators $\bar{\partial}_P$ and ∇_P. The quotient $P = (\mathcal{H}^1 \times X \times \mathbb{C})/\Gamma$ with the induced operators will then be the desired universal bundle.

Given a base point $x_0 \in X$ and $\gamma \in \Gamma$, define

$$f_\gamma(x) = \exp\left(\int_{x_0}^{x} (-\gamma)\right);$$

the condition on the periods of γ are equivalent to this being a well-defined $\mathcal{C}^\infty$-function on X (recall that this is the function we used to trivialize the flat bundle associated to γ in Section 2.1.2). Then define the action of Γ on $\mathcal{H}^1 \times X \times \mathbb{C}$ by

$$\gamma \cdot (\epsilon, x, z) = (\epsilon + \gamma, x, f_\gamma(x) z)$$

It is now a direct computation to check that the of γ on sections of $\mathcal{H}^1 \times X \times \mathbb{C} \to \mathcal{H}^1 \times X$, given by the pullback

$$(\gamma^* s)(\epsilon, x) = f_\gamma(x) \cdot s(\epsilon - \gamma, x),$$

commutes with $\bar{\partial}_P$ and ∇_P. For this, combine the fact that $df_\gamma = -f_\gamma \cdot \gamma$ by construction with the Leibniz rules for $\bar{\partial}_P$ and ∇_P with respect to $\bar{\partial}$ and ∂. It follows that $\bar{\partial}_P$ and ∇_P descend to differential operators on the quotient P, which then gets the structure of a holomorphic line bundle with a relative flat connection.

Note that as $\bar{\partial}_P$ only involves the coordinates of $\mathcal{H}^{0,1}$, the line bundle P (without the connection) is actually the pullback of a line bundle on $\mathrm{Pic}^0(X) \times X$, defined in the same way by descending $\tilde{P}$ restricted to $\mathcal{H}^{0,1} \times X$ along the projection $\mathcal{H}^{0,1} \times X \to \mathrm{Pic}^0(X) \times X$.

By slight abuse of notation, we will also denote this line bundle on $\operatorname{Pic}^0(X) \times X$ by P; it is the Poincaré bundle, the universal family of line bundles on X with $c_1 = 0$.

Consider now the projection $p_1\colon M^0_{\mathrm{dR}}(X) \times X \to M^0_{\mathrm{dR}}(X)$. To compute the direct image $\mathbf{R}p_{1*}\,\mathrm{DR}(P, \nabla_P)$ of the relative de Rham complex, take the complex of sheaves of relative P-valued $\mathcal{C}^\infty$-forms

$$(\mathscr{A}^{\bullet,\bullet}(M^0_{\mathrm{dR}}(X) \times X/M^0_{\mathrm{dR}}(X), P), \bar\partial_P, \nabla_P).$$

As this is a complex of soft sheaves, we can compute the pushforward as the (simple complex associated to the) direct image

$$p_{1*}(\mathscr{A}^{\bullet,\bullet}(M^0_{\mathrm{dR}}(X) \times X/M^0_{\mathrm{dR}}(X), P), \bar\partial_P, \nabla_P).$$

This is a bounded complex of $\mathcal{O}_{M^0_{\mathrm{dR}}(X)}$-modules; we will argue that it has coherent cohomology.

To see this, consider the double complex spectral sequence where we first take cohomology along $\bar\partial_P$. Then the E_1-page has (p,q)-term

$$p^* R^q p_{1*}(\Omega^p_{\operatorname{Pic}^0(X)\times X/\operatorname{Pic}^0(X)} \otimes P),$$

where $p\colon M^0_{\mathrm{dR}}(X) \to \operatorname{Pic}^0(X)$ is the projection and P, by abuse of notation as above, is also the Poincaré line bundle on $\operatorname{Pic}^0(X) \times X$, so $(p \times \mathrm{id}_X)^*P = P$. Hence the E_1-page of the spectral sequence is a complex of $\mathcal{O}_{M^0_{\mathrm{dR}}(X)}$-coherent sheaves. As all differentials in the spectral sequence are $\mathcal{O}_{M^0_{\mathrm{dR}}(X)}$-linear, the spectral sequence hence has coherent limit.

As $\mathbf{R}p_{1*}\,\mathrm{DR}(P, \nabla_P)$ is now bounded with coherent cohomology, it is (at least locally) quasi-isomorphic to a bounded complex of locally free sheaves $(E^\bullet, d)$. But by base change, the fibre of $(E^\bullet, d)$ at a point $(L, \nabla_L) \in M^0_{\mathrm{dR}}(X)$ computes the de Rham cohomology of the flat line bundle (L, ∇_L) on X. Hence the cohomology jump locus $\Sigma^k_m(X)_{\mathrm{dR}}$ is locally identified with the locus

$$\{p \in M^0_{\mathrm{dR}}(X) \mid \dim H^k(E^\bullet|_p, d) \geq m\}$$

This is analytic. Indeed,

$$\dim H^k(E^\bullet|_p, d) = \dim\ker d^k|_p - \dim\operatorname{im} d^{k-1}|_p,$$

which is bounded below by m if and only if $\operatorname{rk} d^{k-1}|_p + \operatorname{rk} d^k|_p \leq \operatorname{rk} E^k - m$. Rank bounds on maps of vector bundles are defined analytically by the vanishing of certain matrix minors (in local charts where the bundles involved are trivial), so we are done. $\square$

Proposition 2.2.3. *If $Z_{\mathrm{dR}} \subseteq M^0_{\mathrm{dR}}(X)$ is an irreducible analytic subvariety such that $Z_{\mathrm{Dol}} \subseteq M^0_{\mathrm{Dol}}(X)$ is stable under $\mathbb{C}^\times$, then Z is a linear subvariety.*

Proof. First, translate Z such that Z_{dR} contains the origin as a smooth point. Recall that $M^0_{\mathrm{dR}}(X)$ was constructed as a discrete quotient of $\mathcal{H}^1(X)$, which is then the tangent space at the origin of $M^0_{\mathrm{dR}}(X)$. Similarly, $M^0_{\mathrm{Dol}}(X) = \operatorname{Pic}^0(X) \times H^0(X, \Omega^1_X)$ is a discrete quotient of $\mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$, which is then the tangent space of $M^0_{\mathrm{Dol}}(X)$. The correspondence $M^0_{\mathrm{dR}}(X) \to M^0_{\mathrm{Dol}}(X)$ induces the isomorphism of real vector spaces

$$h\colon \mathcal{H}^1(X) \to \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)\colon \epsilon \mapsto \left(\frac{\epsilon^{0,1} - \overline{\epsilon^{1,0}}}{2}, \frac{\epsilon^{1,0} + \overline{\epsilon^{0,1}}}{2}\right),$$

with inverse $h^{-1}(\tau, \theta) = (\theta - \bar{\tau}) + (\tau + \bar{\theta})$.

Consider the real tangent space $W \subseteq \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X)$ of Z_{Dol} at the origin. By the assumption of $\mathbb{C}^\times$-stability of Z, W is stable under multiplication by $\mathbb{C}^\times$ on $\mathcal{H}^{1,0}(X)$. Further, since Z_{dR} is a complex analytic subvariety of $M^0_{\mathrm{dR}}(X)$, $h^{-1}(W)$ is actually a $\mathbb{C}$-vector subspace of $\mathcal{H}^1(X)$.

Consider the quotient map $\pi \colon \mathcal{H}^{0,1}(X) \times \mathcal{H}^{1,0}(X) \to M^0_{\mathrm{Dol}}(X)$. This is really just the product of the quotient map $\mathcal{H}^{0,1}(X) \to \mathrm{Pic}^0(X)$ and the identification $\mathcal{H}^{1,0}(X) \cong H^0(X, \Omega^1_X)$.

We will argue in steps that

1. $W = \bar{V} \times V$ for a $\mathbb{C}$-vector subspace of $\mathcal{H}^{1,0}(X)$, hence W is $\mathbb{C}$-linear;
2. $\{0\} \times V \subseteq \pi^{-1}(Z_{\mathrm{Dol}}$ using $\mathbb{C}^\times$-stability;
3. $W \subseteq \pi^{-1}(Z_{\mathrm{Dol}})$ using the complex analytic structure of Z_{dR}; and finally concluding that
4. $Z_{\mathrm{Dol}} = \pi(W)$, finishing the proof with a small remark.

Let us show that $W = \bar{V} \times V$ for a $\mathbb{C}$-vector subspace of $\mathcal{H}^{1,0}(X)$. Let $W^{1,0} = W \cap \mathcal{H}^{1,0}(X)$ and $W^{0,1} = W \cap \mathcal{H}^{0,1}(X)$, which are respectively $\mathbb{C}$- and $\mathbb{R}$-vector subspaces. Stability of W under the action of $\mathbb{C}^\times$ on $\mathcal{H}^{1,0}(X)$ implies that $W = W^{0,1} \times W^{1,0}$. But now $\mathbb{C}$-linearity of $h^{-1}(W)$ implies that $h(i \cdot h^{-1}(W)) = W$. As $h(i \cdot h^{-1}(\tau, \theta)) = (i\bar{\theta}, -i\bar{\tau})$, it follows that $W^{0,1}$ and $W^{1,0}$ are complex conjugate to each other, so W is $\mathbb{C}$-linear. Write $V = W^{1,0}$ in the following.

We now prove that $\{0\} \times V \subseteq \pi^{-1}(Z_{\mathrm{Dol}}$. Indeed given $\theta \in V$, $(0, \theta)$ is in the tangent space W, hence there is a sequence of points $(\tau_n, \theta_n) \in \pi^{-1}(Z_{\mathrm{Dol}})$ converging to the origin $(0,0)$, but such that the secant lines $\mathbb{C} \cdot (\tau_n, \theta_n)$ converge to $\mathbb{C} \cdot (0, \theta)$. Now there's a sequence $\lambda_n \in \mathbb{C}^\times$ such that $\lambda_n \theta_n$ converges to θ, and since $\pi^{-1}(Z_{\mathrm{Dol}})$ is closed and $\mathbb{C}^\times$-stable (under the action on $\mathcal{H}^{1,0}(X)$, we get that $(0, \theta) \in \pi^{-1}(Z_{\mathrm{Dol}})$.

Now we use again the complex analytic structure of Z_{dR}. The real vector subspace $h^{-1}(\{0\} \times V)$ consists of elements $\theta + \bar{\theta}$ for $\theta \in V$, but is also contained in the preimage of Z_{dR} under the quotient $\mathcal{H}^1(X) \to M^0_{\mathrm{dR}}(X)$, which is complex analytic. Any holomorphic function vanishing on $h^{-1}(\{0\} \times V)$ also vanishes on the complex vector space spanned thereby, hence $\pi^{-1}(Z_{\mathrm{dol}})$ contains also

$$h(\lambda\theta + \lambda\bar{\theta}) = \left(\frac{(\lambda - \bar{\lambda})\bar{\theta}}{2}, \frac{(\lambda + \bar{\lambda})\theta}{2} \right),$$

for $\lambda \in \mathbb{C}$. Thus $\bar{V} \times V = W \subseteq \pi^{-1}(Z_{\mathrm{Dol}})$.

Finally, $\pi(W) \subseteq Z_{\mathrm{Dol}}$. The corresponding sets in $M^0_{\mathrm{dR}}(X)$ are complex analytic of the same dimension, and irreducible, so $\pi(W) = Z_{\mathrm{Dol}}$. Remember that π is the product of the quotient $\mathcal{H}^{0,1}(X) \to \mathrm{Pic}^0(X)$ and the identity on $\mathcal{H}^{1,0}(X)$. The image of $\bar{V}$ under the former projection is now a subtorus T of $\mathrm{Pic}^0(V)$ with tangent space V, and $\pi(W) = T \times V$. Since V is also the space of holomorphic one-forms on the dual torus $\hat{T}$, we conclude that Z_{Dol} is the pullback of $M^0_{\mathrm{Dol}}(\hat{T})$ under the induced mapping $X \to \mathrm{Alb}(X) \to \hat{T}$, the second map being dual to the inclusion $T \to \mathrm{Pic}^0(X)$. $\square$

Remark 2.2.4. This proof also illustrates very well the structure of linear subvarieties in $M^0_{\mathrm{Dol}}(X)$, like the cohomology jump loci $\Sigma^k_m(X)_{\mathrm{Dol}}$. Namely, a linear subvariety is of the form

$$(L \otimes T, \theta + V) = \{(L \otimes K, \theta + v) \mid K \in T, v \in V\},$$

where $(L, \theta) \in M^0_{\mathrm{Dol}}(X)$, T is a subtorus of $\mathrm{Pic}^0(X)$, and V is the tangent space of T. This has even dimension $2 \dim T$, and in particular, $(L \otimes T, \theta + V) = M^0_{\mathrm{Dol}}(X)$ if and only if $T = \mathrm{Pic}^0(X)$. This last fact is the critical input from non-abelian Hodge theory used in the proof of Theorem 3.0.1 in Chapter 3.

