Symplectic Involutions: from Cubic Fourfolds to $OG10$ type Manifolds

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Abstract of the Dissertation

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Irreducible holomorphic symplectic (IHS) manifolds are one of the building blocks of Kähler manifolds with trivial first Chern class, but very few examples are known. One strategy for potentially producing new examples is to study the fixed locus of finite groups of symplectic birational transformations of the known examples. We classify symplectic birational involutions of IHS manifolds deformation equivalent to O'Grady's 10 dimensional example ($OG10$ type). In particular, we show that there are 6 possible involutions, classified by their action on the second cohomology. From a cubic fourfold $V \subset \mathbb{P}^5$, one can construct a IHS manifolds of $OG10$ type, equipped with a Lagrangian fibration $X \to \mathbb{P}^5$. Three of the possible symplectic involutions can be obtained via this construction; explicitly, one starts with a cubic fourfold admitting an involution that induces a symplectic birational involution of X. We complete the classification of involutions of a cubic fourfold Hodge theoretically, which is equivalent to identifying a sublattice $A(V)_{\text{prim}} \subset H^4(V, \mathbb{Z})$ spanned by classes of surfaces contained in such a cubic. As a byproduct, we show that cubic fourfolds with involutions exhibit the full range of behaviour with regards to rationality conjectures. In particular, a cubic fourfold V with an involution that fixes a plane $P \subset V$ point-wise is rational.

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Introduction

The main topic of this thesis is the study and classification of birational transformations of **irreducible holomorphic symplectic manifolds** (IHS manifolds) of a specified deformation type. A compact, simply connected Kähler manifold X is an IHS manifold if it admits a unique nowhere degenerate holomorphic 2-form. Such manifolds are also known as compact **hyperkähler** manifolds, and are higher dimensional generalisations of $K3$ surfaces. According to a celebrated result of Beauville and Bogomolov, they are one of three building blocks of manifolds with trivial first Chern class, alongside abelian varieties and strict Calabi-Yau varieties.

IHS manifolds have very rich geometry, and have been the focus of a great deal of study. Despite this, very few examples are known: in every even dimension, there are two deformation types, discovered by Beauville [Bea83]. The first of these is the Hilbert scheme of points on a $K3$ surface (an IHS of $K3^{[n]}$ type), the second the generalised Kummer manifolds. In addition, there are two sporadic examples: one in dimension 6 and one in dimension 10, both discovered by O'Grady [O'G99, O'G03]. We call manifolds deformation equivalent to these examples IHS manifolds of $OG6$ type or $OG10$ type respectively - this thesis will focus primarily on the later example. Both examples were originally obtained as a symplectic desingularisation of certain moduli spaces, parametrising semistable sheaves on either a $K3$ or an abelian surface. Interestingly, one can construct examples of IHS manifolds from a cubic fourfold $V \subset \mathbb{P}^5$. Beauville and Donagi showed that the Fano variety of lines on a cubic fourfold is an IHS manifold of $K3^{[2]}$ type [BD85]. C. Lehn, M. Lehn, Sorger and van Straten constructed an 8-dimensional example of $K3^{[4]}$ type by considering twisted cubics on V [LLSvS17]. Laza, Saccà and Voisin constructed manifolds of $OG10$ type, by compactifying an intermediate Jacobian fibration associated with V [LSV17]. In this thesis we will heavily exploit this connection between IHS manifolds of $OG10$ type and cubic fourfolds.

The search for new examples of IHS manifolds is what motivated the study of symplectic automorphisms. If we consider a finite group of symplectic automorphisms of a known IHS manifold (that is, an automorphism that preserves the holomorphic symplectic form), then both the fixed locus and the quotient inherit induced holomorphic symplectic forms. The fixed point set of a bi-regular morphism is necessarily smooth, and so the fixed locus is an example of a holomorphic symplectic manifold - one hopes that by choosing interesting groups of symplectic automorphisms we could find new examples of IHS. Similarly, the quotient is also holomorphic symplectic, but necessarily acquires singularities. One would like a classification of such automorphisms for each deformation type, this is an interesting problem even in the case of $K3$ surfaces.

In his celebrated paper [Muk88], Mukai classified symplectic automorphisms of a K3 surface. This was further streamlined by Kondō [Kon98], who related automorphisms of K3 surfaces to automorphisms of the Niemeier lattices. For a $K3$ surfaces S, the fixed locus of a finite group of symplectic automorphisms G is necessarily a collection of isolated points. The resolution of singularities of the quotient S/G is again a $K3$ surfaces.

Adapting Kondō's approach, Mongardi obtained a classification of prime order symplectic automorphisms of manifolds of $K3^{[n]}$ type [Mon13, Mon16]; similar results were obtained by Huybrechts [Huy16]. A full classification of symplectic automorphisms of manifolds of OG6 type was obtained in [GOV20]. Once such a classification is obtained, a next step is to study the fixed loci; this is hard in general, as one needs a suitable geometric model. Kamenova, Mongardi and Oblomkov classify the fixed locus for symplectic involutions of $K3^{[n]}$, by studying the involution induced from a symplectic involution of the underlying $K3$ surface. They also studied the case of manifolds of generalised Kummer type [KMO18]. Unfortunately (from the point of view of the search for new IHS manifolds), in both cases the fixed loci consists of IHS manifolds of $K3^{[m]}$ type of lower dimension, and finitely many isolated points.

The remaining case is that of IHS manifolds of $OG10$ type; recently, it was shown that such a manifold can never admit a non-trivial symplectic automorphism [GOV20]. It seems that restricting ourselves to symplectic automorphisms is too strong a notion: one can instead consider symplectic **birational** involutions. Again, the fixed locus will inherit a holomorphic symplectic form, but now may possibly acquire some mild singularities. This idea is due to Markushevich and Tikhomirov [MT07], who provide a new construction of an IHS variety as a fixed locus for a birational symplectic involution of a manifold of $K3^{[6]}$ type. Their construction and the study of the singularities exploits an explicit Lagrangian fibration structure whose general fiber is an abelian variety. These techniques were generalised and developed in [ASF15].

In this thesis we focus on the classification of symplectic birational involutions of IHS manifolds of $OG10$ type. We use two main techniques. The first is a generalisation of the method of Kondō, Mongardi and Huybrechts to the case of $OG10$ type; more specifically, we relate certain involutions to involutions of the Leech Lattice. Our second technique is more geometric: we consider birational symplectic involutions that are induced from a cubic fourfold, via the construction of ([LSV17], [Sac21]). These involutions have the best chance of producing new IHS varieties as the fixed locus, and we are able to study them effectively utilising the well-understood geometry of cubic fourfolds. However, this is not quite enough to complete the classification - there are two more possible involutions that do not occur via the above methods. In joint work with S. Muller [MM23a], we use lattice enumeration techniques to show the existence of these unexpected involutions, and complete the classification.

The main result of this thesis is thus the classification of symplectic birational involutions for IHS manifolds of $OG10$ type. We classify an involution of an IHS manifold X by classifying the induced action on the second cohomology: a birational involution induces an involution $f^* \in O(H^2(X,\mathbb{Z}))$. We denote by $H^2(X,\mathbb{Z})_+, H^2(X,\mathbb{Z})_-$ the invariant and coinvariant sublattices; identifying these sublattices determines the action of f^*. More precisely:

Theorem A (Theorem 4.0.1)**.** *Let X be an IHS manifold of $OG10$ type, $f \in \mathrm{Bir}(X)$ a symplectic birational involution. Then the pair $H^2(X,\mathbb{Z})_+, H^2(X,\mathbb{Z})_-$ appears below:*

$H^2(X,\mathbb{Z})_-$	$H^2(X,\mathbb{Z})_+$
$E_8(2)$	$U^3 \oplus E_8(2) \oplus A_2$
$D_{12}^+(2)$	$E_6(2) \oplus U^2(2)$
$E_6(2)$	$U^3 \oplus D_4^3$
M	$U^2 \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$
G_{12}	$\langle 2 \rangle^3 \oplus \langle -2 \rangle^9$
G_{16}	$\langle 2 \rangle^3 \oplus \langle -2 \rangle^5$

Here, the lattices M, G_{12}, G_{16} are explicitly described, and are of rank $10, 12, 16$ *respectively. Moreover, for each involution of $H^2(X,\mathbb{Z})$ as above, there exists a manifold of $OG10$ type with a symplectic birational involution whose induced action is the prescribed involution.*

The geometry of an IHS manifold X is governed by its second cohomology - we classify each involution by classifying the induced action on $H^2(X,\mathbb{Z})$. The Global Torelli theorem due to Markman, Huybrechts and Verbitsky provides us with a way to determine when a manifold of $OG10$ type with a prescribed involution exists, but it is purely theoretical. It is much harder to get a geometric description of such a manifold; nevertheless, this is possible for three of the involutions.

The idea is to induce involutions of manifolds of $OG10$ type via the known constructions. We focus on the LSV construction: starting with a cubic fourfold $V \subset \mathbb{P}^5$, one can construct a compactification of the intermediate Jacobian of V, denoted X_V (see [LSV17], [Sac21]). In particular, X_V is an IHS manifold with a Lagrangian fibration over $\mathbb{P}^5$, whose general fiber is the intermediate Jacobian of a smooth hyperplane section of V. An involution of $V \subset \mathbb{P}^5$ induces a birational involution of X_V, preserving this fibration. More precisely:

Theorem B (Theorem 4.3.3)**.** *There exists three symplectic birational involutions of a manifold of $OG10$ type that are induced from an involution of a cubic fourfold. Conversely, a symplectic birational involution of a manifold of $OG10$ type that preserves a Lagrangian fibration arises from an involution of a cubic fourfold.*

Both Theorem A and B will be proved in Chapter 4. In order to do so, we must first embark on a study of involutions for cubic fourfolds Hodge theoretically - we complete this classification in Chapter 3. Similarly to the situation for IHS manifolds, automorphisms of cubic fourfolds can be detected by isometries of the primitive middle cohomology. Classifying such automorphisms is equivalent to identifying a sublattice $A(V)_{\mathrm{prim}} \subset H^4(V, \mathbb{Z})$ spanned by algebraic cycles. As a consequence, we show that cubics with involutions exhibit the full range of behaviour in relation to rationality conjectures. It was conjectured by Harris and Hassett that if the transcendental cohomology $T(V) := A(V)^{\perp}_{\mathrm{prim}}$ does not embed into the $K3$ lattice, then V is not rational [Has99, Has16]. If no such embedding exists, we say that V is **potentially irrational**.

Theorem C (Theorems 3.0.1, 3.0.2)**.** *Let V be a general cubic fourfold with ϕ_i an involution of V fixing a linear subspace of $\mathbb{P}^5$ of codimension i. Then either:*

1. *$i = 1$, ϕ_1 is anti-symplectic and $A(V)_{\mathrm{prim}} \cong E_6(2)$, $T(V) \cong U^2 \oplus D_4^3$. The algebraic lattice is spanned by classes of planes contained in X. Here, $T(V)$ **does not** embed into the $K3$ lattice, and V is **potentially irrational**.*

2. *$i = 2$, ϕ_2 is symplectic and $A(V)_{\mathrm{prim}} \cong E_8(2)$, $T(V) \cong A_2 \oplus U^2 \oplus E_8(2)$. The algebraic lattice is spanned by classes of cubic scrolls contained in V. Here, $T(V)$ **does not** embed into the $K3$ lattice, and V is **potentially irrational**.*

3. *$i = 3$, ϕ_3 is anti-symplectic and*

$$A(V)_{\mathrm{prim}} \cong M,\ T(V) \cong U \oplus A_1 \oplus A_1(-1) \oplus E_8(2).$$

*The algebraic lattice contains an index 2 sublattice spanned by classes of planes contained in V. Here, $T(V)$ **does** embed into the $K3$ lattice, and V is predicted to be rational.*

Here M is the unique rank 10 even lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24 \rangle$.

Hassett [Has00] defined a countably infinite union of irreducible divisors $\mathcal{C}_d$ in the moduli space of cubic fourfolds, parametrising special cubic fourfolds that contain additional surface classes. It turns out that cubic fourfolds with additional symmetries contain many algebraic surfaces, and are contained in many of the divisors $\mathcal{C}_d$. In particular, we show:

Theorem D. *Let $\mathcal{M}$ be the 10-dimensional moduli space of cubic fourfolds $V \subset \mathbb{P}^5$ with $\phi \in \mathrm{Aut}(V)$ an involution fixing a plane $P \subset V$ point-wise. Then*

$$\mathcal{M} \subset \bigcap_{\mathcal{C}_d \neq \varnothing} \mathcal{C}_d.$$

In particular, $V \in \mathcal{M}$ is rational.

This geometrically meaningful locus is the largest family known to be contained in $\bigcap_{\mathcal{C}_d \neq \varnothing} \mathcal{C}_d$, and is a 10-dimensional family of rational cubic fourfolds. In contrast, a cubic fourfold with one of the other two possible involutions is conjecturely irrational.

Structure of the Thesis

Chapter 1 summarizes relevant background material on both IHS manifolds and cubic fourfolds. We review the known constructions in more detail, as well as collecting the structure of the Beauville-Bogomolov-Fujiki lattice for each deformation type. We review local and global deformation behaviour for IHS manifolds, as well as the monodromy group. We discuss the structure of the Kähler and birational Kähler cone, before reformulating the Global Torelli theorem in a way that is suitable for the study of symplectic birational involutions. We review the period map for cubic fourfolds, and describe the construction outlined in [LSV17] in slightly more detail.

Chapter 2 contains the lattice theoretic results needed to undertake the study of symplectic involutions. We follow [Nik79b], giving clean exposition of the results needed and providing proofs where we could not find references in the literature. In particular, we focus on 2-elementary lattices and the theory of overlattices. Most of these results can be found in [Mar23, Appendix A].

Chapter 3 begins the study of involutions of a cubic fourfold $V \subset \mathbb{P}^5$. We say an involution is symplectic if it acts trivially on $H^{3,1}(V)$. We discuss the case of anti-symplectic involutions, and show that the lattice of primitive algebraic cycles $A(V)_{\mathrm{prim}}$ is generated by classes of planes contained in V. We classify $A(V), A(V)_{\mathrm{prim}}, T(V)$ for cubics with an anti-symplectic involution fixing a plane $P \subset V$ point-wise, and explore the geometry of such cubics. In particular, we show that such a cubic fourfold is rational. We also discuss the case of symplectic involutions, showing that in this case $A(V)_{\mathrm{prim}}$ is spanned by classes of rational normal cubic scrolls. We discuss how a such a cubic fourfold with a symplectic involution defines a 6 dimensional IHS orbifold.

Chapter 4 is devoted to the proof of our main Theorem A. We begin the classification of symplectic birational involutions of IHS manifolds of $OG10$ type by studying involutions that act trivially on the discriminant group of the Beauville-Bogomolov-Fujiki lattice, using the knowledge of involutions of the Leech lattice. We make the connection between involutions of a cubic fourfold and birational involutions of an IHS manifold of $OG10$ type more precise, and a criteria for when such an involution is induced in this way. We describe the two unexpected involutions, and how we completed the classification using lattice enumeration techniques. Finally, we briefly outline the strategy to study the fixed locus for the symplectic birational involutions - this is ongoing work.

Chapter 1

IHS manifolds of $OG10$ type

Introduction

The main aim of this thesis is to study symplectic birational involutions of IHS manifolds of $OG10$ type; that is, birational involutions that preserve the symplectic form. In this Chapter, we introduce the main objects of study and collect various technical and classical results in the theory of IHS manifolds. In §1.1 we give the definition of an IHS manifold precisely, and introduce the key properties they satisfy: the existence of the Beauville Bogomolov Fujiki form, the local and global Torelli theorems, and the structure of the birational Kähler cone. In §1.2 we discuss the group of finite symplectic birational automorphisms for an IHS manifold, and rephrase the Global Torelli theorem for manifolds of $OG10$ type in a way suited for this study.

Cubic fourfolds are closely linked to the study of IHS manifolds; we make this connection precise in §1.3. We recall the Strong Torelli theorem for cubic fourfolds, and discuss several constructions of IHS manifolds using the geometry of such a cubic.

1.1 Irreducible holomorphic symplectic manifolds

Definition 1.1.1. An **irreducible holomorphic symplectic manifold** (or IHS manifold) is a simply connected, compact, Kähler manifold X such that $H^0(X, \Omega^2_X)$ is generated by a non-degenerate holomorphic 2-form σ.

Such manifolds are often referred to as **compact hyperkähler** manifolds, due to an equivalent definition in the Riemannian geometry context.

It follows from the above definition that the canonical bundle for an IHS manifold X is trivial, the complex dimension of X is even, and the abelian group $H^2(X,\mathbb{Z})$ is a torsion-free $\mathbb{Z}$ module. A fundamental tool in the study of IHS manifolds is the existence of a canonical, integral, non-divisible quadratic form q_X on the free abelian group $H^2(X,\mathbb{Z})$. This form first appeared in [Bea83, §8] and [Fuj87, Thm 4.7]. We call this form the **Beauville-Bogomolov Fujiki form** (BBF form); its signature is $(3, b_2(X)-3)$, and is proportional to the quadratic form:

$$x \mapsto \int_X \sqrt{\mathrm{td}(X)}x^2.$$

The BBF form satisfies the **Fujiki** relation: for all $x \in H^2(X,\mathbb{Z})$, we have that

$$x^{2m} = c_X q_X(x)^m,$$

where c_X is a positive rational number (the Fujiki constant) and $m := \frac{1}{2}\dim X$. Further, we note that $q_X(x) > 0$ for all Kähler (e.g ample) classes $x \in H^2(X,\mathbb{Z})$. When the context is clear, we denote $q_X(x)$ by x^2.

1.1.1 Examples

We study IHS manifolds up to deformation equivalence - indeed, one of the starting points of the study of IHS manifolds is the existence and unobstructedness of their universal deformations. For an IHS manifold X, we denote by $\mathcal{X} \to \mathrm{Def}(X)$ its universal deformation. We say that two IHS manifolds X and X' are deformation equivalent if there exists a flat family $\mathcal{X} \to B$ and $0, 1 \in B$ such that $\mathcal{X}_0 \cong X$ and $\mathcal{X}_1 \cong X'$.

At the time of writing, there are very few known deformation types of IHS manifolds. In dimension two, the only example is a $K3$ surface, and the BBF form coincides with the standard intersection form. In every even dimension ≥ 4, there are two known deformation types, both due to Beauville [Bea83, §6, §7], which we describe momentarily. There are two exceptional examples, one in dimension 6 and one in dimension 10, which were found by O'Grady [O'G99, O'G03].

We will briefly outline some constructions below, and record the lattice structure of $(H^2(X,\mathbb{Z}), q_X)$ for each deformation type. Recall that the lattices A_k, D_k, E_k denote the standard negative definite root lattices of rank k, and U denotes the hyperbolic plane. The lattice $\langle n \rangle$ denotes the lattice $\mathbb{Z}$ with $e^2 = n$ for a generator e. Further properties of integral lattices will be recalled in Chapter 2.

Example 1.1.1 (Hilbert schemes of points on a $K3$ surface)**.** Let S be a $K3$ surface, not necessarily projective, and consider the Hilbert scheme (or Douady space) $S^{[n]}$ of n points on

S. It is an IHS manifold of dimension $2n$. For $n \geq 2$, Beauville computed both the Fujiki constant and the second cohomology group:

$$H^2(S^{[n]}, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta.$$

Here, 2δ is the class of the divisor in $S^{[n]}$ parameterising non-reduced subschemes. The BBF form of $S^{[n]}$ restricts to the intersection pairing on $H^2(S, \mathbb{Z})$ and in particular

$$(H^2(S^{[n]}, \mathbb{Z}), q_{S^{[n]}}) \cong U^3 \oplus E_8^2 \oplus \langle -2(n-1) \rangle.$$

We say that an IHS manifold X that is deformation equivalent to $S^{[n]}$ is a manifold of $K3^{[n]}$ type.

Example 1.1.2 (Generalised Kummer varieties)**.** Let A be an abelian surface, $A^{[n+1]}$ the Hilbert scheme of $n+1$ points of A. This carries a holomorphic symplectic form, however is not simply connected. Instead, consider the sum morphism

$$A^{[n+1]} \to A, \text{ where } (a_1, \ldots a_{n+1}) \mapsto \sum_{i=1}^{n+1} a_i.$$

We let $K_n(A)$ denote the fiber of this morphism over $0 \in A$; this is an IHS manifold as shown by Beauville. For $n = 1$, this recovers the Kummer $K3$ surface associated to A, thus $K_n(A)$ is called a Generalised Kummer variety. Again, there is a decomposition of second cohomology:

$$H^2(K_n(A), \mathbb{Z}) \cong H^2(A, \mathbb{Z}) \oplus \mathbb{Z}\delta,$$

orthogonal for the BBF form, and as a lattice

$$(H^2(K_n(A), \mathbb{Z}), q_{K_n(A)}) \cong U^3 \oplus \langle -2(n+1) \rangle.$$

We say that an IHS manifold that is deformation equivalent to $K_n(A)$ is a manifold of Kum_n type.

Example 1.1.3 (Moduli of sheaves)**.** Let S be a projective $K3$ surface, and consider a primitive class $v = (r, l, s) \in H^*(S, \mathbb{Z})$, where $r \in H^0(S, \mathbb{Z})$, $l \in \mathrm{NS}(S)$ and $s \in H^4(S, Z)$. There is a pairing $\langle \cdot, \cdot \rangle$ on $H^*(S, \mathbb{Z})$ called the Mukai pairing; choose such a v with $\langle v, v \rangle = 2n-2$. One can construct a projective IHS manifold of $K3^{[n]}$ type as a coarse moduli space X of stable coherent sheaves on S with Mukai vector v, with respect to a fixed v-generic ample line bundle on S [O'G97]. One can construct manifolds of Kum_n-type in a similar manner; we omit the details.

Example 1.1.4 ($OG6$ and $OG10$)**.** Let S be a projective $K3$ surface, and consider the Mukai vector $v = 2w$, where $w \in H^*(S, \mathbb{Z})$ is primitive with $\langle w, w \rangle = 2$. The moduli space $\overline{X}$ of stable coherent sheaves on S with Mukai vector $2v$ is a projective irreducible holomorphic symplectic variety (IHS variety) of dimension 10, singular along a subvariety of codimension 2. In particular, there exists a holomorphic symplectic two form on the smooth part of $\overline{X}$, which extends to a holomorphic two form on any resolution. For this particular choice of Mukai vector, O'Grady showed that there exists a resolution of singularities $X \to \overline{X}$ such that the holomorphic symplectic form extends to a form that is everywhere nondegenerate on X. Such a resolution is called a **symplectic resolution**; here X is an IHS manifold. Moreover, X is a new deformation type of IHS manifold; an example that is deformation equivalent to such an X is a manifold of $OG10$ type. A similar construction can be done starting with an abelian surface; we omit the details and refer the reader to [O'G99, O'G03].

In [Rap08], Rapagnetta computed the BBF form for an IHS manifold of $OG10$ type. Indeed, there is an isometry $(H^2(X, \mathbb{Z}), q_X) \cong \Lambda$, where

$$\Lambda := U^3 \oplus E_8^2 \oplus A_2.$$

We shall see another construction of manifolds of $OG10$ type in §1.3.5.

1.1.2 Moduli spaces and the period map

Let X be a IHS manifold whose Beauville-Bogomolov lattice $\eta : (H^2(X, \mathbb{Z}), q_X) \cong L$ for a fixed lattice L. We call a choice of isometry η a **marking** of X - two marked IHS manifolds $(X, \eta), (X', \eta')$ are isomorphic if there exists an isomorphism $f : X \to X'$ compatible with the markings.

Recall that deformations of X are unobstructed, and we denote the universal deformation of X by $\mathcal{X} \to \mathrm{Def}(X)$. By Ehresmann's theorem, if $\eta : H^2(X, \mathbb{Z}) \to L$ is a marking of X, then there exists a family of markings $\eta_t : H^2(\mathcal{X}_t, \mathbb{Z}) \to L$ such that $\eta := \eta_0$.

We let

$$\mathcal{D}_L := \{x \in \mathbb{P}(L \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \bar{x} > 0\}^+$$

denote the period domain associated to L, and consider the local period map

$$\mathscr{P} : \mathrm{Def}(X) \to \mathcal{D}_L,$$

associating $\mathscr{P}(\mathcal{X}_t) = [\eta_t(H^{2,0}(\mathcal{X}_t))]$.

Theorem 1.1.5 (Local Torelli theorem)**.** *[Bea83] The period map $\mathscr{P} : \mathrm{Def}(X) \to \mathcal{D}_L$ is a local isomorphism.*

It is true that IHS manifolds satisfy a global Torelli theorem, as proved by Huybrechts, Markman and Verbitsky. We will state Markman's Hodge theoretic version in 1.1.7, but in order to do so we need to introduce the notion of a **monodromy operator**.

Definition 1.1.2. Let X, X' be IHS manifolds. An isomorphism $f : H^*(X,\mathbb{Z}) \to H^*(X',\mathbb{Z})$ is a **parallel-transport operator** if there exists a smooth and proper family $\pi : \mathcal{X} \to B$ of IHS manifolds over an analytic base B, with $0, 1 \in B$, isomorphisms $\mathcal{X}_0 \cong X$, $\mathcal{X}_1 \cong X'$ and a continuous path $\gamma : [0,1] \to B$ such that the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism f.

A parallel transport operator $f : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is called a **monodromy operator**. The **monodromy group** $\mathrm{Mon}^2(X)$ is the subgroup of $O(H^2(X,\mathbb{Z}))$ consisting of all monodromy operators. Such isometries are always contained in the index two subgroup of $O(H^2(X,\mathbb{Z}))$ consisting of orientation preserving isometries, denoted by $O^+(H^2(X,\mathbb{Z}))$.

The monodromy group has been completely described for all known deformation types of IHS manifolds. In particular, for manifolds of $OG10$ this was computed recently by Onorati:

Theorem 1.1.6. *[Ono21, Theorem 5.4] Let X be of $OG10$ type. Then*

$$\mathrm{Mon}^2(X) \cong O^+(H^2(X,\mathbb{Z})).$$

We can now state Markman's Hodge theoretic version of the global Torelli theorem.

Theorem 1.1.7. *[Mar11, Theorem 1.3] Let X, X' be IHS manifolds that are deformation equivalent.*

1. *X and X' are birational if and only if there exists a monodromy operator $f : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ that is a Hodge isometry.*

2. *Let $f : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ be a monodromy operator which is a Hodge isometry. Then there exists an isomorphism $\tilde{f} : X' \to X$ such that $f = \tilde{f}^*$ if and only if f maps some Kähler class on X to a Kähler class on X'.*

In order to study birational transformations, we are reduced to understanding monodromy operators that preserve the Hodge structure. We will introduce the notion of the birational Kähler cone in the next section, enabling us to produce a numerical criterion for when an isometry $f \in O(H^2(X,\mathbb{Z}))$ is such a monodromy operator.

