Topological Quantum Field Theories in Dimension Four

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Jin-Cheng Guu

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Jin-Cheng Guu

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Alexander Kirillov - Dissertation Advisor
Professor, Department of Mathematics, Stony Brook University

Oleg Viro - Dissertation Examining Committee Chair
Professor, Department of Mathematics, Stony Brook University

Leon Takhtajan - Third Inside Member
Professor, Department of Mathematics, Stony Brook University

Scott Carter - Outside Member
Professor Emeritus, Department of Math and Statistics, University of South Alabama

This dissertation is accepted by the Graduate School

Celia Marshik
Dean of the Graduate School

Abstract of the Dissertation

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Abstract

This thesis is compiled from the previous works of the author ([Guu22], [GT21], [Guu21]). It examines the Crane-Yetter theory in three different contexts. The first context involves proving its equivalence to the Turaev shadow state sum as a 4-manifold invariant. In the second context, we focus on its values at 2-manifolds. The values are linear categories. We describe the structures of the linear categories for all oriented surfaces with at least one puncture. Lastly, an application of Crane-Yetter theory to a problem in tensor categories is provided.

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1 Introduction

1.1 Invariants as data representations

When exploring a mathematical concept, one typically begins with its presentation, which describes the object using simpler objects. However, this presentation merely offers a distant image of the object's true nature. For instance, a finite set of generators and relations that present a finite group appears vastly different from the group itself. Similarly, a finite set of triangles and gluing data presenting a manifold may fully describe the space but obfuscate its topological structure with intricate details. Hence, it is natural to strive for deeper comprehension. One common approach is to represent the presented object in alternative forms.

When it comes to representing data, two primary concerns arise. The first is the accuracy of the representation: does it lose any information, and if so, how much? Secondly, as the representation becomes a mathematical object in its own right, it is reasonable to ask how well we understand the represented object. Can we easily characterize or compute it? In many cases, a balance must be struck between these two considerations. A more precise representation may provide less insight into the matter, as exemplified by the presentation itself, which accurately and tautologically represents the object. Conversely, a less accurate representation simplifies the object to a greater extent and may be more illuminating, as demonstrated by the size of a finite set. Although it does not represent the set faithfully as a mathematical object, it does provide significant information about the set.

Representations are also known as invariants. In topology, they are referred to as topological invariants. The dimension is the simplest example of an integer-valued invariant that assigns a positive integer to a given space. Another example is the genus of a surface, which is also an integer-valued invariant. However, invariants need not be restricted to numerical values. Although numbers are more straightforward to understand, they have limited capacity to hold information. The homology functor $H(-;\mathbb{Q})$, on the other hand, represents topological spaces as vector spaces and provides more information than just the dimension. More sophisticated examples represent the objects as even more intricate algebraic objects, such as groups, algebras, Hopf algebras, and so on. A concrete example of this is given below.

Theorem 1.1 [Man06] Finite type nilpotent spaces X and Y are weakly equivalent if and only if the E^{∞}-algebras $C^{\star}(X)$ and $C^{\star}(Y)$ are quasi-isomorphic. ◇

The transition from numbers to more complex objects is a groundbreaking shift in thinking. Not only does it permit more precise representations of objects, but it also enables us to capture relationships between objects through the use of (higher) functors. This ability to represent relationships is arguably even more important than the accuracy of the representation itself.

1.2 TQFTs as higher invariants

Another example of an invariant that takes vector spaces as its values is a topological quantum field theory (TQFT), as developed by Michael Atiyah [Ati88]. TQFTs were originally motivated by theoretical physics. In essence, a TQFT assigns a vector space to a manifold, much like homology. However, it also assigns a linear transformation to the manifold of one higher dimension that 'bridges' the lower manifolds in a compatible manner. Interestingly, this results in a number-valued invariant for closed manifolds, since each closed manifold 'bridges' the empty manifolds, which must be assigned the trivial vector space.

This generalization from numbers to vector spaces does not stop here. By viewing vector spaces as objects in a category, one can bring this process to assigning objects in higher categories. An instance of such is called an extended topological quantum field theory (extended TQFT) [Lur09].

Example 1.2 (Witten-Reshetikhin-Turaev model) One-dimensional and two-dimensional manifolds have been completely classified. While a presentation for three-dimensional manifolds exists through the use of knots and surgery theory (see [Lic12, chap.12] and [Kir78]), this classification is dependent on the classification of knots, which is a complicated and ongoing research field. Other presentations suffer from the same issue. As a result,invariants of other types for 3-manifolds are of great interest.

In the 1980s, a new kind of invariant emerged from the intersection of mathematics and physics, starting with Vaughan Jones' groundbreaking Jones polynomials. These invariants can be constructed using a variety of mathematical structures such as von Neumann algebras, quantum groups, and rational conformal theories, as described in [CY93b, sec.1]. It is natural to wonder if these different types of invariants can be unified, and indeed they can. They all turn out to be special cases of the Witten-Reshetikhin-Turaev (WRT) model, which takes as its input algebraic data a modular tensor category, as discussed in [Bar+15].

As an extended TQFT, WRT model assigns a number to each closed 3-manifold, a vector space to each closed 2-manifold, and a linear category to each closed 1-manifold. However, it does not go on and assign a linear 2-category in the 0-th dimension. See 1.6 for more discussion. $\diamond$

Example 1.3 (Turaev-Viro model) The Turaev-Viro (TV) model was originally introduced as an invariant for 3-manifolds using state sum construction [TV92]. It was then realized that the TV model provides a 3-dimensional TQFT.

Later, it was established in [KB10] that the TV model is actually a fully extended TQFT. This means that it assigns a number to each closed 3-manifold, a vector space to each closed 2-manifold, a linear category to each closed 1-manifold, and a linear 2-category to each 0-manifold (i.e., the point) in a compatible manner. It was also demonstrated that

$$\mathrm{TV}_C(M) \simeq \mathrm{WRT}_{Z(C)}(M)$$

where M is a closed manifold of dimension 2 or 3, C is a spherical category and $Z(C)$ is its Drinfeld categorical center ([Bal10] and [Bal11]). $\diamond$

Example 1.4 (Crane-Yetter model) As a 4-dimensional analogue of the TV model, the Crane-Yetter (CY) model has premodular categories as its input algebraic data. It is the main topic of this paper, and will be treated in its own section (1.3). $\diamond$

Theorem 1.5 ($\partial\mathrm{CY} = \mathrm{WRT}$) The WRT model is the boundary theory of CY, while the input algebraic data is a modular category. More precisely, for a modular category C and a 4-dimensional manifold W possibly with boundary,

$$\mathrm{CY}_C(W) = \kappa^{\sigma(W)}\mathrm{WRT}_C(\partial W),$$

where σ denotes the signature and κ denotes a constant based on the input algebraic data (the modular category given in 2.15). This extends to manifolds with colored graphs [BGM07, Theorem 2].

For a 3-dimensional manifold M possibly with boundary, the vector spaces $\mathrm{CY}_C(M)$ (cf 4.9) and $\mathrm{WRT}_C(\partial M)$ are widely believed to be equivalent. However, to our best knowledge a rigorous proof has not been provided. $\diamond$

Remark 1.6 In 1.2 we mentioned that the WRT model does not extend to the 0-th dimension. There are two explanations for this fact.

1. From the Turaev-Viro model perspective [Bal10], for WRT to extend to a point, one needs the input modular category C to be the Drinfeld center of a spherical fusion category D, which is not always the case.
2. From the Crane-Yetter (CY) model perspective, the WRT model is a boundary theory of CY (1.5). However, a single point is not the boundary of any 1-manifold: one needs at least two points.

From the second point of view of 1.2, one sees that the WRT model, though successful and fruitful, is a part of larger theory - the Crane-Yetter model, which is the main TQFT we will focus on in this work. $\diamond$

1.3 The Crane-Yetter TQFT

Originated in [CY93b], the Crane-Yetter model was first defined as a state-sum. In particular, for a triangulated 4-manifold M and a modular tensor category C, one define

$$\mathrm{CY}(M) = \Sigma_c D^{(n_0-n_1)}\Pi_\sigma \dim c(\sigma)\Pi_t \dim c(t)\Pi_\xi \underline{15j}(c,\xi),$$

where c runs through all "colorings", n_i is the number of simplices of dimension i in the triangulation, σ runs through all triangles, t runs through all tetrahedra, ξ runs through all 4-simplices, and $\underline{15j}$ denotes the so called 15j-symbols. The upshot is that the sum is independent of the triangularization, therefore defines an invariant of 4-manifolds. This holds even when the C is a premodular category.

Later in [CKY97], it was known that this seemingly complicated sum can be expressed in terms of the Euler characteristic and the signature, both being old and well-known topological invariants. While this provides a combinatorial formula for the signature of 4-folds, it also means that the CY model with the input data being a modular tensor category C somehow trivializes. A reason why it trivializes is that the WRT model is the boundary theory of the CY model (1.5). Indeed, the input algebraic data for 3-manifolds (modular tensor categories) are too 'ideal' for 4-manifolds. Other types of algebraic data should be considered. Premodular categories are such examples, on which we focus in this paper.

On the other hand, the CY model is expected to be a fully extended TQFT. In particular, it gives a number to each closed 4-manifold, a vector space to each closed 3-manifold, a linear category to each closed 2-manifold (or a 2-manifold with boundary but with empty boundary condition) [AT22] [Tha21], and conjecturally a linear 2-category to each closed 1-manifold (the circle), and conjecturally a linear 3-category to each closed 0-manifold (the point).

1.4 Summary of each section

In this work, we focus on the Crane-Yetter theory. We examine it in three different contexts. The first context involves proving its equivalence to the Turaev shadow state sum as a 4-manifold invariant (see section 3). In the second context, we focus on its values at 2-manifolds. The values are linear categories. We describe the structures of the linear categories for all oriented surfaces with at least one puncture (see section 4). Lastly, an application of Crane-Yetter theory to a problem in tensor categories is provided (see section 5).

2 Preliminaries

Convention 2.1 (manifold, field) Throughout this paper, by a manifold of dimension n we mean an oriented smooth manifold without boundary of real-dimension n; we also work over a fixed field $\Bbbk$ that is algebraically closed and with characteristic 0. $\diamond$

2.1 Algebra

2.1.1 Premodular categories

We will define (pre)modular categories assuming familiarity with a fusion category, a braided category, ribbon structure, and the (Drinfeld) categorical center. A complete and recommended source is [Eti+15]. For definitions written in a dictionary-style starting from "scratch" (additive categories and abelian categories), please refer to section 6.3. Other useful sources are [BK02], [Kas12], [Tur10]. Examples can be found in section 2.1.2.

Definition 2.2 (Braided Fusion Category) A *braided fusion category* is a braided category whose underlying monoidal category is a fusion category. $\diamond$

Definition 2.3 (Muger center) Given a braided fusion category C with braided structure $c_{-,\star}$, we say an object X in C is *transparent* (and otherwise *opaque*) if

$$c_{-,X} \circ c_{X,-} = \mathrm{id}_{X,-}.$$

We define the Muger center $Mu(C)$ of C to be the full tensor subcategory of C consisting of transparent objects. Note that in some other literature, the Muger center is also called a Muger centralizer or an E_2-center. $\diamond$

Recall that if $c_{-,\star}$ is a braided structure of a braided fusion category C, then $c^{-1}_{\star,-}$ is also a braided structure for the underlying fusion category. This produces an opposite braided fusion category, which we denote by C^{bop}. Directly by the definition of a (Drinfeld) categorical center, there is a tautological functor from $C \boxtimes C^{bop}$ to $Z(C)$.

Definition 2.4 (Tautological functor F) Given a braided fusion category C, there is a natural functor $C \boxtimes C^{bop} \xrightarrow{F} Z(C)$ that maps each object $X \boxtimes Y$ to $(X \otimes Y, c_{-,X} \otimes c^{-1}_{Y,-})$ and each morphism $(f \boxtimes g)$ to $(f \otimes g)$. $\diamond$

Definition 2.5 (Factorizable category) Given a braided fusion category C, if its tautological functor F is an equivalence of categories, we say that C is *factorizable*, and call any of its inverse functor a factorization of the Drinfeld center $Z(C)$. $\diamond$

Notice that the structure of $Z(C)$ is in general opaque. For example, even the fusion ring of $Z(C)$ is hard to identify. Factorizability reduces the complexity of $Z(C)$ to that of C.

Definition 2.6 (Premodular Category (6.68, 6.72)) A premodular category is a ribbon fusion category (equivalently, a braided fusion category equipped with a spherical structure). $\diamond$

Definition 2.7 (Complete set of simple objects) Let C be a premodular category. By a complete set of simple objects $\mathcal{O}(C)$ we mean a set $\mathcal{O}(C) = \{i, j, \ldots\}$ of simple objects in C that exhausts all simple types and that satisfies $(i \neq j) \Rightarrow (i \not\simeq j)$. Define its dual set to be

$$\mathcal{O}(C)^\star = \{i^\star \,|\, i \in \mathcal{O}(C)\},$$

where $i^\star$ denotes the (left) dual object of i. $\diamond$

Notice that by the axiom of premodular category, any $\mathcal{O}(C)$ is a finite set. From now on, we assume that any premodular category C comes with a fixed complete set of simple objects $\mathcal{O}(C)$.

Definition 2.8 (S-matrix) Let C be a premodular category with the braided structure c. The S-matrix of C is defined by

$$S := (s_{XY})_{X,Y \in \mathcal{O}(C)}$$

where $s_{XY} = \mathrm{Tr}(c_{Y,X}c_{X,Y}) \in \Bbbk$, where Tr denotes the (left) quantum trace that depends on the spherical structure of C. $\diamond$

Definition 2.9 (Modular Category) [Eti+15, p. 8.13.14] A modular category is a premodular category C whose S-matrix is non-degenerate. $\diamond$

Fact 2.10 (Characterization of Modularity) [Eti+15, 8.20.12 and 8.19.3] The following conditions are equivalent for a premodular category C:

1. C is modular.
2. $Mu(C) \simeq (\text{Vect.})$
3. C is factorizable.

$\diamond$

Surprisingly, the fact indicates that modularity and factorizability are equivalent for premodular categories, so as a consequence modularity reduces the complexity of $Z(C)$. While this is desirable from the algebraic point of view, it is not the case from the topological point of view: The power of the topological quantum field theory is largely reduced by modularity exactly due to this fact.

2.1.2 Examples

Example 2.11 (Finite group) Let G be a finite group. Then the category $\mathsf{Rep}(G)$ of finite-dimensional linear representations of G over $\Bbbk$ has a natural structure of a premodular category. $\diamond$

Example 2.12 (Drinfeld double) Let G be a finite group and $D(G)$ its Drinfeld double over $\Bbbk$. Then the category $\mathsf{Rep}(D(G))$ of finite-dimensional linear representations of $D(G)$ over $\Bbbk$ has a natural structure of a premodular category. $\diamond$

Example 2.13 (Crossed module) Let $\mathfrak{X}$ be a finite 2-group (or called a finite crossed-module) [Ban10]. Then the category $\mathsf{Rep}(\mathfrak{X})$ of finite-dimensional linear representations of $\mathfrak{X}$ over $\Bbbk$ has a natural structure of a premodular category. $\diamond$

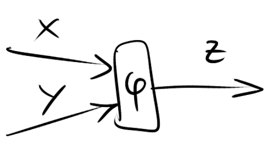
Remark 2.14 Let G be a finite group. Both G and $D(G)$ can be viewed as special cases of finite crossed modules. Hence, 2.13 generalizes 2.11 and 2.12. $\diamond$

Example 2.15 (Quantum group) In the case $\Bbbk = \mathbb{C}$, let $\mathfrak{g}$ be a semisimple Lie algebra and q a root of unity. The semisimplified category $\mathsf{Rep}(U_q(\mathfrak{g}))$ of the category of finite-dimensional representations of the quantum group $U_q(\mathfrak{g})$ has a natural structure of a premodular category. For explicit constructions, see [Kas12] and [BK02]. $\diamond$

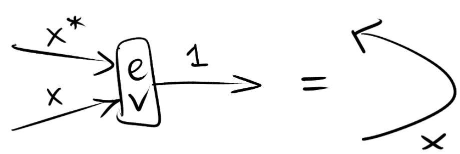
Example 2.16 (Even part of the quantum $\mathfrak{sl}_2$) In the case $\Bbbk = \mathbb{C}$, let q be a root of unity. The semisimplified category $\mathcal{C} = \mathsf{Rep}(U_q(\mathfrak{sl}_2))$ of the category of finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$ has a structure of a modular category. The even part $\mathcal{C}_0$ of $\mathcal{C}$ has a structure of a premodular category [KO01]. $\diamond$

2.1.3 Graphical calculus

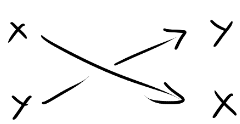
We will use the technique of graphical calculus ([BK02] and [Kas12]) while dealing with premodular categories.



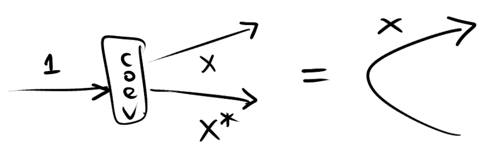
$X \otimes Y \xrightarrow{\phi} Z$



$X^\star \otimes X \xrightarrow{ev_X} 1$



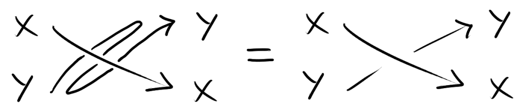
$X \otimes Y \xrightarrow{c_{X,Y}} Y \otimes X$



$1 \xrightarrow{coev_X} X \otimes X^\star$

An advantage of this is that many equalities among morphisms can be proved graphically, thanks to the work of Reshetikhin and Turaev [Res90] [BK02, Theorem 2.3.10]. For example, to prove

$$eval_Y \circ c_{X,Y} \circ c_{X,Y^\star} \circ c_{X,Y} \circ coev_Y = c_{X,Y},$$



it suffices to establish an isotopy of ribbon tangles, which is a trivial task, and translate the procedure back into the equations in the syntactic equations. Such feature of graphical calculus provides sophisticated quantum link invariants (e.g. Jones polynomials). An interesting exercise left for the unconvinced reader is to turn all graphical equations in this paper into syntactic equations.

In the rest of the section, we provide some useful lemmas and notations for graphical calculus.

Lemma 2.17 Let C be a premodular category with spherical structure a. Let X, Y be C-objects. Define a pairing of $\Bbbk$-linear spaces

$$Hom_C(X,Y) \otimes Hom_C(Y,X) \xrightarrow{(,)} \Bbbk$$

that sends $\phi \otimes \psi$ to

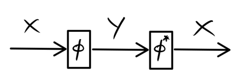
$$Tr(\psi \circ \phi) = eval_X \circ ((a_X \circ \psi \circ \phi) \otimes 1_{X^\star}) \circ coev_X \in End_C(\mathbb{1}) \simeq \Bbbk.$$

Then the pairing is nondegenerate by the semisimplicity of C, identifying the linear space with its linear dual $Hom_C(Y,X) \simeq Hom_C(X,Y)^\star$ ◇

Define the Casimir element

$$\omega_{X,Y} := \Sigma_i \phi_i \otimes \phi^i \in Hom_C(X,Y) \otimes Hom_C(Y,X)$$

where the ϕ_i's is any basis of the former multiplicand and the ϕ^i's is its dual basis under the identification given in 2.17. Graphically, we use dummy variables ϕ and $\phi^\star$ as a short-hand notation:



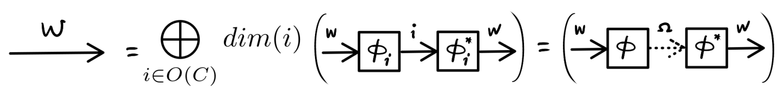
Lemma 2.18 Let C be a premodular category and W be a C-object. Then

$$1_W = \Sigma_{i\in\mathcal{O}(C)}\Sigma_l \dim(i)\phi^l \circ \phi_l,$$

where the ϕ_l's and the ϕ^l's form a pair of dual bases for the vector spaces $\mathrm{Hom}_C(X, Y)$ and $\mathrm{Hom}_C(Y, X)$ respectively, and $\dim(i)$ denotes the (left) quantum trace of id_i. $\diamond$

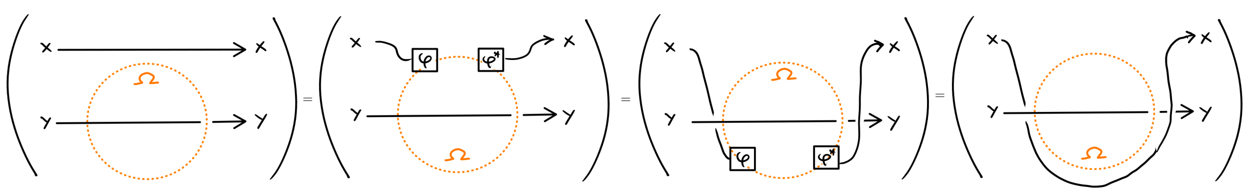
Notation 2.19 (regular color) Let C be a premodular category and $O(C)$ a complete (up to isomorphism) set of simple objects of C. We use Ω in the graphics to represent the regular color $\oplus_{i\in\mathcal{O}(C)}\dim(i)\mathrm{id}_i : i \to i$. We also denote $\dim(\Omega)$ by $\Sigma_{i\in\mathcal{O}(C)}\dim(i)^2$, which is nonzero [ENO09]. $\diamond$

With this shorthand notation Ω, we can present the lemma graphically by



We recall the sliding lemma (lemma 4.28) above.

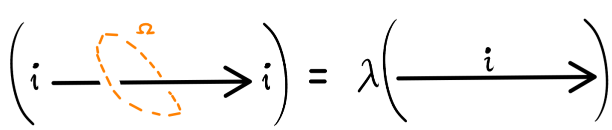
Lemma 2.20 (Sliding lemma) Let C be a premodular category. Then the following morphisms are all equal, where Ω is the shorthand notation given in 2.19.



$\diamond$

Heuristically, the moral of this lemma is that Ω protects everything "inside" it by making it transparent.

Lemma 2.21 (Censorship of Opacity) [Mue03, Lemma 2.13] Let C be a premodular category, $Mu(C)$ its Muger center, i a simple C-object, and $\lambda = \dim(\Omega)\delta_{i\in Mu(C)}$, we have the following equality.

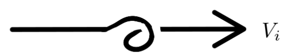


$\diamond$

Heuristically, Ω only allows transparent objects to pass. Note that by the characterization of modularity (2.10), only the identity object is allowed to pass when C is modular.

2.1.4 Coordinates

Recall that a premodular category C is semisimple, $\Bbbk$-linear, and fusion. Define its set of simple objects to be the set I of simple C-objects up to isomorphism. Denote $0 \in I$ so that the monoidal identity $\mathbb{1} \in 0$. As taking monoidal dual preserves simplicity, for each $i \in I$ there is a unique element $i^\star$ in I such that $V_i^\star \in i^\star$ whenever $V_i \in i$. Thus the set I is finite and it has an involution $I \xrightarrow{\star} I$. Using the spherical structure, we can define for each $i \in I$ the number $\dim_C(i) = \dim(i) \in \Bbbk$ as the trace of id_{V_i} and the number $\nu_i \in \Bbbk^\star$ as the twisting coefficient $tr(\theta_{V_i})/tr(id_{V_i})$, where θ_{V_i} denotes the endomorphism of V_i depicted in the following graph.



We further define the Gauss sum of C to be

$$\Delta_C = \sum_{i \in I} \nu_i^{-1} \dim(i)^2.$$

In order to do computations with a premodular category we need to choose and fix some extra data (called a coordinate). All intrinsic results are independent of the choice (except the square root D of the global dimension).

Definition 2.22 (coordinated premodular category) Let C be a premodular category and I its set of simple objects. Choose and fix the following:

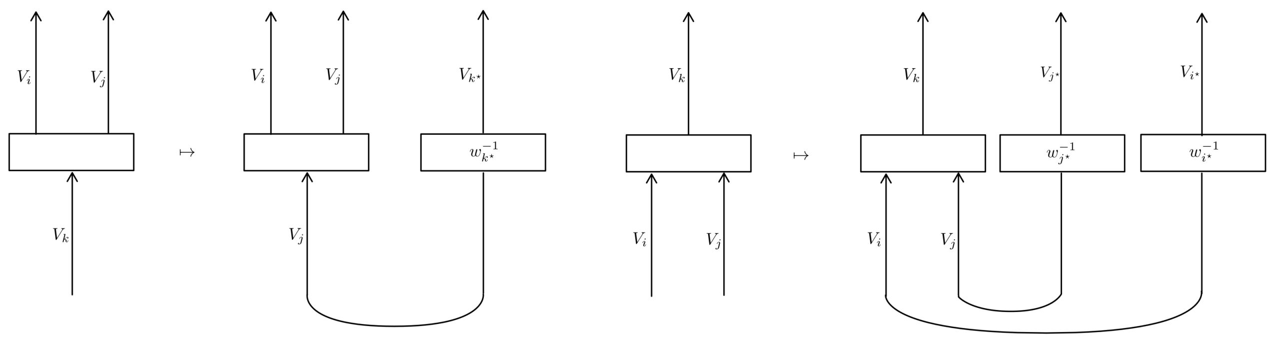
- A number $D \in \Bbbk$ such that $D^2 = \sum_{i \in I} \dim_C(i)^2$ (the global dimension of C).
- A set of C-objects $\{V_i\}_{i \in I}$ such that $V_i \in i$ and that $V_0 = \mathbb{1}$.
- A set of isomorphisms [Tur10, p.313] $\{\omega_i : V_i \to (V_{i^\star})^\star\}_{i \in I}$.
- A set of numbers $\{\dim'_C(i) = \dim'(i) \in \Bbbk\}_{i \in I}$ such that $\dim'_C(0) = 1$, $\dim'_C(i)^2 = \dim_C(i)$, and $\dim'_C(i^\star) = \dim'_C(i)$.
- A set of numbers $\{\nu'_i \in \Bbbk\}_{i \in I}$ such that $\nu'_0 = 1$, $(\nu'_i)^2 = \nu_i$, and $\nu'_{i^\star} = \nu'_i$ [Tur10, p.313].

Such a 5-tuple $\vec{d} = (D, \{V_i\}, \{\omega_i\}, \{\dim'(i)\}, \{\nu'_i\})$ is called a coordinate of the premodular category C. Such a pair $(C, \vec{d})$ is called a coordinated premodular category.
$\diamond$

We will often confuse a premodular category with a coordinated premodular category.

Definition 2.23 (multiplicity module) Let C be a coordinated premodular category and I its set of simple objects. Respectively, define H^{ijk}, H^{ij}_k, and H^k_{ij} to be the $\Bbbk$-modules $\mathrm{Hom}_C(\mathbb{1}, V_i \otimes V_j \otimes V_k)$, $\mathrm{Hom}_C(V_k, V_i \otimes V_j)$, and $\mathrm{Hom}_C(V_i \otimes V_j, V_k)$. $\diamond$

We identify H^{ij}_k with $H^{ijk^\star}$ and H^k_{ij} with $H^{kj^\star i^\star}$ by the linear maps induced by the following graph and call them the canonical identifications:



Recall that the natural pairing

$$H^{ij}_k \otimes_{\Bbbk} H^k_{ij} \to \mathrm{Hom}_C(V_k, V_k) \xrightarrow{\mathrm{tr}} \Bbbk)$$

is nondegenerate by the semisimplicity of C. The braided structure of C guarantees that the $\Bbbk$-modules H^{ijk}, H^{ikj}, H^{jik}, H^{jki}, H^{kij}, H^{kji} are all isomorphic. In category theory, we must carefully distinguish equalities from isomorphicities, hence we introduce a way to keep track of the isomorphisms among the H^{ijk}'s.

Definition 2.24 (canonical isomorphisms) Let C be a premodular category, c its braided structure, I its set of simple objects, and $i, j, k \in I$. Define the canonical isomorphisms $H^{ijk} \xrightarrow{\sigma_1(ijk)} H^{jik}$ and $H^{ijk} \xrightarrow{\sigma_2(ijk)} H^{ikj}$ by

$$\sigma_1(ijk) : \phi \mapsto \nu'_i \nu'_j (\nu'_k)^{-1} (c_{V_i,V_j} \otimes \mathrm{id}_{V_k})\phi,$$
$$\sigma_2(ijk) : \phi \mapsto \nu'_j \nu'_k (\nu'_i)^{-1} (\mathrm{id}_{V_i} \otimes c_{V_j,V_k})\phi.$$

$\diamond$

It is a simple exercise in the theory of tensor categories to check that

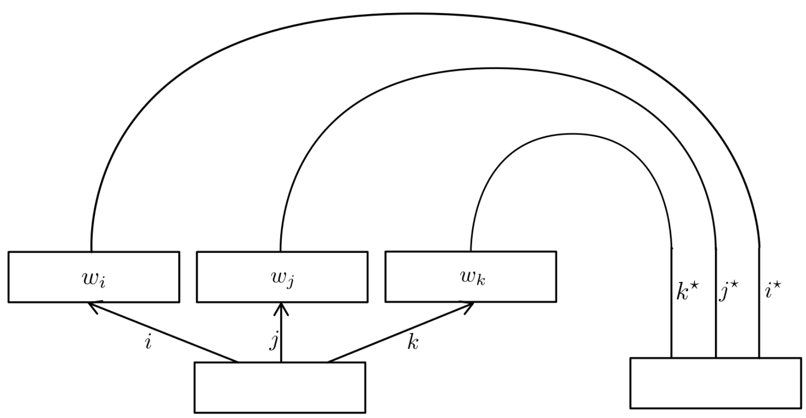
$$\begin{aligned} \sigma_1(jik)\sigma_1(ijk) &= \mathrm{id}, \\ \sigma_2(ikj)\sigma_2(ijk) &= \mathrm{id}, \\ \sigma_1(jki)\sigma_2(jik)\sigma_1(ijk) &= \sigma_2(kij)\sigma_1(ikj)\sigma_2(ijk) \end{aligned} \tag{2.25}$$

so σ_1 and σ_2 specify the isomorphisms among the six $\Bbbk$-modules.

Definition 2.26 (symmetrized multiplicity module) Let C be a premodular category, I its set of simple objects, and $i, j, k \in I$. Define the symmetrized multiplicity module $H(i, j, k)$ to be the $\Bbbk$-module consisting of functions ϕ that assign an element $\phi^{i_1 i_2 i_3} \in H^{i_1 i_2 i_3}$ to each ordering (i_1, i_2, i_3) of the set $\{i, j, k\}$. $\diamond$

The point is that all the symmetrized modules $H(i, j, k)$, $H(i, k, j)$, $H(j, i, k)$, $H(j, k, i)$, $H(k, i, j)$, $H(k, j, i)$ are *equal as sets*. By definition, there is a canonical identification between $H(i, j, k)$ and H^{ijk}.

Definition 2.27 (contraction) Let C be a coordinated premodular category, I its set of simple objects, and $i, j, k \in I$. Define the contraction map $H^{ijk} \otimes H^{k^\star j^\star i^\star} \to \Bbbk$ by the following diagram [Tur10, figure VI.3.5]



Denote the canonically induced contraction map on the symmetrized modules to be ([Tur10, p.334])

$$*_{ijk} : H(i, j, k) \otimes_{\Bbbk} H(i^\star, j^\star, k^\star) \to \Bbbk.$$

This defines a nondegenerate pairing and thus induces a canonical element $\mathrm{Id}(i, j, k)$ in the domain of $*_{ijk}$ ([Tur10, p.333]). $\diamond$

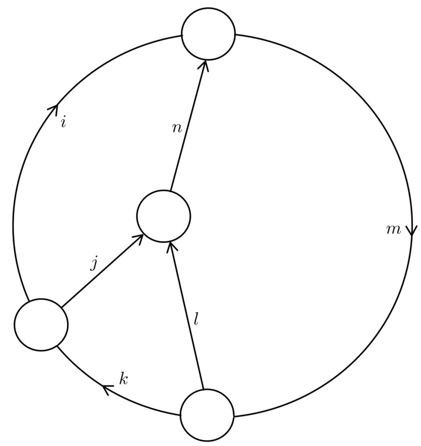
We will abuse notation by denoting natural contractions from the non-ordered tensor products $V \otimes_{\Bbbk} H(i, j, k) \otimes_{\Bbbk} H(i^\star, j^\star, k^\star)$ to $\Bbbk$ by $*_{ijk}$ for any $\Bbbk$-module V.