2.2.1 Application to generic vanishing on abelian varieties

Theorem 2.2.1 has applications to understanding certain jump loci of coherent cohomology. Namely, for each p, q, m, define the jump loci

$$V^q_m(X, \Omega^p_X) = \{L \in \mathrm{Pic}^0(X) \mid \dim H^q(X, \Omega^p_X \otimes L) \geq m\}$$

These were studied by Green and Lazarsfeld [GL87; GL91] in the case that $p = 0$ (or dually $p = n$), who gave certain codimension estimates on these loci in $\mathrm{Pic}^0(X)$ (the Generic Vanishing Theorem, recalled in one form in Section 4.1.1), and showed that they are translates of subtori.

Simpson generalized and strengthened this theorem by embedding $\mathrm{Pic}^0(X)$ into $M^0_{\mathrm{Dol}}(X)$ and applying Theorem 2.2.1.

Theorem 2.2.5. *Let X be a compact Kähler manifold. Then every irreducible component of $S^q_m(X, \Omega^p_X)$ is a translate of a subtorus of* $\mathrm{Pic}^0(X)$.

If X is projective, then these translations are furthermore by torsion points.

By the work of Botong Wang [Wan16], the projectivity assumption is also superfluous here.

Remark 2.2.6. In various applications, including in Chapter 4, the fact that the translations are by torsion points is crucial. In Chapter 4, this is used as follows: If translation by a torsion point $\tau \in \mathrm{Pic}^0(X)$ occurs, consider the multiplication map $N \colon \mathrm{Alb}\, X \to \mathrm{Alb}\, X$ where $N \in \mathbb{N}$ is sufficiently divisible that under the dual map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$, τ is mapped to the origin. Replace then X with the base change $X' \to X$ along the multiplication-by-N map. Since $X' \to X$ is étale, it is easy to relate the cohomology jump loci of the two spaces; but in X', the jump locus associated to τ is now just a subtorus.

In this way, considerations about the cohomology jump loci can often be reduced to the case where all the cohomology jump loci are just subtori (or, in the projective case, abelian subvarieties).

Proof. Recall that for a Higgs bundle $(L, \theta) \in M^0_{\mathrm{Dol}}(X)$, we have

$$H^i_{\mathrm{Dol}}(X; L, \theta) = \bigoplus_{p+q=i} H^{p,q}(X; L, \theta),$$

and that $H^{p,q}(X; L, \theta)$ is the degree p cohomology of the sequence

$$H^q(X, L) \xrightarrow{\theta} H^q(X, \Omega^1_X \otimes L) \to \cdots \to H^q(X, \Omega^n_X \otimes L)$$

Under the embedding $\operatorname{Pic}^0(X) \to M^0_{\mathrm{Dol}}(X) \colon L \mapsto (L, 0)$, we see that $H^{p,q}(X; L, 0)$ recovers the cohomology groups $H^q(X, \Omega^p_X \otimes L)$ that define the cohomology jump locus $V^q_m(X, \Omega^p_X)$.

Define then the jump loci

$$V^{p,q}_m(X) = \{(L, \theta) \in M^0_{\mathrm{Dol}}(X) \mid \dim H^{p,q}(X; L, \theta) \geq m\},$$

so we get $V^q_m(X, \Omega^p_X) = V^{p,q}_m(X) \cap \operatorname{Pic}^0(X)$. It now suffices to show that any irreducible component Z of $V^{p,q}_m(X)$ is a linear subvariety of $M^0_{\mathrm{Dol}}(X)$ (containing a torsion point in the projective case), given that the jump loci $\Sigma^k_m(X)_{\mathrm{Dol}}$ are.

Let then $k = p + q$, and let $m(Z)$ be the generic value of $\dim H^k_{\mathrm{Dol}}(X; L, \theta)$ for $(L, \theta) \in Z$. Then $Z \subseteq \Sigma^k_{m(Z)}(X)_{\mathrm{Dol}}$, by semicontinuity of cohomology. But for the same reason, Z must actually be an irreducible component. Indeed, in a neighbourhood of a general point $(L, \theta) \in Z$, any (L', θ') must have $\dim H^{a,b}(X; L', \theta') \leq \dim H^{a,b}(X; L, \theta)$ for all a, b, but if also $(L', \theta') \in \Sigma^k_{m(Z)}(X)_{\mathrm{Dol}}$, then $m(Z) \leq \sum_{a+b=k} \dim H^{a,b}(X; L', \theta') \leq \sum_{a+b=k} \dim H^{a,b}(X; L', \theta') = m(Z)$. Thus $\dim H^{p,q}(X; L', \theta') = \dim H^{p,q}(X; L, \theta) \geq m$, so $(L', \theta') \in Z$.

It follows that Z is a linear subvariety, and contains a torsion point in the projective case. $\square$

Chapter 3

Kodaira dimension and zeros of holomorphic one-forms, revisited

In [PS14], Popa and Schnell showed that any holomorphic one-form on a smooth projective variety of general type must vanish at some point, a conjecture of Hacon-Kovács and Luo-Zhang [HK05; LZ05]. Wei [Wei20] later gave a slightly simplified argument (as well as a generalization to log-one-forms). Both proofs use the decomposition theorem and various vanishing theorems for Hodge modules. We give a new approach using only classical Hodge theory, namely the rank one case of Simpson's correspondence between Higgs bundles and local systems, and his results on the structure of cohomology jump loci of local systems. Our approach should thus be much more accessible than either of the two previous proofs.

As in [PS14], we will prove the following more precise result.

Theorem 3.0.1 ([PS14, Theorem 2.1])**.** *Let X be a smooth complex projective variety and $f\colon X \to A$ a morphism to an abelian variety. If $H^0(X, \omega_X^{\otimes d} \otimes f^*L^{-1}) \neq 0$ for some integer $d \geq 1$ and some ample line bundle L on A, then for every holomorphic one-form ω on A, the pullback $f^*\omega$ vanishes at some point of X.*

The following conjecture of Luo and Zhang [LZ05] follows as in [PS14]. For varieties of general type, this shows that every holomorphic one-form must vanish at some point.

Corollary 3.0.2 ([PS14, Conjecture 1.2])**.** *Let X be a smooth complex projective variety and $W \subseteq H^0(X, \Omega_X^1)$ be a linear subspace such that every element of $W \setminus \{0\}$ is everywhere non-vanishing. Then* $\dim W \leq \dim X - \kappa(X)$*.*

Our proof of Theorem 3.0.1 goes as follows. Let $V = H^0(A, \Omega_A^1)$ and

$$Z_f = \{(x, \omega) \in X \times V \mid f^*\omega(T_x X) = 0\}.$$

The goal is to show that the restriction of the projection $p_2\colon X \times V \to V$ to Z_f is surjective.

We borrow the idea in [PS14], going back to work of Viehweg-Zuo [VZ01], of constructing two separate sheaves on V. The first is an ambient sheaf coming from a cyclic cover of X, which we will show to be locally free. The second is a non-zero subsheaf coming from X and supported on $p_2(Z_f)$. As the subsheaf is necessarily torsion free, it must have support equal to V, hence $p_2(Z_f) = V$ as desired.

Let us outline the argument for why the ambient sheaf is locally free.

After base change by an isogeny of A, we can assume that $(\omega_X \otimes f^*L^{-1})^{\otimes d}$ has a non-zero section s. Let Y be a resolution of singularities of the associated degree d cyclic cover of X branched along s, and consider the composition $h\colon Y \to A$.

The ambient sheaf is a higher direct image of a complex of sheaves on $Y \times V$, and the fibres of the complex over points in V are Dolbeault complexes of certain Higgs bundles on Y. Using Simpson's results [Sim92; Sim93] relating Higgs bundles to local systems and analyzing cohomology jump loci in the moduli space of local systems, we show that the hypercohomology groups of these Dolbeault complexes have constant dimension over V. This gives the result by Grauert's theorem on locally free direct images.

3.1 Proof of Corollary 3.0.2

Let us first show how Corollary 3.0.2 follows from Theorem 3.0.1. This is the same argument as that given in [PS14].

Proof of Corollary 3.0.2. We only need to consider the case $\kappa(X) \geq 0$. Let $g\colon X' \to Z$ be a smooth model of the Iitaka fibration of X such that $\mu\colon X' \to X$ is a birational modification. We will argue that there exists a map $Z \to A$, where A is a quotient abelian variety of the Albanese variety $\operatorname{Alb} X$, such that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{\mu} & X \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Z & \longrightarrow & A \end{array}$$

Here $X \to A$ is the composition $X \to \operatorname{Alb} X \to A$, where the first map is the Albanese morphism and the second is the quotient map.

To see this, we will argue that under the composition $X' \to X \to \operatorname{Alb} X$, the fibres of g map to translates of a fixed abelian subvariety $B \subset \operatorname{Alb} X$, and then let $A = \operatorname{Alb} X/B$.

This follows from a results of Kawamata [Kaw81, Theorem 1], combined with a standard argument that Kawamata spells out in the same paper [Kaw81, Proof of Theorem 13]. These results rely on some birational geometry that we will not discuss in detail here, but the idea is as follows. Let $G = g^{-1}(z)$ be a general fibre of g; as g is an Iitaka fibration, $\kappa(G) = 0$. By [Kaw81, Theorem 1], G surjects onto its own Albanese variety, hence the image of G in $\operatorname{Alb} X$ is a translate $\tau_G + B_G$ of an abelian subvariety B. But an abelian variety contains at most countably many abelian subvarieties (this can be seen by writing $A = V/\Gamma$ for $\Gamma \subset V$ a discrete lattice of full rank in the complex vector space V; any abelian subvariety of A corresponds to a vector subspace $W \subset V$ such that $W \cap \Gamma$ is a lattice of full rank in W). The abelian subvarieties B_G will vary continuously as G varies, but then it follows that B_G is actually a fixed abelian subvariety B independent of G.

Kawamata argues that this gives a rational map $Z \to A = \operatorname{Alb} X/B\colon z \mapsto \tau_{g^{-1}(z)}$. After possibly modifying the Iitaka fibration further to resolve the indeterminacy locus, we get the desired morphism.

Now note that $\dim G = \dim X - \kappa(X)$, and $\dim A \geq \dim \operatorname{Alb} X - \dim G$, hence $\dim A \geq \dim H^0(X, \Omega^1_X) - (\dim X - \kappa(X))$. We will argue that the image of $f^*\colon H^0(A, \Omega^1_A) \to$

$H^0(X, \Omega^1_X)$ consists of one-forms with a non-empty zero locus, hence completing the proof since this space has codimension at most $\dim X - \kappa(X)$ in $H^0(X, \Omega^1_X)$.

To see this, let L be a very ample line bundle on A. Then $f: X \to A$ is the map induced by the line bundle f^*L. On the other hand, the rational map $X \dashrightarrow Z$ is induced by the line bundle $\omega_X^{\otimes m}$ for sufficiently big and divisible m, by definition of the Iitaka fibration. That $X \to A$ factors rationally as $X \dashrightarrow Z \to A$ implies that there is a non-zero map $f^*L \to \omega_X^{\otimes m}$, hence a non-zero section of $\omega_X^{\otimes m} \otimes f^*L^{-1}$, and we conclude by Theorem 3.0.1. $\square$

3.2 Proof of Theorem 3.0.1

Fix a smooth projective variety X over the complex numbers and a morphism $f: X \to A$ to an abelian variety throughout. Let $V = H^0(A, \Omega^1_A)$ be the vector space of holomorphic one-forms on A, and let $S = \operatorname{Sym} V^*$ be the graded coordinate ring of the vector space V. For an integer i, let $S_{\bullet+i}$ denote S as a graded module over itself, with grading shifted by i, and let $C_{X,\bullet}$ be the complex of graded $\mathcal{O}_X \otimes S$-modules given by

$$\mathcal{O}_X \otimes S_{\bullet-g} \to \Omega^1_X \otimes S_{\bullet-g+1} \to \cdots \to \Omega^n_X \otimes S_{\bullet-g+n},$$

in degrees $-n$ to 0, where $n = \dim X, g = \dim A$, and the differential is induced by the map $\mathcal{O}_X \otimes V \to \Omega^1_X$ given by $\phi \otimes \omega \mapsto \phi f^*\omega$. In a basis $\omega_1, \ldots, \omega_g$ of V with dual basis $s_1, \ldots, s_g$ of S_1, the differential is given by

$$\theta \otimes s \mapsto \sum_{i=1}^{g} (\theta \wedge f^*\omega_i) \otimes s_i s.$$

We denote the associated complex of vector bundles on $X \times V$ by C_X.

Lemma 3.2.1 ([PS14, Lemma 14.1])**.** *The support of C_X is equal to*

$$Z_f = \{(x, \omega) \in X \times V \mid f^*\omega(T_x X) = 0\}.$$

Proof. Let $p: T^*X \to X$ be the projection from the cotangent bundle of X. On T^*X, $p^*\Omega^1_X$ has a tautological section, corresponding to a map $\mathcal{O}_{T^*X} = p^*\mathcal{O}_X \to p^*\Omega^1_X$. Using this map to define the differential, we get a complex

$$p^*\mathcal{O}_X \to p^*\Omega^1_X \to \cdots \to p^*\Omega^n_X$$

But this is just the standard Koszul resolution of the structure sheaf $\mathcal{O}_Z$ of the zero section $Z \subset T^*X$.