1.1.3 Kähler and birational Kähler cones

Throughout, we let X be a IHS manifold of a fixed deformation type. The BBF form allows us to define several cones contained in $H^{1,1}(X,\mathbb{R})$ that are useful for studying birational transformations. These are sub-cones of the **positive cone** $\mathcal{C}_X$, which is the connected component of $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid q_X(\alpha) > 0\}$ that contains a Kähler class. The **Kähler cone** $\mathcal{K}_X \subset \mathcal{C}_X$ is the subset containing the Kähler classes. The **birational Kähler cone** $\mathcal{BK}(X)$ is the union of the Kähler cones of all IHS manifolds birational to X, i.e

$$\mathcal{BK}(X) := \bigcup_{f: X \dashrightarrow X'} f^*\mathcal{K}(X').$$

Here, f runs over all birational maps from X to an IHS manifold X'. When X is projective, the closure of $\mathcal{BK}(X)$ is the movable cone. We can describe both cones more precisely using the following result:

Theorem 1.1.8. *[Huy03],[Bou01] The closure of $\mathcal{K}_X$ is the set of all classes $\alpha \in \overline{\mathcal{C}}_X$ such that $\int_C \alpha \geq 0$ for all rational curves C The closure of $\mathcal{BK}(X)$ is the set of all classes $\alpha \in \overline{\mathcal{C}}_X$ such that $q_X(\alpha, D) \geq 0$ for all uniruled divisors $D \subset X$.*

The BBF form gives us a way to identify uniruled divisors numerically.

Proposition 1.1.9. *[Bou04, Prop. 4.7] Let $D \subset X$ be an prime effective divisor such that $q_X(D,D) < 0$. Then D is uniruled.*

Such a divisor is said to be **prime exceptional**. A divisor is **stably prime exceptional** if it is prime exceptional on a very general deformation of (X, D). By [Mar11, §5], the hyperplanes $D^\perp \subset \mathcal{C}_X$ for D stably prime exceptional define the walls of the birational Kähler cone. We denote by $\mathcal{W}^{pex} \subset H^2(X,\mathbb{Z})$ the set of stably prime exceptional divisors.

Proposition 1.1.10. *The birational Kähler cone $\mathcal{BK}(X)$ is an open set in a connected component of*

$$\mathcal{C}_X \setminus \bigcup_{D \in \mathcal{W}^{pex}} D^\perp.$$

To obtain the Kähler cone of X, one has to further subdivide the birational Kähler cone with more hyperplanes, or walls.

Definition 1.1.3. Let $D \in \mathrm{Pic}(X)$ be a divisor, not necessarily effective. Then D is called a **wall divisor** if $D^2 < 0$ and $\gamma(D^\perp) \cap \mathcal{BK}(X) = \varnothing$ for all $\gamma \in \mathrm{Mon}^2(X)_{\mathrm{Hdg}}$.

Here $\mathrm{Mon}^2(X)_{\mathrm{Hdg}}$ is the subgroup of monodromy operators that preserve the Hodge structure. We let $\mathcal{W} \subset H^2(X,\mathbb{Z})$ denote the set of wall divisors. Notice that a stably prime exceptional divisor is the multiple of a wall divisor, and wall divisors which have an effective multiple are stably prime exceptional.

Remark 1.1.11. We note that via the BBF form q_X we can identify $H_2(X,\mathbb{Z})$ with the dual of $H^2(X,\mathbb{Z})$, and consider $H_2(X,\mathbb{Z}) \subset H^2(X,\mathbb{Q})$. Let R be an extremal ray of the Mori cone of X, that is the cone spanned by classes of effective curves in X. Then any divisor $D \in \mathbb{Q}R$ is a wall divisor (see [Mon15, Lemma 1.4]). One can instead use extremal rays to cut out the Kähler cone; this is the notion of an MBM class as defined by Amerik and Verbitsky (see [AV15, AV21] for more details).

1.2 Symplectic birational transformations of IHS manifolds

We are now ready to discuss symplectic birational transformation of manifolds of $OG10$ type, and to reformulate the Global Torelli theorem in a way that is more useful for our study. In this section we will specialise to the case of $OG10$ type manifolds; similar result hold in the other cases (see [Mon13, Mon16] for the case of $K3^{[n]}$, [KMO18] for symplectic involutions of $K3^{[n]}, Kum_n$ type, and [GOV20] for the case of $OG6$).

1.2.1 Symplectic birational transformations

We denote by $\mathrm{Aut}(X)$, $\mathrm{Bir}(X)$ the groups of automorphisms and birational transformations of X respectively. For an IHS manifold X, a birational transformation $f \in \mathrm{Bir}(X)$ is well defined in codimension one. We thus obtain an isometry $f^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$.

Definition 1.2.1. We say a birational transformation $f \in \mathrm{Bir}(X)$ is **symplectic** if the induced action $f^* : H^2(X,\mathbb{C}) \to H^2(X,\mathbb{C})$ acts trivially on σ. Otherwise, f is non-symplectic (or anti-symplectic if f has order 2).

Assume now that X is an IHS manifold of $OG10$ type, and consider the associated representation map

$$\eta_* : \mathrm{Bir}(X) \to O(\Lambda); \quad f \mapsto \eta \circ f^* \circ \eta^{-1}.$$

Note that for a nontrivial birational transformation $f \in \mathrm{Bir}(X)$ the induced isometry $\eta_*(f) \in O(\Lambda)$ is non-trivial. Indeed, if $\eta_*(f)$ was trivial, by [Fuj81] we see that f is a regular automorphism of X. By [MW17, Theorem 3.1], the associated representation $\mathrm{Aut}(X) \to O(\Lambda)$ is injective, and so f is trivial.

Definition 1.2.2. An isometry $g \in O(\Lambda)$ is **induced** by a birational transformation if there exists a $f \in Bir(X)$ such that $\eta_*(f) = g$.

A birational transformation of X preserves the birational Kähler cone $\mathcal{BK}(X)$. This in turn imposes restrictions on which involutions of the lattice Λ are induced by birational involutions of such a manifold X. The structure of the birational Kähler cone for a manifold of $OG10$ type is now fully understood [MO20]; from Proposition 1.1.10, this amounts to understanding the set of stably prime exceptional divisors $\mathcal{W}^{pex}$ for a manifold of $OG10$ type.

Proposition 1.2.1. *[MO20, Proposition 3.1] Let X be a IHS manifold of $OG10$ type, $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ a marking. Then $D \in \mathrm{Pic}(X)$ is stably prime exceptional if and only if $\eta(D) \in \mathcal{W}^{pex}$, where*

$$\mathcal{W}^{pex} = \{v \in \Lambda : v^2 = -2\} \cup \{v \in \Lambda : v^2 = -6, div_\Lambda(v) = 3\}.$$

We call $v \in \mathcal{W}^{pex}$ with $v^2 = -2$ (resp. $v^2 = -6$) a **short root** (resp. a **long root**). A chamber defined by $\mathcal{W}^{pex}$ is called an **exceptional chamber**.

Theorem 1.2.2. *[MO20, Theorem 3.2] Let X be a manifold of OG10 type. Then, the birational Kähler cone $\mathcal{BK}(X)$ of X is an open set inside one of the components of*

$$\mathcal{C}(X) \setminus \bigcup_{v \in \mathcal{W}^{pex}} v^\perp$$

where $\mathcal{C}(X)$ is the connected component of the positive cone containing a Kähler class.

Using this description for the birational Kähler cone, we can rephrase the Global Torelli Theorem in a way that is more suited for the study of symplectic birational transformations of X. We denote by $O^+_{hdg}(\Lambda)$, and $O^+_{sp}(\Lambda)$ the group of signed Hodge isometries and signed symplectic isometries respectively. The group $O^+_s p(\Lambda, C)$ is the subgroup of $O^+_{sp}(\Lambda)$ leaving C invariant.

Theorem 1.2.3. *(Torelli Theorem for OG10) A subgroup of $G \subset O(\Lambda)$ is induced by a group of symplectic birational transformations on a manifold of $OG10$ type if and only if there exist a signed Hodge structure on Λ and an exceptional chamber C such that $G \subset O^+_{sp}(\Lambda, C)$.*

Proof. The proof follows almost identically to that of [GOV20, Theorem 2.15] in the case of manifolds of $OG6$-type. We reproduce the argument in the case of $OG10$ type for convenience.

Suppose that G is induced by a group of symplectic birational transformations on a manifold X of $OG10$ type. It follows that G is contained in the monodromy group of X, $\mathrm{Mon}^2(X)$. By [Ono21, Theorem 5.4], $\mathrm{Mon}^2(X) \cong O^+(H^2(X,\mathbb{Z}))$. The Hodge decomposition and the canonical choice of the positive cone induce a signed Hodge structure on Λ. All birational transformations of X preserve the birational Kähler cone, and by Theorem 1.2.2 this is a chamber C cut out by the vectors in $\mathcal{W}^{pex}$. Therefore, $G \subset O^+_{sp}(\Lambda, C)$.

Conversely, suppose there exists a signed Hodge structure on Λ and a Kähler chamber C such that $G \subset O^+_{sp}(\Lambda, C)$. Then, by Huybrechts' theorem on the surjectivity of the period map [Huy97, Theorem 8.1] there exists a manifold X of type $OG10$ whose Hodge decomposition induces the given signed Hodge structure. Moreover, we can assume that C is the birational Kähler cone of X. As G consists of monodromy transformations and respects the birational Kähler cone, it is induced by birational transformations of X by Markman's Hodge theoretic version of the Torelli theorem [Mar11, Theorem 1.3]. The birational transformation must be symplectic. □

1.3 Cubic fourfolds

Cubic fourfolds are intimately linked with IHS manifolds through various constructions. In order to study symplectic birational transformations of manifolds of $OG10$ type, we will exploit this connection. We must first understand automorphisms of cubic fourfolds from a Hodge theoretic perspective. In this section we review the relevant results that will allow us to conduct this study. In §1.3.1 we review the period map for cubic fourfolds, and in §1.3.2 the Torelli Theorem for cubic fourfolds V, and reduce the classification of involutions of V to the classification of involutions of Hodge structures. This is essentially a lattice theoretic question; we will follow the same approach as in the case of K3 surfaces (Nikulin, Mukai, Kondō and others). Finally, in §1.3.3, we review some constructions of IHS manifolds starting from a cubic fourfold.

Remark 1.3.1. In this section all ADE lattices are assumed to be positive definite for ease of notation.

1.3.1 The period map for cubic fourfolds

Let V be a smooth cubic fourfold. The middle cohomology $H^4(V, \mathbb{Z})$ with the natural intersection pairing (denoted by $x \cdot y$ for $x, y \in H^4(V, \mathbb{Z})$) is the unique unimodular odd lattice of signature $(21, 2)$. We denote by $\eta_V \in H^4(V, \mathbb{Z})$ the square of the hyperplane class of V, with $\eta_V^2 = 3$. The primitive cohomology $H^4(V, \mathbb{Z})_{\mathrm{prim}} := \langle \eta_V \rangle^{\perp}$ carries a polarized Hodge structure of K3 type (i.e. Hodge numbers (0,1,20,1,0)). As a lattice, $H^4(V, \mathbb{Z})_{\mathrm{prim}} \cong L$, where

$$L := (E_8)^2 \oplus U^2 \oplus A_2.$$

Denote by $\mathcal{M}$ the moduli space of smooth cubic fourfolds, constructed using GIT [Laz09]. Denote by $\mathcal{D}/\Gamma = \{x \in \mathbb{P}(L \otimes \mathbb{C}) \mid x^2 = 0, x \cdot \overline{x} < 0\}^+/\Gamma$ the global period domain for cubic fourfolds. Here the global monodromy group Γ is the subgroup of $O(L)$ that preserves the period domain $\mathcal{D}$ and acts trivially on the discriminant group $L^*/L \cong \mathbb{Z}/3\mathbb{Z}$. Starting with a cubic fourfold V, we associate the Hodge structure on its middle cohomology, obtaining a period map:

$$\mathscr{P} : \mathcal{M} \to \mathcal{D}/\Gamma.$$

In order to discuss the image of this map, we define two divisors in $\mathcal{D}/\Gamma$.

Definition 1.3.1.

1. A norm 2 vector $v \in L$ is called a **short root**. The set of short roots in L determines a Γ-invariant hyperplane arrangement $\mathcal{H}_6$ in $\mathcal{D}$. Let $\mathcal{C}_6 := \mathcal{H}_6/\Gamma \subset \mathcal{D}/\Gamma$ be the associated divisor.

2. A norm 6 vector $v \in L$ with divisibility 3 is called a **long root**. The set of long roots in L determines a Γ-invariant hyperplane arrangement $\mathcal{H}_2$ in $\mathcal{D}$. Let $\mathcal{C}_2 := \mathcal{H}_2/\Gamma \subset \mathcal{D}/\Gamma$.

The two divisors $\mathcal{C}_2, \mathcal{C}_6$ above are the complement of the image of the period map. Geometrically, $\mathcal{C}_6$ corresponds to singular cubic fourfolds, where as $\mathcal{C}_2$ corresponds to degenerations of cubics to the secant to the Veronese surface in $\mathbb{P}^5$. We recall that cubic fourfolds contained in $\mathcal{C}_2$ are determinantal.

Theorem 1.3.2 (Voisin, Hassett, Laza, Looijenga)**.** *The period map for cubic fourfolds is an isomorphism*

$$\mathscr{P} : \mathcal{M} \to \mathcal{D}/\Gamma \setminus (\mathcal{C}_2 \cup \mathcal{C}_6).$$

1.3.2 Strong Torelli and automorphisms

Let $\phi \in \mathrm{Aut}(V)$ be an automorphism of a cubic fourfold V. By considering the induced action ϕ^* on $H^4(V,\mathbb{Z})_{\mathrm{prim}} \cong L$, we obtain a map

$$\mathrm{Aut}(V) \to O(L).$$

We note that this map is injective (combine [JL17, Prop. 2.12], [MM64]). Further, the Strong Global Torelli Theorem [Zhe19] holds for cubic fourfolds. In other words, any isomorphism between polarized Hodge structures of two smooth cubic fourfolds is induced by a unique isomorphism between the cubic fourfolds themselves.

Proposition 1.3.3 (Strong Global Torelli Theorem)**.** *Let V_1, V_2 be two smooth cubic fourfolds. Assume that there is an isomorphism $\phi : H^4(V_2,\mathbb{Z}) \xrightarrow{\cong} H^4(V_1,\mathbb{Z})$ of polarized Hodge structures. Then there exists a unique isomorphism $f : V_1 \xrightarrow{\cong} V_2$ such that $\phi = f^*$. In particular, for any smooth cubic fourfold V,*

$$\mathrm{Aut}(V) \cong \mathrm{Aut}_{HS}(V, \eta_V),$$

where $\mathrm{Aut}_{HS}(V,\eta_V)$ denotes the group of Hodge isometries fixing the class η_V.

Thus it is natural to study automorphisms of a cubic fourfold via the induced Hodge isometry of $H^4(V,\mathbb{Z})_{\mathrm{prim}}$.

Definition 1.3.2. We say an automorphism $\phi \in \mathrm{Aut}(V)$ is **symplectic** if ϕ acts trivially on $H^{3,1}(V)$. Otherwise, ϕ is non-symplectic (or anti-symplectic in the case of involutions).

Possible cyclic groups of symplectic automorphisms of a cubic fourfold were classified [Fu16], followed by a classification of possible symplectic automorphism groups [LZ22]. The action on $H^4(V,\mathbb{Z})_{\mathrm{prim}}$ was identified in each symplectic case by a lattice theoretic argument, however the geometry was not explored.

1.3.3 Constructions via cubic fourfolds

Cubic fourfolds lead to IHS manifolds through various constructions, and one can study birational transformations that are induced by automorphisms of the cubic. In this section we will discuss two examples; we note that this is not an extensive list, and we only mention constructions related to our study.

Example 1.3.4 (The Fano variety of lines)**.** Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold. Then the Fano variety of lines on V, denoted by $F(V)$ is a smooth IHS manifold of dimension 4, as shown in [BD85]. The manifold $F(V)$ is deformation equivalent to the Hilbert scheme of two points on a $K3$ surface, and hence is of $K3^{[2]}$ type.

Automorphisms of the Fano variety of lines induced from a cubic fourfold were first studied by Camere [Cam12] in her work on symplectic involutions. Using lattice theoretic techniques and the classification of automorphisms of the Leech lattice, Hōhn and Mason determined the possible finite groups of symplectic automorphisms for manifolds of $K3^{[2]}$ type [HM19].

Example 1.3.5 (The LSV construction)**.** Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, and denote by $B := |H| = (\mathbb{P}^5)^\vee$ the linear system of hyperplane sections of V. Let $f : \mathcal{Y} \to B$ be the universal family of hyperplane sections: for generic $b \in B$, the fiber $f^{-1}(b) = Y_b$ is a smooth hyperplane section of V. For such a section, the associated intermediate Jacobian

$$J(Y_b) := H^{2,1}(Y_b)^\vee / H_3(Y_b, \mathbb{Z})$$

is a principally polarized abelian variety of dimension 5 [CG72]. Let $U \subset B$ be the open set parametrising smooth hyperplane sections, and let $\pi_U : J_U \to U \subset (\mathbb{P}^5)^\vee$ be the Donagi-Markman fibration; i.e the family of intermediate Jacobians of the smooth hyperplane sections of V. The total space J_U admits a holomorphic symplectic form, by [DM96].

The main result of [LSV17] is the construction, for a general V, of a smooth projective irreducible holomorphic symplectic compactification $\mathcal{J}_V$ of J_U, with a Lagrangian fibration $\pi : \mathcal{J}_V \to (\mathbb{P}^5)^\vee$ extending π_U. It was shown that $\mathcal{J}_V$ is an irreducible holomorphic symplectic manifold, deformation equivalent to O'Grady's 10 dimensional exceptional example [O'G99] and is thus of $OG10$ type. Using both degeneration techniques of [KLSV18] and results of birational geometry, Saccá proved the existence of such a symplectic compactification $\mathcal{J}_V$ for any cubic fourfold, not necessarily general, in [Sac21]. In this case, the compactification may not be unique. With these two results, we have the following theorem:

Theorem 1.3.6. *([LSV17],[Sac21]) Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, and let $\pi_U : J_U \to U \subset (\mathbb{P}^5)^\vee$ be the Donagi-Markman fibration. Then there exists a smooth projective irreducible symplectic compactification $\mathcal{J}_V$ of J_U of $OG10$ type with a morphism $\pi : \mathcal{J}_V \to (\mathbb{P}^5)^\vee$ extending π_U.*

We note that the same result holds for the irreducible holomorphic symplectic compactification $\mathcal{J}_V^T$ of the non trivial J_U-torsor $J_U^T \to U$ of [Voi18].

Chapter 2

Lattice theory

Introduction

A key technique in our study of involutions for both cubic fourfolds and for IHS manifolds is to study isometries of certain lattices. To a cubic fourfold V, we consider the middle primitive cohomology $H^4(V, \mathbb{Z})_{\text{prim}}$, equipped with the intersection pairing. For an irreducible holomorphic symplectic manifold X, we consider the second cohomology $H^2(X, \mathbb{Z})$ equipped with the BBF form. An involution of $\phi \in \text{Aut}(V)$ (respectively $f \in \text{Bir}(X)$) induces an isometry of the associated lattice.

In this section we will review necessary results about lattices. We mainly follow [Nik79b] and we provide proofs to results that we could not find references for in the literature. Most of these results can be found in [Mar23, Appendix A]. In §2.1 we discuss Nikulin's theory of overlattices, followed by the theory of primitive embeddings in §2.2. As we are most interested in involutions of lattices, we recall some results regarding 2-elementary lattices in §2.3.

2.0.1 Notation

A lattice of rank r is a free finitely generated $\mathbb{Z}$-module $L \cong \mathbb{Z}^r$ equipped with a non-degenerate symmetric bilinear pairing $L \otimes L \to \mathbb{Z}$, denoted $e \otimes f \mapsto e \cdot f$. In particular, all of the lattices we consider are integral. We write $e^2 = e \cdot e$; occasionally we denote this by $q_L(e)$. We assume that L is even ($e^2 \in 2\mathbb{Z}$ for each $e \in N$) unless otherwise stated.

We denote by $div_L(v) = \max\{\lambda \in \mathbb{Z} \mid v \cdot L \in \lambda\mathbb{Z}\}$ the divisibility of an element $v \in L$.

The lattice $L(n)$ is obtained from L by multiplying the pairing by n. The lattice $\langle n \rangle$ denotes the lattice $\mathbb{Z}$ with $e^2 = n$ for a generator e. The lattices A_k, D_k, E_k denote the standard negative definite root lattices of rank k, and U denotes the hyperbolic plane.

The dual lattice L^* is given by

$$L^* := \{y \in L \otimes \mathbb{Q} \mid y \cdot x \in \mathbb{Z} \text{ for all } x \in L\}.$$

The finite abelian group $A_L := L^*/L$ is called the **discriminant group** of L, with induced quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ (or symmetric bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ if L is odd).

Example 2.0.1. Consider the lattice A_2; let α_1, α_2 be generators with $\alpha_i^2 = 2$ and $\alpha_1 \cdot \alpha_2 = -1$. Then $A_{A_2} \cong \mathbb{Z}/3\mathbb{Z}$, and is generated by the class

$$\gamma := \left[\frac{2\alpha_1 + \alpha_2}{3}\right].$$

One can see that $q_{A_2}(\gamma) = \frac{2}{3}$.

We say that two lattices L and M are in the same **genus** if they are isomorphic over the ring of p-adic integers $\mathbb{Z}_p$ for each prime p, and also isomorphic over $\mathbb{R}$. Two lattices in the same genus are not always isomorphic over $\mathbb{Z}$: for example, there are two non-isomorphic rank 16 negative definite unimodular lattices D_{16}^+ and $E_8 \oplus E_8$, which are necessarily in the same genus.

2.1 Overlattices

Let L be an even lattice. An **overlattice** of L is an even lattice Γ where $L \subset \Gamma$ is an embedding for which Γ/L is a finite abelian group.

Let $H_\Gamma = \Gamma/L$ We have a chain of embeddings:

$$L \subset \Gamma \subset \Gamma^* \subset L^*,$$

and so $H_\Gamma \subset L^*/L = A_L$, and $(\Gamma^*/L)/H_\Gamma = A_\Gamma$.

We have the following result due to Nikulin [Nik79b, Props. 1.4.1, 1.4.2].

Proposition 2.1.1. *Let L be an even lattice.*

1. *The correspondence $\Gamma \mapsto H_\Gamma$ determines a bijection between even overlattices of L and isotropic subgroups of A_L (a subgroup $H \subset A_L$ is isotropic if $q_L|_H = 0$).*

2. *We have $(H_\Gamma)^\perp = \Gamma^*/L \subset A_L$ and $(q_L|_{(H_\Gamma)^\perp})/H_\Gamma = q_\Gamma$.*

3. *Two even overlattices $L \subset \Gamma_1$ and $L \subset \Gamma_2$ are isomorphic if and only if the isotropic subgroups $H_{\Gamma_1}, H_{\Gamma_2} \subset A_L$ are conjugate under some automorphism of L.*

2.2 Primitive embeddings

An embedding of lattices $M \subset L$ is called **primitive** if L/M is a free group; we denote this by $M \hookrightarrow L$, and denote $N = M^\perp$ the **orthogonal complement** of M in L. Then L is an overlattice of $M \oplus N$, and $H_L = L/(M \oplus N) \subset A_M \oplus A_N$ is the isotropic subgroup corresponding to $q_M \oplus q_N$. The embeddings $M \hookrightarrow L$ and $N \hookrightarrow L$ being primitive is equivalent to the condition that the projections $H_L \to A_M$ and $H_L \to A_N$ are embeddings.

Lemma 2.2.1. *[Nik79b, Prop 1.15.1] Primitive embeddings of M with signature (m_+, m_-) into an even lattice L with signature (l_+, l_-) are determined by the data $(H_M, H_L, \gamma, N, \gamma_N)$ where*

- *H_M, H_L are subgroups of A_M, A_L respectively and $\gamma : q|_{A_M|H_M} \to -q|_{A_L|H_L}$ is an isometry;*
- *N is an even lattice with signature $(l_+ - m_+, l_- - m_-)$ and discriminant form $-\delta$ where $\delta \equiv (q_{A_M} \oplus -q_{A_L})|_{\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_N : q_N \to (-\delta)$ is an isometry.*

The group H_M is called the **gluing group**; indeed, if L is unimodular, and $N = M^\perp$, then $H_M \cong A_M \cong A_N$ and $\gamma_N : q_N \to -q_M$ is the gluing map. More precisely:

Lemma 2.2.2. *Let M be an even lattice of signature (m_+, m_-). The existence of a primitive embedding of M into some unimodular lattice L of signature (l_+, l_-) is equivalent to the existence of a lattice N of signature (n_+, n_-) and discriminant form q_{A_N} such that the following are satisfied:*

- *$m_+ + n_+ = l_+$ and $m_- + n_- = l_-$.*
- *$A_N \equiv A_M$ and $q_{A_N} = -q_{A_M}$.*

2.3 2-elementary lattices

We say that an even lattice L is **2-elementary** if the discriminant group A_L is a 2-elementary group i.e $A_L \cong (\mathbb{Z}/2\mathbb{Z})^a$ for some $a \geq 1$. We obtain examples of 2-elementary lattices by considering involutions.

Let L be any even lattice. An involution $\iota \in O(L)$ determines two eigenspaces

$$L_\pm := \{v \in L : \iota(v) = \pm v\}.$$

We call L_- (respectively L_+) the **coinvariant lattice** (resp. **invariant lattice**). The following hold:

- L_- and L_+ are primitive, mutually orthogonal lattices;
- $H := L/(L_+ \oplus L_-)$ is a 2-elementary group (i.e $H = (\mathbb{Z}/2)^a$);
- H admits embeddings into the discriminant groups A_{L_+} and A_{L_-} such that the diagonal embedding $H \subset A_{L_+} \oplus A_{L_-}$ is isotropic with respect to the finite quadratic form $q_{L_+} \oplus q_{L_-}$ [[Nik79b] Sect. 1.5].

Lemma 2.3.1. *If Λ is a unimodular lattice and $\iota \subset O(\Lambda)$ is an involution, then Λ_- and Λ_+ are 2-elementary lattices.*

Lemma 2.3.2. *Let L be a lattice, and $\iota \subset O(L)$ an involution acting as the identity on A_L. Then L_- is a 2-elementary lattice.*

Proof. Let Λ be a unimodular lattice such that there exists a primitive embedding $L \hookrightarrow \Lambda$, and let $R = L^\perp$. Since ι acts as the identity on A_L, we can extend ι by the identity on R. Since $R_- = 0$, it holds that $L_- \cong \Lambda_-$. We deduce by Lemma 2.3.1 that L_- is 2-elementary. $\square$

In fact, the converse is true: given a primitive embedding of a 2-elementary lattice M into an even lattice, we can always ensure it is the coinvariant sublattice for some involution. More precisely:

Lemma 2.3.3. *Let M be a 2-elementary lattice with a primitive embedding $M \hookrightarrow L$ into a lattice L such that $N := (M)^\perp_L$. Then there exists an involution $\iota \in O(L)$ such that the coinvariant lattice $L_- = M$ and the invariant lattice $L_+ = N$.*

Proof. By assumption we have that

$$M \oplus N \hookrightarrow L \hookrightarrow L \otimes \mathbb{Q} \cong (M \oplus N) \otimes \mathbb{Q}.$$

We can define $\iota_\mathbb{Q} : L_\mathbb{Q} \to L_\mathbb{Q}$ by $\iota(x) = -x$ for $x \in M$, and $\iota(x) = x$ for $x \in N$. We want to show that $\iota_\mathbb{Q}$ is defined over L. By assumption $L/(M \oplus N) \cong (\mathbb{Z}/2\mathbb{Z})^a$, and thus for all $x \in L$, we have that $2x \in M \oplus N$. Let $x \in L$. By above, we can write $x = \frac{x_- + x_+}{2}$, with $x_- \in M, x_+ \in N$. Thus $\iota_\mathbb{Q}(x) = x \mod M \oplus N$, and $[\iota_\mathbb{Q}(x)] = [x]$ in $L/(M \oplus N)$; thus $\iota_\mathbb{Q}(x) \in L$. $\square$

Let M be a 2-elementary lattice. Nikulin proved that M is classified by three natural invariants: the signature (m_+, m_-), the number of generators a of A_M and $\delta \in \{0, 1\}$ with $\delta = 0$ if and only if the finite quadratic form $q_M : A_M \to \mathbb{Q}/2\mathbb{Z}$ takes values in $\mathbb{Z}/2\mathbb{Z}$.

