Constructing dual Lagrangian fibrations of compact hyper-Kähler manifolds

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Abstract of the Dissertation

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Let $\pi : X \to \mathbb{P}^n$ be a Lagrangain fibration of a compact hyper-Kähler manifold X. We construct its dual Lagrangian fibration $\check{\pi} : \check{X} \to \mathbb{P}^n$ in this dissertation, under a single assumption that X is deformation equivalent to one of the currently known constructions of compact hyper-Kähler manifolds. This realizes the Strominger–Yau–Zaslow conjecture for all known types of hyper-Kähler manifolds X. Our main inputs are the use of (1) the neutral component of the relative automorphism scheme of π and its torsors, and (2) the automorphisms of π acting trivially on $H^2(X, \mathbb{Z})$. More specifically, we relate these two by realizing the latter as special global sections of the former. The dual Lagrangian fibration $\check{\pi}$ is then constructed by taking the quotient of π by part of these global sections, resulting a Lagrangian fibration of a new compact hyper-Kähler orbifold $\check{X}$.

To my parents.

사랑하는 부모님께.

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Six years of grad student life in Stony Brook was certainly more difficult than I imagined. But ironically, I can also say it was the happiest period in my life. I thank to the Stony Brook Math department for making this possible. This dissertation is dedicated to my parents.

Stony Brook, New York
May 2022

Introduction

Let Y be a compact Calabi–Yau manifold, a compact Kähler manifold with a trivial canonical bundle. The famous Beauville–Bogomolov decomposition theorem says that up to finite étale covering, Y must be decomposed into a product of several "irreducible" Calabi–Yau manifolds. A (compact) hyper-Kähler manifold is one of the irreducible factors arising in this decomposition theorem. It is a natural higher dimensional generalization of a K3 surface. A Lagrangian fibration of a hyper-Kähler manifold is a generalization of an elliptic fibration of a K3 surface. It is a certain holomorphic surjective map $\pi : X \to B$ from a hyper-Kähler manifold X. Lagrangian fibrations provide an extra flexibility to study hyper-Kähler manifolds, as many geometric properties of hyper-Kähler manifolds are reflected to their Lagrangian fibrations.

There are two main topics in this dissertation. The first and foremost topic is a construction of the *dual* of a Lagrangian fibration of a hyper-Kähler manifold. The motivation for this problem came from the Strominger–Yau–Zaslow (SYZ) conjecture [SYZ96], which predicts that a Lagrangian fibration of a Calabi–Yau manifold should admit an appropriate notion of a dual. More specifically, given a special Lagrangian fibration $f : Y \to B$ of a Calabi–Yau manifold Y, we expect there exists a nice Lagrangian fibration $\check{f} : \check{Y} \to B$ of another Calabi–Yau orbifold $\check{Y}$ that is generically fiberwise dual to the original f.[1]

Hyper-Kähler manifolds are Calabi–Yau, so the SYZ conjecture should apply to Lagrangian fibrations of hyper-Kähler manifolds. Given a Lagrangian fibration of a hyper-Kähler manifold $\pi : X \to B$, we believe there exists a new hyper-Kähler orbifold $\check{X}$ and a holomorphic Lagrangian fibration $\check{\pi} : \check{X} \to B$ that is generically fiberwise dual to π. Unfortunately, the conjecture neither tells us how to construct a new space $\check{X}$ nor how to make sense $\check{\pi}$ is a Lagrangian fibration.

Our main result is an explicit construction of one distinguished candidate $\check{\pi} : \check{X} \to B$

[1]Strictly speaking, we need to consider both the complex / symplectic structures of Y and $\check{Y}$ to correctly formulate the SYZ conjecture. However, for hyper-Kähler manifolds, it is expected that we can stick to the complex structure side by applying hyper-Kähler rotations. See [Huy04, §7.3–7.4] and [GTZ13].

that satisfies all the expected properties of the SYZ conjecture. Before stating our main result, we define a group that will play an important role throughout the article

$$\mathrm{Aut}^\circ(X/B) = \{f : X \to X : \ \pi \circ f = \pi, \ f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \text{ is the identity}\}. \quad (*)$$

Theorem A (Theorem 5.2.3 + 5.2.5)**.** *Let X be a hyper-Kähler manifold of dimension $2n$ and $\pi : X \to B$ its Lagrangian fibration to $B = \mathbb{P}^n$. Assume X is either of $K3^{[n]}$, Kum_n, OG10 or OG6-type. Then there exists a finite abelian group K acting on π such that*

$$\check{X} = X/K, \qquad \check{\pi} : \check{X} \to B$$

satisfies the following:

(i) *$\check{X}$ is a compact hyper-Kähler orbifold and $\check{\pi}$ is its Lagrangian fibration.*
(ii) *$\check{\pi}$ is generically fiberwise dual to π.*

Moreover, the group K is invariant under deformations of π and is a subgroup of the group $\mathrm{Aut}^\circ(X/B)$ *in* $(*)$.

The theorem is stated in a slightly weaker form, as we omitted some relevant definitions to minimize our discussion in the introduction. However, we believe the theorem is quite unexpected even in this weaker form. One may notice that this is quite similar to the duality between Hitchin fibrations of the moduli sapces of SL and PGL-Higgs bundles on a curve [HT03], which are Lagrangian fibrations of certain *non-compact* hyper-Kähler manifolds. We do not know why such a similar phenomenon happens. Nevertheless, we do understand that our technical main result (Theorem 5.2.1) directly implies Theorem A. Numerical evidence on topological invariants of hyper-Kähler manifolds suggests that this may hold for all (compact) hyper-Kähler manifolds.

Conjecture A (Conjecture 4.3.2)**.** *Theorem A holds for any hyper-Kähler manifold X.*

The main theorem will be proved in Chapter 5. There are two main ingredients needed for the proof of Theorem A. The first is a detailed study of the group $(*)$. This will be done in Chapter 4. The second is the notion of a relative automorphism scheme, a fine moduli space of automorphisms of π. This will be discussed in Chapter 3.

The second topic of this dissertation is about computations of the Looijenga–Lunts–Verbitsky (LLV) structure on the cohomology of hyper-Kähler manifolds. This second topic is joint work with Mark Green, Radu Laza and Colleen Robles [GKLR, KL20]. Looijenga–Lunts [LL97] and Verbitsky [Ver95] discovered that the cohomology of every hyper-Kähler manifold admits a certain $\mathfrak{g}$-module structure for a simple Lie algebra $\mathfrak{g}$. We call the Lie algebra $\mathfrak{g}$ the *LLV algebra* and the $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$ the *LLV structure* on the cohomology. The complete reduciblity of a $\mathfrak{g}$-module yields a cohomology decomposition

$$H^*(X, \mathbb{Q}) \cong \bigoplus_{\lambda} m_\lambda V_\lambda, \tag{LLV}$$

which turns out to be an extremely useful topological invariant of X. Our second main result is an explicit computation of the decomposition (LLV) for all known deformation types of hyper-Kähler manifolds.

Theorem B (Theorem 2.2.1–2.2.6)**.** *The cohomology decomposition* (LLV) *is explicitly computed for hyper-Kähler manifolds of $K3^{[n]}$, Kum_n, OG10 and OG6-types.*

We give several applications of this theorem. The first application is an explicit computation of the Hodge structure of all known types of hyper-Kähler manifolds, generalizing the previous results of [GS93], [MRS18], etc. in a completely different flavor. The second application is a partial solution to a conjecture of Nagai [Nag08] about the monodromy operators of one-parameter degenerations of hyper-Kähler manifolds. The third application is a possible numerical bound on the second cohomology of hyper-Kähler manifolds that can be considered as evidence for a boundedness of deformation types of hyper-Kähler manifolds. The second and third applications motivated us to question the following simple yet nontrivial inequality on the cohomology of hyper-Kähler manifolds:

Conjecture B (Conjecture 2.5.1)**.** *Every dominant weight $\lambda = (\lambda_0, \cdots, \lambda_r)$ appearing in* (LLV) *satisfies*

$$\lambda_0 + \cdots + \lambda_{r-1} + |\lambda_r| \le n.$$

We believe this conjecture should have a deeper meaning and can motivate further studies on the cohomology of hyper-Kähler manifolds. The conjecture is verified for all currently known deformation types (Theorem 2.5.2).

Let us now briefly summarize the contents of this article. The following is a dependency diagram between the chapters. The reader who is only interested in the construction of the dual Lagrangian fibration can safely skip Chapter 2.

$$\begin{tikzcd}
\text{Chapter 1} \arrow[r] \arrow[dr] & \text{Chapter 3} \arrow[r] & \text{Chapter 4} \arrow[r] & \text{Chapter 5} \\
\text{Appendix A} \arrow[r] & \text{Chapter 2} & &
\end{tikzcd}$$

Chapter 1 summarizes some background materials on hyper-Kähler manifolds and their Lagrangian fibrations. To start, the most basic and important topological properties of hyper-Kähler manifolds are reviewed. Local deformation behaviors of hyper-Kähler manifolds and their Lagrangian fibrations are also reviewed. The subtle problem on the base of a Lagrangian fibration is carefully documented. Finally, many of the known constructions of hyper-Kähler manifolds / Lagrangian fibrations are collected.

Chapter 2 starts with a crash course on the definition of the LLV structure. The main computational results (Theorem B) are stated in §2.2. The following section §2.3 presents similar computational results for the reduced LLV structures and Hodge structures. Finally, two applications of the LLV structures are briefly presented in the last two sections.

Chapter 3 discusses how the relative automorphism scheme can be used to study Lagrangian fibrations. In fact, we will use only the most manageable part of the relative automorphism scheme due to a technical difficulty for the non-projective case. This will be discussed in §3.1. Even this weak discussion already gives us some useful information, such as the notion of a polarization type and polarization scheme of a Lagrangian fibration. This will be discussed in §3.2.

Chapter 4 studies the group $\mathrm{Aut}^\circ(X/B)$ in $(*)$. The first section §4.1 proves it is deformation invariant. The third section §4.3 gets into our core question on relating the group $\mathrm{Aut}^\circ(X/B)$ with the polarization scheme in Chapter 3. Explicit computation of $\mathrm{Aut}^\circ(X/B)$ for all known types of hyper-Kähler manifolds is given in §4.4. The remaining section §4.2 is slightly parallel to our discussion but can be of independent interest; it is about the geometric origin of many H^2-trivial automorphisms.

Chapter 5 is devoted to the more precise statement of Theorem A, its proof and some related results. The first section §5.1 proposes a new definition of the dual torus fibration. The second section §5.2 then constructs a dual Lagrangian fibration as a compactification of a dual torus fibration.

Appendix A provides some representation theory backgrounds. It will collect the facts on irreducible representations of special orthogonal Lie algebras and their field of definitions. Restriction representations between two different Lie algebras are discussed as well. Nothing in this appendix should be new, but some of them were hard to locate in the algebraic geometry literature so we provide it as an appendix.

Chapter 1

Hyper-Kähler manifolds and their Lagrangian fibrations

Introduction

In this chapter, we summarize some basic facts on compact hyper-Kähler manifolds and their Lagrangian fibrations. Only minimal backgrounds necessary for our future discussions will be presented. Most of the materials in this chapter are not original, but still there will be a few new results filling the missing parts in the literature.

A compact hyper-Kähler manifold X is a higher dimensional generalization of a K3 surface. The intersection form on the middle cohomology of a K3 surface correspondingly generalizes to a quadratic form on the second cohomology of X. This is the Beauville–Bogomolov lattice, the most basic and important topological invariant of X. A Lagrangian fibration $\pi : X \to B$ of a compact hyper-Kähler manifold is a generalization an elliptic fibration of a K3 surface. They are of particular interest due to the following reasons.

(i) Any morphism $f : X \to Y$ (that is neither constant nor generically finite) factors through a Lagrangian fibration $X \xrightarrow{\pi} B \to Y$, the Stein factorization of f.

(ii) They provide an extra flexibility to study hyper-Kähler manifolds X, as many properties of X are reflected into π.

(iii) Largangian fibrations have rich geometry.

At the moment we are writing this article, there are only few known deformation types of hyper-Kähler manifolds. They are called the $\mathrm{K3}^{[n]}$, Kum_n, OG10 and OG6-types. We collect some of their known constructions and if possible, collect the known constructions of their Lagrangian fibrations.

1.1 Basic properties of hyper-Kähler manifolds

Definition 1.1.1. A *compact hyper-Kähler manifold*, or an *irreducible symplectic manifold*, is a compact Kähler manifold X with the following properties:

(i) $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$ for an everywhere nondegenerate global holomorphic 2-form σ.

(ii) X is simply connected.

Any hyper-Kähler manifold in this article will be assumed to be compact, but will not be assumed to be projective unless explicitly stated. We will typically drop the adjective "compact" to prevent our terminology becoming too long. The letter X in this article will always reserved for a hyper-Kähler manifold.

There is another commonly used definition of a hyper-Kähler manifold in the Riemannian geometry context. Although this second definition will not be extensively used in this article, we spell it out explicitly because it is equally fundamental to the study of hyper-Kähler manifolds.

Definition 1.1.2. A *compact Riemannian hyper-Kähler manifold* is a compact Riemannian manifold (M, g) of real dimension $4n$ with the following properties:

(i) Its holonomy group $\mathrm{Hol}(M, g) \subset \mathrm{O}(T_pM, g)$ is isomorphic to the compact symplectic group $\mathrm{Sp}(n) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$.

(ii) M is simply connected.

Definition 1.1.1 and 1.1.2 are describing a topologically same object. The following result makes this relation more precise [Joy03, Thm 5.11]. Write $X = (M, I)$ where M is an underlying C^∞ real manifold and I a complex structure on M.

Proposition 1.1.3. (i) *Let $X = (M, I)$ be a hyper-Kähler manifold. Then attached to any Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$, there exists a unique Ricci-flat metric g making (M, g) a Riemannian hyper-Kähler manifold.*

(ii) *Let (M, g) be a Riemannian hyper-Kähler manifold. Then there exists an S^2-family of complex Kähler structures I on (M, g) making each (M, I) a hyper-Kähler manifold.*

In other words, we have a one-to-one correspondence

$$\begin{aligned}
&\text{A hyper-Kähler manifold } X = (M, I) \text{ with the choice of a Kähler class } \omega \\
\Leftrightarrow\ &\text{A Riemannian hyper-Kähler manifold } (M, g) \\
&\qquad\qquad\qquad\qquad \text{with the choice of a complex (Kähler) structure } I
\end{aligned}$$

1.1.1 The Beauville–Bogomolov form and Fujiki relation

One of the characteristic properties of hyper-Kähler manifolds is the existence of a canonical quadratic form on its second cohohomology $H^2(X, \mathbb{Z})$. This important quadratic form first appeared in [Bea83b, §8] and [Fuj87, Thm 4.7].

Theorem-Definition 1.1.4. *Let X be a hyper-Kähler manifold. Then there exist*

(i) *a unique primitive nondegenerate symmetric bilinear form $\bar{q} : H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z}$; and*

(ii) *a unique constant $c_X \in \mathbb{Q}_{>0}$*

such that the following identity holds:

$$\int_X x^{2n} = c_X \cdot \frac{(2n)!}{2^n \cdot n!} \cdot \bar{q}(x,x)^n \qquad \textit{for} \quad x \in H^2(X, \mathbb{Z}). \tag{1.1.5}$$

The form $\bar{q}$ and the constant c_X are called the Beauville–Bogomolov(–Fujiki) form *and the* Fujiki constant*, respectively. The identity* (1.1.5) *is called the* Fujiki relation*.*

The Beauville–Bogomolov form $\bar{q}$ and the Fujiki constant c_X are the most important basic topological invariants of X. The expression $\bar{q}(x, x)$ will be often shorten to $\bar{q}(x)$. Note that even without the primitiveness condition on $\bar{q}$, the Beauville–Bogomolov form and Fujiki constant are uniquely defined up to scaling. Usually the Fujiki relation is stated as in (1.1.5), but one can formally generalize this to the following more useful relation. This identity can be found, e.g., in O'Grady's note [O'G13, §4.2]. This explains the role of the constant $\frac{(2n)!}{2^n \cdot n!}$.

Proposition 1.1.6. *There exists an identity*

$$\int_X x_1 \cdots x_{2n} = c_X \sum_{\sigma} \bar{q}(x_{\sigma(1)}, x_{\sigma(2)}) \cdots \bar{q}(x_{\sigma(2n-1)}, x_{\sigma(2n)}) \qquad \textit{for} \quad x_1, \cdots, x_{2n} \in H^2(X, \mathbb{Z}), \tag{1.1.7}$$

where $\sigma \in \mathfrak{S}_{2n}$ runs through all the $2n$-permutations but up to $2^n \cdot n!$ ambiguities inducing the same expression in the summation. We call (1.1.7) *again the* Fujiki relation*.*

Proof. This is a formal consequence of the polarization process. The map $P : H^2(X, \mathbb{Q}) \to \mathbb{Q}$ defined by $x \mapsto \int_X x^{2n}$ is a homogeneous polynomial of degree $2n$ in the sense that $P(tx) = t^{2n}P(x)$ and the "polarization" of P is a symmetric $2n$-multilinear map. The claim follows by the original Fujiki relation in Theorem-Definition 1.1.4 and the uniqueness of the polarization multilinear map. See, e.g., [Dol03, §1.2] for details on the polarization process. □

Given the importance of the Beauville–Bogomolov form and Fujiki constant, it is intriguing that even the following basic question in [Bea11] can be already hard to answer. We can answer the question positively for all currently known deformation types of hyper-Kähler manifolds, but to our knowledge it is open in general.

Conjecture 1.1.8. *Let X be a hyper-Kähler manifold. Then*

(i) *The Beauville–Bogomolov form $\bar{q}$ is even, i.e., $\bar{q}(x)$ is even for all $x \in H^2(X, \mathbb{Z})$.*

(ii) *The Fujiki constant c_X is a positive integer.*

We give a final remark that the Beauville–Bogomolov form and the Hodge–Riemann bilinear form are not too different. The following discussion can be found in [Bea83b, Thm 5], but has not been emphasized enough since. Fix any Kähler class $\omega \in H^2(X, \mathbb{R})$. Associated to it we may define the Hodge–Riemann bilinear form and the primitive decomposition of the cohomology

$$\begin{aligned} &\bar{q}_\omega : H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}, \qquad x \otimes y \mapsto \frac{\int_X xy\omega^{2n-2}}{\int_X \omega^{2n}}, \\ &H^2(X, \mathbb{R}) = H^2_{\mathrm{prim}}(X, \mathbb{R}) \oplus \omega H^0(X, \mathbb{R}). \end{aligned} \tag{1.1.9}$$

Note that the primitive decomposition is orthogonal with respect to $\bar{q}_\omega$.