Now consider the pullback map $df: X \times V \to T^*X$. The differential of C_X is induced by the tautological section of $p_1^*\Omega^1_X$ induced by pulling back one-forms from A, so C_X is the pullback along df of the Koszul complex discussed above. Hence C_X computes the derived pullback of $\mathcal{O}_Z$, whose support is exactly Z_f. $\square$

Remark 3.2.2. For a possibly more elementary alternative argument, observe that if a one-form ω is nowhere vanishing, then $\mathcal{O}_X \xrightarrow{\omega} \Omega^1_X \to \cdots \to \Omega^n_X$ is exact. By restricting C_X to the sets $X \times \{\omega\}$ as ω varies, and using that C_X is a complex of vector bundles, this proves that the support of C_X is contained in Z_f, which suffices for our purposes.

For $\alpha \in \mathrm{Pic}^0(A)$, let $C^\alpha_X = C_X \otimes p_1^* f^* \alpha$ and $C^\alpha_{X,\bullet} = C_{X,\bullet} \otimes f^*\alpha$, where p_1 is the projection $X \times V \to X$. The sheaves $R^i p_{2*} C^\alpha_X$ on V are then supported on $p_2(Z_f)$ for all i; recall that we are trying to show that $p_2(Z_f) = V$. We will show that these sheaves are locally free for general α in Proposition 3.2.3 below. As we will see, the fibres of C^α_X over V are related to certain Higgs bundles on X.

Recall that a Higgs bundle on X is a vector bundle E together with a morphism of coherent sheaves $\theta\colon E \to \Omega^1_X \otimes E$, the Higgs field, satisfying $\theta \wedge \theta = 0$. Given a Higgs bundle, we get a holomorphic Dolbeault complex

$$E \xrightarrow{\theta\wedge} E \otimes \Omega^1_X \to \cdots \to E \otimes \Omega^n_X.$$

Simpson's non-abelian Hodge theorem [Sim92] associates to each Higgs bundle (E, θ) (satisfying some conditions on stability and Chern classes) a local system $\mathbb{C}_{(E,\theta)}$ of complex vector spaces, and shows that Dolbeault cohomology

$$H^k_{\mathrm{Dol}}(X, E, \theta) = \mathbf{H}^k(X, E \xrightarrow{\theta\wedge} E \otimes \Omega^1_X \to \cdots \to E \otimes \Omega^n_X),$$

the hypercohomology of the Dolbeault complex, is isomorphic to the cohomology of $\mathbb{C}_{(E,\theta)}$.

We will only need the rank one case; see also the lecture notes [Sch13, Lectures 17-18] for a concrete treatment of this case, and the associated Hodge theory.

In the rank one case, a Higgs bundle is just a line bundle together with a holomorphic one-form. The stability condition in Simpson's theorem is always satisfied for line bundles, and the condition on Chern classes is simply that the first Chern class vanishes in $H^2(X, \mathbb{C})$; let $\mathrm{Pic}^\tau(X)$ be the space of line bundles satisfying this condition. Let then $M_{\mathrm{Dol}}(X) = \mathrm{Pic}^\tau(X) \times H^0(X, \Omega^1_X)$, and let $M_{\mathrm{B}}(X)$ denote the moduli space of local systems of one-dimensional complex vector spaces on X. Then Simpson's correspondence, mapping a rank one Higgs bundle to the associated local system, takes the form of a real analytic isomorphism $M_{\mathrm{Dol}}(X) \cong M_{\mathrm{B}}(X)$.

For each k and m, consider the cohomology jump loci

$$\Sigma^k_m(X) = \{\mathcal{L} \in M_{\mathrm{B}}(X) \mid \dim H^k(X, \mathcal{L}) \geq m\}$$
$$\Sigma^k_m(X)_{\mathrm{Dol}} = \{(E, \theta) \in M_{\mathrm{Dol}}(X) \mid \dim H^k_{\mathrm{Dol}}(X, E, \theta) \geq m\}$$

of local systems and Dolbeault cohomology of Higgs bundles. These loci get mapped to each other under Simpson's correspondence.

Using this relationship, Simpson [Sim93] proves that every irreducible component of these loci is a linear subvariety or, in his terminology, a translate of a triple torus (in fact a torsion translate, though we will not need that). A triple torus is a closed, connected, algebraic subgroup N of $M_{\mathrm{B}}(X)$ such that the corresponding subgroup of $M_{\mathrm{Dol}}(X)$ (which we will also refer to as a linear subvariety) is also algebraic (this is equivalent to the usual definition, involving also the de Rham moduli space, by [Sim93, Lemma 2.1]). A linear subvariety is thus a subset of $M_{\mathrm{B}}(X)$ of the form

$$\{\mathcal{L} \otimes \mathcal{N} \mid \mathcal{N} \in N\}$$

where N is a triple torus and $\mathcal{L} \in M_{\mathrm{B}}(X)$ a local system.

Simpson [Sim93, Lemma 2.1] shows that a triple torus is of the form $g^* M_{\mathrm{B}}(T)$ for a map $g\colon X \to T$ to an abelian variety, where $g^*\colon M_{\mathrm{B}}(T) \to M_{\mathrm{B}}(X)$ denotes pullback of local systems. It follows that a linear subvariety in $M_{\mathrm{Dol}}(X)$ is a translate of a subset of the form $g^* \operatorname{Pic}^0(T) \times g^* H^0(T, \Omega^1_T)$ for $g\colon X \to T$ a morphism to an abelian variety. In particular, a linear subvariety is either the entire moduli space, or maps to a proper subvariety of $\operatorname{Pic}^0(X)$ under the projection $M_{\mathrm{Dol}}(X) \to \operatorname{Pic}^0(X)$ that forgets the Higgs field.

The following proposition is the main new ingredient in the proof. Note that this proposition is valid for an arbitrary morphism $f\colon X \to A$, not just those that satisfy the hypotheses of Theorem 3.0.1.

Proposition 3.2.3. *For general $\alpha \in \operatorname{Pic}^0(A)$, the higher direct image sheaves $R^i p_{2*} C^\alpha_X$ are locally free on V for all i.*

Proof. We will show that for general α, the dimensions of the hypercohomology $H^i(X \times \{v\}, C^\alpha_X|_{X\times\{v\}})$ of C^α_X on fibres of p_2 are constant in v. The result follows by a version of Grauert's theorem on locally free direct images for complexes of sheaves [EGAIII, Proposition 7.8.4]

Note that for any fibre $X \times \{v\}$ of p_2 for $v \in V$, the restriction of C^α_X to the fibre is the Dolbeault complex

$$f^*\alpha \xrightarrow{\wedge f^* v} f^*\alpha \otimes \Omega^1_X \to \cdots \to f^*\alpha \otimes \Omega^n_X.$$

of the Higgs bundle $(f^*\alpha, f^*v)$. The Dolbeault cohomology of these Higgs bundles is governed by the cohomology jump loci $\Sigma^k_m(X)_{\mathrm{Dol}}$, of which only finitely many are nonempty by algebraicity. Each irreducible component of the nonempty ones is a linear subvariety, so it suffices to show that for each linear subvariety S of $M_{\mathrm{Dol}}(X)$, the set $\{\alpha\} \times f^*V$ is either entirely contained in S or entirely disjoint from it, for general α.

Let then $\phi = f^*\colon M_{\mathrm{Dol}}(A) \to M_{\mathrm{Dol}}(X)$, and suppose $S \subset M_{\mathrm{Dol}}(X)$ is a linear subvariety. We observe that if N is a triple torus, then the connected component of the identity in $\phi^{-1}(N)$ is again a triple torus; it follows that $\phi^{-1}(S)$ is either empty, or a finite union of linear subvarieties. If $\phi^{-1}(S) = M_{\mathrm{Dol}}(A)$ then $\{\alpha\} \times f^*V \subset S$ for any $\alpha \in \operatorname{Pic}^0(A)$. If S is a proper subset of $M_{\mathrm{Dol}}(A)$, it suffices to take α to be outside the image of $\phi^{-1}(S)$ in $\operatorname{Pic}^0(A)$ under the projection $M_{\mathrm{Dol}}(A) \to \operatorname{Pic}^0(A)$. □

If we could show that one of the sheaves $R^i p_{2*} C^\alpha_X$ were nontrivial on V, under the hypotheses of Theorem 3.0.1, we would now be done. Unfortunately we cannot, but instead we make use of a covering construction as in [PS14].

Lemma 3.2.4 ([PS14, Lemma 11.1])**.** *Suppose $\omega_X^{\otimes d} \otimes f^*L^{-1}$ has a nonzero section for some d and some ample line bundle L on A. For an isogeny $\phi\colon A' \to A$, define $f'\colon X' \to A'$ by base change of f. For an appropriately chosen ϕ, there exists an ample line bundle L' on A' such that $(\omega_{X'} \otimes f'^* L'^{-1})^{\otimes d}$ has a nonzero section.*

Proof. Let $A' = A$, and $\phi\colon A \to A$ be given by multiplication by $2d$. Then $\phi^* L$ is the dth power of some line bundle L', which is hence ample. Further, the pullback of ω_X along the étale map $X' \to X$ is $\omega_{X'}$, so $(\omega_{X'} \otimes f'^* L'^{-1})^{\otimes d}$ has a non-zero section. $\square$

Assume now the hypotheses of Theorem 3.0.1. Note that zero loci of one-forms are not affected by étale covers, so if we can prove the theorem for $f'\colon X' \to A'$, then the desired conclusion also follows for $f\colon X \to A$.

In particular, replacing f by this f', we can now assume without loss of generality that $B^{\otimes d}$ has a nonzero section s for $B = \omega_X \otimes f^* L^{-1}$. Let Y be a resolution of singularities of the d-fold cyclic cover $\pi\colon X_d \to X$ ramified along $Z(s)$, giving us the following maps:

$$\begin{tikzcd} Y \arrow[r] \arrow[rr, bend left, "\phi"] \arrow[drr, "h"'] & X_d \arrow[r, "\pi"] & X \arrow[d, "f"] \\ & & A \end{tikzcd}$$

By construction, $X_d = \operatorname{Spec} \bigoplus_{i=0}^{d-1} B^{-i}$, so $\pi_* \pi^* B = \bigoplus_{i=-1}^{d-2} B^{-i}$. This has a section in the $i = 0$ term, and the corresponding section of $\pi^* B$ gives a morphism $\phi^* B^{-1} \to \mathcal{O}_Y$, an isomorphism away from $Z(s)$. Together with pullback of forms, this gives injective morphisms $\phi^*(B^{-1} \otimes \Omega_X^k) \to \Omega_Y^k$. As $\mathcal{O}_X \to \phi_* \mathcal{O}_Y$ is injective, the corresponding morphisms

$$B^{-1} \otimes \Omega_X^k \to \phi_* \Omega_Y^k$$

on X are also injective.

Note that we get a complex $C_{Y,\bullet}$ of graded $\mathcal{O}_Y \otimes S$-modules using the morphism $h\colon Y \to A$, constructed in the same way that $C_{X,\bullet}$ was constructed starting from f above Lemma 3.2.1.

We give a slightly modified version of [PS14, Lemma 13.1].

Lemma 3.2.5. *The morphisms above induce a morphism of complexes of graded $\mathcal{O}_X \otimes S$-modules*

$$B^{-1} \otimes C_{X,\bullet} \to \mathbf{R}\phi_* C_{Y,\bullet}$$

Proof. The morphisms $\phi^*(B^{-1} \otimes \Omega_X^k) \to \Omega_Y^k$ commute with the differentials on Y, giving

$$\phi^*(B^{-1} \otimes C_{X,\bullet}) \to C_{Y,\bullet}$$

Using the projection formula and the morphism $\mathcal{O}_X \to \mathbf{R}\phi_* \mathcal{O}_Y$, pushing forward to X gives the desired composition

$$B^{-1} \otimes C_{X,\bullet} \to (B^{-1} \otimes C_{X,\bullet}) \otimes^{\mathbf{L}} \mathbf{R}\phi_* \mathcal{O}_Y \to \mathbf{R}\phi_* C_{Y,\bullet}$$

$\square$

Proof of Theorem 3.0.1. We must show that Z_f surjects onto V under the projection $p_2\colon X \times V \to V$.

Let $\alpha \in \operatorname{Pic}^0(A)$ be a general element. Then Lemma 3.2.5 gives, after twisting by $f^*\alpha$ and pushing forward to V, a morphism $\mathbf{R}p_{2*}(p_1^*B^{-1} \otimes C_X^\alpha) \to \mathbf{R}p_2^*C_Y^\alpha$ where $p_1\colon X \times V \to X$ is the first projection, and p_2, by abuse of notation, is used for both of the projections $X \times V \to V$ and $Y \times V \to V$. Let $\mathscr{F}$ be the image of the induced map $R^0p_{2*}(p_1^*B^{-1} \otimes C_X^\alpha) \to R^0p_{2*}C_Y^\alpha$.

As α is general, each $R^ip_{2*}C_Y^\alpha$ is locally free by Proposition 3.2.3. In particular $\mathscr{F}$ is torsion free. Since C_X is supported on Z_f, $\mathscr{F}$ is supported on $p_2(Z_f)$, so it suffices to show that $\mathscr{F}$ is non-zero.

Let $k = g-n$. Then $C_{X,k} = \omega_X$ and $C_{Y,k} = \omega_Y$, and the morphism $B^{-1} \otimes C_{X,k} \to \mathbf{R}\phi_*C_{Y,k}$ from Lemma 3.2.5 induces the morphism of sheaves $f^*L = B^{-1} \otimes \omega_X \to \phi_*\omega_Y$ constructed before the lemma after taking cohomology sheaves (by results of Kollár [Kol86b], we actually have $\mathbf{R}\phi_*\omega_Y = \phi_*\omega_Y$ since ϕ is generically finite, but we do not need this).