Theorem 2.3.4. *(Classification of 2-elementary lattices) The genus of an even 2-elementary lattice is determined by the invariants $\delta_M, l(M)$ and $sign(M)$. If M is indefinite, then the genus consists of one isomorphism class.*

An even 2-elementary lattice M with $\delta_M = \delta$, $l(M) = a$ and $sign(M) = (m_+, m_-)$ exists if and only if the following conditions are satisfied:

1. $m_+ + m_- \geq a$;
2. $m_+ + m_- + a \equiv 0 \mod 2$;
3. $m_+ - m_- \equiv 0 \mod 4$ *if* $\delta = 0$;
4. $\delta = 0$, $m_+ - m_- \equiv 0 \mod 8$ *if* $a = 0$;
5. $m_+ - m_- \equiv 1 \mod 8$ *if* $a = 1$;
6. $\delta = 0$ *if* a=2, $m_+ - m_- \equiv 4 \mod 8$;
7. $m_+ - m_- \equiv 0 \mod 8$ *if* $\delta = 0$ *and* $a = m_+ + m_-$.

Note that the genus of a **definite** 2-elementary lattice may consist of more than one isomorphism class. We will use the above classification to determine the existence of primitive embeddings in specific situations. In particular, we will be interested in embeddings of 2-elementary lattices into the lattice

$$\Lambda := A_2 \oplus (E_8)^2 \oplus U^3.$$

Recall that for an IHS manifold X of $OG10$ type we have that $H^2(X, \mathbb{Z}) \cong \Lambda$.

Lemma 2.3.5. *Let M be a 2-elementary, negative definite lattice of rank r with invariants (r, a, δ), where $a = l(A_M)$. Then there exists a primitive embedding $M \hookrightarrow \Lambda$ if and only if there exists a lattice N of signature $(3, 21 - r)$ satisfying the following properties:*

1. $A_N \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/3\mathbb{Z}$;
2. $q_N|_{(\mathbb{Z}/2\mathbb{Z})^a} \cong -q_M$;
3. $q_N|_{\mathbb{Z}/3\mathbb{Z}} \cong q_\Lambda$.

We say $\delta := \delta_N = 0$ or 1 if and only if $\delta_M = 0$ or 1.

Proof. This follows from Lemma 2.2.1 and 2.3.4. □

Lemma 2.3.6. *Let M be an even lattice of rank r such that $(\mathbb{Z}/2\mathbb{Z})^r \subset A_M$. Then $M(1/2)$ is a well-defined integral lattice.*

Proof. For $T := M(1/2)$ to be well-defined, we must check that the induced symmetric bilinear form takes integral values. Recall that $(x,y)_T = \frac{1}{2}(x,y)_M$, and thus this is equivalent to showing that $div_M(x)$ is divisible by 2 for all $x \in M$. Let G be the Gram matrix of M; it has integral entries, and has Smith Normal form

$$diag(d_1, \dots d_r),$$

with $d_1 d_2 \dots d_r \neq 0$, and $d_i | d_{i+1}$ for $i = 1, \dots r-1$. Further, $d_1 d_2 \dots d_i$ is the greatest common divisor all the $i \times i$ minors of G. In particular, d_1 is the greatest common divisor of all entries of G. Thus in order to prove that $div_M(x)$ is divisible by 2 for all $x \in M$ it suffices to show that $2|d_1$.

Since M is a free module over a PID, there exists a basis $u_1, \dots u_r$ of M^* such that $v_1 = d_1 u_1, \dots, v_r = d_r u_r$ is a basis of M. Hence

$$M^*/M = \frac{\mathbb{Z}u_1 \oplus \dots \mathbb{Z}u_r}{\mathbb{Z}v_1 \oplus \dots \mathbb{Z}v_r} \equiv (\mathbb{Z}/d_1\mathbb{Z}) \oplus \dots (\mathbb{Z}/d_r\mathbb{Z}).$$

Since $(\mathbb{Z}/2\mathbb{Z})^r$ is a subgroup of A_M, we must have that d_1 is divisible by 2; if not, A_M can only contain $(\mathbb{Z}/2\mathbb{Z})^{r-1}$.

□

Lemma 2.3.7. *Let M be an even, 2-elementary lattice with $l(A_M) = r$ where r is the rank and $\delta = 1$. Consider the lattice $T := M(\frac{1}{2})$. Then T is an odd unimodular lattice.*

Proof. Assume that $T := M(\frac{1}{2})$ is even. Then for all $x \in T$, we have that $(x \cdot x)_T = 2k$ for some integer k. Thus $(x \cdot x)_M = 4k$.

Since $\delta = 1$, there exists $\gamma \in A_M$ of order 2 with $q(\gamma)_M \in (\mathbb{Q} \setminus \mathbb{Z})/2\mathbb{Z}$. We can write $\gamma = \alpha + M$ for $\alpha \in M^*$, and so $(\alpha \cdot \alpha)_M \in \mathbb{Q} \setminus \mathbb{Z}$. Since γ has order 2, we have that $2\gamma \in M$, i.e $2\alpha \in M$ and so $\alpha = \frac{x}{2}$ for some $x \in M$. Finally we see that

$$(\alpha \cdot \alpha)_M = \frac{(x \cdot x)_M}{4} = k$$

with k an integer, providing a contradiction.

□

Lemma 2.3.8. *Let T be an odd lattice. Then $M := T(2)$ has $\mathbb{Z}/2\mathbb{Z} \subset A_M$ and $q_M(\xi) \notin \mathbb{Z}/2\mathbb{Z}$, where ξ is a generator of $\mathbb{Z}/2\mathbb{Z} \subset A_M$.*

Proof. Since T is odd, there exists an element $v \in T$ with $(v,v)_T = 2k+1$ for some integer k. Thus $(v,v)_M = 4k+2 \equiv 2 \mod 4$.

Write $div_N(v) = d$; then $div_M(v) = 2d$. Let $v^* = \frac{v}{2d}$, and consider the class $[v^*] \in A_M$. Then $d[v^*] = [\frac{v}{2}]$ is an element of order 2 in A_M, and thus generates a subgroup

$$\langle d[v^*] \rangle \cong \mathbb{Z}/2\mathbb{Z} \subset A_M.$$

Now

$$q(d[v^*]) = d^2 q(v^*) = d^2 \frac{(v,v)_M}{4d^2} = \frac{4k+2}{4} \notin \mathbb{Z}/2\mathbb{Z}.$$

□

2.3.1 p-elementary lattices

Let $p \neq 2$ be prime. A p-elementary lattice M is a lattice such that $A_M \cong (\mathbb{Z}/p\mathbb{Z})^a$. We will need the following existence result:

Theorem 2.3.9. *[RS81, Section 1]*

1. *An even, hyperbolic, p-elementary lattice M of rank r with $p \neq 2$, $r \geq 3$ is uniquely determined by the number $l(A_M)$.*

2. *For $p \neq 2$, a hyperbolic p-elementary lattice with invariants $r, a := l(A_M)$ exists if and only if the following conditions are satisfied: $a \leq r$, $r \equiv 0 \mod 2$, and*

$$\begin{cases} \text{if } a \equiv 0 \mod 2, \text{ then } r \equiv 2 \mod 4 \\ \text{if } a \equiv 1 \mod 2, \text{ then } p \equiv (-1)^{r/2-1} \mod 4. \end{cases}$$

Moreover, if $r \not\equiv 2 \mod 8$, then $r > a > 0$.

Chapter 3

Cubic fourfolds with an involution

Recently, there has been renewed interest in the classification of automorphisms of a cubic fourfold. This has been motivated by two main problems: the irrationality of a general cubic fourfold, and the search for new constructions of IHS manifolds. Adapting the techniques of Nikulin, Mukai and Kondō ([Muk88], [Kon98], [Nik79a])who studied symplectic automorphisms of $K3$ surfaces, one can hope to classify automorphisms of a cubic fourfold $V \subset \mathbb{P}^5$ Hodge theoretically. The Strong Torelli theorem 1.3.3 asserts that automorphisms of V are equivalent to (polarized) Hodge isometries of the middle cohomology $H^4(V, \mathbb{Z})$. Such an isometry in turn determines the lattice of algebraic primitive cycles $A(V)_{\text{prim}} := H^{2,2}(V, \mathbb{C}) \cap H^4(V, \mathbb{Z})_{\text{prim}}$ contained in the cubic V.

The purpose of this chapter is to study the case of involutions of a cubic fourfold V, and to determined the lattice $A(V)_{\text{prim}}$. Our main motivation is to the study of birational transformations of IHS manifolds of $OG10$ type, which will be discussed in Chapter 4. We say that an involution of a cubic fourfold is symplectic if it acts trivially on $H^{3,1}(V)$; such an involution will induce a symplectic birational involution of the associated manifold of $OG10$ type through the construction detailed in 1.3.5. Thus a Hodge theoretic classification of the involutions of V will be one major tool in order to classify symplectic birational involutions in the $OG10$ setting.

Any automorphism of a cubic fourfold $V \subset \mathbb{P}^5$ can be lifted to one of the ambient projective space: there are three possible involutions of V, denoted ϕ_1, ϕ_2, ϕ_3. Here ϕ_i is uniquely characterised by the dimension of the fixed linear subspaces of $\mathbb{P}^5$; ϕ_i fixes complementary linear spaces of codimension $i, 6-i$ respectively. The involutions ϕ_1, ϕ_3 are easily seen to be anti-symplectic, where as ϕ_2 is symplectic. It is easy to write down the equation for a cubic fourfold with specified involution ϕ_i. Identifying the lattice $A(V)_{\text{prim}}$

and the transcendental lattice $T(V) := (A(V)_{\mathrm{prim}})^{\perp}$ is more subtle; it roughly corresponds to identifying a basis of algebraic cycles. The main result of this chapter is the classification of these lattices.

Theorem 3.0.1. *Let V be a general cubic fourfold with ϕ_i an involution of V fixing a linear subspace of $\mathbb{P}^5$ of codimension i. Then either:*

1. *$i = 1$, ϕ_1 is anti-symplectic and $A(V)_{\mathrm{prim}} \cong E_6(2)$, $T(V) \cong U^2 \oplus D_4^3$. The algebraic lattice is spanned by classes of planes contained in V;*
2. *$i = 2$, ϕ_2 is symplectic and $A(V)_{\mathrm{prim}} \cong E_8(2)$, $T(V) \cong A_2 \oplus U^2 \oplus E_8(2)$. The algebraic lattice is spanned by classes of cubic scrolls contained in V;*
3. *$i = 3$, ϕ_3 is anti-symplectic and*

$$A(V)_{\mathrm{prim}} \cong M,\ T(V) \cong U \oplus A_1 \oplus A_1(-1) \oplus E_8(2).$$

The algebraic lattice contains an index 2 sublattice spanned by classes of planes contained in V.

Here M is the unique rank 10 even lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24 \rangle$, as described in §3.2.14.

Case (1) is studied in detail in [LPZ18]; the existence of the involution ϕ_1 is equivalent to the cubic having an Eckardt point. In case (2), the lattice $A(V)_{\mathrm{prim}} \cong E_8(2)$ was identified as part of [LZ22] by purely lattice theoretic considerations. Here we show that this lattice is generated by classes corresponding to cubic scrolls contained in V; in particular V contains 120 pairs of families of cubic scrolls. Case (3) was previously unknown, we approach this case using both lattice theoretic and geometric techniques.

The study of automorphisms Hodge theoretically has many consequences with regards to rationality. It was conjectured by Harris that if the transcendental cohomology $T(V)$ is not induced from a K3 surface, then V is not rational; evidence was given by Hassett [Has99, Has16]. Indeed, all known rational cubic fourfolds have an associated $K3$ surface. Kuznetsov [Kuz10a] proposed another criteria for rationality via the derived category, which was verified for the known examples of rational cubic fourfolds [Kuz16]. This criteria is satisfied exactly when V has an associated $K3$ ([AT14], [BLM$^+$21]). A more generalised notion, introduced in ([Huy17],[Bra20]), is an associated *twisted* $K3$ surface (S, α), where $\alpha \in Br(S)$ (see §3.4.1 for more details). Our next result identifies which cubic fourfolds with an involution have an associated $K3$ or twisted $K3$ surface.

Theorem 3.0.2. *Let V be a cubic fourfold with an involution ϕ_i fixing a linear subspace of $\mathbb{P}^5$ of codimension i.*

1. *For the symplectic involution ϕ_2, V does not have an associated $K3$ surface, or a visible twisted $K3$ surface.*
2. *For the antisymplectic involution ϕ_1, V has an associated twisted $K3$ surface (S, α) for $\alpha \in Br(S)_2$, but does not have an associated $K3$ surface.*
3. *For the antisymplectic involution ϕ_3, V has an associated $K3$ surface.*

It follows that the cubic fourfolds of case (1) and (2) of Theorem 3.0.1 are conjecturely irrational, whereas a cubic V with involution ϕ_3 is expected to be rational. We indeed verify rationality for such a cubic:

Theorem 3.0.3. *Let $\mathcal{M}_{\phi_3}$ be the moduli space of cubic fourfolds with the involution of type ϕ_3. Then $[V] \in \mathcal{M}_{\phi_3}$ is rational.*

The results contained in this chapter can be found also in [Mar23]. Let us briefly outline the content. In §3.1, we recall the necessary results on automorphisms of cubic fourfolds and set up notation. We begin by investigating the geometry of a cubic fourfold with anti-symplectic involutions in §3.2, showing that the lattice $A(V)$ is generated (over $\mathbb{Q}$) by the classes of planes, before identifying the lattices $A(V)_{\text{prim}}, T(V)$ for the cubic in Case (3). In §3.3, we make some complementary remarks, showing that a cubic with symplectic involution cannot contain a plane, but rater $A(V)$ is generated by classes of cubic scrolls. Finally, in §3.4, we discuss the implications of rationality for each family of cubic fourfolds; in particular we prove Corollary 3.0.3.

Remark 3.0.4. Throughout this chapter, all ADE lattices are assumed to be positive definite for ease of notation.

3.1 Preliminaries

Let V be a cubic fourfold, $f \in \text{Aut}(V)$ an involution. Denote the induced action on

$$L := (E_8)^2 \oplus U^2 \oplus A_2$$

by ϕ, via the isomorphism $L \cong H^4(V, \mathbb{Z})_{\text{prim}}$. We denote by $L_{\pm}$ the invariant and coinvariant sublattices of L respectively.

We define the following sublattices of $H^4(V,\mathbb{Z})$:

$$A(V) := H^{2,2}(V) \cap H^4(V,\mathbb{Z}),$$
$$T(V) := (A(V))^{\perp}_{H^4(V,\mathbb{Z})},$$

the lattice of algebraic cycles, and the transcendental lattice, respectively. Note that the integral Hodge conjecture holds for cubic fourfolds [Voi13, Thm 1.4], so every class $v \in A(V)$ is indeed algebraic. We denote by $A(V)_{\mathrm{prim}}$ the lattice $A(X) \cap H^4(X,\mathbb{Z})_{\mathrm{prim}} \subset H^4(X,\mathbb{Z})_{\mathrm{prim}}$.

Proposition 3.1.1. *Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold and $\phi \in \mathrm{Aut}(V)$. Let r denote the rank of L_-.*

1. *The covariant sublattice L_- is 2-elementary, with discriminant group of length $0 \leq a \leq \min\{r, 22-r\}$. Thus*

$$A_{L_-} \cong (\mathbb{Z}/2\mathbb{Z})^a; A_{L_+} \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a.$$

2. *If ϕ is symplectic, L_- has signature $(r,0)$.*

3. *If ϕ is anti-symplectic, L_- has signature $(r-2,2)$.*

Proof. Since $\phi \in \mathrm{Aut}(V) \cong \mathrm{Aut}_{HS}(V,\eta_V)$, ϕ acts as the identity on $\langle \eta_V \rangle$, and thus acts trivially on $A_{\langle \eta_V \rangle} \cong \mathbb{Z}/3\mathbb{Z} \cong A_L$. We see that $H^4(V,\mathbb{Z})_- \cong L_-$, and since $H^4(V,\mathbb{Z})$ is an unimodular lattice, L_- is 2-elementary by Lemma 2.3.1. The description of the discriminant group of L_+ follows from Proposition 2.1.1.

Suppose now that ϕ is symplectic; by definition, ϕ acts trivially on $T(V)$, thus $L_- \subset A(V)$. The class $\eta_V \in A(V)$ is clearly invariant under ϕ, and so $L_- \subset A(V) \cap L$. The claim on the signature follows. Similarly, if ϕ is anti-symplectic, then $T(V) \subset L_-$, but η_V is invariant. $\square$

Definition 3.1.1. We say that a cubic with symplectic involution is general if $A(V)$ is as small as possible, i.e $A(V)_{\mathrm{prim}} \cong L_-$. Similarly, a cubic with anti-symplectic involution is general if $A(V)_{\mathrm{prim}} \cong L_+$.

3.1.1 Moduli of cubic fourfolds with an involution

Let $\mathcal{M}_\phi$ denote the moduli space of cubic fourfolds with an involution of type ϕ, constructed via GIT. Similarly to the theory developed by Dolgachev [Dol96] for K3 surfaces, one can also define $\mathcal{M}_\phi$ as a moduli space of lattice polarised cubic fourfolds. More precisely, let M be a positive definite lattice with a fixed primitive embedding into the primitive lattice L. Assume further that M does not contain any short or long roots. One can define a moduli space $\mathcal{M}_M$ of cubics such that $M \subset H^{2,2}(V) \cap H^4(V,\mathbb{Z})_{\text{prim}} \subset H^4(V,\mathbb{Z})_{\text{prim}} \cong L$ and the composition $M \subset L$ is equivalent to the fixed embeddings. In other words, $\mathcal{M}_M$ is the moduli space of cubics with primitive algebraic lattice equivalent to the prescribed lattice M. We say such a cubic is **M-polarized** [YY21]. Up to passing to a normalization, $\mathcal{M}_M$ is (possibly the complement of some divisors in) a locally symmetric variety $\mathcal{D}_M/\Gamma_M$, where $\mathcal{D}_M$ is the type IV domain associated with the transcendental lattice $T = M_L^{\perp}$. Thus $\dim \mathcal{M}_M = 20 - \text{rank}(M)$. Further, if $M \subset M' \subset L$ (all primitive embeddings), then $\mathcal{M}_{M'} \subset \mathcal{M}_M$; the more algebraic cycles contained in X, the smaller the moduli. To construct $\mathcal{M}_\phi$, for ϕ anti-symplectic, we apply the above construction for $M = L_+$.

In particular, one can consider the loci of *special* cubic fourfolds [Has00] as lattice polarised cubic fourfolds.

Definition 3.1.2. A cubic fourfold V is **special** if it admits a surface S not homologous to a complete intersection, i.e $\mathbb{Z}[\eta_V] \subsetneq A(V)$. A **labeling** of V consists of a rank 2 saturated sublattice $K \subset A(V)$ containing η_V. The discriminant of K is the determinant of the intersection form of K, denoted by d.

Theorem 3.1.2. *[Has00] Let $\mathcal{M}$ denote the moduli of smooth cubic fourfolds. Let $\mathcal{C}_d \subset \mathcal{M}$ denote special cubic fourfolds admitting a labeling of discriminant d. Then $\mathcal{C}_d$ is non-empty if and only if $d > 6, d \equiv 0, 2 \mod 6$. Moreover, $\mathcal{C}_d$ is an irreducible divisor.*

We will see that cubic fourfolds with involutions are very special cubic fourfolds - in other words, the rank of $A(V)_{\text{prim}}$ is large. In particular, we can compute the dimension of $\mathcal{M}_\phi$ by computing the dimension of invariant classes in $H^{3,1}(V)$. This is easy due to the fact that any automorphism of V is induced by a projective transformation of the ambient $\mathbb{P}^5$ that leaves V invariant [MM64, Theorem 1 and 2]. One can linearise such an automorphism, and obtain a complete classification of cyclic automorphisms of cubic fourfolds [GAL11]. One can identify which are symplectic [Fu16]. In particular, for involutions we have the following classification.

Proposition 3.1.3. *[GAL11], [Fu16] Let $V = V(F)$ be a smooth cubic fourfold in $\mathbb{P}^5$, $\phi \in \text{Aut}(V)$ an involution of V. Applying a linear change of co-ordinates, we can diagonalise ϕ, so that*

$$\phi : \mathbb{P}^5 \to \mathbb{P}^5, [x_0, \dots x_5] \mapsto [(-1)^{a_0} x_0, \dots, (-1)^{a_5} x_5],$$

with $a_i \in \{0, 1\}$. Let $a := (a_0, \dots a_5)$, and $d = \dim \mathcal{M}_\phi$. Then either:

1. *$a = (0,0,0,0,0,1)$ and $\phi := \phi_1$ fixes linear subspaces of $\mathbb{P}^5$ of codimension 1 and 5 respectively. We have that $\phi_1$is anti-symplectic, $d = 14$, and*

$$F = g(x_0, x_1, x_2, x_3, x_4) + x_5^2 l_1(x_0, x_1, x_2, x_3, x_4)$$

2. *$a = (0,0,0,0,1,1)$ and $\phi := \phi_2$ fixes linear subspaces of $\mathbb{P}^5$ of codimension 2 and 4 respectively. We have that ϕ_2 is symplectic, $d = 12$ and*

$$F = g(x_0, x_1, x_2, x_3) + x_4^2 l_1(x_0, x_1, x_2, x_3) + \\ + x_4 x_5 l_2(x_0, x_1, x_2, x_3) + x_5^2 l_3(x_0, x_1, x_2, x_3)$$

3. *$a = (0,0,0,1,1,1)$ and $\phi := \phi_3$ fixes two linear subspaces of $\mathbb{P}^5$ of codimension 3. We have that ϕ_3 is anti-symplectic, $d = 10$ and*

$$F = g(x_0, x_1, x_2) + x_0 q_0(x_3, x_4, x_5) + x_1 q_1(x_3, x_4, x_5) + x_2 q_2(x_3, x_4, x_5)$$

Here l_i, q_i, g denote homogeneous polynomials of degree 1, 2 and 3, respectively.

Proof. The classification and dimension of the families of such cubics can be found in either [GAL11] or [Fu16]. Let $\Omega = \sum_i (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_5$. A basis for $H^{3,1}(V)$ is given by $\{Res \frac{\Omega}{F^2}\}$, where F is the defining equation for V. One can check whether this is invariant or not for each involution above. $\square$

Corollary 3.1.4. *Let V be a general cubic fourfold with an involution ϕ. Then:*

1. *If ϕ is anti-symplectic, the sublattice $A(V)_{\text{prim}} \cong L_+ \hookrightarrow L$ is a positive definite lattice of rank*

$$r(L_+) = \begin{cases} 6, & \textit{if } \phi = \phi_1 \\ 10, & \textit{if } \phi = \phi_3. \end{cases}$$

2. *If ϕ is symplectic, the sublattice $A(V)_{\text{prim}} \cong L_- \hookrightarrow L$ is a positive definite lattice of rank 8.*

Proof. The claims follow immediately from Proposition 3.1.1 and Proposition 3.1.3. $\square$

Despite both containing many algebraic surfaces, the geometry of cubic fourfolds with a symplectic involution is very different to that of one with an anti-symplectic involution. We will discuss each in turn.

3.2 Geometry of anti-symplectic involutions

In this section, we focus on cubic fourfolds V admitting an anti-symplectic involution. After a change of co-ordinates, we can assume that (V, ϕ_i) has the form given by Proposition 3.1.3, for $i = 1, 3$. In §3.2.1 we will show that such a cubic contains many planes, whose classes span the algebraic cohomology $A(V)$ (over $\mathbb{Q}$). Determining the integral structure is slightly more nuanced; for the case of ϕ_1 this was done in [LPZ18], and we review in §3.2.1.1. The case of ϕ_3 had not been previously studied - we identify the lattices $A(V), A(V)_{\text{prim}}, T(V)$ explicitly in §3.2.2, proving Theorem 3.0.1 for the case (V, ϕ_3).

3.2.1 Existence of planes

We prove that in the anti-symplectic case the algebraic lattice $A(V)$ is spanned (over $\mathbb{Q}$) by the classes of planes. It is straightforward to see that a cubic fourfold with an anti-symplectic involution contains many invariant planes.

Lemma 3.2.1. *Let V be a general cubic fourfold, ϕ an anti-symplectic involution. Then V contains an invariant plane P. Further, (V, P) determines a plane sextic curve C_P and a theta-characteristic on C_P. More precisely:*

1. *if $\phi = \phi_1$, $C_P = L \cup Q$ where L is a line, Q is a smooth quintic curve, κ a non-trivial odd theta characteristic on Q.*

2. *if $\phi = \phi_3$ and $C_P = C \cup D$ where C, D are smooth cubic curves, κ a non-trivial two torsion line bundle on D.*

Conversely, such a triple (L, Q, κ) (respectively (C, D, κ)) determines a cubic fourfold with an anti-symplectic involution of type (V, ϕ_1) (respectively (V, ϕ_3)).

Proof. Keeping the notation of Theorem 3.1.3, we see that we can choose co-ordinates for $\mathbb{P}^5$ such that ϕ is either ϕ_1 or ϕ_3 and the equation for V is as in the theorem.

For the involution ϕ_1, studied in [LPZ18], V contains the cone over a cubic surface S, where S is fixed by the involution. Each of the 27 lines on the cubic surface gives a plane contained in V that is invariant under ϕ_1.

In the case of ϕ_3, one can see from the equation of V that the plane $V(x_0, x_1, x_2)$ is contained in V, and is point-wise fixed by ϕ_3.

Choose this invariant plane P. The linear projection with center P expresses V as a quadric fibration over $\mathbb{P}^2$; indeed, $\mathbb{P}^2$ parametrises the space of $\mathbb{P}^3$-sections of V containing P

(see for example [Voi86]). The blow up $\mathbb{P}^5_P$ of the ambient projective space along P gives a commutative diagram:

$$\begin{tikzcd} V_P \arrow[r] \arrow[dr, "\pi"'] & \mathbb{P}^5_P \arrow[d, "\tau"] \\ & \mathbb{P}^2 \end{tikzcd}$$

where V_P is the strict transform of V, and τ and π are the linear projections with center P. The generic fiber of π is a smooth quadric surface; the degenerate fibers of π are parametrised by a plane sextic C_P, the discriminant curve, where $C_P = V(\det A_i)$ for a matrix A_i depending on ϕ_i for $i = 1, 3$ respectively. A simple analysis of the cases gives:

$$A_1 = \begin{pmatrix} l_3 & l_{34} & 0 & q_3 \\ l_{34} & l_4 & 0 & q_4 \\ 0 & 0 & l_1 & 0 \\ q_3 & q_4 & 0 & f \end{pmatrix}; \ A_3 = \begin{pmatrix} l_3 & l_{34} & l_{35} & 0 \\ l_{34} & l_4 & l_{45} & 0 \\ l_{35} & l_{45} & l_5 & 0 \\ 0 & 0 & 0 & g \end{pmatrix},$$

where each l_i, l_{ij} are linear, q_i quadratic and f, g cubic polynomials. We see that C_P is the union of two curves of the correct degree respectively. We discuss the case of ϕ_3 in detail; the other case is similar.