2.1.5 6j-symbols, 10j-symbols, and 15j-symbols

Definition 2.28 (6j-symbol) For each $i, j, k, l, m, n \in I$, we define the 6j-symbol

$$\begin{bmatrix} i & j & k \\ l & m & n \end{bmatrix} : H^{ij}_k \otimes H^{kl}_m \otimes H^n_{jl} \otimes H^m_{in} \to \Bbbk$$

to be the linear map induced by the partial tensor network on the 2-sphere S^2:



Using the canonical identifications, we define the induced map

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} : H(i,j,k^\star) \otimes H(k,l,m^\star) \otimes H(n,l^\star,j^\star) \otimes H(m,n^\star,i^\star) \to \Bbbk$$

to be the normalized 6j-symbol. ◇

Proposition 2.29 (basic equalities of 6j symbols) Let C be a coordinated premodular category, I its set of simple objects, $i,j,k,k',l,m \in I$, $j_0,j_1,\dots,j_8 \in I$, and δ be the Kronecker delta. Then we have the degenerated 6j symbol

$$\begin{vmatrix} i & j & k \\ l & m & 0 \end{vmatrix} = \delta_{m,i}\delta_{l,j^\star}\dim'(i)^{-1}\dim'(j)^{-1}\mathrm{Id}(i,j,k^\star) \in H(i,j,k^\star) \otimes_{\Bbbk} H(i^\star,j^\star,k). \quad (2.30)$$

We also have the so called Biedenharn-Elliott identity as an equality in the non-ordered tensor product of the $\Bbbk$-modules

$$H(j_3^\star,j_5^\star,j_6) \otimes H(j_1^\star,j_2^\star,j_5) \otimes H(j_4^\star,j_6^\star,j_0) \otimes H(j_0^\star,j_1,j_7) \otimes H(j_7^\star,j_2,j_8) \otimes H(j_8^\star,j_3,j_4)$$

(in the context of state sum over a triangulation, this corresponds to the Pachner $(2,3)$-move):

$$*_{j_0^\star j_5 j_8}\left(\begin{vmatrix} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{vmatrix} \otimes \begin{vmatrix} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{vmatrix}\right) = \sum_{j\in I} \dim(j) *_{j^\star j_2 j_3} *_{j j_4 j_7^\star} *_{j j_1 j_6^\star} \left(\begin{vmatrix} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{vmatrix} \otimes \begin{vmatrix} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{vmatrix} \otimes \begin{vmatrix} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{vmatrix}\right). \quad (2.31)$$

We also have the orthonormality relation

$$\delta_{k,k'}\mathrm{Id}(i,j,k^\star)\otimes\mathrm{Id}(k,l,m^\star) = \dim(k) \sum_{n\in I} \dim(n) *_{im^\star n} *_{jln^\star} \left(\begin{vmatrix} i^\star & j^\star & k^\star \\ l^\star & m^\star & n^\star \end{vmatrix} \otimes \begin{vmatrix} i & j & k' \\ l & m & n \end{vmatrix}\right). \quad (2.32)$$

Finally, we have the Racah identity

$$\nu'_{j_3}\nu'_{j_6}(\nu'_{j_1}\nu'_{j_2}\nu'_{j_4}\nu'_{j_5})^{-1}\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} = \sum_{j\in I}(\nu'_j)^{-1}\dim(j) *_{j^\star j_1 j_4} *_{j j_2 j_5^\star}\left(\begin{vmatrix} j_1 & j_4 & j \\ j_2 & j_5 & j_6 \end{vmatrix} \otimes \begin{vmatrix} j_2 & j_1 & j_3 \\ j_4 & j_5 & j \end{vmatrix}\right) \tag{2.33}$$

◇

Proof. Proofs and references for the modular case can be found in [Tur10, section VI.5.4]. The proof does not use modularity at all, so it carries through for the premodular case verbatim. ■

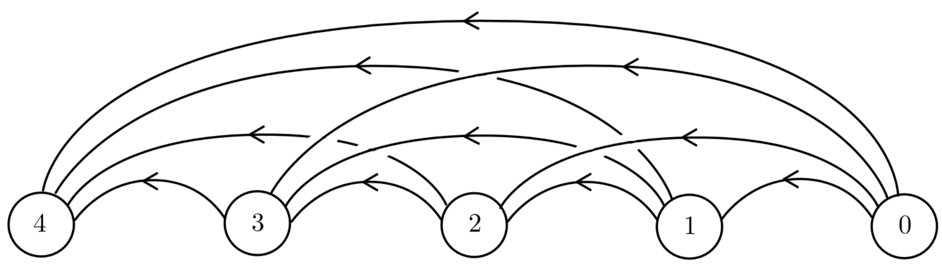
Definition 2.34 (10j-symbol) Let C be a coordinated premodular category, I its set of simple objects, and $j_{ab} \in I$ with $j_{ab} = j_{ba}^\star$ for $0 \leqslant a, b \leqslant 4$. Denote $[x, y, z, w]$ to be the vector space $\mathrm{Hom}_C(V_0, V_{j_x} \otimes V_{j_y} \otimes V_{j_z} \otimes V_{j_w})$. Then define the 10j symbol (and its mirror, resp.)

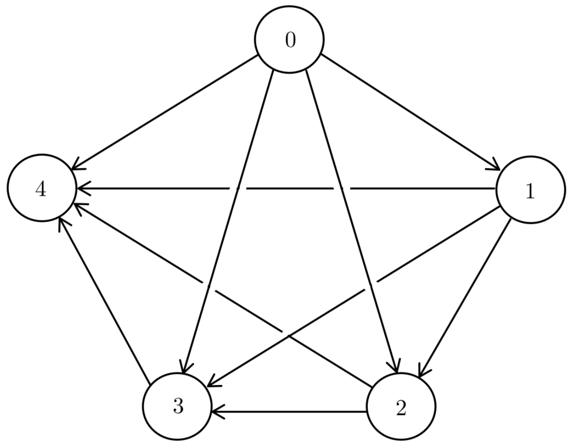
$$\begin{vmatrix} j_{01} & j_{02} & j_{03} & j_{04} \\ \cdot & j_{12} & j_{13} & j_{14} \\ \cdot & \cdot & j_{23} & j_{24} \\ \cdot & \cdot & \cdot & j_{34} \end{vmatrix}_{10j} \qquad \left(\begin{vmatrix} j_{01} & j_{02} & j_{03} & j_{04} \\ \cdot & j_{12} & j_{13} & j_{14} \\ \cdot & \cdot & j_{23} & j_{24} \\ \cdot & \cdot & \cdot & j_{34} \end{vmatrix}_{\overline{10j}}, \text{resp.}\right)$$

to be the $\mathbb{k}$-linear map from the non-ordered tensor product of $\mathbb{k}$-modules

$$[01, 02, 03, 04] \otimes [12, 13, 14, 10] \otimes [23, 24, 20, 21] \otimes [34, 30, 31, 32] \otimes [40, 41, 42, 43]$$

to $\mathbb{k}$ induced by the following (equivalent) C-colored graphs (resp., the same gadget but with the underlying graph mirrored and all arrows reversed).





◇

Remark 2.35 (15j-symbol) A 15j-symbol is an equivalent variant of a 10j-symbol. It was used in the older literature to make sure the morphism spaces are 1-dimensional. The 10j-symbols are more intrinsic, so we use them instead of the 15j-symbols. ◇

2.2 Topology

2.2.1 4-manifolds

Manifolds in real dimension 4 are interesting because of their wildness, witnessed in the following examples:

1. Real dimension 4 is the smallest dimension where the topological structures and the smooth structures disagree.
2. For $n \in \mathbb{N} \setminus \{4\}$, the euclidean space $\mathbb{R}^n$ as a topological space admits exactly one diffeomorphism type, while $\mathbb{R}^4$ admits infinitely many [Sco05][Mil62, p.2].
3. The (smooth) Poincare conjecture for the n-dimensional sphere S^n has been resolved except for $n = 4$, which remains widely open to date despite several attempts.
4. The Universe where we live seems to be best-modeled by a 4-manifold.

Despite its wildness, in dimension 4 the notion of smooth manifolds coincides with the notion of piecewise-linear (PL) manifolds [Tur10, sec.IX.1.1]. The data of the later can be made combinatorial and concrete.

2.2.2 Triangulations

This section is standard [RS72, chap.1] [Man16, sec.2] but included for completeness.

Definition 2.36 (simplicial complex) An abstract simplicial complex is a pair $K = (V, S)$ of finite sets V and $S \subset 2^V$, such that $\tau \in S$ whenever $\sigma \in S$ and $\tau \subset \sigma$. For a subset $S' \subset S$, its closure is

$$\overline{S'} = \{\tau \in S \,|\, \tau \subset \sigma \in S'\}.$$

Given a simplex τ, its star and its link are

$$\mathrm{Star}(\tau) := \{\sigma \in S \,|\, \tau \subset \sigma\}, \quad \mathrm{Link}(\tau) := \{\sigma \in \overline{\mathrm{Star}(\tau)} \,|\, \tau \cap \sigma = \phi\}.$$

We say that K is an abstract combinatorial manifold (possibly with boundary) of dimension n if the link of each of its simplices (or equivalently each of its vertices) is PL homeomorphic to either a sphere or a disk, and if top cell has dimension n. The geometric realization $|K|$ of K is defined as usual by gluing k-dimensional simplices inducively on $k \geqslant 0$.

An orientation of a combinatorial manifold is an ordering of the vertices up to even permutations. We define the standard n-simplex to be $\Delta_n = \{0, 1, 2, \ldots, n\}$, with $[0 < 1 < \ldots < n]$ being its standard orientation. Its kth face is defined to be

$$\Delta_n(\widehat{k}) = (\text{-}1)^k \Delta(012 \ldots \widehat{k} \ldots n).$$

For example, the standard oriented 4-simplex $\Delta_4(01234)$ has a 3-dimensional face being $\Delta_4(\widehat{1}) = \text{-}\Delta_4(0234) = \Delta_4(2034) = \ldots$. This face, in turn, has another 2-dimensional face $\Delta_4(\widehat{12}) = \Delta_4(034)$. In general, for $i < j$, denote $\Delta(\widehat{ij}) = (\text{-}1)^{i+j\text{-}1}(0 \ldots \widehat{i} \ldots \widehat{j} \ldots 4)$. $\diamond$

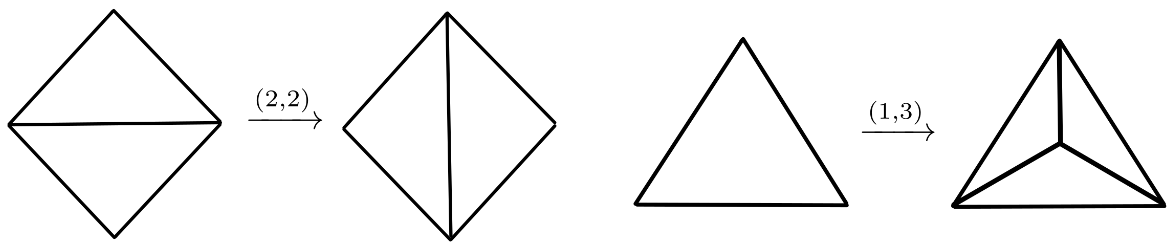
Definition 2.37 (Pachner move) Let an abstract combinatorial manifold $K = (V, S)$ of dimension n. A Pachner $(1, n+1)$-move along a top-simplex $\tau \in S$ is defined to be $K \rightsquigarrow K'$, where

$$K' = \left(V \coprod \{\star\}, \quad (S \setminus \{\tau\}) \coprod (\coprod_f (f \cup \{\star\})) \right),$$

where f funs through each face of τ. A Pachner $(2, n)$-move along two top-simplices $\tau, \tau' \in S$ that share a face $f \in S$ is defined to be $K \rightsquigarrow K'$, where

$$K' = \left(V, \quad (S \setminus \{\tau, \tau'\}) \coprod (\coprod_g (g \cup \{\star, \star'\})) \right),$$

where g runs through each face of f, and $\star$ ($\star'$, resp.) denotes the opposite vertex of f in τ (τ', resp.). We say the inverses are Pachner $(n, 2)$-moves and $(n+1, 1)$-moves respectively. Denote $K \sim K'$ if K' can be obtained by K via a finite sequence of Pachner moves.



$\diamond$

Notice that a Pachner move $K \rightsquigarrow K'$ induces naturally a PL homeomorphism $K \xrightarrow{\sim} K'$.

Definition 2.38 (triangulation of PL-manifolds) Let X be a piecewise-linear manifold. A triangulation of X is a PL-homeomorphism $X \xrightarrow{\phi} |K|$ for some combinatorial manifold K. $\diamond$

Fact 2.39 [Pac87] Any piecewise-linear manifold X has a triangulation $\phi : X \simeq |K|$. Any other triangulation $\phi' : X \simeq |K'|$ satisfies $K \sim K'$. Finally, an orientation of X restricts to a coherent orientation for each top cell of K. $\diamond$

2.2.3 Handle decompositions

By Morse's theory of extremal points, any smooth manifold admits a handle decomposition. By Cerf theory, two handle decompositions present the same manifold (up to diffeomorphism) if and only if both decomposition data are related by a finite sequence of handle creations, handle annihilations, and handle slides [GS01]. A triangulation of a manifold admits a natural handle decomposition by taking dual (each k-dimensional simplex corresponds to a $(4-k)$-handle; each (co)face relation corresponds to an attachment). The correct state sum based on this datum is the universal state sum [Wal21]; it transforms a handle decomposition into a number. A useful fact to notice is that closed 4-manifolds are reconstructible from their handles of indices 0, 1, and 2 (3.19).

2.3 Crane-Yetter state sum

Throughout this section, let C to be a coordinated premodular category and I be the set of simple C-objects.

Definition 2.40 (colored combinatorial manifold) A C-coloring of a combinatorial manifold X is a map $\beta : X_2 \to I$, where X_2 denotes the set of oriented 2-simplices of X, such that $\beta(\text{-}x) = \beta(x)^\star$ for all $x \in X_2$. A C-colored combinatorial manifold is a pair of a combinatorial manifold and a C-coloring of X. $\diamond$

Definition 2.41 (10j symbol for a colored simplex) Let Δ be a 4-simplex with a total ordering on the set of vertices, and let β to be a C-coloring for Δ. C-colored simplex. Denote $\beta_{\widehat{ab}}$ to be the color $\beta(\Delta_4(\widehat{ab})) \in I$ assigned to the oriented 2-cell $\Delta_4(\widehat{ab})$. We define the 10j symbols for (Δ, β) to be the 10j-symbols (2.34)

$$10j(\Delta) = \begin{vmatrix} \beta_{\widehat{01}} & \beta_{\widehat{02}} & \beta_{\widehat{03}} & \beta_{\widehat{04}} \\ \cdot & \beta_{\widehat{12}} & \beta_{\widehat{13}} & \beta_{\widehat{14}} \\ \cdot & \cdot & \beta_{\widehat{23}} & \beta_{\widehat{24}} \\ \cdot & \cdot & \cdot & \beta_{\widehat{34}} \end{vmatrix}_{10j}, \quad \overline{10j}(\Delta) = \begin{vmatrix} \beta_{\widehat{01}} & \beta_{\widehat{02}} & \beta_{\widehat{03}} & \beta_{\widehat{04}} \\ \cdot & \beta_{\widehat{12}} & \beta_{\widehat{13}} & \beta_{\widehat{14}} \\ \cdot & \cdot & \beta_{\widehat{23}} & \beta_{\widehat{24}} \\ \cdot & \cdot & \cdot & \beta_{\widehat{34}} \end{vmatrix}_{\overline{10j}}.$$

$\diamond$

Definition 2.42 (Crane-Yetter state sum for a closed 4-manifold) Let X be an connected, oriented, closed piecewise-linear manifold, $\phi : X \xrightarrow{\sim} |K|$ a triangulation, $\beta : K_2 \longrightarrow I$ a C-coloring of K, and τ a total ordering on the set of vertices of K.

For each 4-simplex Δ of K, we assign a 10j-symbol $10J(\beta, \Delta)$ as follows. If the orientation restricted from X agrees with that from τ (i.e. $[X]|_\Delta = \tau|_\Delta$, or say of coherent orientation), then we assign $10J(\beta, \Delta) = 10j(\Delta, \beta|_\Delta)$; otherwise, if $[X]|_\Delta = -\tau|_\Delta$ (or say decoherent orientation), then we assign $10J(\beta, \Delta) = \overline{10j}(\Delta, \beta_\Delta)$.

Now each 4-simplex has a 10j-symbol, which is just a linear map. Recall that the oriented 3-simplices correspond to morphism spaces. We will contract the linear maps (taking a huge trace) using the fact that each 3-simplex Δ' is the face of exactly two 4-simplices. More concretely, observe that there are two cases.

- Both of them have coherent (or decoherent) orientations.
- One of them has coherent orientation, while the other has decoherent orientation.

In the first case, the corresponding vertices (2.34) of the C-colored graphs that underly the assigned 10j-symbols have the incoming and outgoing arrows exchanged. In the second case, the orientations of the arrows are the same but the colors are dual. Hence in both cases, we can contract the 10j-symbols along Δ' as usual (2.27). Since X is a closed 4-manifold, the final result is an element in the underlying field (i.e. a number).

Finally, we define the Crane-Yetter state sum of X to be the number

$$\int_X^{CY} C := D^{2(n_0 - n_1)} \sum_\beta \prod_f \dim(\beta(f)) \left(* \bigotimes_\Delta 10J(\beta; \Delta) \right),$$

where D^2 denotes the global dimension of C, n_0 denotes the amount of vertices, n_1 denotes the amount of edges, the sum runs over all possible C-colorings β of K, the product runs through all faces f of K (recall $\dim(x) = \dim(x^\star)$ for all x in C), the tensor product runs through all 4-simplices of K, and $*$ denotes the large contraction specified above.

The result only depends on the PL-homeomorphism type of X due to the invariance under Pachner moves. We refer the curious readers to the original paper [CY93a] [CKY97]. $\diamond$

The original state sum uses 15j-symbols and therefore involves a product running through the 3-simplices. The term is absent here because it is absorbed into the 10j symbols. The state sum is expected to be extended to a fully extended topological quantum field theory ([BJS21, section 1.5] [Coo23] [BBJ18] [AT22]). For explicit evaluations of the Crane-Yetter model see [Bär21] (for numerical values on 4-folds) and [Guu21] (for categorical values on 2-folds).

3 Crane-Yetter Theory and Turaev Shadow

3.1 Introduction

Topology is the wildest in dimension 4. For example, the smooth Poincare conjecture remains far from proven only for $n = 4$, and the topological $\mathbb{R}^n$ admits exactly one diffeomorphism type unless $n = 4$, in which case uncountably many are available. There are gauge-theoretic tools which, to some extent, are sensitive to exotic smooth phenomena, such as the Donaldson and Seiberg-Witten invariants. Despite their successes, they are unable to tackle a large class of problems including the smooth Poincare conjecture for $n = 4$.

In the 90s, a simpler invariant of smooth 4-manifolds was proposed by Crane and Yetter (CY). The original CY invariant could only detect homotopy type but its simplicity leaves room for modifications. Despite several attempts at modification (e.g. [Bär21]), to date, there has not been much success at detecting exotic smooth phenomena. A recent work by Reutter [Reu20] explains the failure, and suggests the need for a non-semisimple or derived variant of the CY model.

Before moving into that direction, the author aims to settle another issue first. There is another invariant of 4-dimensional smooth manifolds, the shadow model a la V. Turaev [Tur91] [Tur10], from statistical mechanics. Moreover, it was known that the shadow model coincides with the CY model when both degenerate [Tur10, X.3.2 & theorem X.3.3] [BGM07] to the 3D Witten-Reshetikhin-Turaev model (also known as the quantized Chern-Simons theory). It is thus necessary to clarify their relationship in the general semisimple case. Despite the difference of their origins and formal definitions, this paper shows them equal, suggesting once again that semisimple models have reached their limit in terms of detecting exotic smooth phenomena.

Along proving the equivalence of the two models, we make heavy use of the construction of the shadow model given in [Tur10]. We include the essential details of the construction in this paper which serve as a digestible survey of the shadow model.

3.2 Turaev shadows

A shadow is another type of structure that encodes closed 4-manifolds. Roughly speaking, a shadow is a 2-polyhedron with extra decorations (called gleams) that remember the twisting data. A 2-polyhedron is a topological and combinatorial object that encodes 3-dimensional manifolds [Mat]. It is called a pre-foam in the literature of Khovanov homology (from foams) [KR21].

Definition 3.1 (tripod) Define the standard tripod to be the topological subspace of $\mathbb{R}^3$ consisting of the points (x, y, z) such that at least two of the entries are zero, and the last entry belongs to $[0, 1)$. Define a tripod to be any topological space homeomorphic to the standard tripod. $\diamond$

Definition 3.2 (cone) For each topological space X, define its standard open cone $cone(X)$ to be the quotient space $(X \times \mathbb{R}_{\geqslant 0})/((x, 0) \sim (x', 0))$. Define an open cone of X to be any topological space homeomorphic to $cone(X)$. $\diamond$

Definition 3.3 (local shape) Let X be a topological space and $x \in X$. Denote by T the standard tripod and S the 1-skeleton of the boundary of the standard tetrahedron (a trivalent graph with 4 vertices and 6 edges). Respectively, we say that x is a smooth point, a line point, a tetrahedral point, a boundary smooth point, or a boundary line point of X if it has a relative neighborhood homeomorphic to $(\mathbb{R}^2, 0)$, $(T \times \mathbb{R}, (0, 0))$, $(cone(S), (*, 0))$, $(\mathbb{R} \times \mathbb{R}_{\geqslant 0}, (0, 0))$, or $(T \times \mathbb{R}_{\geqslant 0}, (0, 0))$. $\diamond$

Definition 3.4 (simple 2-polyhedron) A simple 2-polyhedron with boundary is defined to be a piecewise-linear compact CW-complex P of real dimension two, such that each of its point p is either a smooth point, a line point, a tetrahedral point, a boundary smooth point, or a boundary line point. If only the first three types are involved, we call P a simple 2-polyhedron without boundary. $\diamond$

Definition 3.5 (components of a simple 2-polyhedron) Let P be a simple 2-polyhedron with boundary. Define the set of smooth points (or called interior points) of P to be $Int(P)$. Define the set of line points, tetrahedral points, and boundary line points to be $sing(P)$. Define the set of boundary line points and boundary smooth points to be ∂P. Call a connected component of $Int(P)$ to be a region of P; define the set of regions to be $Region(P)$. P is said to be orientable if each region of P is orientable. An orientation of P is an assignment of orientations to each of the region. $\diamond$

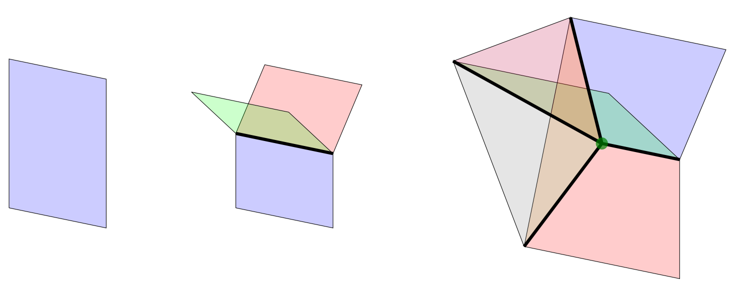


Figure 1: The graphic is taken from [KR21].

Definition 3.6 (shadowed 2-polyhedron) Let P be a simple 2-polyhedron, and A an abelian group with a distinguished element $\omega \in A$. We define a shadow to be a pair of an orientable 2-polyhedron P and a map (called gleam) $gl : Region(P) \to A$.

Unless specified further, we assume that $A = \mathbb{Z}\left[\frac{1}{2}\right]$ and $\omega = \frac{1}{2}$. We denote $-P$ to be the same simple 2-polyhedron but with all gleams flipped by $(a \mapsto -a)$. $\diamond$

For each connected oriented closed surface Σ and each $a \in A$, there is a shadowed 2-polyhedron Σ_a which consists of Σ with the gleam a assigned to the only region. For example, S^2_0 denotes the 0-gleamed 2-sphere.

Definition 3.7 (nullity of a shadowed 2-polyhedron) [Tur10, section VIII.5.1] Let P be an oriented shadowed 2-polyhedron. For each region Y of P, the contraction map $(P/\partial P) \to P/(P \setminus Y)$ and the orientation of Y induces a map

$$H_2(P; \partial P) \to \mathbb{Z}; h \mapsto \langle h|Y\rangle.$$

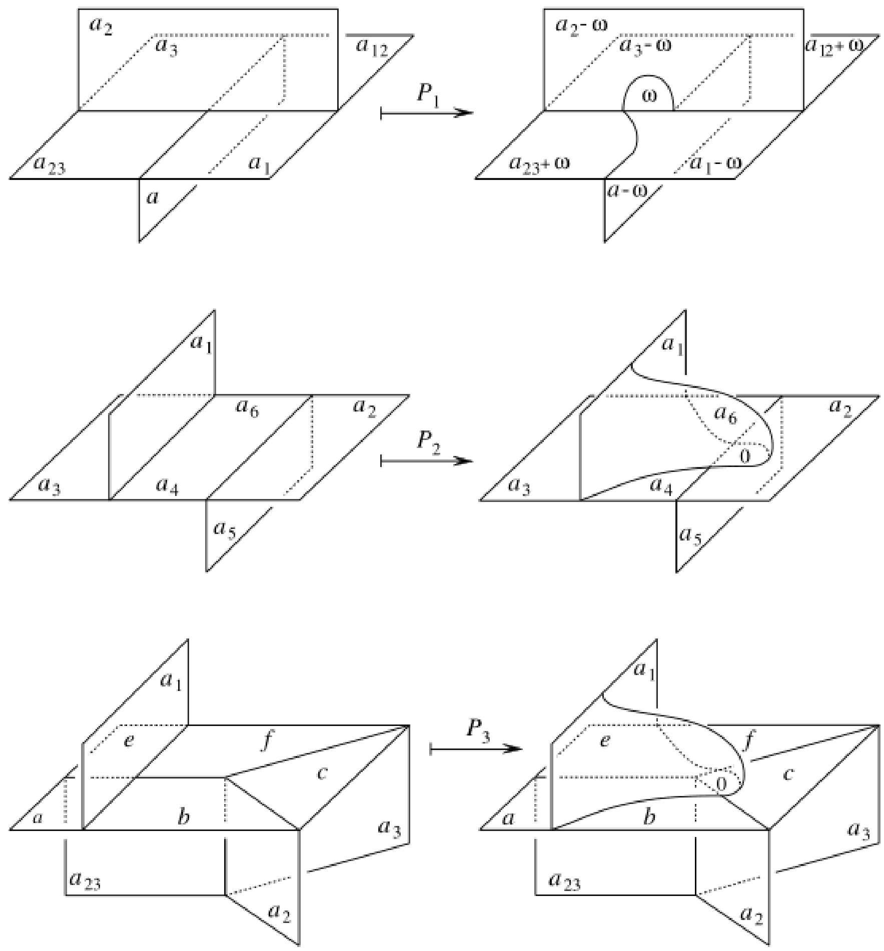
Define the symmetric bilinear form $\tilde{Q}_P$ on $H_2(P; \partial P)$ by summing over all regions of Y

$$\tilde{Q}_P(h_1, h_2) = \sum_Y \langle h_1|Y\rangle\langle h_2|Y\rangle gl(Y) \in A$$

and restrict it to Q_P along the natural map $H_2(P) \to H_2(P; \partial P)$ (which is injective by a usual argument using long exact sequence). $H_2(P)$ is a free abelian group, and so is $Ann(Q_P)$. Finally, define the nullity of P to be $null(P) = rank(Ann(Q_P))$. $\diamond$

We remark that if the shadowed polyhedron comes from a 4-manifold X, then the bilinear form defined in the previous definition coincide with the intersection form of X [Tur10, section IX.5].

Definition 3.8 (shadow moves) [Tur10, section VIII.1.3, p.369]
The basic shadow moves P_1, P_2, P_3 are given in the following graphics (taken from [Tur10]). A shadow move is a finite composition of the $P_i^{\pm 1}$'s.



◇

Definition 3.9 (shadow) A shadow is an equivalence class of shadowed 2-polyhedron P up to a shadow move. We denote the shadow by $[P]$, and say that P represents the shadow $[P]$ [Tur10, p.370]. ◇

For two connected shadow $[P]$ and $[P']$, we construct the shadow $[P] + [P']$ as follows. Arbitrarily identify two arbitrarily chosen closed disks $D \subset \mathrm{Int}(P)$ and $D' \subset \mathrm{Int}(P')$ in $P \coprod P'$, and equip the interior of D (a new region) with gleam 0. So defines a simple 2-polyhedron and we say that it represents $[P] + [P']$. It is well-defined by [Tur10, lemma VIII.2.1.1]. For an integer $m \in \mathbb{Z}_{\geqslant 0}$, we define $m[P]$ as the sum of m-many $[P]$.

Definition 3.10 (stable shadow) Two connected shadowed polyhedra P, P' are called stably shadow equivalent if there exists $n, n' \in \mathbb{Z}_{\geqslant 0}$ such that $[P]+m[S^2_0] = [P']+m'[S^2_0]$. Extend the definition to non-connected ones in an obvious fashion. A stable shadow is defined to be a shadowed polyhedron up to stable shadow equivalence. Denote the stable shadow of $[P]$ to be $stab([P])$. $\diamond$

We are ready to present a closed 4-manifold in terms of shadows.

Definition 3.11 (locally flat 2-polyhedron in a 4-manifold) Let X be a closed 4-manifold. A 2-polyhedron P in X is flat at a point $p \in P$ if there exists a neighborhood U of p in X such that $U \cap P$ lies in a 3-dimensional submanifold of X. We say that P is locally flat if it is flat at all $p \in P$ ([Tur10, p.394]). $\diamond$

Definition 3.12 (skeleton of a 4-manifold) Let X be a closed 4-manifold. A skeleton [Tur10, p.395] of X is a locally flat orientable simple 2-polyhedron without boundary P such that a closed regular neighborhood of it with some 3- and 4-handles form X. $\diamond$

For example, $\mathbb{C}P^1 = \{[x : y : 0]\}$ is a skeleton of $\mathbb{C}P^2 = \{[x : y : z]\}$. By [Tur10, theorem IX.1.5], every 4-manifold has a skeleton (by compressing the $(0, 1, 2)$-handles in an arbitrary handle decomposition).

Definition 3.13 (stable shadow of a 4-manifold) Let X be a closed 4-manifold. Take a skeleton P of X and construct a shadowed simple 2-polyhedron by assigning gleams to the regions Σ in the following way.

1. If Σ is homeomorphic to a closed surface, define the gleam to be the self-intersection (which is independent of the orientation of Σ)

$$([\Sigma] \cdot [\Sigma]) \in H_0(X; \mathbb{Z}) = \mathbb{Z} \subset \mathbb{Z}[1/2].$$

2. Otherwise, Σ is non-compact. Deformation retract it to a compact subsurface Σ_0. Denote N to be the normal bundle of Σ_0 in X. Consider the line bundle l over $\partial\Sigma_0$ by [Tur10, section VIII.6.2, p.397], which may be regarded as a sub-bundle of $N|_{\partial\Sigma_0}$. The circle bundle $\mathbb{P}(N)$ is trivial over Σ_0 since the later is a homotopy 1-type. With a choice of an orientation of Σ_0 and X, l induces a section of $\mathbb{P}(N)|_\partial$. The obstruction class of this section to the whole $\mathbb{P}(N)$ is an element of $H^2(\Sigma_0, \partial\Sigma_0; \pi_1(S^1)) = \mathbb{Z}$. Finally, define the gleam to be the half of the resulting integer (which is independent to the choice of Σ_0).