After twisting by α, the morphism $B^{-1} \otimes C_{X,k}^\alpha \to \mathbf{R}\phi_*C_{Y,k}^\alpha$ thus induces $f^*(L \otimes \alpha) \to \phi_*\omega_Y \otimes f^*\alpha$. For the graded S-module $\mathscr{F}_\bullet = H^0(V, \mathscr{F})$, it follows that $\mathscr{F}_k \cong H^0(X, f^*(L \otimes \alpha))$ since the pushforward to V preserves injectivity. But $f^*(L \otimes \alpha)$ has non-zero sections, hence $\mathscr{F}$ is non-zero: Otherwise all sections of its pushforward $f_*\mathcal{O}_X \otimes L \otimes \alpha$ to A would vanish, which would imply that X is contained in a general translate of a hyperplane section of A, a contradiction. $\square$

Chapter 4

Chen-Jiang decompositions for projective varieties, without Hodge modules

Given a morphism $f\colon X \to A$ from a smooth projective variety over $\mathbb{C}$ to an abelian variety, the direct image $f_*\omega_X$ is known by work of Green and Lazarsfeld [GL87] to be a GV-sheaf, that is, the cohomology support locus

$$V^k(A, f_*\omega_X) = \{\alpha \in \hat{A} \mid H^k(A, f_*\omega_X \otimes \alpha) \neq 0\}$$

has codimension at least k in the dual abelian variety $\hat{A}$ for each $k \geq 0$. Moreover, the precise structure of these loci is well understood: The components of $V^k(A, f_*\omega_X)$ are translates of abelian subvarieties by [GL91], and are in fact translates by points of finite order by work of Simpson [Sim93]. We recall these results more precisely in Section 4.1.1.

In the case where f is generically finite, Chen and Jiang [CJ18] prove a semi-positivity result for $f_*\omega_X$ corresponding to the structure of the cohomology support loci. Namely, they prove that there exists a decomposition

$$f_*\omega_X \cong \bigoplus_i \alpha_i \otimes p_i^*\mathscr{F}_i,$$

since called a Chen-Jiang decomposition in [PPS17], where each $\alpha_i \in \hat{A}$ is a point of finite order, $p_i\colon A \to A_i$ is a surjective homomorphism of abelian varieties with connected fibres, and each $\mathscr{F}_i$ is an M-regular coherent sheaf on A_i, i.e. for each $k > 0$ we have $\operatorname{codim}_{\hat{A}_i} V^k(A_i, \mathscr{F}_i) > k$. The dual of each p_i is an inclusion $\hat{p}_i\colon \hat{A}_i \to \hat{A}$, and the codimension k components of $V^k(A, f_*\omega_X)$ for each k are exactly the translates by α_i of $\hat{A}_i$, so the failure of $f_*\omega_X$ itself to be M-regular is explained by this decomposition. The proof by Chen and Jiang relies on the structural results on $V^k(A, f_*\omega_X)$, but is otherwise algebraic in nature.

Using Hodge modules, this theorem was widely generalized by Pareschi, Popa and Schnell [PPS17]. They prove a Chen-Jiang decomposition result for the associated graded pieces of the Hodge filtration on any polarizable real Hodge module on a compact complex torus. Since direct images of canonical bundles arise in this way, the result of Chen-Jiang is thus extended to arbitrary morphisms (and even to the Kähler setting).

Building on this result, Lombardi, Popa and Schnell [LPS20] prove that direct images of pluricanonical bundles of smooth projective varieties likewise admit Chen-Jiang decompositions, by showing that for $f\colon X \to A$ and any $m \geq 2$, there exists a smooth projective variety X_m and a morphism $f_m\colon X_m \to A$ such that $f_*\omega_X^{\otimes m}$ is a direct summand in $f_{m*}\omega_{X_m}$. In the case where (X, Δ) is a klt pair, Jiang and Meng [Jia21; Men21] independently give results for direct images of line bundles with divisor rationally equivalent to $m(K_X + \Delta)$, Jiang for integral $m \geq 1$ with the condition that f be primitive for $m \geq 2$, and Meng unconditionally for any rational $m \geq 1$. All of these results in turn have applications to the birational theory of irregular varieties.

The proof in [PPS17] relies heavily on Hodge modules and the decomposition theorem, but in the geometric case where $f\colon X \to A$ is a morphism from a smooth projective variety to an abelian variety, it is reasonable to expect a more direct proof along the lines of the original work by Chen-Jiang. We give such a proof (Section 4.2), relying only on the theory of variations of Hodge structure, removing the dependence of the previously mentioned results on Hodge modules.

Theorem 4.0.1. *For any morphism $f\colon X \to A$ from a smooth projective variety to an abelian variety, the sheaf $f_*\omega_X$ admits a Chen-Jiang decomposition.*

Following the method of [CJ18], the key part of the proof is the following. Assume that the components of the vanishing loci $V^k(A, f_*\omega_X)$ pass through the origin, and are hence abelian subvarieties; this can be arranged via an isogeny of A. Suppose then that $\hat{B} \subset \hat{A}$ is a codimension k component of $V^k(A, f_*\omega_X)$, and let $p\colon A \to B$ be the projection dual to the inclusion of $\hat{B}$. We must then produce an appropriate M-regular sheaf $\mathscr{F}$ on B such that $p^*\mathscr{F}$ is a direct summand of $f_*\omega_X$. Given such M-regular sheaves for each such B, the theorem follows for formal reasons (see Lemma 4.2.1).

In the case where f is generically finite, Chen and Jiang show that $R^kp_*f_*\omega_X$ is actually the pushforward to B of the canonical bundle of a lower-dimensional variety, hence admits a Chen-Jiang decomposition by dimensional induction. The M-regular summand of this decomposition serves as $\mathscr{F}$. More precisely, they construct the following diagram (note that the notation here differs slightly from the paper [CJ18]).

$$\begin{tikzcd}
X \arrow[dr, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "q"'] & & \\
& Y \arrow[r, "f'"] \arrow[d, "r"] & A \arrow[d, "p"] \\
& Z \arrow[r, "h"] & B
\end{tikzcd}$$

Here $X \xrightarrow{q} Z \xrightarrow{h} B$ is a modified Stein factorization where Z is smooth and h generically finite, and Y is the pullback of p along h. Then q is a fibration of relative dimension k, so $R^kq_*\omega_X = \omega_Z$ hence $R^kp_*f_*\omega_X = h_*\omega_Z$. Furthermore r is a pullback of a morphism of abelian varieties so $r^*\omega_Z = \omega_Y$, and since g is generically finite, ω_Y is a direct summand of $g_*\omega_X$, hence $\mathscr{F}$ is a direct summand of $f_*\omega_X$ by base change.

In the general case of arbitrary f, we use results of Kollár [Kol86a; Kol86b] on variations of Hodge structures and higher direct images of canonical bundles to prove that $p^*R^kp_*f_*\omega_X$

is a direct summand of $f_*\omega_X$, otherwise finishing the proof in the same manner as Chen and Jiang. The more technical proof is deferred to Section 4.3.

Theorem 4.0.2. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are surjective morphisms of smooth projective varieties, and that g is a smooth fibration of relative dimension k with $\omega_{Y/Z}$ trivial. Then $f_*\omega_X$ admits $g^*R^k g_* f_*\omega_X$ as a direct summand.*

The idea is to construct, by Grothendieck duality, a morphism

$$\Psi_{X/Y} \colon f_*\omega_{X/Y} \to g^*R^k g_* f_*\omega_{X/Z}$$

which, fibrewise, encodes certain Gysin morphisms. This is most easily described when Z is a point. Let $y \in Y$ be a general point and $F = f^{-1}(y)$ the corresponding fibre of f. Then the fibre of $\Psi_{X/Y}$ at y is a morphism $H^0(F, \omega_F) \to H^k(Y, f_*\omega_X)$. When composed with the edge map $H^k(Y, f_*\omega_X) \to H^k(X, \omega_X)$ of the Leray spectral sequence we get a morphism $\Psi'_{X/Y}|_y \colon H^0(F, \omega_F) \to H^k(X, \omega_X)$. On the other hand, the inclusion $F \to X$ gives a Gysin morphism $H^d(F, \mathbb{C}) \to H^{2k+d}(F, \mathbb{C})$ where $d = \dim F$, and the restriction to $H^0(F, \omega_F)$ under the Hodge decomposition coincides with $\Psi'_{X/Y}|_y$.

To prove the theorem, we construct a morphism of variations of Hodge structure over an open locus which encodes the topological Gysin morphisms on fibres (although for technical reasons, we actually split the direct image $g_*\Psi_{X/Y}$ on Z instead and then use the push-pull adjunction). The splitting then comes from the semisimplicity of the category of polarizable VHS, and we use Kollár's results on higher direct images of canonical bundles [Kol86b, Theorem 2.6] to extend from the open locus. The resulting more precise statement, describing the morphism $f_*\omega_X \to g^*R^k g_* f_*\omega_X$, is given as Theorem 4.3.4.

4.1 Preliminaries

We work throughout with smooth varieties over $\mathbb{C}$.

4.1.1 Generic vanishing

Fix an abelian variety A throughout this section, and let $\hat{A}$ be the dual abelian variety. Let us recall some basic notions related to GV-sheaves.

For a coherent sheaf $\mathscr{F}$ on A, let $V^k(A, \mathscr{F}) = \{\alpha \in \hat{A} \mid H^k(A, \mathscr{F} \otimes \alpha) \neq 0\}$ denote its kth cohomology support locus. This is a closed subvariety of $\hat{A}$.

Definition 4.1.1. A coherent sheaf $\mathscr{F}$ on an abelian variety A is a *GV-sheaf* if

$$\operatorname{codim}_{\hat{A}} V^k(A, \mathscr{F}) \geq k$$

for every $k \geq 0$, and *M-regular* if

$$\operatorname{codim}_{\hat{A}} V^k(A, \mathscr{F}) > k$$

for every $k > 0$.

Following [Sch19], define the symmetric Fourier-Mukai transform to be the contravariant functor

$$\mathsf{FM}_A\colon D^b_{\mathrm{coh}}(A) \to D^b_{\mathrm{coh}}(\hat{A})$$

given by the formula

$$\mathsf{FM}_A(K) = \mathbf{R}(\mathrm{pr}_2)_*(P \otimes \mathrm{pr}_1^* D_A(K))$$

where $\mathrm{pr}_1\colon A \times \hat{A} \to A$ and $\mathrm{pr}_2\colon A \times \hat{A} \to \hat{A}$ are the two projections, P is the Poincaré line bundle on $A \times \hat{A}$ normalized by requiring that its fibres over $0 \in A$ respectively $0 \in \hat{A}$ are trivial, and

$$D_A(K) = \mathbf{R}\,\underline{\mathrm{Hom}}(K, \omega_A[\dim A])$$

is the Grothendieck duality functor on A. Then FM_A is an equivalence of categories with inverse $\mathsf{FM}_{\hat{A}}$, and is a version of the Fourier-Mukai transform particularly well-adapted to talking about generic vanishing.

GV-sheaves and M-regular sheaves can be defined in terms of the Fourier-Mukai transform. This goes back to Hacon for GV-sheaves, and Pareschi and Popa [PP11] for M-regular sheaves.

Proposition 4.1.1 ([Hac04, Theorem 1.2],[PP11, Proposition 2.8]). *A coherent sheaf $\mathscr{F}$ on an abelian variety A is a GV-sheaf if and only if $\mathsf{FM}_A(\mathscr{F})$ is a sheaf (i.e. a complex with cohomology only in degree 0), and $\mathscr{F}$ is M-regular if and only if $\mathsf{FM}_A(\mathscr{F})$ is furthermore a torsion-free sheaf.*

Proposition 4.1.2 ([Sch19, Proposition 4.1]). *Let $f\colon A \to B$ be a morphism of abelian varieties. Denoting by $\hat{f}\colon \hat{B} \to \hat{A}$ the dual morphism, there are natural isomorphisms*

$$\mathsf{FM}_B \circ \mathbf{R}f_* = \mathbf{L}\hat{f}^* \circ \mathsf{FM}_A$$
$$\mathsf{FM}_A \circ \mathbf{L}f^* = \mathbf{R}\hat{f}_* \circ \mathsf{FM}_B$$

Proposition 4.1.3 ([Sch19, Proposition 5.1]). *For $a \in A$ let $t_a\colon A \to A$ be the translation morphism and P_a the corresponding line bundle on $\hat{A}$. For $a \in A$ and $\alpha \in \hat{A}$, there are natural isomorphisms*

$$\mathsf{FM}_A \circ (t_a)_* = (P_a \otimes -) \circ \mathsf{FM}_A$$
$$\mathsf{FM}_A \circ (P_\alpha \otimes -) = (t_\alpha)_* \circ \mathsf{FM}_A$$

The following characterization of the vanishing loci arising from canonical bundles is due to Green-Lazarsfeld and Simpson for ordinary direct images.