Let V be a cubic fourfold with the involution ϕ_3 - we can rewrite the equation of V as:

$$l_3x_3^2 + l_4x_4^2 + l_5x_5^2 + 2l_{34}x_3x_4 + 2l_{35}x_3x_5 + 2l_{45}x_4x_5 + g = 0 \tag{3.2.1}$$

where l_i, l_{ij}, g are homogeneous polynomials in x_0, x_1, x_2, with l_i, l_{ij} linear and g degree 3. The discriminant sextic C_P is given by the determinant of the matrix A_3, and so C_P is the union of two smooth cubic plane curves C and D, where

$$\begin{aligned} C &= V(g) \\ D &= V(l_3l_4l_5 - l_{45}^2l_3 + 2l_{34}l_{35}l_{45} - l_{34}^2l_5 - l_{35}^2l_4). \end{aligned}$$

Since D is a determinantal curve, by [Bea00, Prop 4.2] this determines an even theta characteristic κ on the corresponding curve. In particular, since D is an elliptic curve $\kappa^{\otimes 2} = \mathcal{O}_D$. Conversely, suppose we have the triple (C, D, κ). Then by [Bea00, Prop 4.2] we can write D as the determinant of a 3×3 matrix of linear forms in 3 variables. Using this matrix and $C = V(g)$, we can write an equation for a cubic fourfold of the form of equation (3.2.1). This is clearly invariant under the involution ϕ_3. $\square$

The general fiber of $\pi : V_P \to \mathbb{P}^2$ is a smooth quadric surface, and the fiber over a smooth point of the discriminant curve C_P is a quadric cone. We claim that the fiber over a node of

C_P is the union of two planes. This follows from the following elementary lemma, we omit the proof.

Lemma 3.2.2. *Let $P \subset V$ be a cubic fourfold containing a plane, $\pi : V_P \to \mathbb{P}^2$ be the quadric fibration obtained via projection from P. Suppose that $\pi^{-1}(p)$ was a plane with multiplicity 2 (a double plane). Then V is a singular cubic fourfold.*

3.2.1.1 The involution ϕ_1

Cubic fourfolds with the involution of type ϕ_1 were studied intensely in [LPZ18]; admitting such an involution is equivalent to the existence of an Eckardt point $p \in V$. In particular, a cubic fourfold of this kind contains the cone over a cubic surface S. The authors identify generators of the algebraic cohomology as cones over the 27 lines on a cubic surface.

Theorem 3.2.3. *[LPZ18] For a cubic fourfold with an involution (V, ϕ_1) as in [LPZ18], the following hold:*

1. *V is geometrically equivalent to a pair (Y, H) where Y is a cubic threefold, H a hyperplane in $\mathbb{P}^4$.*
2. *V contains 27 planes Π_i passing through the Eckardt point p, corresponding to the 27 lines on the cubic surface $Y \cap H$;*
3. *The primitive algebraic cohomology $A(V)_{\mathrm{prim}} \cong E_6(2)$ (spanned by classes $[\Pi_i] - [\Pi_j]$);*
4. *The transcendental cohomology of V is $T \cong (D_4)^3 \oplus U^2$.*

From Lemma 3.2.1, we see 5 pairs of planes corresponding to the singular points of the discriminant curve $C_P = L \cup Q$. Indeed, the plane P corresponds to a line on the cubic surface, and each pair of planes corresponds to the residual lines of a tritangent plane containing this line.

3.2.1.2 The involution ϕ_3

We wish to prove a similar theorem to Theorem 3.2.3 for cubic fourfolds with the anti-symplectic involution ϕ_3. The situation is this case is slightly more delicate - although the primitive algebraic cohomology is spanned by differences of classes of planes over $\mathbb{Q}$, this is no longer true as integral cohomology.

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold, $\phi := \phi_3$ an antisymplectic involution. Denote by $P \subset V$ the unique point-wise fixed plane, and $\{F_i, F_i'\}_{i=1}^9$ the 9 pairs of planes occurring as the fibers of $\pi : V_P \to \mathbb{P}^2$ over the singular points of the discriminant curve $C_P = C \cup D$. Let the corresponding classes in $H^4(V, \mathbb{Z})$ be denoted by $[P], [F_i], [F_i'] \in A(V)$ for $i = 1, \ldots 9$. Notice that

$$\eta_V \sim [P] + [F_i] + [F_i'].$$

Proposition 3.2.4. *Let V be a cubic fourfold with involution ϕ_3. Then:*

1. *V contains at least 19 invariant distinct planes $P, \{F_i, F_i'\}_{i=1}^9$, where additionally P is point-wise fixed by ϕ_3.*
2. *The classes $\{\eta_V, [P], [F_1], \ldots [F_9]\} \subset H^{2,2}(V)$ span $A(V) \otimes \mathbb{Q}$.*

Proof. One can check that ϕ_3 leaves each plane F_i, F_i' invariant be considering equations for these planes.

We have identified 11 linearly independent algebraic classes, thus $rank A(V) \geq 11$. By Corollary 3.1.4, we see this must be equality. Thus $A(V)$ is determined up to finite index by these classes. □

The identification of the lattices $A(V), A(V)_{\text{prim}}$ and $T(V)$ is done in §3.2.2. In contrast to the previous case, the lattice spanned by the differences of two planes forms an index 2 sublattice of $A(V)_{\text{prim}}$.

3.2.2 The algebraic lattice of (V, ϕ_3)

We will prove Theorem 3.0.1 by identifying the lattices $A(V)$, $L_+ \cong A(V)_{\text{prim}}$, and $T(V)$ for a general such cubic. We briefly outline the strategy. We first consider the lattice spanned by planes contained in V. We show that the primitive lattice spanned by differences of planes $\tilde{K}$ is an index 2 sublattice of $A(V)_{\text{prim}}$. In particular, we will show that a class $y = \frac{[P]+\sum_{i=1}^9 [F_i]}{2}$ belongs to $A(V)$. In §3.2.2.1 we show that the lattice spanned by $\langle \eta_V, [F_1], \ldots [F_9], y \rangle$ is in fact isomorphic to $A(V)$. This allows us to identify the lattice $A(V)_{\text{prim}}$ and $T(V)$ in §3.2.2.2.

For convenience, we collect the lattice invariants for $A(V), A(V)_{\text{prim}}$ and $T(V)$ below - these follow from results in §3.1.1.

Lemma 3.2.5. *Let V be a cubic fourfold with the involution $\phi := \phi_3$. Then:*

1. *$A(V)$ is an odd, positive definite, 2-elementary lattice of rank 11 and discriminant group $A_{A(V)} \cong (\mathbb{Z}/2\mathbb{Z})^a$, where $1 \leq a \leq 10$.*

2. $A(V)_{\text{prim}}$ *is a positive definite even lattice of rank 10, with discriminant group* $A_{A(V)_{\text{prim}}} \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a$.

3. $T(V)$ *is a 2-elementary lattice of signature* $(10, 2)$, *with discriminant group* $A_{T(V)} \cong (\mathbb{Z}/2\mathbb{Z})^a$.

By Proposition 3.2.4, the lattice $A(V) \otimes \mathbb{Q}$ is generated by the cohomology classes $\{\eta_V, [P], [F_1], \ldots [F_9]\}$. We first wish to compute the intersection products of these classes, in order to identify the lattice they generate. We will use the following lemma:

Lemma 3.2.6. *[Voi86] Let* P_1, P_2 *be planes contained in* V, *and denote by* $p_i = [P_i] \in H^4(V, \mathbb{Z})$. *Then:*

1. $\eta_V \cdot p = 1$,

2. $p^2 = 3$,

3. $p_1 \cdot p_2 = \begin{cases} 0 & \text{if } P_1 \cap P_2 = \emptyset, \\ 1 & \text{if } P_1 \cap P_2 = \text{ a point }, \\ -1 & \text{if } P_1 \cap P_2 = \text{ a line.} \end{cases}$

This has the following important consequence:

Corollary 3.2.7. *Let* $p \in H^4(V, \mathbb{Z})$ *such that* $p^2 = 3$, *and* $p \cdot \eta_V = 1$. *Then* p *is represented by a unique plane.*

Using these results as well as our geometrical descriptions for the planes, we can compute the intersection numbers directly.

Lemma 3.2.8. *The intersection products of the classes* $\{\eta_V, [P], [F_1], \ldots [F_9]\}$ *above are given as follows:*

1. $[P] \cdot [P] = [F_i] \cdot [F_i] = \eta_X \cdot \eta_X = 3$ *for* $1 \leq i \leq 9$,

2. $\eta_X \cdot [P] = \eta_X \cdot [F_i] = 1$,

3. $[P] \cdot [F_i] = -1$,

4. $[F_i] \cdot [F_j] = 1$ *for* $1 \leq i \neq j \leq 9$.

Proof. One can compute the intersections using Lemma 3.2.6. □

Remark 3.2.9. Consider the lattice $N = \langle \eta_V, [P], [F_1], \dots [F_9] \rangle \subset A(V)$. This has determinant 2^{12}, and rank 11. By Lemma 3.2.5, N cannot be isomorphic to $A(V)$. Indeed, if $N \cong A(V)$, then the number of generators $l(T(V)) = a$ of the discriminant group of $T(V)$ satisfies $l(T(V)) > 10$; this contradicts Corollary 3.1.4.

Proposition 3.2.10. *Consider the lattice K spanned by the classes $\alpha_i = [F_i]-[F_{i+1}], 1 \leq i \leq 8$ and $\alpha_9 = [P] + [F_8] + [F_9] - \eta_V$. Then K is a sublattice of $A(V)_{\text{prim}}$ and is isomorphic to $D_9(2)$.*

Proof. Clearly $\alpha_i \in A(V)_{\text{prim}}$ for $1 \leq i \leq 9$; we compute the intersection matrix for the lattice spanned by the $\{\alpha_i\}$ using Lemma 3.2.8. In particular, we see that for $1 \leq i \leq j \leq 9:$

$$\alpha_i \cdot \alpha_j = \begin{cases} 4 & \text{if } i = j, \\ -2 & \text{if } j = i+1, i \neq 8 \\ -2 & \text{if } i = 7, j = 9 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we see the lattice spanned by $\{\alpha_i\}_{i=1}^{9}$ is isomorphic to $D_9(2)$. $\square$

Consider the class $\delta := \eta_V - 3[P]$; we see that $\delta \in A(V)_{\text{prim}}$, and the lattices $\langle \delta \rangle$ and $K \cong D_9(2)$ are mutually orthogonal in $A(V)_{\text{prim}}$. Thus $A(V)_{\text{prim}}$ is an overlattice of $\langle \delta \rangle \oplus D_9(2):$

$$\langle \delta \rangle \oplus D_9(2) \subset A(V)_{\text{prim}}.$$

We can calculate that $\delta^2 = 24$; it follows that $\tilde{K} := \langle \delta \rangle \oplus D_9(2) \cong \langle 24 \rangle \oplus D_9(2)$. The discriminant group of $\tilde{K}$ is

$$A_{\tilde{K}} = A_\delta \oplus A_{D_9(2)} \cong \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^8,$$

whereas $A_{A(V)_{\text{prim}}} = \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^a$. By Proposition 2.1.1, $A(V)_{\text{prim}}$ is a nontrivial overlattice of $\tilde{K}$, corresponding to a nontrivial isotropic subgroup $H \subset A_{\tilde{K}}$. Since $D_9(2) \hookrightarrow A(V)_{\text{prim}}$ is a primitive embedding with $D_9(2)^\perp \cong \langle \delta \rangle$, we see that the projections

$$H \to A_\delta \cong \mathbb{Z}/24\mathbb{Z}$$
$$H \to A_{D_9(2)} \cong \mathbb{Z}/8\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^8$$

are also embeddings (see 2.2 for a discussion). It follows that $H \cong \mathbb{Z}/4\mathbb{Z}$, or $\mathbb{Z}/8\mathbb{Z}$.

Lemma 3.2.11. *There exists a non trivial isotropic subgroup $H \cong \mathbb{Z}/4\mathbb{Z}$ of $A_{\tilde{K}}$ corresponding to an overlattice $\tilde{K} \subset M \subset A(V)_{\text{prim}}$.*

Proof. The discriminant group of $\langle\delta\rangle$ is generated by $\xi = [\frac{\eta_V - 3P}{24}]$, and $q_\delta(\xi) = \frac{1}{24}$. Let G_D be the Gram matrix for $D_9(2)$ with respect to the basis $\{\alpha_i\}_{i=1}^9$. To find explicit generators of the discriminant group $A_{D_9(2)}$, we proceed as follows: first consider the inverse matrix G_D^{-1}, given below. We consider the linear combinations of $\alpha_1, \dots \alpha_9$ with coefficients given by the columns of G_D^{-1}. Denote them by $\alpha_1^*, \dots \alpha_9^*$, and their image in $A_{D_9(2)}$ by $[\alpha_i^*]$. We see that $\beta := [\alpha_9^*]$ has order 8, thus we can consider β as a generator of $\mathbb{Z}/8\mathbb{Z}$. Note that $q_{D_9(2)}(\beta) = \frac{9}{8}$.

$$G_D^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & 2 & 2 & 2 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & \frac{5}{2} & \frac{5}{2} & \frac{5}{4} & \frac{5}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} & \frac{7}{4} & \frac{7}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{3}{2} & \frac{7}{4} & \frac{9}{8} & \frac{7}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{3}{2} & \frac{7}{4} & \frac{7}{8} & \frac{9}{8} \end{pmatrix}$$

Let $H = \mathbb{Z}/4\mathbb{Z} = \langle 6\xi + 2\beta\rangle$; indeed one can see that $q(6\xi + 2\beta) = 0 \mod 2\mathbb{Z}$, hence H is an isotropic subgroup, and thus corresponds to some overlattice $\tilde{K} \subset M \subset A(V)_{\text{prim}}$. $\square$

Proposition 3.2.12. *Notations as above.*

1. *The class*

$$x = \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + [F_9] - [P]}{2}$$

belongs to $M \subset A(V)_{\text{prim}}$; in particular x is an integral algebraic class in $A(V)$.

2. *The Gram matrix of M (denoted G_M) with respect to the basis $\{x, \alpha_i\}_{i=1}^9$ is given below.*

In particular, the discriminant group of M is $\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$.

$$G_M = \begin{pmatrix} 6 & 2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & 0 \\ 2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & -2 & -2 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 4 \end{pmatrix}.$$

Proof. For the first claim, we have that M is the overlattice of $\tilde{K}$ corresponding to the isotrivial subgroup given by $H = \langle 6\xi + 2\beta \rangle \subset A_{\tilde{K}}$. We see that

$$\begin{aligned} 6\xi + 2\beta &= \frac{\eta_X - 3[P] + 2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7) - \alpha_8 + \alpha_9}{4} \\ &= \frac{2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7) + 2[F_9] - 2[P]}{4} \\ &= \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_7 + [F_9] - [P]}{2} \mod \tilde{K} \end{aligned}$$

It follows that class x belongs to M; in particular, $x \in A(V)_{\mathrm{prim}}$.

For the second claim, we can calculate the intersection matrix with respect to the basis $\{x, \alpha_i\}_{i=1}^{9}$ using Lemma 3.2.8. Let G_M be the matrix with respect to this basis. We see that $\det G_M = 3 \times 2^{10}$; in order to compute the discriminant group, we compute the inverse matrix G_M^{-1}, and consider the column vectors x^*, α_i^*. It is clear to see that there are no elements of order 4; thus the discriminant group of $A_M \cong \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$. □

Remark 3.2.13. Consider the linear combination $y := x + [F_2] + [F_4] + [F_6] + [F_8] + [P] \in A(V)$. We see that

$$y := \frac{[P] + [F_1] + [F_2] + [F_3] + [F_4] + [F_5] + [F_6] + [F_7] + [F_8] + [F_9]}{2}$$

is thus an element of the integral lattice $A(V)$.

We are now ready to identify the lattices $A(V), A(V)_{\mathrm{prim}}, T(V)$; more specifically we prove the following:

Theorem 3.2.14. *Let N be the lattice generated by $\{\eta_V, y, [F_1], \dots [F_9]\}$; and M be the lattice generated by $\{x, \alpha_i\}$ for $1 \leq i \leq 9$. Then:*

1. *The lattice of algebraic cycles $A(V)$ is isomorphic to N.*
2. *The lattice $A(V)_{\mathrm{prim}} \cong L_+$ is isomorphic to the lattice M.*
3. *The lattice $T(V)$ is isomorphic to the lattice $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$.*

This will conclude the proof of Theorem 3.0.1.

3.2.2.1 The lattice $A(V)$

Consider the lattice $A(V) \subset H^4(V, \mathbb{Z})$; this is an odd, 2-elementary lattice of rank 11. Let N be the lattice spanned by $\eta_X, y, [F_1], \dots [F_9]$, where y is the class as in Remark 3.2.13; we claim that $N = A(V)$. Let G_N be the Gram matrix of N (with respect to $\eta_V, y, [F_1], \dots [F_9]$), with entries calculated according to Lemma 3.2.8. The inverse matrix is given below. Denote by $\eta^*, y^*, [F_1]^*, \dots [F_9]^*$ the dual basis of N^*, given as linear combinations of the elements $\{\eta_X, y, [F_1], \dots [F_9]\}$ with coefficients given by the column vectors of G_N^{-1}. By abuse of notation, $\eta^*, y^*, [F_1]^*, \dots [F_9]^*$ also denote the corresponding elements in $A_N = N^*/N$. It is straightforward to check that A_N is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{10}$ and $\{\eta^*, [F_1]^*, \dots [F_9]^*\}$ is a basis.

$$G_N^{-1} := \begin{pmatrix} \frac{3}{2} & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -\frac{5}{2} & 6 & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} \\ 1 & -\frac{5}{2} & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} & 1 \\ 1 & -\frac{5}{2} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \frac{3}{2} \end{pmatrix}$$

Lemma 3.2.15. *The discriminant group A_N of N is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{10}$. The discriminant bilinear form*

$$b_N : A_N \times A_N \to \mathbb{Q}/\mathbb{Z}$$

is given by the matrix with every entry equal to $1/2$ *(with respect to the basis* $\{\eta^*, [F_1]^*, \dots [F_9]^*\}$ *of* A_N*).*

Proposition 3.2.16. *The lattice* N *generated by* $\eta_V, y, [F_1], \dots [F_9]$ *is saturated in* $A(V) = H^4(V, \mathbb{Z}) \cap H^{2,2}(V)$*.*

Proof. Assume the natural embedding $N \subset A(V)$ is not saturated. Then it factors as $N \subsetneq \mathrm{Sat}(N) = A(V) \subset H^4(V, \mathbb{Z})$, where $\mathrm{Sat}(N)$ denotes the saturation of N in $H^4(V, \mathbb{Z})$. Thus, $\mathrm{Sat}(N)$ is a nontrivial overlattice of N and corresponds to a nontrivial isotropic subgroup of $(A_N, b_N : A_N \times A_N \to \mathbb{Q}/\mathbb{Z})$. Elements in A_N are given by linear combinations of $\eta^*, [F_1]^*, \dots [F_9]^*$ with coefficients either 0 or 1; isotropic elements must have an even number of non-zero coefficients. There are 9 cases to consider. For example, let us suppose that $[F_i]^* + [F_j]^*$ is contained in the isotropic subgroup. From the expression of G_N^{-1} it is easy to see that $[F_i]^* + [F_j]^* \equiv \frac{1}{2}([F_i] + [F_j]) \mod N$. It follows that the element $\frac{1}{2}([F_i] + [F_j])$ belongs to $A(V)$ (i.e $[F_i] + [F_j]$ is divisible by 2). The other cases are similar. In conclusion, $N \neq \mathrm{Sat}(N)$ if and only if at least one of the following elements is 2-divisible in $A(V)$:

1. $[F_i] + [F_j]$
2. $[F_i] + [F_j] + [F_k] + [F_l]$
3. $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n]$
4. $[F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p] + [F_q]$
5. $\eta + [F_i]$
6. $\eta + [F_i] + [F_j] + [F_k]$
7. $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m]$
8. $\eta + [F_i] + [F_j] + [F_k] + [F_l] + [F_m] + [F_n] + [F_p]$
9. $\eta + [F_1] + [F_2] + [F_3] + [F_4] + [F_5] + [F_6] + [F_7] + [F_8] + [F_9]$.

for $1 \leq i, j, k, l, m, n, p, q \leq 9$ distinct. Let us do a case by case analysis.

1. Write $[F_i] + [F_j] = 2\sigma$ for some $\sigma \in A(V)$. It is easy to see that $\sigma^2 = 2$; by [Voi86, Sect. 4 Prop. 1], this implies that V is a singular cubic fourfold, clearly a contradiction.

2. Write $[F_i]+[F_j]+[F_k]+[F_l]=2\sigma$, and consider the element $2\sigma-2[F_j]-[2F_l]\in A(V)$. This is clearly divisible by 2; write $2\widetilde{\sigma}=2\sigma-2[F_j]-2[F_l]$. Then $\widetilde{\sigma}^2=2$, again a contradiction.

3. Write $[F_i]+[F_j]+[F_k]+[F_l]+[F_m]+[F_n]=2\sigma$, and consider the element $2\sigma-2[F_j]-2[F_l]-2[F_n]\in A(V)$. This is clearly divisible by 2; write $2\widetilde{\sigma}=2\sigma-2[F_j]-2[F_l]-2[F_n]$. It is easy to see that $\eta_V\cdot\widetilde{\sigma}=0$, implying that $\widetilde{\sigma}$ is even. On the other hand, $\widetilde{\sigma}^2=3$, a contradiction.

4. Let r be the index such that $\{i,j,k,l,m,n,p,q,r\}=\{1,\dots 9\}$. Write $[F_i]+[F_j]+[F_k]+[F_l]+[F_m]+[F_n]+[F_p]+[F_q]=2\sigma$; note that $2y-2\sigma=([P]+[F_r])$. Thus $[P]+[F_r]$ is divisible by 2; write $[P]+[F_r]=2\widetilde{\sigma}$. It is easy to see that $\widetilde{\sigma}\cdot\eta_V=1=\widetilde{\sigma}^2$. Then V is a special cubic fourfold labeled by the rank 2 lattice generated by η_V and $\widetilde{\sigma}$. The corresponding discriminant is $d=2$. By [Has00, Sect 4.4] this cannot happen for the smooth cubic V.

5. Write $\eta+[F_i]=2\sigma$ for some $\sigma\in A(V)$. Again, $\sigma^2=2$, a contradiction.

6. Write $\eta+[F_i]+[F_j]+[F_k]=2\sigma$. It is easy to see that $\eta_V\cdot\sigma=3,\sigma^2=6$. Then V is a special cubic fourfold labeled by the rank 2 lattice generated by η_V,σ, with determinant $d=9$. Since $d\neq 0,2 \mod 6$, no such V exists, by [Has00].

7. Write $\eta+[F_i]+[F_j]+[F_k]+[F_l]+[F_m]=2\sigma$. Let n,p,r,s be the indices such that $\{i,j,k,l,m,,n,p,r,s\}=\{1,\dots 9\}$. Consider the element

$$2\sigma-2y+2[F_s]=\eta-[P]-[F_n]-[F_p]-[F_r]+[F_s];$$

we see that $\eta-[P]-[F_n]-[F_p]-[F_r]+[F_s]$ must be divisible by 2. Write $\eta-[P]-[F_n]-[F_p]-[F_r]+[F_s]=2\widetilde{\sigma}$. We see that $\eta_V\cdot\widetilde{\sigma}=0$, and $\widetilde{\sigma}^2=2$, again a contradiction.

8. Let r,s be the indices such that $\{i,j,k,l,m,,n,p,r,s\}=\{1,\dots 9\}$, and write $\eta+[F_i]+[F_j]+[F_k]+[F_l]+[F_m]+[F_n]+[F_p]=2\sigma$. Consider the element $2\sigma-2y=\eta-[F_r]-[F_s]-[P]\in A(V)$. Thus $\eta-[F_r]-[F_s]-[P]$ is divisible by 2; write $\eta-[F_r]-[F_s]-[P]=2\widetilde{\sigma}$. One can check that $\eta_V\cdot\widetilde{\sigma}=0$, implying that $\widetilde{\sigma}$ is even. On the other hand, $\widetilde{\sigma}^2=1$, a contradiction.

9. Write $\eta+[F_1]+[F_2]+[F_3]+[F_4]+[F_5]+[F_6]+[F_7]+[F_8]+[F_9]=2\sigma$. Then $2\sigma-2y=\eta-[P]$; thus $\eta-[P]$ is divisible by 2. Write $2\widetilde{\sigma}=\eta-[P]$. It is clear that $\eta\cdot\widetilde{\sigma}=1=\widetilde{\sigma}^2$, again a contradiction as in (4).

$\square$

Corollary 3.2.17. *We have that $A(V) \cong N$ where N is the lattice given above. In particular,*

$$A_{A(V)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}.$$

3.2.2.2 The primitive algebraic cohomology

The above description of $A(V)$ allows us to identify the discriminant group of $A(V)_{\text{prim}}$, and in turn the lattice almost immediately.

Proposition 3.2.18. *The discriminant group of $A(V)_{\text{prim}}$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$. Further, the lattice $A(V)_{\text{prim}}$ is isomorphic to the lattice M.*

Proof. By definition, $A(V)_{\text{prim}} = \langle \eta_V \rangle^{\perp} \subset A(V) \cong N$. Since $A_N \cong (\mathbb{Z}/2\mathbb{Z})^{10}$ and $A_{\langle \eta \rangle} \cong \mathbb{Z}/3\mathbb{Z}$, the first claim follows by Proposition 2.1.1. For (2); if $M \neq A(V)_{\text{prim}}$, then $A(V)_{\text{prim}}$ is a non-trivial overlattice of M, corresponding to a non-trivial isotropic subgroup $H \subset A_M \equiv \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$ by Proposition 2.1.1. This would imply that $A_{A(V)_{\text{prim}}} \cong H^{\perp}_{A_M}/H$, which is impossible. $\square$

3.2.2.3 The transcendental cohomology

Next we study the transcendental lattice $T(V) \cong L_{-}$. Recall that for a 2-elementary lattice S, we define the invariant $\delta(S) \in \{0, 1\}$ to be 0 if $q_S : A_S \to \mathbb{Q}/2\mathbb{Z}$ takes values in $\mathbb{Z}$, and 1 otherwise.

Lemma 3.2.19. *The invariants of the transcendental lattice $T(V)$ are computed as follows:*

1. *$T(V)$ is an even lattice of rank 12. The signature is $(10, 2)$.*

2. *$A_{T(V)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. In particular, $T(V)$ is 2-elementary and $\mathrm{discr}(T(V)) = 1024$.*

3. *$T(V)$ has $\delta(T(V)) = 1$.*

The lattice $T(V)$ is isomorphic to the orthogonal direct sum $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$.