Proposition 1.1.10. *Let $\omega \in H^2(X, \mathbb{R})$ be any Kähler class and consider the Hodge–Riemann bilinear form and primitive decomposition in* (1.1.9)*. Then*

(i) *The primitive decomposition is orthogonal with respect to the Beauville–Bogomolov form $\bar{q}$.*

(ii) *$\bar{q}$ is precisely $(2n-1)\bar{q}(\omega)$ times $\bar{q}_\omega$ on $H^2_{\mathrm{prim}}(X, \mathbb{R})$, and $\bar{q}(\omega)$ times $\bar{q}_\omega$ on $\omega H^0(X, \mathbb{R})$.*

That is, the Beauville–Bogomolov form $\bar{q}$ is essentially $\bar{q}_\omega$ on each components $H^2_{\mathrm{prim}}(X, \mathbb{R})$ and $\omega H^0(X, \mathbb{R})$, but with different scaling factors.

Proof. Let $x \in H^2_{\mathrm{prim}}(X, \mathbb{R})$ and $s \in \mathbb{R}$. Apply the following lemma to $x + s\omega$ and ω in place of x and y. We get the desired identity

$$\frac{\bar{q}(x + s\omega)}{\bar{q}(\omega)} = (2n-1)\bar{q}_\omega(x) + \bar{q}_\omega(s\omega). \qquad \square$$

Lemma 1.1.11. *There exists an identity*

$$\frac{\bar{q}(x)}{\bar{q}(y)} = (2n-1)\frac{\int_X x^2 y^{2n-2}}{\int_X y^{2n}} - (2n-2)\left(\frac{\int_X xy^{2n-1}}{\int_X y^{2n}}\right)^2 \qquad \text{for} \quad x, y \in H^2(X, \mathbb{Q}).$$

Proof. Transform all the integrations on the right hand side to the Beauville–Bogomolov forms, using the Fujiki relation (1.1.7). □

Remark 1.1.12. The Kähler property of the class ω is only used to construct a primitive decomposition of $H^2(X, \mathbb{R})$, i.e., it is used in the hard Lefscehtz property of ω. As we will see in Chapter 2, the hard Lefschetz property holds for any cohomology class $\omega \in H^2(X, \mathbb{Q})$ with $\bar{q}(\omega) \neq 0$, so Proposition 1.1.10 generalizes to any such ω.

The advantage of using the Beauville–Bogomolov form $\bar{q}$ instead of $\bar{q}_\omega$ is that (1) it does not depend on the choice of ω and is a topological invariant of X, and (2) it is defined over $\mathbb{Q}$ (or $\mathbb{Z}$). The Hodge–Riemann bilinear relation computes the signature of $\bar{q}_\omega$, and thus the signature of $\bar{q}$.

Corollary 1.1.13. *The signature of the Beauville–Bogomolov form $\bar{q}$ is* $(3, b_2(X) - 3)$.

1.1.2 Deformation of hyper-Kähler manifolds

A *family of hyper-Kähler manifolds* is a smooth proper morphism $\mathcal{X} \to S$ over a complex space (or a germ) S with hyper-Kähler manifold fibers. A *deformation of* X is a connected family of hyper-Kähler manifolds whose fiber at a distinguished point $0 \in S$ is isomorphic to X.

Definition 1.1.14. Let X and X' be hyper-Kähler manifolds. We call X and X' are *deformation equivalent* if there exists a finite sequence of hyper-Kähler manifolds $X = X_0, X_1, \cdots, X_k = X'$ such that each adjacent X_i and X_{i+1} can be realized by two fibers of a family of hyper-Kähler manifolds $\mathcal{X}_i \to \Delta$ over a complex open disc.

Given a family of hyper-Kähler manifolds $f : \mathcal{X} \to S$ over a *simply connected* base S, one can associate a holomorphic map, called the (second) period map

$$\Phi : S \to D, \qquad t \mapsto [H^{2,0}(X_t)],$$

where the target D is the period domain of weight 2 Hodge structures with Hodge numbers $(1, b_2(X) - 2, 1)$. More concretely, we can define[1]

$$D = \{[\mathbb{C}\sigma] \in \mathbb{P}H^2(X, \mathbb{C})^\vee : \bar{q}(\sigma) = 0,\ \bar{q}(\sigma + \bar{\sigma}) > 0\} \quad \subset \mathbb{P}^{b_2(X)-1}.$$

The *universal deformation of* X is a family of hyper-Kähler manifolds $\mathcal{X} \to \mathrm{Def}(X)$ over a germ $\mathrm{Def}(X)$ such that: for any family of hyper-Kähler manifolds $\mathcal{X}_S \to S$ over a germ of a complex space, there exists a unique holomorphic map $S \to \mathrm{Def}(X)$ realizing $\mathcal{X}_S$ as a

[1]Throughout this article, the projectivization $\mathbb{P}E$ of a vector space (vector bundle) E is the set of 1-dimensional quotients of E. Thus, $\mathbb{P}E^\vee$ is the set of 1-dimensional subspaces of E.

pullback of $\mathcal{X}$. Hyper-Kähler manifolds enjoy excellent deformation behaviors. The following is the Tian–Todorov unobstructedness theorem together with [Bea83b].

Theorem 1.1.15 (Local Torelli theorem)**.** *Let X be a hyper-Kähler manifold. Then*

(i) *X admits a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ over a smooth germ $\mathrm{Def}(X)$ of dimension $b_2(X) - 2$.*

(ii) *The period map $\Phi : \mathrm{Def}(X) \to D$ associated to the universal deformation is a local isomorphism.*

In fact, hyper-Kähler manifolds satisfy a much stronger theorem called the global Torelli theorem, but we will not state this as it is not necessary for our future discussions. The interested reader may consult [Ver13], [Huy12], [Mar11] and [BL18].

Besides the universal deformation, there is one more distinguished family of hyper-Kähler manifolds of prime interest. Recall from Definition 1.1.2 a notion of a Riemannian hyper-Kähler manifold (M, g).

Definition 1.1.16. The *twistor family* of a Riemannian hyper-Kähler manifold (M, g) is a smooth proper morphism $\mathcal{X} \to \mathbb{P}^1$ of complex manifolds parametrizing the S^2-family of hyper-Kähler manifolds (M, I) in Proposition 1.1.3.

If one starts from our usual hyper-Kähler manifold X in Definition 1.1.1, for each Kähler class $\omega \in H^{1,1}(X, \mathbb{R})$ there exists an associated twistor family $\mathcal{X}_\omega \to \mathbb{P}^1$ containing X as a special fiber. Note that the twistor family is a global family, meaning the base $\mathbb{P}^1$ is compact.

1.2 Lagrangian fibrations

Definition 1.2.1. Let X be a hyper-Kähler manifold of dimension $2n$.

(i) A *Lagrangian fibration of X to a normal base* is a holomorphic surjective map $\pi : X \to B$ with connected fibers to a normal complex space B with $0 < \dim B < 2n$.[2]

(ii) A *Lagrangian fibration of X (to a smooth base)* is a holomorphic surjective map $\pi : X \to B$ with connected fibers to a complex manifold B with $0 < \dim B < 2n$.

A *Lagrangian fibartion* in this article will always mean a Lagrangian fibration to a smooth base. We will often call $\pi : X \to B$ a *Lagrangian fibered hyper-Kähler manifold* for short. By the surprising result of [Hwa08] and [GL14], the base B of any Lagrangian fibration will automatically be a projective space.

[2] One may simplify the condition "π is surjective, π has connected fibers and B is normal" into $\pi_* \mathcal{O}_X = \mathcal{O}_B$. See [Laz04, Ex 2.1.15].

Theorem 1.2.2 (Hwang, Greb–Lehn)**.** *Let X be a hyper-Kähler manifold of dimension $2n$ and $\pi : X \to B$ its Lagrangian fibration. Then B is isomorphic to $\mathbb{P}^n$.*

Proposition 1.2.3. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then π is flat.*

Proof. We will later see in Theorem 1.2.13 that π is equidimensional. The claim follows by the miracle flatness theorem for holomorphic maps between complex manifolds (e.g., [Fis76, §3.20]). In fact, π is flat if and only if B is smooth. See [HMa, Rmk 1.18] and [Mat89, Thm 23.7]. □

Some of our discussions in this article will depend on the fact that B is a projective space. For example, our notion of a family of Lagrangian fibrations will assume the base B is always $\mathbb{P}^n$, and our proof of the deformation invariance of $\mathrm{Aut}^\circ(X/B)$ in §4.1 will use $\mathrm{Aut}(B) = \mathrm{PGL}_{n+1}(\mathbb{C})$. In the end, we expect the base B will be always $\mathbb{P}^n$, even for Lagrangian fibrations to normal bases. This is the famous Conjecture 1.2.12, a generalization of Theorem 1.2.2. We will later discuss the current status of the conjecture in §1.2.3.

The *discriminant locus* $\Delta \subset B$ is the set of all points over which π has singular fibers. It is a Zariski closed subset of pure codimension 1 in B by [HO09, Prop 3.1]. We will extensively use its complement $B_0 = B \setminus \Delta$ and the restriction of the Lagrangian fibration to this smaller base

$$\pi : X_0 = \pi^{-1}(B_0) \to B_0,$$

called the associated *torus fibration* of π. Indeed, it is a smooth proper family of complex tori (abelian varieties) by Theorem 1.2.13.

Throughout the article, we denote by H the pullback of an ample line bundle and h its cohomology class

$$H = \pi^*\mathcal{O}_B(1) : \text{line bundle on } X, \qquad h = c_1(H) \quad \in \mathrm{NS}(X). \tag{1.2.4}$$

The line bundle H and its cohomology class h will play an important role in our study of Lagrangian fibered hyper-Kähler manifolds. For the future use, we define the *divisibility* of h by a positive integer

$$\mathrm{div}(h) = \gcd\{\bar{q}(h, x) : x \in H^2(X, \mathbb{Z})\}. \tag{1.2.5}$$

Lemma 1.2.6. *The cohomology class $h \in H^2(X, \mathbb{Z})$ in (1.2.4) satisfies $\bar{q}(h) = 0$.*

Proof. Use the Fujiki relation $\int_X h^{2n} = (\mathrm{const.})\bar{q}(h)^n$ and $h^{2n} = 0$ as h is pulled back from B, which has dimension only n. □

1.2.1 Rational sections and multisections

There are four different notion of sections we need to consider for Lagrangian fibered hyper-Kähler manifolds: (1) sections, (2) multisections, (3) rational sections, and (4) rational multisections. One always needs to keep in mind that a Lagrangian fibration may or may not admit these. The questions about existence / count of the various notion of sections can be quite technical. For example, the set of rational sections of π is called the Mordell–Weil group and has been an independent topic of interest. See [Huy16, §11.3] for the case of elliptic K3 surfaces, and [Ogu09b] or [Sac20, §5] for higher dimensional Lagrangian fibrations.

A precise relation between rational sections and sections (resp. rational multisections and multisections) is unclear from the current literature. Our future discussions will only need the notion of *rational* (multi)sections, so we provide the following two basic results on them.

Proposition 1.2.7. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then the following are equivalent.*

(i) *X is projective.*

(ii) *There exists $x \in \mathrm{NS}(X)$ with $\bar{q}(x, h) \neq 0$.*

(iii) *π admits at least one rational multisection.*

Proof. Set $H = \pi^*\mathcal{O}_B(1)$ and $h = c_1(H)$ as in (1.2.4). If X is projective then any ample class $x \in \mathrm{NS}(X)$ and a smooth fiber F satisfies $\int_F (x_{|F})^n > 0$. Use the Fujiki relation $\int_F (x_{|F})^n = \int_X x^n h^n = (\text{const.})\bar{q}(x, h)^n$ to conclude $\bar{q}(x, h) \neq 0$ (in fact, one can further prove it is positive). Conversely, assume there exists $x \in \mathrm{NS}(X)$ with $\bar{q}(x, h) \neq 0$. The quadratic subspace $\mathbb{Q}\{x, h\} \subset \mathrm{NS}(X)_{\mathbb{Q}}$ is the hyperbolic plane, so it contains an element y with $\bar{q}(y) > 0$. The projectiveness of X follows by Huybrechts's projectiveness criterion [Huy99, Huy03].

Again assume X is projective. Then $\pi : X \to B$ becomes an algebraic morphism. Over any smooth point $b \in B$ of π, there exists an étale local section of π [Sta, Tag 054L]. Its image in X is the desired rational multisection of π. Conversely if π has a rational multisection,[3] then Campana–Oguiso's result in [Saw09, Lemma 2] claims X is projective. □

The undefined locus of any rational section of π is of codimension ≥ 2 in B. However, it is a priori unclear that the undefined locus is contained in the discriminant locus Δ.

Proposition 1.2.8. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then any rational section of π is necessarily defined over the smooth locus B_0 of π.*

[3]Here rational multisection means a closed analytic subvariety $S \subset X$ such that π induces a generically finite morphism $S \to B$.

Proof. The hyper-Kähler manifold X is projective and the discussion becomes algebraic, thanks to Proposition 1.2.7. Restrict the Lagrangian fibration to a torus fibration $\pi : X_0 = \pi^{-1}(B_0) \to B_0$. Assume $s : B_0 \dashrightarrow X_0$ is a rational section of π undefined at least one point in B_0 and set $B_0' \subset X_0$ the closure of the image of s. The morphism π restricted to B_0' defines a map $f : B_0' \to B_0$, a projective birational morphism to a smooth base but not an isomorphism.

By the classical Abhyankar lemma [Kol96, Thm VI.1.2], the exceptional locus of f is of pure codimension 1 and ruled.[4] We need to use its variant for our purpose, e.g., [Deb01, Prop 1.43]: there exists a rational curve on B_0' that is contracted by f to a point b. This means the fiber $F = \pi^{-1}(b)$, an abelian variety, contains a rational curve. Contradiction. □

Remark 1.2.9. The same proof applies and yields the following more general result:

Let $V \to B_0$ be a morphism and assume V is smooth over $\mathbb{C}$. Then any rational map $V \dashrightarrow X_0$ over B_0 is defined everywhere. For its proof, simply notice that the rational map defines a rational section $V \dashrightarrow X_V$ over V where $X_V = X_0 \times_{B_0} V$. This is a generalization of the result [BLR90, Cor 8.4.6] to any torsor of an abelian scheme.

One more corollary: any birational automorphism of $\pi : X_0 \to B_0$ is an automorphism.

1.2.2 Deformation of Lagrangian fibrations

The notion of a family of Lagrangian fibered hyper-Kähler manifolds needs some care. Since we are assuming the base of the Lagrangian fibration is always $\mathbb{P}^n$, we can make its definition quite intuitive.

Definition 1.2.10. A *family of Lagrangian fibered hyper-Kähler manifolds* is a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p"'] \arrow[dr, "\pi"] & \\ S & \mathcal{B} \arrow[l, "q"] \end{tikzcd}$$

with the following conditions.

(i) $p : \mathcal{X} \to S$ is a smooth proper family of hyper-Kähler manifolds of relative dimension $2n$ over a complex space S.

(ii) $q : \mathcal{B} \to S$ is the projectivization of a rank $n+1$ holomorphic vector bundle on S.

(iii) For all $t \in S$, the fiber $\pi : X_t \to B_t$ is a Lagrangian fibered hyper-Kähler manifold. In other words, $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{B}}$.

[4] Abhyankar's lemma needs a smooth base B_0.

Note that the axiom (ii) claims more than simply saying $\mathcal{B} \to S$ is a $\mathbb{P}^n$-bundle. The obstruction for a $\mathbb{P}^n$-bundle to be the projectivization of a vector bundle lies in the analytic Brauer group $H^2(S, \mathcal{O}_S^*)$. Thus, if $H^2(S, \mathcal{O}_S^*) \neq 0$ then (ii) may be strictly stronger than requiring $\mathcal{B} \to S$ is a $\mathbb{P}^n$-bundle.

If we have a family of hyper-Kähler manifolds as above, the pullback $\mathcal{H} = \pi^* \mathcal{O}_{\mathcal{B}/S}(1)$ can be considered as a flat family of line bundles H_t on X_t in (1.2.4). Therefore, Definition 1.2.10 in particular induces a family of pairs (X, H). Note also that there exists an isomorphism $\mathcal{B} \cong \mathbb{P}_S(p_* \mathcal{H})$. Conversely, one may start from a single Lagrangian fibered hyper-Kähler manifold $\pi : X \to B$ with $H = \pi^* \mathcal{O}_B(1)$. Consider a family of pairs (X, H), i.e., a family of hyper-Kähler manifolds $p : \mathcal{X} \to S$ with a line bundle $\mathcal{H}$ on $\mathcal{X}$ (flat over S). Does it construct an associated family of Lagrangian fibered hyper-Kähler manifolds $\mathcal{X} \to \mathcal{B} \to S$ in the sense of Definition 1.2.10? This question turns out to be equivalent to the following three conditions, which are quite delicate and not always satisfied:

(i) The direct image sheaf $p_* \mathcal{H}$ on S is locally free of rank $n + 1$.
(ii) $\mathcal{H}$ is p-globally generated, i.e., the adjunction map $p^* p_* \mathcal{H} \to \mathcal{H}$ is surjective.
(iii) The induced map $\pi : \mathcal{X} \to \mathbb{P}(p_* \mathcal{H})$ over S is surjective and has connected fibers.

In other words, Definition 1.2.10 is stronger than the notion of a family of pairs (X, H) by precisely the three conditions (i)–(iii). Matsushita in [Mat16] proved all the conditions (i)–(iii) are automatically satisfied when S is a germ of a complex space ([Mat17] may be further needed to handle (iii)). This is Theorem 1.2.11, which is a slight reformulation of Matushita's original result. We will revisit the three conditions above with more technical details in the next subsection.

Two Lagrangian fibrations $\pi : X \to B$ and $\pi' : X' \to B'$ of hyper-Kähler manifolds are called *deformation equivalent* if it connected with a finite sequence of families over complex discs as in Definition 1.1.14. The *universal deformation* of π is a family of Lagrangian fibered hyper-Kähler manifolds $\mathcal{X} \to \mathcal{B} \to \mathrm{Def}(X, H)$ over a germ such that: for any family of Lagrangian fibered hyper-Kähler manifolds $\mathcal{X}_S \to \mathcal{B}_S \to S$ over a germ S, there exists a unique holomorphic map $S \to \mathrm{Def}(X, H)$ realizing $\mathcal{X}_S \to \mathcal{B}_S$ as a pullback of $\mathcal{X} \to \mathcal{B}$. The existence of a universal deformation for a Lagrangian fibered hyper-Kähler manifold is established by Matsushita.

Theorem 1.2.11 (Matsushita)**.** *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Set $H = \pi^* \mathcal{O}_B(1)$ a line bundle on X and $h \in H^2(X, \mathbb{Z})$ its associated cohomology class.*

(i) *There exists a universal deformation $\mathcal{X} \to \mathcal{B} \to \mathrm{Def}(X, H)$ of $\pi : X \to B$ over a* *smooth germ* $\mathrm{Def}(X, H)$ *of dimension* $b_2(X) - 3$.