Theorem 4.1.4 ([GL87; GL91; Sim93]). *Suppose $f\colon X \to A$ is any morphism from a smooth projective variety X to an abelian variety A. For every i, the sheaf $R^i f_* \omega_X$ is a GV-sheaf. Furthermore, for every k, every component of $V^k(A, R^i f_* \omega_X)$ is a translate of an abelian subvariety of $\hat{A}$ by a point of finite order.*

By work of Kollár, for each i the higher direct image $R^i f_* \omega_X$ is a direct summand in $g_* \omega_Y$ for some $g \colon Y \to A$, where Y is smooth projective [Kol86b, Corollary 2.24], and the properties stated in the theorem are inherited by direct summands, so we get the theorem for higher direct images as well.

4.1.2 Variations of Hodge structure and higher direct images of canonical bundles

We will fix notation for, and recall some facts about, variations of Hodge structure. For the full definition see e.g. [PS08]. We will follow the notation of [Kol86b]; see also that paper for an introduction to canonical extensions.

The data of a VHS of weight k on a smooth variety U consists of a local system H, which for us will always have coefficient group $\mathbb{Q}$, together with a filtration by holomorphic subbundles of the vector bundle $\mathscr{H} = H \otimes_{\mathbb{Q}} \mathcal{O}_U$, denoted

$$\mathscr{H} = \mathscr{F}^0(H) \supset \cdots \supset \mathscr{F}^n(H) \supset 0$$

by abuse of notation, such that the fibre of the filtration at a given point x defines a rational Hodge structure of weight k on the rational vector space H_x. We likewise let $\mathrm{Gr}^i(H) = \mathscr{F}^i/\mathscr{F}^{i+1}$, which are again vector bundles. We let $\nabla \colon \mathscr{H} \to \mathscr{H} \otimes \Omega^1_U$ denote the induced Gauss-Manin connection, with respect to which the Hodge filtration is required to satisfy the Griffiths transversality condition

$$\nabla(\mathscr{F}^p) \subset \mathscr{F}^{p-1} \otimes \Omega^1_U$$

Finally, a polarization on H is a map of local systems $H \otimes H \to \mathbb{Q}_U$ which induces polarizations on the rational Hodge structures H_x for all $x \in U$.

Suppose $f \colon X \to Y$ is a smooth projective morphism with fibres of dimension d. For any k, the sheaf $R^k f_* \mathbb{Q}_X$ is then a local system underlying a VHS of weight k. We note that this VHS is polarizable. To see this, choose a class $\eta \in H^2(X, \mathbb{Q})$ whose restriction to fibres of f is Kähler, as granted by the assumption that f is projective. For $i \leq d$, η induces a bilinear form on $R^i f_* \mathbb{Q}_X$ defined fibrewise at $y \in Y$ by $(\alpha, \beta) \mapsto \int_F \eta_{|F}^{d-i} \wedge \alpha \wedge \beta$ where $F = f^{-1}(y)$. This defines a polarization on the primitive part $(R^i f_* \mathbb{Q}_X)_{\mathrm{prim}}$ by the Hodge-Riemann bilinear relations. By the relative Hard Lefschetz theorem and consequent relative Lefschetz decomposition, $R^k f_* \mathbb{Q}_X$ decomposes as a direct sum of (Tate twists of) such primitive pieces, hence admits a polarization.

Note in particular that in middle degree d and up, the bottom piece of the Hodge filtration is

$$\mathscr{F}^{k+d}(R^{k+d} f_* \mathbb{Q}_X) = R^k f_* \omega_{X/Y}.$$

Suppose now that H underlies a VHS on an open subset $X^0 \subset X$ such that $X \setminus X^0$ is a normal crossings divisor. Suppose the monodromy of H around this divisor is quasi-unipotent. A choice of set-theoretic logarithm $\log \colon \mathbb{C}^\times \to \mathbb{C}$ yields a corresponding canonical extension of $\mathscr{H}$ and the filtration $\mathscr{F}^\bullet(H)$ to vector bundles on X. One way to fix such a logarithm is by choosing a fixed length 2π interval for the imaginary values of the logarithm. The choice $[0, 2\pi)$ is called the upper canonical extension in [Kol86b], and will be denoted by ${}^u\mathscr{H}, {}^u\mathscr{F}^\bullet(H)$.

The choice $(-2\pi, 0]$ gives the lower canonical extension, denoted by ${}^l\mathscr{H}, {}^l\mathscr{F}^\bullet(H)$. Similarly, ${}^u\operatorname{Gr}^i(H)$ and ${}^l\operatorname{Gr}^i(H)$ denote the associated graded pieces of the extended filtrations.

We briefly recall the local construction. In an analytic neighbourhood of a point in $X \setminus X^0$, X^0 looks like $(\mathbf{D}^*)^s \times \mathbf{D}^r$ for some s and r, where $\mathbf{D}$ is the unit disk and $\mathbf{D}^*$ the punctured unit disk. The local monodromy on a fixed fibre of H in this neighbourhood is generated by the monodromy operators $T_1, \dots, T_s$ corresponding to the generators of the fundamental groups of each $\mathbf{D}^*$. We will describe how to extend over each $\mathbf{D}^*$ separately, so it suffices for us to assume that $s = 1$ and $r = 0$, so we simply have a VHS on $\mathbf{D}^*$ with quasi-unipotent monodromy generated by an operator T.

Let $\exp\colon \mathbf{H} \to \mathbf{D}^*$ be the universal cover of the unit disk by the left half plane in $\mathbb{C}$. Then T acts on global sections of $\exp^* \mathscr{H}$ by the action of pullback along the translation by i of $\mathbf{H}$. In particular, T acts on the space V of global flat sections.

Now choose coordinates on V such that T decomposes as a product $T = UD$ of a unipotent matrix U and a diagonal matrix D. The unipotent matrix has a logarithm given by the usual power series expansion

$$\log U = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(B - I)^k}{k},$$

while our choice of logarithm gives a logarithm $\log D$; thus we get $N = \log T = \log U + \log D$. If $v \in V$, then the section

$$s(z) = \exp\left(-\frac{1}{2\pi i} N \cdot \log z\right) v(z)$$

is T-invariant, hence descends to a global section of $\mathscr{H}$. This defines a trivialization of $\mathscr{H}$, hence an extension of $\mathscr{H}$ to a free sheaf on $\mathbf{D}$. This also gives an extension of the Gauss-Manin connection on $\mathscr{H}$ to a connection on the extension with logarithmic singularities at 0, though we will not need a detailed description of this. The nilpotent orbit theorem says that the filtration $\mathscr{F}^\bullet(H)$ likewise extends, and these local extensions glue to an extension of $\mathscr{H}$ to X.

One technical obstacle with this theory is that if $H_i, i = 1, 2$ are two VHS on $\mathbf{D}^*$ with monodromy operators T_i, the monodromy operator of $H_1 \otimes H_2$ is $T = T_1 \otimes T_2$, but the chosen logarithms of T and the T_i may not be directly related. Decomposing $T_i = U_i D_i$ and $T = UD$ in a unipotent and diagonal part, the eigenvalues of D are products $d_1 d_2$ where d_i is an eigenvalue of D_i. But it is not necessarily the case that $\log(d_1 d_2) = \log d_1 + \log d_2$, and as a consequence it is not necessarily the case that the canonical extension of H, with this fixed choice of logarithm, is the tensor product of the canonical extensions of the H_i. However, if, say, H_1 has trivial monodromy, then canonical extensions and tensor products will in fact commute in this special case since $d_1 d_2 = d_2$ for every pair of eigenvalues as above. This will be important in the proof of Theorem 4.0.2.

Using this machinery of canonical extensions, Kollár proves the following results.

Theorem 4.1.5 ([Kol86b, Theorem 2.6])**.** *Let $f\colon X \to Y$ be a surjective map of relative dimension d between smooth projective varieties. Suppose $Y^0 \subset Y$ is an open subset such* *that $Y \setminus Y^0$ is a normal crossings divisor and f is smooth over Y^0. Let $X^0 = f^{-1}(Y^0)$ and $f^0 = f|_{X^0}$. Then*

$$R^k f_* \omega_{X/Y} \cong {}^u\mathscr{F}^{k+d}(R^{k+d} f^0_* \mathbb{Q}_{X^0})$$
$$R^k f_* \mathcal{O}_X \cong {}^l \operatorname{Gr}^0(R^k f^0_* \mathbb{Q}_{X^0})$$

In particular, $R^k f_* \mathcal{O}_X$ and $R^k f_* \omega_{X/Y}$ are locally free.

Theorem 4.1.6 ([Kol86b, Theorem 3.4])**.** *Let X, Y, Z be projective varieties, X smooth, and $f: X \to Y, g: Y \to Z$ surjective maps. Then*

1. *$R^p(g \circ f)_* \omega_X \cong \bigoplus_i R^i g_* R^{p-i} f_* \omega_X$;*
2. *$R^i g_* R^j f_* \omega_X$ is torsion-free;*
3. *$R^i g_* R^j f_* \omega_X = 0$ if $i > \dim Y - \dim Z$;*
4. *In the derived category of coherent sheaves on Z,*

$$\mathbf{R} g_* R^j f_* \omega_X = \bigoplus_i R^i g_* R^j f_* \omega_X[-i].$$

Theorem 4.1.5 shows that $R^k f_* \omega_{X/Y}$, hence also $R^k f_* \omega_X$, is globally controlled by a polarizable VHS on an open subset. Observe the following.

1. The formation of canonical extensions is compatible with taking direct sums of VHS.
2. The open subset $Y^0 \subset Y$ in the theorem does not have to be the entire smooth locus of f; any non-empty Zariski-open subset thereof suffices as long as the complement is normal crossings.
3. The category of polarizable VHS is semisimple [PS08, Theorem 10.13].

It follows that one way to get a direct sum decomposition of $R^k f_* \omega_X$ is to construct an appropriate morphism involving the VHS $R^{k+d} f^0_* \mathbb{Q}_{X^0}$, for some appropriate Y^0 as in the theorem. This will be the mechanism for getting the splitting in Theorem 4.0.2.

4.1.3 Chen-Jiang decompositions

Let's recall the following definition and proposition from [LPS20].

Definition 4.1.2 ([LPS20, Definition 4.1])**.** Suppose $\mathscr{F}$ is a coherent sheaf on an abelian variety A. A Chen-Jiang decomposition of $\mathscr{F}$ is a direct sum decomposition

$$\mathscr{F} \cong \bigoplus_i \alpha_i \otimes p_i^* \mathscr{F}_i$$

where each $p_i: A \to A_i$ is a surjective homomorphism of abelian varieties with connected fibres, each $\mathscr{F}_i$ is an M-regular coherent sheaf on A_i and each $\alpha_i \in \hat{A}$ is a line bundle of finite order.

Proposition 4.1.7 ([LPS20, Proposition 4.6])**.** *Suppose $\mathscr{F}'$ and $\mathscr{F}''$ are coherent sheaves on an abelian variety A. If $\mathscr{F}' \oplus \mathscr{F}''$ admits a Chen-Jiang decomposition, so do $\mathscr{F}'$ and $\mathscr{F}''$.*

The following proposition is essentially proven as part of the proof of [CJ18, Theorem 3.5].

Proposition 4.1.8. *Suppose $\mathscr{F}$ is a coherent sheaf on an abelian variety A, and $\phi\colon A' \to A$ is an isogeny. Then $\mathscr{F}$ admits a Chen-Jiang decomposition if and only if $\phi^*\mathscr{F}$ does.*

Proof. If $\mathscr{F}$ admits a Chen-Jiang decomposition then clearly so does $\phi^*\mathscr{F}$.

In the other direction note that by Propositions 4.1.1, 4.1.2, and 4.1.3, a Chen-Jiang decomposition of ϕ^*F is equivalent to a decomposition

$$\mathsf{FM}_{A'}(\phi^*\mathscr{F}) \cong \bigoplus_i \tau_{\alpha_i *}\iota_{i*}\mathscr{G}_i$$

where for each i, τ_{α_i} is a translation of A' by a point α_i of finite order, $\iota_i\colon \hat{A}_i \to \hat{A}'$ the inclusion of an abelian subvariety, and $\mathscr{G}_i$ is a torsion free sheaf on $\hat{A}_i$.

Now for each i, $\hat{\phi}^*\tau_{\alpha_i *}\iota_{i*}\mathscr{G}_i$ is the direct image of a torsion free sheaf on $\hat{\phi}^{-1}(\hat{A}_i)$, which is again a torsion translate of an abelian subvariety of $\hat{A}$; namely the direct image of $\left(\hat{\phi}|_{\hat{\phi}^{-1}(\hat{A}_i)}\right)^*\mathscr{G}_i$ translated by a preimage of α_i.

By Proposition 4.1.2 and since ϕ is an isogeny we have

$$\hat{\phi}^*\mathsf{FM}_{A'}(\phi^*\mathscr{F}) = \mathsf{FM}_A(\phi_*\phi^*\mathscr{F}),$$

so $\phi_*\phi^*\mathscr{F}$ admits a Chen-Jiang decomposition. But $\mathscr{F}$ is a direct summand thereof, hence admits a Chen-Jiang decomposition by Proposition 4.1.7. □

Given a morphism $f\colon X \to A$ to an abelian variety, we will need to understand how the image $f(X)$ relates to the various components of the cohomology support loci of $f_*\omega_X$.