It is the main theorem of [Tur10, section IX.1.7] that all shadowed polyhedra chosen in such fashion above are all stably shadow equivalent. Therefore, it defines the stable shadow $sh(X)$ of the closed 4-manifold X. $\diamond$

Example 3.14 $sh(\pm\mathbb{C}P^2) = stab([S^2_{\pm 1}])$ and $sh(S^4) = stab([S^2_0])$. $\diamond$

A handle decomposition of a closed 4-manifold X gives rise to a shadow of X [Tur10, section IX.4]. The explicit construction will be recalled below in 3.17, which will be used to prove our main theorem.

Definition 3.15 (skeleton of a 3-manifold) Let Y be a closed 3-manifold. A skeleton of Y is an orientable simple 2-polyhedron without boundary $P \subset Y$ such that $Y \setminus P$ is a disjoint union of open 3-balls [Tur10, p. 400]. ◇

Definition 3.16 (shadow cone of a framed link in a 3-manifold) Every compact 3-manifold Y has a skeleton [Tur10, theorem IX 2.1.1]. For example, the equator S^2 of S^3 is a skeleton. Let P be a skeleton of Y and l be a framed link in Y. Projecting l generically onto P induces a shadow projection. Assign gleams around each crossing point as in [Tur10, figure IX.3.4]. Then construct the shadow by naturally attaching a disk along each projected component on P (as a new region) endowed with zero gleam. Denote the resulting shadow to be $CO(Y, l)$ (well-defined up to stable shadow moves [Tur10, section IX.3.3]). ◇

Definition 3.17 (shadow of a 4-manifold from a handle decomposition) Let X be an oriented 4-manifold and $H = \bigcup_{i=0}^{4} H_i$ be a handle decomposition, where H_i denotes the union of the handles of index i. Define Y to be the closed 3-manifold $\partial(H_0 \cup H_1)$. By the definition of handle decomposition, the gluing datum of H_2 onto the handles with lower indices is encoded as a link l in Y. Define the stable shadow $sh'(X, H)$ to be $CO(Y, l)$. ◇

Remark 3.18 It is a theorem of [Tur10, sec.IX.4.2] that $sh'(X, H)$ does not depend on the choice of H as a stable shadow. In fact, $sh'(X, H)$ equals the stable shadow $sh(X)$ [Tur10, sec. IX.7]. ◇

Remark 3.19 [GS01, section 4.4] The handles of indices $\leqslant 2$ are enough to reconstruct the whole closed 4-manifold. ◇

3.3 Shadow state sum

Throughout this subsection (3.3), we fix an orientable shadowed 2-polyhedron P (over $\mathbb{Z}[\frac{1}{2}]$, with boundary), a coordinated premodular category C, and its set of simple objects I. Our goal is to define the shadow state sum $\int_P^{sh} C$.

Definition 3.20 (module of a trivalent graph) Let K_0 be the empty graph andγ be a trivalent graph. A C-coloring of γ is a map

$$\{\text{oriented edge of } \gamma\} \xrightarrow{\lambda} I, \quad \text{with } \lambda(e) = \lambda(-e)^\star.$$

Define a $\Bbbk$-module

$$H(\lambda) = \bigotimes_x H(\lambda_x, \lambda'_x, \lambda''_x),$$

where H denotes the symmetrized modules (2.26), x runs through all vertices of γ and the λ_x's denote the colors assigned to the nearby edges oriented toward x. Define

$\Bbbk$-modules

$$H(\gamma) = \bigoplus_{\lambda \in \mathrm{color}(\gamma;C)} H(\lambda), \quad H(K_0) = \Bbbk,$$

where $\mathrm{color}(\gamma; C)$ denotes the set of C-colorings of γ. $\diamond$

By a C-coloring of P we mean a map φ from the set of oriented regions of P to I such that $\varphi(\Sigma) = \varphi(-\Sigma)^\star$. Denote by $\mathrm{color}(P; C)$ the set of all C-colorings of P. An orientation of a 2D region induces an orientation on its edges by $(\vec{n} \wedge \text{-})$, where $\vec{n}$ denotes a vector pointing outward from the region. Therefore, a C-coloring φ of P induces a C-coloring $\partial\varphi$ of its boundary ∂P, a trivalent graph.

Definition 3.21 (shadow state sum) [Tur10, section X.1.2]
Every C-coloring on ∂P extends to some C-coloring on P, so $H(\partial P) = \sum_{\varphi \in \mathrm{color}(P;C)} H(\partial\varphi)$. Fix a $\varphi \in \mathrm{color}(P; C)$, and define the following $\Bbbk$-modules and vectors.

- For each oriented edge $\vec{e}$ in $P \setminus \partial P$, define $H_\varphi(\vec{e})$ to be $H(i, i', i'')$ where the i's are the colors assigned to the three adjacent regions compatibly oriented with $\vec{e}$.
- For each (unoriented) edge e in $P \setminus \partial P$, define $H_\varphi(e)$ to be the non-ordered tensor product $H_\varphi(\vec{e}) \otimes H_\varphi(\text{-}\vec{e})$ with an arbitrary orientation $\vec{e}$. The pairing (2.27) defines a canonical vector $|e|_\varphi \in H_\varphi(e)$.
- For each tetrahedral point $x \in P$, pick a small enough neighborhood U of x in P homeomorphic to the cone of the 1-skeleton of the boundary of some tetrahedron. The closure $\overline{U}$ a C-colored 2-polyhedron with four boundary line points x_0, x_1, x_2, x_3 and six C-colored regions. Denote by φ_{ij} the color for the oriented region $\overrightarrow{xx_ix_j}$ (clearly, $\varphi_{ij} = \varphi_{ji}^\star$). Finally, define a vector and a $\Bbbk$-module

  $$|x|_\varphi := \begin{vmatrix} \varphi_{01} & \varphi_{02} & \varphi_{30} \\ \varphi_{32} & \varphi_{13} & \varphi_{21} \end{vmatrix} \in \bigotimes_{i=0}^{3} H_\varphi(\overrightarrow{x_i x}) =: H_\varphi(x),$$

  where $\otimes$ denotes the unordered tensor product of $\Bbbk$-modules. The result is independent to the labeling $0, 1, 2, 3$.

The procedure above defines a vector in the $\Bbbk$-module

$$(\otimes_x |x|_\varphi) \otimes (\otimes_e |e|_\varphi) \in \left(\bigotimes_x H_\varphi(x) \right) \otimes \left(\bigotimes_e H_\varphi(e) \right)$$

where x runs over all tetrahedral points of P, and e runs over all (nonoriented) edges of $P \setminus \partial P$. By contracting the vector along all tetrahedral points x and all edges e whose boundary points are not both in ∂P, we obtain a vector in $|\varphi| \in H(\partial\varphi)$. Finally, we define the shadow state sum to be

$$\left(\int_P^{sh} C \right) = \left(D^{\text{-}b_2(P)\text{-}\mathrm{null}(P)} \sum_{\varphi \in \mathrm{color}(P)} \sigma_\varphi \, |\varphi| \right) \in \sum_\varphi H(\partial\varphi) = H(\partial P),$$

where D denotes the global dimension, b_2 denotes the second betti number, null denotes the nullity (3.7), and $\sigma_\phi \in \Bbbk$ is a normalizing constant defined as

$$\sigma_\phi = \prod_e \dim'_C(\partial\phi(e))^{-1} \prod_Y \dim_C(\phi(Y))^{\chi(Y)} \nu'_\phi(Y)^{2gl(Y)} \prod_g \dim_\Bbbk(\mathrm{Hom}_C(V_0, V_i \otimes V_j \otimes V_k)),$$

where e runs over edges of ∂P (but not circle 1-strata), Y runs over regions of X, g runs over circle 1-strata of $\mathit{sing}(X)$, and χ denotes the Euler characteristics. $\diamond$

Proposition 3.22 (shadow state sum is invariant under stable shadow move) Let C be a premodular category and P, P' be 2-polyhedra that are equal as stable shadows. Then

$$\int_P^{sh} C = \int_{P'}^{sh} C.$$

Namely, shadow state sum is invariant under stable shadow move. $\diamond$

Proof. We start with the special case where C is a modular category. For invariance under basic shadow moves, the essential ingredients are the orthonormality relation, the Racah identity, and the Biedenharn-Elliott identity (2.29); see [Tur10, theorem X.2.1] for a proof. For invariance under addition of S_0^2, it boils down to proving the addition formula

$$|P_1 + P_2| = |P_1| \otimes |P_2|$$

[Tur10, theorem X.2.2] and using the equality $|S_0^2| = D^{-2} \sum_{i\in I} \dim(i)^2 = 1$. Both proofs carry through verbatim to the premodular case. ■

Definition 3.23 (shadow state sum of a 4-manifold) Let X be a closed 4-manifold, C a coordinated premodular category, $\mathit{stab}([P])$ a stable shadow of X represented by a shadowed 2-polyhedron P. Define the shadow state sum $\int_X^{sh} C$ of X to be $\int_P^{sh} C$, which is well-defined by 3.18 and 3.22. $\diamond$

3.4 Main result: equivalence of state sums

Theorem 3.24 (equivalence of state sums) Let X be a closed 4-manifold and C be a coordinated premodular category. Then

$$\int_X^{CY} C = \int_X^{sh} C.$$

Namely, their Crane-Yetter state sum and shadow state sum are equal. $\diamond$

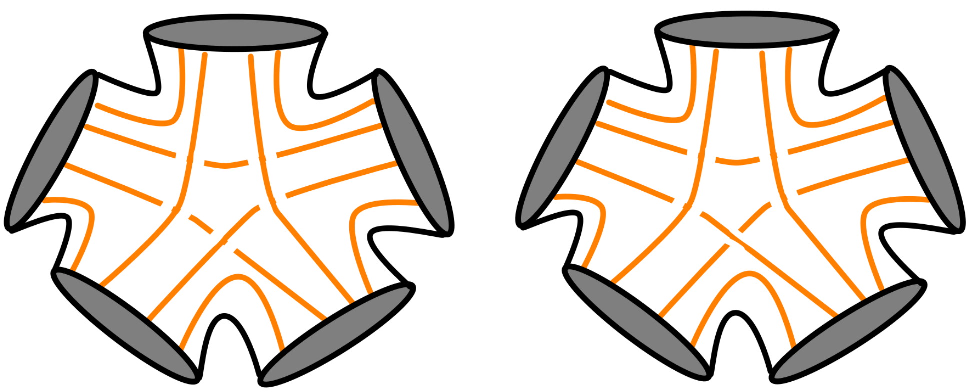
Remark 3.25 The Witten-Reshetikhin-Turaev (quantum Chern-Simons) model is known to be the boundary theory of Crane-Yetter model [BGM07] [Tha21]. It is also shown that the former is the boundary theory of the shadow TQFT [Tur10, X.3.2 & theorem X.3.3]. Therefore, theorem 3.24 provides another proof for the first fact. $\diamond$

Proof. Fix a small $\epsilon > 0$.

We begin by computing the shadow state sum for X. First, fix a triangulation T for X. Denote by T_i to be the set of i-cells of T, and fix a total ordering on T_0. Recall that the ordering induces an orientation of each 4-cell. If it agrees with the orientation from X, we call it an coherently oriented cell; otherwise a decoherently oriented cell. From T we will construct a shadow of a similar "shape". Indeed, the dual of the any triangulation provides a handle decomposition H, in which each k-cell corresponds to an $(n-k)$-handle. The construction in (3.17) constructs a shadow for X, as follows.

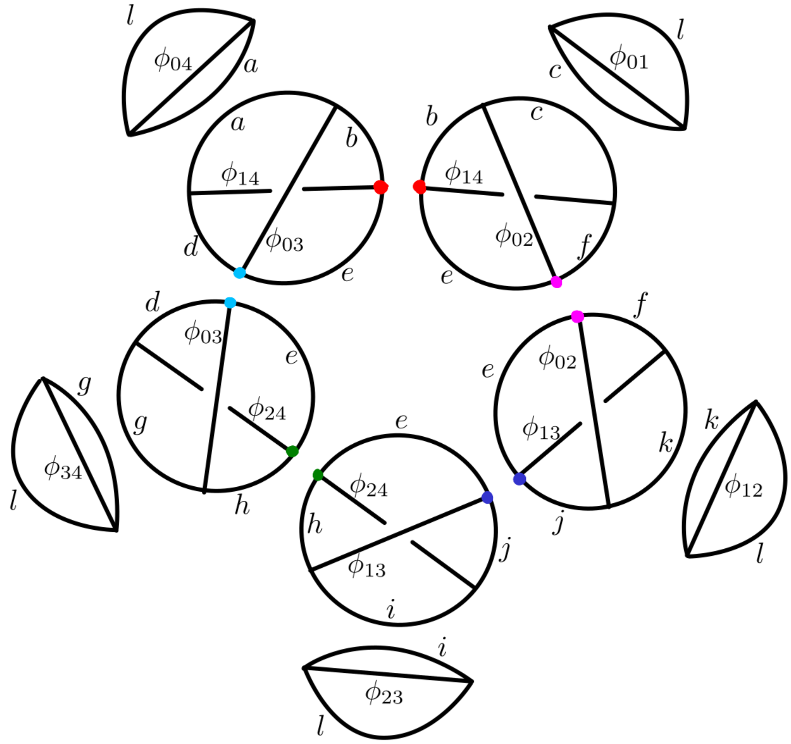
Take the union of the 0-handles and the 1-handles. Its boundary Y is a connected sum of $|T_0|$ 3-spheres. By (3.17 and 3.16), we need to pick a skeleton of Y. We will construct a very concrete one as follows. Within each 4-simplex Δ, $Y \cap \Delta$ is $S^3 \setminus (5 \times B^3)$. Think of this as $\mathbb{R}^3 \setminus \bigcup_{\vec{v}} B_\epsilon(\vec{v})$, where $B_r(x)$ denotes the ball of radius r centered at x, and $\vec{v}$ runs through the set $\{(1,0,0), (0,1,0), (-1,0,0), (0,-1,0)\}$. Denote $S^1(\vec{v})$ to be the equator dual to $\vec{v}$ of each 2-sphere $S^2(\vec{v}) := \partial B_\epsilon(\vec{v})$. The largest component of $B_0(1) \setminus \left(\bigcup_{\vec{v}} S^1(\vec{v})\right)$ is a 4-punctured 2-sphere. Finally, remove an ϵ-disk centered at $(0,0,1)$ from it, and let the boundary straightly stretch to $(0,0,+\infty)$; this is a 5-punctured 2-sphere Σ, which is a local skeleton for Y.

To continue following (3.17), we need to project to the links (the gluing data of the 2-handles in Y) to Σ. It is a geometric exercise to construct a projection so that the projected diagrams look as follows

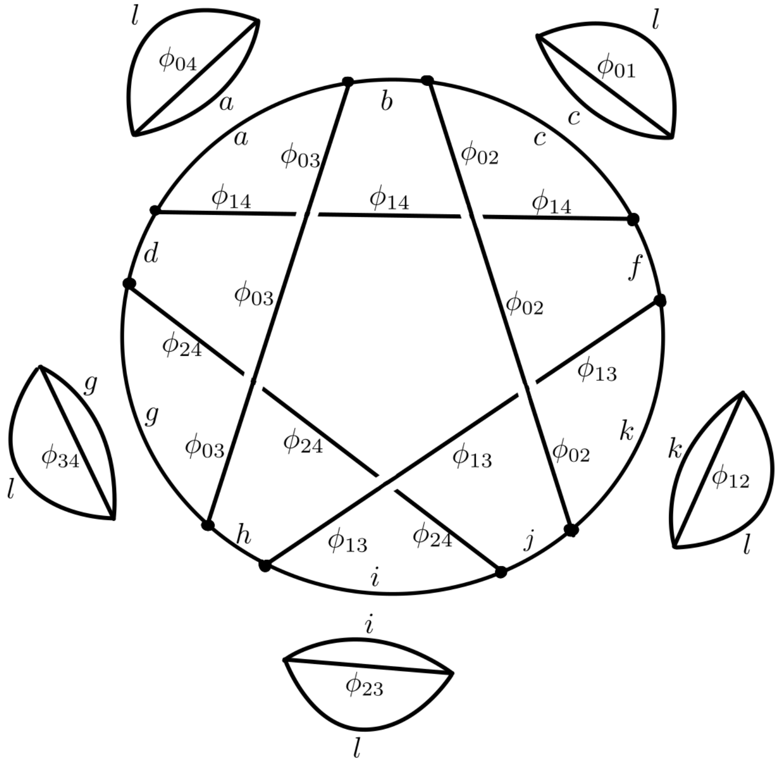


depending on whether the 4-cell is coherently oriented or not. The construction then cones the projected links, and assigns gleams around each intersection of links on Σ. This encodes a local piece of the complete shadow, which is a gleamed 2-polyhedron P without boundary. However, P also looks the same locally within each 4-simplex (up to mirror), so for simplicity we will keep working locally.

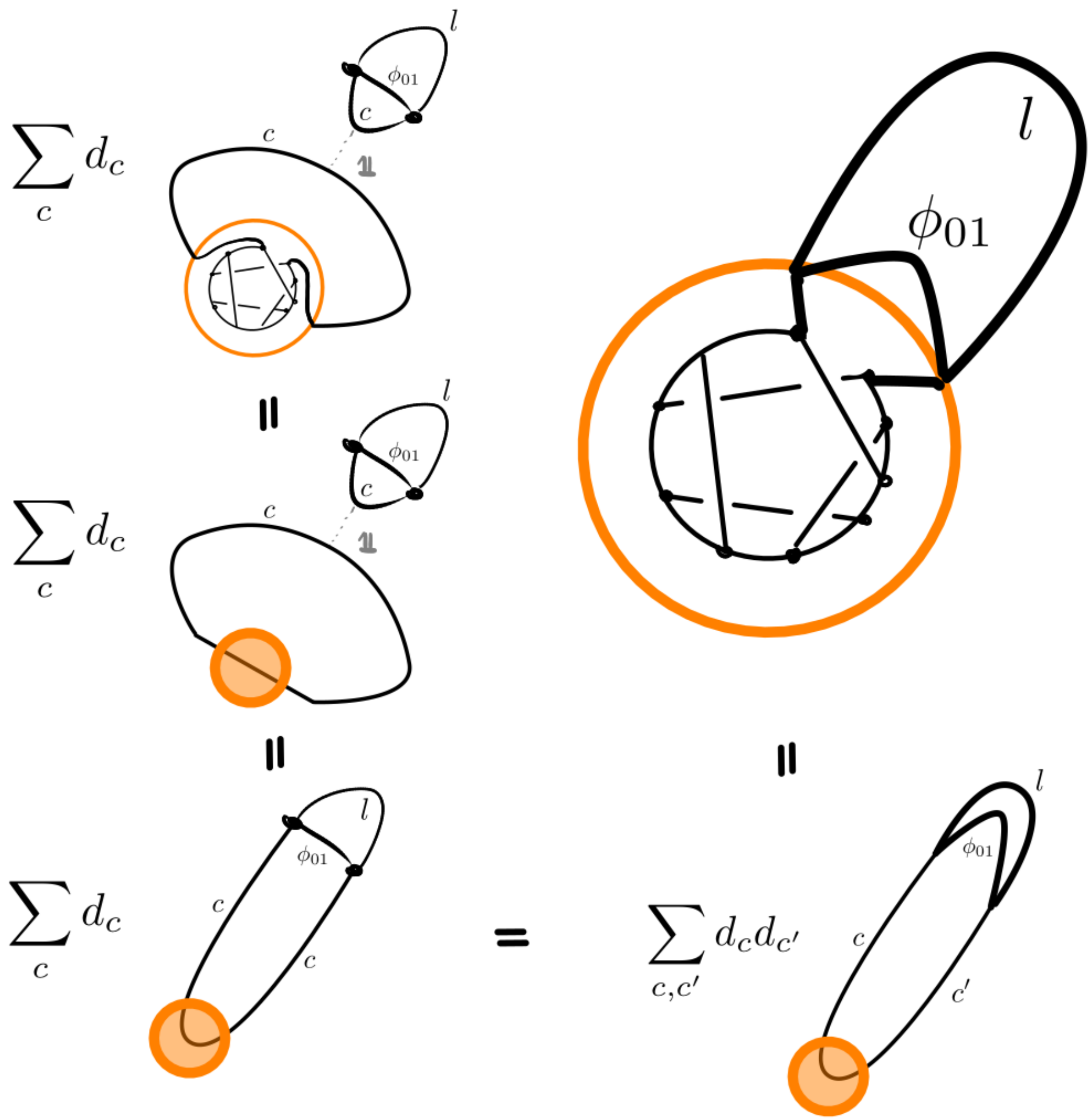
The shadow state sum, locally, is represented by the following diagram and its mirrored image.



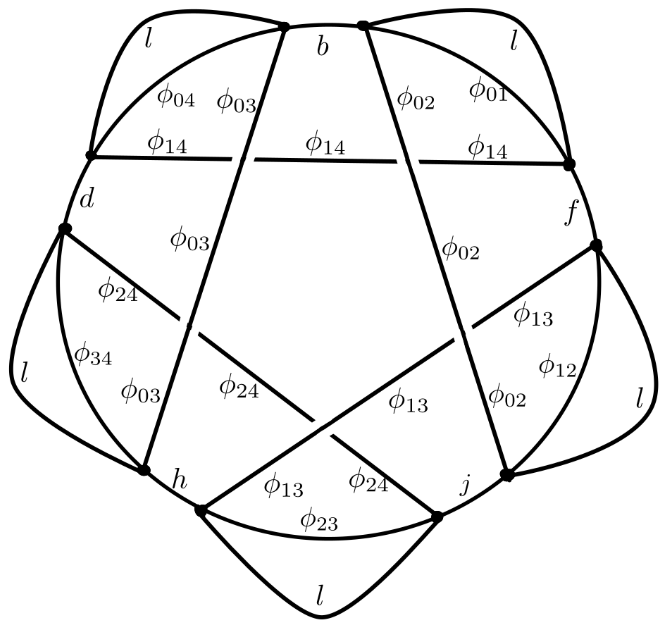
where each of the 5 tetrahedral graphs is obtained by the 6j-symbol twisted with the 4 gleams around the corresponding tetrahedral point. The vertices in the graph are paired (indicated by the colors), and paired vertices are actually labeled by elements of the bases and dual bases of the morphism spaces. We contract them and obtain the following diagram using the techniques given in [KB10, Lemma 1.1, 1.3].



We can further contract each theta graph to the central component using the following procedure.



Repeat for five times, and the result is the following



This is almost the 10j-symbol involved in the definition of the $\int_X^{CY} C$, except the extra edges labeled by b, d, f, h, j, l. However, after contracting the local pieces together, the extra edges form unlinks (colored by the regular coloring $\Omega = \sum_{i \in I} \dim(i)i$) and therefore can be viewed as a factor D^2 and removed from the diagram. The rest of the proof is by counting. ■

4 Categorical Center of Higher Genera

4.1 Overview

Recall that the Crane-Yetter theory is extended to (co)dimension 2 (sec 1.3). So given a premodular category $\mathcal{C}$ and an oriented 2-manifold $Sigma$, its value $\mathrm{CY}_{\mathcal{C}}(Sigma)$ is a linear category. In this section, we compute and describe these the values for all surfaces with at least one puncture.

4.1.1 Previous work of CY in (co)dimension two

Recent developments of the CY model include its Hamiltonian formulation and its higher-codimensional aspect. The former is called the Walker-Wang model (cf [WW12] and [Wal15]). However, while the relation between CY and the Walker-Wang model are commonly believed, to the best knowledge of the author, it has not been explicitly proved.

The later, on the other hand, is currently studied by A. Kirillov's school starting around 2018. In particular, Tham and Kirillov correctly defined CY model in (co)dimension 2, computed some examples, and proved the excision property. This in turn showed that the CY model coincides with the factorization homology in the special case that the input category is premodular.

Remark 4.1 In a recent paper [AT22], it is stated in Corollary 7.6 that the Crane-Yetter theory in (co)dimension 2 coincides with the factorization homology given in [BBJ18] for certain algebraic input data. Therefore, we expect that both approaches actually study the same theory. Please note that [BBJ18] wrote their work earlier than [AT22]. However, we will still quote the results from [AT22] as the author is more familiar with it. ◇

In the rest of this section, we recall the results from Kirillov and Tham. The main statement of this paper will follow in the next section.

Σ	Disk	Cylinder	Sphere	1-punctured torus	General
$\mathrm{CY}_{\mathcal{C}}(\Sigma)$	$\mathcal{C}$	$Z(\mathcal{C})$	$Mu(\mathcal{C})$	$Z^{el}(\mathcal{C})$	$Z_{\sigma}(\mathcal{C})$
		Drinfeld center	Muger center	Elliptic center	Categorical center of higher genera

Theorem 4.2 [AT22, Section 5] Let $\mathcal{C}$ be a premodular category, and $\Sigma = D^2$ be the open disk. Then as abelian categories

$$\mathrm{CY}_{\Sigma}(\mathcal{C}) \simeq \mathcal{C}.$$

◇

Theorem 4.3 (Drinfeld center) [AT22, Example 8.2] Let C be a premodular category, $Z(C)$ its Drinfeld categorical center, and $\Sigma = S^1 \times I = S^1 \times (0,1)$ be the cylinder. Then as abelian categories

$$\mathrm{CY}_\Sigma(C) \simeq Z(C).$$

Moreover, as multifusion categories, the topological nature of Σ induces a so called reduced tensor product $\overline{\otimes}$ ([Tha22], [Was19]) for $Z(C)$. Indeed, stacking two cylinders together produces another cylinder $S^1 \times (0,2) \simeq S^1 \times I$. ◇

Notice that the reduced tensor product is in general different from the usual tensor product of the Drinfeld center.

Theorem 4.4 (Excision principle) [AT22, Theorem 2.3]

Denote the Deligne tensor product by $\boxtimes$. Let Σ be an open surface with n punctures. Then the category $\mathrm{CY}_\Sigma(C)$ has a structure of module category over $\mathrm{CY}_\Sigma(S^1 \times I)^{\boxtimes n}$, which is $(Z(C), \overline{\otimes})^{\boxtimes n}$ by 4.3.

Let Σ_1 and Σ_2 be smooth oriented surfaces possibly with punctures. And let CY be Crane-Yetter theory in dimension two. Then we have an equivalence of abelian categories.

$$\mathrm{CY}_{(\Sigma_1 \cup \Sigma_2)} \simeq \mathrm{CY}_{\Sigma_1} \boxtimes_{\mathrm{CY}_{(\Sigma_1 \cap \Sigma_2)}} \mathrm{CY}_{\Sigma_2},$$

where $\boxtimes_D$ denotes the balanced (Deligne) tensor product over D [DSS19]. ◇

Remark 4.5 For a related result on the excision principle, see [Coo23] and [BBJ18]. In particular, we mentioned above that the CY model is a special case of the factorization homology. In [Coo23], it is further shown that a factorization homology coincides with the skein category for ribbon categories, which are more general than premodular categories; it is also shown that the ribbon categories satisfy the excision principle, providing a more general result than the theorem above. Notice that in [BJS21], a fully-extended 4-dimensional TQFT (based on a braided fusion category) is constructed, and is expected to be an extension of the CY model. Please see discussions in section 1.5.2 and 1.5.3 therein. ◇

Theorem 4.6 (Muger centralizer) [AT22, Corollary 8.5] Let C be a premodular category, $Z'(C)$ its Muger centralizer, and $\Sigma = S^2$ be the 2-sphere. Then as abelian categories

$$\mathrm{CY}_\Sigma(C) \simeq Z'(C).$$

In particular, if C is modular, then the result trivializes as in

$$\mathrm{CY}_\Sigma(C) \simeq (\mathrm{Vect}).$$

◇

Tham defined for a premodular category C an associated category $Z^{el}(C)$, coined the elliptic Drinfeld center. Its objects are the triples (X, γ_1, γ_2), where X is an object of C and the γ_i's are half-braidings of X that satisfy certain relations. See [Tha19] for a full definition. The name is justified by the following theorem.

Theorem 4.7 (elliptic Drinfeld center) [AT22, Corollary 9.5+6] Let C be a premodular category, $Z^{el}(C)$ its elliptic Drinfeld center, and $\Sigma = \Sigma_{1,0}$ be a once-punctured torus. Then as abelian categories

$$CY_\Sigma(C) \simeq Z^{el}(C).$$

In particular, if C is modular, then there is an equivalence of left $CY_{S^1 \times I}(C)$-modules

$$C \simeq CY_{D^2}(C) \simeq CY_\Sigma(C).$$

$\diamond$

Theorem 4.8 [Tha22, Corollary 4.5] Let C be a premodular category, Z the Drinfeld center construction, $\overline{\otimes}$ the stacking tensor product, and $\Sigma = S^1 \times S^1$ be the standard torus. Then

$$CY_\Sigma(C) \simeq Z((Z(C), \overline{\otimes})).$$

$\diamond$

One of the main theorem of [AT22] is that Crane-Yetter theorem in dimension two also trivializes when the input data is modular.

Theorem 4.9 [AT22, Remark 9.8] Let C be a modular category, and Σ an n-punctured surface of genus g. Then up to equivalence $CY_\Sigma(C)$ is independent of g. In fact, we have an equivalence of module categories over $(Z(C), \overline{\otimes})^{\boxtimes n}$

$$CY_\Sigma(C) \simeq C^{\boxtimes n}.$$

Notice that when $n = 0$ the power is the category of finite dimensional vector spaces.

$\diamond$

An easy proof of this fact due to the author of this paper uses the excision principle and a basic equivalence of braided categories $C \boxtimes C^{bop} \simeq Z(C)$ [Eti+15], where C^{bop} denotes the same braided category as C but with the inverse braiding.

4.1.2 Main result: $CY \simeq Z$

The main result of this paper is an explicit calculation of the Crane-Yetter theory for all smooth oriented surfaces with at least one puncture. In particular, we obtain a description of the value on the 3-punctured 2-sphere which was missing in the literature.

With the excision principle, this provides us several combinatorial constructions of the same category. Also with the excision principle, this provides us a fairly compact description for the values on **all** smooth oriented surfaces. In this section, we overview the statement and its consequences, leaving a detailed proof to section 4.4.

Crane-Yetter theory in dimension two is defined as a linear category whose spaces of morphisms are vector spaces presented by many complicated generators and relations. By a calculation we mean to describe it as a category whose description is much smaller. An analogy of this is that the first homology of the circle "calculated" to be the group of integers

$$H_1(S^1) \simeq \mathbb{Z}.$$

Given any open surface Σ, we present it by an oriented 2-disk and some segments of its boundary glued. Which segments are glued together are described by some combinatorial data, called the admissible gluings. With a choice of an admissible gluing σ, the categorical center of higher genera $Z = Z_\sigma$ is a specific category explicitly constructed in 4.49, with some basic properties given in 4.3.3. Roughly, given a premodular category C and an admissible gluing σ of rank n, the categorical center of higher genera $Z = Z_\sigma(C)$ is a category with objects of the form $(X, \gamma_1, \ldots, \gamma_n)$ where X is a C-object and the γ_i's are half-braidings satisfying specific relations. The main statement is that Z is equivalent to $CY(\Sigma)$ as a finite semisimple abelian category.

Theorem 4.10 (Main statement) Let n be a nonnegative integer, σ an admissible gluing of rank n, Σ_σ the surface constructed from σ, C a premodular category, $Z_\sigma(C)$ the categorical center of higher genera of C with respect to σ, and $CY_{\Sigma_\sigma}(C)$ the Crane-Yetter theory (over C) of the surface Σ_σ.

Then $CY_{\Sigma_\sigma}(C)$ only depends on C and the oriented topological type of the surface Σ_σ. Moreover, we have an equivalence of finite semisimple abelian categories

$$CY_{\Sigma_\sigma}(C) \simeq Z_\sigma(C).$$

$\diamond$

A detailed proof of the main statement is given in section 4.4.