Proof. The first claim follows from Lemma 3.2.5. Recall that $A_{A(V)_{\text{prim}}} = \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{10}$, and that $A(V)_{\text{prim}} \hookrightarrow A_2 \oplus U^3 \oplus E_8^2$ primitively. By Lemma 2.2.1, this is equivalent to

$$q_{A(V)_{\text{prim}}}|_{\mathbb{Z}/3\mathbb{Z}} \cong q_{A_2},$$
$$q_{A(V)_{\text{prim}}}|_{(\mathbb{Z}/2\mathbb{Z})^{10}} \cong -q_{T(V)};$$

in particular $A_{T(V)} \cong (\mathbb{Z}/2\mathbb{Z})^{10}$. We see that $\delta(T(V)) = 1$ by computing the values of $q_T = -q_{A(V)_{\text{prim}}}|_{(\mathbb{Z}/2\mathbb{Z})^{10}}$.

The indefinite, 2-elementary lattice $T(V)$ is classified uniquely up to isomorphism by the signature, $\delta(T(V))$, and $l(T(V))$, where $l(T(V)) = 10$ is the minimum number of generators of A_T (Theorem 2.3.4). The lattice $E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U$ has the same invariants, and thus they are isomorphic. □

This concludes the proof of Theorem 3.0.1, in particular for the case of a cubic fourfold V with the anti-symplectic involution ϕ_3.

3.3 Geometry of symplectic involutions

A cubic fourfold V with the symplectic involution ϕ_2 was studied briefly as part of [LZ22], where they identified $A(V)_{\text{prim}} \cong E_8(2)$ via lattice theory. The geometry was not explored - here we make a couple of complementary remarks, distinguishing this case from the anti-symplectic situation. In particular, we prove that such a cubic cannot contain a plane, and further the lattice $A(V)_{\text{prim}}$ is generated by classes that correspond to cubic scrolls contained in V.

3.3.1 Non-existence of planes

Let V be a general cubic fourfold with a symplectic involution $\phi := \phi_2$; we first show that V contains no planes. We can detect the existence of a plane via the discriminant group of the algebraic primitive cohomology.

Lemma 3.3.1. *Let V be a cubic fourfold containing a plane P. Then P determines a non trivial class $\bar{\delta} \in A_{A(V)_{\text{prim}}}$ of order 3. Moreover, any two planes P_1, P_2 determine the same class, i.e $\bar{\delta}_1 = \bar{\delta}_2 \in A_{A(V)_{\text{prim}}}$.*

Proof. Let p denote the class of the plane P in $H^4(V, \mathbb{Z})$. Consider the class $\delta := 3p - \eta_V$; then $\delta \in A(V)_{\text{prim}}$ with $\delta^2 = 24$. Let $\alpha \in A(V)_{\text{prim}}$; we have that $\alpha \cdot (3p - \eta) = 3\alpha \cdot p$. Thus $div_{A(V)_{\text{prim}}}(\delta) = 3$.

Note that δ is primitive: if $\delta = kw$ for some $w \in A(V)_{\text{prim}}, k \in \mathbb{Z}$, then k must divide $div_{A(V)_{\text{prim}}}(\delta) = 3$. On the other hand, $k^2 = 9$ does not divide 24; thus $k = 1$.

Thus δ is primitive of divisibility 3, and so $\delta^* = \frac{\delta}{3} \in A(V)^*_{\text{prim}}$, where

$$A(V)^*_{\text{prim}} = \{y \in A(V)_{\text{prim}} \otimes \mathbb{Q} | y \cdot x \in \mathbb{Z} \text{ for all } x \in A(V)_{\text{prim}}\}.$$

Let $\bar{\delta}$ be the image of δ^* in $A_{A(V)_{\text{prim}}}$; then $\bar{\delta}$ is a nontrivial element of order three.

Now suppose V contains two planes; denote their cohomology classes by p_1, p_2. Then in turn this determines δ_1^*, δ_2^* as above. Note that $\delta_1^* - \delta_2^* = p_1 - p_2 \in A(V)_{\text{prim}}$; it follows that the images of δ_1^* and δ_2^* in $A_{A(V)_{\text{prim}}}$ coincide. $\square$

Corollary 3.3.2. *Let V be a general smooth cubic fourfold with the symplectic involution ϕ. Then V does not contain a plane.*

Proof. From [LZ22], the general such cubic has $A(V)_{\text{prim}} \cong E_8(2)$; the determinant is 2^8. In particular, there are no non-trivial elements of $A_{A(V)_{\text{prim}}}$ of order 3. $\square$

3.3.2 Existence of cubic scrolls

Let $v \in A(V)_{\text{prim}} \cong E_8(2)$ be an element with $v^2 = 4$. Then $K = \langle \eta_V, v \rangle$ gives a labeling of V with determinant 12; the lattice $E_8(2)$ has 240 such elements. Indeed, the elements v correspond exactly to the roots of the unscaled lattice E_8. Thus a general cubic fourfold with a symplectic involution belongs to the Hassett divisor $\mathcal{C}_{12}$; this is the closure of the locus of cubic fourfolds containing a cubic scroll. Equivalently, a general $[V] \in \mathcal{C}_{12}$ has a hyperplane section with at least 6 double points in linear position [HT10, Proposition 23]. We will show that this is true for the general cubic with symplectic involution:

Theorem 3.3.3. *Let V be a general cubic fourfold with the symplectic involution ϕ.*

1. *V contains 120 pairs of families of cubic scrolls $\{T_i, T_i'\}_{i=1}^{120}$ whose cohomology classes satisfy $[T_i] + [T_i'] = 2\eta_V$.*

2. *The lattice $A(V)_{\text{prim}} \cong E_8(2)$ is generated by classes $\alpha_i := [T_i] - \eta_V$.*

In order to prove Theorem 3.3.3, we will need to investigate the geometry of such a cubic fourfold V more closely. The involution ϕ fixes a line $l \subset V$, and a cubic surface $S := \Pi \cap V$, where $\Pi \cong \mathbb{P}^3$ is the complimentary subspace of $\mathbb{P}^5$ also point-wise fixed by the involution (Proposition 3.1.3). Let $\pi : Bl_l V \to \Pi$ be the linear projection from l to the disjoint linear subspace Π; this is a conic fibration. The discriminant locus parametrising singular fibers is given by:

$$\det \begin{pmatrix} l_1(x_0, \ldots, x_3) & l_2(x_0, \ldots, x_3) & 0 \\ l_2(x_0, \ldots, x_3) & l_3(x_0, \ldots, x_3) & 0 \\ 0 & 0 & g(x_0, \ldots x_3) \end{pmatrix} = 0,$$

and is thus the union of the fixed cubic surface $S = V(g) \subset \Pi$ and a quadric surface $Q = V(l_1 l_3 - l_2^2)$. Denote the intersection $S \cap Q = C$; this is then a genus 4 space curve, parametrising the fibers that are double lines.

Lemma 3.3.4. *Let V be a cubic fourfold with symplectic involution ϕ, l the point-wise fixed line contained in V. Let $H \subset \mathbb{P}^5$ be a general hyperplane containing l, and denote $Y := H \cap V$ the hyperplane section. Then Y is smooth, and the discriminant locus of the linear projection of Y from $l \subset Y$ is the union of a smooth conic Z and a smooth cubic plane curve E.*

Proof. Notations as above. Since Π, l are complimentary linear subspaces, $\Gamma := H \cap \Pi \cong \mathbb{P}^2$. Note that ϕ induces an involution $\phi|_H : H \to H$ whose fixed locus is $l \cup \Gamma$. The cubic threefold Y is invariant under the involution, and from the equation of Y one can check that Y is smooth. In particular, Y has equation

$$x_4^2 L_1(x_0, x_1, x_2) + 2x_4 x_5 L_2(x_0, x_1, x_2) + x_5 L_3(x_0, x_1, x_2) + G(x_0, x_1, x_2) = 0,$$

where L_i are linear and G has degree 3. Consider the restriction of $\pi : Bl_l V \to \Pi$ to the proper transform of Y; we have a conic bundle $\pi_Y : Bl_l Y \to \Gamma$, with discriminant curve

$$\det \begin{pmatrix} L_1 & L_2 & 0 \\ L_2 & L_3 & 0 \\ 0 & 0 & G \end{pmatrix} = 0.$$

This is the union of the plane cubic $E := V(G(x_0, x_1, x_2)) = S \cap \Gamma$ and $Z := V(L_1 L_3 - L_2^2) := Q \cap \Gamma$; both are smooth for a general hyperplane H. □

We claim that if we choose a hyperplane containing $l \subset H \subset \mathbb{P}^5$ such that the discriminant locus of π_Y consists of curves E, Z that are tangent at a point p, then $Y := V \cap H$ is singular.

Lemma 3.3.5. *Let Y be a cubic threefold with an involution fixing a line l and consider the discriminant locus $E \cup Z$ of $\pi_Y : Bl_l Y \to \mathbb{P}^2$ as above. Suppose that the conic Z is tangent to the cubic curve E at a point $p \in \mathbb{P}^2$. Then Y has (at least) two nodes interchanged by the involution.*

Proof. Without loss of generality, assume that Z is given by $x_0 x_1 - x_2^2 = 0$, and suppose that Z and E are tangent at the point $p = [p_0, p_1, p_2]$. Note that this implies that:

$$\frac{\partial G}{\partial x_0}(p) = p_1,\ \frac{\partial G}{\partial x_1}(p) = p_0,\ \frac{\partial G}{\partial x_2}(p) = -2p_2. \tag{3.3.1}$$

The equation for Y is given as

$$x_4^2x_0 + 2x_4x_5x_2 + x_5x_1 + G(x_0, x_1, x_2) = 0.$$

Taking partial derivatives, we see that Y has two nodes, interchanged by the involution, at the points $[p_0, p_1, p_2, \pm\sqrt{-p_1}, \pm\sqrt{-p_0}]$. $\square$

Proposition 3.3.6. *Let V be a general smooth cubic fourfold with a symplectic involution. Then V contains 120 pairs of families of cubic scrolls T_i, T_i' whose classes satisfy $[T_i] + [T_i'] = 2\eta_X$.*

Proof. Notations as above. Let $C \subset \Pi$ be the intersection of $Q \cap S \subset \Pi \subset \mathbb{P}^5$, the genus 4 curve as above. Then there are 120 tritangent planes to C, denoted by $\Gamma_i \subset \Pi \cong \mathbb{P}^3$ with $\Gamma_i \cong \mathbb{P}^2$. Since the intersection points of $\Gamma_i \cap C$ are the intersection points $\Gamma \cap Q \cap S$, we must have that the conic $Z := \Gamma \cap Q$ is tangent to the cubic curve $E := \Gamma \cap S$ in three points. Let $H_i = span\{l, \Gamma_i\} \subset \mathbb{P}^5$; by Lemma 3.3.5, $Y_i := H_i \cap V$ has three pairs of nodes $\{p_i, q_i\}$ with $\iota(p_i) = q_i$. Using the involution, one can check that these nodes are in general position, and so the existence of the cubic scrolls follows by [HT10, Proposition 23]. $\square$

Remark 3.3.7. The existence of a cubic scroll implies the existence of a rational curve on the Fano variety of lines $F(V)$ parametrising the lines in the ruling of T_i. The induced symplectic involution on $F(V)$ was studied in [Cam12].

Proposition 3.3.8. *Let M be the lattice spanned by $\{\alpha_i\}_{i=1}^{120}$, where $\alpha_i := [T_i] - \eta_V$. Then the lattice M is isomorphic to $A(V)_{\text{prim}} \cong E_8(2)$.*

In order to prove Proposition 3.3.8, we must first look at the possible intersection numbers of two cubic scrolls contained in a cubic fourfold. Note two homologous cubic scrolls are necessarily contained in the same hyperplane section (see [Has96, Lemma 2.11]).

Lemma 3.3.9. *Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold containing two non-homologous cubic scrolls T_1, T_2. Then $[T_1] \cdot [T_2] = \tau$ for $\tau \in \{1, 3, 5\}$.*

Proof. The cubic fourfold V has a sublattice $K_\tau := \langle \eta_V, T_1, T_2 \rangle \subset A(V)$, with Gram matrix:

	η_X	T_1	T_2
η_X	3	3	3
T_1	3	7	τ
T_2	3	τ	7

for some $\tau \in \mathbb{Z}$ depending on V. The lattice $A(V)$ is positive definite; it follows that the discriminant of K_τ should be positive. We see that $d(K_\tau) = 3(7 - \tau^2 + 6\tau)$, the only values ensuring this is positive are $\tau \in \{0, 1, 2, 3, 4, 5, 6\}$.

Let $\alpha_1 := \eta_V - T_1, \alpha_2 := \eta_V - T_2$; this is a basis for $\langle \eta_V \rangle^\perp_{K_\tau}$. Note that $\alpha_i^2 = 4$, and $\alpha_1 \cdot \alpha_2 = \tau - 3$. Let $v = x\alpha_1 + y\alpha_2$ with $x, y \in \mathbb{Z}$. We will exclude $\tau = 0, 2, 4, 6$ by exhibiting either a short or long root in $\langle \eta_V \rangle^\perp \subset K_\tau$. We have that

$$v^2 = 2(2x^2 + 2y^2 + xy(\tau - 3)). \tag{3.3.2}$$

Let $\tau = 0$, then $v = \alpha_1 + \alpha_2$ has $v^2 = 2$. Similarly for $\tau = 6$, $v = \alpha_1 - \alpha_2$ is a short root. Now let $\tau = 2$; we see that $v = \alpha_1 + \alpha_2$ satisfies $v^2 = 6$. Note that $v = 2\eta_V - T_1 - T_2$; it is easy to check that v has divisibility 3, and is thus a long root. Similarly, for $\tau = 4$ we see that $v = \alpha_1 - \alpha_2$ is also a long root.

For the remaining values of τ, one can check that $\langle \eta_V \rangle^\perp \subset K_\tau$ contains no long or short roots by using standard Diophantine equation techniques. □

3.4 Associated $K3$ surfaces, Hassett divisors, and rationality

In this section we investigate the consequences of Theorem 3.0.1 in terms of rationality. In §3.4.1 we investigate the existence of associated $K3$ surfaces; a cubic fourfold is conjectured to be rational if and only if such an associated $K3$ surface exists. Next in §3.4.2 we investigate whether a cubic with the anti-symplectic involution ϕ_3 is trivially rational, i.e contains two disjoint planes. Finally in §3.4.3 we show that such a cubic fourfold is Hassett maximal; in particular such a cubic is rational.

3.4.1 Associated and twisted $K3$ surfaces

Let V be a smooth cubic surface with a labeling $K_d \subset A(V)$, as in Definition 3.1.2.

Definition 3.4.1. A polarised $K3$ surface (S, L) of degree d is **associated** to V if there exists an isomorphism of Hodge structures

$$K_d^\perp \cong H^2(S, \mathbb{Z})_{\text{prim}}(-1),$$

where $H^2(S, \mathbb{Z})_{\text{prim}}$ is orthogonal to L in $H^2(S, \mathbb{Z})$.

A cubic fourfold $V \in \mathcal{C}_d$ has such an associated $K3$ surface if and only if d satisfies the following condition [Has00]:

$$d \text{ even and not divisible by } 4, 9 \text{ or any odd prime } p \equiv 2 \mod 3. \tag{3.4.1}$$

Notice that this implies that as lattices $T(S) \cong T(V)(-1)$, and so a necessary condition for the existence of an associated $K3$ surface is that $T(V)$ embeds primitively into the $K3$ lattice $\Lambda_{K3} \cong U^3 \oplus E_8^2$. It is conjectured that a cubic fourfold is rational if and only if there exists an associated $K3$ surface [Has00, Kuz10a], we investigate the existence of such $K3$'s for cubics with involutions.

Lemma 3.4.1. *Let V be a general cubic fourfold with involution either ϕ_1 or ϕ_2. Then there does not exist an associated $K3$ surface.*

Proof. Consider the involution ϕ_1; the statement is proved in [Laz21, Theorem 1.8]. In fact, the lattice $A(V)_{\text{prim}}$ for such a cubic V is maximal in a certain sense.

Consider next the involution ϕ_2; we show there does not exist a primitive embedding $T(V) \hookrightarrow \Lambda_{K3}$. Recall that in this case $A(V)_{\text{prim}} \cong E_8(2)$, and so $T(V)$ has signature $(2, 12)$ with discriminant group $A_{T(V)} \cong (\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/3\mathbb{Z}$. We also have that $q_{T(V)}|_{\mathbb{Z}/3\mathbb{Z}} = q|_{A_2}$, and $q_{T(X)}|_{(\mathbb{Z}/2\mathbb{Z})^8} = -q_{E_8(2)}$.

Suppose that there exists such a primitive embedding. Since Λ_{K3} is the unique even unimodular lattice with signature $(3, 19)$, by [Nik79b, Prop 1.15.1] the existence of this embedding is equivalent to the existence of an even lattice K of signature $(1, 7)$, discriminant group $A_K \cong A_{T(V)}$ such that $q_K = -q_{T(X)}$. Since $(\mathbb{Z}/2\mathbb{Z})^8 \subset A_K$, by Lemma 2.3.6 the lattice $M := K(1/2)$ is a well-defined integral lattice. Now M is a rank $(1, 7)$ lattice with $A_M \cong \mathbb{Z}/3\mathbb{Z}$.

First, suppose that M is an odd lattice. Then by Lemma 2.3.8, there exists a generator $\xi \in \mathbb{Z}/2\mathbb{Z} \subset A_K$ such that $q_K(\xi) \notin \mathbb{Z}/2\mathbb{Z}$. Since $K^\perp = E_8(2)$ is a 2-elementary lattice such that $q_{E_8(2)}(v) \in \mathbb{Z}/2\mathbb{Z}$ for all $[v] \in A_{E_8(2)}$, this is a contradiction - thus M must be an even lattice. By Theorem 2.3.9, M is an even 3-elementary lattice and is uniquely determined by the rank and $l(A_M) = 1$. Consider the lattice $U \oplus E_6$. This is a 3-elementary lattice with the same invariants as M; thus $M \cong U \oplus E_6$. Hence $K \cong U(2) \oplus E_6(2)$; however $q_M|(\mathbb{Z}/3\mathbb{Z}) \neq -q_{A_2}$; it follows that no such lattice K exists. □

The Kuznetsov conjecture would then imply that a cubic fourfold V with involution ϕ_1 or ϕ_2 is irrational. On the other hand, for V with involution ϕ_3, we can construct a $K3$ surface with transcendental lattice $T(S) \cong T(V)(-1)$ geometrically.

Lemma 3.4.2. *Let V be a general cubic fourfold with involution ϕ_3. Then there exists a primitive embedding $T(V) \hookrightarrow \Lambda_{K3}$.*

Proof. First, assume that $\phi = \phi_3$. We show there exists such a primitive embedding by exhibiting a $K3$ surface with transcendental lattice $T(S) \cong T(V)(-1)$. Let $C \subset \mathbb{P}^2$ be an irreducible general sextic curve with 9 nodes. Let $\pi : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the 9 nodes, and let $r : S \to \widetilde{\mathbb{P}}^2$ be the double cover ramified along the strict transform of C. Then S is a K3 surface; we claim $T(S) \cong E_8(-2) \oplus U \oplus A_1 \oplus A_1(-1)$. Let $h = r^*\pi^*(l)$, where l is the class of a line in $\mathbb{P}^2$, and e_i for $i = 1, \dots 9$ the pullback of the exceptional curves of π. Let $D \in \widetilde{\mathbb{P}^2}$ be the strict transform of C, and denote also by D the ramified curve on S. In particular, $2D \sim 6h - \sum_{i=1}^{9} 2e_i$, and the $NS(S)$ is spanned by the classes $h, e_1, \dots e_9$. Notice that $e_i^2 = -2, h^2 = 2$ and $h \cdot e_i = 0$. Thus the lattice $NS(S)$ is a 2-elementary lattice, with signature $(1, 9)$, $l(A_{NS(S)}) = 10$, and $\delta = 1$ (see Theorem 2.3.2). This determines $NS(S)$ uniquely, and $NS(S) \cong E_8(2) \oplus A_1 \oplus A_1(-1)$. Now by definition $T(S) = (NS(S))^\perp$, and by Lemma 2.2.2 has signature $(2, 10), l(A_{T(S)}) = 10, \delta = 1$. This uniquely determines $T(S)$; in particular, $T(S) \cong E_8(-2) \oplus U \oplus A_1 \oplus A_1(-1)$. $\square$

Recently, Brakkee considered instead associated twisted $K3$ surfaces [Bra20]. Recall that the Brauer group of a scheme S is the group of sheaves of Azumaya algebras modulo Morita equivalence, with multiplication given by tensor product. For references, see [Huy05, Huy09].

For S a complex $K3$ surface, we have that

$$\mathrm{Br}(S) \cong H^2(S, \mathcal{O}_S^*)_{tors} \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho(X)}.$$

Any Brauer class $\alpha \in \mathrm{Br}(S)$ is of the form $\exp(B^{0,2})$, where $B \in H^2(S, \mathbb{Q})$. We can consider the linear map $f_\alpha : T(S) \to \mathbb{Q}/\mathbb{Z}$ given by intersecting with B. This map depends only on α, and one can show that the identification $\alpha \mapsto f_\alpha$ yields an isomorphism $\mathrm{Br}(S) \cong \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$.

Definition 3.4.2. A **twisted $K3$ surface** is a pair (S, α) where S is a $K3$ surface, $\alpha \in \mathrm{Br}(S)$.

More precisely, consider the following condition:

$$d' = dr^2 \text{ for some } d, r \text{ satisfying condition 3.4.1.} \tag{3.4.2}$$

Theorem 3.4.3. *[Bra20, Theorem 2] A cubic fourfold V belongs to the divisor $\mathcal{C}_{d'}$ for d' satisfying (3.4.2) if and only if for every decomposition $d' = dr^2$, V has an associated polarized twisted $K3$ surface (S, L, α) of degree d and order r.*

Here $r := order(\alpha) \in \mathrm{Br}(S)$ and the triple (S, L, α) is associated to V if there is a Hodge isometry $K_d^{\perp} \cong \mathrm{Ker} f_{\alpha}$.

Consider the case $d = r = 2$, so $d' = 8$. The cubic fourfolds contained in $\mathcal{C}_8$ contain a plane, and there is an associated twisted $K3$ surface (S, α). As discussed in [Kuz10b], there is a geometric construction for (S, α) obtained by projecting the cubic fourfold from the plane, and letting S be the double cover of $\mathbb{P}^2$ branched in the discriminant sextic. We call this twisted $K3$ the **visible** twisted $K3$ surface associated to $P \subset V$. By Lemma 3.2.1, a cubic fourfold with anti-symplectic involution contains a plane - we immediately see the existence of such a $K3$ surface.

Corollary 3.4.4. *Let V be a cubic fourfold with anti-symplectic involution ϕ. Then there exists an associated visible twisted $K3$ surface (S, α) with $order(\alpha) = 2$.*

On the other hand, let V be a general cubic fourfold with the symplectic involution ϕ_2. By Corollary 3.3.2, we cannot associate to V a visible twisted $K3$ surface.

3.4.2 The divisor $\mathcal{C}_8$

The cubic fourfolds that contain a plane have been well studied and are central to the original proof of the Torelli theorem [Voi86]. They have a quadric bundle structure, and rationality would follow provided the bundle has a rational section. Here, we note that this is not the case for cubics V with the involution ϕ_3.

Definition 3.4.3. A cubic fourfold V containing a plane P is called trivially rational (see [Gal17]) if the associated quadric bundle $\pi : V_P := Bl_P V \to \mathbb{P}^2$ has a rational section.

In particular, one sees immediately that if a cubic fourfold V contains two disjoint planes, then we have such a section and V is trivially rational. The following theorem gives a criteria for the existence of such a section.

Theorem 3.4.5. *[Has99, Theorem 3.1] A cubic fourfold V containing a plane is trivially rational if and only if there exists a class $T \in A(V)$ with $T \cdot Q$ odd for a smooth fiber Q of π.*

We can use this criteria in our situation; our complete description of the lattice $A(V)$ allows us to conclude no such class exists.

Corollary 3.4.6. *Let V be a general cubic fourfold with the involution $\phi = \phi_3$, and let $[P]$ be the class of the plane as in §3.2.2. Then V is not trivially rational with respect to the quadric fibration $\pi : V_P \to \mathbb{P}^2$.*

Proof. Let Q denote a general fiber of π, i.e a smooth quadric surface. The class $[Q]$ in $H^4(V, \mathbb{Z})$ satisfies $[P] + [Q] = \eta_V$. One can easily see that a $\mathbb{Z}$-linear combination of the basis of $A(V)$ intersects the quadric $[Q]$ evenly, using the intersection matrix in Proposition 3.2.16. $\square$

Let $V \in \mathcal{C}_8$, and suppose that the discriminant curve C_P associated to $V_P \to \mathbb{P}^2$ is smooth. Following Voisin [Voi86], we let (S, α) be associated visible $K3$ surface, where $\alpha \in Br(S)[2]$. This in turn defines a 2-torsion Brauer class $\alpha_V \in \mathrm{Br}(X)$; we have the following result of Kuznetsov [Kuz16, Sect. 4.3].

Lemma 3.4.7. *Let V be a cubic fourfold containing a plane P as above. The following are equivalent:*

1. *there exists a rational section of the quadric fibration $\pi : V_P \to \mathbb{P}^2$;*

2. *the associated Brauer class is trivial, i.e. $\alpha_V = 1 \in \mathrm{Br}(V)$.*

Moreover, the conditions above imply that V is rational.

Corollary 3.4.8. *Let V be a cubic fourfold with anti-symplectic involution ϕ_3. Then the associated Brauer class is non-trivial.*

3.4.3 Hassett maximal cubic fourfolds

Cubic fourfolds with involutions ϕ_1 or ϕ_2 have no associated $K3$ surfaces, and are conjecturely irrational. On the other hand, a cubic fourfold V with the involution ϕ_3 has transcendental lattice coming from a $K3$ surface, and is potentially rational. Despite the rationality not following from the obvious quadric bundle structure, we will establish that V is indeed rational by investigating which Hassett divisors such an V belongs to. Recall the following definition:

Definition 3.4.4. We say that a cubic fourfold V is **Hassett maximal** if

$$V \in \bigcap_{\substack{d>6 \\ d\equiv 0,2 \pmod 6}} \mathcal{C}_d.$$

We denote the locus of Hassett maximal cubic fourfolds by $\mathcal{Z}$.

Lemma 3.4.9. *A Hassett maximal cubic fourfold is rational.*

Proof. A Hassett maximal cubic fourfold necessarily belongs to the divisor $\mathcal{C}_{14}$, which is the closure of the Pfaffian locus. We can conclude that such a cubic is rational by results of [BD85] and the fact that rationality specialises in families [KT19]. $\square$

In this section we prove the following result:

Theorem 3.4.10. *Let $\mathcal{M}_{\phi_3}$ be the moduli space of cubic fourfolds with the involution of type ϕ_3. Then $\mathcal{M}_{\phi_3}$ is contained in the Hassett maximal locus $\mathcal{M}_{\phi_3} \subset \mathcal{Z}$ In particular, $V \in \mathcal{M}_{\phi_3}$ is rational.*

It is known that $\mathcal{Z}$ is non-empty; it contains the Fermat cubic fourfold [YY21, Theorem 1.2]. Further, the authors show that $\dim \mathcal{Z} \geq 13$, by illustrating a lattice M of rank 7 such that the moduli of M-polarized cubic fourfolds $\mathcal{M}_M$ is non-empty, and M contains a labeling of determinant d for every $d > 6, d \equiv 0, 2 \mod 6$. We adapt the method in [YY21] to our situation.