Lemma 4.1.9. *Suppose given $f\colon X \to A$ where X is a smooth projective variety, A an abelian variety, and suppose $\hat{B} \subset \hat{A}$ is a codimension k component of $V^k(A, f_*\omega_X)$ which passes through $0 \in \hat{A}$ (and is hence an abelian subvariety by Theorem 4.1.4). Let $p\colon A \to B$ be dual to the inclusion. Then all fibres of $f(X)$ over B are of dimension k, hence $f(X)$ is the preimage of $p(f(X))$. In particular $p|_{f(X)}\colon f(X) \to p(f(X))$ is a smooth fibration with trivial relative canonical bundle.*

Proof. Observe that p is smooth of relative dimension k, so it suffices to show that a general fibre of $p|_{f(X)}$ has dimension k. Suppose $\beta \in \hat{B}$. By Kollár's result (Theorem 4.1.6), we have

$$h^k(A, f_*\omega_X \otimes p^*\beta) = \sum_{i=0}^{k} h^i(B, R^{k-i}p_*f_*\omega_X \otimes \beta)$$

The left hand side is non-zero by the assumptions on B, while for general $\beta \in \hat{B}$, the terms with $i > 0$ on the right hand side vanish since $R^{k-i}p_*f_*\omega_X$ is a direct summand of

$R^{k-i}(p\circ f)_*\omega_X$ (by Theorem 4.1.6 again), which is a GV-sheaf by Theorem 4.1.4. It follows that $h^0(B, R^kp_*f_*\omega_X\otimes\beta)$ is non-zero, hence that $R^kp_*f_*\omega_X$ is non-zero.

To conclude, recall that $R^k(p\circ f)_*\omega_X$, hence the summand $R^kp_*f_*\omega_X$, is torsion-free over the image of f in B (Theorem 4.1.6). By base change over the smooth locus of f, this k^{th} higher direct image would vanish if the general fibre had dimension smaller than k, hence the general fibre actually has dimension k. □

4.1.4 Generic base change

Finally we will need the following generic base change theorem. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are proper morphisms of schemes of finite type over a field, that Z is generically reduced, and that $h = g\circ f$ is surjective. Let $\mathscr{F}$ be a coherent sheaf on X. For $z\in Z$, let $G = h^{-1}(z)$ and $H = g^{-1}(z)$ be the fibres over z, and consider $f|_G\colon G\to H$.

Proposition 4.1.10 ([LPS20, Proposition 5.1])**.** *In the setting above, there is a non-empty Zariski-open subset $U\subset Z$ such that the base change morphism*

$$\big(R^if_*\mathscr{F}\big)|_H \to R^i(f|_G)_*(\mathscr{F}|_G)$$

is an isomorphism of sheaves on H for every $z\in U$ and every i.

In particular, if X, Y and Z are smooth projective and $\mathscr{F}$ is the relative canonical bundle $\omega_{X/Y} = \omega_X\otimes f^*\omega_Y^{-1}$, this says that the restriction $(f_*\mathscr{F})|_H$ is, for general $z\in Z$, isomorphic to the relative canonical bundle $\omega_{G/H}$ of the morphism $f|_G$, and similarly for the higher direct images.

4.2 Chen-Jiang decompositions for direct images of canonical bundles

The goal of this section is to prove that Chen-Jiang decompositions always exist for direct images of canonical bundles, following the approach originally used in [CJ18] to give the decompositions for generically finite morphisms.

Let us first recall the original proof of [CJ18, Theorem 3.5], namely that if $f\colon X\to A$ is a generically finite morphism to an abelian varietythen $f_*\omega_X$ admits a Chen-Jiang decomposition. First, by Theorem 4.1.4 and Proposition 4.1.8 it suffices to assume that all components of the cohomology support loci $V^k(A, f_*\omega_X)$ for every k passes through the origin of A. Indeed choose a finite order point in each such component, then choose an isogeny $\phi\colon A'\to A$ such that each of those finite points get mapped to the origin of $\hat{A}'$ under ϕ^*. If $X' = X\times_A A'$ and $f'\colon X'\to A'$ is the second projection, then $\phi^*f_*\omega_X = f'_*\omega_{X'}$, and the components of $V^k(A', f'_*\omega_{X'})$ all pass through the origin of A'.

Chen and Jiang's proof of [CJ18, Theorem 3.4] is then essentially to show the following result, and prove that the conditions are satisfied when f is generically finite.

Lemma 4.2.1. *Assume all components of each $V^k(A, f_*\omega_X)$ pass through $0\in A$. Suppose that for each $k>0$, and for each component $\hat{B}$ of $V^k(A, f_*\omega_X)$ of codimension k, there exists an M-regular sheaf $\mathscr{F}_B$ on B with the following properties.*

1. *If $p_B\colon A \to B$ is dual to the inclusion $\hat{B} \to \hat{A}$, then $f_*\omega_X$ admits $p_B^*\mathscr{F}_B$ as a direct summand.*

2. *For general $\beta \in \hat{B}$, $h^k(A, f_*\omega_X \otimes p_B^*\beta) = h^0(B, \mathscr{F}_B \otimes \beta)$*

Then $f_*\omega_X$ admits a Chen-Jiang decomposition.

Proof. Following the notation of [CJ18], let S_X^k denote the set of codimension k components of $V^k(A, f_*\omega_X)$. Then Step 2 of the proof of [CJ18, Theorem 3.4] applies verbatim to show that there exists a decomposition

$$f_*\omega_X \cong \mathscr{W} \oplus \bigoplus_{k>0, \hat{B}\in S_X^k} p_B^*\mathscr{F}_B$$

It remains to show that $\mathscr{W}$ is M-regular. This follows from the arguments of Step 3 of the proof of [CJ18, Theorem 3.4] and the second point in the statement of this lemma. □

As outlined in the introduction, Chen and Jiang critically use the fact that if $f\colon X \to Y$ is generically finite and surjective, then $f_*\omega_X$ is a direct summand of ω_Y. The main new result is Theorem 4.0.2, which serves as a generalization of this statement to arbitrary morphisms. We defer the proof of this to Section 4.3, but recall the statement here.

Corollary 4.2.2. *Suppose $f\colon X \to A$ is a morphism from a smooth projective variety X to an abelian variety A. Then $R^if_*\omega_X$ admits a Chen-Jiang decomposition on A for all i.*

Proof. For $i > 0$, there exists by work of Kollár [Kol86b, Corollary 2.24] a smooth variety Z with $\dim Z = \dim X - i$ and a morphism $\phi\colon Z \to A$ such that $R^if_*\omega_X$ is a direct summand of $\phi_*\omega_Z$ (the claim about the dimension of Z follows from the proof of Kollár's corollary; in fact Z is a generic intersection of i hyperplane sections of some birational model of X). The result follows by Proposition 4.1.7. □

Remark 4.2.3. The proof of Corollary 4.2.2 for a fixed X and $i > 0$ only relies on Theorem 4.0.1 in the case of varieties with strictly smaller dimension than X. In the proof of Theorem 4.0.1, we can thus assume that $R^if_*\omega_X$ admits a Chen-Jiang decomposition for all $i > 0$ by induction on $\dim X$.

Proof of Theorem 4.0.1. By Theorem 4.1.4 and Proposition 4.1.8, we can assume that all components of the cohomology support loci $V^k(A, f_*\omega_X)$ are in fact abelian subvarieties. It suffices to verify the conditions of Lemma 4.2.1. Namely, suppose $\hat{B}$ is a codimension k component of $V^k(A, f_*\omega_X)$ with $k > 0$, and $p\colon A \to B$ is dual to the inclusion $\hat{B} \subset \hat{A}$. Then we must show that there exists an M-regular sheaf $\mathscr{F}$ on B such that $h^k(A, f_*\omega_X \otimes p^*\beta) = h^0(B, \mathscr{F} \otimes \beta)$ for general $\beta \in \hat{B}$, and that $f_*\omega_X$ admits $p^*\mathscr{F}$ as a direct summand.

Let $Y \subset A$ be the image of f, and $p(Y) = Z \subset B$. Let $g\colon Y \to Z$ denote the restriction of p and $q = g \circ f$. By Lemma 4.1.9, g is a smooth fibration of relative dimension k, and $\omega_{Y/Z}$ is trivial. Now let $\pi_Z\colon Z' \to Z$ be a resolution of singularities of Z, and construct the following diagram by pullback.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "\pi_X"] & Y' \arrow[r, "g'"] \arrow[d, "\pi_Y"] & Z' \arrow[d, "\pi_Z"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Then g' is again a smooth fibration of relative dimension k, and $\omega_{Y'/Z'}$ is trivial. By Theorem 4.0.2 applied to the sequence $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$, we can now conclude that $f'_*\omega_{X'}$ admits $g'^*R^k g'_* f'_*\omega_{X'}$ as a direct summand. Pushing forward to Y it follows that $f_*\omega_X$ admits $g^*R^k g_* f_*\omega_X$ as a direct summand, by flat base change along g and the fact that $\pi_{X*}\omega_{X'} = \omega_X$ since π_X is birational.

To apply Lemma 4.2.1, it then suffices to show that $R^k g_* f_*\omega_X$ admits as direct summand an M-regular sheaf $\mathscr{F}$ such that $h^0(B, R^k p_* f_*\omega_X \otimes \beta) = h^0(B, \mathscr{F} \otimes \beta)$ for general $\beta \in \hat{B}$. It suffices to show that $R^k g_* f_*\omega_X$ admits a Chen-Jiang decomposition as a sheaf on B, as we can then take $\mathscr{F}$ to be the M-regular summand of the decomposition.

But $R^k g_* f_*\omega_X$ is a direct summand of $R^k q_*\omega_X$ by Kollár's result (Theorem 4.1.6). By dimensional induction and Corollary 4.2.2 (see Remark 4.2.3), $R^k q_*\omega_X$ admits a Chen-Jiang decomposition on B, hence so does $R^k g_* f_*\omega_X$ by Proposition 4.1.7. □

4.3 Splitting of direct images of canonical bundles

The goal of this section is to prove Theorem 4.0.2. More precisely, suppose given surjective morphisms of smooth varieties $X \xrightarrow{f} Y \xrightarrow{g} Z$ where g is flat. Let $q = g \circ f$. We will construct a morphism $\Psi_X \colon f_*\omega_X \to g^! R^k g_* f_*\omega_X[-k]$ where $k = \dim Y - \dim Z$; as g is flat, this is actually a map of sheaves. We will then show that this morphism is split surjective in the setting of Theorem 4.0.2, using the results by Kollár outlined in Section 4.1.2.

4.3.1 Relative Gysin morphism for canonical bundles

To construct the desired morphism, note first that $R^i g_* f_*\omega_X$ vanishes for $i > k$ by Kollár's result (Theorem 4.1.6), hence $\mathbf{R}g_* f_*\omega_X$, as an object of the derived category of coherent sheaves on Z, is concentrated in degrees 0 to k. Thus the projection to the k^{th} cohomology sheaf gives a map $\mathbf{R}g_* f_*\omega_X \to R^k g_* f_*\omega_X[-k]$. By adjunction this corresponds to a morphism

$$\Psi_X \colon f_*\omega_X \to g^! R^k g_* f_*\omega_X[-k].$$

Since Y and Z are smooth and g is flat of relative dimension k, we have

$$g^! R^k g_* f_*\omega_X[-k] \cong \omega_{Y/Z} \otimes g^* R^k g_* f_*\omega_X.$$

Note that there's a canonical morphism $R^k g_* f_*\omega_X \to R^k q_*\omega_X$, namely the edge map from the composed functor spectral sequence $R^i g_* R^j f_*\omega_X \implies R^{i+j} q_*\omega_X$. By Kollár's result (Theorem 4.1.6), this map is an inclusion of a direct summand. Pulling this back to Y, and applying twists by canonical bundles and composing with twists of Ψ_X, yields the following morphisms.

$$f_*\omega_X \xrightarrow{\Psi_X} \omega_{Y/Z} \otimes g^* R^k g_* f_* \omega_X$$

$$\begin{tikzcd}
f_*\omega_{X/Z} \arrow[r, "\Psi_{X/Z}"] \arrow[rr, bend right, "\Psi'_{X/Z}"'] & \omega_{Y/Z} \otimes g^* R^k g_* f_* \omega_{X/Z} \arrow[r] & \omega_{Y/Z} \otimes g^* R^k q_* \omega_{X/Z}
\end{tikzcd}$$

$$\begin{tikzcd}
f_*\omega_{X/Y} \arrow[r, "\Psi_{X/Y}"] \arrow[rr, bend right, "\Psi'_{X/Y}"'] & g^* R^k g_* f_* \omega_{X/Z} \arrow[r] & g^* R^k q_* \omega_{X/Z}
\end{tikzcd}$$

Lemma 4.3.1. *For a general point $y \in Y$, let $F = f^{-1}(y), G = q^{-1}(g(y))$ and $H = g^{-1}(g(y))$.*

1. *The fibre $\Psi'_{X/Y}|_y \colon H^0(F, \omega_F) \to H^k(G, \omega_G)$ of $\Psi'_{X/Y}$ is the Gysin morphism of the inclusion $F \subset G$.*

2. *The fibre of $\Psi_{X/Y}$ at y is a surjective morphism*
$$\Psi_{X/Y}|_y \colon H^0(F, \omega_F) \to H^k(H, (f|_G)_*\omega_G).$$

3. *Furthermore, let $z = g(y)$ and assume g is a fibration. Then the fibre $(g_*\Psi_{X/Z})|_z$ of $g_*\Psi_{X/Y} \colon q_*\omega_{X/Y} \to R^k g_* f_* \omega_{X/Z}$ at z is an isomorphism for general y.*

The notation can be summarized in the following commuting diagram, where all squares are cartesian.