Example 4.11 ($n = 0$) For $n = 0$, the surface Σ is the open disk, the categorical center of higher genera reduces to the underlying premodular category C, so the theorem recovers that $CY_\Sigma(C) \simeq C$ as shown in 4.2. $\diamond$

Example 4.12 ($n = 1$) For $n = 1$, the only possible surface is the cylinder, the categorical center of higher genera reduces to the Drinfeld center $Z(C)$. Hence, the theorem recovers that $CY_\Sigma(C) \simeq Z(C)$ as shown in 4.3. $\diamond$

Example 4.13 ($n = 2$) For $n = 2$, there are two possible surfaces: the 1-punctured torus and the 3-punctured disk. In the former case, the categorical center of higher genera reduces to the elliptic center $Z^{el}(C)$, so the theorem recovers that $CY_\Sigma(C) \simeq Z^{el}(C)$ as shown in 4.7. In the later case, the theorem provides a new result. $\diamond$

Remark 4.14 In $H_1(S^1) \simeq \mathbb{Z}$, one sees the algebra of the shape S^1 and the shape of the algebra $\mathbb{Z}$. Our main result should be viewed as a higher analogue. That is, one sees the (higher) algebra of the shape Σ_σ and the shape of the (higher) algebra Z_σ. $\diamond$

Remark 4.15 A full definition for premodular categories is given in 6.68. $\diamond$

Remark 4.16 These categories have their tensor structures and module categorical structures [Eti+15] coming from their topological nature. This will be treated in the author's following work. $\diamond$

Remark 4.17 The smoothness condition is not necessary for our theory, but is included for the sake of simplicity. Indeed, later we will see that the Crane-Yetter theory in dimension 2 can be defined based on stringnets. With the smooth structure, it is easier to regulate how they meet each other. On the other hand, Crane-Yetter theory works also in the PL-setting, parallel to its 3-dimensional analogue, the Turaev-Viro theory. Curious readers are refer to a setup given in [KB10]. $\diamond$

4.1.3 Summary of each subsection

- Section 4.2: The relevant topological theory, namely the Crane-Yetter theory in dimension two, is treated formally in terms of string-nets ([Kir11]).
- Section 4.3: The relevant algebraic theory, namely the categorical center of higher genera $Z_\sigma(C)$, is constructed. We prove some of its basic properties, such as its finite semisimple abelianess and its ambidextrous adjunction with the underlying C.
- Section 4.4: The proof for the main theorem is given, which bridges the topological theory and the algebraic theory.
- Section 4.5: Outlook and remarks.

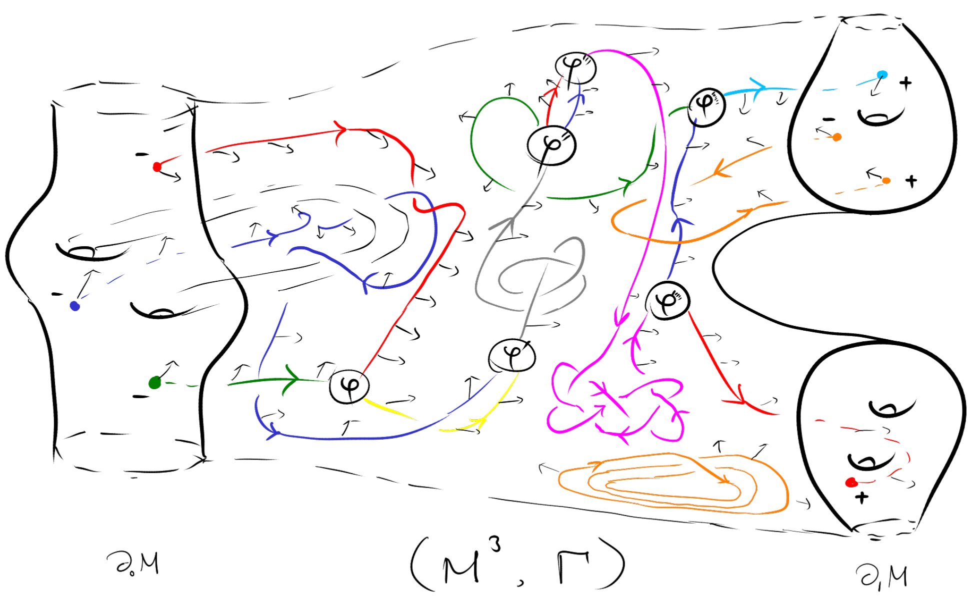
4.2 Topological theory

In this section, we describe the topological side of our main statement (cf 4.10 and 4.14), namely the Crane-Yetter theory in dimension two, is treated formally in terms of string nets. This includes a definition of Crane-Yetter in dimension two, and a combinatorial description of oriented smooth surfaces. The former requires the notion of string nets (also called tensor nets or tensor networks), which will be treated in 4.2.1. A definition of Crane-Yetter theory in dimension two follows in 4.2.2. Finally, the combinatorial description of smooth surfaces (σ-construction) is given in 4.2.3.

4.2.1 String nets

Originated from Penrose combinatorial description of space-time [Pen71], string nets are the building stone of Crane-Yetter theory. They are also called (quantum) tensor nets or tensor networks in other contexts. In dimension two, they were first explicitly written by the physicists Levin and Wen in [LW05]. For Crane-Yetter theory, however, we need string nets in dimension three. Following [AT22], we provide a formal definition 4.27 of them in this section.

Before the formal definition, keep in mind that it aims to formalizes the pictures of the following sort.



Definition 4.18 (2-folds) A 2-fold is either a compact oriented smooth manifold without boundary of real dimension 2, or such a manifold with finitely many points removed (punctures). A 2-fold is also called a surface. $\diamond$

Definition 4.19 (Extended 2-folds) An extended 2-fold is a 2-fold M with the extra data

$$\{(p_1, v_1, or_1) \dots (p_n, v_n, or_n)\},$$

where $n < \infty$, the points $p_i \in M$ are disjoint to each other, the tangent vectors $v_i \in T_{p_i}M$ are nonzero, and the orientations or_i are in the set $\{+, -\}$. ◇

Definition 4.20 (3-folds) A 3-fold is an oriented smooth manifold with boundary of real dimension 3. ◇

Definition 4.21 (Framed arcs in a 3-fold) Let M be a 3-fold (4.20). An arc α in M is a smooth embedding of the standard interval $I = [0, 1]$ (with orientation from 0 to 1) into M. We require that if an end-point is sent by α to the boundary ∂M, then α has to intersect the boundary transversally.

A framing of an arc α in M is a non-vanishing smooth section s of the normal bundle of $\alpha(I) \subseteq M$. A framed arc is an arc equipped with a framing. ◇

Remark 4.22 For our theory, the smoothness condition is not necessary but included for the sake of simplicity. Crane-Yetter theory works also in the PL-setting, parallel to its 3-dimensional analogue, the Turaev-Viro theory. Curious readers are referred to the setup given in [KB10].

The framing, on the other hand, is necessary. Such structure is expressed in slightly different way in related works. For example, in the context of skein modules, people use the notion of ribbons instead of that of arcs. The "width" of a ribbon corresponds to the normal vector from the section. ◇

Definition 4.23 (Framed graphs in a 3-fold) Let M be a 3-fold (4.20). A framed graph Γ in M is a finite collection of framed arcs α_i 4.21 satisfying the following conditions.

- Denote the set of arcs of Γ by $E(\Gamma)$.
- The images of the embeddings α_i's do not meet each other, with an exception at their endpoints which must be in the interior of M.
- Let p be a point in the interior of M. Denote $Out(p)$ ($In(p)$, resp.) be the set of the α_i's with $p = \alpha_i(0)$ ($\alpha_i(1)$, resp.). Denote the set of all arcs (edges) $E(p) := In(p) \cup Out(p)$. The directions of the tangent vectors of the α's that end at p must be different, i.e. for each $i \neq j$, there is no positive real number r such that $v_i = rv_j$. In the case where $E(p)$ is nonempty, we call p a vertex of Γ. Denote the set of vertices by $V(\Gamma)$.
- Let $\alpha \in E(\Gamma)$. If an end of α ends on the boundary of M, we call the end-point a boundary point of Γ. Denote the set of boundary points by $B(\Gamma)$.

◇

Definition 4.24 (Extended 3-folds) An extended 3-fold is a pair of a 3-fold M and a framed graph Γ in M. ◇

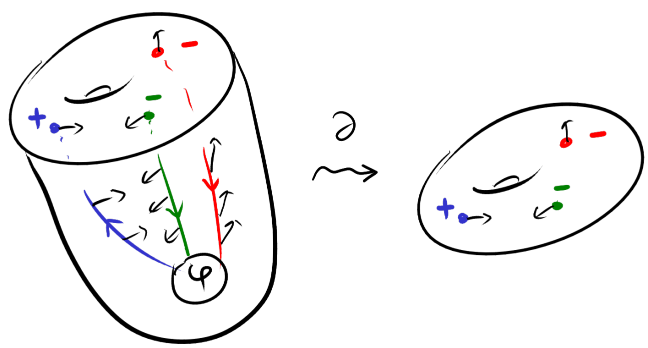
Notice that an extended 3-fold (M, Γ) naturally induces an extended 2-fold $\partial(M, \Gamma) := (\partial M, \partial\Gamma)$, where $\partial\Gamma$ denotes the set

$$\{(p_i, \nu_i, or_i) \,|\, p_i \in B(\Gamma)\},$$

where ν_i is naturally identified via

$$N_{p_i} = T_{p_i}M / T_{p_i}\alpha \simeq T_{p_i}(\partial M)$$

with the framing vector of Γ at the point p_i, and $or_i \in \{+, -\}$ is $+$ ($-$, resp.) if the framed arc α that passes through p_i is oriented such that $\alpha(1) = p_i$ ($\alpha(0) = p_i$, resp.).



Definition 4.25 (C-extended 2-folds) Given a premodular category C, a C-extended 2-fold, or a C-colored extended 2-fold, is an extended 2-fold with an extra data: a C-object X_i is assigned to each oriented framed point (p_i, ν_i, or_i). We call X_i the "color" assigned to the point p_i. ◇

To define a C-extended 3-fold, we need the Reshetikhin-Turaev theory ([Tur10], [BK02], [Kas12]) for genus-0 surfaces which we recall here. Let C be a premodular category. To each C-extended 2-fold

$$M = (M, \{(X_1, p_1, \nu_1, or_1) \dots (X_n, p_n, \nu_n, or_n)\})$$

diffeomorphic to a sphere, the first part of Reshetikhin-Turaev theory functorially assigns a vector space $RT(M)$ that is (non-canonically) isomorphic to

$$\langle X_1^\epsilon, \dots, X_n^\epsilon \rangle := \mathrm{Hom}_C(\mathbb{1}, X_1^\epsilon \otimes \dots \otimes X_n^\epsilon),$$

where X_i^ϵ denotes X_i if $(or_i = +)$ or X_i^* if $(or_i = -)$.

Let C be a premodular category, and V be a real vector space of dimension 3. Let S be a finite collection of distinct framed oriented rays from the origin of V, with an assignment $S \xrightarrow{\phi} \mathrm{Obj}(C)$. In this case, we say V has a finite collection of distinct C-colored rays. Then the Reshetikhin-Turaev theory for genus-0 surfaces naturally assigns a vector space $RT_C(V, S, \phi)$ to the sphere $(V - 0)/\mathbb{R}_+$.

Definition 4.26 (C-extended 3-folds) Let C be a premodular category, and M be an extended 3-fold (M, Γ). A C-coloring of (M, Γ) is an assignment as follows:

- To each arc α of Γ, assign a C-object $X(\alpha)$.
- After such assignment, to each vertex p of Γ, the tangent space at p naturally has a finite collection of C-colored rays (S, ϕ).
- The Reshetikhin-Turaev theory for genus-0 surfaces assigns a vector space $RT(p) := RT_C(T_p(M), S, \phi)$ as above. Note that $RT(p)$ is (non-canonically) isomorphic to

  $$\mathrm{Hom}_C(\otimes X_i, \otimes X_o)$$

  where the X_i runs through the objects assigned to all incoming arcs, and the X_o runs through the objects assigned to all outgoing arcs.
- After such assignment, to each vertex p of γ assign a vector $v \in RT(p)$.

A C-extended 3-fold M is a 3-fold with a C-colored graph inside. This gives the boundary ∂M a C-extended surface structure. Conversely, we call such a C-colored graph a framed graph that satisfies the boundary condition posed by the C-extended surface ∂M. $\diamond$

Let C be a premodular category. While the first part of Reshetikhin-Turaev theory for genus-0 surfaces assigns a vector space to a C-extended genus 0 surface, the second part of it assigns a C-extended 3-fold (M, Γ) diffeomorphic to a ball to a vector $RT(M, \Gamma) \in RT(\partial M)$.

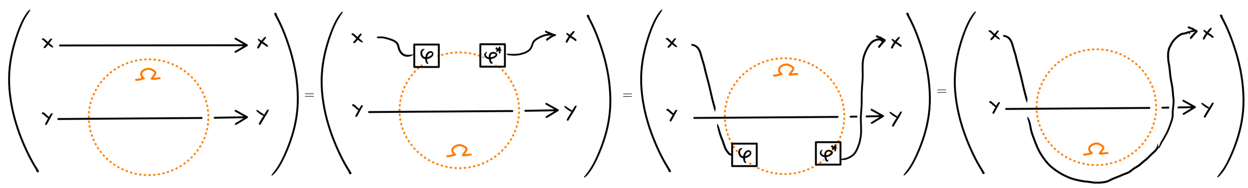
Definition 4.27 (String nets in 3D) Let C be a premodular category and M a 3-fold whose boundary ∂M is a C-extended surface. Let F be the free vector space over $\Bbbk$ generated by all C-colored graphs that satisfy the boundary condition posed by ∂M. Let N be the subspace generated by either of the following

- The difference $\Gamma - \Gamma'$ of two C-colored graphs that are smoothly isotopic to each other.
- A linear combination $v = \Sigma c_i \Gamma_i$ such that there exists a closed region $B \subseteq M$ diffeomorphic to a ball such that the vector assigned by the Reshetikhin-Turaev theory for genus-0 surfaces of $v|_B$ is the zero vector.

We call $S(M) := F/N$ the space of string nets of M with the given boundary condition, and we call an element of $S(M)$ a string net. $\diamond$

We conclude this section with a useful lemma.

Lemma 4.28 (Sliding lemma) Let C be a premodular category. Then the following string nets are equal, where Ω is the shorthand notation given in 2.19. Heuristically, the moral is that Ω protects anything "inside" it by making it transparent.



◇

Proof. Apply 2.18 locally with $W = X \otimes \Omega$. Use isotopy and the naturality of the braidings. And then apply 2.18 locally again with $W = \Omega \otimes X$. ∎

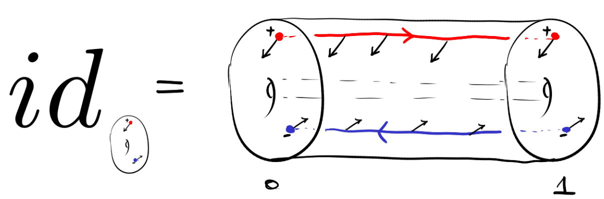
Notice that in fact Y can be more general than an object - the lemma works even when Y is a puncture.

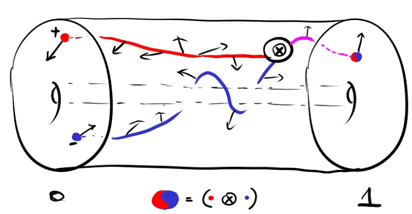
4.2.2 Crane-Yetter theory in dimension two (CY)

We define the Crane-Yetter theory in dimension two in this subsubsection, following [AT22, section 5]. Let Σ be a smooth oriented 2-manifold and $\mathcal{C}$ a premodular category. To define $\mathrm{CY}_\Sigma(\mathcal{C})$, we first define an auxiliary category $\mathrm{cy}_\Sigma(\mathcal{C})$.

Definition 4.29 ($\mathrm{cy}_\Sigma(\mathcal{C})$, an auxiliary category) Given a premodular category $\mathcal{C}$ and a 2-fold Σ, we define the $\mathbb{k}$-linear category $\mathrm{cy}_\Sigma(\mathcal{C})$ as follows. An object is a collection c of $\mathcal{C}$-colored points and tangent vectors, such that (Σ, c) is a $\mathcal{C}$-extended 2-fold. Given two objects c and c', the morphism space $\mathrm{Hom}_{\mathrm{cy}_\Sigma(\mathcal{C})}(c, c')$ between c and c' is defined to be the space of string nets for the 3-fold $\Sigma \times [0,1]$ satisfying the boundary condition $(\bar{c} \times \{0\}) \cup (c' \times \{1\})$, where $\bar{c}$ denotes the same collection of $\mathcal{C}$-colored points as c does but with all orientations flipped. ◇

Two examples of morphisms in $\mathrm{cy}_{\Sigma_{1,0}}(\mathcal{C})$ is depicted as follows.





Definition 4.30 (Karoubi envelope) Given an additive category C, its Karoubi envelope (Karoubi completion) $\mathrm{Kar}(C)$ is defined to be the category as follows. The objects are pairs (X, p), where $X \in \mathrm{Obj}(C)$ and $p \in \mathrm{Hom}_C(X, X)$, such that $p^2 = p$. Given objects $\bar{X} = (X, p)$ and $\bar{Y} = (Y, q)$, the space of morphisms $\mathrm{Hom}_{\mathrm{Kar}(C)}(\bar{X}, \bar{Y})$ is defined to be the subspace of $\mathrm{Hom}_C(X, Y)$ consisting of those f such that $qfp = f$. $\diamond$

The Karoubi envelope is the pre-abelian completion in our context.

Definition 4.31 ($CY_\Sigma(C)$, Crane-Yetter theory in dimension 2) With the notations above, we define

$$CY_\Sigma(C) := \mathrm{Kar}(cy_\Sigma(C)). \tag{4.32}$$

$\diamond$

Remark 4.33 The definition given in 4.31 was first given in [AT22, section 5]. That it extends the original Crane-Yetter theory is proved in [Tha21]. $\diamond$

It is immediate from the definition that $cy_\Sigma(C)$ is additive. On the other hand, $CY_\Sigma(C)$ is in fact finite semisimple abelian for all surfaces with at least one puncture (cf 4.37, 4.57, 4.58). It's conjectured that it holds in fact for all surfaces.

4.2.3 A presentation of surfaces (σ-construction)

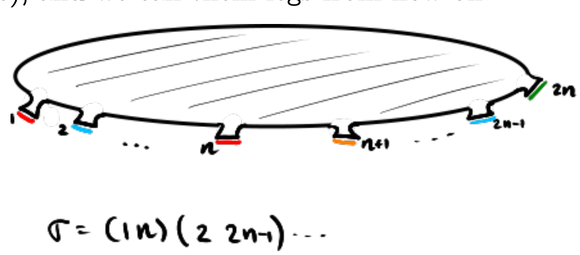
In this paper, we construct a surface Σ from the standard disk and an additional data $\sigma \in \mathrm{Adm}_{2n}$., give some examples, and prove that such construction produces all oriented surfaces with at least one puncture.

Definition 4.34 (Adm_{2n}, admissible gluings) Let n be a nonnegative integer. An element σ in the permutation group S_{2n} on $2n$ elements is called an admissible gluing (of rank n), if σ satisfies the following conditions.

- σ has no fixed points.
- σ is an involution; i.e. $\sigma^2 = 1$.

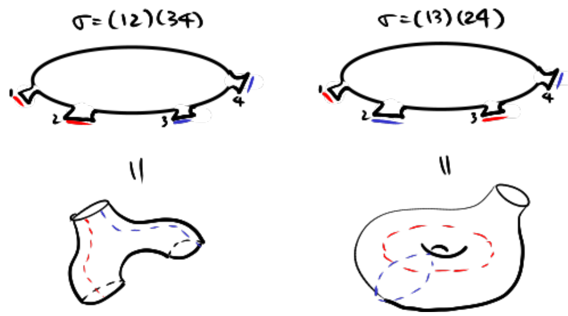
We denote the subset of admissible gluings by $\mathrm{Adm}_{2n} \subseteq S_{2n}$. $\diamond$

Definition 4.35 (Σ_σ, σ-construction) For each admissible gluing $\sigma \in \mathrm{Adm}_{2n}$, we construct a smooth surface Σ_σ. Start from the standard oriented disk. We choose $2n$ closed segments with the same length from the boundary. To make the presentation easier, we emphasize them by drawing them like legs (without changing the diffeomorphism type), and we call them legs from now on.



Glue the end of the legs in pairs according to σ with the orientation preserved. Finally, removed the boundary the the result to be an open surface. The result is denoted by Σ_σ. $\diamond$

Example 4.36 The only element (12) in Adm_2 constructs the cylinder $\Sigma_{(12)} \simeq$ Cylinder. The elements $\sigma_{0,3} = (12)(34)$ and $\sigma_{1,1} = (13)(24)$ in Adm_4 construct a 3-punctured sphere Σ and a 1-punctured torus respectively.



$\diamond$

So constructed surfaces must have at least one puncture, thus the procedure does not give all surfaces. However, the following theorem shows that this is the only case it misses.

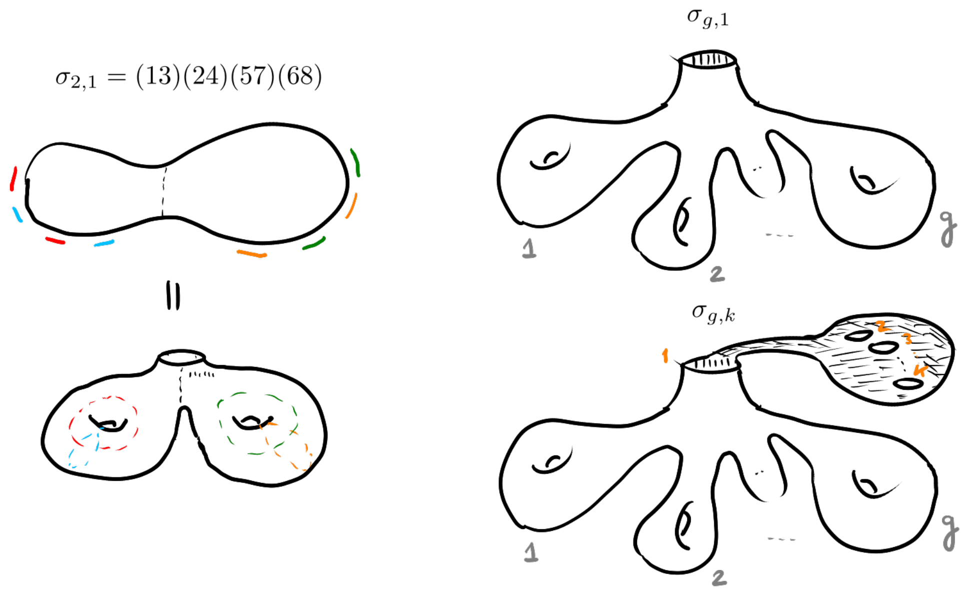
Theorem 4.37 The σ-constructions produce all oriented surfaces with at least one puncture (i.e. all open surfaces). $\diamond$

Proof. Indeed, the admissible gluing $\sigma_{1,1} = (13)(24) \in \mathrm{Adm}_4$ gives an once-punctured torus $\Sigma_{\sigma_{1,1}}$. Similarly, the admissible gluing

$$\sigma_{2,1} = \sigma_{1,1} \circ (57)(68) = (13)(24)(57)(68) \in \mathrm{Adm}_8$$

gives an once-punctured surface of genus two. Following this fashion, for any $g \in \mathbb{N}$ one can construct an once-punctured surface of genus g by using the admissible gluing

$$\sigma_{g,1} = (13)(24)(57)(68)\ldots((4g-3)(4g-1))((4g-2)(4g)) \in \mathrm{Adm}_{4g}\,.$$



To add k punctures to the surface, use the admissible gluing

$$\sigma_{g,(k+1)} = \sigma_{g,1} \circ ((4g+1)(4g+2))((4g+3)(4g+4)) \ldots ((4g+2k-1)(4g+2k)) \in \mathrm{Adm}_{4g+2k}.$$

Then the statement follows from the well-known classification of oriented smooth surfaces. ■

Notation 4.38 (σ-orbit) We take this opportunity to introduce a later useful notation. Let $\sigma \in \mathrm{Adm}_{2n}$. Denote $[i]$ to be the orbit of

$$i \in \{1, 2 \ldots, 2n\}$$

under the action of σ, $[i]'$ the smaller number in the set $[i]$, and $[i]''$ the larger number in the set $[i]$. Note that the set $\{[1], [2], \ldots, [2n]\}$ has exactly n elements.

As an example, for $\sigma = (13)(24)$, we have

$$\begin{aligned} &[1] = \{1, 3\}, \quad [1]' = 1, \quad [1]'' = 3; \\ &[2] = \{2, 4\}, \quad [2]' = 2, \quad [2]'' = 4; \\ &[3] = \{1, 3\}, \quad [3]' = 1, \quad [3]'' = 3; \\ &[4] = \{2, 4\}, \quad [4]' = 2, \quad [4]'' = 4. \end{aligned}$$

◇

4.3 Algebraic theory

In this section, we describe the algebraic side of our main statement (cf 4.10 and 4.14), namely the categorical center of higher genera Z.

Its definition is quite algebraic and abstract, so some motivation is supplemented in 4.3.1. The formal definition is given in 4.3.2. Finally, some basic properties of Z are proved in 4.3.3. In particular, we show that Z is finite abelian semisimple, and that there is a strictly ambidextrous adjunction between $Z = Z(C)$ and the underlying premodular category C.

4.3.1 Motivation: Drinfeld categorical center

Abstract algebraic theories (groups, rings, modules.. etc) are ubiquitous in modern mathematics. Among the algebraic objects, the abelian ones are simpler, and are often first treated. One then builds the theory toward the generic cases. In group theory, for example, one can study a group G by starting with its center $Z(G) \subseteq G$ and then apply induction.

Drinfeld's categorical center is an analogue in the categorical setting. There, algebras are replaced by categorical algebras (more precisely, by monoidal categories [Eti+15]), and centers are replaced by categorical centers. As in the classical theory, the theory of the one side helps that of the other.

In contrast to the classical case, categorical centers need not be smaller nor easier. This is due to the fact that equalities are replaced by equivalences in the categorical settings. Therefore, the condition $ab = ba$ is replaced by $ab \simeq ba$. That is to say, a categorical commutativity not only remembers both sides being identified, but also **how** they are identified. Therefore, a typical object in the Drinfeld center $Z(C)$ is a pair $(X \in \mathrm{Obj}(C), \gamma)$, where γ is a half-braiding that encodes how X commutes with all the others. To be more precise, a **half-braiding** γ of X is a natural equivalence

$$(-) \otimes X \xrightarrow{\gamma} X \otimes (-)$$

satisfying some compatibility conditions 6.62. It is worthwhile to mention that such construction has been successful in many contexts, e.g. representation theory, statistical physics, knot theory, .. etc.

Categorical center of higher genera $Z = Z_\sigma(C) = Z_{\Sigma_\sigma}(C)$, on the other hand, generalizes the Drinfeld center. Instead of remembering how X commutes with others, an object (X, γ) remembers how X commutes in **multiple** different ways. The amount of ways depends on the underlying surface $\Sigma = \Sigma_\sigma$. Therefore, an object of $Z_\sigma(C)$ is a pair (X, γ), where γ is a collection of half-braidings

$$\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}.$$

However, extra conditions must be carefully imposed in order to keep track of the underlying topological data. In contrast to the case of Drinfeld center, multiple half-braidings give essentially infinite ways to fuse via tensors, e.g. $\gamma_1\gamma_2\gamma_2\gamma_1\gamma_1\gamma_1\gamma_2\ldots$. Therefore, suitable commutative relations among the half-braidings are needed. This is given in the formal definition of $Z_\sigma(C)$ as `(:comm 1)`, `(:comm 2)`, and `(:comm 3)` (cf 4.40).

Before moving on to the formal definition of $Z_\sigma(C)$, let us remark on the premodular condition on C. As a classical analogue, it does not make sense to talk about the center $Z(S)$ for a set S; one needs a few extra structures on S. In the categorical setting, in order to define the Drinfeld center, merely a plain category C is not enough. Essentially, a monoidal structure is required. Similarly, for categorical centers of higher genera, we need essentially the braided structures (cf 6.60), which are included in the premodular condition. Note that we will assume premodularity for other purposes, but the categorical center of higher genera can certainly be defined for other less restricted categories.

4.3.2 Categorical center of higher genera (Z)

In this section, we formally define the categorical center of higher genera $Z_\sigma(C)$ for a premodular category C and an admissible gluing $\sigma \in \mathrm{Adm}_{2n}$. Assume C to be a premodular category throughout this section.

Definition 4.39 (σ-pair) Let $\sigma \in \mathrm{Adm}_{2n}$, i.e. σ an admissible gluing. Define a σ-pair of C to be a pair (X, γ), where X is a C-object and γ is a set of half-braidings for X (cf 6.62)

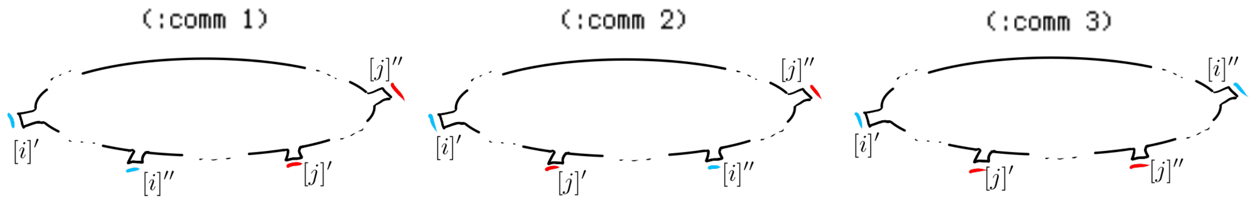
$$\gamma = \{\gamma_{[1]}, \gamma_{[2]}, \ldots, \gamma_{[2n]}\}$$

satisfying pairwise commutative relations in 4.40. $\diamond$

Recall the notation $[i]$ from 4.38, and note that γ contains exactly n elements instead of $2n$.

Definition 4.40 (`(:comm)`, technical commutative relations) Let Z_1 and Z_2 be objects in C, and c be the braided structure of C (so $a \otimes b \xrightarrow{c_{a,b}} b \otimes a$). Given $[i]$ and $[j]$, there are three possible cases without loss of generality

- $[i]' < [i]'' < [j]' < [j]''$
- $[i]' < [j]' < [i]'' < [j]''$
- $[i]' < [j]' < [j]'' < [i]''$



We will give the technical conditions that the γ's should obey, following by their graphical versions.
(1) In the first case, $\gamma_{[i]}$ and $\gamma_{[j]}$ are required to satisfy the following commutative relation `(:comm 1)`, functorial in Z_1 and Z_2.

$$(\gamma_{[j],Z_2} \otimes 1)(1 \otimes (c_{Z_1,X}c_{X,Z_1}\gamma_{[i],Z_1})) \tag{4.41}$$
$$= (1 \otimes c_{Z_1,Z_2})((c_{Z_1,X}c_{X,Z_1}\gamma_{[i],Z_1}) \otimes 1)(1 \otimes \gamma_{[j],Z_2})(c^{(-1)}_{Z_1,Z_2} \otimes 1) \tag{4.42}$$

(2) In the second case, $\gamma_{[i]}$ and $\gamma_{[j]}$ are required to satisfy the following commutative relation `(:comm 2)`, functorial in Z_1 and Z_2.