In order to prove Theorem 3.4.10, we will need the following classical results of number theory.

Lemma 3.4.11. *(Lagrange's 4-square theorem) Any positive integer can be expressed in the form $x^2 + y^2 + z^2 + u^2$ for some integers x, y, z, u.*

Lemma 3.4.12. *(Ramanujan) Any positive integer except for 1 and 17 can be expressed in the form $2x^2 + 2y^2 + 2z^2 + 3u^2$ for some integers x, y, z, u.*

Proof of Theorem 3.4.10. We will exhibit a primitive sublattice $\eta_X \in K_d \hookrightarrow A(V)$ with determinant d for every $d > 6, d \cong 0, 2 \mod 6$. Recall that a basis for $A(V)$ is given by

$$\{\eta_V, y, [F_1], [F_2], \ldots [F_9]\}$$

keeping notations of §3.2.2. Denote by $\alpha_1 = [F_1] - [F_2], \alpha_3 = [F_3] - [F_4], \alpha_5 = [F_5] - [F_6], \alpha_7 = [F_7] - [F_8], \beta = [\eta_V] - [P] - [F_9]$, and $\gamma = y - [F_5] - [F_6] - [F_7] - [F_8] - [F_9]$. It is easy to see that the sublattice lattice $\langle \eta_x, \alpha_1, \alpha_3, \alpha_5, \alpha_7, \beta, \gamma \rangle \subset A(V)$ is primitive; indeed, writing each class in the basis of $A(V)$ we see they are linearly independent.

Suppose that $v = x_1\alpha_1 + x_3\alpha_3 + x_5\alpha_5 + x_7\alpha_7 + s\beta + t\gamma$ for integers $x_1, \ldots x_7, s, t$. One can see that $\eta_X \cdot v = s$, and

$$v^2 = 4x_1^2 + 4x_3^2 + 4x_5^2 + 4x_7^2 + 3s^2 + 6t^2.$$

Consider the rank two sublattice $\langle \eta_V, v \rangle \subset A(V)$: its discriminant is given by:

$$d = 3(4x_1^2 + 4x_3^2 + 4x_5^2 + 4x_7^2 + 6t^2) + 8s^2.$$

We will show we can obtain every $d \equiv 0, 2 \mod 6$.

Case 1: $d = 6k$**,** for $k \geq 2$. Let $s = 0$. We need to find suitable integers such that

$$k = 2x_1^2 + 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

- If $k = 2m$, let $x_1 = 1$. Then the lattice $\langle \eta_V, v \rangle$ is primitive, and by Ramanujan's theorem we can find suitable integers such that

$$2(m-1) = 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

- If $k = 2m + 1$, let $t = 1$. Then the lattice $\langle \eta_V, v \rangle$ is primitive, and by Lagrange's 4-square theorem we can find suitable integers such that

$$m - 1 = x_1^2 + x_3^2 + x_5^2 + x_7^2.$$

Case 2: $d = 6k + 2$**.** Since we know $[V] \in \mathcal{C}_8$ (V contains a plane), we can assume that $k \geq 2$. Let $s = 1$. The lattice $\langle \eta_V, v \rangle$ is primitive, and we need to find suitable integers such that

$$k - 1 = 2x_1^2 + 2x_3^2 + 2x_5^2 + 2x_7^2 + 3t^2.$$

For $k \geq 3$ this reduces to Case 1 - we deal with $k = 2$ below.

Case 3: $d = 14$**.** Consider the class $v = y - [F_2] - [F_4] - [F_6] - [F_8]$; then $v^2 = 5$ and $\eta_V \cdot v = 1$. Thus the lattice $\langle \eta_V, v \rangle$ is primitive and of determinant $d = 14$. It is well known that any $[V] \in \mathcal{C}_{14}$ is rational [BD85]. $\square$

3.4.4 Low degree classes

We have seen that a cubic V with the involution ϕ_3 is rational by showing it belongs to the Hassett maximal locus. In particular, such a cubic belongs to $\mathcal{C}_{14}$, the closure of the locus of Pfaffian cubic fourfolds. The Pfaffian locus has been well studied; Beauville and Donagi [BD85] showed that Pfaffian cubic fourfolds are rational. Further, the Pfaffian condition is equivalent to V containing a smooth degree 5 del Pezzo surface [Bea00]. In this section, we show that such a cubic does indeed belong to the Pfaffian locus inside of $\mathcal{C}_{14}$. Our argument is lattice theoretic; it would be interesting to realise the rationality of a cubic fourfold V with the involution ϕ_3 geometrically.

The complement of the Pfaffian locus has been studied by Auel in [Aue21]. More precisely, the complement of the Pfaffian locus inside $\mathcal{C}_{14}$ is contained in the irreducible locus of cubic fourfolds containing two disjoint planes. Using this, description we can prove our next result:

Proposition 3.4.13. *Let V be a general cubic fourfold with the involution $\phi = \phi_3$. Then V is Pfaffian.*

Proof. Suppose that such a cubic fourfold V is not Pfaffian. Then by [Aue21, Theorem 1], V contains two disjoint planes P, P'. Consider the class $v = [P] - [P']$; clearly $v \in A(V)_{\text{prim}}$. Thus we can write $v = a_0 x + \sum_{i=1}^{9} a_i \alpha_i$, where $\{x, \alpha_1, \dots, \alpha_9\}$ is a basis for $A(V)_{\text{prim}} \cong M$ as in Theorem 3.2.14. Let $w \in A(V)_{\text{prim}}$; we can also write $w = b_0 x + \sum_{i=1}^{9} b_i \alpha_i$. Notice that $x \cdot w$ and $\alpha_i \cdot w$ are even, by using the intersection matrix in Proposition 3.2.12. Thus $v \cdot \alpha \in 2\mathbb{Z}$ for all $\alpha \in A(V)_{\text{prim}}$.

Consider the element $\delta := 3[P] - \eta_V$, we see $\delta \in A(V)_{\text{prim}}$. Since P and P' are disjoint, $[P] \cdot [P'] = 0$ and so $v \cdot \delta = 3[P]^2 = 9$, a contradiction. Thus no two planes are disjoint. $\square$

The cubic fourfolds V admitting the involution ϕ_3 thus belong to the intersection of $\mathcal{C}_8 \cap \mathcal{C}_{14}$, first studied in [ABBVA14] and later in [BRS19]. In particular, $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has 5 irreducible components, indexed by the value $[P] \cdot [T] \in \{-1, 0, 1, 2, 3\}$, where $P \subset V$ is a plane and $[T]$ is the class of a small OADP surface (for a general point in $\mathbb{P}^5$ there exists a unique secant line to T; see [BRS19, Def. 1.5]) such that $[T]^2 = 10$ and $[T] \cdot \eta_V = 4$ [BRS19, Theorem 3.4].

Corollary 3.4.14. *[BRS19, Corollary 3.5] Let $V \in \mathcal{C}_{14}$, and $[T] \in H^{2,2}(V, \mathbb{Z})$ such that $[T] \cdot \eta_V = 4$ and $[T]^2 = 10$. Then $[T]$ is represented by a small OADP surface T contained in V.*

Theorem 3.4.15. *Let V be a cubic fourfold with an involution $\phi := \phi_3$. Then V contains a smooth quartic rational normal scroll.*

Proof. Let $[T] = 2\eta_V - y + [F_7] + [F_8] + [F_9]$. One can easily compute that $[T] \cdot \eta_V = 4$ and $[T]^2 = 10$; thus $[T]$ is represented by a small OADP surface. By the proof of [BRS19, Theorem 3.4], there are three possibilities for T:

1. $T = S \cup P'$ where S is a cubic rational normal scroll and P' is a plane;

2. T contains only irreducible components of degree less than or equal to 2;

3. T is an irreducible smooth quartic rational normal scroll.

Let $[P]$ be the class of any plane in V; thus $[P]$ is equivalent to a $\mathbb{Z}$-linear combination of the basis of $A(X)$ given in Proposition 3.2.16. One can compute that $[P] \cdot T = 2k$ for some integer k. We will use this to rule out (1) and (2) above.

1. Suppose that $T = S \cup P'$. Since T is a small OADP, the surfaces S and P' intersect along a line; thus $[S] \cdot [P'] = 0$. Hence we see that $[P'] \cdot [T] = [P']^2 = 3$, a contradiction.

2. Suppose that T contains only irreducible components of degree less than or equal to 2. By the proof of [BRS19, Theorem 3.4], this implies that V contains a pair of skew planes. One can compute that for one such plane $[P] \cdot [T] = -1$, again a contradiction.

Thus $[T]$ is represented by a smooth quartic rational normal scroll. □

Remark 3.4.16. In fact, it follows that a cubic fourfold V with an involution ϕ_3 is contained in the intersection of two of these components. Indeed, for the fixed plane $P \subset V$, we see that $[P] \cdot [T] = 2$, where as $[F_1] \cdot [T] = 0$.

3.5 An associated IHS variety

We saw in §1.3.3 two examples of constructions of IHS manifolds associated to a smooth cubic fourfold $V \subset \mathbb{P}^5$. In this section, we return to the setting of a cubic fourfold with **symplectic** involution $\phi := \phi_2$, and show that such a pair (V, ϕ) determines a IHS variety of a specific type, a so-called Matteini orbifold. More precisely, in [Mat16] Matteini constructs an IHS orbifold from the data of a $K3$ surface equipped with a non-symplectic involution whose quotient is a cubic surface. Such an IHS orbifold is singular; there does not exist a symplectic resolution, and the dimension of the moduli space parametrising IHS varieties of the same deformation type is 13 dimensional. A pair (V, ϕ) determines such an IHS orbifold - we thus describe a codimension 1 family of IHS orbifolds of this type.

In §3.5.1, we first describe how the pair (V, ι) determines such a $K3$ double cover. Next, we outline the construction of the six dimensional IHS variety $\mathcal{P}$ in §3.5.2; we follow [Mat16]. The construction follows the strategy of [MT07], where the authors constructed a four dimensional example of an IHS variety in a similar manner. In particular, both examples are relative compactified Prym varieties, and are equipped with a Lagrangian fibration in Prym varieties. These constructions were generalised in [ASF15]. We describe the locus of IHS varieties $\mathcal{P}$ that are obtained from a cubic fourfold $V \subset \mathbb{P}^5$ with a symplectic involution.

3.5.1 Determining a $K3$ surface

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold with symplectic involution $\phi := \phi_2$. First, we notice that the pair (V, ϕ) is equivalent to the data of a $K3$ surface Z with anti-symplectic involution

τ, such that $S := Z/\langle\tau\rangle$ is a cubic surface. The associated double cover $Z \to S$ is branched in the intersection of the cubic surface S with a quadric cone.

Proposition 3.5.1. *Let (V, ϕ) be a cubic fourfold with symplectic involution. Then V determines a $K3$ double cover $Z \to S$, where $S \subset V$ is the point-wise fixed cubic surface, and the double cover is branched in the intersection of S with a quadric cone.*

Conversely, such a double cover determines a cubic fourfold with a symplectic involution.

Proof. Recall that such a pair (V, ϕ) can be written (after a change of co-ordinates) with equation

$$g(x_0, \dots x_3) + x_4^2 l_1(x_0, \dots x_3) + 2x_4 x_5 l_2(x_0, \dots x_3) + x_5^2 l_3(x_0, \dots x_3) = 0,$$

where $\phi : x_4, x_5 \mapsto -x_4, -x_5$. We have that $\mathrm{Fix}(\phi) = L \sqcup S$ where S is a smooth cubic surface, L the line determined by $x_0 = \dots x_3 = 0$.

Consider the linear projection of $f : Bl_L V \to \Pi$ from the line L onto the disjoint linear subspace $\Pi \cong \mathbb{P}^3$; this induces a conic fibration structure. The discriminant locus is the union of the fixed cubic surface S and a quadric cone $Q \subset \mathbb{P}^3$, where Q has equation

$$q := \det \begin{pmatrix} l_1 & l_2 \\ l_2 & l_3 \end{pmatrix} = 0.$$

The fiber of f over a smooth point of $S \cup Q$ is a pair of lines, and the fiber of a point on the intersection $S \cap Q = \Gamma$ is a double line. Since the cubic fourfold was general, the curve Γ is a smooth genus 4 space curve (the cubic surface does not contain the cone point of Q).

Let $\mathcal{L} := \mathcal{O}_S(1)$, and notice that $q = 0$ restricted to S defines a section of $\mathcal{O}_S(2)$, vanishing on Γ. Since $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_S(\Gamma)$, the data $(S, \Gamma, \mathcal{L}, q)$ determines a double cover

$$\mu : Z \to S$$

branched along Γ. One can verify that Z is a smooth $K3$ surface, and the covering involution is necessarily an anti-symplectic involution of Z. Notice this is the discriminant double cover restricted to S.

Conversely, the data of a double cover $\mu : Z \to S$ branched in the intersection of the cubic surface S with a quadric cone Q is equivalent to the pair (S, Q). Choose co-ordinates on $\mathbb{P}^3$ such that Q has equation $l_1 l_2 - l_3^2 = 0$, and write $S = V(g)$. One can reconstruct the equation of V, and notice that V admits a symplectic involution ϕ. It is easy to see that such a cubic fourfold V is smooth. $\square$

3.5.2 Determining a Matteini orbifold

Let Z be a $K3$ surface with an antisymplectic involution τ such that $Z/\tau = S$ is a cubic surface. In particular,

$$\mu : Z \to S$$

is a double cover branched along $B \subset S$ a genus 4 curve. Note that $B \in |-2K_S|$, and is thus the intersection $B = S \cap Q$ for a quadric surface Q. The moduli space of such $K3$ surfaces is 13 dimensional.

From this data, Matteini constructs a six dimension IHS variety $\mathcal{P}$ of dimension 6, with a Lagrangian fibration $\mathcal{P} \to \mathbb{P}^3$ whose general fiber is a $(1, 1, 2)$ polarised abelian variety [Mat16]. We briefly recall the construction.

Let $C \in \mu^*|-K_S|$ be a generic curve on Z. We consider the sub-linear system $|C| \subset |\mu^*(-K_S)|$ of curves on S. Let $\mathcal{J} := \overline{\mathrm{Jac}}(|C|)$ be the compactified Jacobian of the linear system $|C|$. One can construct $\mathcal{J}$ equivalently as a moduli space of semistable sheaves on the $K3$ surface Z; we refer to [Mat16] for references and more details. In particular, $\mathcal{J}$ is symplectic, with a Lagrangian fibration $\mathcal{J} \to \mathbb{P}^3 \cong |C|$, whose general fiber is the compactified Jacobian over the curve. The covering involution τ induces a regular involution τ^* on $\mathcal{J}$. Matteini defines the relative compactified Prym variety $\mathcal{P}$ as a connected component of the fixed part of an involution η on $\mathcal{J}$; here η restricts to $-\tau^*$ on the smooth fibers of $\mathcal{J} \to |C|$. In particular, he has the following result:

Theorem 3.5.2. *[Mat16] $\mathcal{P}$ is an irreducible holomorphic symplectic orbifold of dimension 6, with $\pi : \mathcal{P} \to |C| \cong \mathbb{P}^3$ a Lagrangian fibration whose general fiber is an abelian threefold with polarisation of type $(1, 1, 2)$. The singularities of the variety $\mathcal{P}$ are explicitly described, and $\chi(\mathcal{P}) = 2283$.*

We call such a variety a **Matteini orbifold**. For a smooth curve $D \in |C|$, the fiber of π over D is the Prym variety $\mathrm{Prym}(D/D')$, where $D \to D'$ is the double cover obtained by restricting $\mu : Z \to S$ to $D \subset Z$.

Corollary 3.5.3. *Let (V, ι) be a cubic fourfold with symplectic involution ι. Then one can construct a unique associated Matteini orbifold $\mathcal{P}_V \to \mathbb{P}^3$.*

Proof. By Proposition 3.5.1, the pair (V, ι) is equivalent to a $K3$ double cover $\mu : Z \to S$ branched in the intersection $S \cap Q$ where Q is the quadric cone. We apply the Matteini construction to obtain $\pi : \mathcal{P} \to \mathbb{P}^3$. Note that the base of this fibration is isomorphic to $\mathbb{P}(|\mathcal{O}_S(1)|)$. For each hyperplane H section of S, we obtain a double cover of curves $\mu : \tilde{C} \to C$

where $C = H \cap S$, and the cover is branched in the six points of $H \cap S \cap Q$. The corresponding fiber of $\pi : \mathcal{P} \to \mathbb{P}(|\mathcal{O}_S(1)|)$ is isomorphic to $\mathrm{Prym}(\tilde{C}, C)$. $\square$

Remark 3.5.4. The moduli space of cubic fourfolds with a symplectic involution is 12 dimension - we obtain a codimension 1 subfamily of Matteini orbifolds.

Chapter 4

Symplectic birational involutions of IHS manifolds of $OG10$ type

As discussed in the introduction, one strategy to discover new deformation classes of IHS manifolds is to study the fixed locus (or closure of the fixed locus) of birational symplectic transformations of a known IHS manifold. As the first step, one can classify possible birational symplectic transformations, by classifying the action on the second cohomology.

In this Chapter, we investigate this question for IHS manifolds of $OG10$ type. We restrict ourselves to the case of involutions: in this setting, the associated moduli space of $OG10$ manifolds with an involution of a given type is a type IV period domain. In this way we obtain variations of Hodge structures of $K3$ type - this is strong evidence that the fixed locus is at least an IHS variety. This does not occur for higher order cyclic groups (see [YZ20], [LPZ18]). We will obtain a full classification of possible symplectic birational involutions of IHS manifolds of $OG10$ type.

Let X be manifold of $OG10$ type, $f \in \mathrm{Bir}(X)$ a symplectic birational involution. As discussed in §1.2.1, we obtain an induced involution on the second cohomology, determining two sublattices $H^2(X,\mathbb{Z})_+, H^2(X,\mathbb{Z})_-$, the invariant and the coinvariant lattice respectively. Vice versa, specifying such sublattices (subject to certain lattice theoretic conditions) determines a symplectic birational transformation of some manifold of $OG10$ type via the Global Torelli Theorem (Theorem 1.2.3). Our main theorem is a classification of symplectic birational involutions of manifolds of $OG10$ type.

Theorem 4.0.1. *Let X be a manifold of $OG10$ type, $f \in \mathrm{Bir}(X)$ a symplectic birational involution.*

1. *Assume that f acts trivially on the discriminant group. Then the pair $H^2(X,\mathbb{Z})_-, H^2(X,\mathbb{Z})_+$ appears below:*

$H^2(X,\mathbb{Z})_-$	$H^2(X,\mathbb{Z})_+$
$E_8(2)$	$U^3 \oplus E_8(2) \oplus A_2$
$D_{12}^+(2)$	$E_6(2) \oplus U^2(2) \oplus A_1 \oplus A_1(-1)$

2. *Assume that f acts non-trivially on the discriminant group, and such that* $\mathrm{rank}(H^2(X,\mathbb{Z})_-) < 12$*. Then the pair $H^2(X,\mathbb{Z})_-, H^2(X,\mathbb{Z})_+$ appears below:*

$H^2(X,\mathbb{Z})_-$	$H^2(X,\mathbb{Z})_+$
$E_6(2)$	$U^3 \oplus D_4^3$
M	$E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2$

Here M is the unique index two overlattice of $D_9(2) \oplus \langle -24 \rangle$ as described in §3.2.12.

3. *Assume that f acts non-trivially on the discriminant group, and such that* $\mathrm{rank}(H^2(X,\mathbb{Z})_-) \geq 12$*. Then the pair $H^2(X,\mathbb{Z})_-, H^2(X,\mathbb{Z})_+$ appears below:*

$H^2(X,\mathbb{Z})_-$	$H^2(X,\mathbb{Z})_+$
G_{12}	$\langle 2 \rangle^3 \oplus \langle -2 \rangle^9$
G_{16}	$\langle 2 \rangle^3 \oplus \langle -2 \rangle^5$

The lattices G_{12}, G_{16} are explicitly described in terms of their Gram matrix, and are of rank $12, 16$ respectively.

Moreover, each involution listed above exists.

We approach the proof of this classification from three distinct vantage points. First we classify symplectic birational involutions acting trivially on the discriminant group by using the same techniques of [Huy16],[Mon16]; we relate these involutions to involutions of the Leech lattice. This recovers the Nikulin type involution with coinvariant lattice $E_8(2)$, and more interestingly, we obtain an involution with coinvariant lattice $D_{12}^+(2)$ that cannot be realised in the case of $K3$ surfaces. Thus we obtain the two involutions listed in (1).

Secondly, we begin our classification of symplectic birational involutions acting non-trivially on the discriminant group by studying involutions that are obtained from cubic fourfolds. We have already seen a Hodge theoretic classification of involutions for a cubic fourfold in Chapter 3. Using the observation of Saccá [Sac21, §3.1] (see also [LPZ18]), one can see that an involution of a cubic fourfold V induces a birational transformation of the corresponding compactified intermediate Jacobian $X := \mathcal{J}_V$, a manifold of $OG10$ type constructed via the method described in §1.3.5. Using the results of Chapter 3, we first exhibit three symplectic birational involutions geometrically. In fact more is done: we use these results to classify all the involutions with the assumptions of (2).

Finally, it remains to be seen whether there exists birational symplectic involutions that act both non-trivially on the discriminant group, with $\mathrm{rank}(H^2(X,\mathbb{Z})_-) \geq 12$. In joint work with Stevell Muller, we show that there are two such possibilities, completing the classification in (3) [MM23a]. The proof is computer aided; we provide brief details here and refer the interested reader to [MM23a, Appendix].

The results contained in this chapter can be found also in [MM23a]. Let us briefly outline the content. We recall the relevant definitions and previous results in §4.1. In §4.2 we begin the proof of Theorem 4.0.1 by considering birational symplectic involutions of an IHS manifold X of $OG10$ type acting trivially on the discriminant group of $H^2(X,\mathbb{Z}) \cong \Lambda$. We embed the lattice Λ_- into the Leech lattice $\mathbb{L}$ and use the classification of involutions of $\mathbb{L}$ [HL90]. Using a corollary of Theorem 1.2.3, we obtain the classification in Theorem 4.0.1 (1). In §4.3.1 we make the relationship between involutions of a cubic fourfold and that of manifolds of $OG10$ type more precise. We obtain a classification of birational symplectic involutions that fix a copy of U. In §4.3.3, we identify a criteria for when this occurs, completing the proof of Theorem 4.0.1(2). In §4.4, we briefly discuss the remaining two involutions, concluding the proof of Theorem 3.

Remark 4.0.2. Throughout this chapter, all ADE lattices are assumed to be **negative definite**, for ease of notation.

4.1 Preliminaries

Let X be an IHS manifold of $OG10$ type, $f \in \mathrm{Bir}(X)$ a symplectic involution. We denote the induced action on

$$\Lambda := (E_8)^2 \oplus U^3 \oplus A_2$$

by ι, via the fixed marking $\eta : H^2(X,\mathbb{Z}) \cong \Lambda$; more precisely, $\iota := \eta_*(f) \in O(\Lambda)$ as in §1.2.1. We denote by $\Lambda_\pm$ the invariant and coinvariant sublattices of Λ respectively. Since f is symplectic, we see that via the marking $\Lambda_- \subset NS(X)$; we say such a pair (X, f) is **general** if $\Lambda_- \cong \mathrm{NS}(X)$. We consider the moduli space of pairs as a moduli space of lattice polarised manifolds of $OG10$ type.

Recall that we define the set of vectors, corresponding to the stably prime exceptional divisors (see Proposition 1.2.1):

$$\mathcal{W}^{pex} := \{v \in \Lambda \mid v^2 = -2\} \cup \{v \in \Lambda \mid v^2 = -6, div_\Lambda(v) = 3\}.$$

Via the marking of X, these vectors correspond to stably prime exceptional divisors of X, which cut out the birational Kähler cone by Theorem 1.2.2. Using the group theoretic version of the Global Torelli Theorem 1.2.3, we have the following Lemma for the structure of Λ_-.

Lemma 4.1.1. *Let X be an irreducible holomorphic symplectic manifold of $OG10$ type. Let $f \in \mathrm{Bir}(X)$ be a birational symplectic involution. Then let $\iota := \eta_*(f) \in O(\Lambda)$ be the induced action. Then the coinvariant lattice Λ_- has the following properties:*

1. *Λ_- is negative definite.*

2. *Λ_- does not contain any short or long roots; i.e $\Lambda_- \cap \mathcal{W}^{pex} = \varnothing$.*

Proof. The first statement follows from the fact that f is symplectic, and that $\Lambda_+ \otimes \mathbb{R}$ contains a big and nef class $\kappa + \iota(\kappa)$ for κ a Kähler class.

We prove (2): suppose that Λ_- contains such a class $v \in \mathcal{W}^{pex}$. The isometry ι preserves the birational Kähler cone (see [Deb20, Prop 3.15] for a more detailed discussion). Let $x \in \mathcal{BK}(X)$. Then $x + \iota(x)$ is also in the interior of $\mathcal{BK}(X) \subset \Lambda_+$. Thus since $\Lambda_+ \perp \Lambda_-$, we have that $((x + \iota(x)), v) = 0$; this implies that $x + \iota(x)$ belongs to a wall. By Theorem 1.2.2, this is a contradiction. □

Theorem 4.1.2. *An involution $\iota \subset O(\Lambda)$ is induced by a symplectic birational transformation if and only if Λ_- is negative definite and*

$$\Lambda_- \cap \mathcal{W}^{pex} = \varnothing.$$

Proof. By Lemma 4.1.1, the conditions are sufficient. Suppose that Λ_- is negative definite and $\Lambda_- \cap \mathcal{W}^{pex} = \varnothing$. By Lemma [GOV20, Lemma 2.13], there exists a signed Hodge structure on Λ such that $\Lambda^{1,1} \cap \Lambda = \Lambda_-$ and $\iota \subset O^+_{sp}(\Lambda)$. Since $\Lambda^{1,1} \cap \mathcal{W}^{pex} = \Lambda_- \cap \mathcal{W}^{pex} = \varnothing$, the decomposition 1.2.2 is trivial, and $\mathcal{C}(X)$ is the Kähler chamber. Therefore, $O^+_{sp}(\Lambda) = O^+_{sp}(\Lambda, \mathcal{C}(X))$. We conclude by Theorem 1.2.3. □

In order to classify symplectic birational involutions of manifolds of $OG10$ type, we will consider two cases corresponding to the induced action of $\iota \in O(\Lambda)$ on the discriminant group

$$A_\Lambda := \Lambda^*/\Lambda \cong \mathbb{Z}/3\mathbb{Z}.$$

It follows that an involution acts by $\iota|_{A_\Lambda} = \pm id_{A_\Lambda}$.