(ii) *The period map* $\Phi : \mathrm{Def}(X, H) \to D$ *maps* $\mathrm{Def}(X, H)$ *locally isomorphically into a hyperplane* $D \cap h^{\perp}$, *where* $h^{\perp} = \{[\mathbb{C}\sigma] \in \mathbb{P}H^2(X, \mathbb{C}) : \bar{q}(\sigma, h) = 0\}$.

As one can guess from the notation, the germ $\mathrm{Def}(X, H)$ coincides with the universal deformation space of pairs (X, H). The existence of the universal deformation of pairs is proved in [Bea83b, Cor 1].

1.2.3 Lagrangian fibration to a normal base

This subsection is devoted to summarizing the known facts about Lagrangian fibrations to normal bases. The current status of this topic is quite delicate, so we need an extra care. The story begins from a series of foundational papers [Mat99, Mat01, Mat00] of Matsushita, where he observed any Lagrangian fibered hyper-Kähler manifold to a normal base has to satisfy exceptionally rigid properties as if the base B is a projective space. Ultimately, we expect the following conjecture to hold.

Conjecture 1.2.12. *Let* X *be a hyper-Kähler manifold of dimension* $2n$ *and* $\pi : X \to B$ *its Lagrangian fibration to a normal base. Then* B *is isomorphic to* $\mathbb{P}^n$.

Theorem 1.2.2 would thus be a special case of this conjecture when B is smooth. At this point, the conjecture is not proved in its full generality. The following theorem collects various partial progress made to this direction.

Theorem 1.2.13. *Let* X *be a hyper-Kähler manifold of dimension* $2n$ *and* $\pi : X \to B$ *its Lagrangian fibration to a normal base. Then*

(i) B *is a simply connected,* $\mathbb{Q}$*-factorial, log terminal and projective Fano variety of dimension* n.

(ii) $\mathrm{Pic}(B) = H^2(B, \mathbb{Z}) \cong \mathbb{Z}$.

(iii) $IH^k(B, \mathbb{Q}) = H^k(B, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$.

(iv) *Every smooth fiber of* π *is an abelian variety of dimension* n.

(v) *Every fiber of* π *is projective of pure dimension* n *and all its irreducible components are Lagrangian subvarieties of* X.

Remark 1.2.14. The statements and proofs of Theorem 1.2.13 are scattered in the literature. The primitive form of the theorem is first proved by Matsushita under an additional assumption that X and B are projective [Mat99, Mat01, Mat00]. Matsushita later in [Mat03] dropped the projectiveness assumptions on X and B, but still assumed B to be Kähler. In fact, he showed the Kähler condition on B implies the projectiveness of B. Amerik–Campana [AC13, Thm 1, footnote 1] again dropped the Kähler condition on B by proving B is always Kähler (hence always projective). See also [Mat16, Rmk 1.4].

The smooth fibers are well-known to be complex tori by the holomorphic Arnold–Liouville theorem. Any fiber, smooth or not, is projective by [Leh16, Thm 1.1] or more generally [Cam21]. Simple connectedness of B and the statement (ii) are proved in [HMa, Prop 1.6, Rmk 1.15]. Finally, the statement (iii) is a combination of [SY22, Thm 0.4] together with [HMa, Prop 1.10].

Theorem 1.2.13 gives strong evidence to Conjecture 1.2.12. In a different direction, the conjecture is verified in many special cases. As mentioned, Theorem 1.2.2 can be interpreted as a special case of the conjecture when B is smooth. Let us collect some other special cases of the conjecture. The references for the following two theorems are [CMSB02, Thm 7.2], [Ou19] and [HX20].

Theorem 1.2.15 (Cho–Miyaoka–Shepherd-Barron)**.** *If π admits at least one section then Conjecture 1.2.12 holds.*

Theorem 1.2.16 (Ou, Huybrechts–Xu)**.** *If $\dim X = 4$ then Conjecture 1.2.12 holds.*

The final special case for the conjecture is when X is one of the currently known deformation types of hyper-Kähler manifolds; these are $\mathrm{K3}^{[n]}$, Kum_n, OG10 and OG6-type hyper-Kähler manifolds that will be defined in §1.3. The theorem should be well-known to experts in the field; for example, it was stated in [Mat15, Thm 1.4] for $\mathrm{K3}^{[n]}$ and Kum_n-type hyper-Kähler manifolds (but without a proof). We were not able to locate its written proof, so the rest of this subsection will be devoted to collecting and documenting the proof of the following theorem.

Theorem 1.2.17. *If X is either of $K3^{[n]}$, Kum_n, OG10 or OG6-type then Conjecture 1.2.12 holds.*

The approach to the theorem requires a similar but separate conjecture about Lagrangian fibrations. We start with the following definition, which can be found in, e.g., [Mar14, Def 1.2].

Definition 1.2.18. Let X be a hyper-Kähler manifold of dimension $2n$ and H a line bundle on it. We say *H defines a Lagrangian fibration of X* if the following conditions are satisfied.

(i) $h^0(X,H) = n+1$.

(ii) H is globally generated.

(iii) The morphism $\pi : X \to |H|^\vee \cong \mathbb{P}^n$ associated to the complete linear system of H is surjective and has connected fibers. That is, it is a Lagrangian fibration of X.

In such a case, H is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^n}(1)$. Conversely, if we start from a Lagrangian fibration $\pi : X \to \mathbb{P}^n$ then $H = \pi^*\mathcal{O}_{\mathbb{P}^n}(1)$ defines a Lagrangian fibration of X.

If H defines a Lagrangian fibration then it is clearly nef and isotropic (Lemma 1.2.6). It is also expected to be primitive (see [KV19] or Theorem 1.2.23 below). The following conjecture predicts the converse. It is often referred as the "SYZ conjecture for hyper-Kähler manifolds" and has some variants.

Conjecture 1.2.19. *Let X be a hyper-Kähler manifold and H a line bundle on it. If H is primitive, nef and isotropic then H defines a Lagrangian fibration of X.*

We prove the desired Theorem 1.2.17 in two steps. First we prove that Conjecture 1.2.19 is stronger than Conjecture 1.2.12. Next we prove Conjecture 1.2.19 for all known deformation types of hyper-Kähler manifolds.

Lemma 1.2.20. *Conjecture 1.2.19 implies Conjecture 1.2.12.*

Proof. Let $\pi : X \to B$ be a Lagrangian fibration of a hyper-Kähler manifold to a normal base B. By Theorem 1.2.13, $\mathrm{Pic}(B)$ is generated by an ample line bundle $\mathcal{O}_B(1)$. Consider its pullback $H = \pi^*\mathcal{O}_B(1)$. It is nef and isotropic (note that Lemma 1.2.6 does not use the smoothness of B). It may not be a primitive line bundle, so let us assume $H = dH'$ for d a positive integer and H' a primitive nef isotropic line bundle on X.

If Conjecture 1.2.19 holds, then H' defines a Lagrangian fibration $\pi' : X \to |H'|^\vee \cong \mathbb{P}^n$. In particular, H' is globally generated and so is $H = dH'$. This implies $\mathcal{O}_B(1)$ is a globally generated line bundle on B. The complete linear system of H and $\mathcal{O}_B(1)$ are identical due to the identity $\pi_*H = \pi_*\pi^*\mathcal{O}_B(1) = \mathcal{O}_B(1)$. Hence we have a sequence of morphisms $X \xrightarrow{\pi} B \xrightarrow{\phi} |H|^\vee$. On the other hand, H' has Iitaka dimension $\kappa(X,H') = n$ and π' has connected fibers, so π' becomes an Iitaka fibration (see [Laz04, Thm 2.1.27]). We correspondingly have a Stein factorization $X \to |H'|^\vee \hookrightarrow |H|^\vee$ where the former map is the Lagrangian fibration π' and the latter map is the d-th Veronese embedding. The result is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "\pi'"] & B \arrow[d, "\phi"] \\ |H'|^\vee \arrow[r, hook, "\nu_d"] & |H|^\vee \end{tikzcd} .$$

We claim the morphism ϕ is finite. If so, then the uniqueness of the Stein factorization of $X \to |H|^\vee$ implies $\pi = \pi'$, $\phi = \nu_d$ and $B = |H'|^\vee \cong \mathbb{P}^n$, proving our desired lemma. Note that B is proper, so we only need to prove ϕ is quasi-finite, i.e., it has 0-dimensional fibers.[5] Assume on the contrary that ϕ has at least one fiber of dimension ≥ 1. Since π was equidimensional of relative dimension n by Theorem 1.2.13, this means $\phi \circ \pi$ has at least one fiber of dimension $\geq n+1$. But π' was a Lagrangian fibration so every fiber of $\nu_d \circ \pi'$ is either an empty set or of dimension n. Contradiction. □

We next prove Conjecture 1.2.19 for all known deformation types of hyper-Kähler manifolds. The following lemma, which is essentially proved in [Mat17], claims that Conjecture 1.2.19 is invariant under deformations of a pair (X, H).

Lemma 1.2.21 (Matsushita)**.** *Let (X, H) and (X', H') be hyper-Kähler manifolds with primitive nef isotropic line bundles on them. If (X, H) is deformation equivalent to (X', H'), then H defines a Lagrangian fibration of X if and only H' defines a Lagrangian fibration of X'.*

Proof. This result is essentially [Mat17, Thm 1.2]. The lemma in this form is stated in [Mar14, Rmk 1.8] but without a proof. It is proved in the first paragraph of Theorem 7.2 in [MR21] but in a different context of *rational* Lagrangian fibrations (a variant of Conjecture 1.2.19). Starting from Mongardi–Rapagnetta's version we can further argue as follows.

Assume H defines a Lagrangian fibration of X. By [MR21, Thm 7.2], we can say H' defines a rational Lagrangian fibration of X', i.e., there exists a composition

$$X' \overset{f}{\dashrightarrow} X'' \xrightarrow{\pi''} \mathbb{P}^n ,$$

where f is a birational map to another hyper-Kähler manifold X'' and $H'' = f_* H'$ is a line bundle defining a Lagrangian fibration π'' of X''. Any birational map between hyper-Kähler manifolds is isomorphic in codimension 1, so by Serre's condition S2 (Hartog's theorem) we obtain $h^0(X', kH') = h^0(X'', kH'') = \binom{n+k}{k}$ for all $k \geq 0$. This implies H' defines a Lagrangian fibration of X' by [Mat17, Lem 3.1].

The reader should be aware of a subtlety on the assumption of the lemma; the primitiveness and isotropic property of a line bundle is invariant under deformations of (X, H). However, the nefness of a line bundle is not deformation invariant, so one must assume both H and H' are nef. □

Theorem 1.2.22 (Markman, Matsushita, Mongardi–Rapagnetta, Mongardi–Onorati)**.** *If X is either of $K3^{[n]}$, Kum_n, OG10 or OG6-type then Conjecture 1.2.19 holds.*

[5] A morphism is finite if and only if it is proper and quasi-finite.

Proof. The theorem is proved for K3$^{[n]}$-type in [Mar14, Thm 1.3, Rmk 1.8]. The same proof can be applied to Kum$_n$-types; any Lagrangian fibration of a Kum$_n$-type hyper-Kähler manifold can be deformed to the construction in Example 1.3.14 by [Wie18]. One then uses Lemma 1.2.21 to conclude. The theorem for OG10 and OG6-type hyper-Kähler manifolds is [MO22, Thm 2.2] and [MR21, Thm 7.2], respectively. Notice their results are about rational Lagrangian fibrations, but we can again use [Mat17, Lem 3.1] as in the proof of Lemma 1.2.21 to conclude. □

This completes the proof of Theorem 1.2.17. Incidentally during our argument, we have also proved the following theorem. Let us explicitly document it.

Theorem 1.2.23. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. If X is either of $K3^{[n]}$, Kum_n, OG10 or OG6-type then the line bundle $H = \pi^*\mathcal{O}_B(1)$ is primitive.* □

1.3 Some known examples of hyper-Kähler manifolds and Lagrangian fibrations

At this point, there are essentially four types of known construction of hyper-Kähler manifolds. The first two constructions exist in every even dimension $2n$ and called the K3$^{[n]}$-type and Kum$_n$-type. The latter two constructions only exist in dimension 10 and 6, respectively, and called the OG10-type and OG6-type. The goal of this section is to collect some known constructions of hyper-Kähler manifolds of such deformation types.

The most important topological invariants of such deformation types are already computed. Before we get into their constructions/definitions, we collect their topological invariants for reader's convenience. The following is obtained in [Bea83b], [Rap07] and [Rap08].

Theorem 1.3.1 (Beauville, Rapagnetta)**.** (i) *The second Betti numbers of the currently known types of hyper-Kähler manifolds are*

$$b_2(X) = \begin{cases} 23 & \text{if } X \text{ is of } K3^{[n]}\text{-type,} \\ 24 & \text{if } X \text{ is of OG10-type,} \\ 7 & \text{if } X \text{ is of } Kum_n\text{-type,} \\ 8 & \text{if } X \text{ is of OG6-type.} \end{cases}$$

(ii) *The Beuaville–Bogomolov forms of the currently known types of hyper-Kähler manifolds* *are*

$$\big(H^2(X,\mathbb{Z}),\bar{q}\big) \cong \begin{cases} U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus\langle -2(n-1)\rangle & \textit{if } X \textit{ is of } K3^{[n]}\textit{-type},\\ U^{\oplus 3}\oplus E_8(-1)^{\oplus 2}\oplus A_2(-1) & \textit{if } X \textit{ is of OG10-type},\\ U^{\oplus 3}\oplus\langle -2(n+1)\rangle & \textit{if } X \textit{ is of } Kum_n\textit{-type},\\ U^{\oplus 3}\oplus\langle -2\rangle^{\oplus 2} & \textit{if } X \textit{ is of OG6-type.}\end{cases}$$

(iii) *The Fujiki constants of the currently known types of hyper-Kähler manifolds are*

$$c_X = \begin{cases} 1 & \textit{if } X \textit{ is of } K3^{[n]} \textit{ or OG10-type},\\ n+1 & \textit{if } X \textit{ is of } Kum_n\textit{-type},\\ 4 & \textit{if } X \textit{ is of OG6-type.}\end{cases}$$

1.3.1 Some known constructions of hyper-Kähler manifolds

This subsection collects some of the known constructions of hyper-Kähler manifolds. Most examples need highly nontrivial justifications and developed by several people. We will make no attempt for their proofs but provide references. If there are multiple references, we will give only one. Many of the examples below are also collected (with proofs) in [HL10].

Example 1.3.2 (Hilbert scheme of points on a K3 surface [Bea83b])**.**

Let S be a K3 surface, not necessarily projective. Consider the Hilbert scheme (or Douady space) $S^{[n]}$ of n points on S. It is a hyper-Kähler manifold of dimension $2n$. Any hyper-Kähler manifold deformation equivalent to $S^{[n]}$ will be called a *$K3^{[n]}$-type* hyper-Kähler manifold.■

Example 1.3.3 (Moduli of sheaves on a K3 surface [Muk84])**.**

Let S be a projective K3 surface and

$$v=(r,l,s)\quad\in H^*(S,\mathbb{Z}),\qquad r\in H^0(X,\mathbb{Z}),\quad l\in \mathrm{NS}(S),\quad s\in H^4(S,\mathbb{Z})$$

a primitive effective cohomology class with $\langle v,v\rangle = 2n-2$ with respect to the Mukai pairing $\langle,\rangle$. Consider the coarse moduli space X of stable coherent sheaves on S with Mukai vector v, with respect to a fixed v-generic ample line bundle on S. Then X is a smooth projective variety of dimension $2n$ and becomes a K3$^{[n]}$-type hyper-Kähler manifold by [O'G97]. ■

Example 1.3.4 (Hilbert scheme of points on an abelian surface; generalized Kummer variety [Bea83b])**.**

Let A be a 2-dimensional complex torus. Consider the Hilbert scheme $A^{[n+1]}$ of $n+1$

points on A, a smooth compact symplectic variety of dimension $2n+2$. Its Albanese variety becomes isomorphic to A, and its Albanese morphism

$$\mathrm{Alb} : A^{[n+1]} \to A$$

becomes an étale trivial surjective fibration. Any fiber X of the fibration is a hyper-Kähler manifold of dimension $2n$, called the *generalized Kummer variety*. Any hyper-Kähler manifold deformation equivalent to X is called a *Kum$_n$-type* hyper-Kähler manifold. ∎

Example 1.3.5 (Moduli of sheaves on an abelian surface [Yos01])**.**

Let A be an abelian surface and $v = (r, l, s) \in H^*_{\mathrm{even}}(A, \mathbb{Z})$ a primitive effective even cohomology class with $\langle v, v\rangle = 2n+2$. Consider the moduli space M of stable coherent sheaves on A with Mukai vector v, with respect to a fixed v-generic ample line bundle on A. Then M becomes a smooth projective symplectic variety of dimension $2n+4$. Its Albanese variety is isomorphic to $A \times \check{A}$, and its Albanese morphism

$$\mathrm{Alb} : M \to A \times \check{A}$$

is an étale trivial surjective fibration. The fiber X of the Albanese fibration is a Kum$_n$-type hyper-Kähler manifold. ∎

Example 1.3.6 (Singular moduli of sheaves on a K3 surface; O'Grady's 10-dimensional example [O'G99])**.**

Let S be a projective K3 surface and $v \in H^*(S, \mathbb{Z})$ a primitive effective cohomology class with $\langle v, v\rangle = 2$. Consider the moduli space $\bar{X}$ of stable coherent sheaves on S with Mukai vector $2v$, with respect to a v-generic ample line bundle on S. It is an irreducible projective symplectic variety of dimension 10, singular along a codimension 2 irreducible subvariety $\bar{X}_{\mathrm{sing}}$ (isomorphic to the symmetric product of the moduli of sheaves on S with Mukai vector v). A single blowup along $\bar{X}_{\mathrm{sing}} \subset \bar{X}$ will symplectically resolve $\bar{X}$ and yield a 10-dimensional hyper-Kähler manifold X. Any hyper-Kähler manifold deformation equivalent to X is called an *OG10-type* hyper-Kähler manifold. See [LS06, KLS06] for details. ∎

Example 1.3.7 (Singular moduli of sheaves on an abelian surface; O'Grady's 6-dimensional example [O'G03])**.**

Let A be an abelian surface and $v \in H^*_{\mathrm{even}}(A, \mathbb{Z})$ a primitive effective cohomology class with $\langle v, v\rangle = 2$. Consider the moduli space $\bar{M}$ of stable coherent sheaves on S with Mukai vector $2v$, with respect to a v-generic ample line bundle on A. It is again a projective symplectic variety of dimension 10, singular along a codimension 2 irreducible subvariety. Blowing up the singular locus will again symplectically resolve $\bar{M}$ and yield a smooth projective symplectic variety M. The Albanese fibration of M becomes

$$\mathrm{Alb} : M \to \bar{M} \to A \times \check{A},$$

an étale trivial surjective fibration. Its fiber is a hyper-Kähler manifold X of dimension 6. Any hyper-Kähler manifold deformation equivalent to X is called an *OG6-type* hyper-Kähler manifold. ■

Example 1.3.8 (Fano variety of lines on a cubic fourfold [BD85]).