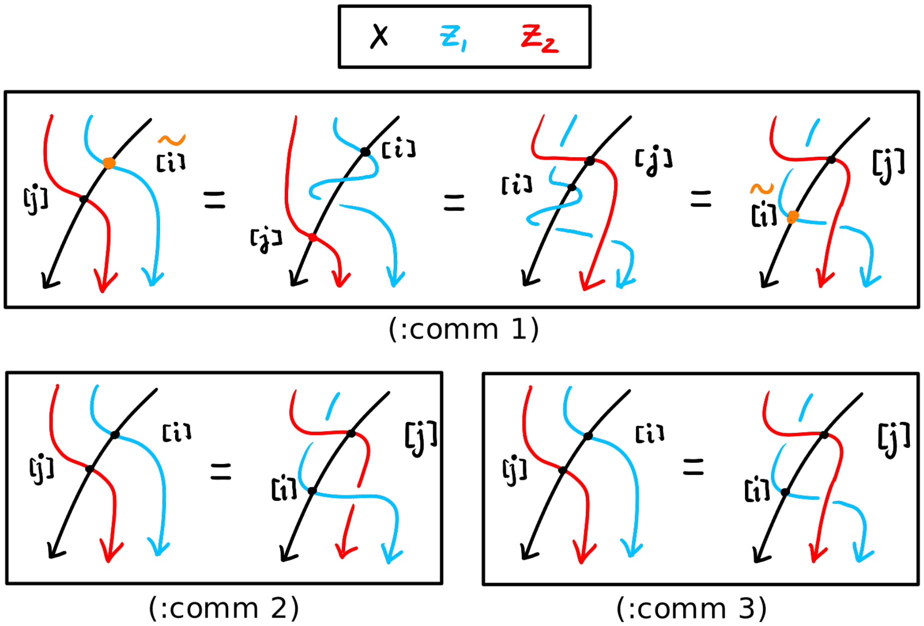
$$(\gamma_{[j],Z_2} \otimes 1)(1 \otimes \gamma_{[i],Z_1}) \tag{4.43}$$
$$= (1 \otimes c^{(-1)}_{Z_2,Z_1})(\gamma_{[i],Z_1} \otimes 1)(1 \otimes \gamma_{[j],Z_2})(c^{(-1)}_{Z_1,Z_2} \otimes 1) \tag{4.44}$$

(3) In the third case, $\gamma_{[i]}$ and $\gamma_{[j]}$ are required to satisfy the following commutative relation `(:comm 3)`, functorial in Z_1 and Z_2.

$$(\gamma_{[j],Z_2} \otimes 1)(1 \otimes \gamma_{[i],Z_1}) \tag{4.45}$$
$$= (1 \otimes c_{Z_1,Z_2})(\gamma_{[i],Z_1} \otimes 1)(1 \otimes \gamma_{[j],Z_2})(c^{(-1)}_{Z_1,Z_2} \otimes 1) \tag{4.46}$$

Notice that the first and the third case are almost the same, which is not surprising given their topological meaning. To make them look alike, define

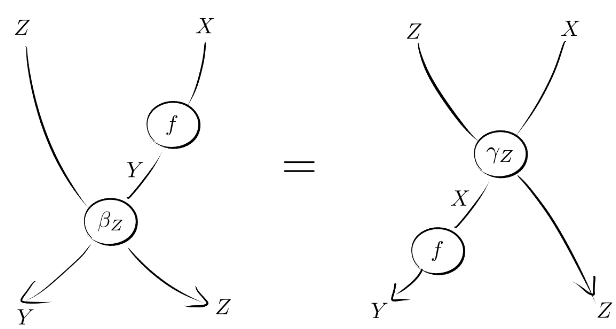
$$\tilde{\gamma}_{[i],-} = c_{-,X}c_{X,-}\gamma_{[i],-}.$$



◇

Definition 4.47 (σ-morphism) Given two σ-pairs $\bar{X} = (X, \gamma)$ and $\bar{Y} = (Y, \beta)$ of C, define $[\bar{X}, \bar{Y}]_\sigma$ to be the linear subspace of $\mathrm{Hom}_C(X, Y)$ consisting of the morphisms $(X \xrightarrow{f} Y)$ compatible with all the half-braidings in the following sense. For any $Z \in C$, we have for each $1 \leqslant i \leqslant 2n$,

$$\beta_{[i],Z}(1 \otimes f) = (f \otimes 1)\gamma_{[i],Z}. \tag{4.48}$$



Finally, let the identity maps and the compositions be inherited from that of C. ◇

Definition 4.49 (categorical center of higher genera) Let C be a premodular category, and $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing. The categorical center of higher genera $Z_\sigma(C)$ of C is defined to be the category with objects the σ-pairs of C and with morphisms the σ-morphisms. $\diamond$

4.3.3 Properties of Z

In this section, we establish some basic properties of categorical centers of higher genera. In particular, we show that they are finite semisimple abelian categories, and that there is a strictly ambidextrous adjunction between it and the underlying premodular category C.

4.3.3.1 Connecting functors

In this subsubsection, we establish the relation between C and its categorical center of higher genera $Z_\sigma(C)$, where C is a premodular category and σ is an admissible gluing. More precisely, there exist two additive functors I_σ and F_σ.

$$I_\sigma : C \rightleftarrows Z_\sigma(C) : F_\sigma.$$

We will see that I_σ is both a right and a left adjoints of F_σ (thus vice versa) in section 4.3.3.2. Such a pair of adjunction is called a (strictly) ambidextrous adjunction in the literature.

Definition 4.50 (forgetful functor) The forgetful functor

$$C \xleftarrow{F_\sigma} Z_\sigma C$$

is defined to send objects (X, γ) to X, and to send morphisms by inclusion (recall that the morphism space of $Z_\sigma(C)$ is defined as a subspace of that of C). Clearly, it is an additive functor. $\diamond$

Definition 4.51 (induction functor) The induction functor

$$C \xrightarrow{I_\sigma} Z_\sigma(C)$$

is more complicated, so will be defined step-by-step. Define $I_\sigma(X)$ to be (X_σ, γ), which is given below.
Denote by $\mathcal{O}(C)$ the set of isomorphism classes of simple objects of C. Let $o(C)$ be a set of representatives. To each C-object X, define another C-object

$$X_\sigma := \bigoplus_{o(C)^n} (X_1 \otimes X_2 \otimes \ldots \otimes X_n \otimes X \otimes X_{n+1} \otimes \ldots \otimes X_{2n-1} \otimes X_{2n}), \tag{4.52}$$

where X_k runs through $o(C)$ for those $k = [k]'$, and $X_k = X^{\star}_{([k]')}$ for those $k = [k]''$. Here, recall that the first case means that k is the smaller member in the set $[k]$, while the second case means that k is the larger member (cf 4.38). For example,

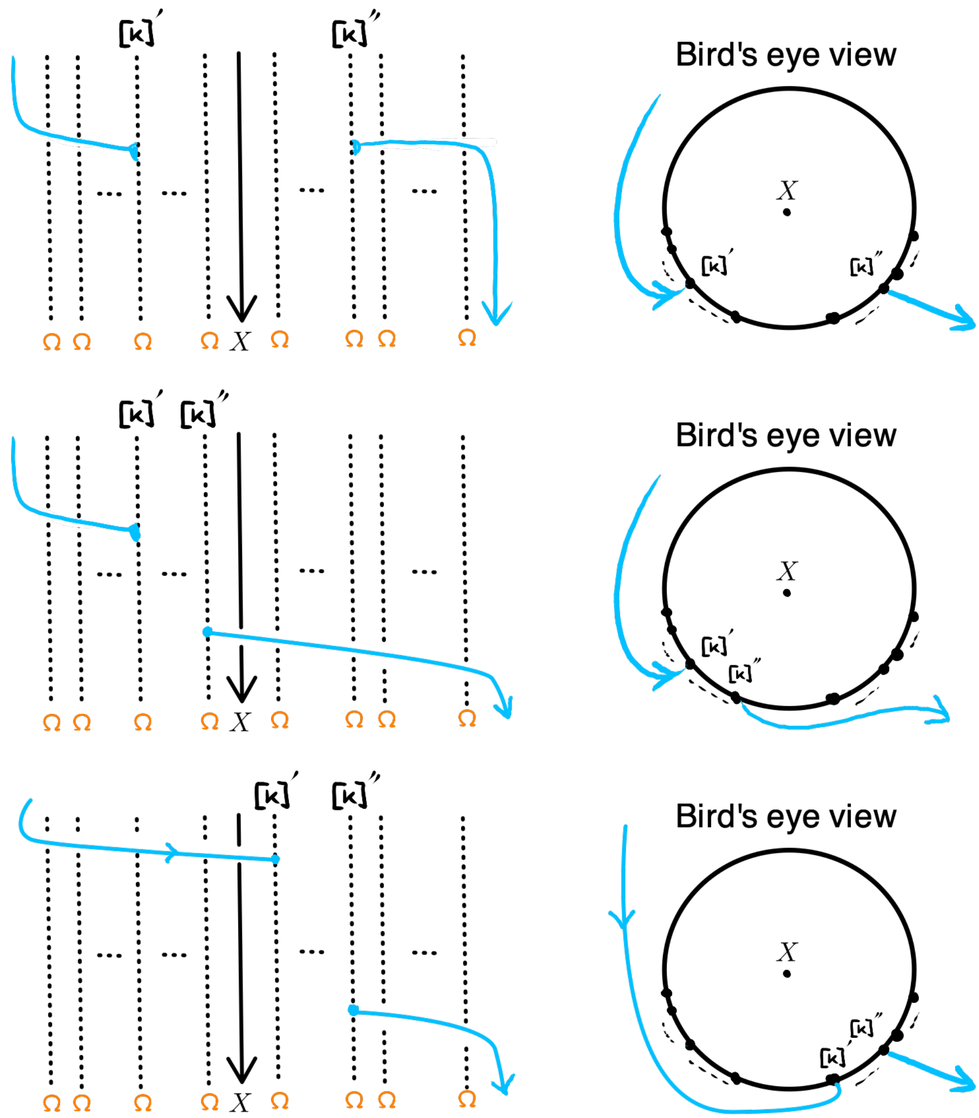
$$1_{(12)} = \oplus_{X \in o(C)} X \otimes X^{\star}.$$

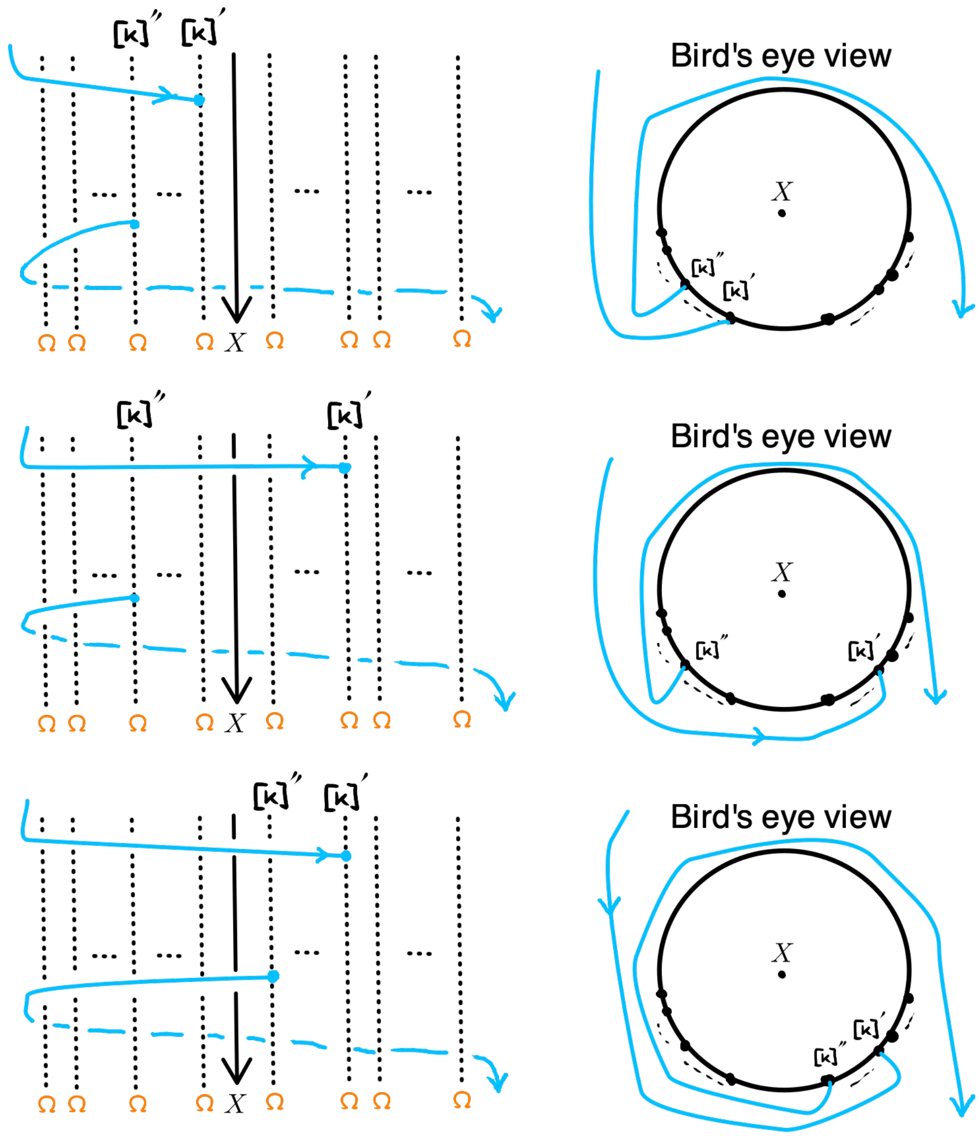
Notice that it does not depend on the choice of $o(C)$ up to canonical isomorphisms. Similarly, neither does X_σ for general admissible gluings σ.

Next, to each k, we define the $[k]$-th half-braiding $\gamma_{[k]}$ for X_σ

$$(-) \otimes X_\sigma \xrightarrow{\gamma_{[k]}} X_\sigma \otimes (-) \tag{4.53}$$

as in the following pictures.





To be more precise, denote the braiding of $\mathcal{C}$ by β. The morphism in the first picture above is

$$\beta_{-,\Omega_{2n}} \circ \beta_{-,\Omega_{2n-1}} \circ \ldots \circ \beta_{-,\Omega_{[k]''+1}} \circ \phi^a \circ \phi_a \circ \beta_{-,\Omega_{[k]'-1}} \circ \ldots \circ \beta_{-,\Omega_2} \circ \beta_{-,\Omega_1}$$

where Ω_i denotes the ith component of Ω in the picture counted from the left, and where

$$\phi^a \otimes \phi_a \in \mathrm{Hom}_{\mathcal{C}}(\Omega_{[k]''}, \Omega_{[k]''} \otimes (-)) \otimes \mathrm{Hom}_{\mathcal{C}}((-) \otimes \Omega_{[k]'}, \Omega_{[k]'})$$

is the canonical element (a sum of any dual basis) given similarly as in 2.18.

We contend that so given (X_σ, γ) is indeed an object of $Z_\sigma(C)$. One needs to show that each element in γ is a half-braiding, and that γ satisfies the commutative relations (4.40). A proof of this can be found in (6.91). It remains to define the map on the morphism spaces $\mathrm{Hom}_C(X, Y)$. Given a morphism $X \xrightarrow{f} Y$, one defines

$$I_\sigma(f) := \overline{f} := \mathrm{id}^{\otimes n} \otimes f \otimes \mathrm{id}^{\otimes n}. \tag{4.54}$$

To conclude, it remains to show that

- The morphism $\overline{f}$ is compatible with the half-braidings γ and β.
- The construction $(\overline{-})$ preserves the identities and the compositions.

The first point is shown in (6.92). The second point is clear. $\diamond$

4.3.3.2 Ambidextrous adjunction

Both functors F_σ and I_σ are additive immediately by definition. We are ready to state and prove the main statement of this subsubsection.

Theorem 4.55 The functors

$$I_\sigma : C \rightleftarrows Z_\sigma(C) : F_\sigma$$

so defined in 4.50 and 4.51 are (strictly) ambidextrous adjoint to each other. In other words, I_σ is both a left adjoint and a right adjoint of F_σ, thus vice versa. $\diamond$

Proof. We will prove that F_σ is right adjoint to I_σ. Namely, we need to show that for each C-object X and for each $Z_\sigma(C)$-object (Y, β), there is a vector space isomorphism

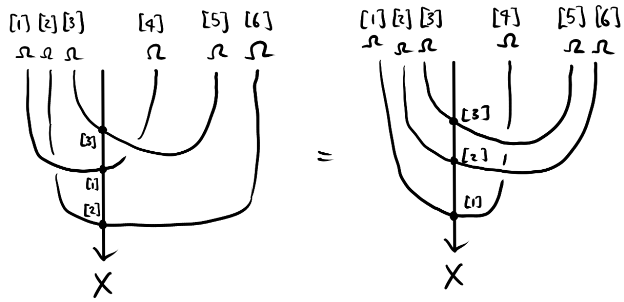
$$F : \mathrm{Hom}_C(X, F_\sigma(Y, \beta)) \rightleftarrows \mathrm{Hom}_{Z_\sigma(C)}(I_\sigma(X), (Y, \beta)) : G.$$

It will then be obvious that the other side can be proved verbatim by taking duals (or by flipping the graph, in terms of graphical calculus). To prove such equivalence, we construct explicit maps for both sides, and argue that each composition equals the identity map.

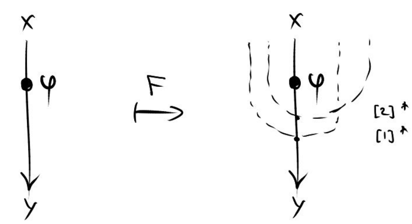
Given $\phi \in \mathrm{Hom}_C(X, F_\sigma(Y, \beta))$, define its image on the other side to be

$$F(\phi) := \frac{1}{\dim(\Omega)^n}(\widehat{\prod}_{k\in\{[1],\ldots,[2n]\}}\beta^\star_{k,\Omega}) \circ (1 \otimes \ldots \otimes 1 \otimes \phi \otimes 1 \otimes \ldots \otimes 1),$$

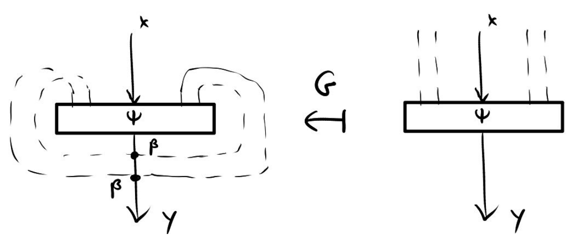
where Ω is the shorthand notation given in 2.19, the index set $\{[1] \ldots, [2n]\}$ consists of exactly n elements instead of $2n$ (cf. 4.38), and the term $\widehat{\prod}$ is explained below: The term $\widehat{\prod}$ is a C-morphism $I_\sigma(Y) \to Y$. Each $\beta^\star_{[i],\Omega}$ is a C-morphism $\Omega \otimes Y \otimes \Omega \to Y$, induced from $\Omega \otimes Y \xrightarrow{\beta_{[i],\Omega}} Y \otimes \Omega$ by composing the evaluation map (note that $\Omega^\star \simeq \Omega$). So the $\beta^\star_{[i],\Omega}$'s are maps that kills the $[i]'$-th and the $[i]''$-th component of Ω by using $\beta_{[i]}$. However, depending on the combinatorial nature of $\sigma \in \mathrm{Adm}_{2n}$, one should insert suitable braidings for it to make sense. For example, if $n = 3$, $[1] = [4]$, $[2] = [6]$, and $[3] = [5]$, we define the $\widehat{\Pi}$ term as in the following diagram – the order of the $[i]$'s does not really matter, thanks to 4.40.



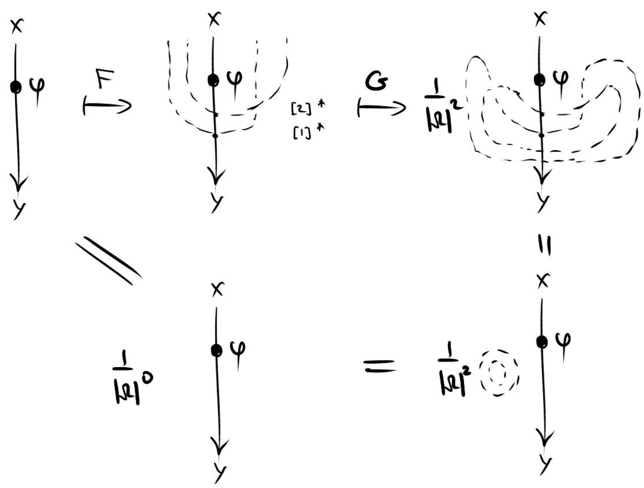
That $F(\phi)$ is indeed a morphism in $Z_\sigma(C)$ follows directly from the commutative relation 4.40, that half-braidings are by definition monoidal, and the sliding lemma 4.28.



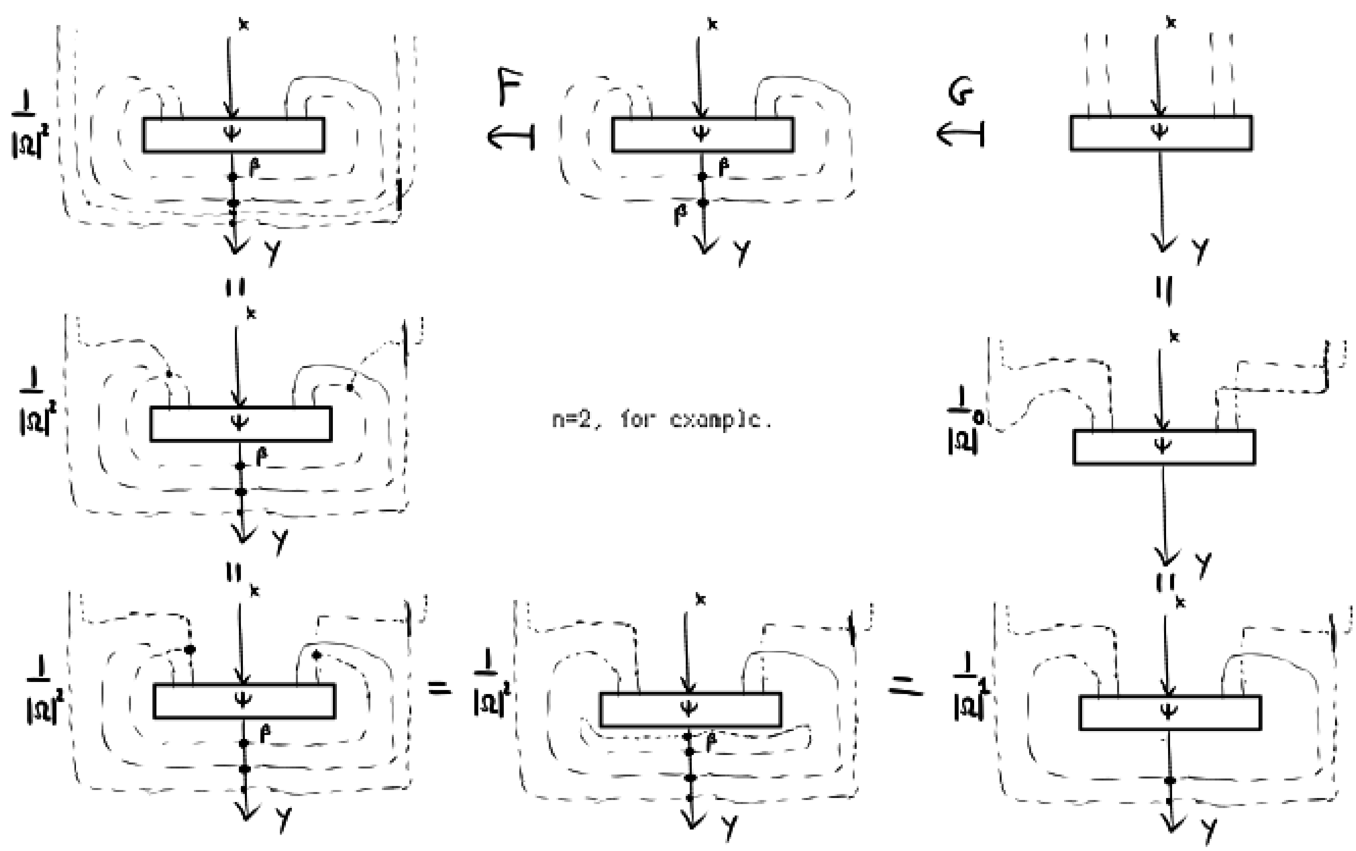
On the other hand, given $\psi \in \mathrm{Hom}_{Z_\sigma(C)}(I_\sigma(X), (Y, \beta))$, define its image $G(\psi)$ on the other side to be as indicated in the graph below.



To prove that GF is the identity map, use the fact that the half-braidings are by definition functorial. Hence one can slide the Ω's out the axis. Finally, the product of the dimensions of the Ω's cancel with the denominator.



To prove that FG is the identity map, use the sliding lemma again. Then use the assumption that ψ is a $Z_\sigma(C)$-morphism to drag Ω down. Finally, slide the Ω's away from the axis as in the case for $GF = 1$.



■

4.3.3.3 Z is finite semisimple abelian

In this section, we show that the categorical centers of higher general over premodular categories are finite semisimple abelian categories.

Lemma 4.56 ((monadic) projection) Let $\sigma \in \mathrm{Adm}_{2n}$ be an admissible gluing, C be a premodular category, $Z = Z_\sigma(C)$ be the categorical center of higher genera of C with respect to σ, and $\overline{X} = (X, \gamma_\cdot)$ and $\overline{Y} = (Y, \beta_\cdot)$ be Z-objects.

Recall that the morphism space $Z(\overline{X}, \overline{Y})$ is by definition a subspace of $C(X, Y)$. Then there is a natural projection $\pi_{\gamma_\cdot, \beta_\cdot} \in \mathrm{End}(\mathrm{Mor}_C(X, Y))$ to the subspace $Z(\overline{X}, \overline{Y})$ that respects the composition. ◇

Proof. The full proof is tedious and postponed to 6.4. In particular, see 6.87. Roughly, the statement follows from the monadic nature of the strictly ambidextrous adjunctions and a condition called the "unity trace condition" (6.82). ■

Theorem 4.57 Let C be a premodular category, $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing. Then the categorical center of higher genera $Z_\sigma(C)$ is a finite semisimple abelian category. ◇

Proof. The complete proof is tedious and thus postponed to the appendix. See 6.88, 6.89, and 6.90. The main idea is to make heavy use of the projection 4.56. ■

4.4 Proof of the main theorem

In this section, we prove the main statement of this paper.

Theorem 4.58 (Main Statement) Let C be a premodular category, $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing, $\Sigma = \Sigma_\sigma$ the surface constructed from σ. Then the Crane-Yetter theory of Σ_σ over C and the categorical center of higher genera $Z_\sigma(C)$ are equivalent as $\Bbbk$-linear categories

$$CY_{\Sigma_\sigma}(C) \simeq Z_\sigma(C).$$

As the $Z_\sigma(C)$'s are proven to be finite semisimple abelian 4.57, the Crane-Yetter theory for each open surface is also a finite semisimple abelian category. ◇

To stress the informal aspect again, we recall 4.14.

Remark 4.59 In $H_1(S^1) \simeq \mathbb{Z}$, one sees the algebra of the shape S^1 and the shape of the algebra $\mathbb{Z}$. Our main result should be viewed as a higher analogue. That is, one sees the (higher) algebra of the shape Σ_σ and the shape of the (higher) algebra Z_σ. $\diamond$

Example 4.60 **($n = 0$)** For $n = 0$, the surface Σ is the open disk, the categorical center of higher genera reduces to the underlying premodular category C, so the theorem recovers that $CY_\Sigma(C) \simeq C$ as shown in 4.2. $\diamond$

Example 4.61 **($n = 1$)** For $n = 1$, the only possible surface is the cylinder, the categorical center of higher genera reduces to the Drinfeld center $Z(C)$. Hence, the theorem recovers that $CY_\Sigma(C) \simeq Z(C)$ as shown in 4.3. $\diamond$

Example 4.62 **($n = 2$)** For $n = 2$, there are two possible surfaces: the 1-punctured torus and the 3-punctured disk. In the former case, the categorical center of higher genera reduces to the elliptic center $Z^{el}(C)$, so the theorem recovers that $CY_\Sigma(C) \simeq Z^{el}(C)$ as shown in 4.7. $\diamond$

Example 4.63 **(General case)** In the later case, the theorem provides a new result by providing an explicit value $Z_{0,3}(C)$ for the 3-punctured sphere. Together with the excision principle and the value for the disk and the cylinder, this allows us to actually compute the values for **all** oriented surfaces, in several different ways. For example, a 4-punctured sphere has values $Z_{0,4}(C)$ and $Z_{0,3}(C) \boxtimes_{(Z(C), \overline{\otimes})} Z_{0,3}(C)$. By the invariance of tqfts, we know that they are equivalent. We can then use this to compute the value for the surface of genus 2, say, and so on. $\diamond$

4.4.1 Strategy

$$
\begin{tikzcd}
CY_{\Sigma_\sigma}(C) \arrow[d, ":="] & & Z_\sigma(C) \arrow[dl, bend right, "G"'] \\
Kar(cy_{\Sigma_\sigma}(C)) \arrow[r, dash, "\simeq"] & Kar(ho.cy_{\Sigma_\sigma}(C)) \arrow[ur, "J"', "\simeq"] & \\
cy_{\Sigma_\sigma}(C) \arrow[u, hook] & ho.cy_{\Sigma_\sigma}(C) \arrow[u, hook] \arrow[l, hook', "\simeq"] \arrow[uur, bend right, "j"'] &
\end{tikzcd}
$$

1. (Condensation of string nets) By definition, $CY_{\Sigma_\sigma}(C)$ is the Karoubi envelope of $cy_{\Sigma_\sigma}(C)$. Find an equivalent subcategory $ho.cy_{\Sigma_\sigma}(C)$ of cy_{Σ_σ} by reducing topological data. Then of course
$$Kar(cy_{\Sigma_\sigma}) \simeq Kar(ho.cy_{\Sigma_\sigma}(C)).$$
2. (top $\to$ alg) Construct a functor
$$ho.cy_{\Sigma_\sigma}(C) \xrightarrow{j} Z_\sigma(C),$$

and extend it to

$$\mathrm{Kar}(\mathrm{ho.cy}_{\Sigma_\sigma}(C)) \xrightarrow{J} Z_\sigma(C).$$

3. (top $\leftarrow$ alg) Construct a functor

$$\mathrm{Kar}(\mathrm{ho.cy}_{\Sigma_\sigma}(C)) \xleftarrow{G} Z_\sigma(C).$$

4. Argue that the compositions $J \circ G$ and $G \circ J$ are equivalent to the identity functors.
5. Show that the equivalence is of finite semisimple abelian categories.

4.4.2 Proof

In this section, we give the full proof of the main statement. Each subsection corresponds to each step in the outlined strategy.

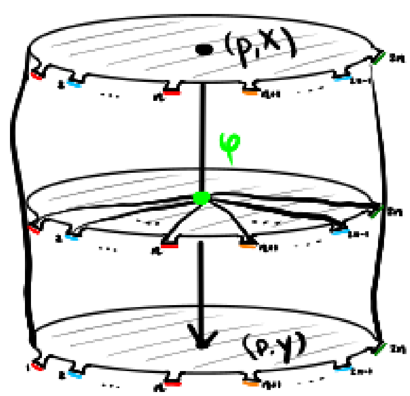
4.4.3 Reducing topological data

In this subsection, we reduce the topological data by constructing a smaller yet equivalent subcategory

$$\mathrm{ho.cy}_{\Sigma_\sigma}(C) \underset{\sim}{\overset{\subset}{\to}} \mathrm{cy}_{\Sigma_\sigma}(C). \tag{4.64}$$

Definition 4.65 ($\mathrm{ho.cy}_{\Sigma_\sigma}(C)$) Let C be a premodular category and σ an admissible gluing. The subcategory $\mathrm{ho.cy}_{\Sigma_\sigma}(C)$ of $\mathrm{cy}_{\Sigma_\sigma}(C)$ is defined as follows.

Let p be the central point of the standard disk. An object of $\mathrm{ho.cy}_{\Sigma_\sigma}(C)$ is defined to be the single C-colored point (p, X) for some $X \in \mathrm{Obj}(C)$. A morphism from (p, X) to (p, Y) is the equivalence class in which the following string net lives.