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[d] & G \arrow[r, hook] \arrow[d] & X \arrow[d] \\
\{y\} \arrow[r, hook] & H \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
 & \{z\} \arrow[r, hook] & Z
\end{tikzcd}$$

Proof. Step 1: Identifying fibres of $\Psi'_{X/Y}$ with Gysin morphisms.
By generic base change (Proposition 4.1.10), we can assume Z is a point, so $G = X, H = Y$ and F is a general fibre of f. Then $\dim Y = k, \dim X = d + k$, and $\dim F = d$.

Taking a log resolution of Y, we can furthermore assume that the the discriminant locus of f is normal crossings. By Kollár's result (Theorem 4.1.5) all the higher direct images $R^i f_* \omega_X$ are then locally free. At a general $y \in Y$, we get a sequence

$$H^0(F, \omega_F) \to H^k(Y, f_*\omega_X) \to H^k(X, \omega_X),$$

and the linear dual is a sequence

$$H^d(X, \mathcal{O}_X) \to H^0(Y, R^d f_* \mathcal{O}_X) \to H^d(F, \mathcal{O}_F)$$

by Serre duality on the spaces involved. We claim that the first morphism is the edge map from the second page of the Leray spectral sequence for $\mathcal{O}_X$, the second the base change morphism to the fibre, and the composition as a result the restriction to a fibre.

Let

$$D_Y(-) = \mathbf{R}\,\underline{\mathrm{Hom}}(-, \omega_Y[\dim Y])$$

be the Serre duality functor for Y and similarly D_X for X. Then

$$D_Y(\mathbf{R}f_*\omega_X) = \mathbf{R}f_*\mathcal{O}_X[\dim X],$$

and since the discriminant locus of f is a normal crossings divisor, the $R^if_*\omega_X$ are locally free and

$$D_Y(R^if_*\omega_X[-i]) = R^{d-i}f_*\mathcal{O}_X[i + \dim Y],$$

recalling that $d = \dim X - \dim Y$.

To compute the dual of the fibre

$$\Psi_{X/Y}|_y \colon H^0(F, \omega_F) \to H^k(Y, f_*\omega_X),$$

we can apply $\underline{\mathrm{Hom}}(-, \mathcal{O}_Y)$ and compute fibres of the resulting morphism, since the sheaves involved are locally free. But since $\Psi_{X/Y} = \Psi_X \otimes \omega_Y^{-1}$ we have

$$\underline{\mathrm{Hom}}(\Psi_{X/Y}, \mathcal{O}_Y) = D_Y(\Psi_X)[-\dim Y].$$

Now the morphism Ψ_X is constructed as the composition

$$f_*\omega_X \to g^!\mathbf{R}g_*f_*\omega_X \to g^!R^kg_*f_*\omega_X[-k]$$

where the first morphism is the unit of adjunction, and the second is the projection to the highest cohomology sheaf. The (Serre) dual of the former is the counit of adjunction

$$g^*\mathbf{R}g_*R^df_*\mathcal{O}_X \to R^df_*\mathcal{O}_X,$$

while the dual of the latter is the inclusion

$$g_*R^df_*\mathcal{O}_X \to \mathbf{R}g_*R^df_*\mathcal{O}_X$$

of the lowest direct image along g. The composition is thus just the counit of the non-derived adjunction, namely $g^*g_*R^df_*\mathcal{O}_X \to R^df_*\mathcal{O}_X$. Since Z is a point, $g^*g_*R^df_*\mathcal{O}_X = H^0(Y, R^df_*\mathcal{O}_X) \otimes \mathcal{O}_Y$, and the fibre at a general $y \in Y$ is just given by restricting a global section to the fibre over y, so $H^0(Y, R^df_*\mathcal{O}_X) \to H^d(F, \mathcal{O}_F)$ in the sequence above is given as claimed.

The morphism

$$H^k(Y, f_*\omega_X) \to H^k(X, \omega_X)$$

is an edge map in the Leray spectral sequence for ω_X with respect to f, induced by the canonical map $f_*\omega_X \to \mathbf{R}f_*\omega_X$. The edge map

$$H^d(X, \mathcal{O}_X) \to H^0(Y, R^df_*\mathcal{O}_X)$$

in the Leray spectral sequence for $\mathcal{O}_X$ is similarly induced by the projection $\mathbf{R}f_*\mathcal{O}_X \to R^d f_*\mathcal{O}_X[-d]$ to the highest direct image (the ones in degree $> d$ vanishing by duality). We claim that these maps get identified under D_Y.

Let $K^0 \xrightarrow{d} K^1 \to \cdots$ be any locally free resolution of $\mathbf{R}f_*\omega_X$. Then the inclusion $f_*\omega_X \to \mathbf{R}f_*\omega_X$ is canonically identified with the inclusion $\ker d \to K^0$. Applying D_Z (and dropping the index shifts from the notation) gives a resolution $(\cdots \to K^{1\vee} \to K^{0\vee}) \otimes \omega_Y$ of $\mathbf{R}f_*\mathcal{O}_X$. The inclusion $\ker d \to K^0$ gets mapped under D_Y to the surjection $(K^{0\vee} \to \operatorname{coker} d^\vee) \otimes \omega_Y$. But this is just the canonical map $\mathbf{R}f_*\mathcal{O}_X \to R^d f_*\mathcal{O}_X[-d]$ as desired. This proves the claim that $\Psi'_{X/Y}$ is the Gysin morphism on general fibres.

Step 2: Surjectivity of general fibres of $\Psi_{X/Y}$.

Let us now show that the restriction

$$H^0(Y, R^d f_*\mathcal{O}_X) \to H^d(F, \mathcal{O}_F)$$

is injective. Suppose that $\alpha \in H^0(Y, R^d f_*\mathcal{O}_X)$ vanishes when restricted to some point y_0 in the smooth locus $Y^0 \subset Y$ of f. Lift α to an element $\tilde{\alpha}$ of $H^d(X, \mathcal{O}_X)$, and let $X^0 = f^{-1}(Y^0)$ and $f^0 \colon X^0 \to Y^0$ be the restriction of f. Since f^0 is smooth, $R^d f^0_* \mathbb{C}_{X^0}$ is a local system, and the Leray spectral sequence for $\mathbb{C}_{X^0}$ with respect to f^0 degenerates, we get a map

$$\pi \colon H^d(X, \mathbb{C}_X) \to H^0(Y^0, R^d f^0_* \mathbb{C}_{X^0}).$$

Furthermore, the fibre of $R^d f^0_* \mathbb{C}_{X^0}$ at $y \in Y^0$ is exactly $H^d(F, \mathbb{C})$. Applying the Hodge decomposition for X and F, the restriction of $\pi(\tilde{\alpha})$ to y equals the restriction of α to y, which vanishes for $y = y_0$. But $\pi(\tilde{\alpha})$ is then a section of a local system which vanishes at a point, and since Y^0 is connected, $\pi(\hat{\alpha})$ must thus be identically 0. It follows that α vanishes at every $y \in Y^0$. As $R^d y_* \mathcal{O}_X$ is locally free, and since α vanishes on a dense open set, we get $\alpha = 0$. This dually gives the desired surjectivity.

Step 3: Surjectivity of general fibres of $g_*\Psi_{X/Y}$.

For the final statement of the lemma, consider the monodromy action of $\pi_1(Y^0)$ on $H^d(F, \mathbb{C})$, and the subspace $H^d(F, \mathbb{C})^{\pi_1(Y^0)}$ of invariants under this action.

Define $H^0(F, \omega_F)^{\pi_1(Y^0)}$ as the preimage of $H^d(F, \mathbb{C})^{\pi_1(Y^0)}$ under the inclusion $H^0(F, \omega_F) \hookrightarrow H^d(F, \mathbb{C})$. Note that $\pi_1(Y^0)$ does not act on $H^0(F, \omega_F)$; we are considering invariants under the action on the larger space $H^d(F, \mathbb{C})$.

Consider the following commuting diagram.

$$\begin{tikzcd}
H^0(Y, f_*\omega_{X/Y}) \arrow[r, "H^0(\Psi_{X/Y})"] \arrow[d] & H^k(Y, f_*\omega_X) \\
H^0(F, \omega_F) \arrow[ur, "\Psi_{X/Y}|_y"'] \arrow[r, phantom, "\supseteq"] & H^0(F, \omega_F)^{\pi_1(Y^0)} \arrow[u]
\end{tikzcd}$$

We claim that the right side vertical map is an isomorphism, while the image of the restriction morphism $H^0(Y, f_*\omega_{X/Y}) \to H^0(F, \omega_F)$ contains $H^0(F, \omega_F)^{\pi_1(Y^0)}$; this would yield the desired surjectivity.

Define dually $H^d(F,\mathcal{O}_F)^{\pi_1(Y^0)} \subset H^d(F,\mathcal{O}_F)$ as the image of $H^d(F,\mathbb{C})^{\pi_1(Y^0)}$ under the canonical projection $H^d(F,\mathbb{C}) \to H^d(F,\mathcal{O}_F)$. Note again that $\pi_1(Y^0)$ does not act on $H^d(F,\mathcal{O}_F)$ by itself, only on the larger $H^d(F,\mathbb{C})$.

Then $H^d(F,\mathcal{O}_F)^{\pi_1(Y^0)}$ is exactly the image of the restriction morphism

$$H^0(Y, R^d f_*\mathcal{O}_X) \to H^d(F,\mathcal{O}_F).$$

Indeed the image of the restriction map $H^d(X,\mathcal{O}_X) \to H^d(F,\mathcal{O}_F)$ is exactly $H^d(F,\mathcal{O}_F)^{\pi_1(Y^0)}$ by the global invariant cycles theorem and the fact that the restriction map in singular cohomology is a morphism of Hodge structures, and the coherent restriction map factors through $H^0(Y, R^d f_*\mathcal{O}_X)$.

We conclude that the restriction morphism gives an isomorphism

$$H^0(Y, R^d f_*\mathcal{O}_X) \xrightarrow{\sim} H^d(F,\mathcal{O}_F)^{\pi_1(Y^0)}$$

by the injectivity from the previous step of this proof. Since $H^d(F,\mathcal{O}_F)^{\pi_1(Y^0)}$ and $H^0(F,\omega_F)^{\pi_1(Y^0)}$ are dual, we conclude that the restriction of $\Psi_{X/Y}|_y$ to $H^0(F,\omega_F)^{\pi_1(Y^0)}$ is an isomorphism.

Finally, observe that by the global invariant cycles theorem, the image of the restriction $H^0(X,\Omega^d_X) \to H^0(F,\omega_F)$ is exactly $H^0(F,\omega_F)^{\pi_1(Y^0)}$, and that this restriction factors through $H^0(Y, f_*\omega_{X/Y})$; in fact

$$H^0(Y, f_*\omega_{X/Y}) \to H^0(F,\omega_F)^{\pi_1(Y^0)}$$

is an isomorphism by the same type of argument as in step 2. It follows that

$$H^0(\Psi_{X/Y})\colon H^0(Y, f_*\omega_{X/Y}) \to H^k(Y, f_*\omega_X)$$

is an isomorphism as desired. □

The case where Z is a point immediately gives the following.

Corollary 4.3.2. *Suppose $f\colon X \to Y$ is a surjective morphism of relative dimension k between smooth projective varieties. If $H^k(Y, f_*\omega_X) \neq 0$ then $\omega_{X/Y}$ is effective.*

4.3.2 Morphism of VHS

Taking the direct image of $\Psi'_{X/Z}$ along g yields

$$g_*\Psi_{X/Z}\colon q_*\omega_{X/Z} \to g_*\omega_{Y/Z} \otimes R^k q_*\omega_{X/Z}.$$

The goal is to recover this morphism from a map of VHS, at least over the locus where g and q are smooth.

Let $Z^0 \subset Z$ be a Zariski-open subset over which q and g are smooth, and let $q^0\colon X^0 \to Z^0$ and $g^0\colon Y^0 \to Z^0$ be the corresponding restrictions. Let $d = \dim X - \dim Y$. We construct a morphism of VHS $R^k g^0_*\mathbb{Q}_{Y^0} \otimes R^d q^0_*\mathbb{Q}_{X^0} \to R^{d+k} q^0_*\mathbb{Q}_{X^0}$ as follows. A section of $R^k g^0_*\mathbb{Q}_{Z^0}$ is locally a cohomology class $\alpha \in H^k(g^{-1}(U),\mathbb{Q})$ and a section of $R^d q^0_*\mathbb{Q}_X$ is locally a class $\beta \in H^d(q^{-1}(U),\mathbb{Q})$ for small open $U \subset Z$. Thus we get an element $f^*\alpha \wedge \beta \in H^{d+k}(q^{-1}(U),\mathbb{Q})$, which defines a local section of $R^{d+k} q^0_*\mathbb{Q}_{X^0}$. As this is compatible with the Hodge filtrations, we get a morphism of VHS.