Let Y be a smooth cubic fourfold in $\mathbb{P}^5$. The Fano variety of lines on Y becomes a hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$-type. ■

Example 1.3.9 (Twisted cubics on a cubic fourfold; LLSvS eightfold [LLSvS17]).

Let Y be a smooth cubic fourfold in $\mathbb{P}^5$ not containing any plane. Consider the Fano variety M of twisted cubics contained in Y. The variety M is smooth projective, and in fact a $\mathbb{P}^2$-bundle over an 8-dimensional smooth projective variety $\tilde{X}$. This space $\tilde{X}$ turns out to be a blowup of a hyper-Kähler eightfold X along a smooth subvariety isomorphic to the original Y. It is shown in [AL17] that X is of $\mathrm{K3}^{[4]}$-type. ■

1.3.2 Some known constructions of Lagrangian fibrations

Example 1.3.10 (Hilbert scheme of points on an elliptic K3 surface).

Apply the construction in Example 1.3.2 to an elliptic K3 surface $f : S \to \mathbb{P}^1$. Then the composition $\pi : S^{[n]} \to S^{(n)} \to (\mathbb{P}^1)^{(n)}$ of the Hilbert–Chow morphism and the symmetric power of f becomes a Lagrangian fibration of $S^{[n]}$. Note that the base $(\mathbb{P}^1)^{(n)}$ is isomorphic to $\mathbb{P}^n$. Also, the fiber of π at a general point $b = (b_1, \cdots, b_n) \in (\mathbb{P}^1)^{(n)}$ is

$$\pi^{-1}(b) = S_{b_1} \times \cdots \times S_{b_n}, \qquad S_{b_i} = f^{-1}(b_i),$$

a product of n elliptic curves, which is a principally polarizable abelian variety of dimension n. ■

Example 1.3.11 (Moduli of torsion sheaves on a K3 surface [Mar14, Ex 3.1]).

Apply the construction in Example 1.3.3 to the Mukai vector $v = (0, l, s)$, where l is an ample class with $\int_S l^2 = 2n - 2$. Any closed point $[F] \in X$ is a torsion coherent sheaf whose cohomological first Chern class is l. Let $L \in \mathrm{Pic}(S) = \mathrm{NS}(S)$ be a *unique* line bundle associated to l. We can consider the (Fitting) support map

$$\pi : X \to |L|, \qquad [F] \mapsto [\mathrm{Fitt}_0(F)],$$

This is a Lagrangian fibration of a K3$^{[n]}$-type hyper-Kähler manifold X.

Consider a Zariski open set $B_0 \subset |L|$ parametrizing smooth curves C on S, and the smooth projective universal family of $\mathcal{C} \to B_0$ over it. The smooth (abelian variety) fibers of π consist of the relative Jacobian $\mathrm{Pic}^m_{\mathcal{C}/B_0} \to B_0$ for $m = s + n - 1$. In particular, every smooth fiber of π is the Jacobian of a curve, meaning they are principally polarizable. ■

Example 1.3.12 (Compactified relative Picard scheme of a degree 2 K3 surface [Mar95]).

Let $f : S \to \mathbb{P}^2$ be a degree 2 K3 surface branched over a smooth sextic $D \subset \mathbb{P}^2$. Consider the universal family of lines $\mathbb{P}^2 \leftarrow \mathcal{L} \to (\mathbb{P}^2)^\vee$ on $\mathbb{P}^2$. The fiber product $\mathcal{C} = \mathcal{L} \times_{\mathbb{P}^2} S$ mapped into $(\mathbb{P}^2)^\vee$ models the family of degree 2 cyclic coverings of lines in $\mathbb{P}^2$ branched along D. A generic curve C in this family is a genus 2 hyperelliptic curve. If we assume any C in this family has at worst nodes or cusps, then the compactified relative Jacobian of $\mathcal{C} \to (\mathbb{P}^2)^\vee$ exists and becomes a Lagrangian fibered hyper-Kähler fourfold of K3$^{[2]}$-type. ■

Example 1.3.13 (Hilbert scheme of points on a non-simple abelian surface [Mat15, §2]).

Let $f : A \to E$ be a surjective homomorphism from an abelian surface A to an elliptic curve E. Apply the construction in Example 1.3.4 to A and form a commutative diagram

$$\begin{array}{ccc} A^{[n+1]} & \longrightarrow & A \\ \downarrow g & & \downarrow f \\ E^{(n+1)} & \longrightarrow & E \end{array},$$

where the horizontal maps are summation maps and the vertical map g is induced from f. Both horizontal maps are isotrivial; the first row is an étale trivial (Albanese) fiber bundle, and the second row is a Zariski locally trivial $\mathbb{P}^n$-bundle. The induced map between their fibers is a map $\pi : X \to \mathbb{P}^n$, where X is a generalized Kummer variety of dimension $2n$.

To compute the fibers of π, consider the kernel $E' = \ker f$. One can show the fiber of $g : A^{[n+1]} \to E^{(n+1)}$ over a general point $b = (b_1, \cdots, b_{n+1}) \in E^{(n+1)}$ is isomorphic to $(E')^{n+1}$. The fiber $F = \pi^{-1}(b)$ of the Lagrangian fibration sits in a short exact sequence

$$0 \longrightarrow F \longrightarrow (E')^{n+1} \xrightarrow{s} E' \longrightarrow 0 \ ,$$

where the map s is the summation map. Computation shows the polarization type of F is $(1, \cdots, 1, n+1)$. ■

Example 1.3.14 (Moduli of torsion sheaves on an abelian surface [Yos01] [Wie18]).

Apply the construction in Example 1.3.5 to the Mukai vector $v = (0, l, s)$, where l is an ample class with $\int_A l^2 = 2n + 2$. Any closed point $[F] \in M$ is a torsion coherent sheaf whose cohomological first Chern class is l. Contrary to Example 1.3.11, there are $\mathrm{Pic}^l_A \cong \check{A}$

choices of line bundles with associated cohomology class l. The support map in this situation correspondingly becomes

$$\mathrm{Supp} : M \to P, \qquad [F] \mapsto [\mathrm{Fitt}_0 F],$$

where P is a Zariski locally trivial $\mathbb{P}^n$-bundle over $\check{A}$. The Albanese fibration factors into

$$\begin{tikzcd} M \arrow[d, "\mathrm{Alb}"'] \arrow[dr] & \\ A \times \check{A} & A \times P \arrow[l] \end{tikzcd} .$$

This is a family of Lagrangian fibered hyper-Kähler manifolds. Its fiber is a Lagrangian fibration of a Kum_n-type hyper-Kähler manifold. The fibers of the Lagrangian fibrations have the polarization type $(1, \cdots, 1, d, \frac{n+1}{d})$. Here d is an integer measuring the non-primitiveness of the ample class l, i.e., $l = dl'$ for a primitive ample class l'. ■

Example 1.3.15 (Compactified intermediate Jacobian of a cubic fourfold [LSV17])**.**

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. Consider the universal family of hyperplanes $\mathbb{P}^5 \leftarrow \mathcal{L} \to (\mathbb{P}^5)^\vee$. The fiber product $\mathcal{C} = \mathcal{L} \times_{\mathbb{P}^5} Y$ mapped into $(\mathbb{P}^5)^\vee$ models the family of hyperplane sections of Y. Over a Zariski open set $U \subset (\mathbb{P}^5)^\vee$ parametrizing hyperplanes that intersect with Y transversally, the fiber C is a smooth cubic threefold. The associated relative intermediate Jacobian is an abelian scheme $\mathcal{J} \to U$ of relative dimension 5. There exists a smooth projective compactification $X \to (\mathbb{P}^5)^\vee$ of it, where X is a hyper-Kähler manifold of OG10-type. ■

Chapter 2

The Looijenga–Lunts–Verbitsky structure on the cohomology

Introduction

Cohomology of hyper-Kähler manifolds enjoys an exceptional amount of symmetries, well capturing its surface-like behaviors. Its symmetry is captured by a Lie algebra (or algebraic group) acting on the cohomology. Verbitsky [Ver95] and independently Looijenga–Lunts [LL97] attached a $\mathbb{Q}$-Lie algebra $\mathfrak{g} = \mathfrak{g}(X)$ to any hyper-Kähler manifold X, naturally acting on its rational cohomology $H^*(X, \mathbb{Q})$. We call $\mathfrak{g}$ the *Looijenga–Lunts–Verbitsky (LLV) algebra*, and the $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$ the *LLV structure*. They are topological invariants of X.

The main result of this chapter is collected in §2.2, where we explicitly compute the LLV structure for all currently known deformation types of hyper-Kähler manifolds. Such a computation can be extremely useful to compute other structures on the cohomology of hyper-Kähler manifolds. As an example, we show how LLV structure can be used to compute the Hodge structure of $H^*(X, \mathbb{Q})$. In particular, one can compute the Hodge classes on any even cohomology of hyper-Kähler manifolds. We provide some explicit examples to demonstrate this computation.

We give two applications of the LLV decomposition. The first is a positive answer to the conjecture of Nagai for all deformation types of hyper-Kähler manifolds. The second is an upper bound on the second Betti number of hyper-Kähler manifolds. It depends on Conjecture 2.5.1, which we believe should be of independent interest. The results in this chapter are joint work with Mark Green, Radu Laza and Colleen Robles [GKLR] [KL20].

2.1 The LLV structure

2.1.1 Definition of the LLV algebra

Let Y be a compact Kähler manifold of dimension m. We review the original definition of Looijenga–Lunts and Verbitsky's Lie algebra $\mathfrak{g}$ attached to Y. Let $x \in H^{1,1}(Y, \mathbb{R})$ be a Kähler class and consider two operators acting on the cohomology

$$\begin{aligned} h : H^*(Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \qquad & \xi \mapsto (k - \dim Y)\xi \quad \text{on} \quad H^k(Y, \mathbb{Q}), \\ L_x : H^*(Y, \mathbb{R}) \to H^*(Y, \mathbb{R}), \qquad & \xi \mapsto x \cdot \xi. \end{aligned} \tag{2.1.1}$$

The operators h and L_x are called the degree operator and the Lefschetz operator, respectively. The hard Lefschetz theorem proves there exists a unique operator Λ_x on $H^*(Y, \mathbb{R})$ making $\{h, L_x, \Lambda_x\}$ an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(H^*(Y, \mathbb{R}))$.

Looijenga–Lunts in [LL97] observed that the existence of a (unique) operator Λ_x completing $\{h, L_x, \Lambda_x\}$ an $\mathfrak{sl}_2$-triple is a Zariski open property on $x \in H^2(Y, \mathbb{Q})$ by the Jacobson–Morozov lemma. If Y is Kähler then the classical hard Lefschetz theorem guarantees at lest one $x \in H^2(Y, \mathbb{R})$ with an associated $\mathfrak{sl}_2$-triple, so almost all $x \in H^2(Y, \mathbb{Q})$ must automatically admit their associated $\mathfrak{sl}_2$-triples. Collecting all of them will generate a *semisimple* Lie subalgebra of $\mathfrak{gl}(H^*(Y, \mathbb{Q}))$.

Theorem-Definition 2.1.2 (Looijenga–Lunts)**.** *Let Y be a compact Kähler manifold. The* total Lie algebra $\mathfrak{g}$ *of Y is defined to be the Lie subalgebra of $\mathfrak{gl}(H^*(Y, \mathbb{Q}))$ generated by all possible Lefschetz $\mathfrak{sl}_2$-triples $\{L_x, h, \Lambda_x\}$ for $x \in H^2(Y, \mathbb{Q})$. It is a semisimple Lie algebra over $\mathbb{Q}$ and is a topological invariant on Y.*

Let us now assume X is a hyper-Kähler manifold. Throughout, we will use the following two quadratic spaces associated to it

$$\begin{aligned} (\bar{V}, \bar{q}) &= \big(H^2(X, \mathbb{Q}),\ \text{Beauville–Bogomolov form}\big), \\ (V, q) &= (\bar{V}, \bar{q}) \oplus (\mathbb{Q}^2, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)). \end{aligned} \tag{2.1.3}$$

The following theorem was proved by Verbistky [Ver95] and Looijenga–Lunts [LL97] over $\mathbb{R}$. The result is strengthened to $\mathbb{Q}$ in [GKLR], using results in more extensive studies of cohomology of hyper-Kähler manifolds in [KSV19]. We also note that [KSV19, Cor 3.13] deduced that the inverse Lefschetz operator Λ_x exists for $x \in H^2(X, \mathbb{Q})$ if and only if $\bar{q}(x) \neq 0$.

Theorem-Definition 2.1.4 (Looijenga–Lunts, Verbitsky)**.** *Let X be a hyper-Kähler manifold. Then its total Lie algebra is isomorphic to*

$$\mathfrak{g} \cong \mathfrak{so}(V, q).$$

To emphasize their contribution, we call $\mathfrak{g}$ *the* Looijenga–Lunts–Verbitsky (LLV) algebra. *The canonical* $\mathfrak{g}$*-module structure on* $H^*(X, \mathbb{Q})$ *is called the* LLV structure.

Base changing over $\mathbb{R}$, the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ becomes isomorphic to $\mathfrak{so}(4, b_2(X) - 2)$ because real quadratic spaces are classified by their signature up to isomorphism. Over $\mathbb{C}$, the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is isomorphic to $\mathfrak{so}(b_2(X) + 2, \mathbb{C})$. We note that quadratic spaces over $\mathbb{Q}$ are *not* classified by their signature, so the best we can say is $\mathfrak{g} \cong \mathfrak{so}(V, q)$.

The LLV algebra $\mathfrak{g}$ itself can be considered as a $\mathfrak{g}$-module under the adjoint action. The semisimple degree operator $h \in \mathfrak{g}$ induces an eigenspace decomposition of $\mathfrak{g}$. In the case of hyper-Kähler manifolds, only degrees 2, 0, and -2 occur:

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}.$$

The 0-eigenspace $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$, which can be decomposed further as

$$\mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{Q}h, \qquad \bar{\mathfrak{g}} = [\mathfrak{g}_0, \mathfrak{g}_0] : \text{semisimple part}, \quad \mathbb{Q}h = \mathfrak{z}(\mathfrak{g}_0) : \text{center}.$$

We call $\bar{\mathfrak{g}}$ the *reduced LLV algebra* of X. It is isomorphic to

$$\bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q}).$$

Since $\bar{\mathfrak{g}} \subset \mathfrak{g}_0$ consists of degree 0 operators, the induced $\bar{\mathfrak{g}}$-action on $H^*(X)$ preserves the degree. That is, each $H^k(X, \mathbb{Q})$ admits a $\bar{\mathfrak{g}}$-module structure. The second cohomology $H^2(X, \mathbb{Q}) = \bar{V}$ becomes a standard $\bar{\mathfrak{g}}$-module.

2.1.2 The LLV decomposition

The LLV structure is a $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$. Since $\mathfrak{g} \cong \mathfrak{so}(V, q)$ is simple, we can formally consider its irreducible $\mathfrak{g}$-module decomposition

$$H^*(X, \mathbb{Q}) \cong \bigoplus_{\lambda} m_\lambda V_\lambda. \tag{2.1.5}$$

Here λ is a dominant weight of $\mathfrak{g}_{\mathbb{C}}$, V_λ is an irreducible $\mathfrak{g}$-module of highest weight λ (however, see the remark below) and m_λ is its multiplicity. We provide Appendix A to summarize representation theory facts of the Lie algebra $\mathfrak{g}$ and our notation for it. The isomorphism (2.1.5) will be called the *LLV decomposition* of the cohomology of X. It is a topological invariant of a hyper-Kähler manifold.

Remark 2.1.6. We address two technical subtleties of the expression (2.1.5).

(i) The $\mathfrak{g}$-module isomorphism (2.1.5) is noncanonical. However, the component $m_\lambda V_\lambda$ for each λ is canonically defined as a submodule of $H^*(X, \mathbb{Q})$. It is called the λ-isotypic component.

(ii) Irreducible $\mathfrak{g}$-modules over $\mathbb{Q}$ are not always classified by dominant weights λ (see Appendix A). Hence, strictly speaking, using the notation V_λ in (2.1.5) is incorrect. What is correct instead is that we have a decomposition $H^*(X, \mathbb{C}) \cong \bigoplus m_\lambda V_{\lambda,\mathbb{C}}$ of the complex cohomology. This subtlety is not clearly addressed in the original paper [GKLR]. However, we claim that at least for the known deformation types of hyper-Kähler manifolds, (2.1.5) makes sense as it is *over* $\mathbb{Q}$.

If X is of $\mathrm{K3}^{[n]}$-type then X does not have any odd cohomology. Hence every irreducible $\mathfrak{g}$-module contained in $H^*(X, \mathbb{Q})$ is absolutely irreducible and classified by dominant weights by Proposition A.1.2 (note that $b_2(X) = 23$ is odd). This proves we can use (2.1.5) over $\mathbb{Q}$. If X is of Kum_n-type then the quadratic space (V, q) is isomorphic to

$$(V, q) \cong U^{\oplus 4} \oplus \langle -2(n+1) \rangle,$$

which has a maximal Witt index 4. The special orthogonal Lie algebra $\mathfrak{g} \cong \mathfrak{so}(V, q)$ correspondingly becomes a *split* simple Lie algebra and every irreducible $\mathfrak{g}$-module is absolutely irreducible, whence classified by dominant weights λ. See, e.g., [Mil17, §24.j]. Finally, if X is of OG10 or OG6-type then we first need to show the main result Theorem 2.2.6 over $\mathbb{C}$. But then one can notice every irreducible $\mathfrak{g}$-modules arising in the theorem is defined over $\mathbb{Q}$ by Proposition A.1.2 and A.1.3, so the result can be strengthened into $\mathbb{Q}$ and becomes the form (2.1.5).

Set $b_2(X) = \dim \bar{V}$ and $r = \lfloor \frac{1}{2} b_2(X) \rfloor$. The LLV algebra $\mathfrak{g} \cong \mathfrak{so}(V, q)$ is a simple Lie algebra of type B_{r+1} or D_{r+1} depending on the parity of $b_2(X)$. Either case, dominant weights λ can be expressed as a tuple $\lambda = (\lambda_0, \cdots, \lambda_r)$ of certain half-integers $\lambda_0, \cdots, \lambda_r$. We will typically drop the 0's at the end of this tuple for simplicity. For example, the dominant weight $\lambda = (1, 0, \cdots, 0)$ will be simply written as (1), and $V_{(1)} = V$ is the standard representation of $\mathfrak{g}$. Again see Appendix A for more details. The following was observed in [Ver96] and [Bog96].