Proposition 4.1.3. *Let $f \in \mathrm{Bir}(X)$ be a symplectic birational involution, and let $\iota = \eta_*(f) \in O(\Lambda)$ the induced isometry of f. Then Λ_- is a negative definite lattice of rank $r \leq 21$, with $\Lambda_- \cap \mathcal{W}_{OG10} = \varnothing$, and the following hold:*

1. *If ι acts trivially on A_Λ, then Λ_- is a 2-elementary, negative definite even lattice determined by the invariants $(r, l(A_{\Lambda_-}), \delta)$.*

2. *If ι acts by $-id|_{A_\Lambda}$ on A_Λ, then Λ_+ is a 2-elementary lattice with signature $(3, 21-r)$.*

Proof. The negative definiteness and the claim that $\Lambda_- \cap \mathcal{W}_{OG10} = \varnothing$ follows from 4.1.1. To prove claim (1), we consider a unimodular lattice L such that $\Lambda \hookrightarrow \Lambda$ is a primitive embedding - for instance, we could take $L \cong \mathbb{B} \oplus U$ where $\mathbb{B}$ is the Borcherds lattice. Let $K = \Lambda^\perp$; since ι acts trivially on the discriminant group A_Λ, we can extend ι to an isometry $\tilde{\iota} \in \varnothing(L)$ acting as the identity on K. It follows that $L_- \cong \Lambda_-$ is 2-elementary by Lemma 2.3.1. Claim (2) is similar: instead consider $\iota' := -\iota$; it follows that Λ_+ is 2-elementary. $\square$

4.2 Involutions acting trivially on the discriminant

Throughout, we let (X, η) be a marked IHS manifold of $OG10$ type. In this section we describe all possible symplectic birational involutions of for such an IHS manifold X. More precisely, we will prove the following:

Theorem 4.2.1. *Let X be an IHS manifold of $OG10$ type, and $f \in \mathrm{Bir}(X)$ be a symplectic birational involution. Suppose that $\eta_*(f)$ acts trivially on the discriminant group A_Λ. Then one of the following holds:*

1. *$H^2(X, \mathbb{Z})_- \cong E_8(2)$ and $H^2(X, \mathbb{Z})_+ \cong U^3 \oplus E_8(2) \oplus A_2$; or*

2. *$H^2(X, \mathbb{Z})_- \cong D_{12}^+(2)$ and $H^2(X, \mathbb{Z})_+ \cong E_6(2) \oplus U^2(2) \oplus A_1 \oplus A_1(-1)$.*

Moreover, both involutions exist.

The strategy to prove Theorem 4.2.1 is as follows: we first consider arithmetic involutions $\iota \in O(\Lambda)$ such that ι acts trivially on A_Λ, and Λ_- is negative definite. We then use techniques of Kondō and Mongardi to embed the covariant lattice Λ_- into the Leech lattice $\mathbb{L}$. Next, we extend the involution ι to one of the Leech lattice $\mathbb{L}$, and use the classification of involutions [HL90] to obtain three candidates. We then discuss case by case and show that only $E_8(2)$ and $D_{12}^+(2)$ are realised as coinvariant lattices Λ_- for an involution of Λ. We then show that they contain no short or long roots, i.e $\Lambda_- \cap \mathcal{W}^{pex} = \varnothing$, and conclude by Theorem 4.1.2 that such an involution ι is induced by a geometric symplectic birational involution $f \in \mathrm{Bir}(X)$ of a manifold X of $OG10$ type.

4.2.1 The Leech Lattice

We reduce the classification of involutions $\iota \in O(\Lambda)$ acting trivially on A_Λ to classifications of involutions of the Leech lattice $\mathbb{L}$. The following result is originally due to [GHV12]; the argument was then reproduced by Huybrechts [Huy16, §2.2].

Proposition 4.2.2. *Let $\iota \in O(\Lambda)$ be an involution acting trivially on A_Λ and such that Λ_- is negative definite and does not contain any short roots. Then there exists a primitive embedding of Λ_- into the Leech lattice $\mathbb{L}$.*

Proof. This is the same argument as in [Huy16, Prop. 2.2]. By Proposition 4.1.3, Λ_- is a negative-definite 2-elementary lattice, with rank $r \leq 21$, and $l(A_{\Lambda_-}) = a$, with $a \leq \min\{r, 24 - r\}$. Our strategy is to first primitively embed Λ_- into $\Pi_{1,25}$, the unique even unimodular lattice of signature $(1, 25)$. By [Nik79b, Cor.1.12.3], such a primitive embedding exists if $a < 24 - r$. In the case $a = 24 - r$, we still have such an embedding provided that Λ_- splits off an A_1 summand (see [Nik79b, Theorem 1.12.2]). This in general is hard to check; instead we follow [GHV12] and [Huy16] and apply this criteria to the lattice

$$S := \Lambda_- \oplus A_1.$$

Note that $l(S) = a + 1 \leq 25 - r = 26 - (r + 1)$, where S has rank $r + 1$. Again by [Nik79b, Theorem 1.12.2], this gives us a primitive embedding

$$\Lambda_- \hookrightarrow S \hookrightarrow \Pi_{1,25} =: \Gamma.$$

Since ι acts trivially on the discriminant group A_{Λ_-}, we can extend to an action ι on Γ by acting as the identity on $(\Lambda_-)^\perp_\Gamma$ by using [Nik79b, Cor. 1.5.2]. Then $\Gamma_+ = \Lambda_-^\perp$ is a nondegenerate even lattice of signature $(1, 25 - r)$.

Consider the positive cone $\mathcal{C} \subset \Gamma \otimes \mathbb{R}$. Let Δ_Γ be the set of (-2)-vectors of Γ, and consider the wall-chamber decomposition of $\mathcal{C}$ with respect to Δ_Γ. We claim that $\Gamma_+ \otimes \mathbb{R}$ intersects one of these chambers. Indeed, if not Γ_+ must be contained in a wall, and there exists a vector $\delta \in \Delta_\Gamma$ such that $\Gamma_+ \subset \delta^\perp$. This implies that $\delta \in (\Gamma_+)^\perp = \Lambda_-$, a contradiction.

Fix an isomorphism $\Gamma \cong \mathbb{L} \oplus U$, where $\mathbb{L}$ is the Leech lattice, and let $w \in \Gamma$ be a standard isotropic generator of U. Call the set of (-2)-vectors with $\delta \cdot w = 1$ the Leech roots. Then the Weyl group of Γ is generated by the standard reflections s_δ for each Leech root δ. Thus there exists a chamber $\mathcal{C}_0 \subset \mathcal{C}$ defined by $\delta \cdot \mathcal{C}_0 > 0$ for all Leech roots δ.

Applying the Weyl group to the embedding $\Lambda_- \hookrightarrow \Gamma$ if needed, we can assume that $\mathcal{C}_0$ is fixed by ι.

Let Co_∞ denote the subgroup of $O(\Gamma)$ that fix $\mathcal{C}_0$. Clearly $\iota \in Co_\infty$, and it is known that Co_∞ fixes $w \in \Gamma$, thus $w \in \Gamma_+$.

Thus we see that we get a primitive embedding $\Lambda_- \hookrightarrow \mathbb{L}$ as the composition:

$$\Lambda_- \hookrightarrow w^\perp \twoheadrightarrow w^\perp / \mathbb{Z} \cdot w \cong \mathbb{L}.$$

$\square$

Corollary 4.2.3. *Assumptions as in Prop. 4.2.2. Then there exists an involution of the Leech lattice $\mathbb{L}$ such that $\mathbb{L}_- \cong \Lambda_-$.*

Proof. Consider the primitive embedding $\Lambda_- \hookrightarrow \mathbb{L}$. We apply [Nik79b, Cor. 1.5.2] to extend ι: indeed, since ι acts trivially on A_{Λ_-}, we can extend ι to an involution of $\mathbb{L}$, with $\mathbb{L}_- \cong \Lambda_-$, acting by the identity on $\Lambda_-^\perp = \mathbb{L}_+$. $\square$

The non-trivial involutions $\iota \in O(\mathbb{L})$ are classified [HL90]:

Proposition 4.2.4. *There exists three conjugacy classes of non-trivial involutions of the Leech lattice $\mathbb{L}$. They are classified by specifying the invariant/anti-invariant sublattices:*

1. $\mathbb{L}_- \cong E_8(2)$, $\mathbb{L}_+ \cong BW_{16}$;
2. $\mathbb{L}_- \cong BW_{16}$, $\mathbb{L}_+ \cong E_8(2)$;
3. $\mathbb{L}_- \cong D_{12}^+(2)$, $\mathbb{L}_+ \cong D_{12}^+(2)$.

We have three possible candidates for Λ_- as above. It remains to be seen whether there exists an involution $\iota \in O(\Lambda)$ whose coinvariant lattice is the given candidate. By Lemma 2.3.3, this is equivalent to the existence of a primitive embedding $\Lambda_- \hookrightarrow \Lambda$. We consider each case in Proposition 4.2.4 in turn.

4.2.2 Case (1)

We can easily see the existence of a primitive embedding $E_8(2) \hookrightarrow \Lambda$, and exhibit an involution of Λ with $\Lambda_- \cong E_8(2)$. Consider the involution defined by interchanging the two copies of E_8, and identity elsewhere. Then $\Lambda_- \cong E_8(2)$ and we have the following result (see [Mor84]).

Proposition 4.2.5. *There exists a primitive embedding $E_8(2) \hookrightarrow \Lambda$. In particular, there exists an involution of Λ such that $\Lambda_- = E_8(2)$.*

Proof. We can explicitly define an involution $\iota \in O(\Lambda)$ with $\Lambda_- \cong E_8(2)$ following Morrison [Mor84]. Let $\phi : E_8^2 \hookrightarrow \Lambda := A_2 \oplus U^3 \oplus E_8^2$ be the primitive embedding. Let $\{c_j^i\}$ with $i = 1, 2, 1 \leq j \leq 8$ be a basis of E_8^2, such that $c_j^1 \in E_8 \oplus (0)$ and $c_j^2 \in (0) \oplus E_8$.

Define an involution ι by:

$$\iota(\phi(c_j^1)) = \phi(c_j^2)$$
$$\iota(\phi(c_j^2)) = \phi(c_j^1)$$

and $\iota(e) = e$ for all elements $e \in (\phi(E_8^2))^\perp$. This is well defined since ϕ is a primitive embedding, and E_8 is unimodular. The coinvariant lattice Λ_- is generated by $\{\phi(c_j^1 - c_j^2)\}_{j=1}^8$, and thus $\Lambda_- \cong E_8(2)$.

□

4.2.3 Case (2)

For the two remaining cases, we will use Lemma 2.3.5 to establish whether or not there exists a primitive embedding of Λ_- into Λ.

Proposition 4.2.6. *There does not exist a primitive embedding $BW_{16} \hookrightarrow \Lambda$. In particular, there is no involution of Λ such that $\Lambda_- \cong BW_{16}$.*

Proof. The Barnes-Wall lattice BW_{16} is an even 2-elementary lattice of signature $(0, 16)$, $a = 8$ and $\delta = 0$. Suppose there exists such an embedding. By above, this is equivalent to the existence of an even lattice N of signature $(3, 5)$, $A_N = (\mathbb{Z}/2\mathbb{Z})^8 \oplus \mathbb{Z}/3\mathbb{Z}$, with $q_N|_{(\mathbb{Z}/2\mathbb{Z})^8}$ taking values in $\mathbb{Z}/2\mathbb{Z}$. We also have that $q_N|_{\mathbb{Z}/3\mathbb{Z}} = q_{A_2}$.

Since $(\mathbb{Z}/2\mathbb{Z})^8 \subset A_N$, we can deduce that $K := N(1/2)$ is a well-defined integral lattice by Lemma 2.3.6. Notice that K has signature $(3, 5)$, and $A_K = \mathbb{Z}/3\mathbb{Z}$.

We claim that K is even; indeed, suppose that K was odd. Then by Lemma 2.3.7, there exists an element $\xi \in \mathbb{Z}/2\mathbb{Z} \subset A_N$ such that $q(\xi) \notin \mathbb{Z}/2\mathbb{Z}$, contradicting our assumption on N.

Thus K is an even lattice with $A_K = \mathbb{Z}/3\mathbb{Z}$, and $(A_K, q_K) \cong (A_{E_6}, q_{E_6})$. Since $q_K = q_{E_6} = -q_{A_2}$ and $A_K \cong A_{A_2}$, by [Nik79b, Prop 1.15.1] there exists a primitive embedding of K into some even unimodular lattice Γ of signature $(3, 7)$. By Milnor's theorem on unimodular forms, we see that no such even unimodular lattice Γ exists. Thus such an N cannot exist. $\square$

4.2.4 Case (3)

Here we show the somewhat surprising result that the lattice $D_{12}^+(2)$ does primitively embed into Λ. Recall that the lattice $D_{12}^+(2)$ is an even, 2-elementary lattice with signature $(0, 12), a = 12$ and $\delta = 1$.

Proposition 4.2.7. *There exists a primitive embedding $D_{12}^+(2) \hookrightarrow \Lambda$. In particular, there exists an involution of Λ such that $\Lambda_- \cong D_{12}^+(2)$.*

Proof. By Lemma 2.3.5, this is equivalent to the existence of a lattice N with signature $(3, 9)$, satisfying the conditions of the Lemma. We will show such a lattice exists. Consider the lattice

$$N = E_6(2) \oplus U^2(2) \oplus A_1 \oplus A_1(-1).$$

This lattice satisfies condition (1) of Lemma 2.3.5; it remains to show the conditions (2), (3) hold. More specifically, we need to show that

1. $q_N|_{\mathbb{Z}/3\mathbb{Z}} \cong q_\Lambda = q_{A_2}$, and that
2. $q_N|_{(\mathbb{Z}/2\mathbb{Z})^{12}} \cong -q_{D_{12}^+(2)}$.

In order to do so, we will calculate the values of the discriminant form for both $A_{D_{12}^+(2)}$ and A_N. We will use the fact that N is also isomorphic to $E_6(2) \oplus U(2) \oplus (A_1 \oplus A_1(-1))^2$, by Nikulin's classification of 2-elementary lattices.

First, note that $A_{E_6(2)} = \mathbb{Z}/6\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^5$. Let $\alpha_1, \ldots \alpha_6$ be a basis for $K := E_6(2)$, with Gram matrix:

$$G_{E_6(2)} := \begin{bmatrix} -4 & 2 & 0 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 & 2 \\ 0 & 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 2 & 0 & 0 & -4 \end{bmatrix}$$

The inverse matrix below allows us to compute the dual lattice K^* and the discriminant group $A_K = K^*/K$. More specifically, we consider the linear combinations of $\alpha_1, \ldots \alpha_6$ with coefficients given by the columns of $G^{-1}_{E_6(2)}$. Denote them by $\alpha_1^*, \dots \alpha_6^*$, and their image in A_K by $[\alpha_i^*]$.

$$G^{-1}_{E_6(2)} := \begin{bmatrix} -\frac{2}{3} & -\frac{5}{6} & -1 & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{5}{3} & -2 & -\frac{4}{3} & -\frac{2}{3} & -1 \\ -1 & -2 & -3 & -2 & -1 & -\frac{3}{2} \\ -\frac{2}{3} & -\frac{4}{3} & -2 & -\frac{5}{3} & -\frac{5}{6} & -1 \\ -\frac{1}{3} & -\frac{2}{3} & -1 & -\frac{5}{6} & -\frac{2}{3} & -\frac{1}{2} \\ -\frac{1}{2} & -1 & -\frac{3}{2} & -1 & -\frac{1}{2} & -1 \end{bmatrix}$$

Notice that $[\alpha_1^*], [\alpha_2^*], [\alpha_4^*]$ and $[\alpha_5^*]$ all have order 6. Let $\beta := [\alpha_1^*]$; then $\langle\beta\rangle \cong \mathbb{Z}/6\mathbb{Z}$. We look for generators of $(\mathbb{Z}/2\mathbb{Z})^5$; order two elements not contained in $\langle\beta\rangle$. We find the following generators:

$$\gamma_1 := [3\alpha_2^*] = \left[\tfrac{\alpha_1}{2}\right]; \quad \gamma_4 := [3\alpha_5^* - \alpha_3^*] = \left[\tfrac{\alpha_4}{2}\right];$$

$$\gamma_2 := [\alpha_3^*] = \left[\tfrac{\alpha_6}{2}\right]; \quad \gamma_5 := [\alpha_6^* - 3\alpha_2^* - 3\alpha_4^*] = \left[\tfrac{\alpha_3}{2}\right].$$

$$\gamma_3 := [3\alpha_4^*] = \left[\tfrac{\alpha_5}{2}\right];$$

Thus 2β is a generator of $\mathbb{Z}/3\mathbb{Z}$, and we calculate that $q_N(2\beta) = -\frac{8}{3} \equiv -\frac{2}{3} \mod 2\mathbb{Z}$. Note that $q_{A_2}(\gamma) = -\frac{2}{3}$ for a generator γ of A_{A_2}, and so we have shown that N satisfies (1).

Next, we need to show that $q_N|_{(\mathbb{Z}/2\mathbb{Z})^{12}} = -q|_{D_{12}^+(2)}$. We see that $q_N(\gamma_i) = 1$. Let v, w be a basis for $A_{U(2)}$ with $v^2 = w^2 = 0$. Then $[\frac{v}{2}], [\frac{w}{2}]$ are generators for $U(2)$ with

$$q_N([\frac{v}{2}]) = q_N([\frac{w}{2}])) = 0.$$

Let e, f be a basis for $A_1 \oplus A_1(-1)$ with $e^2 = -2, f^2 = 2$ and $e \cdot f = 0$. Then $[\frac{e}{2}], [\frac{e+f}{2}]$ generate $A_{A_1 \oplus A_1(-1)} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, with

$$q_N([\frac{e}{2}]) = -\frac{1}{2}, \; q_N([\frac{e+f}{2}]) = 0.$$

We now look at the discriminant form of $D_{12}^+(2)$. Let $F_1, \dots F_{12}$ be a basis for $D_{12}^+(2)$, where:

$$\begin{aligned} F_i^2 &= -4 \text{ for } i \neq 12, \\ F_{12}^2 &= -6; \\ F_i \cdot F_{i+1} &= 2 \text{ for } i = 2, \dots 10; \\ F_1 \cdot F_3 &= -2, \\ F_1 \cdot F_{12} &= 2. \end{aligned}$$

Calculating the inverse matrix provides a basis $\{F_i^*\}_{i=1}^{12}$ for the dual lattice, where each column is viewed as the coefficients for F_i^* written as a linear combination of $F_1, \dots F_{12}$. This allows us to find generators for $A_{D_{12}^+(2)} = (\mathbb{Z}/2\mathbb{Z})^{12}$. We calculate the value of the discriminant form on each generator below.

$$\begin{array}{lll}
q_{D_{12}^+(2)}([F_1^*]) = \frac{1}{2} & q_{D_{12}^+(2)}([F_5^*]) = 0 & q_{D_{12}^+(2)}([F_9^*]) = 0 \\
q_{D_{12}^+(2)}([F_2^*]) = \frac{1}{2} & q_{D_{12}^+(2)}([F_6^*]) = 1 & q_{D_{12}^+(2)}([F_{10}^*]) = 1 \\
q_{D_{12}^+(2)}([F_3^*]) = 1 & q_{D_{12}^+(2)}([F_7^*]) = 1 & q_{D_{12}^+(2)}([F_{11}^*]) = 1 \\
q_{D_{12}^+(2)}([F_4^*]) = 0 & q_{D_{12}^+(2)}([F_8^*]) = 0 & q_{D_{12}^+(2)}([F_{12}^*]) = 0
\end{array}$$

Using the generators described above we see that $q_N|_H = -q_{D_{12}^+(2)}$, where $H = (\mathbb{Z}/2\mathbb{Z})^{12}$. Thus N satisfies all the conditions of Lemma 2.3.5 and so the existence of the embedding $D_{12}^+(2) \hookrightarrow \Lambda$ follows. The existence of the involution follows by Lemma **??** □

Remark 4.2.8. Note that one can complete the above calculations using the Hecke computer algebra package [FHHJ17] as described in [MM23a], however we prefer to include the original calculations.

4.2.5 Proof of Theorem 4.2.1

We have exhibited involutions $\iota \in O(\Lambda)$ acting trivially on A_Λ with Λ_- isomorphic to either $E_8(2)$ or $D_{12}^+(2)$. In order to conclude that both involutions are induced by symplectic birational involutions of a manifold X of $OG10$ type we must show that neither contain long or short roots.

Lemma 4.2.9. *Let $\iota \in O(\Lambda)$ be an involution and suppose that Λ_- contains a long root. Then A_{Λ_-} contains an element of order 3.*

Proof. Let $v \in \Lambda_-$ be a long root; i.e $v^2 = -6$ and $div_\Lambda(v) = 3$. Then 3 divides the divisibility of v in Λ_-. We can write $div_{\Lambda_-}(v) = 3k$ for some positive integer k. Then $[v^*] = \frac{v}{3k}$ defines a non-zero element of A_{Λ_-}; in particular, kv^* is a non-trivial element of order 3. □

Proof of Theorem 4.2.1. The discriminant group of both $E_8(2)$ and $D_{12}^+(2)$ contains no elements of order three; by Lemma 4.2.9 neither contains any long roots. The maximal norm of both lattices is -4, and so they do not contain short roots. Thus in both cases Λ_- is a negative definite lattice with $\Lambda_- \cap \mathcal{W}^{pex} = \varnothing$; we have also shown these are the only possible negative definite coinvariant lattices for an involution $\iota \in O(\Lambda)$ acting trivially on the discriminant group. By Theorem 4.1.2 the two involutions are induced by symplectic birational involutions of a manifold of $OG10$ type. The classification of the corresponding invariant sublattices follow from the proofs of Proposition 4.2.5 and 4.2.7. $\square$

4.3 Involutions induced from a cubic fourfold

We have classified in the previous section possible symplectic birational involutions that act trivially on the discriminant group A_Λ. It remains to be seen whether geometric involutions can act non-trivially. In this section we will prove the following:

Theorem 4.3.1. *Let X be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \mathrm{Bir}(X)$ a symplectic birational involution, such that the induced action $\eta_*(f)$ is non-trivial on the discriminant group of Λ. Assume further that*

$$\mathrm{rank}\Lambda_- < 12.$$

Then one of the following holds:

1. *$H^2(X,\mathbb{Z})_- \cong E_6(2)$, $H^2(X,\mathbb{Z})_+ \cong U^2 \oplus D_4^3$;*
2. *$H^2(X,\mathbb{Z})_- \cong M$, $H^2(X,\mathbb{Z})_+ \cong E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2$ where M is the unique rank 10 lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24\rangle$.*

Further, both involutions exist and can be geometrically realised via the LSV construction of §1.3.5.

The key insight in proving this classification is to utilise Theorem 3.0.1; the classification of involutions of a cubic fourfold. Let us briefly outline the strategy. In §4.3.1 we first use a lattice theoretic argument to show the existence of two involutions acting as in Theorem 4.3.1, using the knowledge of involutions on the smaller lattice $A_2 \oplus E_8^2 \oplus U^2$, via Theorem 3.0.1. Next, we approach from a geometric point of view in §4.3.2, and consider induced symplectic birational involutions on the compactified intermediate Jacobian starting from a cubic fourfold with an involution. We note that in all of the cases considered above the invariant sublattice contains a U summand, i.e $\Lambda_+ = \Gamma \oplus U$ for some lattice Γ. In §4.3.3 we investigate lattice theoretic criteria for this to be satisfied, and in particular show it is always the case assuming that $\mathrm{rank}\Lambda_- < 12$. This completes the proof of Theorem 4.3.1, and further exhibits a geometric realisation of each case. This completes the proof of Theorem 4.0.1(2).

4.3.1 Existence of symplectic involutions via cubic fourfolds

Recall that all ADE lattices are assumed to be negative definite. In particular for a root lattice R, the lattice $R(-1)$ is positive definite.

Let $V \subset \mathbb{P}^5$ be a smooth cubic fourfold. Recall that $H^4(V,\mathbb{Z}) \cong L(-1)$ where $L := A_2 \oplus E_8^2 \oplus U^2$; in particular

$$\Lambda \cong L \oplus U.$$

We will use the arithmetic classification of involutions of a cubic fourfold to prove the following theorem.

Theorem 4.3.2. *There exist symplectic birational involutions $f \in \mathrm{Bir}(X)$ of an irreducible holomorphic symplectic manifold X of $OG10$ type with either:*

1. *$H^2(X,\mathbb{Z})_- \cong E_6(2)$, $H^2(X,\mathbb{Z})_+ \cong U^2 \oplus D_4^3$;*
2. *$H^2(X,\mathbb{Z})_- \cong M$, $H^2(X,\mathbb{Z})_+ \cong E_8(2) \oplus A_1 \oplus A_1(-1) \oplus U^2$ where M is the unique rank 10 lattice obtained as an index 2 overlattice of $D_9(2) \oplus \langle 24\rangle$.*

Moreover, the induced involution of Λ act non-trivially on the discriminant group in both cases.

Proof. Consider an antisymplectic involution ϕ of a cubic fourfold V. The action on $H^4(V,\mathbb{Z})_{\mathrm{prim}} \cong L(-1)$ has been classified by Theorem 3.0.1; either $L_- \cong E_6(2)$ or $L_- \cong M$. We can extend the involution to an involution ι of $\Lambda \cong L \oplus U$, acting by the identity on the remaining copy of U. Notice now that $\Lambda_- \cong L_-$ and $\Lambda_+ \cong L_+ \oplus U$.

In both cases, the coinvariant lattice Λ_- is negative definite, and

$$\Lambda_- \cap \mathcal{W}^{pex} = \varnothing.$$

Indeed, we know L_- contains no short or long roots since it is the coinvariant lattice for an involution of a smooth cubic. Hence by Theorem 4.1.2, ι is induced geometrically by a symplectic birational transformation $f \in \mathrm{Bir}(X)$ for some manifold X of $OG10$ type. Further we see that such an involution necessarily acts by $-id$ on the discriminant group A_Λ; if ι acted trivially, then Λ_- would be 2-elementary, a contradiction by Proposition 4.1.3. The classification of Λ_+ in both cases follows. $\square$

It is worth stressing that the existence of such symplectic birational involutions of a manifold of $OG10$ type seems to be in direct contrast with $OG6$ type manifolds. For manifolds of $OG6$ type, symplectic automorphisms act trivially on the second cohomology, and further birational symplectic transformations of finite order act trivially on the corresponding discriminant group [GOV20]. We note however that there exists examples of symplectic birational involutions of manifolds of $K3^{[n]}$ type acting non-trivially on the discriminant, as described in [Mar13, §11].

4.3.2 Geometric observations

We notice in the previous section that for both examples of birational symplectic involutions $f \in \mathrm{Bir}(X)$, the invariant lattice $H^2(X,\mathbb{Z})_+ \cong \Lambda_+$ contains a U summand. The compactified intermediate Jacobians of cubic fourfolds are examples of manifolds of $OG10$ type with a U polarisation; it is natural to consider involutions that appear via this construction. In particular, we show that involutions of a cubic fourfold produce symplectic birational involutions of the associated compactified intermediate Jacobian with this property.

Theorem 4.3.3. *Let X be an irreducible holomorphic symplectic manifold of OG10 type. Let $f \in \mathrm{Bir}(X)$ be a symplectic birational involution of X, and suppose that $H^2(X,\mathbb{Z})_+ \cong \Gamma \oplus U$ for some lattice Γ. Then there exist a smooth cubic fourfold V with an involution ϕ whose action is determined by f. In particular, one of the following holds:*

$$H^2(X,\mathbb{Z})_- \cong \begin{cases} E_6(2), \\ E_8(2), \\ M. \end{cases}$$

Conversely, an involution ϕ of a smooth cubic fourfold V induces a birational symplectic involution f on the compactified associated Intermediate Jacobian $\mathcal{J}_V$, that leaves a copy of U invariant and whose action is determined by ϕ.