Clearly, $\mathrm{ho.cy}_\sigma(C)$ is a subcategory of $\mathrm{cy}_\sigma(C)$. $\diamond$

Theorem 4.66 (equivalence of reduction) The inclusion functor

$$\iota : \mathrm{ho.cy}_{\Sigma_\sigma}(C) \xrightarrow{\subset} \mathrm{cy}_{\Sigma_\sigma}(C)$$

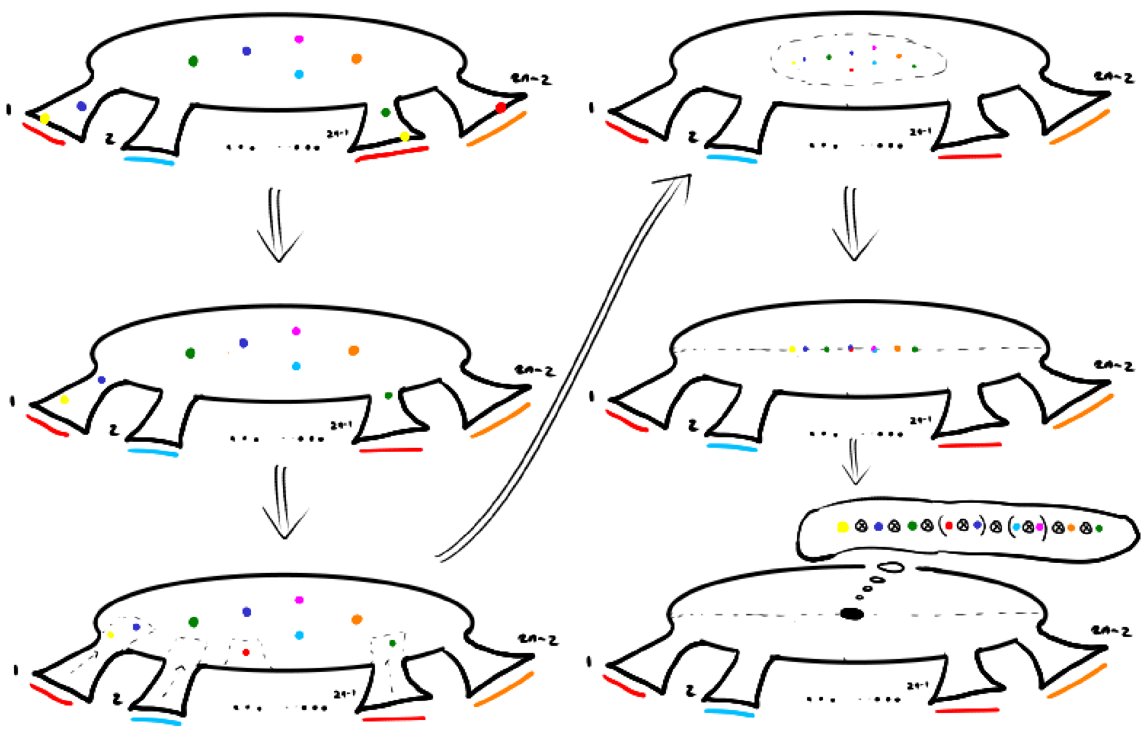
is an equivalence of categories. Clearly, it is additive. $\diamond$

Proof. By a basic lemma in category theory, it is enough to show that ι is fully faithful and essentially surjective.

(Essentially surjective) Recall that a typical object of $\mathrm{cy}_{\Sigma_\sigma}(C)$ is a finite collection of C-colored points on Σ_σ. It suffices to find an equivalent object of the form (p, X), for some $X \in \mathrm{Obj}(C)$. This can be done by the following reductions.

1. Slightly push the points on the boundary into the smaller side.
2. Compress the points from the legs into the disk.
3. Then compress further for the points to stay in a small unit disk in the middle.
4. Project the objects to a fixed line.
5. Take their tensor products.

Each step above can be realized as an isomorphism in $\mathrm{CY}_{\Sigma_\sigma}(C)$, so every object is isomorphic to an object in $\mathrm{ho.cy}_{\Sigma_\sigma}(C)$.

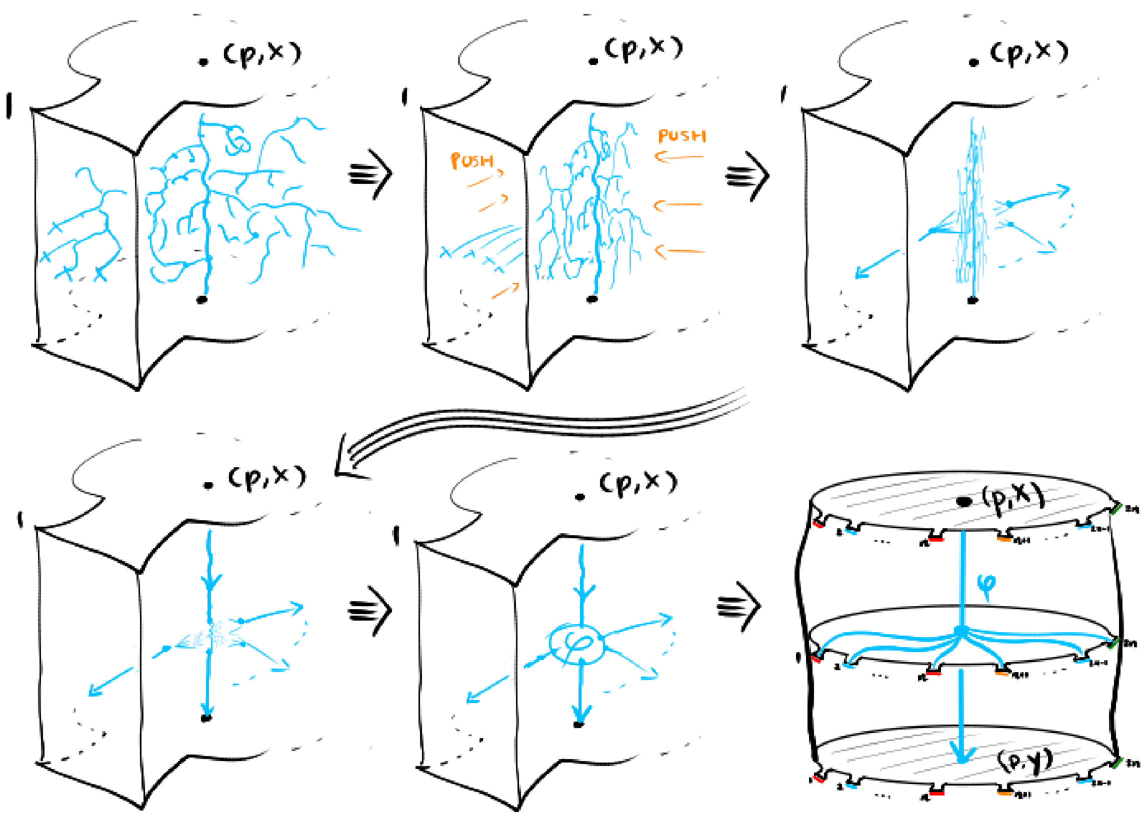


(Fully faithful) We ought to show that

$$\mathrm{Hom}_{\mathrm{ho.cy}_{\Sigma_\sigma}(C)}((p,X),(p,Y)) \xrightarrow{\iota} \mathrm{Hom}_{\mathrm{cy}_{\Sigma_\sigma}(C)}((p,X),(p,Y))$$

is an equivalence of vector spaces. Clearly, it is linear and injective, as the quotient relations on both sides are the same. To prove surjectivity, we have to show that any arbitrary string net with given boundary condition is equivalent to a stringnet given in the definition of $\mathrm{ho.cy}_{\Sigma_\sigma}(\mathcal{C})$. This can be done by a similar compression process as in the proof of essential surjectivity.

1. Push the stringnets away from the end of the legs.
2. Push the stringnets away from the legs.
3. Compress everything into a fixed central bar.
4. Replace the stringnets through boundaries with one strand for each leg by taking tensor products by using the Reshetikhin-Turaev evaluation.
5. Then compress vertically.
6. Then finally replace the tangled mess in the middle by a morphism.



■

4.4.3.1 topology $\rightarrow$ algebra

In this subsubsection, we aim to construct a functor

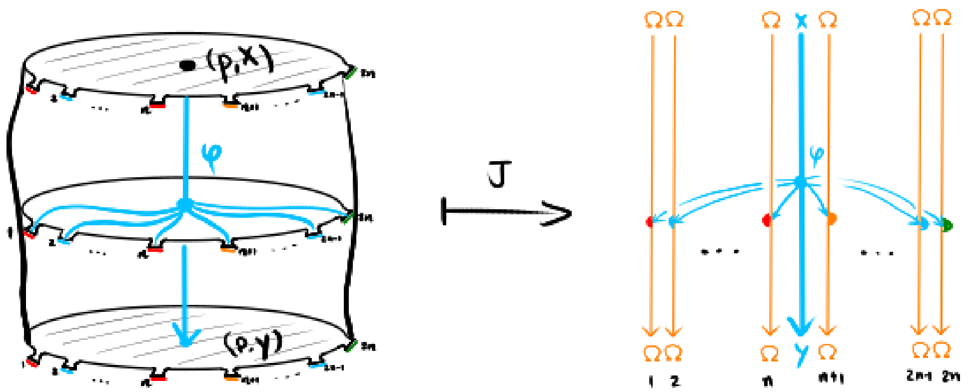
$$\mathrm{Kar}(\mathrm{ho.cy}_{\Sigma_\sigma}(C)) \xrightarrow{J} Z_\sigma(C).$$

As $Z_\sigma(C)$ is abelian (4.57), by 6.93 we only have to construct an additive functor

$$\mathrm{ho.cy}_{\Sigma_\sigma}(C) \xrightarrow{j} Z_\sigma(C).$$

To define j, recall that a typical object of $\mathrm{ho.cy}_{\Sigma_\sigma}(C)$ is a colored point (p, X), where p denotes the central point of the standard disk. Define its image under j to be X_σ as in (4.52). A typical morphism from (p, X) to (p, Y) is a linear combination of the equivalence classes of the stringnets like Γ. Define the image of $[\Gamma]$ under j to be

$[\Gamma] :=$



where Ω is the shorthand notation given in 2.19 crossings mean the braidings of C, and the nontrivial pairs of intertwiners are given in [KB10, (1.8)]. Extend the definition additively, and then we have our desired additive functor j.

4.4.3.2 topology $\leftarrow$ algebra

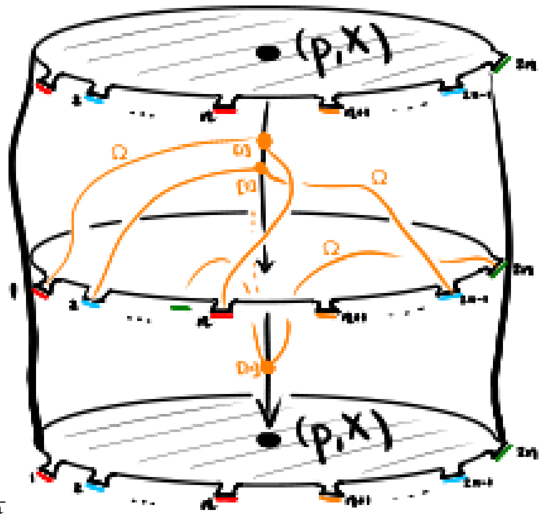
In this subsubsection, we construct a functor

$$\mathrm{Kar}(\mathrm{ho.cy}_{\Sigma_\sigma}(C)) \xleftarrow{G} Z_\sigma(C).$$

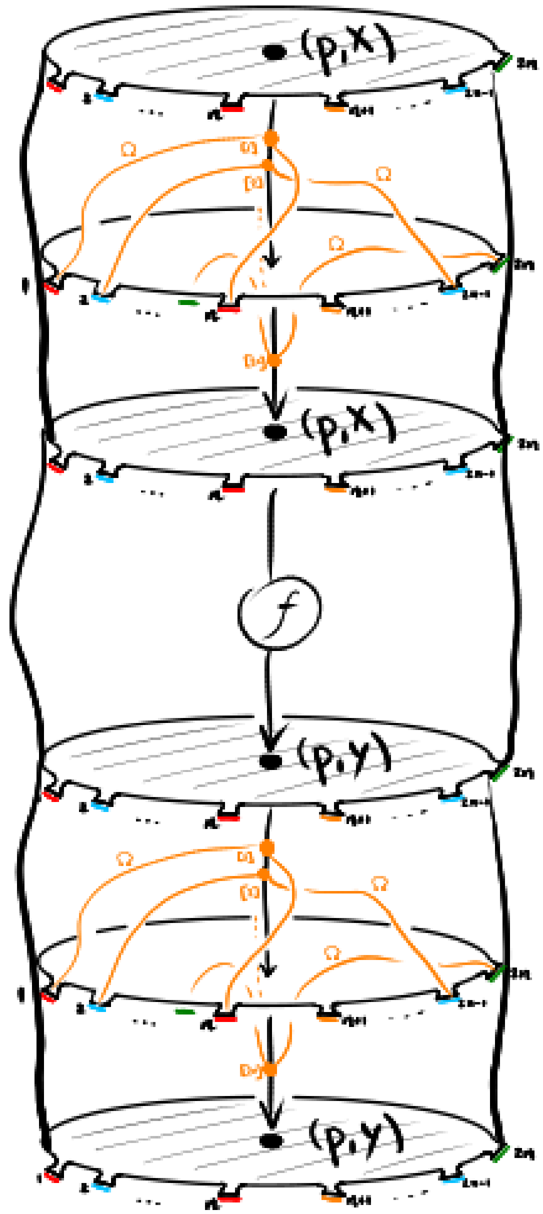
Recall that a typical object in $Z_\sigma(C)$ is (X, γ), where $X \in \mathrm{Obj}(C)$ and γ is a set of half-braidings

$$\gamma = \{\gamma_{[1]}, \gamma_{[2]}, \dots \gamma_{[2n]}\}$$

satisfying some relations 4.40. Define the image of (X, γ) under G to be $((p, X), \pi_\gamma)$, where p denotes the central point of the standard disk, and π_γ to be the equivalence class of the following stringnets.

$$\frac{1}{|\Omega|^n}$$ 

That π_γ is a projection follows from the commutative relations (4.40) and a graphical lemma [Kir11, (3.7)]. For morphisms, define the image of $(X, \gamma) \xrightarrow{f} (Y, \beta)$ under G to be



which is indeed a morphism in the Karoubi envelope because π_γ and π_β are idempotents.

4.4.3.3 topology $\leftrightarrow$ algebra

In this subsection, we will show that $J \circ G$ and $G \circ J$ are equivalent to identity functors.

That $G \circ J \simeq 1$ follows directly from the same argument of [Kir11, Figure 15]; we just have to do it n times. On the other hand, in fact we have $J \circ G = 1$. Indeed, denote $(J \circ G)((X, \gamma)) = (X', \gamma')$. That $X' = X$ follows directly from the sliding lemma

4.28, and that $\gamma' = \gamma$ follows from the sliding lemma, and the fact that half-braidings are by definition monoidal.

Finally, since G and J are additive, this proves the equivalence of both sides as abelian categories. Therefore, $\mathrm{CY}_{\Sigma_\sigma}(\mathcal{C})$ and $Z_\sigma(\mathcal{C})$ are equivalent as finite semisimple abelian categories.

4.5 Outlooks and Remarks

In this section, we describe some open directions and more work in progress.

Surface combinatorics Let Σ be an open surface of a fixed topological type. In general, there are many different admissible gluings σ with $\Sigma_\sigma \simeq \Sigma$. As Crane-Yetter theory is topological, we have many differently-presented categories that are in fact equivalent.

Moreover, the excision property 4.4

$$\mathrm{CY}_{\Sigma_1 \cup \Sigma_2}(\mathcal{C}) \simeq \mathrm{CY}_{\Sigma_1}(\mathcal{C}) \boxtimes_{\mathrm{CY}_{\Sigma_1 \cap \Sigma_2}(\mathcal{C})} \mathrm{CY}_{\Sigma_2}(\mathcal{C})$$

provides more ways to obtain Σ. See 4.63 for a concrete example. It is an interesting work to establish explicit equivalences.

Surfaces without punctures

The main statement of this work provides a nice description for the Crane-Yetter theory of any surface with at least one puncture. While the case without punctures can be taken care easily by patching with the excision principle, the resulting categories are described in terms of balanced (Deligne) tensor products, which are more obscure. The author believes that there should be a better description.

Module categorical structures

The Drinfeld center with the stacking tensor product $\overline{\otimes}$ acts on $\mathrm{CY}_\Sigma(\mathcal{C})$ in possibly multiple ways. We will establish the module categorical structure for $\mathrm{CY}_\Sigma(\mathcal{C})$ explicitly in future work.

Concrete computations

Compute examples for Crane-Yetter theory in dimension two and three explicitly and concretely, especially for premodular categories $\mathcal{C}$ that are neither modular nor symmetric. There are a few candidates. The first is the even part of the semi-simplification of $\mathrm{Rep}(U_q\mathfrak{sl}_2)$ for special q. Another family of examples are given by $\mathrm{Rep}(\mathfrak{X})$, where $\mathfrak{X}$ denotes a finite 2-group [Ban10]. Compute C-Y invariants for 3-folds and 4-folds directly from our result, and seek for insights.

Action of mapping class groups

Crane-Yetter theory is expected to be a fully-extended TQFT. So given a surface Σ, its mapping class group $MCG(\Sigma)$ should act on $CY_C(\Sigma)$ or any premodular category C. Compute the action explicitly.

Minimal data for Crane-Yetter

When C is modular, CY in dimension two trivializes to the number of punctures. In particular, for closed surfaces Σ, $CY_\Sigma(C)$ trivialize to the Muger center of C, which is just $(Vect)$ due to the modularity [Eti+15, Prop 8.20.12]. On the other hand, when C is not modular, $CY_\Sigma(C)$ does no seem to depend on full information from C. Find the minimal data needed in order to determine $CY_\Sigma(C)$.

Piecewise-linear setting

For exposing stringnets with simplicity, we assume smooth structures for our surfaces. Crane-Yetter theory can be made precise in the PL setting. This is not necessary and should be removed in future work. For an analogue in one dimension lower, see [KB10].

5 Explicit Factorization of Categorical Center

Section 5 is written with Ying Hong Tham.

5.1 Introduction

Topological quantum field theories (TQFT) provide sophisticated invariants for manifolds. For example, the Witten-Reshetikhin-Turaev (WRT) 3D model gives rise to the celebrated quantum invariants for knots and 3-manifolds, vastly generalizing the Jones polynomial which disclosed deep connections among low-dimensional topology, quantum algebras, combinatorics, field theories, and statistical mechanics. It is known in recent years that the WRT is a special case of (more precisely, a boundary theory of) a 4D model, the Crane-Yetter model (CY).

Another interesting aspect about TQFTs is that they provide invariants for (higher) algebras. For example, the 2D FHK model "integrates" [BBJ18] an input algebra A over the circle to its (classical) center $Z(A)$. A nontrivial example is the WRT model, which distinguishes higher algebras (e.g. modular tensor categories (MTC)) by examining the expectation values over various Wilson loops. This is a powerful tool for the classification program of MTCs, whose importance in mathematics and physics can be found in [Bar+15] and [Bar+19].

We are interested in the CY model, which produces category-valued invariants of surfaces. Similar to the FHK model, over the cylinder the CY model "integrates" the input tensor category into its categorical center (à la Drinfeld) $Z(A)$ [1]. Moreover, over the 2-sphere the CY model integrates the input to its Muger centralizer, which was a key to study premodular categories [Mue03]. Given their importance, it is therefore natural to ask for the results over other surfaces. The results for the (punctured) tori $\Sigma_{1,0}$ and $\Sigma_{1,1}$ were computed in [Tha19] and [Tha22], while the rest of the surfaces $\Sigma_{g,n}$ were done in [Guu21] by showing that the CY model integrates to the categorical center of higher genera (see a construction therein).

This work focuses on an extension of the computation. To be more precise, we investigate the surgery picture of the CY model in dimension $2+1$ by computing the values (as functors) of the morphisms in the cobordism category Cob_{2+1}. Thanks to the excision principle [Coo23] [AT22], we only have to take care of the basic handle move:

$$CY(D^2 \cup D^2) \longleftrightarrow CY(S^1 \times I)$$
$$F : C \boxtimes C^{bop} \longleftrightarrow Z(C) : G$$

While the functor F is well-known to the tensor categorists [Eti+15, (8.18)], the construction of the functor G has been left open for years. In particular, in the special

[1] The construction was not obvious priori and was used to construct quantum groups in the 80s.

case where the input algebra is modular, the functor G was known to exist by dimension argument. Despite the succinct definition of $Z(C)$, its structure is notoriously hard. Knowing an explicit construction of G helps us understand their structures in future work. In this paper we do four things.

1. Explicitly construct the functor G (section 5.2.1).
2. Show that it is ambidextrous adjoint to F (section 5.2.3).
3. Explicitly construct the natural transformations witnessing the adjunctions. We conjecture that this is the value of the CY model for the corresponding 2-cobordism (section 5.2.2).
4. Show that G is indeed the inverse functor of F in the special case while the input category C is modular (section 5.2.4).

Finally, we remark that the construction of G is fairly nontrivial from the viewpoint of tensor categories, and only becomes obvious when we considere the topological nature of the higher algebras. We hope that the community can utilize the CY model to reveal the inner structures of tensor categories more in the future.

5.2 Main Result

For any premodular category C, we aim to construct a functor

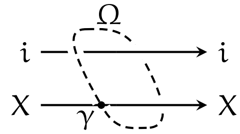
$$Z(C) \xrightarrow{G} C \boxtimes C^{bop},$$

and prove it an inverse functor for $C \boxtimes C^{bop} \xrightarrow{F} Z(C)$ in the case that C is modular by constructing explicit natural isomorphisms.

Throughout this section, we fix a premodular category C, fix a complete set of simple objects $O(C)$ and its dual $O(C)^\star$ (2.7). With a C-object X fixed, $\mathrm{Hom}_C(X, i^\star)$ is a finite dimensional vector space over $\mathbb{C}$ with a natural nondegenerate pairing 2.17 with $\mathrm{Hom}_C(i^\star, X)$. Pick and fix an arbitrary basis $X[i] = \{\alpha_{i,1}, \ldots, \alpha_{i,l_i}\}$ for the former space, and form its dual basis $X[i]^\star = \{\alpha_i^1, \ldots \alpha_i^{l_i}\}$ in the latter. We will drop the super/subfix when there is little danger of confusion. We also identify x and its (left) double dual $x^{\star\star}$ by the spherical structure of C.

5.2.1 Functor: G

Definition 5.1 (The coupling morphism $\Gamma_{i,(X,\gamma)}$) Let (X, γ) be an object of $Z(C)$. For each $i \in O(C)$, define the C-morphism $\Gamma_{i,(X,\gamma)}$ to be the product of $\frac{1}{\dim(\Omega)}$ and the following morphism



$\diamond$

By axiom, C is an abelian category, so there is a canonical object $I_{i,(X,\gamma)}$ and two canonical maps $(i \otimes X) \xrightarrow{\twoheadrightarrow} I_{i,(X,\gamma)}$ and $I_{i,(X,\gamma)} \xrightarrow{\subseteq} (i \otimes X)$ such that $\Gamma_{i,(X,\gamma)} = (\subseteq) \circ (\twoheadrightarrow)$. Both canonical maps depend on i and (X,γ). However for simplicity we often omit mentioning the dependence. Notice also that $\Gamma^2_{i,(X,\gamma)} = \Gamma_{i,(X,\gamma)}$ by the tensoriality of γ and the sliding lemma.

Definition 5.2 (The functor G) Define G to be the functor that sends any object (X,γ) in $Z(C)$ to

$$\bigoplus_{i \in O(C)} i^\star \boxtimes I_{i,(X,\gamma)},$$

and any morphism $(X,\gamma) \xrightarrow{\phi} (Y,\beta)$ to

$$\bigoplus_{i \in O(C)} 1_{i^\star} \boxtimes (\twoheadrightarrow \circ \Gamma_{i,(Y,\beta)} \circ (1_i \otimes \phi) \circ \Gamma_{i,(X,\gamma)} \circ \subseteq).$$

$\diamond$

The construction was motivated by the Crane-Yetter theory and the construction of the categorical center of higher genera [Guu21]. Notice what while its existence was known by (Frobenius-Perron) dimension argument [Eti+15], this construction is new and is expected to provide insight in the difference between two topological quantum field theories, the Witten-Reshetikhin-Turaev theory and the Crane-Yetter theory.

Remark 5.3 It is not hard to prove from the definition of G that G is lax monoidal. When C is modular, lemma 5.12 implies that G is monoidal. However we do not expect this to be the case in general. One might find counterexample in $C = Rep(S_3)$, where S_3 is the nonabelian group of order $(3!)$. $\diamond$

5.2.2 Transformation: b, d, p, q

We will further construct four natural transformations

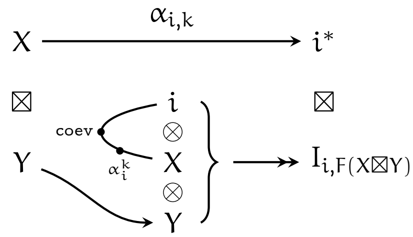
$$1 \xrightarrow{d} GF, \quad GF \xrightarrow{q} 1, \quad 1 \xrightarrow{b} FG, \quad FG \xrightarrow{p} 1.$$

and argue that they witness $FG \simeq 1$ and $GF \simeq 1$ where C is modular.

Definition 5.4 (The transformations d and q) To construct the natural transformation $1 \xrightarrow{d} GF$, it suffices to construct a morphism in $C \boxtimes C^{bop}$ for each object $X \boxtimes Y$. We thus define

$$d = d_{X \boxtimes Y} := \sum_{i \in O(C)} d_i := \sum_{i \in O(C)} \sum_{k=1}^{|X[i]|} d_i(k),$$

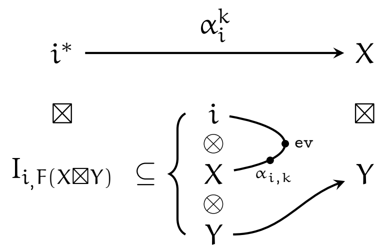
where $d_i(k)$ denotes the product of $\sqrt{\dim(i)}$ and the following morphism:



Similarly, define the natural transformation $GF \xrightarrow{q} 1$ to be the sum

$$q = q_{X\boxtimes Y} := \sum_{i\in O(C)} q_i := \sum_{i\in O(C)} \sum_{k=1}^{|X[i]|} q_i(k),$$

where $q_i(k)$ denotes the product of $\sqrt{\dim(i)}$ and the following morphism:



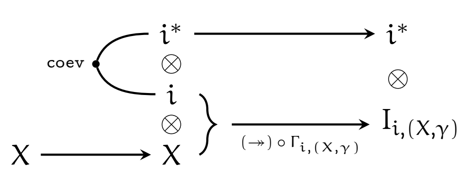
◇

Notice that while $d_i(k)$ and $q_i(k)$ depend on the choice X[i], the morphisms d_i and q_i do not.

Definition 5.5 (The transformations b and p) To construct the natural transformation $1 \xrightarrow{b} FG$, it suffices to construct a morphism in $Z(C)$ from each object (X, γ) to $FG(X, \gamma) = \bigoplus_{i\in O(C)} i^\star \otimes i \otimes X$. We thus define

$$b = b_{(X,\gamma)} := \sum_{i\in O(C)} b_i$$

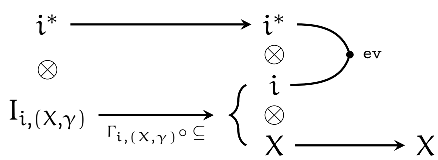
where b_i denotes the product of $\sqrt{\dim(i)}$ and the following morphism:



Similarly, define the natural transformation $FG \xrightarrow{p} 1$ to be the sum

$$p = p_{(X,\gamma)} := \sum_{i \in O(C)} p_i$$

where p_i denotes the product of $\sqrt{\dim(i)}$ and the following morphism:

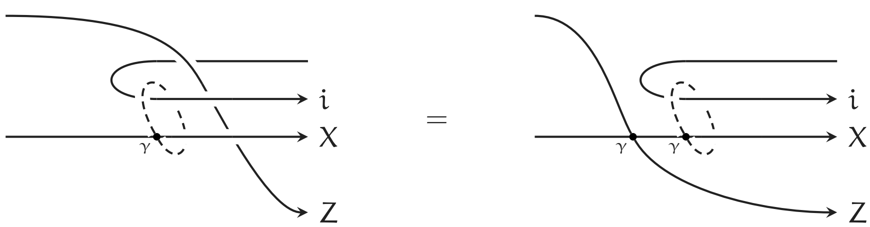


$\diamond$

It requires some effort to check that so defined transformations b and p are indeed morphisms in $Z(C)$. We prove that in the following lemma.

Lemma 5.6 Given an $Z(C)$-object (X, γ), the definition of the natural transformations $b = b_{(X,\gamma)}$ and $p = p_{(X,\gamma)}$ are indeed morphisms in $Z(C)$. $\diamond$

Proof. By the definition of $Z(C)$, it suffices to show that b and d respect the half-braidings γ and $c \otimes c^{-1}$. We provide a graphical proof for this fact.
Since any proof for b also works similarly for p, so we shall only prove for b. By definition of $Z(C)$, it suffices to prove the following equality (functorial in $Z \in \mathrm{Obj}(C)$)



However, this follows directly from the tensoriality of the half-braiding γ and the sliding lemma 4.28. ■

5.2.3 $F \vdash G \vdash F$

In this section we prove that F and G are ambidextrous adjunctions witnessed by the natural transformations b, d, p, q. In the next section, we will prove that F and G are inverse to each other if the underlying category is modular.

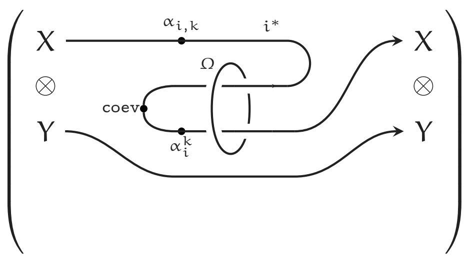
Proposition 5.7 ($G \vdash F$) The functor G is a left adjoint to the functor F, witnessed by the natural transformations d and p. $\diamond$

Proof. It amounts to proving two things, in which $(\cdot)$ denotes the whiskering product in the 2-category of categories.

- $(F \xrightarrow{1_F} F) = (F \xrightarrow{F\cdot d} FGF \xrightarrow{p\cdot F} F)$
- $(G \xrightarrow{1_G} G) = (G \xrightarrow{d\cdot G} GFG \xrightarrow{G\cdot p} G)$

To prove the first equation, evaluate the term on the right on $(X \boxtimes Y) \in \mathrm{Obj}(\mathcal{C} \boxtimes \mathcal{C}^{bop})$. The result is

$$\Sigma_{i\in O(C)} \frac{\dim(i)}{\dim(\Omega)} \qquad (5.8)$$



which is exactly $1_{F(X\boxtimes Y)}$ by canceling $\dim(\Omega)$ with the Ω-loop and by absorbing $\dim(i)$ to the α-pair and creating 1_X using lemma 2.18. The second equation can be proved similarly. ■

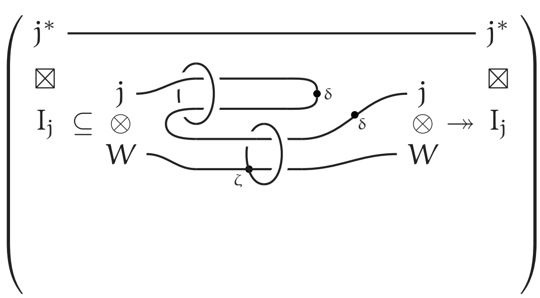
Proposition 5.9 ($F \vdash G$) The functor F is a left adjoint to the functor G, witnessed by the natural transformations b and q. ◇

Proof. It amounts to proving two things, in which $(\cdot)$ denotes the whiskering product in the 2-category of categories.

- $(G \xrightarrow{1_G} G) = (G \xrightarrow{G\cdot b} GFG \xrightarrow{q\cdot G} G)$
- $(F \xrightarrow{1_F} F) = (F \xrightarrow{b\cdot F} FGF \xrightarrow{F\cdot q} F)$

To prove the first equation, evaluate the term on the right on $(W, \zeta) \in \mathrm{Obj}(Z(\mathcal{C}))$. The result is

$$\bigoplus_{i,j\in O(C)} \frac{\dim(i)}{\dim(\Omega)^2} \qquad (5.10)$$



which is indeed $i_{G(W,\zeta)}$ by absorbing $\dim(i)$ into the δ-pair to create 1_j and by canceling the Ω's. The second equation can be proved similarly. ■

Remark 5.11 (**G is a $\mathcal{C} \boxtimes \mathcal{C}^{bop}$-bimodule map**) Since F is monoidal, the fact that G is adjoint to F makes G a $\mathcal{C} \boxtimes \mathcal{C}^{bop}$-bimodule map by the proof of [DSS19, corollary 2.13]. The authors thank Thibault Décoppet for pointing out this fact. ◇

5.2.4 $\mathcal{C}$ modular $\Rightarrow$ G is an inverse of F

We state our main theorem in this paper and will provide a proof after a few lemmas.