Theorem-Definition 2.1.7 (Verbitsky)**.** *The LLV decomposition* (2.1.5) *always contains the irreducible component $V_{(n)}$ with multiplicity $m_{(n)} = 1$. We call this the* Verbitsky component *of $H^*(X, \mathbb{Q})$.*

The Verbitsky component should be considered as a primary component of the cohomology of hyper-Kähler manifolds. We will see in §2.2 that there can be many more complicated

V_λ's occurring in the LLV decomposition (2.1.5)

2.1.3 LLV structure and the Hodge structure

The LLV structure may be of interest because (1) it is a topological invariant on X, and (2) it is "stronger" than all known structures on the cohomology of hyper-Kähler manifolds. We will see in this and the following subsection how the LLV structure dominates the other structures on the cohomology of hyper-Kähler manifolds. We start with discussing the Hodge structure.

Every discussion so far was topological, whereas the Hodge structure on the cohomology certainly depends on the complex structure of X. Let us consider a new operator that depends on the complex structure

$$f : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R}), \qquad \xi \mapsto (q-p)\sqrt{-1}\xi \quad \text{on} \quad H^{p,q}(X). \tag{2.1.8}$$

Note that f determines the Hodge structure on each $H^k(X, \mathbb{Q})$ by its eigenspace decomposition. The k-th cohomology $H^k(X, \mathbb{Q})$ itself is determined by the eigenspace decomposition of the degree operator h on $H^*(X, \mathbb{Q})$ defined in (2.1.1). Therefore, the study of the two operators f and h is enough to understand the Hodge structure of $H^*(X, \mathbb{Q})$. It is immediate from the definition that h is a semisimple element in $\mathfrak{g}$ (or even $\mathfrak{g}_0$). The operator f is also a semisimple element in the LLV algebra ([GKLR, Prop 2.24]):

Lemma 2.1.9. *The operator f in* (2.1.8) *is a semisimple element in $\bar{\mathfrak{g}}_{\mathbb{R}}$.*

Therefore, the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(H^*(X, \mathbb{Q}))$ contains the entire information of the Hodge structure of $H^*(X, \mathbb{Q})$. Put differently, the $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$ determines the Hodge structure of $H^*(X, \mathbb{Q})$. The precise way of deducing the Hodge decomposition from a $\mathfrak{g}$-weight decomposition is presented in [GKLR, §2.2.1], but we will not copy it here. A more theoretical interpretation of this fact is a comparison of the Mumford–Tate group (algebra) and the LLV algebra of $H^*(X, \mathbb{Q})$.

Definition 2.1.10. The *special Mumford–Tate algebra* $\overline{\mathfrak{mt}}$ of $H^*(X, \mathbb{Q})$ is a minimal algebraic Lie subalgebra of $\mathfrak{gl}(H^*(X, \mathbb{Q}))$ that contains $f \in \mathfrak{gl}(H^*(X, \mathbb{R}))$ after base change over $\mathbb{R}$. The *Mumford–Tate algebra* of $H^*(X, \mathbb{Q})$ is simply a Lie subalgebra $\mathfrak{mt}_0 = \overline{\mathfrak{mt}} \oplus \mathbb{Q}h$ of $\mathfrak{gl}(H^*(X, \mathbb{Q}))$.

There is a general notion of a (special) Mumford–Tate algebra associated to any pure Hodge structure, the associated Lie algebra of its (special) Mumford–Tate group. See [Moo99], [Zar83] or [GKLR, §2.3]. The following proposition provides an important viewpoint to the study of the LLV algebra; one may think of the degree 0 part of the LLV algebra $\mathfrak{g}_0$ (resp. reduced LLV algebra $\bar{\mathfrak{g}}$) as the generic Mumford–Tate algebra (resp. generic special Mumford–Tate algebra) of the universal deformation $\mathcal{X} \to \mathrm{Def}(X)$ of X. In particular, the LLV algebra always contains the Mumford–Tate algebra, meaning it defines a more rigid structure on $H^*(X, \mathbb{Q})$ compared to the Hodge structure. See [GKLR, Prop 2.38].

Proposition 2.1.11. *Let $\overline{\mathfrak{mt}}$ be the special Mumford–Tate algebra of $H^*(X, \mathbb{Q})$.*

(i) *There exists an inclusion $\overline{\mathfrak{mt}} \subset \bar{\mathfrak{g}}$, and the equality holds for a very general X.*

(ii) *Assume $0 < k < 4n$ and $H^k(X, \mathbb{Q}) \neq 0$. Then the special Mumford–Tate algebra of $H^k(X, \mathbb{Q})$ is isomorphic to $\overline{\mathfrak{mt}}$. In particular, the special Mumford–Tate algebra of $H^2(X, \mathbb{Q})$ is isomorphic to $\overline{\mathfrak{mt}}$.*

Finally, we would also like to mention that the special Mumford–Tate Lie algebra (group) of any projective hyper-Kähler manifold is classified. This is a result of [Zar83].[1] We can use Proposition 2.1.11 to translate this result to the classification of the special Mumford–Tate algebra of any projective hyper-Kähler manifold.

Theorem 2.1.12 (Zarhin)**.** *Let X be a projective hyper-Kähler manifold. Let T be the transcendental Hodge structure of $H^2(X, \mathbb{Q})$ and $K = \mathrm{End}_{\mathsf{HS}}(T)$ its endomorphism algebra over $\mathbb{Q}$. Then*

(i) *K is either a totally real number field or a CM number field.*

(ii) *The special Mumford–Tate algebra $\overline{\mathfrak{mt}}$ of X is isomorphic to*

$$\overline{\mathfrak{mt}} \cong \begin{cases} \mathfrak{so}_K(T, q_K) & \text{if } K \text{ is a totally real number field} \\ \mathfrak{u}_K(T, q_K) & \text{if } K \text{ is CM number field} \end{cases} .$$

Here q_K is a K-symmetric (or K-sesquilinear) bilinear form on T induced by the Beauville–Bogomolov form $\bar{q}$.

See [Zar83] or [Huy16, §3.3] for more details.

The following fact about the generic Mumford–Tate group is very useful. Its proof is an easy application of the descriptions of Noether–Lefschetz loci via the period map. See, e.g., [GGK12].

[1] Zarhin's result was stated in terms of the second cohomology of projective K3 surfaces, but his method applies to any polarizable weight 2 pure Hodge structures of K3 type.

Proposition 2.1.13. (i) *If X is a very general hyper-Kähler manifold, then $\overline{\mathfrak{mt}} \cong \mathfrak{so}(\bar{V}, \bar{q})$.*

(ii) *It X is a very general polarized hyper-Kähler manifold with $T \subset H^2(X, \mathbb{Q})$ the orthogonal complement of the polarization, then $\overline{\mathfrak{mt}} \cong \mathfrak{so}(T, \bar{q})$.*

2.1.4 LLV structure and other structures

Let us briefly mention some other structures on $H^*(X, \mathbb{Q})$ and their relations to the LLV structure. First, Looijenga–Lunts also defined another Lie algebra generated only by the algebraic classes $x \in \mathrm{NS}(X)_{\mathbb{Q}}$. It defines a $\mathbb{Q}$-Lie algebra $\mathfrak{g}_{\mathrm{NS}} \subset \mathfrak{gl}(H^*(X, \mathbb{Q}))$, which turns out to be isomorphic to the special orthogonal Lie algebra of the quadratic space $(\mathrm{NS}(X)_{\mathbb{Q}}, \bar{q})$. We clearly have an inclusion of Lie algebras $\mathfrak{g}_{\mathrm{NS}} \subset \mathfrak{g}$. Hence the $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$ is always more rigid than the $\mathfrak{g}_{\mathrm{NS}}$-module structure on it.

Another interesting Lie algebra is an $\mathbb{R}$-Lie algebra $\mathfrak{g}_g$ attached to each hyper-Kähler metric g on X [Fuj87] [Ver90]. The Lie algebra $\mathfrak{g}_g$ is generated by the Lefschetz / inverse Lefschetz operators associated to all S^2-family of Kähler classes $\omega = g(I-, -)$ of the hyper-Käher metric g in Proposition 1.1.3. It becomes isomorphic to $\mathfrak{so}(4, 1)$. Again we have $\mathfrak{g}_g \subset \mathfrak{g}_{\mathbb{R}}$. Its reduced part $\bar{\mathfrak{g}}_g$ is isomorphic to $\mathfrak{so}(3)$. Thus the corresponding simply connected Lie group $\mathrm{Spin}(3)$ acts on each cohomology $H^k(X, \mathbb{R})$. This action is often referred to an $\mathrm{SU}(2)$-action or an $\mathrm{Sp}(1)$-action, because we have an accidental isomorphism $\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)$ of the compact real forms of the type $B_1 = A_1 = C_1$ simple Lie groups.

2.2 Computing the LLV structures for known deformation types

This section states explicit computations for the LLV structure of all currently known deformation types of hyper-Kähler manifolds. The following results are proved in [GKLR], which is joint work with Mark Green, Radu Laza and Colleen Robles. The statements were stated over $\mathbb{R}$ in the original paper but the arguments prove the statements over $\mathbb{Q}$. See Remark 2.1.6.

Theorem 2.2.1. *The generating series for the formal characters of $K3^{[n]}$-type hyper-Kähler manifolds is*

$$\sum_{n=1}^{\infty} \mathrm{ch}\left(H^*(K3^{[n]}, \mathbb{C})\right) q^n = \prod_{m=1}^{\infty} \frac{1}{\prod_{i=0}^{11}(1 - x_i q^m)(1 - x_i^{-1} q^m)}.$$

The equality in the theorem is taken in the formal power series ring $A[[q]]$, where

$$A = \mathbb{Z}[x_0^{\pm 1}, \cdots, x_{11}^{\pm 1}, (x_0 \cdots x_{11})^{\frac{1}{2}}]^{W_{25}}$$

is the representation ring of the type B_{12} simple Lie algebra $\mathfrak{so}(25, \mathbb{C})$. One can convert this formal character computation into an irreducible decomposition (with the aid of computer). The first few computations are listed in the following corollary for reader's convenience.

Corollary 2.2.2. *The LLV decompositions of some low-dimensional $K3^{[n]}$-type hyper-Kähler manifolds are as follows:*

$$\begin{aligned}
&H^*(K3^{[2]}, \mathbb{Q}) \cong V_{(2)},\\
&H^*(K3^{[3]}, \mathbb{Q}) \cong V_{(3)} \oplus V_{(1,1)},\\
&H^*(K3^{[4]}, \mathbb{Q}) \cong V_{(4)} \oplus V_{(2,1)} \oplus V_{(2)} \oplus \mathbb{Q},\\
&H^*(K3^{[5]}, \mathbb{Q}) \cong V_{(5)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,1)} \oplus V_{(1,1)} \oplus V,\\
&H^*(K3^{[6]}, \mathbb{Q}) \cong V_{(6)} \oplus V_{(4,1)} \oplus V_{(4)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,2)} \oplus V_{(2,1)} \oplus 2V_{(2)} \oplus V_{(1,1,1)} \oplus V \oplus \mathbb{Q},\\
&H^*(K3^{[7]}, \mathbb{Q}) \cong V_{(7)} \oplus V_{(5,1)} \oplus V_{(5)} \oplus V_{(4,1)} \oplus V_{(4)} \oplus V_{(3,2)} \oplus 2V_{(3,1)} \oplus 2V_{(3)} \oplus V_{(2,1,1)},\\
&\qquad\qquad\qquad\quad \oplus {V_{(2,1)}}^{\oplus 2} \oplus V_{(2)} \oplus 2V_{(1,1)} \oplus 2V.
\end{aligned}$$

A similar result for Kum_n-type hyper-Kähler manifolds is slightly more involved. In fact, the intriguing coefficients in the formulas for the Kum_n-type suggests that there should be a better way to formulate the result. We plan to address this issue in our future work with Mirko Mauri. To state the result, let us first define a formal power series

$$B(q) = \prod_{m=1}^{\infty} \frac{\prod_j (1 + x_0^{j_0} x_1^{j_1} x_2^{j_2} x_3^{j_3} q^m)}{\prod_{i=0}^{3} (1 - x_i q^m)(1 - x_i^{-1} q^m)} \cdots, \tag{2.2.3}$$

where $j = (j_0, \cdots, j_3) \in \{-\frac{1}{2}, \frac{1}{2}\}^{\times 4}$ runs through all the eight 4-tuples satisfying $j_0 + \cdots + j_3 \in 2\mathbb{Z}$. Given a positive integer d, we define its *fourth Jordan totient value* by

$$J_4(d) = d^4 \cdot \prod_{p|d} \left(1 - \tfrac{1}{p^4}\right),$$

where p runs through all the prime factors of d.

Theorem 2.2.4. *The generating series for the formal characters of Kum_n-type hyper-Kähler manifolds is*

$$\sum_{n=1}^{\infty} \mathrm{ch}\Big(H^*(Kum_n, \mathbb{C})\Big) q^n = \sum_{d=1}^{\infty} J_4(d) \cdot \frac{B(q^d) - 1}{b_1 q}.$$

Here $B(q)$ is defined in (2.2.3)*, b_1 is the degree* 1 *coefficient of $B(q)$, and $J_4(d)$ is the fourth Jordan totient value of d.*

The equality in the theorem again is taken in the formal power series ring $A[[q]]$, where

$$A = \mathbb{Z}[x_0^{\pm 1}, \cdots, x_3^{\pm 1}, (x_0 \cdots x_3)^{\frac{1}{2}}]^{W_9}$$

is the representation ring of the type B_4 simple Lie algebra $\mathfrak{so}(9, \mathbb{C})$. The computations of the LLV decompositions are the following.

Corollary 2.2.5. *The LLV decompositions of some low-dimensional* Kum_n*-type hyper-Kähler manifolds are as follows:*

$$\begin{aligned}
H^*(Kum_2, \mathbb{Q}) &\cong V_{(2)} \oplus 80\mathbb{Q} \quad \oplus V_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}\\
H^*(Kum_3, \mathbb{Q}) &\cong V_{(3)} \oplus V_{(1,1)} \oplus 16V \oplus 240\mathbb{Q} \quad \oplus V_{(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}\\
H^*(Kum_4, \mathbb{Q}) &\cong V_{(4)} \oplus V_{(2,1)} \oplus V_{(2)} \oplus V_{(1,1,1)} \oplus V_{(1,1)} \oplus 625\mathbb{Q} \quad \oplus V_{(\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \oplus V_{(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \oplus V_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}\\
H^*(Kum_5, \mathbb{Q}) &\cong V_{(5)} \oplus V_{(3,1)} \oplus V_{(3)} \oplus V_{(2,1,1)} \oplus 2V_{(2,1)} \oplus 16V_{(2)} \oplus V_{(1,1,1,1)} \oplus V_{(1,1)} \oplus 82V\\
&\quad \oplus 1200\mathbb{Q} \quad \oplus V_{(\frac{7}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \oplus V_{(\frac{5}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \oplus V_{(\frac{3}{2},\frac{3}{2},\frac{1}{2},\frac{1}{2})} \oplus 2V_{(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})} \oplus 17V_{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}
\end{aligned}$$

O'Grady's two sporadic examples are addressed separately in the following theorem.

Theorem 2.2.6. *The LLV decompositions of the OG10 and OG6-type hyper-Kähler manifolds are*

$$H^*(OG10, \mathbb{Q}) \cong V_{(5)} \oplus V_{(2,2)}, \qquad H^*(OG6, \mathbb{Q}) \cong V_{(3)} \oplus V_{(1,1,1)} \oplus 135V \oplus 240\mathbb{Q}.$$

We only present the proof for the $\mathrm{K3}^{[n]}$-case and refer the remaining proofs to the original article [GKLR]. In fact, proof for the Kum_n-case is essentially the same but only combinatorially more involved. The proofs for OG10 and OG6-types need different approaches. Additional inputs for their proofs are [dCRS21] and [MRS18].

2.2.1 The LLV structure of $\mathrm{K3}^{[n]}$

This subsection is entirely devoted to the proof of Theorem 2.2.1. Our starting point is the result of [GS93] and [dCM00] for the computation of the Hodge structure of $S^{[n]}$ for any compact Kähler surface S.

Theorem 2.2.7 (Göttsche–Soergel, de Cataldo–Migliorini)**.** *Let S be a compact Kähler surface. Then there exists a Hodge structure isomorphism*

$$H^*(S^{[n]}, \mathbb{Q}) = \bigoplus_{\alpha \vdash n} \left[\bigotimes_{i=1}^{n} \mathrm{Sym}^{a_i} H^*(S, \mathbb{Q}) \Big(\sum_{i=1}^{n} a_i - n \Big) \right], \tag{2.2.8}$$

where $\alpha = (1^{a_1}, \cdots, n^{a_n})$ runs through all partitions of n.

Say S is a very general (non-projective) K3 surface. Its special Mumford–Tate algebra and Mumford–Tate algebra are

$$\bar{\mathfrak{m}} = \mathfrak{so}(H^2(S,\mathbb{Q}), \bar{q}_S), \qquad \mathfrak{m}_0 = \bar{\mathfrak{m}} \oplus \mathbb{Q}h,$$

where $\bar{q}_S$ is the intersection pairing of S. By (2.2.8), the Mumford–Tate algebra of $H^*(S^{[n]},\mathbb{Q})$ is also isomorphic to $\mathfrak{m}_0$, and a posteriori the equality (2.2.8) becomes an $\mathfrak{m}_0$-module isomorphism.

Consider the formal "Mukai completion" $\mathfrak{m}$ of $\mathfrak{m}_0$. It is a Lie algebra $\mathfrak{m} = \mathfrak{so}(H^*(S,\mathbb{Q}), q_S)$ containing $\mathfrak{m}_0 = \bar{\mathfrak{m}} \oplus \mathbb{Q}h$ by precisely the degree 0 part with respect to the h-eigenspace decomposition:

$$\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_0 \oplus \mathfrak{m}_2, \qquad \mathfrak{m}_0 = \bar{\mathfrak{m}} \oplus \mathbb{Q}h, \quad \mathfrak{m}_{-2} = \mathfrak{m}_2 = H^2(S,\mathbb{Q}).$$

The cohomology $H^*(S,\mathbb{Q})$ becomes the standard $\mathfrak{m}$-module whose restriction to $\mathfrak{m}_0$ induces the original $\mathfrak{m}_0$-module structure on it. Correspondingly the RHS of (2.2.8) admits a unique $\mathfrak{m}$-module structure whose restriction to $\mathfrak{m}_0$ recovers the original $\mathfrak{m}_0$-module structure on it. On the other hand, consider the LLV algebra $\mathfrak{g}$ of the hyper-Kähler manifold $S^{[n]}$. By Proposition 2.1.11, we have an inclusion of Lie algebras

$$\mathfrak{m} \ \subset \ \mathfrak{g} = \mathfrak{so}\big((H^*(S,\mathbb{Q}), q_S) \oplus \langle -2(n-1)\rangle\big).$$

As a result, the LHS of (2.2.8) also admits a unique $\mathfrak{m}$-module structure by restricting its $\mathfrak{g}$-module structure, the LLV structure. By Lemma A.2.2, we can upgrade (2.2.8) to an $\mathfrak{m}$-module isomorphism.