Proof. Denote by $\iota := \eta_*(f) \in O(\Lambda)$ the induced involution on Λ, and let $U_1 := U$ be such that $\Lambda_+ = \Gamma \oplus U_1 \hookrightarrow \Lambda$. Denote by $L = (U_1^\perp)_\Lambda$; then L is an even, indefinite lattice with signature $(2,20)$ and discriminant group $A_L \cong \mathbb{Z}/3\mathbb{Z} \cong A_{A_2}$. By [Nik79b, Cor. 1.13.3], L is unique; thus we see that

$$L \cong U^2 \oplus E_8^2 \oplus A_2.$$

Since ι acts as the identity on $\Gamma \oplus U_1$, ι restricts to an isometry of L with

$$L_+ \cong \Gamma \text{ and } L_- \cong \Lambda_-.$$

Note that Λ_- is negative definite of rank $r \leq 20$, and Γ has signature $(2, 20-r)$. We can choose a Hodge structure H on L of type $(0,1,20,1,0)$ such that

$$H^{2,2} \cap L = \Lambda_-.$$

Notice this implies $H^{3,1} \subset L_+$. By assumption, Λ_- contains no prime exceptional vectors (Prop. 4.1.1); in particular it contains no long or short roots. The Global Torelli Theorem for cubic fourfolds (Theorem 1.3.3) implies that there exists a smooth cubic fourfold V with $H^4(V,\mathbb{Z})_{\text{prim}} \cong H(-1)$ as Hodge structures.

Let $\eta_V \in H^4(V,\mathbb{Z})$ be the square of the hyperplane class. We wish to extend ι to an isometry of $H^4(V,\mathbb{Z})$ fixing η_V. We have

$$L(-1) \oplus \langle \eta_X \rangle \subset H^4(V,\mathbb{Z});$$

in order to extend ι, ι must act trivially on $A_{L(-1)} \cong \mathbb{Z}/3\mathbb{Z} \cong A_{\langle \eta_V \rangle}$. Note that $O(A_{L(-1)}) \cong \mathbb{Z}/2\mathbb{Z}$; thus ι can act as $\pm id_{A_{L(-1)}}$.

Suppose first that ι acts by $id_{A_{L(-1)}}$. Then $\iota \oplus id_{\langle \eta_V \rangle}$ extends to an isometry of $H^4(V,\mathbb{Z})$, denoted by ι_V. Thus $\iota_V \in \mathrm{Aut}_{HS}(V, \eta_V)$, and by the Strong Global Torelli Theorem 1.3.3 there exists a unique automorphism $\phi \in \mathrm{Aut}(V)$ such that $\iota_V = \phi^*$. Notice that by construction, ϕ is necessarily a symplectic involution of V, and by Theorem 3.0.1

$$(H^4(V,\mathbb{Z})_{\text{prim}})_- = L_-(-1) \cong E_8(-2)$$

(see also [LZ22, Theorem 1.2 (1)]). Thus necessarily $\Lambda_- \cong E_8(2)$.

Next suppose that ι acts by $-id_{A_{L(-1)}}$ Set $\sigma := -\iota$; notice now that for the action of σ on $H^4(V,\mathbb{Z})_{\text{prim}}$ we have:

$$H^4(V,\mathbb{Z})_- = L_+(-1) \cong \Gamma(-1), \text{ and } H^4(V,\mathbb{Z})_+ = L_-(-1) \cong \Lambda_-(-1).$$

Now $\sigma \oplus id_{\langle \eta_V \rangle}$ extends to an isometry of $H^4(V,\mathbb{Z})$; let us denote this by σ_V. Thus $\sigma_V \in \mathrm{Aut}_{HS}(V,\eta_V)$, and again by the Strong Global Torelli theorem there exists a unique automorphism $\phi \in \mathrm{Aut}(V)$ such that $\sigma_V = \phi^*$. Notice that σ_V acts non-trivially on $H^{3,1}$; the involution ϕ is anti-symplectic, and by Theorem 3.0.1

$$\Lambda_- \cong (H^4(V,\mathbb{Z})_{\text{prim}})_+(-1) = \begin{cases} E_6(2), \\ M. \end{cases}$$

Conversely, suppose we have an involution $\phi \in \mathrm{Aut}(V)$ of a smooth cubic fourfold $V \subset \mathbb{P}^5$; let $\sigma := \phi^*$ be the induced involution on $H^4(V,\mathbb{Z})_{\text{prim}}$. By ([LSV17], [Sac21, Theorem 1.6]), we can associate to V an irreducible holomorphic symplectic manifold $\mathcal{J}_V$ of OG10 type, with a Lagrangian fibration $\pi : \mathcal{J}_V \to \mathbb{P}^5$ that compactifies the intermediate Jacobian fibration of V. Note that the compactification $\mathcal{J}_V$ is not unique; the cubic fourfold V is a special cubic fourfold containing either a plane or a cubic scroll [Mar23], and so may have many birational compactifications, as discussed in [Sac21]. Let Θ denote the relative theta-divisor of $\mathcal{J}_V$; then the sublattice $\langle \Theta, \pi^*\mathcal{O}(1)\rangle$ is isomorphic to the hyperbolic lattice U [Sac21, Lemma 3.5, communicated by K.Hulek, R.Laza].

To obtain an involution of $\mathcal{J}_V$, we follow [Sac21, Sect. 3.1]. The automorphism $\phi \in \mathrm{Aut}(V)$ acts on the universal family of hyperplane sections of V, and thus on the Donagi-Markman fibration $\mathcal{J}_U \to U$, where $U \subset (\mathbb{P}^5)^*$ parametrises smooth hyperplane sections of V [Sac21, Section 3.1]. We thus obtain in this way a birational transformation $f : \mathcal{J}_V \dashrightarrow \mathcal{J}_V$, that leaves the sublattice $\langle \Theta, \pi^*\mathcal{O}(1)\rangle \cong U$ invariant. If $\phi \in \mathrm{Aut}(V)$ is symplectic (i.e acts trivially on $H^{3,1}(V)$), then the induced birational involution $f \in \mathrm{Bir}(\mathcal{J}_V)$ is symplectic, by [Sac21, Lemma 3.2]. If not, there exists a regular anti-symplectic involution $\tau \in \mathrm{Aut}(\mathcal{J}_V)$ given geometrically by sending $x \mapsto -x$ on the fibers of $\mathcal{J}_V \to \mathbb{P}^5$. Further this involution τ commutes with the induced anti-symplectic involution $f \in \mathrm{Bir}(\mathcal{J}_V)$. It follows that $\tau \circ f$ is a non-trivial symplectic birational involution of $\mathcal{J}_V$. Set $\widetilde{f} := f$ if f is symplectic, $\widetilde{f} := \tau \circ f$ otherwise. Note that $\widetilde{f}$ leaves $\langle \Theta, \pi^*\mathcal{O}(1)\rangle$ invariant in both cases.

Finally, note that if ϕ and thus f is anti-symplectic, then the symplectic birational involution $\tilde{f}$ acts by $-id|_{A_\Lambda}$ on the discriminant group of Λ. □

Remark 4.3.4. The proof of the previous theorem classifies the invariant and coinvariant lattices for a symplectic birational involution such that $H^2(X,\mathbb{Z})_+ \cong \Gamma \oplus U$; we see that Γ necessarily is either the coinvariant or invariant sublattice for the induced involution ϕ on $H^4(V,\mathbb{Z})$. Moreover, such involutions exist by Theorem 4.3.2. To complete the proof of Theorem 4.3.1, it remains to show the assumption $\mathrm{rank}\Lambda_- < 12$ implies that Λ_+ contains a U summand.

4.3.3 Criteria for splitting a U summand

The aim of this subsection is to identify a lattice theoretic criteria to complete the proof of Theorem 4.3.1.

Assume that $f \in \mathrm{Bir}(X)$ is a symplectic birational involution of a manifold X of $OG10$-type, such that $\iota := \eta_*(f) \in O(\Lambda)$ is an involution that acts by $-id$ on A_Λ. By Lemma 4.1.1,

Λ_- is negative definite of rank $1 \leq r \leq 21$, and $\Lambda_- \cap \mathcal{W}_M = \varnothing$. By Proposition 4.1.3, Λ_+ is a 2-elementary lattice. We establish a numerical criteria for Λ_+ to split of a U summand.

Lemma 4.3.5. *Let $r = rank\Lambda_-$, a, δ as above. Then Λ_+ splits of a U summand if and only if:*

1. *$r \leq 20$, and $a \leq 22 - r$,*

2. *If $a = 22 - r$ and $\delta = 0$, then $r \equiv 2 \mod 8$*

Proof. Assume Λ_+ splits of a U summand, i.e $\Lambda_+ \cong N \oplus U$. Applying Nikulin's classification of 2-elementary lattices to the lattice N with invariants $(2, 20 - r), a, \delta)$, we see the above conditions are necessary for the existence of such a lattice N. Conversely, assume the conditions in the theorem hold. Then again by the classification, there exists a 2-elementary lattice N with invariants $((2, 20 - r), a, \delta)$. Then $N \oplus U$ has the same invariants as Λ_+, and thus are in the same genus. Since Λ_+ is indefinite, it is unique and the claim holds. ☐

Corollary 4.3.6. *Let X be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \mathrm{Bir}(X)$ a birational involution of X acting by $-id_{A_\Lambda}$ and such that the coinvariant lattice Λ_- has rank $r < 12$. Then $\Lambda_+ \cong \Gamma \oplus U$ for some lattice Γ, and Theorem 4.3.3 applies.*

Proof. Since $r < 12$, then by assumption $a \leq r \leq 22 - r$ and the conditions of Lemma 4.3.5 are satisfied. ☐

This completes the proof of Theorem 4.3.1

4.4 Unexpected involutions

In joint work with S. Muller [MM23a], we complete the classification of birational symplectic involutions of manifolds of $OG10$ type. In particular, we show that there exists two unexpected involutions that act non-trivially on the discriminant group and are not induced from a cubic fourfold. In particular, we show:

Theorem 4.4.1. *Let X be an irreducible holomorphic symplectic manifold of $OG10$ type, and $f \in \mathrm{Bir}(X)$ a symplectic birational involution such that the induced action $\eta_*(f)$ is non-trivial on the discriminant group of Λ. Assume further that $rank\Lambda_- \geq 12$. Then either*

$$\Lambda_- = \begin{cases} G_{12}, & or \\ G_{16} & \end{cases},$$

where G_i has rank i and Gram matrix displayed in §4.4.3 . Moreover, each involution exists.

In particular, this implies Theorem 4.0.1 (3). To show the existence of these involutions, we use lattice enumeration techniques. Under the hypothesis of Theorem 4.4.1, it is fairly easy to identify possible lattices Λ_+, however the corresponding lattices Λ_- are no longer unique in their genera. There are 12 possible genera for Λ_+; we enumerate the possible lattices Λ_- in each genus, totaling over 13,000 lattices. In each case, we check for long and sort roots. There are two lattices that do not contain any such roots, and thus two involutions induced by geometric symplectic birational involutions of a manifold of $OG10$ type, by Theorem 4.1.2. More details can be found in [MM23a, §6 and Appendix A.].

4.4.1 Genus of the remaining possible cases

We wish to classify the possible genera of the coinvariant lattice Λ_- for an involution $\iota \in O(\Lambda)$ with non-trivial action on A_Λ, such that Λ_- is negative definite, and such that Λ_+ does not split a U summand, i.e. Λ_+ fails the criteria of Lemma 4.3.5.

Proposition 4.4.2. *Let $\iota \in O(\Lambda)$ acting non-trivially on A_Λ such that Λ_- is negative definite. Assume that Λ_+ does not split a U summand. Let $r := \mathrm{rank}\Lambda_-$. Then one of the following holds:*

1. $\Lambda_+ \cong U(2)^3$ *and* $r = 18$*;*

2. $\Lambda_+ \cong U(2)^3 \oplus D_4$ *and* $r = 14$*;*

3. $\Lambda_+ \cong \langle 2\rangle^3 \oplus \langle -2\rangle^{21-r}$ *and* $r \geq 12$.

Proof. For ease of notation, let $M := \Lambda_+$. Since M does not split of a U summand, we have that $r \geq 12$. Assume first that $r \neq 21$, then M is an indefinite, 2-elementary lattice and is classified uniquely by the invariants (r, a, δ). There are two cases to consider by Lemma 4.3.5: either $22 - r < a$, or $a = 22 - r, \delta = 0$ and $r \not\equiv 2 \mod 8$.

Case 1: Assume that $22 - r < a$; we necessarily have that $22 - r < a \leq 24 - r$. Since M is 2-elementary, we have that $a \equiv r \mod 2$; we can exclude $a = 23 - r$. Thus $a = 24 - r = rk(M)$. The lattice $N := M(1/2)$ is well defined by Lemma 2.3.6. Further, $A_N = \{1\}$, and so N is unimodular.

Assume that $\delta = 0$; this implies that N is an even unimodular lattice (see for example Lemma 2.3.8). By Milnor's theorem on unimodular forms (see [Nik79b, Thm 0.2.1] for a precise statement), N exists if and only if

$$\begin{aligned} 3 + r - 21 &\equiv 0 \mod 8; \\ r &\equiv 2 \mod 8. \end{aligned}$$

Since $r \geq 12$, we have that $r = 18$. Thus N has signature $(3, 3)$, and hence $N \cong U^3$. Thus $M \cong U(2)^3$.

Now assume that $\delta = 1$. It follows that N is an odd indefinite unimodular lattice (see for example Lemma 2.3.7). By Milnor's theorem again, N exists and is isomorphic to $\langle 1 \rangle^3 \oplus \langle -1 \rangle^{21-r}$, thus

$$M \cong \langle 2 \rangle^3 \oplus \langle -2 \rangle^{21-r}.$$

Case 2: Assume that $a = 22 - r$, with $\delta = 0$ and $r \neq 2 \mod 8$. Note again that $r \geq 12$; if $r \leq 11$,, since $22 - r = a \leq r \leq 22 - r$, we must have that $r = 11$. But since $\delta = 0$, for M to exist $r \equiv 2 \mod 4$, a contradiction.

So $r \geq 12$, and since $r \equiv 2 \mod 4$, $r \in \{14, 18\}$. By assumption, $r \neq 2 \mod 8$, thus $r = 14$. Hence M has signature $(3, 7)$ with $a = 8, \delta = 0$. Consider the lattice $U(2)^3 \oplus D_4$; it has the same signature and invariants. Since indefinite 2-elementary lattices are unique up to isomorphism, we necessarily have $M \cong U(2)^3 \oplus D_4$.

Finally, assume that $r = 21$. In this case, M has signature $(3, 0)$. Since $a \equiv r \mod 2$, $a = 1$ or 3. If $a = 1$, no such lattice exists by Theorem 2.3.4 ; thus $a = 3$. Again, the lattice $N := M(1/2)$ is well defined. Further, $A_N = \{1\}$, so N is unimodular. If $\delta = 0$, N is an even unimodular lattice: once more, Milnor's Theorem on unimodular forms gives an immediate contradiction with the rank of N. Thus $\delta = 1$ and N is an odd unimodular lattice, thus $N \cong \langle 1 \rangle^3$, and $M \cong \langle 2 \rangle^3$. □

The above result classifies the genus of the coinvariant lattice Λ_-; the lattice Λ_- is negative definite of rank r, with discriminant group $A_\Lambda \cong A_{\Lambda_+} \oplus A_{A_2}$, with quadratic form

$$q_{\Lambda_-} = (-q_{\Lambda_+}) \oplus q_{A_2}.$$

Unfortunately, these invariants are not enough to classify Λ_- uniquely; there are many isometry classes of lattices with these invariants.

In order to conclude our classification of symplectic birational involutions of $OG10$ type, we need to see whether an involution $\iota \in O(\Lambda)$ as in Proposition 4.4.2 is induced by a birational symplectic involution. By Theorem 4.1.2, only the involutions with $\Lambda_- \cap \mathcal{W}^{pex} = \varnothing$ are induced. One possible strategy is to classify the isometry classes Λ_- for each case in

Proposition 4.4.2, and check for the existences of vectors in $\mathcal{W}^{pex}$, i.e. for short or long roots. This turns out to be a difficult problem; these lattices have both large rank and discriminant (the methods of Conway-Sloane have not been extended [CS88]), and have not been enumerated. We illustrate this difficulty with an example.

Example 4.4.3. Consider $(\Lambda)_+) \cong U(2)^3$. Then $(\Lambda)_-$ has rank 18, and discriminant group

$$A_{(\Lambda)_-} \cong \mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^6,$$

and $q_{(\Lambda)_-}|_{\mathbb{Z}/3\mathbb{Z}} = q_{A_2}$. There are two easily identifiable possibilities for $(\Lambda)_-$:

$$A_2 \oplus K;$$
$$A_2 \oplus E_8 \oplus N,$$

where K is the Kummer lattice and N is the Nikulin lattice (see [Mor84] for a description of these lattices). Both of these embed into the lattice Λ and are orthogonal to $U(2)^3$. Although both examples contain short roots and thus cannot be realised by a geometric birational involution, there may be other lattices in the same genus without short or long roots.

Example 4.4.4. Consider $(\Lambda)_+ \cong U(2)^3 \oplus D_4$. Then $(\Lambda)_-$ has rank 14, and discriminant group

$$A_{(\Lambda)_-} \cong \mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^8.$$

Thus $(\Lambda)_-$ is in the same genera as the lattice $A_2 \oplus N \oplus D_4$. Again this example contains short roots.

4.4.2 Enumeration of lattices

By Proposition 4.4.2; there are 12 possible genera for the lattice Λ_-. For each genus, we must first enumerate all isometry classes of lattices in each genus - this enumeration was undertaken joint with Stevell Muller and was computer aided. Our results of this enumeration and analysis are summarised in Table 4.1; for more detail see the database [MM23b]. We refer to [MM23a, §6, Appendix A] for more details on the techniques and implementation of our enumeration algorithm.

We represent each isometry class of lattices obtained by the enumeration by their Gram matrix, which are available in the files [MM23b]. Each of them represent a coinvariant lattice Λ_- for an involution $\iota \in O(\Lambda)$. It remains to verify whether they are induced from a geometric involution or not.

Let L be one the lattices enumerated; L is the coinvariant lattice for an involution $\iota \in O(\Lambda)$. We verify whether L is induced by a geometric involution by verifying if L contains a short or long root. We use the method **short_vectors** on Hecke [FHHJ17], which allows us to compute all vectors in $L(-1)$ of a given norm. For each such vector y, we then compute the positive generator d of the $\mathbb{Z}$-ideal $(y, L(-1))$: if $d = 3$, then y is a long root in L. Finally, if L contains no short or long roots, then by Theorem 4.1.2 the involution ι is induced, and L is isometric to the co-invariant lattice associated to a symplectic birational involution on an irreducible symplectic manifold of $OG10$ type.

4.4.3 Results

Table 4.1 contains a summary of our enumeration and analysis. For each possible genus, we give the number N of isometry classes of lattices it contains. In the column WITH S.R. we record the number of classes that have a representative with a short root. In the column WITHOUT S.R., WITH L.R. we record how many classes have a representative without any short roots but with at least one long root. Finally, the last column GEOMETRIC CASES presents all possible isometry classes that are induced by a symplectic birational involution of a manifold X of $OG10$ type, and thus are isometric to $H^2(X, \mathbb{Z})_-$. This concludes the proof of Theorem 4.4.1, and thus completes the Proof of Theorem 4.0.1(3).

The two lattices admitting a geometric realization from Table 4.1 are determined respectively by the following Gram matrices:

$$G_{12} = \begin{bmatrix} -4 & 2 & -2 & -2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ 2 & -4 & 2 & 0 & 2 & 0 & -2 & 2 & -2 & -2 & 2 & 2 \\ -2 & 2 & -4 & -2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ -2 & 0 & -2 & -4 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -2 & -2 & -4 & -2 & 2 & -2 & 2 & 2 & -2 & -2 \\ -2 & 0 & -2 & -2 & -2 & -4 & 2 & -2 & 2 & 2 & -2 & -2 \\ 2 & -2 & 2 & 0 & 2 & 2 & -4 & 2 & -2 & -2 & 2 & 2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -6 & 4 & 4 & -2 & -4 \\ 2 & -2 & 2 & 0 & 2 & 2 & -2 & 4 & -6 & -2 & 4 & 4 \\ 2 & -2 & 2 & 0 & 2 & 2 & -2 & 4 & -2 & -6 & 2 & 2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -2 & 4 & 2 & -6 & -2 \\ -2 & 2 & -2 & 0 & -2 & -2 & 2 & -4 & 4 & 2 & -2 & -6 \end{bmatrix} \quad G_{16} = \begin{bmatrix} -4 & 2 & -2 & 2 & -1 & 1 & 2 & 1 & 2 & 1 & -2 & 1 & -2 & -2 & 1 & -2 \\ 2 & -4 & 0 & -1 & 2 & -2 & 0 & -2 & 0 & -2 & 0 & -2 & 1 & 2 & 1 & 1 \\ -2 & 0 & -4 & 2 & 1 & 1 & 1 & -1 & 2 & -1 & -1 & 1 & -1 & -1 & 2 & -2 \\ 2 & -1 & 2 & -4 & -1 & -2 & -2 & -1 & 0 & 1 & 1 & 0 & 2 & 0 & -2 & 1 \\ -1 & 2 & 1 & -1 & -4 & 1 & -1 & 2 & 1 & 2 & 1 & 2 & 1 & -1 & -2 & 0 \\ 1 & -2 & 1 & -2 & 1 & -4 & -1 & -2 & 0 & -1 & -1 & -2 & 1 & 1 & -1 & 1 \\ 2 & 0 & 1 & -2 & -1 & -1 & -4 & -1 & 0 & 0 & 2 & 1 & 2 & 1 & -2 & 1 \\ 1 & -2 & -1 & -1 & 2 & -2 & -1 & -4 & 0 & -2 & -1 & -1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -4 & -1 & 0 & -1 & 1 & 1 & -1 & 2 \\ 1 & -2 & -1 & 1 & 2 & -1 & 0 & -2 & -1 & -4 & 0 & -2 & 0 & 2 & 1 & 0 \\ -2 & 0 & -1 & 1 & 1 & -1 & 2 & -1 & 0 & 0 & -4 & -1 & -1 & -1 & 1 & 0 \\ 1 & -2 & 1 & 0 & 2 & -2 & 1 & -1 & -1 & -2 & -1 & -4 & -1 & 2 & 1 & 1 \\ -2 & 1 & -1 & 2 & 1 & 1 & 2 & 1 & 1 & 0 & -1 & -1 & -4 & 0 & 2 & -2 \\ -2 & 2 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & 2 & -1 & 2 & 0 & -4 & 0 & -1 \\ 1 & 1 & 2 & -2 & -2 & -1 & -2 & 1 & -1 & 1 & 1 & 1 & 2 & 0 & -4 & 1 \\ -2 & 1 & -2 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & -2 & -1 & 1 & -4 \end{bmatrix}$$

They correspond respectively to the 3rd and the 472nd lattices in the respective files **c3r12** and **c3r16** of our database, available in [MM23b].

Case	rank	N	with s.r.	without s.r., with l.r.	geometric cases
(1)	18	430	430	0	None
(2)	14	21	21	0	None
(3)	12	5	4	0	1: G_{12}
	13	23	22	1	None
	14	70	70	0	None
	15	211	211	0	None
	16	617	616	0	1: G_{16}
	17	1291	1291	0	None
	18	2524	2524	0	None
	19	3682	3682	0	None
	20	3375	3375	0	None
	21	1316	1316	0	None

Table 4.1: Genus enumeration and geometric cases of Theorem 4.4.1

4.5 The fixed locus: future directions

Our desire for classifying symplectic birational involutions was motivated by the search for new examples of IHS varieties. The natural next step is to study the fixed locus for such an involution; this becomes difficult unless we have an explicit geometric model for the action. In this sense, the three symplectic birational involutions induced from a cubic fourfold have the advantage; one can exploit the geometry of the associated intermediate Jacobian fibration to study the fixed locus. We saw in §3.4 that a cubic fourfold with an involution that fixes a plane point-wise (ϕ_3 in the notation of §3) is rational, and has many associated $K3$ surfaces. In contrast, the cubic fourfolds V with the other two involutions are potentially irrational, and the transcendental cohomology $T(V)$ does not embed into the $K3$ lattice. From the point of view of producing new examples of IHS varieties, these involutions have the best chance of success - we do not expect IHS of $K3^{[n]}$ type to appear as components of the fixed locus.

We focus on a symplectic involution induced from a cubic fourfold $V \subset \mathbb{P}^5$ with a symplectic involution $\phi := \phi_2$ as in Chapter 3. This is ongoing work which will appear elsewhere - we briefly outline the strategy here. The goal is to exploit the structure of the Lagrangian fibration $\pi : X \to B := (\mathbb{P}^5)^\vee$ of a compactified intermediate Jacobian fibration of V in order to obtain a description of the fixed locus. The involution ϕ acts on the ambient $\mathbb{P}^5$; it fixes a line L and a linear space $\Pi \cong \mathbb{P}^3$ point-wise. The line L is contained in V. Since $B \cong (\mathbb{P}^5)^\vee$, the involution ϕ also acts on the base B of the associated Lagrangian fibration, fixing point-wise two complimentary linear subspaces $\check{L} \sqcup \check{\Pi} \cong \mathbb{P}^3 \sqcup \mathbb{P}^1 \subset B$. The image of the fixed locus $\pi(\mathrm{Fix}(\iota))$ is contained in this locus - we obtain two disjoint varieties, fibered over $\check{L}$ and $\check{\Pi}$. A point $b \in \check{L} \sqcup \check{\Pi}$ corresponds to an invariant hyperplane section $Y_b := H_b \cap X$; the cubic threefold Y_b obtains an induced involution ϕ_b. In turn, this involution acts on the intermediate Jacobian $J(Y_b)$; in obtain to describe the fiber of the fixed locus over $\check{L} \sqcup \check{\Pi}$, one needs to identify the fixed abelian subvariety $J(Y_b)^{\phi_b} \subset J(Y_b)$.

Fortunately, the intermediate Jacobians of cubic threefolds Y with an involution have been well studied. There are two possibilities for an involution ϕ_b of Y_b; either Y_b is an Eckardt cubic, where ϕ_b fixes the Eckardt point and a disjoint hyperplane section, or the involution ϕ_b fixes a line $l \subset Y_b$ point-wise. An involution of the second type is called a **non-Eckardt involution** (see [CMMZ22, §1.1]). For a general point $b \in \check{\Pi}$, one can see that Y_b is a smooth Eckardt cubic, and the invariant subabelian variety $J(Y_b)^{\phi_b}$ is isomorphic to an elliptic curve associated to Y_b (see [CMZ21]). On the other hand, a general point $p \in \check{L}$ corresponds to a cubic with a non-Eckardt involution; indeed, the hyperplane section Y_b contains the point-wise fixed line L. In this case, $J(Y_b)^{\iota}$ is a 3-dimensional abelian variety with a $(1, 2, 2)$ polarisation, as studied in [CMMZ22].

The difficulty with this study is the fact that the induced symplectic involution is strictly birational. We define the $\mathrm{Fix}(\iota) \subset X$ as the closure of the fixed locus of ι restricted to the largest open subset where it is regular. It is clear the open locus of $\mathrm{Fix}(\iota)$ is smooth, and admits a holomorphic symplectic form, but taking the closure may introduce singularities. The structure of the singularities and identifying the components is on going work.

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