Theorem 5.12 (Main Theorem) If $\mathcal{C}$ is modular, then the functor G is a factorization of the Drinfeld center $Z(\mathcal{C})$. More precisely, G is an inverse functor for F witnessed by the natural transformations b, d, p, q. $\diamond$

Proof. It amounts to proving that the compositions

$$\begin{aligned}
qd &: 1 \to GF \to 1, \\
dq &: GF \to 1 \to GF, \\
pb &: 1 \to FG \to 1, \\
bp &: FG \to 1 \to FG
\end{aligned} \tag{5.13}$$

are equal to identity natural transformations. These are proved respectively in lemma 5.14, 5.16, 5.18, and 5.19. ■

Lemma 5.14 Let $X \boxtimes Y$ be an object in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{bop}}$. Then the morphism

$$X \boxtimes Y \xrightarrow{q \circ d} X \boxtimes Y$$

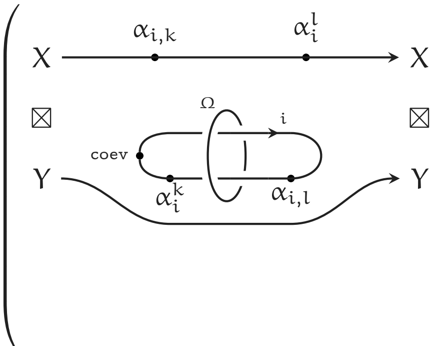
is equal to the identity morphism $\mathrm{id}_{X \boxtimes Y}$. $\diamond$

Note that this lemma does not assume modularity.

Proof. We prove the equality by direct computation.

$$\begin{aligned}
q \circ d &= (\sum_i q_i) \circ (\sum_j d_j) \\
&= \sum_i q_i \circ d_i \\
&= \sum_i \sum_{k=1}^{|X[i]|} \sum_{r=1}^{|X[i]|} \dim(i) q_i(k) \circ d_i(r) \\
&= \sum_i \sum_{k=1}^{|X[i]|} \dim(i) q_i(k) \circ d_i(k)
\end{aligned}$$

$$= \frac{1}{\dim(\Omega)} \sum_i \sum_{k=1}^{|X[i]|} \dim(i) \left(\quad \right) \tag{5.15}$$



$$= \mathrm{id}_{X \boxtimes Y}$$

The first pair of sums collapse to a single sum because $\mathrm{Hom}_C(i, j)$ is zero unless $i = j$. The second pair of sums collapse to a single sum by the simplicity of i and the definition of the pairing between $\mathrm{Hom}_C(X, i^\star)$ and $\mathrm{Hom}_C(i^\star, X)$. The factor $\dim(i)$ is absorbed to the upper pair of α to make the identity of X by 2.18. The lower pair of α traces to δ_l^k by definition. ■

Lemma 5.16 Suppose C is modular. Let $X \boxtimes Y$ be an object in $C \boxtimes C^{bop}$. Then $(d \circ q)_{X \boxtimes Y}$ is equal to the identity isomorphism $\mathrm{id}_{GF(X \boxtimes Y)}$. ◇

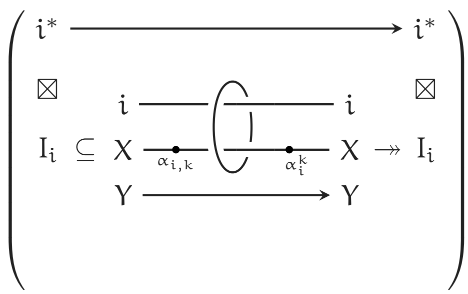
Proof. Recall that the image of $X \boxtimes Y$ under GF is

$$\bigoplus_{i \in O(C)} i^\star \boxtimes I_i,$$

where I_i denotes $I_{i,F(X\boxtimes Y)}$. We will prove the equality by direct computation.

$$
\begin{aligned}
d \circ q &= \Big(\sum_i d_i\Big) \circ \Big(\sum_j q_j\Big) \\
&= \sum_i d_i \circ q_i \\
&= \sum_i \sum_{k=1}^{|X[i]|} \sum_{r=1}^{|X[i]|} \dim(i)\, d_i(r) \circ q_i(k) \\
&= \sum_i \sum_{k=1}^{|X[i]|} \dim(i)\, d_i(k) \circ q_i(k)
\end{aligned} \tag{5.17}
$$

$$
= \frac{1}{\dim(\Omega)} \sum_i \sum_{k=1}^{|X[i]|} \dim(i)
$$

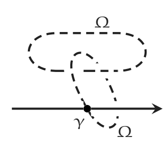


$$
= \mathrm{id}_{GF(X\boxtimes Y)}
$$

The pairs of sums collapse as in the proof of the last lemma. The cut skeins are connected and protected by a Ω-circle by lemma 2.21. And the factor $\dim(i)$ is absorbed to make the identity of X by 2.18. ■

Lemma 5.18 Suppose C is modular. Let (X, γ) be an object in $Z(C)$. Then $(p \circ b)_{(X,\gamma)}$ is equal to the identity isomorphism $\mathrm{id}_{(X,\gamma)}$. ◇

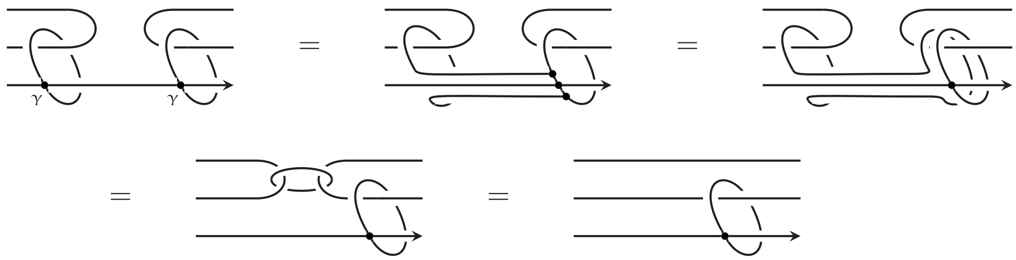
Proof. It is not hard to check that $(p \circ b)_{(X,\gamma)}$ is equal to the product of $\frac{1}{\dim(\Omega)}$ and the following morphism



By lemma 2.21, the horizontal Ω kills off all nontrivial components in the vertical Ω providing the desired equality. ■

Lemma 5.19 Suppose C is modular. Let (X, γ) be an object in $Z(C)$. Then $(b \circ p)_{(X,\gamma)}$ is equal to the identity isomorphism $\mathrm{id}_{FG(X,\gamma)}$. ◇

Proof. We prove the equality by direct computation.



The tensoriality of γ allows the left circle to attach on the right; thus follows the first equality. The sliding lemma allows us to slide one strand to the background, and then again the tensoriality of γ allows detachment; thus follows the second equality. Finally, we sear the two strands and use lemma 2.21 to smooth it out; thus follows the third equation. ■

5.3 Discussion & Prospect

Using the topological insight from the Crane-Yetter TQFT, we provided an explicit equivalence between $C \boxtimes C^{bop}$ and $Z(C)$ for modular categories C. With the same idea, we can also provide explicit equivalences (and witnessing natural isomorphisms) between the categorical centers of higher genera $Z_\Sigma(C)$ [Guu21] and C^n, where Σ is an oriented surface with n punctures. In particular, this provides an explicit equivalence between C and the elliptic Drinfeld center $Z^{el}(C)$ [Tha19].

We stress again that this only works in the case where C is modular. This happens for a good reason. Over modular categories, the Crane-Yetter theory is expected to trivialize to the Witten-Reshetikhin-Turaev theory by taking boundaries. It is interesting to investigate the situation where C is not modular. In fact, this is the motivation of the current paper. We expect that by measuring how the adjoint functors F and G fail to be an inverse of the other, the difference between both theories will become clear, leading to a better understanding of the full power of the Crane-Yetter theory. Moreover, this will also help understand the structures of the categorical center of higher genera (note that the Drinfeld center is hard enough).

One way to attack this problem is to look for a general tool in category theory that measures the failure of the invertibility of a pair of adjoint functors. As F and G are adjoint to each other, the failure can be measure by the induced (co)monads. We expect that the effect of the (co)monads should coincide with tensoring with some object whose size is controlled by the size of $Mu(C)$. This seems to be the case for GF but not FG. Successfully characterizing the effect of both GF and FG in such a way will provide a better proof for theorem 5.12 and a better understanding of the failure of invertibility. Before such characterization is known, there is hope to make guesses based on explicit computations for various genuine premodular categories: (super)-groups, crossed modules, and the even part of the semisimplification of $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ [KO01]. We leave this for future work.

6 Appendix

Most sections in the appendix are added for the sake of completeness.

6.1 Abelian categories

A complete definition of an abelian category is given in this subsection. In particular, see 6.25.

Definition 6.1 (pre-additive category) [Mac10, I.8. p.28] A pre-additive category, or called an Ab-category, is a category A in which each hom-set is an (additive) abelian group, with respect to which the composition maps are bilinear. $\diamond$

Definition 6.2 (biproduct) [Mac10, VIII.2. Definition] Let A be an pre-additive category (6.1). For each pair of A-objects (a, b), define their biproduct to be the pair $(c, \{p_a, p_b, i_a, i_b\})$, where c is an A-object, p_a and p_b are morphisms from c to x, i_a and i_b are morphisms from a and b to c, with the equations satisfied:

$$1_a = p_a i_a \tag{6.3}$$

$$1_b = p_b i_b \tag{6.4}$$

$$1_c = i_a p_a + i_b p_b. \tag{6.5}$$

$\diamond$

Definition 6.6 (initial object) [Mac10, p.20] Let C be a category. An initial object s in C is a C-object such that to each C-object a there is exactly one C-morphism $s \to a$. $\diamond$

Definition 6.7 (terminal object) [Mac10, p.20] Let C be a category. A terminal object t in C is an C-object such that to each object a there is exactly one morphism $a \to t$. $\diamond$

Definition 6.8 (null object) [Mac10, p.20] Let C be a category. A null object z is a C-object which is both initial (6.6) and terminal (6.7). $\diamond$

Definition 6.9 (additive category) [Mac10, VIII.2. p.196] An additive category A is an pre-additive category (6.1) that satisfies the following conditions

- A has a null object (6.8).
- A has a binary biproduct for each pair of A-objects (6.2).

$\diamond$

Definition 6.10 (zero morphism) [Mac10, p. VIII.1.] Let C be a category with a null object z (6.8). Let a, b be C-objects. The zero morphism from a to b is defined to be the composition of the morphism from a to z and the morphism from z to b

$$(a \xrightarrow{0} b) := (a \to z \to b).$$

$\diamond$

Definition 6.11 (monic morphism) [Mac10, p.19] Let C be a category. A monic morphism is a C-morphism $a \xrightarrow{m} b$ such that the left cancellation rule holds:

$$(mf = mg) \Rightarrow (f = g). \tag{6.12}$$

$\diamond$

Definition 6.13 (epi morphism) [Mac10, p.19] Let C be a category. An epi morphism is a C-morphism $a \xrightarrow{m} b$ such that the right cancellation rule holds:

$$(fe = ge) \Rightarrow (f = g). \tag{6.14}$$

$\diamond$

Definition 6.15 (diagonal functor) Let C and J be categories. The diagonal functor Δ from C to C^J is defined to send each C-object c to the constant functor Δ_c, and to send each C-morphism $c \xrightarrow{f} c$ to the constant natural transform Δ_f. (cf [Mac10, p.67]). $\diamond$

Definition 6.16 (universal morphism) [Mac10, p. III.1.] Let C and D be categories. Let c be an C-object. Let $D \xrightarrow{S} C$ be a functor. A universal morphism from c to S is a pair (r, u), where r is D-object and $c \xrightarrow{u} Sr$ is an C-morphism that satisfies the following condition:

For each pair $(d, f) \in \mathrm{Obj}(D) \times C(c, Sd)$, there is a unique D-morphism $r \xrightarrow{f'} d$ with $Sf' \circ u = f$. $\diamond$

Definition 6.17 (categorical limit) [Mac10, p. III.4.] Let C and J be categories and $J \xrightarrow{F} C$ be a functor. A limit for the functor F is defined to be a universal morphism (6.16) (r, ν) from Δ to F, where Δ is the diagonal functor (6.15) from C to C^J. $\diamond$

Definition 6.18 (equalizer) [Mac10, p. III.4.] Let C be a category, a, b be C-objects, and f, g be C-morphisms from a to b. The equalizer for the pair (f, g) is defined to be the limit (6.17) of the corresponding functor $P \xrightarrow{J} C$, where P denotes the category with exactly two objects $0, 1$ and two non-identity morphisms $0 \rightrightarrows 1$. $\diamond$

Definition 6.19 (kernel) [Mac10, p. VIII.1.] Let C be a category with a null object (6.8). A kernel of a morphism $a \xrightarrow{f} b$ is defined to be an equalizer (6.18) for the pair $(f, a \xrightarrow{0} b)$, where 0 denotes the zer morphism (6.10). $\diamond$

Definition 6.20 (cokernel) [Mac10, p.192] The notion of a cokernel is the dual of the notion of a kernel (6.19). $\diamond$

Definition 6.21 (pre-abelian category) An abelian category is an additive category (6.1) in which ever morphism has a kernel (6.19) and a cokernel (6.20). $\diamond$

Lemma 6.22 Let $\mathcal{A}$ be a pre-abelian category. Then each morphism $X \xrightarrow{f} Y$ in $\mathcal{A}$ has a canonical factorization [Ive, p. I.1]

$$f = \Big(X \to \mathrm{cok}(\ker(f)) \xrightarrow{f'} \ker(\mathrm{cok}(f)) \to Y\Big). \tag{6.23}$$

$\diamond$

Definition 6.24 (exact category) An exact category is a pre-abelian category in which the middle morphism of the canonical factorization (6.22) f' of each morphism $X \xrightarrow{f} Y$ is an isomorphism. $\diamond$

Definition 6.25 (abelian category) [Mac10, VIII.3. Definition] An abelian category is an pre-abelian category (6.1) in which every monic morphism (6.11) is a kernel, and every epi morphism (6.13) is a cokernel. $\diamond$

Lemma 6.26 Let $\mathcal{A}$ be a pre-abelian category. Then the followings are equivalent.

- $\mathcal{A}$ is abelian.
- $\mathcal{A}$ is exact.

$\diamond$

Lemma 6.27 To summarize, an abelian category is a category such that the followings are satisfied.

- (pre-additivity) Every hom set is an abelian group such that every composition is bilinear.
- (additivity) A null object and binary biproducts exist.
- (pre-abelianity) Every morphism has a kernel and cokernel.
- (exactness) Canonical factorizations induce isomophisms between the images and coimages
$$\mathrm{cok} \circ \ker(-) \to \ker \circ \mathrm{cok}(-).$$

$\diamond$

Lemma 6.28 Let C be an abelian category and D be an additive category. Suppose there is an additive equivalence of categories $C \xrightarrow{F} D$. Then D is abelian. $\diamond$

Proof. By 6.27, we have to show that the additive category D is pre-abelian and exact.

Let $X' \xrightarrow{\phi'} Y'$ be a D-morphism. Pick C-objects X and Y such that FX and FY are isomorphic to X' and Y' respectively. Since

$$C(X, Y) \simeq D(FX, FY) \simeq D(X', Y'),$$

ϕ' has kernels and cokernels, and its canonical factorization induces isomorphisms between images and coimages. ■

6.2 Semisimple categories

Throughout the whole section, assume that $\Bbbk$ is an algebraically closed field of characteristic 0.

Definition 6.29 (subobject) [Eti+15, p. 1.3.5] Let C be a category and X be a C-object. A subobject of X is a monic C-morphism $Y \xrightarrow{f} X$. ◇

Definition 6.30 (simple object) [Eti+15, p. 1.5.1] Let C be an abelian category. A simple object X of C is a nonzero C-object whose only subobjects are $\mathbb{0} \xrightarrow{0} X$ and $X \xrightarrow{\mathrm{id}_X} X$. ◇

Definition 6.31 (semisimple object) [Eti+15, p. 1.5.1] Let C be an abelian category. A semisimple object X of C is a direct sum of some simple objects of C. ◇

Definition 6.32 (semisimple category) [Eti+15, p. 1.5.1] A semisimple category is an abelian category whose objects are all semisimple. ◇

Definition 6.33 (object of finite length) [Eti+15, p. 1.5.3] Let X be an object of an abelian category C. We say that X is of finite length if there exists a positive integer n and a sequence of monic morphisms

$$\mathbb{0} = X_0 \to X_1 \to \cdots \to X_n = X \tag{6.34}$$

each of whose cokernel object X_{i+1}/X_i is simple. Call n the length of this sequence. ◇

Remark 6.35 Such a sequence is called a Jordan-Holder series for X. By Jordan-Holder theorem [Eti+15, p. 1.5.4], all Jordan-Holder series of X have the same length. ◇

Definition 6.36 (length of an object) [Eti+15, p. 1.5.5] Let C be an abelian category and X a C-object. The length X is defined to be the length of one, thus all, of its Jordan-Holder series. ◇

Definition 6.37 (linear category over a field) [Eti+15, p. 1.2.2] Let $\mathbb{k}$ be a field. A $\mathbb{k}$-linear category is an additive category C whose hom-spaces are $\mathbb{k}$-vector spaces, such that all compositions of morphisms are $\mathbb{k}$-linear maps. $\diamond$

Definition 6.38 (locally finite abelian category over a field) [Eti+15, p. 1.8.1] A locally finite category (or an Artinian category) over $\mathbb{k}$ is a $\mathbb{k}$-linear abelian category C that satisfies the following conditions.

- Every object has finite length.
- Every hom space is a finite dimensional $\mathbb{k}$-vector space.

$\diamond$

Definition 6.39 (finite abelian category over a field) [Eti+15, p. 1.8.6] A finite category abelian category C over $\mathbb{k}$ is a locally finite abelian category over $\mathbb{k}$ such that

- C has enough projectives, i.e. every simple object of C has a projective cover.
- The set of isomorphism classes of simple objects is finite.

$\diamond$

Remark 6.40 By discussion before [Eti+15, p. 1.8.6], a finite $\mathbb{k}$-linear abelian category C is equivalent to the category of finite dimensional modules over a finite dimensional $\mathbb{k}$-algebra A. $\diamond$

6.3 Tensor categories

Recall that the Crane-Yetter theory CY comes in a family, of which member depends on a type of algebraic data called the premodular categories. Despite its technical definition (finite semisimple ribbon braided rigid tensor category), it does not hurt too much to think of a premodular category as a higher categorical analogue of a finite abelian group: the "braided tensor" structure encodes the (higher) group operation, the "rigid" structure encodes the (higher) inverses, and the "ribbon" structure ensures that $(g^{-1})^{-1}$ is equivalent to g.

In this section, a complete definition for a premodular category 6.68 is collected from [Eti+15]. Throughout the whole section, assume that $\mathbb{k}$ is an algebraically closed field of characteristic 0.

Definition 6.41 (monoidal category) [Eti+15, p. 2.1.1] A monoidal category is a septuple

$$(C, \otimes, a, \mathbb{1}, \iota, l, r),$$

that satisfies the pentagon axiom and the triangle axiom [Eti+15, (2.2)], where C is a category, $C \times C \xrightarrow{\otimes} C$ is a bifunctor,

$$((-_1 \otimes -_2) \otimes -_3) \xrightarrow{a} (-_1 \otimes (-_2 \otimes -_3))$$

is a natural equivalence, $\mathbb{1}$ is an object in C, $\mathbb{1} \otimes \mathbb{1} \xrightarrow{\iota} \mathbb{1}$ is an isomorphism, $(-) \xrightarrow{l} (\mathbb{1} \otimes -)$ and $(-) \xrightarrow{r} (- \otimes \mathbb{1})$ are natural equivalences.

We will abuse notations and denote the septuple by C. The bifunctor $\otimes$ is called the tensor product bifunctor, the pair $(\mathbb{1}, \iota)$ is called the unit object, and the natural equivalence a is called the associativity isomorphism $\diamond$

Definition 6.42 (duals of an object) [Eti+15, 2.10.1 and 2.10.2] Let X be an object of a monoidal category $(C, \otimes, \mathbb{1}, a, \iota, l, r)$. A left dual of X is an object L with two morphisms

$$L \otimes X \xrightarrow{ev} \mathbb{1} \tag{6.43}$$

$$\mathbb{1} \xrightarrow{coev} X \otimes L \tag{6.44}$$

such that the compositions of the following the identity morphisms

$$X \xrightarrow{coev \otimes 1} (X \otimes L) \otimes X \xrightarrow{a} X \otimes (L \otimes X) \xrightarrow{1 \otimes ev} X, \tag{6.45}$$

$$L \xrightarrow{1 \otimes coev} L \otimes (X \otimes L) \xrightarrow{a^{(-1)}} (L \otimes X) \otimes L \xrightarrow{ev \otimes 1} L. \tag{6.46}$$

Similarly, a right dual of X is an object R with two morphisms

$$X \otimes R \xrightarrow{ev'} \mathbb{1} \tag{6.47}$$

$$\mathbb{1} \xrightarrow{coev'} R \otimes X \tag{6.48}$$

such that the compositions of the following are the identity morphisms

$$X \xrightarrow{1 \otimes coev'} X \otimes (R \otimes X) \xrightarrow{a^{(-1)}} (X \otimes R) \otimes X \xrightarrow{ev' \otimes 1} X, \tag{6.49}$$

$$R \xrightarrow{coev' \otimes 1} (R \otimes X) \otimes R \xrightarrow{a} R \otimes (X \otimes R) \xrightarrow{1 \otimes ev'} R. \tag{6.50}$$

$\diamond$

Remark 6.51 It can be proved that the left (resp., right) dual, if exists, is unique up to isomorphism [Eti+15, p. 2.10.5]. We will denote it by X^* (resp., ${}^*X$). $\diamond$

Definition 6.52 (rigid object) [Eti+15, p. 2.10.11] Let C be a monoidal category. A rigid object X of C is a C-object that has a left dual and a right dual. $\diamond$

Definition 6.53 (rigid category) [Eti+15, p. 2.10.11] A rigid category C is a monoidal category all of whose objects are rigid. $\diamond$

Definition 6.54 (multitensor category) [Eti+15, p. 4.1.1] A multitensor category C over $\Bbbk$ is a locally finite $\Bbbk$-linear abelian rigid monoidal category if the bifunctor $\otimes$ in the monoidal structure is $\Bbbk$-bilinear on morphisms. $\diamond$

Lemma 6.55 [Eti+15, p. 4.2.1] Let C be a multitensor category. Then the bifunctor $\otimes$ is exact in both factors. $\diamond$

Proof. It is a fun exercise to prove. A sketch is as follows. Let X be a C-object. The rigidity says that $X \otimes (-)$ is a left and right adjoint functor. In general category theory, adjoint functors preserve all (co)limits essentially because Hom does and Yoneda lemma. In particular, they preserve (co)kernels. ∎

Definition 6.56 (multifusion category) [Eti+15, p. 4.1.1] A multifusion category over $\Bbbk$ is a multitensor category that is finite over $\Bbbk$ and semisimple. $\diamond$

Definition 6.57 (fusion category) [Eti+15, p. 4.1.1] A fusion category C is a multifusion category with $\mathrm{End}_C(\mathbb{1}) \simeq \Bbbk$. $\diamond$

Definition 6.58 (braiding) [Eti+15, p. 8.1.1] A braiding of a monoidal category $(C, \otimes, \mathbb{1}, a, \iota, l, r)$ is a natural equivalence

$$(-_1 \otimes -_2) \xrightarrow{c} (-_2 \otimes -_1) \tag{6.59}$$

such that the hexagon diagram [Eti+15, (8.1)] holds. $\diamond$

Definition 6.60 (braided category) [Eti+15, p. 8.1.2] A braided category is a monoidal category with a braiding. $\diamond$

Remark 6.61 The Yang-Baxter equation holds automatically in a braided category [Eti+15, p. 8.1.10]. $\diamond$

Definition 6.62 (half-braiding) [Eti+15, (7.41)] A half-braiding for an object X in a monoidal category $(C, \otimes, \mathbb{1}, a, \iota, l, r)$ is a natural equivalence

$$(X \otimes -) \xrightarrow{c} (- \otimes X) \tag{6.63}$$

such that the hexagon diagram [Eti+15, (7.41)] holds. $\diamond$

Definition 6.64 (twist) [Eti+15, p. 8.10.1] Let C be a braided rigid monoidal category. A twist of C is an element $\theta \in \mathrm{Aut}(\mathrm{id}_C)$ such that for each C-object X, Y

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \tag{6.65}$$

$\diamond$

Definition 6.66 (ribbon structure) [Eti+15, p. 8.10.1] Let C be a braided rigid monoidal category. A twist θ is called a ribbon structure if $(\theta_X)^\star = (\theta_{(X^\star)})$, where the first dual is taken in a rigid category. ◇

Definition 6.67 (ribbon tensor category) [Eti+15, p. 8.10.1] A ribbon tensor category is a braided rigid monoidal category equipped with a ribbon structure. ◇

Definition 6.68 (premodular category) [Eti+15, p. 8.13.1] A premodular category is a ribbon fusion category. ◇

Definition 6.69 (pivotal structure) [Eti+15, p. 4.7.7] Let C be a rigid monoidal category. A pivotal structure of C is a natural isomorphism

$$(-) \xrightarrow[\sim]{a} (-)^{\star\star}$$

such that $a_{X\otimes Y} = a_X \otimes a_Y$ for all C-objects X and Y. We call a rigid monoidal category pivotal if it is equipped with a pivotal structure. ◇

Definition 6.70 (pivotal dimension) Let C be a rigid monoidal category with a pivotal structure a. Let X be a C-object. We define the pivotal dimension with respect to a to be

$$\dim_a(X) := \mathrm{Trace}(a_X) \in \mathrm{End}_C(1).$$

◇

Definition 6.71 (spherical structure) [Eti+15, p. 4.7.14] Let C be a rigid monoidal category with a pivotal structure a. The latter is called a spherical structure if

$$\dim_a(X) = \dim_a(X^\star)$$

for any C-object X. ◇

Remark 6.72 [Eti+15, p. 8.13.1] Equivalently, a premodular category is also a braided fusion category equipped with a spherical structure. ◇

Definition 6.73 (modular category) [Eti+15, 8.14 and 8.20.12] A modular category is a premodular category with a non-degenerate S-matrix. ◇

6.4 Adjunctions as monads

Definition 6.74 Let X be a strict 2-category, C and D be X-objects, and $C \xrightarrow{F} D$ and $C \xleftarrow{G} D$ be morphisms. We say that F is right adjoint to G, that G is left adjoint to F, if there exists 2-morphisms

$$1_D \overset{\eta}{\Rightarrow} FG, \quad GF \overset{\epsilon}{\Rightarrow} 1_C$$

such that the followings

$$G = G \circ 1_D \overset{1_G * \eta}{\Rightarrow} G \circ F \circ G \overset{\epsilon * 1_G}{\Rightarrow} G$$

$$F = 1_D \circ F \overset{\eta * 1_F}{\Rightarrow} F \circ G \circ F \overset{1_F * \epsilon}{\Rightarrow} F$$

equal the identity 2-morphisms 1_G and 1_F respectively. Denote $F \vdash G$ in this case. We call η and ϵ the unit and counit of the monad, and call the coherence condition the rigidity condition. $\diamond$

Definition 6.75 Let X be a strict 2-category, D be an X-object. Then $E = \mathrm{End}_X(D)$ is a 1-category. A monad of D is a monoid object $T = (T, \eta, \mu)$ in E. That is to say, T is an E-object, and $(1_D \overset{\eta}{\Rightarrow} T)$ and $(T^2 \overset{\mu}{\Rightarrow} T)$ are E-morphisms such that

$$(1_T * \eta) = 1_T = \mu \circ (\eta * 1_T), \qquad \mu \circ (\mu * 1_T) = \mu \circ (1_T * \mu).$$

$\diamond$

Theorem 6.76 Let X be a strict 2-category, C and D be X-objects, $(C \xrightarrow{F} D)$ and $(C \xleftarrow{G} D)$ be morphisms such that F is right adjoint to G. Then $T = FG$ has a monad structure given by

$$(1_D \overset{\eta}{\Rightarrow} T), \quad (T^2 \overset{\mu}{\Rightarrow} T)$$

where μ is defined as $FGFG \overset{1_F * \epsilon * 1_G}{\Rightarrow} FG$. Dually, $\perp = GF$ has a comonad structure. $\diamond$

Proof.

$$\begin{aligned}
& 1_T \\
= \quad & 1_F * 1_G \\
= \quad & 1_F * (G \overset{1_G * \eta}{\Rightarrow} GFG \overset{\epsilon * 1_G}{\Rightarrow} G) \\
= \quad & (FG \circ 1_D \overset{1_F * 1_G * \eta}{\Rightarrow} FGFG \overset{1_F * \epsilon * 1_G}{\Rightarrow} FG) \\
= \quad & (T \circ 1_D \overset{1_T * \eta}{\Rightarrow} T \circ T \overset{\mu}{\Rightarrow} T)
\end{aligned}$$

$$\begin{aligned}
& 1_T \\
= \quad & 1_F * 1_G \\
= \quad & (F \overset{\eta * 1_F}{\Rightarrow} FGF \overset{1_F * \epsilon}{\Rightarrow} F) * 1_G \\
= \quad & (1_D \circ FG \overset{\eta * 1_F * 1_G}{\Rightarrow} FGFG \overset{1_F * \epsilon * 1_G}{\Rightarrow} FG) \\
= \quad & (1_D \circ T \overset{\eta * 1_T}{\Rightarrow} T \circ T \overset{\mu}{\Rightarrow} T)
\end{aligned}$$

$$
\begin{aligned}
& (T^3 \overset{1_T \circ \mu}{\Rightarrow} T^2 \overset{\mu}{\Rightarrow} T) \\
= \; & ((FG)(FGFG) \overset{1_{FG} * 1_F * \epsilon * 1_G}{\Rightarrow} FGFG \overset{1_F * \epsilon * 1_G}{\Rightarrow} FG) \\
= \; & 1_F * (GFGF \overset{1_{GF} * \epsilon}{\Rightarrow} GF \overset{\epsilon}{\Rightarrow} 1_D) * 1_G \\
= \; & 1_F * (GFGF \overset{\epsilon * \epsilon}{\Rightarrow} 1_D) * 1_G \\
= \; & \ldots = (T^3 \overset{\mu \circ 1_T}{\Rightarrow} T^2 \overset{\mu}{\Rightarrow} T).
\end{aligned}
$$

■

Theorem 6.77 Let C and D be categories, $C \xrightarrow{F} D$ and $C \xleftarrow{G} D$ be functors such that F is a right adjoint functor to G, i.e. there exists natural equivalence Φ such that

$$C(Gd, c) \xrightarrow[\Phi_{d,c}]{\sim} D(d, Fc).$$

Then F is right adjoint to G the strict 2-category $\mathcal{C}at$ of all categories. ◇

Proof. From the given natural equivalence Φ, we have to construct $1_D \overset{\eta}{\Rightarrow} FG$ and $GF \overset{\epsilon}{\Rightarrow} 1_C$ that satisfy the conditions as in 6.74. We contend that $\eta_d = \Phi_{d,Gd}(1_{Gd})$ and $\epsilon_c = \Phi^{(-1)}_{Fc,c}(1_{Fc})$ are as desired.