Finally, we are in a position to apply Lemma A.2.3. After base change over $\mathbb{C}$, the LHS and RHS of the equality admits a $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_{25}(\mathbb{C})$ and $\mathfrak{m}_{\mathbb{C}} = \mathfrak{so}_{24}(\mathbb{C})$-module structure, respectively. The formal $\mathfrak{so}_{24}(\mathbb{C})$-character of RHS is clearly

$$\sum_{\alpha \vdash n} s_{a_1} \cdots s_{a_n},$$

where s_i denotes the formal character of the i-th symmetric power of $H^*(S,\mathbb{Q})$. This becomes the formal $\mathfrak{so}_{25}(\mathbb{C})$-character of the LHS by Lemma A.2.3.

In conclusion, the generating series for the $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}_{25}(\mathbb{C})$-characters of $H^*(S^{[n]},\mathbb{C})$ becomes

$$\begin{aligned}\sum_{n=0}^{\infty} \operatorname{ch}\Big(H^*(\mathrm{K3}^{[n]},\mathbb{C})\Big) q^n &= \sum_{n=0}^{\infty} \sum_{\alpha \vdash n} s_{a_1} s_{a_2} \cdots s_{a_n} q^{a_1 + 2a_2 + \cdots + na_n} \\ &= \left(\sum_{a_1=0}^{\infty} s_{a_1} q^{a_1}\right) \left(\sum_{a_2=0}^{\infty} s_{a_2} q^{2a_2}\right) \left(\sum_{a_3=0}^{\infty} s_{a_3} q^{3a_3}\right) \cdots .\end{aligned}$$

Each factor simplifies into

$$\sum_{i=0}^{\infty} s_i q^i = 1 + (x_0 + x_1 + \cdots + x_{11}^{-1})q + (x_0^2 + x_0 x_1 + \cdots + x_{11}^{-2})q^2 + \cdots$$
$$= \prod_{i=0}^{11}(1 + x_i q + x_i^2 q^2 + \cdots)(1 + x_i^{-1} q + x_i^{-2} q^2 + \cdots) = \prod_{i=0}^{11} \frac{1}{(1 - x_i q)(1 - x_i^{-1} q)}.$$

This completes the proof of Theorem 2.2.1.

2.3 Computing the reduced LLV structures and Hodge structures

The goal of this section is to present how one can use our main results in §2.2 to compute the Hodge structure of a hyper-Kähler manifold. We present two illustrative examples. The results in this section are not contained in [GKLR].

2.3.1 The reduced LLV structure

The *reduced LLV structure* on the k-th cohomology $H^k(X, \mathbb{Q})$ is its $\bar{\mathfrak{g}}$-module structure. The *reduced LLV decomposition* is a decomposition of a single degree cohomology into a direct sum of irreducible $\bar{\mathfrak{g}}$-modules

$$H^k(X, \mathbb{Q}) \cong \bigoplus_{\bar{\lambda}} m_{\bar{\lambda}} \bar{V}_{\bar{\lambda}}.$$

Here $\bar{\lambda}$ indicates a dominant $\bar{\mathfrak{g}}$-weight and $\bar{V}_{\bar{\lambda}}$ is an irreducible $\bar{\mathfrak{g}}$-module of highest weight $\bar{\lambda}$. The reduced LLV structure can be computed from the original LLV structure of $H^*(X, \mathbb{Q})$ by restriction of scalars. Recall from §2.1.1 that we had a Lie subalgebra $\mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{Q}h$ of the LLV algebra $\mathfrak{g}$. Hence the $\mathfrak{g}$-module structure on $H^*(X, \mathbb{Q})$ induces a $\mathfrak{g}_0$-module structure on $H^*(X, \mathbb{Q})$ by

$$\bar{\mathfrak{g}} \oplus \mathbb{Q}h = \mathfrak{g}_0 \hookrightarrow \mathfrak{g} \to \mathfrak{gl}(H^*(X, \mathbb{Q})).$$

The h-action is deciding the degree of the cohomology, and the $\bar{\mathfrak{g}}$-action is our desired reduced LLV structure. The reduced LLV structure is a topological invariant.

From the results in §2.2, we can deduce the reduced LLV decompositions for known deformation types of hyper-Kähler manifolds. Again the computations were done with the aid of computer and we will make no justification for these.

Proposition 2.3.1. *The reduced LLV decompositions of some low-dimensional $K3^{[n]}$-type hyper-Kähler manifolds are as follows.*

(i) $H^4(K3^{[2]},\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V}$.

(ii) $H^4(K3^{[3]},\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus \bar{V}$, $\qquad H^6(K3^{[3]},\mathbb{Q}) \cong \operatorname{Sym}^3 \bar{V} \oplus \Big(\bar{V}_{(1,1)} \oplus \mathbb{Q}\Big)$.

(iii) $H^4(K3^{[4]},\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus \bar{V} \oplus \mathbb{Q}$,

$H^6(K3^{[4]},\mathbb{Q}) \cong \operatorname{Sym}^3 \bar{V} \oplus \Big(\bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus \bar{V}$,

$H^8(K3^{[4]},\mathbb{Q}) \cong \operatorname{Sym}^4 \bar{V} \oplus \Big(\bar{V}_{(2,1)} \oplus 2\bar{V}\Big) \oplus \Big(\bar{V}_{(2)} \oplus \mathbb{Q}\Big) \oplus \mathbb{Q}$.

(iv) $H^4(K3^{[5]},\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus \bar{V} \oplus \mathbb{Q}$,

$H^6(K3^{[5]},\mathbb{Q}) \cong \operatorname{Sym}^3 \bar{V} \oplus \Big(\bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus \bar{V} \oplus \bar{V}$,

$H^8(K3^{[5]},\mathbb{Q}) \cong \operatorname{Sym}^4 \bar{V} \oplus \Big(\bar{V}_{(3)} \oplus \bar{V}_{(2,1)} \oplus 2\bar{V}\Big) \oplus \Big(\bar{V}_{(2)} \oplus \mathbb{Q}\Big) \oplus \Big(\bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus \bar{V} \oplus \mathbb{Q}$,

$H^{10}(K3^{[5]},\mathbb{Q}) \cong \operatorname{Sym}^5 \bar{V} \oplus \Big(\bar{V}_{(3,1)} \oplus 2\bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus \Big(\bar{V}_{(3)} \oplus \bar{V}\Big) \oplus \Big(\bar{V}_{(2,1)} \oplus 2\bar{V}\Big) \oplus \Big(\bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus \bar{V}$.

The reduced LLV structure of $H^{2n+k}(X,\mathbb{Q})$ is isomorphic to that of $H^{2n-k}(X,\mathbb{Q})$. This is why we only provided computations up to the middle degree cohomology. Expressions in a single parenthesis mean they are from a single irreducible $\mathfrak{g}$-module component. For example, recall from Theorem 2.2.1 that we had an LLV decomposition of a $\mathrm{K3}^{[4]}$-type hyper-Kähler manifold

$$H^*(\mathrm{K3}^{[4]},\mathbb{Q}) \cong V_{(4)} \oplus V_{(2,1)} \oplus V_{(2)} \oplus \mathbb{Q}.$$

The results in the previous proposition is written in a way that, for example, the expression in a single parenthesis $\big(\bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus \mathbb{Q}\big)$ in $H^6(\mathrm{K3}^{[4]},\mathbb{Q})$ is coming from a single component $V_{(2,1)}$.

Remark 2.3.2. Alternatively, for each integer k one can read off the coefficients of $x_0^{k/2}$ in the generating series Theorem 2.2.1. This will give us the formal characters of the reduced LLV structures $H^{2n+k}(\mathrm{K3}^{[n]},\mathbb{Q})$, and hence their irreducible decompositions. We were not able to obtain a closed formula for the generating series of the reduced LLV structures.

Proposition 2.3.3. *The reduced LLV decompositions of some low-dimensional Kum_n-type hyper-Kähler manifolds are as follows.*

(i) $H^3(Kum_2,\mathbb{Q}) \cong \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$, $\qquad H^4(Kum_2,\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus 80\mathbb{Q}$.

(ii) $H^3(Kum_3,\mathbb{Q}) \cong \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$, $\qquad H^4(Kum_3,\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus \bar{V} \oplus 16\mathbb{Q}$,

$H^5(Kum_3,\mathbb{Q}) \cong \bar{V}_{(\frac{3}{2},\frac{1}{2},\frac{1}{2})} \oplus \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$,

$H^6(Kum_3,\mathbb{Q}) \cong \operatorname{Sym}^3 \bar{V} \oplus \Big(\bar{V}_{(1,1)} \oplus \mathbb{Q}\Big) \oplus 16\bar{V} \oplus 240\mathbb{Q}$.

(iii) $H^3(Kum_4,\mathbb{Q}) \cong \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$, $\qquad H^4(Kum_4,\mathbb{Q}) \cong \operatorname{Sym}^2 \bar{V} \oplus \bar{V} \oplus \mathbb{Q}$,

$H^5(Kum_4,\mathbb{Q}) \cong \Big(\bar{V}_{(\frac{3}{2},\frac{1}{2},\frac{1}{2})} \oplus \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}\Big) \oplus \bar{V}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})}$,

$$
\begin{aligned}
H^6(Kum_4,\mathbb{Q}) &\cong \operatorname{Sym}^3\bar V \oplus \left(\bar V_{(2)}\oplus \bar V_{(1,1)}\oplus\mathbb{Q}\right)\oplus \bar V\oplus \bar V_{(1,1)}\oplus \bar V,\\
H^7(Kum_4,\mathbb{Q}) &\cong \left(\bar V_{(\frac52,\frac12,\frac12)}\oplus \bar V_{(\frac32,\frac12,\frac12)}\oplus \bar V_{(\frac12,\frac12,\frac12)}\right)\oplus\left(\bar V_{(\frac32,\frac12,\frac12)}\oplus \bar V_{(\frac12,\frac12,\frac12)}\right)\oplus \bar V_{(\frac12,\frac12,\frac12)},\\
H^8(Kum_4,\mathbb{Q}) &\cong \operatorname{Sym}^4\bar V\oplus\left(\bar V_{(2,1)}\oplus 2\bar V\right)\oplus\left(\bar V_{(2)}\oplus\mathbb{Q}\right)\oplus\left(\bar V_{(1,1,1)}\oplus\bar V\right)\\
&\qquad\oplus\left(\bar V_{(1,1)}\oplus\mathbb{Q}\right)\oplus 625\mathbb{Q}.
\end{aligned}
$$

Proposition 2.3.4. *The reduced LLV decompositions of the OG10-type hyper-Kähler manifolds are*

$$
\begin{aligned}
&H^4(OG10,\mathbb{Q})\cong\operatorname{Sym}^2\bar V, && H^6(OG10,\mathbb{Q})\cong \operatorname{Sym}^3\bar V\oplus\bar V_{(2)},\\
&H^8(OG10,\mathbb{Q})\cong\operatorname{Sym}^4\bar V\oplus\bar V_{(2,1)}\oplus\bar V, && H^{10}(OG10,\mathbb{Q})\cong\operatorname{Sym}^5\bar V\oplus\bar V_{(2,2)}\oplus\bar V_{(2)}\oplus\bar V_{(1,1)}\oplus\mathbb{Q}.
\end{aligned}
$$

Proposition 2.3.5. *The reduced LLV decompositions of the OG6-type hyper-Kähler manifolds are*

$$
\begin{aligned}
H^4(OG6,\mathbb{Q})&\cong\operatorname{Sym}^2\bar V\oplus\bar V_{(1,1)}\oplus 135\mathbb{Q},\\
H^6(OG6,\mathbb{Q})&\cong\operatorname{Sym}^3\bar V\oplus\left(\bar V_{(1,1,1)}\oplus\bar V\right)\oplus 135\bar V\oplus 240\mathbb{Q}.
\end{aligned}
$$

2.3.2 Computing the Hodge structures

The reduced LLV structure can be useful for computing the Hodge structure of a hyper-Kähler manifold. From the results The results in §2.1.1 says the Mumford–Tate algebra $\mathfrak{mt}_0 = \overline{\mathfrak{mt}}\oplus\mathbb{Q}h$ is always a Lie subalgebra of $\mathfrak{g}_0$, and thus the reduced LLV structure determines the Hodge structure. In particular, all the reduced LLV decompositions computed in the previous subsection are Hodge structure isomorphisms once you consider $\bar V = H^2(X,\mathbb{Q})$ as the Hodge structure of the second cohomology of X.

If one wants to know the full irreducible decomposition of the Hodge structure of $H^k(X,\mathbb{Q})$ (e.g., when computing the Hodge cycles) then the reduced LLV decomposition needs to be decomposed further. The goal of this subsection is to present two examples to show how these discussions can give us an explicit computation of the Hodge structure of X.

Example 2.3.6 ($\mathrm{K3}^{[2]}$-type)**.** Let X be a hyper-Kähler manifold of $\mathrm{K3}^{[2]}$-type. We have a computation of its reduced LLV structure

$$
H^4(X,\mathbb{Q})\cong\operatorname{Sym}^2\bar V=\bar V_{(2)}\oplus\mathbb{Q}, \tag{2.3.7}
$$

which in particular becomes a Hodge structure isomorphism if we consider $\bar V = H^2(X,\mathbb{Q})$ as a Hodge structure.

Assume X is a very general non-projective hyper-Kähler manifold, so that the special Mumford–Tate algebra is $\overline{\mathfrak{mt}}=\bar{\mathfrak{g}}\cong\mathfrak{so}(\bar V,\bar q)$ by Proposition 2.1.13. Then the $\bar{\mathfrak{g}}$-module decomposition (2.3.7) is the irreducible Hodge structure decomposition. In particular, there are 1-dimensional Hodge cycles in $H^4(X,\mathbb{Q})$, generated by the Beauville–Bogomolov class.

Assume now X is a very general polarized hyper-Kähler manifold. The special Mumford–Tate algebra is $\overline{\mathfrak{mt}} \cong \mathfrak{so}(W,\bar{q})$ again by Proposition 2.1.13, and the irreducible Hodge structure decomposition of $H^2(X,\mathbb{Q})$ is $\bar{V} = T \oplus \mathbb{Q}$. The $\bar{\mathfrak{g}}$-module decomposition (2.3.7) correspondingly is not an irreducible $\overline{\mathfrak{mt}}$-module decomposition; its irreducible $\overline{\mathfrak{mt}}$-module decomposition becomes

$$H^4(X,\mathbb{Q}) \cong \mathrm{Sym}^2(W \oplus \mathbb{Q}) = W_{(2)} \oplus W \oplus 2\mathbb{Q}.$$

In particular, the Hodge cycles in $H^4(X,\mathbb{Q})$ are 2-dimensional.

If we do not assume X is very general then one needs to understand the special Mumford–Tate algebra $\overline{\mathfrak{mt}}$ of X to do similar computations. By Theorem 2.1.12, we can compute $\overline{\mathfrak{mt}}$ and understand everything about the tensor construction of Hodge structures arising from $\bar{V}$. If we have $K = \mathbb{Q}$ in the theorem, then $\overline{\mathfrak{mt}}$ becomes the special orthogonal Lie algebra as usual and the computation will be the same. If $K \neq \mathbb{Q}$, then in principle one can still understand the representation theory of $\overline{\mathfrak{mt}}$, but with a more extensive use of representation theory of type BD simple Lie algebras over $\mathbb{Q}$. See, e.g., [Mil17]. ■

Example 2.3.8 (OG6-type)**.** Let X be a hyper-Kähler manifold of OG6-type. The reduced LLV structures of X are

$$H^4(X,\mathbb{Q}) \cong \bar{V}_{(2)} \oplus \bar{V}_{(1,1)} \oplus 136\mathbb{Q}, \qquad H^6(X,\mathbb{Q}) \cong \bar{V}_{(3)} \oplus \bar{V}_{(1,1,1)} \oplus 137\bar{V} \oplus 240\mathbb{Q}.$$

Once we consider $\bar{V} = H^2(X,\mathbb{Q})$ as a Hodge structure, these isomorphisms are Hodge structure isomorphisms. However, they may not be an irreducible decomposition of Hodge structures.

If X is very general then $\overline{\mathfrak{mt}} = \bar{\mathfrak{g}}$, meaning the above isomorphisms are indeed irreducible Hodge structure decompositions. In particular, there are precisely 136-dimensional and 240-dimensional Hodge cycles in $H^4(X,\mathbb{Q})$ and $H^6(X,\mathbb{Q})$, respectively. If X is a very general polarized hyper-Kähler manifold then the special Mumford–Tate algebra $\overline{\mathfrak{mt}} \cong \mathfrak{so}(T,\bar{q})$ becomes smaller and we have a further decomposition of irreducible Hodge structures

$$\begin{aligned} H^4(X,\mathbb{Q}) &\cong T_{(2)} \oplus T_{(1,1)} \oplus 2T \oplus 138\mathbb{Q}, \\ H^6(X,\mathbb{Q}) &\cong T_{(3)} \oplus T_{(1,1,1)} \oplus T_{(2)} \oplus T_{(1,1)} \oplus 139T \oplus 379\mathbb{Q}. \end{aligned}$$

In particular, there are 138-dimensional and 379-dimensional Hodge cycles in $H^4(X,\mathbb{Q})$ and $H^6(X,\mathbb{Q})$, respectively. ■

2.4 Digression: Nagai's conjecture

We provide two applications of our results in §2.2. In this section, we discuss its first application to a conjecture raised by Nagai [Nag08], following [GKLR, §4–6].