As q^0 and g^0 are smooth, dualizing gives the desired map $\Phi\colon R^{d+k} q^0_*\mathbb{Q}_{X^0} \to R^k g^0_*\mathbb{Q}_{Y^0} \otimes R^{d+2k} q^0_*\mathbb{Q}_{X^0}$.

Lemma 4.3.3. *Suppose g is smooth and $\omega_{Y/Z}$ is trivial. On the lowest graded piece of the Hodge filtration, the morphism*

$$R^{d+k}q^0_*\mathbb{Q}_{X^0}\otimes\mathcal{O}_{Z^0}\to R^k g^0_*\mathbb{Q}_{Y^0}\otimes R^{d+2k}q^0_*\mathbb{Q}_{X^0}\otimes\mathcal{O}_{Z^0}$$

induced by Φ agrees with the restriction to Z^0 of

$$g_*\Psi'_{X/Z}\colon q_*\omega_{X/Z}\to g_*\omega_{Y/Z}\otimes R^k q_*\omega_{X/Z}.$$

Proof. As q^0 and g^0 are smooth, base change applies to the direct images of the line bundles. By proper base change for the direct images of constant sheaves, it thus suffices to assume that Z is a point. Then $g_*\Psi'_{X/Z}$ is just a map

$$H^0(X,\omega_X)\to H^0(Y,\omega_Y)\otimes H^k(X,\omega_X),$$

and we must identify the dual

$$H^k(Y,\mathcal{O}_Y)\otimes H^d(X,\mathcal{O}_X)\to H^{d+k}(X,\mathcal{O}_X)$$

with the cup product map, by definition of Φ.

By assumption, $\omega_{Y/Z}=\omega_Y$ is trivial, so fix an isomorphism by choosing a non-zero $\tau\in H^0(Y,\omega_Y)$. Suppose given

$$\alpha\in H^k(Y,\mathcal{O}_Y),\beta\in H^d(X,\mathcal{O}_Y),\gamma\in H^0(X,\omega_X).$$

Since $H^k(Y,\mathcal{O}_Y)$ is dual to $H^0(Y,\omega_Y)$, we can assume that α is dual to τ under Serre duality. The claim is that

$$f^*\alpha\wedge\beta\wedge\gamma=(\alpha\otimes\beta,\Psi'_{X/Z}(\gamma))$$

in $H^{k+d}(X,\omega_X)$, where the right hand side is the Serre duality pairing of $H^k(Y,\mathcal{O}_Y)\otimes H^d(X,\mathcal{O}_X)$ with $H^0(Y,\omega_Y)\otimes H^k(X,\omega_X)$. To see this, note that triviality of ω_Y implies that the natural map

$$H^0(Y,f_*\omega_{X/Y})\otimes H^0(Y,\omega_Y)\to H^0(Y,f_*\omega_X)=H^0(X,\omega_X)$$

is an isomorphism. Thus there's a global section $\psi\in H^0(Y,f_*\omega_{X/Y})$ such that $\gamma=\tau\otimes\psi$. At least over the smooth locus of Y, ψ is nothing but a holomorphic d-form on X such that $\gamma=f^*\tau\wedge\psi$.

For general $y\in Y$, let $F=f^{-1}(y)$. Then $\Psi'_{X/Z}(\gamma)$ is exactly the Gysin morphism of the inclusion $F\subset X$ applied to the restriction $\psi|_F$, tensor τ, by Lemma 4.3.1. To compute, we note now that

$$\begin{aligned}(\alpha\otimes\beta,G(\gamma))&=(\alpha,\tau)\cdot(\beta|_F,\psi|_F)\\&=(\beta|_F,\psi|_F)\end{aligned}$$

since α and τ are dual. In particular, $(\beta|_F,\psi|_F)$ is independent of y (this is related to monodromy invariance of $\psi|_F$). On the other hand,

$$f^*\alpha\wedge\beta\wedge\gamma=f^*(\alpha\wedge\tau)\wedge\beta\wedge\psi$$

To integrate the right hand side, we integrate $\beta\wedge\psi$ over fibres F, then integrate $\alpha\wedge\tau$ over Y; but that gives exactly the desired result. $\square$

Finally, we can state and prove the following more precise version of Theorem 4.0.2.

Theorem 4.3.4. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are surjective morphisms of smooth projective varieties, and let $q = g \circ f$. Suppose further that g is a smooth fibration and $\omega_{Y/Z}$ is trivial. Then the morphism $\Psi_X \colon f_*\omega_X \to g^*R^kg_*f_*\omega_X$ is split surjective.*

Proof. By generic base change (Proposition 4.1.10), we can fix an open $Z^0 \subset Z$ over which q is smooth and base change to fibres over Z applies to the sheaves $f_*\omega_X$ and $g^*R^kg_*f_*\omega_X$.

We can in fact assume that $Z \setminus Z^0$ is a normal crossings divisor. If not, consider a log resolution $\pi_Z \colon Z' \to Z$ of $Z \setminus Z^0$; by pullback we get the following diagram, where the vertical maps are birational.

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "\pi_X"] & Y' \arrow[r, "g'"] \arrow[d, "\pi_Y"] & Z' \arrow[d, "\pi_Z"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Assuming $f'_*\omega_{X'} \cong g'^*R^kg'_*f'_*\omega_{X'} \oplus \mathscr{Q}$, we get $f_*\omega_X \cong g^*R^kg_*f_*\omega_X \oplus \pi_{Y*}\mathscr{Q}$. Indeed

$$\begin{aligned}
\pi_{Y*}f'_*\omega_{X'} &= f_*\pi_{X*}\omega_{X'} \\
&= f_*\omega_X
\end{aligned}$$

since π_X is birational, and on the other hand

$$\begin{aligned}
\pi_{Y*}g'^*R^kg'_*f'_*\omega_{X'} &= g^*\pi_{Z*}R^kg'_*f'_*\omega_{X'} \\
&= g^*R^kg_*f_*\omega_X
\end{aligned}$$

where the first line is by flat base change along g, and the second by Kollár's result (Theorem 4.1.6).

Assume thus that $Z \setminus Z^0$ is a normal crossings divisor. In particular, q is smooth away from a normal crossings divisor, which implies that $R^kq_*\omega_X$ and its direct summand $R^kg_*f_*\omega_X$ are locally free by Kollár's result (Theorem 4.1.5). Consider then

$$g_*\Psi_{X/Z} \colon q_*\omega_{X/Z} \to g_*\omega_{Y/Z} \otimes R^kq_*\omega_{X/Z}$$

which, by Lemma 4.3.3, is induced over Z^0 by a morphism of VHS

$$\Phi \colon R^{d+k}q^0_*\mathbb{Q}_{X^0} \to R^kg^0_*\mathbb{Q}_{Y^0} \otimes R^{d+2k}q^0_*\mathbb{Q}_{X^0}.$$

Since the category of polarizable VHS is semisimple [PS08, Theorem 10.13], there is a direct sum decomposition of VHS $R^{d+k}q^0_*\mathbb{Q}_{X^0} \cong I \oplus K$ where K is the kernel of Φ, and I maps isomorphically to the image of Φ. We also have a decomposition $R^kg^0_*\mathbb{Q}_{Z^0} \otimes R^{d+2k}q^0_*\mathbb{Q}_{X^0} \cong I \oplus C$, and the resulting $I \oplus K \to I \oplus C$ is just the identity map on I while vanishing on K.

Again by Theorem 4.1.5, the lowest piece of the Hodge filtration of the upper canonical extension of $R^{d+k}q^0_*\mathbb{Q}_{X^0}$ is exactly $q_*\omega_{X/Z}$, while the same construction applied to $R^kg^0_*\mathbb{Q}_{Y^0} \otimes R^{d+2k}q^0_*\mathbb{Q}_{X^0}$ yields $g_*\omega_{Y/Z} \otimes R^kq_*\omega_{X/Z}$. Note in the latter case that $R^kg^0_*\mathbb{Q}_{Y^0}$ has trivial monodromy around the complement of Z^0, since g is smooth, so taking canonical extensions and tensor products does in fact commute in this case by the discussion in Section 4.1.2.

Let $\mathscr{I}, \mathscr{K}, \mathscr{C}$ be the lowest pieces of the Hodge filtration on the upper canonical extensions of I, K and C respectively. The formation of canonical extensions is compatible with direct sums, so $q_*\omega_{X/Z} \cong \mathscr{I} \oplus \mathscr{K}$. By Lemma 4.3.3, the image of $g_*\Psi'_{X/Z}$ inside $g_*\omega_{Y/Z} \otimes R^k q_*\omega_{X/Z}$ agrees with $\mathscr{I}$ over Z^0. Since all sheaves involved are locally free, it follows that $\mathscr{I}$ is in fact the image of $g_*\Psi'_{X/Z}$.

Back on Y, the push-pull adjunction for g applied to $\Psi'_{X/Z}$, together with the projection formula, gives the following commuting diagram.

$$\begin{tikzcd} g^*q_*\omega_{X/Z} \arrow[r, "g^*g_*\Psi'_{X/Z}"] \arrow[d] & g^*g_*\omega_{Y/Z} \otimes g^*R^k q_*\omega_{X/Z} \arrow[d] \\ f_*\omega_{X/Z} \arrow[r, "\Psi'_{X/Z}"] & \omega_{Y/Z} \otimes g^*R^k q_*\omega_{X/Z} \end{tikzcd}$$

As $\omega_{Y/Z}$ is trivial and g is a fibration, the right side vertical map is an isomorphism. Moreover, $g^*\mathscr{I}$ is a direct summand of both the top left and bottom right corners, and the composition through $f_*\omega_{X/Z}$, when restricted to $g^*\mathscr{I}$, is the identity. It thus remain only to show that $\mathscr{I} = R^k g_* f_*\omega_{X/Z}$, as it would then follow that $g^*\mathscr{I} = g^*R^k g_* f_*\omega_{X/Z}$ is the image of $\Psi'_{X/Z}$, hence also of $\Psi_{X/Z}$, and the previous diagram yields a splitting of $\Psi_{X/Z}$ as desired.

Thus we must show that $\Psi_{X/Z}$ remains surjective after pushing forward to Z. On Z, $\mathscr{I}$ and $R^k g_* f_*\omega_{X/Z}$ are both locally free subsheaves of $R^k q_*\omega_{X/Z}$ (in fact direct summands). Thus it suffices to show that for general $z \in Z$, the fibre of $g_*\Psi_{X/Z}$ at z is surjective. By generic base change (Proposition 4.1.10) we can assume that Z is just a point, in which case we are to show that the map induced by $\Psi_{X/Z}$ on global sections is surjective.

Since $\Psi_{X/Z}$ and $\Psi_{X/Y}$ are related by twisting by $\omega_{Y/Z}$, fixing a trivialization of $\omega_{Y/Z}$ identifies the two maps, so we are done by Lemma 4.3.1. □

It seems more natural to consider VHS on Y rather than Z to get the splitting, but there's a technical issue with that approach. Namely, one ends up having to take a resolution $\pi\colon Y' \to Y$ of Y that doesn't come from a resolution of Z by pullback. One then wants to express $\pi^*g^*R^k g_* f_*\omega_{X/Z}$ as a direct summand of the canonical extension of a VHS pulled back from an open subset of Z, with the hope of splitting $\Psi_{X/Y}$. While g is smooth, so functoriality of canonical extensions is not an issue there, the composition $\pi \circ g$ is not smooth, so it's not clear what the canonical extension on Y' gives. This functoriality issue can be fixed by appealing to Hodge modules, which would give a proof along these lines even without the assumptions on g and $\omega_{Y/Z}$.

4.3.3 Effectiveness of relative canonical bundles and fibres of the Albanese morphism

Corollary 4.3.2 can be used to give a variation of a proof of a theorem by Jiang. For a smooth projective variety X, let $P_n = h^0(X, \omega_X^n)$ denote the plurigenera of X.

Theorem 4.3.5 ([Jia11, Theorem 3.1])**.** *Suppose X be a smooth projective variety with $P_1(X) = P_2(X) = 1$. Then the fibres of the Albanese mapping are connected.*

Proof. By [HP02], it is known that the Albanese mapping $a_X \colon X \to \mathrm{Alb}(X)$ is surjective in this case. Taking the Stein factorization and resolving singularities of the middle term (replacing X with a birational modification) yields a factorization $X \xrightarrow{g} V \xrightarrow{b} \mathrm{Alb}(X)$ of a_X where b is generically finite. It is a theorem of Chen and Hacon [CH01] that if, in this case, $P_1(V) = P_2(V) = 1$, then the Albanese mapping of V is birational, and as a consequence also b is birational so a_X has connected fibres. To prove the theorem it thus suffices to show that $\omega_{X/V}$ is effective.

However, it follows from the theory of GV-sheaves, by an observation of Ein and Lazarsfeld [EL97], that $H^g(\mathrm{Alb}(X), a_{X*}\omega_X) \neq 0$ since $P_1(X) = P_2(X) = 1$, where $g = \dim \mathrm{Alb}(X) = \dim V$. Since b is generically finite, and by Kollár's result (Theorem 4.1.6), $H^g(\mathrm{Alb}(X), a_{X*}\omega_X) = H^g(V, g_*\omega_X)$. Then Corollary 4.3.2 gives the conclusion. $\square$

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