Let us first show that η_d is indeed a natural transformation from 1_D to FG. So is ϵ_c similarly, whose proof will be omitted. Let $d \xrightarrow{\phi} d'$ be a D-morphism. We shall prove that the following commutative diagram commute.

$$
\begin{array}{ccc}
d & \xrightarrow{\eta_d} & FGd \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle FG\phi} \\
d' & \xrightarrow{\eta_{d'}} & D
\end{array}
$$

First notice that

$$
\begin{aligned}
\Phi^{(-1)}_{d,Gd'}(\eta_{d'} \circ \phi) &= (G\phi)^\star \circ \Phi^{(-1)}_{d',Gd'} \circ (\phi^\star)^{(-1)}(\eta_{d'} \circ \phi) \\
&= (G\phi)^\star \circ \Phi^{(-1)}_{d',Gd'}(\eta_{d'}) \\
&= (G\phi)^\star(1_{Gd'}) \\
&= G\phi
\end{aligned}
$$

As Φ is an equivalence, it suffices to prove that $\Phi^{(-1)}_{d,Gd'}((FG\phi) \circ \eta_d)$ is also $G\phi$.

$$
\begin{aligned}
\Phi^{(-1)}_{d,Gd'}((FG\phi)\circ\eta_d) &= \Phi^{(-1)}_{d,Gd'}((FG\phi)\circ\Phi_{d,Gd}(1_{Gd}))\\
&= (G\phi)_\star\circ\Phi^{(-1)}_{d,Gd}\circ(FG\phi)^{(-1)}_\star((FG\phi)\circ\Phi_{d,Gd}(1_{Gd}))\\
&= (G\phi)_\star\circ\Phi^{(-1)}_{d,Gd}(\Phi_{d,Gd}(1_{Gd}))\\
&= (G\phi)_\star(1_{Gd})\\
&= G\phi
\end{aligned}
$$

which is due to the naturality of Φ

$$
\begin{array}{ccc}
C(Gd,Gd) & \xrightarrow{\Phi_{d,Gd}} & D(d,FGd)\\
\big\downarrow{\scriptstyle (G\phi)_\star} & & \big\downarrow{\scriptstyle (FG\phi)_\star}\\
C(Gd,Gd')' & \xrightarrow{\Phi_{d,Gd'}} & D(d,FGd')
\end{array}
$$

Next, we have to show the rigidity conditions of η and ϵ. Indeed, from the naturality of Φ we have the commutative diagram

$$
\begin{array}{ccc}
C(GFGd,Gd) & \xrightarrow{\Phi_{FGd,Gd}} & D(FGd,FGd)\\
\big\downarrow{\scriptstyle (G(\eta_d)^\star)} & & \big\downarrow{\scriptstyle \eta_d^\star}\\
C(Gd,Gd) & \xrightarrow{\Phi_{d,Gd}} & D(d,FGd)
\end{array}
$$

and thus the following holds

$$
\begin{aligned}
[(\epsilon*1_G)\circ(1_G*\eta)]_d &= (\epsilon*1_G)_d\circ(1_G*\eta)_d\\
&= (GFGd\xrightarrow{\epsilon_{Gd}}Gd)\circ G(d\xrightarrow{\eta_d}FGd)\\
&= Gd\xrightarrow{G(\eta_d)}GFGd\xrightarrow{\epsilon_{Gd}}Gd\\
&= Gd\xrightarrow{G(\eta_d)}GFGd\xrightarrow{\Phi^{(-1)}_{FGd,Gd}(1_{(FGd)})}Gd\\
&= (G(\eta_d))^\star\Big(GFGd\xrightarrow{\Phi^{(-1)}_{FGd,Gd}(1_{(FGd)})}Gd\Big)\\
&= \big(\Phi^{(-1)}_{d,Gd}\circ\eta_d^\star\big)(1_{FGd})\\
&= \Phi^{(-1)}_{d,Gd}(\eta_d)=1_{Gd}.
\end{aligned}
$$

The other rigidity condition follows similarly from the commutative diagram

$$
\begin{array}{ccc}
C(GFc,GFc) & \xrightarrow{\Phi_{Fc,GFc}} & D(Fc,FGFc)\\
\big\downarrow{\scriptstyle (\epsilon_c)_\star} & & \big\downarrow{\scriptstyle (F\epsilon_c)_\star}\\
C(GFc,c) & \xrightarrow{\Phi_{Fc,c}} & D(Fc,Fc)
\end{array}
$$

■

Therefore, adjoint functors give adjoint pairs in the 2-category $\mathcal{C}at$, which in turn gives a monad and a comonad in $\mathcal{C}at$. Let's summarize the result in the following theorem.

Theorem 6.78 Let $C \xrightarrow{F} D$ and $C \xleftarrow{G} D$ be functors such that F is right adjoint to G. Then there is a D-monad $(T = FG, \eta, \mu)$ and a C-comonad $(\perp = GF, \epsilon, \Delta)$, where

$$\eta_d = \Phi_{d,Gd}(1_{(Gd)}), \mu = 1_F * \epsilon * 1_G,$$

$$\epsilon_c = \Phi^{(-1)}_{Fc,c}(1_{Fc}), \Delta = 1_G * \eta * 1_F.$$

◇

For the rest of this subsection, assume that $C \xrightarrow{F} D$ is a right adjoint functor of $C \xleftarrow{G} D$ with the natural transformation Φ. The categories C and D are intimately tided together by the adjoint functors between them. For example, a part of compositions in C can be identified as monadic composition in D.

Theorem 6.79 The usual composition map

$$C(Gx, Gy) \times C(Gy, Gz) \xrightarrow{\circ} C(Gx, Gz)$$

is identified under Φ as the Kleisi composition

$$D(x, Ty) \times D(y, Tz) \xrightarrow{\circ_T} D(x, Tz)$$

$$\circ_T(f, g) \mapsto \mu_Z \circ (Tg) \circ f.$$

◇

Proof. Since Φ is an equivalence, and since

$$\Phi(\circ(\Phi^{(-1)}(f, g))) = \Phi_{x,Gz}(\Phi^{(-1)}_{y,Gz}(g) \circ \Phi^{(-1)}_{x,Gy}(f)),$$

it suffices to prove that

$$\Phi^{(-1)}_{x,Gz}(\mu_Z \circ (Tg) \circ f) = \Phi^{(-1)}_{y,Gz}(g) \circ \Phi^{(-1)}_{x,Gy}(f).$$

The main task would be to express μ_Z in terms of Φ.

From the commutative diagram

$$\begin{tikzcd}
C(GFGz, Gz) \arrow[r, "\Phi_{FGz,Gz}"] \arrow[d, "(-)\circ G(g)"] & D(FGz, FGz) \arrow[d, "(-)\circ g"] \\
C(Gy, Gz) \arrow[r, "\Phi_{y,Gz}"] & D(y, FGz)
\end{tikzcd}$$

we have

$$
\begin{aligned}
& \mu_Z \circ (Tg) \circ f \\
= \; & F(\Phi^{(-1)}_{FGz,Gz}(1_{FGz}) \circ F(G(g)) \circ f) \\
= \; & F(\Phi^{(-1)}_{FGz,Gz}(1_{FGz}) \circ G(g)) \circ f) \\
= \; & F(\Phi^{(-1)}_{y,Gz}(1_{FGz} \circ g)) \circ f) \\
= \; & F(\Phi^{(-1)}_{y,Gz}(g)) \circ f)
\end{aligned}
$$

It remains to prove that

$$\Phi^{(-1)}_{x,Gz}(F(\Phi^{(-1)}_{y,Gz}) \circ f) = \Phi^{(-1)}_{y,Gz}(g) \circ \Phi^{(-1)}_{x,Gy}(f),$$

which directly follows from the commutative diagram obtained from the naturality of Φ:

$$
\begin{array}{ccc}
C(Gx, Gy) & \xrightarrow{\Phi_{x,Gy}} & D(x, FGy) \\
\downarrow{\scriptstyle \Phi^{(-1)}(g) \circ (-)} & & \downarrow{\scriptstyle F(\Phi^{(-1)}) \circ (-)} \\
C(Gx, Gz) & \xrightarrow{\Phi_{x,Gz}} & D(x, FGz)
\end{array}
$$

■

The natural equivalence Φ isn't as easy to manipulate as the (co)monads it induces. We collect more statements that express the former in terms of the later.

Lemma 6.80 In terms of (co)monad, Φ can be expressed as follows.

$$\Phi_{d,c}(\phi) = F(\phi) \circ \eta_d,$$

$$\Phi^{(-1)}_{d,c}(\psi) = \epsilon_c \circ G(\psi).$$

◇

Proof. These are evident from the commutative diagrams below respectively.

$$
\begin{array}{ccc}
C(Gd, Gd) & \xrightarrow{\Phi_{d,Gd}} & D(d, FGd) \\
\downarrow{\scriptstyle \Phi \circ (-)} & & \downarrow{\scriptstyle (F\phi) \circ (-)} \\
C(Gd, c) & \xrightarrow{\Phi_{d,c}} & D(d, Fc)
\end{array}
\qquad
\begin{array}{ccc}
C(GFc, Fc) & \xrightarrow{\Phi_{Fc,c}} & D(Fc, Fc) \\
\downarrow{\scriptstyle (-) \circ \psi} & & \downarrow{\scriptstyle (-) \circ G(\psi)} \\
C(Gd, c) & \xrightarrow{\Phi_{d,Fc}} & D(d, Fc)
\end{array}
$$

■

Instances arise where two functors are adjoint to each other from both sides. We call them (strict) ambidextrous functors.

Definition 6.81 (ambidextrous adjunctions) Let $C \xrightarrow{F} D$ and $C \xleftarrow{G} D$ be functors. We call F and G a pair of (strict) ambidextrous functors if F is both left-adjoint and right-adjoint to G. $\diamond$

Two monads and two comonads arise from a pair of ambidextrous functors. More precisely, that $F \vdash G$ gives a natural D-monad $(T = FG, \eta, \mu)$ and a natural C-comonad $(\perp = GF, \epsilon, \Delta)$. Similarly, that $F \dashv G$ gives a natural D-comonad $(T = FG, \eta', \mu')$ and a natural C-monad $(\perp = GF, \epsilon', \Delta')$. In particular, we have a bimonad structure on $T = (T, \eta, \mu, \epsilon, \Delta)$, with unit η, multiplication μ, counit ϵ, and co-multiplication Δ.

Definition 6.82 We say the bimonad T is of unity trace if

$$(1_D \xrightarrow{\eta} T \xrightarrow{\eta'} 1_D) = (1_D \xrightarrow{1_{(1_D)}} 1_D).$$

We say the bimonad T is of collapsable diamond if

$$(T \xrightarrow{\mu'} T^2 \xrightarrow{\mu} T) = (T \xrightarrow{1_T} T).$$

$\diamond$

By the following lemma, the second condition is superseded by the first one.

Lemma 6.83 If such adjunction is of unity trace for $\perp$, then T is of collapsable diamond. $\diamond$

Proof.

$$\begin{aligned}
& (T \xrightarrow{\mu'} T^2 \xrightarrow{\mu} T) \\
= \quad & (FG \xrightarrow{1_F * \epsilon' * 1_G} F(GF)G \xrightarrow{1_F * \epsilon * 1_G} FG) \\
= \quad & 1_F * (1 \xrightarrow{\epsilon'} \perp \xrightarrow{\epsilon} 1) * 1_G \\
= \quad & 1_F * 1_{(1_C)} * 1_G = 1_T
\end{aligned}$$

■

The unity trace condition turns out to be crucial for our work - essentially it guarantees an averaging map analogue to that in the theory of finite group representations.

Lemma 6.84 Let $T = (T, \eta, \mu, \epsilon, \Delta)$ be a D-bimonad of unity trace. Then for each D-object x and y, the morphism $[D(x, y) \xrightarrow{\eta_y \circ (-)} D(x, Ty)]$ is monic, the arrow $[D(x, Ty) \xrightarrow{\eta'_y \circ (-)} D(x, y)]$ is epic, and moreover the map $(\eta_y \circ \eta'_y)$ is a projection map onto the image of $(\eta_y \circ -)$. $\diamond$

Proof. The unity trace condition says that $\eta'_y \circ \eta_y = 1_y$, so the first two conditions follow. The last statement is also evident since

$$(\eta'_y \eta_y)^2 = \eta'_y 1_y \eta_y = \eta'_y \eta_y.$$

■

Therefore, in the context of (strict) ambidextrous adjoint functors, the unity trace condition yields a projection map

$$C(Gx, Gy) \xrightarrow{\pi_{x,y}} D(x, y)$$

from the equivalence $C(Gx, Gy) \simeq D(x, Ty)$. In the next lemma, we see that this projection is functorial without any extra assumption.

Theorem 6.85 Let $F : C \leftrightarrow D : G$ be a pair of strictly ambidextrous adjoint functors, and $T = FG$ be the naturally induced bimonad on D. If T is of unity trace, then $\pi_{x,y}$ is functorial in the sense that

1. $\pi_{x,x}(Gx \xrightarrow{1_{Gx}} Gx) = 1_x$.
2. For C-morphisms $(Gx \xrightarrow{\phi} Gy \xrightarrow{\sigma} Gz)$, we have

$$(x \xrightarrow{\pi_{x,y}\phi} y \xrightarrow{\pi_{y,z}\sigma} z) = (x \xrightarrow{\pi_{x,z}(\sigma\phi)} z).$$

◇

Proof. By the unity trace condition and 6.80,

$$\pi_{x,x}(1_{Gx}) = \eta'_x \circ (1_{FGx} \circ \eta_x) = 1_X,$$

proving the first statement. It remains to prove that

$$(\eta'_z \circ F(\sigma) \circ \eta_y) \circ (\eta'_y \circ F(\phi) \circ \eta_x) = (\eta'_z \circ F(\sigma\phi) \circ \eta_x).$$

Indeed,

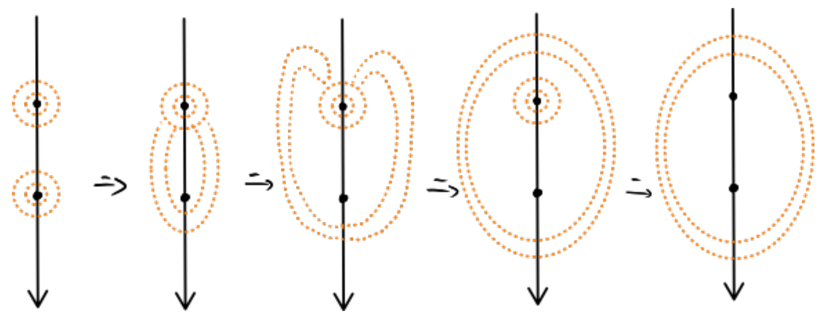
$$\begin{aligned}
& \left(z \xleftarrow{\eta'_z} Tz \xleftarrow{F(\sigma)} Ty \xleftarrow{\eta_y} y \xleftarrow{\eta'_y} Ty \xleftarrow{F(\phi)} Tx \xleftarrow{\eta_x} x\right) \\
= & \left(z \xleftarrow{\eta'_z} Tz \xleftarrow{F(\sigma)} Ty \xleftarrow{T\eta'_y} T^2y \xleftarrow{\eta_{Ty}} Ty \xleftarrow{F(\phi)} Tx \xleftarrow{\eta_x} x\right) \\
= & \left(z \xleftarrow{\eta'_z} Tz \xleftarrow{F(\sigma)} Ty \xleftarrow{T\eta'_y} T^2y \xleftarrow{T(F(\phi))} T^2x \xleftarrow{\eta_{Tx}} Tx \xleftarrow{\eta_x} x\right) \\
= & \left(z \xleftarrow{\eta'_z} Tz \xleftarrow{F(\sigma)} Ty \xleftarrow{T\eta'_y} T^2y \xleftarrow{T(F(\phi))} T^2x \xleftarrow{T\eta_x} Tx \xleftarrow{\eta_x} x\right) \\
= & \left(z \xleftarrow{\eta'_z} Tz \xleftarrow{F(\sigma)} Ty \xleftarrow{F(\phi)} Tx \xleftarrow{\eta_x} x\right)
\end{aligned}$$

The first two equalities follow from the naturality of η. The third equality $T_{(\eta_x)} = \eta_{(Tx)}$ follows from the Eckmann-Hilton argument

$$\eta * 1_T = 1_T * \eta.$$

Finally, the last equality follows from all the tricks and conditions: that η is natural, that $T \circ \eta = \eta \circ T$, that T is functorial, and the unity trace condition. ■

Remark 6.86 In the context of categorical center of higher genera, the proof above translates into the following graphical proof, where the orange dotted lines represent the shorthand notation Ω given in 2.19.



◇

In the proof, it is tempting to demand $\eta_y \circ \eta'_y$ to be identity, which would have finished the proof right away. However, it is not necessarily true. In fact, it is false in our context. The best one can say about it is that it is idempotent.

Example 6.87 Let C be a premodular category, σ be an admissible gluing, D be the categorical center of higher genera $Z_\sigma(C)$, F be the induction functor $C \xrightarrow{I_\sigma} Z_\sigma(C)$, and G be the forgetful functor $C \xleftarrow{F_\sigma} Z_\sigma(C)$.

By 4.55, both functors are strictly ambidextrous to each other. Moreover, it is clear by their definitions that the bi-monads they form are of unity trace. Therefore, by 6.85, we have the followings.

1. $D((X,\gamma),(Y,\beta))$ embeds into $C(X,Y)$ naturally, with a natural projection $\pi_{\gamma,\beta}$ onto the subspace.
2. $C(X,Y)$ embeds into $D(I_\sigma(X), I_\sigma(Y))$ naturally, with a natural projection $\pi_{X,Y}$ onto the subspace.

◇

This is an analogue of the averaging map one has in the theory of finite dimensional complex linear representations of finite groups.

6.5 Misc proofs

Statements and proofs that could break the flow of are collected in this section. The readers are advised to use it as a reference.

Lemma 6.88 Let C be a premodular category and $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing. Then the categorical center of higher genera $Z_\sigma(C)$ is an abelian category. $\diamond$

Proof. By 6.27 we need to show that $Z_\sigma(C)$ is an additive category such that

- every morphism in $Z_\sigma(C)$ has a kernel and a cokernel.
- $Z_\sigma(C)$ is an exact category.

By its definition, $Z_\sigma(C)$ is additive. To prove that every morphism has a kernel and a cokernel, first let $(X, \gamma) \xrightarrow{f} (Y, \beta)$ be a $Z_\sigma(C)$-morphism. Recall by definition that f is a C-morphism $X \xrightarrow{f} Y$ that respects both sets of half-braidings γ and β. Since C is premodular thus abelian, f has a kernel $K \xhookrightarrow{m} X$ in C. We will construct a $Z_\sigma(C)$-object $(K, m^\star\gamma)$ such that $(K, m^\star) \xrightarrow{m} (X, \gamma)$ is a $Z_\sigma(C)$-morphism and is in fact a kernel of f.

To construct $m^\star\gamma$, notice that all we need is a set of half-braidings for K that work compatibly with γ. As

$$K \xrightarrow{m} X \xrightarrow{f} Y$$

is exact and that $\otimes$ is bi-exact 6.55, we see that $1_{(-)} \otimes m$ and $m \otimes 1_{(-)}$ are kernels of $1_{(-)} \otimes f$ and $f \otimes 1_{(-)}$ respectively. Therefore, $\gamma_{[i]} \circ (m \otimes 1_{(-)})$ factors through $1_{(-)} \otimes m$ uniquely. Ditto for the other direction. So defines an natural isomorphism

$$K \otimes (-) \xrightarrow{(m^\star\gamma)_{[i]}} (-) \otimes K.$$

Define so for all other i's. it's straightforward to prove that $m^\star\gamma$ is a σ-pair 4.39 from that γ is also one.

From the construction above, clearly

$$(K, m^\star\gamma) \xrightarrow{m} (X, \gamma)$$

is a $Z_\sigma(C)$-arrow. It remains to show that m is indeed a kernel of f in $Z_\sigma(C)$. Let $(W, \alpha) \xrightarrow{h} (X, \gamma)$ be a $Z_\sigma(C)$-morphism such that $fh = 0$. Then h uniquely factors through m by some C-morphism k. The crux is to show that k is indeed a $Z_\sigma(C)$-morphism. But indeed, by the projection 6.87 we have

$$\begin{aligned} & h = m \circ k \\ \Rightarrow \quad & \pi(h) = \pi(m \circ k) = \pi(m) \circ \pi(k) \\ \Rightarrow \quad & h = m \circ \pi(k) \end{aligned}$$

But since k is unique, we have $k = \pi(k)$, which is indeed a morphism in $Z_\sigma(C)$. Therefore, m is a kernel of f. The argument works for the cokernel, and is thus omitted. A corollary of this construction is that the (co)kernels are really the same as in C, so $Z_\sigma(C)$ is clearly exact since C is exact. ∎

Lemma 6.89 Let C be a premodular category and $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing. Recall from 6.88 that the categorical center of higher genera $Z_\sigma(C)$ is abelian. Moreover, it is semisimple. ◇

Proof. Let $(X, \gamma) \xrightarrow{f} (Y, \beta)$ be a monic morphism in $Z_\sigma(C)$. It suffices to show that f has a left inverse. Recall that f is also a C-morphism $X \xrightarrow{f} Y$. We contend that $X \xrightarrow{f} Y$ is monic in C. Indeed, assume

$$(W \xrightarrow{g} X \xrightarrow{f} Y) = (W \xrightarrow{h} X \xrightarrow{f} Y)$$

then

$$(I_\sigma W \xrightarrow{\overline{g}} (X, \gamma) \xrightarrow{f} (Y, \beta)) = (I_\sigma W \xrightarrow{\overline{h}} (X, \gamma) \xrightarrow{f} (Y, \beta))$$

by the construction of I_σ. Then $\overline{g} = \overline{h}$, and thus $g = h$.

Since C is semisimple, we get a left inverse $X \xleftarrow{p} Y$ for free. However, p lives in C, so we need to find another candidate that does the job in $Z_\sigma(C)$. This is again taken care by the projection 6.87 We contend that it is a left inverse of f in $Z_\sigma(C)$. Indeed, as $\pi_{\beta,\gamma}$ is a projection, $\pi_{\beta,\gamma}(f) = f$. So,

$$\pi_{\beta,\gamma}(p) \circ f = \pi_{\beta,\gamma}(p) \circ \pi_{\gamma,\beta}(f) = \pi_{\gamma,\gamma}(p \circ f) = \pi_{\gamma,\gamma}(1_X) = 1_X.$$

∎

Lemma 6.90 Let C be a premodular category and $\sigma \in \mathrm{Adm}_{2n}$ an admissible gluing. Recall from 6.88 and 6.89 that the categorical center of higher genera $Z_\sigma(C)$ is semisimple abelian. Moreover, it is finite. ◇

Proof. To prove that $Z_\sigma(C)$ is finite, we turn to the finiteness of C. Since $Z_\sigma(C)$ is a $\Bbbk$-linear abelian category by construction, from 6.38 and 6.39 we only have to show four things.

- Every object has finite length.
- Every hom space is a finite dimensional $\Bbbk$-vector space.
- $Z_\sigma(C)$ has enough projectives, i.e. every simple object of $Z_\sigma(C)$ has a projective cover.
- The set of isomorphism classes of simple objects is finite.

To prove that every object has finite length, pass a simple filtration of an object in $Z_\sigma(C)$ to one in C by the forgetful functor F_σ. Extend the latter to a simple filtration in C, which has finite length as C is assumed finite. Thus the former is also of finite length. To prove that every hom space is of finite dimensional, recall that the morphism spaces of $Z_\sigma(C)$ are defined as subspaces of those of C. Therefore the dimension of the former is bounded by the dimension of the later, which is finite the finiteness assumption of C.

To prove that $Z_\sigma(C)$ has enough projectives, it suffices to show that $Z_\sigma(C)$ is semisimple, as then each epic morphism admits a left inverse. But this fact has been shown in 6.89. To prove that there are only finitely many simple objects (up to isomorphism), we utilize the ambidextrous adjunction of F_σ and I_σ. Let (X, γ) be a simple object of $Z_\sigma(C)$. From

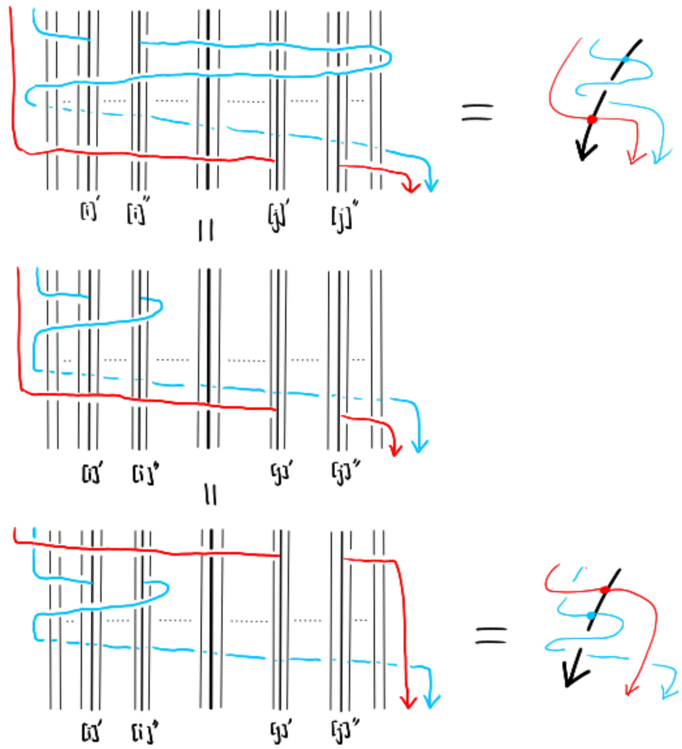
$$\mathrm{Hom}_C(X, Y) \simeq \mathrm{Hom}_{Z_\sigma(C)}((X, \gamma), I_\sigma(Y))$$

we know that (X, γ) appears as a summand in $I(Y)$ for any Y that appears as a summand in X. Since C has finitely many simple objects (up to isomorphism), it follows that there are finitely many such (X, γ). ∎

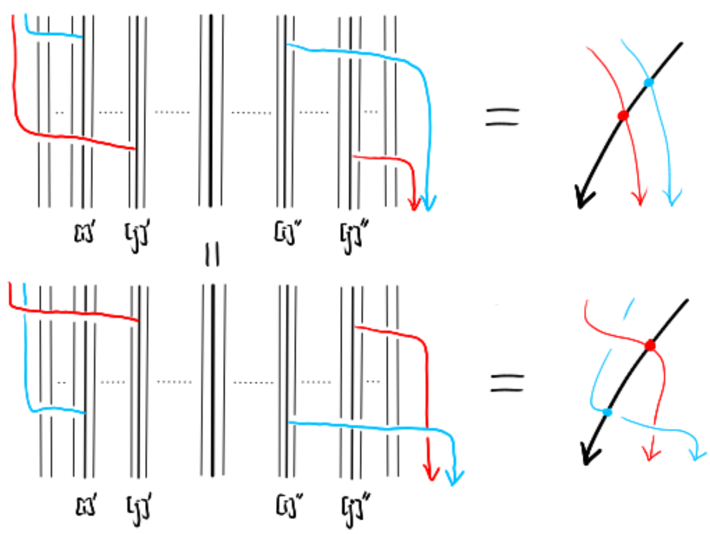
Lemma 6.91 The set of half-braidings defined in 4.53 satisfies the pairwise commutative relations 4.40. ◇

Proof. By definition, we have to prove that for each i, j, γ_i and γ_j satisfies the commutative relation posed in 4.40. In this pictorial proof, we use the color light-blue to indicate γ_i and the color red to indicate γ_j. Recall that with out loss of generality, there are three cases to consider

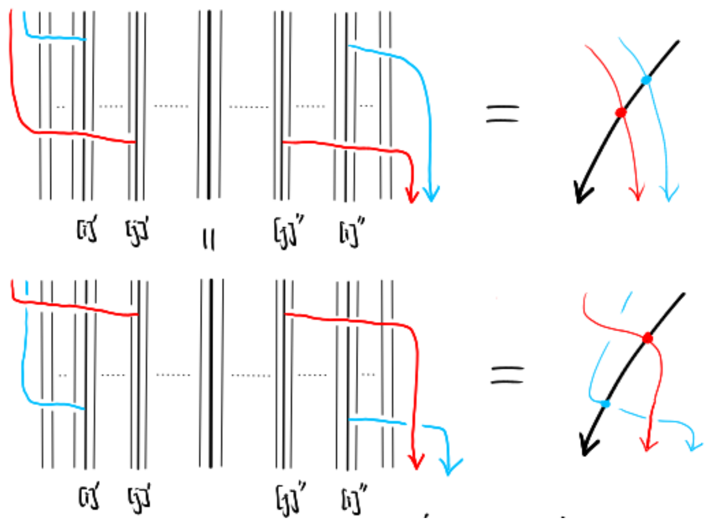
$$1. \quad [i]' < [i]'' < [j]' < [j]''$$



2.$[\mathrm{i}]' < [\mathrm{j}]' < [\mathrm{i}]'' < [\mathrm{j}]''$



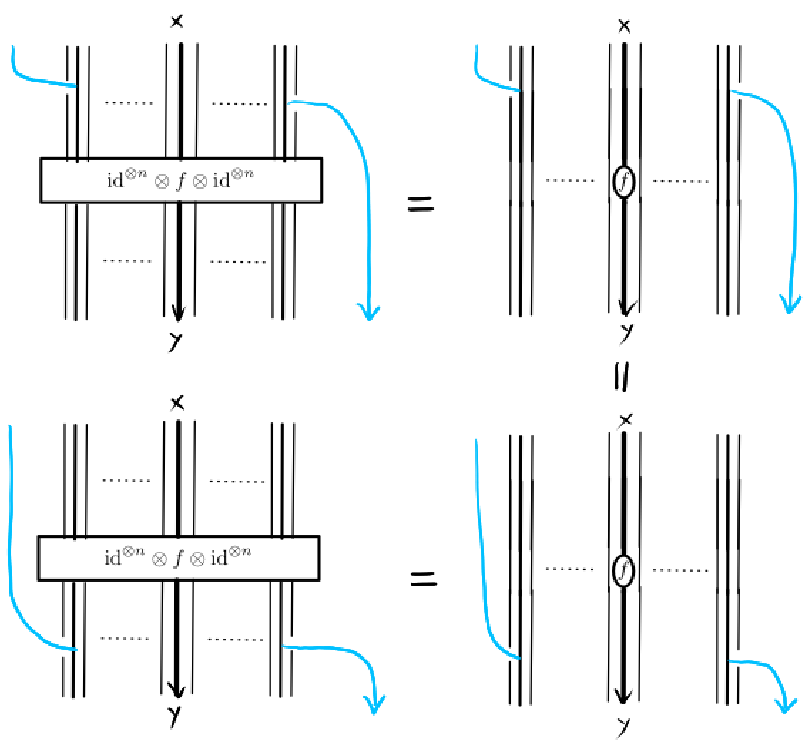
3.$[\mathrm{i}]' < [\mathrm{j}]' < [\mathrm{j}]'' < [\mathrm{i}]''$



■

Lemma 6.92 The induced morphisms in (4.54) is compatible with the sets of half-braidings γ and β given in (4.53). ◇

Proof. Clearly it holds from the following figure.



$\blacksquare$

Lemma 6.93 Let A be an additive category and B be an abelian category. Suppose

$$A \xrightarrow{\phi} B$$

is an additive functor. Then ϕ lifts additively to the Karoubi completion $\mathrm{Kar}(A)$ of A:

$$\mathrm{Kar}(A) \xrightarrow{\Phi} B.$$

$\diamond$

Proof. Given the assumptions, we must construct Φ explicitly. Recall that a typical object of $\mathrm{Kar}(A)$ is $\bar{X} := (X, p)$ of $X \in \mathrm{Obj}(A)$ and an idempotent $p \in \mathrm{End}_A(X)$. Define $\Phi(\bar{X})$ to be $\mathrm{im}_B(\phi(p))$. Recall also that a typical morphism

$$(X, p) \xrightarrow{f} (Y, q)$$

is an A-morphism $X \xrightarrow{f} Y$ such that $f = qfp$. Hence $\Phi(f)$ induces a B-morphism

$$\mathrm{im}(\phi(p)) \xrightarrow{\phi(f))} \mathrm{im}(\phi(q)).$$

Define it to be $\Phi(f)$. So defined map Φ is clearly an additive functor that extends ϕ. $\blacksquare$

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