To state the conjecture, we need to set up some additional notation. Let $\mathcal{X} \to \Delta^*$ be a smooth projective family of hyper-Kähler manifolds over a punctured disc and fix a single fiber X, a projective hyper-Kähler manifold of dimension $2n$. Associated to the family we can define a notion of the k-th monodromy operator $T_k \in \mathrm{GL}(H^k(X, \mathbb{Q}))$ (in fact, in $\mathrm{GL}(H^k(X, \mathbb{Z}))$) on each cohomological degree $0 \le k \le 4n$. The monodromy theorem [Sch73, (6.1)] claims T_k is always a quasi-unipotent operator on $H^k(X, \mathbb{Q})$. One can thus define its logarithm

$$N_k = \log T_k \quad \in \mathfrak{gl}(H^k(X, \mathbb{Q})) \qquad \text{for} \quad 0 \le k \le 4n,$$

which becomes a nilpotent operator on $H^k(X, \mathbb{Q})$. Its *index of nilpotency* ν_k is the minimal nonnegative integer satisfying

$$(N_k)^{\nu_k} = 0.$$

A general result in Hodge theory gives us a bound $0 \le \nu_k \le k$. In particular, $\nu_2 = 0, 1$ or 2.

Conjecture 2.4.1 (Nagai)**.** *Let $\mathcal{X} \to \Delta^*$ be a smooth projective family of hyper-Kähler manifolds of dimension $2n$ over a punctured disc. If ν_k denotes the index of nilpotency of its k-th log monodromy operator, then we have*

$$\nu_{2k} = k \cdot \nu_2 \qquad \textit{for} \quad 0 \le k \le n.$$

In fact, Nagai himself solved his conjecture when the fiber X of $\mathcal{X} \to \Delta^*$ is of $\mathrm{K3}^{[n]}$-type. The hope of the conjecture is to establish a relation between the second and k-th monodromy operators of a family of hyper-Kähler manifolds. In this sense, the following theorem discovered by Soldatenkov [Sol20] should be considered as a conceptually more important result than the conjecture itself. Soldatenkov's theorem is later reproved in [GKLR, Thm 4.9]. Both proofs crucially rely on the reduced LLV structure on the cohomology $H^k(X, \mathbb{Q})$. Recall that the reduced LLV structure was nothing but a $\bar{\mathfrak{g}}$-module structure on each cohomology $H^k(X, \mathbb{Q})$. Since $\bar{\mathfrak{g}} \cong \mathfrak{so}(\bar{V}, \bar{q})$, this meant each cohomology $H^k(X, \mathbb{Q})$ had the $\mathfrak{so}(\bar{V}, \bar{q})$-module structure

$$\rho_k : \mathfrak{so}(\bar{V}, \bar{q}) \to \mathfrak{gl}(H^k(X, \mathbb{Q})).$$

The monodromy operator is defined by a diffeomorphism of X, meaning it respects the Beauville–Bogomolov form $\bar{q}$. Hence $T_2 \in \mathrm{O}(\bar{V}, \bar{q})$ and $N_2 = \log T_2 \in \mathfrak{so}(\bar{V}, \bar{q})$.

Theorem 2.4.2 (Soldatenkov)**.** *In the setting of Conjecture 2.4.1, the k-th log monodromy operator N_k is determined by the second log monodromy operator N_2 via the reduced LLV structure map:*

$$N_k = \rho_k(N_2).$$

One may naturally wonder Conjecture 2.4.1 may be a formal consequence of this theorem. Surprisingly, the conjecture turns out to claim more. The following theorem reduces the conjecture into a numerical criterion on the dominant weights appearing in the LLV decomposition. Strictly speaking, since the conjecture is solely about the even cohomology, we need to consider the LLV decomposition of the even cohomology

$$H^*_{\mathrm{even}}(X, \mathbb{Q}) \cong \bigoplus_{\lambda} m_\lambda V_\lambda. \tag{2.4.3}$$

Proposition 2.4.4. (i) *If $b_2(X) \le 4$ then Conjecture 2.4.1 holds.*

(ii) *Assume $b_2(X) \ge 5$ and consider the condition*

$$\lambda_0 + \lambda_1 + \lambda_2 \le n \qquad \text{for all } \lambda \text{ appearing in the even LLV decomposition (2.4.3).} \tag{2.4.5}$$

Then (2.4.5) *implies Conjecture 2.4.1. Conversely, Conjecture 2.4.1 implies* (2.4.5) *if there exists a type II degeneration of X.*

A *type II degeneration* is a smooth projective family $\mathcal{X} \to \Delta^*$ of hyper-Kähler manifolds whose index of nilpotency of the second log monodromy is $\nu_2 = 1$. In other words, modulo the technical issue on the existence of a type II degeneration, Conjecture 2.4.1 is equivalent to the numerical criterion (2.4.5) about the LLV structure on $H^*_{\mathrm{even}}(X, \mathbb{Q})$. The results in §2.2 readily apply to verify such a numerical criterion.

Theorem 2.4.6. *Assume the fiber X of the family $\mathcal{X} \to \Delta^*$ is either of $K3^{[n]}$, Kum_n, OG10 or OG6-type. Then Conjecture 2.4.1 holds.*

The proof of Proposition 2.4.4 is almost purely representation theoretic, but with a single additional geometric input that the second log monodromy operator N_2 is compatible to the limit mixed Hodge structure of a K3-type Hodge structure $\bar{V} = H^2(X, \mathbb{Q})$. The main technical lemma is [GKLR, Lem 5.10], which uses this fact to obtain a preferred basis of $\bar{V}$ with which the computation can be easily done. This goes back to the idea of [FS86]. We will neither give a proof for Proposition 2.4.4 nor state the main technical lemma in the original paper.

Let us conclude this section with a further progress on Nagai's conjecture. The following is proved in [HMb].

Theorem 2.4.7 (Huybrechts–Mauri)**.** *Conjecture 2.4.1 holds for $k = n$.*

2.5 Digression: The second Betti number of hyper-Kähler manifolds

Nagai's conjecture is essentially equivalent to the cohomological condition (2.4.5). The condition is verified for all known deformation types of hyper-Kähler manifolds by Theorem 2.4.6. In fact, the following stronger condition is satisfied for all known deformation types of hyper-Kähler manifolds. Recall that we are using the notation $\lambda = (\lambda_0, \cdots, \lambda_r)$ to denote dominant weights of the LLV algebra $\mathfrak{g}_{\mathbb{C}}$. If $b_2(X) = 2r + 1$ is odd then λ_r is always nonnegative, but if $b_2(X) = 2r$ is even then λ_r may be negative. See Appendix A.

Conjecture 2.5.1. *Every dominant weight λ appearing in the LLV decomposition* (2.1.5) *satisfies*

$$\lambda_0 + \cdots + \lambda_{r-1} + |\lambda_r| \le n.$$

Theorem 2.5.2. *Conjecture 2.5.1 holds for $K3^{[n]}$, Kum_n, OG10 or OG6-type hyper-Kähler manifolds.*

Our feeling is that Conjecture 2.5.1 is quite a strong claim (for example, it implies Nagai's conjecture). The goal of this section is to present the following consequence of the conjecture to the second Betti number $b_2(X)$ of X. This is joint work with Radu Laza in [KL20].

Theorem 2.5.3. *Assume Conjecture 2.5.1 holds for a hyper-Kähler manifold X of dimension $2n$. Then its second Betti number $b_2(X) = \dim H^2(X, \mathbb{Q})$ is bounded above by*

$$b_2(X) \le \begin{cases} \frac{1}{2}\left(21 + \sqrt{96n + 433}\right) & \textit{if } \; H^*_{\text{odd}}(X, \mathbb{Q}) = 0, \\ 2k + 1 & \textit{if } \; H^k(X, \mathbb{Q}) \neq 0 \textit{ for some odd } k. \end{cases}$$

A weaker version of the bound can be written as

$$b_2(X) \le \max\left\{\tfrac{1}{2}\left(21 + \sqrt{96n + 433}\right),\ 4n - 1\right\},$$

or equivalently

n	1	2	3	4	5	6	7	≥ 8
$b_2(X) \le$	22	23	23	24	25	26	27	$4n - 1$

.

The idea is to attach a rational invariant to each $\mathfrak{g}$-modules and use Salamon's result. We briefly sketch the proof. Let E be any $\mathfrak{g}$-module and $h \in \mathfrak{g}$ a degree operator defined in (2.1.1). Consider the eigenspace decomposition $E = \bigoplus_{k\in\mathbb{Z}} E_k$ with respect to h. We attach to E a formal power series

$$S(E) = \sum_{k\in\mathbb{Z}} (-1)^k \dim E_k \cdot \exp(kt) \quad \in \mathbb{Q}[[t]]. \tag{2.5.4}$$

It is easy to show

$$S(E \oplus E') = S(E) + S(E'), \qquad S(E \otimes E') = S(E) \cdot S(E'), \tag{2.5.5}$$

meaning S defines a ring homomorphism from the representation ring of $\mathfrak{g}$ to $\mathbb{Q}[[t]]$. Moreover, it is easy to observe $S(E)$ does not have any odd degree term on t.

Definition 2.5.6. The *slope* $s(E)$ of a $\mathfrak{g}$-module E is twice the ratio between the constant and t^2-coefficients of $S(E) \in \mathbb{Q}[[t]]$ in (2.5.4). More explicitly, we define

$$s(E) = \frac{\sum_k (-1)^k k^2 \dim E_k}{\sum_k (-1)^k \dim E_k} \quad \in \mathbb{Q}.$$

The reader should be aware that the slope is undefined when E has the "Euler characteristic" 0. One nice thing about the notion of a slope is that we can restate the result of Salamon [Sal96] in the following better way:

Theorem 2.5.7 (Salamon)**.** *Let X be a hyper-Kähler manifold of dimension $2n$ and $\mathfrak{g}$ its LLV algebra. Assume the topological Euler characteristic of X is nonzero. Then the slope of the $\mathfrak{g}$-module $H^*(X,\mathbb{Q})$ is $\frac{n}{3}$.*

The core ingredient for the proof will be a computation of the slope for every irreducible $\mathfrak{g}$-module. This proved in [KL20, Thm 3.4] and we will not reproduce its proof here. Once we have this, the bound on the second Betti number is a formal consequence of the properties of the slope function.

Theorem 2.5.8. *Let V_λ be an irreducible $\mathfrak{g}$-module of highest weight $\lambda = (\lambda_0, \cdots, \lambda_r)$. If $\lambda_r \geq 0$ then*

$$s(V_\lambda) = 8 \cdot \frac{\left(\sum_{i=0}^r \lambda_i\right) b_2(X) + \left(\sum_{i=0}^r (\lambda_i - i)^2 - i^2\right)}{(b_2(X)+1)(b_2(X)+2)}.$$

If $\lambda_r < 0$, then $s(V_\lambda) = s(V_{\lambda'})$ where $\lambda' = (\lambda_0, \cdots, \lambda_{r-1}, -\lambda_r)$.

Proof of Theorem 2.5.3. We only consider the case when X does not have any odd cohomology. Start from the result $s(H^*(X,\mathbb{Q})) = \frac{n}{3}$ in Theorem 2.5.7. The additivity of (2.5.5) bounds it above by

$$\frac{n}{3} = s(H^*(X,\mathbb{Q})) \leq \max\{s(V_\lambda) : \lambda \text{ appearing in the LLV decomposition (2.1.5)}\}.$$

Among the irreducible components V_λ in the LLV decomposition, we always have the Verbitsky component $V_{(n)}$. Now Theorem 2.5.8 together with Conjecture 2.5.1 guarantees $V_{(n)}$ has the maximum slope among all V_λ's appearing in the LLV decomposition. We conclude

$$\frac{n}{3} = s(H^*(X,\mathbb{Q})) \leq s\big(V_{(n)}\big) = \frac{8n(b_2(X)+n)}{(b_2(X)+1)(b_2(X)+2)}.$$

The desired bound on $b_2(X)$ follows by solving the inequality. $\square$

Chapter 3

Relative automorphism scheme of a Lagrangian fibration

Introduction

Any flat projective morphism between algebraic varieties $Y \to B$ admits the notion of a relative automorphism scheme $\mathrm{Aut}_{Y/B} \to B$. It is a group scheme over B. In particular, a Lagrangian fibered *projective* hyper-Kähler manifold admits a group scheme $\mathrm{Aut}_{X/B} \to B$. [AF16] observed its neutral component $\nu : P \to B$ almost makes π a torsor under it. Unfortunately, we do not know this discussion on the automorphism scheme can be generalized into non-projective hyper-Kähler manifolds.

The main result Theorem 3.1.1 of this section uses Hodge theory to overcome this issue. It proves the smooth part $\pi : X_0 \to B_0$ of the Lagrangian fibration is a torsor under a unique polarized abelian scheme $\nu : P_0 \to B_0$. It is a much weaker result than Arinkin–Fedorov's in the projective setting, but has an advantage that it also works for the non-projective setting and has more contents on polarizations.

One consequence of the main theorem is the notion of a polarization type associated to π. This recovers the result of [Wie16]. The polarization type is deformation invariant on π (Theorem 3.3.1) and already computed for all known deformation types of hyper-Kähler manifolds (Theorem 3.2.3). We will also define the notion of a polarization scheme. The polarization scheme a priori contains additional monodromy information than the polarization type, but we conjecture this monodromy information is always trivial. This will be discussed in the next chapter and more specifically in Conjecture 4.3.2.

3.1 Abelian scheme associated to a Lagrangian fibration

Let us start with the main theorem of this section.

Theorem 3.1.1. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold, B_0 a Zariski open subset of B over which π is smooth, and $X_0 = \pi^{-1}(B_0)$. Then*

(i) *There exists a unique abelian scheme $\nu : P_0 \to B_0$ making $\pi : X_0 \to B_0$ an analytic torsor under ν. The abelian scheme P_0 is simple and projective.*

(ii) *P_0 admits a unique primitive polarization*

$$\lambda : P_0 \to \check{P}_0. \tag{3.1.2}$$

Definition 3.1.3. The abelian scheme $\nu : P_0 \to B_0$ in Theorem 3.1.1 is called the *abelian scheme associated to π*.

Our theorem is highly motivated by [AF16, Thm 2], which is stronger than our theorem when X is projective. The theorem is also motivated by [vGV16] and [Saw04]. We first prove Theorem 3.1.1 in the first subsection, and next talk about more properties and examples of the abelian scheme P_0 in the following subsections. The reader should be aware that π is only an analytic torsor in general. If we further assume X is projecitve then π becomes an étale torsor by Proposition 1.2.7.

3.1.1 Proof of the theorem

Recall from Theorem 1.2.13 that every smooth fiber F of π is an abelian variety. It would be helpful to first reproduce the proof of this fact. The key idea is the following cohomological lemma, which has been discovered several times independently in [Voi92, Ogu09a, Mat16] and recently generalized into higher degree cohomologies by [SY22]. We copy Matsushita's proof here.

Lemma 3.1.4 (Matsushita)**.** *Let F be any smooth fiber of π and $h \in H^2(X, \mathbb{Z})$ the cohomology class of $\pi^*\mathcal{O}_B(1)$. Then the restriction map*

$$-_{|F} : H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$$

has $\ker(-_{|F}) = h^{\perp}$. Consequently, it has $\operatorname{im}(-_{|F}) \cong \mathbb{Z}$.

Proof. The smooth fiber F is a complex torus. Thus both $H^2(X,\mathbb{Z})$ and $H^2(F,\mathbb{Z})$ are torsion-free, so we may prove the claim over $\mathbb{R}$. Also, if $\sigma \in H^{2,0}(X)$ is a holomorphic symplectic form then its restriction $\sigma_{|F}$ vanishes because F is Lagrangian. Hence we may prove the statement for the real $(1,1)$-classes.

Fix any Kähler class $\omega \in H^{1,1}(X,\mathbb{R})$ of X, so that its restriction $\omega_{|F}$ becomes a Kähler class on F. By the hard Lefschetz theorem combined with the Hodge–Riemann bilinear relation, a cohomology class $y \in H^{1,1}(F,\mathbb{R})$ is zero if and only if $\int_F y(\omega_{|F})^{n-1} = 0$ and $\int_F y^2(\omega_{|F})^{n-2} = 0$. When $y = x_{|F}$ is a restriction of a class $x \in H^{1,1}(X,\mathbb{R})$, we are ready to use the Fujiki relation

$$\begin{aligned}\int_F x_{|F}(\omega_{|F})^{n-1} &= \int_X x\omega^{n-1}h^n = (\text{const.})\bar{q}(x,h)\bar{q}(\omega,h)^{n-1},\\ \int_F (x_{|F})^2(\omega_{|F})^{n-2} &= \int_X x^2\omega^{n-2}h^n = (\text{const.})\bar{q}(x,h)^2\bar{q}(\omega,h)^{n-2}.\end{aligned}$$

Since $\bar{q}(\omega,h) > 0$, $x_{|F} = 0$ if and only if $\bar{q}(x,h) = 0$. □

Proposition 3.1.5 (Voisin)**.** *The image of the restriction map $-_{|F}$ in Lemma 3.1.4 is generated by an ample class of F. As a result, F is an abelian variety.*

Proof. Say y is an integral generator of Lemma 3.1.4. Choose any Kähler class $\omega \in H^2(X,\mathbb{R})$ and consider its restriction $\omega_{|F}$, a Kähler class on F. It has to be a nonzero real multiple of y. This means, up to sign, y has to be a Kähler class on F. Hence y is an integral Kähler class, so it is ample. □

We caution the reader to be aware that the ample generator y of the image of the restriction map need not be primitive. This issue will be later discussed in Proposition 3.2.6. Our choice of the polarization will be a unique primitive ample class in $H^2(F,\mathbb{Z})$ parallel to y, so it may not be contained in the image of the restriction map $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ over $\mathbb{Z}$.

We divide the proof of Theorem 3.1.1 into three parts: (1) an explicit construction of the polarized abelian scheme P_0, (2) proving such a construction makes X_0 a torsor under P_0, and finally (3) its uniqueness. The uniqueness should be a more general fact about arbitrary torsors, at least in the algebraic case (see Moret-Bailly's answer in [MB]). The construction of P_0 works for any proper family of complex tori. The uniqueness of the polarization is the only part that needs the fact X_0 is obtained from a Lagrangian fibered hyper-Kähler manifold X.

Proof of Theorem 3.1.1, construction. The following is presented in [vGV16], and we reproduce their argument here. Apply the global invariant cycle theorem (for proper maps between compact Kähler manifolds [Del71]) and Lemma 3.1.4 to obtain

$$H^0(B_0, R^2\pi_*\mathbb{Q}) = \mathrm{im}(H^2(X,\mathbb{Q}) \to H^2(F,\mathbb{Q})) \cong \mathbb{Q}.$$

Hence, there exists a unique homomorphism $(R^2\pi_*\mathbb{Q})^\vee \to \mathbb{Q}$ of local systems on B_0 up to scalar. This is a homomorphism of $\mathbb{Q}$-VHS: fiberwise, Proposition 3.1.5 proves the image of $H^2(X,\mathbb{Q}) \to H^2(F,\mathbb{Q})$ is an ample class. Restrict it to the morphism of $\mathbb{Z}$-VHS $(R^2\pi_*\mathbb{Z})^\vee \to \mathbb{Z}$. The morphism can be uniquely determined once we assume it to be primitive and represents an ample class on each fiber. Finally, use the fact that $\pi : X_0 \to B_0$ is a family of complex tori (abelian varieties) and obtain an isomorphism $R^2\pi_*\mathbb{Z} = \wedge^2 R^1\pi_*\mathbb{Z}$. The result is a primitive polarization

$$(R^1\pi_*\mathbb{Z})^\vee \otimes (R^1\pi_*\mathbb{Z})^\vee \to \mathbb{Z}. \tag{3.1.6}$$

We have constructed a weight -1 $\mathbb{Z}$-VHS $(R^1\pi_*\mathbb{Z})^\vee$ equipped with a polarization (3.1.6). Now use a formal equivalence of categories between polarized weight -1 $\mathbb{Z}$-VHS and that of polarized abelian schemes (e.g., [Del72, §5.2] [Del71, §4.4]). This constructs our desired abelian scheme $\nu : P_0 \to B_0$ with a unique primitive polarization $\lambda : P_0 \to \check{P}_0$ over B_0. To prove P_0 is simple, we may prove the corresponding VHS $R^1\pi_*\mathbb{Q}$ is simple. This is tacitly proved in [vGV16] and later explicitly stated in [Voi18, Lem 4.5]. We omit its proof here. □

Proof of Theorem 3.1.1, torsor. Consider an analytic open covering $\{B_i : i \in I\}$ of B_0 so that over each B_i, the restriction of the Lagrangian fibration $\pi : X_i \to B_i$ admits at least one holomorphic section $s_i : B_i \to X_i$. Considering s_i as a zero section, $\pi : X_i \to B_i$ becomes an abelian scheme. Hence by the equivalence of abelian schemes and $(R^1\pi_*\mathbb{Z})^\vee$, ν and π are isomorphic over B_i by $\phi_i : X_i \to P_i$ sending s_i to the zero section of P_i.

Use the isomorphism ϕ_i to transform the group law $+ : P_i \times_{B_i} P_i \to P_i$ into a P_i-action on X_i. That is, we define a group action morphism by

$$\rho_i : P_i \times_{B_i} X_i \to X_i, \qquad (p_i, x_i) \mapsto \phi_i^{-1}(\phi_i(x_i) + p_i).$$

We want to patch ρ_i together to define a group action $\rho : P_0 \times_{B_0} X_0 \to X_0$ over the entire B_0. To do so, we need to check whether the definitions of ρ_i and ρ_j coincides over the intersection $B_{ij} = B_i \cap B_j$, i.e.,

$$\phi_i^{-1}(\phi_i(x_{ij}) + p_{ij}) = \phi_j^{-1}(\phi_j(x_{ij}) + p_{ij}) \qquad \text{for all} \quad (p_{ij}, x_{ij}) \in P_{ij} \times_{B_{ij}} X_{ij}. \tag{3.1.7}$$

Over B_{ij}, one has a transition function $\phi_j \circ \phi_i^{-1} : P_{ij} \to X_{ij} \to P_{ij}$, an automorphism of P_{ij}. Recall that the isomorphisms ϕ_i and ϕ_j are constructed by choosing the zero sections s_i and s_j, and the corresponding isomorphisms $\phi_i : X_{ij} \cong P_{ij}$ and $\phi_j : X_{ij} \cong P_{ij}$ are as abelian schemes. From it, we notice the automorphism $\phi_j \circ \phi_i^{-1} : P_{ij} \to P_{ij}$ is a *translation* automorphism. The translation is by $\phi_j \circ \phi_i^{-1}(0)$, the difference of the two zero sections. With this, we have a sequence of identities

$$
\begin{aligned}
\phi_j(x_{ij}) + p_{ij} = \phi_j \circ \phi_i^{-1}(\phi_i(x_{ij})) + p_{ij} &= \big(\phi_i(x_{ij}) + \phi_j \circ \phi_i^{-1}(0)\big) + p_{ij} \\
&= \big(\phi_i(x_{ij}) + p_{ij}\big) + \phi_j \circ \phi_i^{-1}(0) = \phi_j \circ \phi_i^{-1}\big(\phi_i(x_{ij}) + p_{ij}\big).
\end{aligned}
$$

This proves (3.1.7). Hence ρ_i patches together and defines a morphism $\rho : P_0 \times_{B_0} X_0 \to X_0$. The group action axioms are all easily verified. Also, X_0 is clearly a P_0-torsor by construction. □

Proof of Theorem 3.1.1, uniqueness. Let $\nu : P_0 \to B_0$ be a (not necessarily projective) abelian scheme so that π becomes a torsor under ν. We claim $R^1\nu_*\mathbb{Z} \cong R^1\pi_*\mathbb{Z}$ as VHS over B_0. Consider the group scheme action map

$$
\begin{tikzcd}
P_0 \times_{B_0} X_0 \arrow[r, "\rho"] \arrow[dr, "\mu"'] & X_0 \arrow[d, "\pi"] \\
& B_0
\end{tikzcd} .
$$

From the diagram, we have a pullback morphism between the VHS $\rho^* : R^1\pi_*\mathbb{Z} \to R^1\mu_*\mathbb{Z}$. The latter VHS is isomorphic to the direct sum $R^1\nu_*\mathbb{Z} \oplus R^1\pi_*\mathbb{Z}$ by the Künneth formula (e.g., [Ive86, VII.2.7]) and the decomposition theorem for smooth proper morphisms. Hence composing with the first projection, we obtain a morphism $R^1\pi_*\mathbb{Z} \to R^1\nu_*\mathbb{Z}$. Now over a small analytic open subset $U \subset B_0$, fix any holomorphic section of $\pi : X_U \to U$ so that we can identify P_U and X_U. Hence ρ becomes the addition operation of the abelian scheme $X_U \times_U X_U \to X_U$. With this description, the pullback morphism is fiberwise $\rho^* : H^1(F, \mathbb{Z}) \to H^1(F, \mathbb{Z}) \oplus H^1(F, \mathbb{Z})$, $x \mapsto (x, x)$. Hence the morphism $R^1\pi_*\mathbb{Z} \to R^1\nu_*\mathbb{Z}$ is an isomorphism over U, and the claim follows. □

3.1.2 More properties of the abelian scheme

The abelian scheme $\nu : P_0 \to B_0$ should be considered as the neutral component of the automorphism scheme $\mathrm{Aut}_{X/B} \to B$ restricted to B_0. This was the original construction of [AF16] when X is projective. (Note: for the construction of the automorphism scheme for flat projective morphisms, see [Nit05, Thm 5.23]. Keep in mind that π is flat by Proposition 1.2.3.) Unfortunately, if X is non-projective then we do not know a general construction of the automorphism scheme for flat proper morphisms.[1]

Over a smaller base B_0, one can simply avoid this technical difficulty by considering a smaller subsheaf of *translation* automorphisms

$$\underline{\mathrm{Aut}}^{tr}_{X_0/B_0}(U) = \{f : X_U \to X_U : U\text{-automorphism acting by translation on each fibers}\}. \tag{3.1.8}$$

It can be easily showed P_0 represents this sheaf, giving another interpretation for the associated abelian scheme P_0. The following proposition collects the characterizations of P_0.

Proposition 3.1.9. *We have the following three different characterizations of the associated abelian scheme $\nu : P_0 \to B_0$.*

(i) *It represents the translation automorphism sheaf $\underline{\mathrm{Aut}}^{tr}_{X_0/B_0}$ in* (3.1.8)*.*
(ii) *It is a unique abelian scheme associated to the weight -1 VHS $(R^1\pi_*\mathbb{Z})^\vee$ on B_0.*
(iii) *It is a dual abelian scheme of the relative Picard scheme $\mathrm{Pic}^0_{X_0/B_0} \to B_0$.*

The following proposition was first observed in [Ogu09a] in a special case when π admits a (rational) section.

Proposition 3.1.10 (Oguiso)**.** *The generic fiber of the associated abelian scheme $\nu : P_0 \to B_0$ has the Picard number* 1*.*

Proof. Say L is the function field of B and $P_L \to \operatorname{Spec} L$ is the generic fiber of ν, which is an abelian variety over L. Since P_0 has a unique polarization (up to scalar), P_L has a unique polarization. Note that the ampleness is an open condition in $\mathrm{NS}(P_L)_{\mathbb{R}}$. Hence the uniqueness of the polarization implies $\rho(P_L) = 1$. □

Proposition 3.1.11. *Assume π admits a rational section. Then $\nu : P_0 \to B_0$ is in fact isomorphic to $\pi : X_0 \to B_0$.*

Proof. The rational section must be defined over B_0 by Proposition 1.2.8. Hence X_0 becomes a trivial P_0-torsor. □

Proposition 3.1.12. *Any rational section of $\nu : P_0 \to B_0$ can be uniquely extended to an honest section.*

Proof. The proof is identical to Proposition 1.2.8. An alternative proof can be found in [BLR90, Cor 8.4.6]. □

[1] That is, we do not know the representability of the automorphism sheaf. We could not find a reference for the construction of the "relative Douady space" of a flat proper morphism $X \to B$.

3.1.3 Examples of the associated abelian schemes

We provide three examples describing Theorem 3.1.1. Notice that in all the given examples the morphism $\nu : P_0 \to B_0$ is extendable, often into a Lagrangian fibration of another hyper-Kähler manifold.

Example 3.1.13. Let $\pi : X \to B = \mathbb{P}^1$ be an elliptic K3 surface and $\nu : P_0 \to B_0$ the associated abelian scheme over the smooth locus B_0. The theory of elliptic K3 surfaces further claims we can uniquely extend this to the Néron model (semi-abelian scheme) $P \to B$ or even to an elliptic K3 surface $\pi' : X' \to B$. The new elliptic K3 surface π' always admits a section, and is called the relative Jacobian fibration construction of π (e.g., [Huy16, §11.4]). Therefore, Theorem 3.1.1 is far from being its most general form. ■

Example 3.1.14. Start from an elliptic K3 surface $f : S \to \mathbb{P}^1$ and construct $\pi : S^{[n]} \to (\mathbb{P}^1)^{(n)}$ as in Example 1.3.10. Say $V \subset \mathbb{P}^1$ is the smooth locus of f and define

$$U = \{b = (b_1, \cdots, b_n) \in V^{(n)} : b_i \neq b_j \text{ for all } i \neq j\} \subset (\mathbb{P}^1)^{(n)},$$

a Zariski open subset of the base $B = (\mathbb{P}^1)^{(n)}$. At $b \in U$, the Lagangian fibration fibration has an abelian variety fiber $\pi^{-1}(b) = S_{b_1} \times \cdots \times S_{b_n}$. This means $U \subset B_0$.

Consider the relative Jacobian fibration construction $f' : S' \to \mathbb{P}^1$ in the previous example. Apply the same construction and yield *another* Lagrangian fibered hyper-Kähler manifold $\pi' : (S')^{[n]} \to (\mathbb{P}^1)^{(n)}$. Note that π' has a rational section defined at least over U. Hence $P_U = (\pi')^{-1}(U) \to U$ is an abelian scheme and makes $X_U = \pi^{-1}(U)$ a torsor under it. By the uniqueness of the associated abelian scheme, we know $P_U \to U$ is the restriction of $P_0 \to B_0$ over U.[2] We can again see there is at least one compactification of the associated abelian scheme into a new Lagrangian fibered hyper-Kähler manifold $(S')^{[n]} \to (\mathbb{P}^1)^{(n)}$ with a rational section. ■

Example 3.1.15. Let $\pi : X \to B = |L|$ be as in Example 1.3.11. The torus fibration $\pi : X_0 \to B_0$ is isomorphic to a relative Jacobian $\mathrm{Pic}^m_{\mathcal{C}/B_0} \to B_0$ for $m = s + n - 1$. The relative Jacobian $\mathrm{Pic}^m_{\mathcal{C}/B_0}$ is a torsor under the neutral relative Jacobian $\mathrm{Pic}^0_{\mathcal{C}/B_0}$ (e.g., [BLR90, Thm 9.3.1]). By the uniqueness of the associated abelian scheme, P_0 is $\mathrm{Pic}^0_{\mathcal{C}/B_0}$.

Let us now apply the same construction Example 1.3.11 to a different Mukai vector $v' = (0, l, -n + 1)$ instead. If l is not divisible by any d with $d^2 \mid n - 1$ then v' is primitive effective, so this yields a new Lagrangian fibered hyper-Kähler manifold $\pi' : X' \to B$ that

[2] Is $U = B_0$?

compactifies the associated abelian scheme $\mathrm{Pic}^0_{\mathcal{C}/B_0} \to B_0$. If l is divisible by some d with $d^2 \mid n-1$, then the Mukai vector $v = (0, l, -n+1)$ is not primitive anymore. Nevertheless, the moduli space X' still becomes a normal projective and singular symplectic variety by [KLS06]. Hence we can still compactify the abelian scheme $P_0 \to B_0$ to a Lagrangian fibration of a singular symplectic variety. ■

3.2 The polarization scheme and polarization type

Given any Lagrangian fibered hyper-Kähler manifold $\pi : X \to B$, we have constructed a unique abelian scheme $\nu : P_0 \to B_0$ with a unique polairzation (3.1.2). The kernel of this polarization is thus uniquely determined.

Definition 3.2.1. The *polarization scheme* of π is the kernel

$$K = \ker\left(\lambda : P_0 \to \check{P}_0\right)$$

of the unique primitive polarization λ of the associated abelian scheme P_0 in (3.1.2).

Since the morphism λ is finite and étale, its kernel $K \subset P_0$ is a finite étale commutative group scheme over B_0. Any finite étale group scheme is characterized by: (1) a single fiber of K as a finite group, and (2) the monodromy information. We conjecture that the monodromy information for the polarization scheme K is redundant, i.e., K is a constant group scheme. This conjecture will be part of the more refined Conjecture 4.3.2 in the next chapter.[3]

A single fiber of K is the kernel of a polarization on a single abelian variety F. It is therefore necessarily of the form $(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2}$. This defines an invariant of a Lagrangian fibration π.

Definition 3.2.2. The *polarization type* of π is an n-tuple of positive integers $(d_1, \cdots, d_n)$ with $d_1 \mid \cdots \mid d_n$ such that the fibers of the polarization scheme are isomorphic to $(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2}$.

The definition of the polarization type first arose in [Wie16]. We note that our definition is equivalent to this original definition (see [Kim21, §3.2]). We will see in §3.3 that the polarization type is invariant under deformations of π. Note also that we have $d_1 = 1$ because our polarization λ is primitive.

[3] In a different direction, we expect K is extendable to a group scheme over a subset $B_1 \subset B \cong \mathbb{P}^n$ of codimension 2 complement. If this happens, simple connectedness of B_1 implies the monodromy is automatically trivial and hence K becomes a constant group scheme.

The polarization type is already computed for all known deformation types of hyper-Kähler manifolds. The computation is done for $K3^{[n]}$ and Kum_n-types in [Wie16, Wie18], for OG10-type in [MO22] and for OG6-type in [MR21]. To state the result, recall from (1.2.4) and (1.2.5) that we have defined a cohomology class $h = c_1\big(\pi^*\mathcal{O}_B(1)\big) \in H^2(X,\mathbb{Z})$ and its divisibility $\operatorname{div}(h) = \gcd\{q(h,x) : x \in H^2(X,\mathbb{Z})\}$.

Theorem 3.2.3 (Wieneck, Mongardi–Onorati, Mongardi–Rapagnetta)**.** *Let $\pi : X \to B$ be a Lagrangian fibered compact hyper-Kähler manifold. Then the polarization type of π is*

$$\begin{cases} (1,\cdots,1) & \textit{if } X \textit{ is of } K3^{[n]}\textit{-type}, \\ (1,1,1,1,1) & \textit{if } X \textit{ is of OG10-type}, \\ (1,\cdots,1,d_1,d_2) & \textit{if } X \textit{ is of } Kum_n\textit{-type}, \\ (1,2,2) & \textit{if } X \textit{ is of OG6-type}. \end{cases}$$

When X is of Kum_n-type, we set $d_1 = \operatorname{div}(h)$ in (1.2.5) *and $d_2 = \frac{n+1}{d_1}$.*

Comparing the result with §1.3, one immediately observes the degree of the polarization $d_1 \cdots d_n$ coincides with the Fujiki constant c_X for all known deformation types of hyper-Kähler manifolds.

Conjecture 3.2.4. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold and $(d_1,\cdots,d_n)$ its polarization type. Then the Fujiki constant c_X of X is precisely $d_1\cdots d_n$.*

We were not able to prove this conjecture in general, but we could still say the following.

Proposition 3.2.5. *We have $c_X = s^n \cdot d_1 \cdots d_n$ for some $s \in \mathbb{Q}$.*

Proof. Choose any cohomology class $x \in H^2(X,\mathbb{Z})$ with $q(h,x) = a \neq 0$. By Lemma 3.1.4, the class $x_{|F} \in H^2(F,\mathbb{Z})$ must be a positive integer multiple of the primitive polarization class θ. Set $x_{|F} = b\theta$. The claim follows from the Fujiki relation

$$d_1\cdots d_n = \frac{1}{n!}\int_F \theta^n = \frac{1}{n!}\int_X h^n\big(\tfrac{1}{b}x\big)^n = c_X \cdot q\big(h,\tfrac{1}{b}x\big)^n = c_X\left(\frac{a}{b}\right)^n. \qquad \square$$

3.2.1 Divisibility of the cohomology class h

The class $h \in H^2(X,\mathbb{Z})$ and its divisibility $\operatorname{div}(h)$ in (1.2.5) is a topological invariant attached to a Lagrangian fibration π. We claim $\operatorname{div}(h)$ captures the non-primitiveness of the restriction map $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ in Lemma 3.1.4. Unfortunately, our proof depends on the validity of Conjecture 3.2.4. Since Conjecture 3.2.4 holds for known deformation types, the following proposition applies to the known deformation types.

Proposition 3.2.6. *Suppose we have $c_X = d_1 \cdots d_n$. Then the image of the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is generated by $a\theta$, where $a = \operatorname{div}(h)$ and θ is a primitive ample class representing the canonical polarization of F.*

Proof. Use the same method in Proposition 3.2.5. $\square$

The divisibility of h is also a topological obstruction to the existence of a rational section of π.

Proposition 3.2.7. *Assume $c_X = d_1 \cdots d_n$ and π admits at least one rational section. Then* $\operatorname{div}(h) = 1$ *or* 2.

Proof. If π admits a rational section, then $X_0 \cong P_0$ becomes a projective abelian scheme (Proposition 3.1.11). By the general theory of abelian schemes, twice a polarization is always associated to a line bundle (e.g., [MFK94, Prop 6.10] or [FC90, Def I.1.6]). This means $2\theta \in H^2(F, \mathbb{Z})$ is contained in the image of $\operatorname{Pic}(X) \subset H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$. By Proposition 3.2.6, this implies $\operatorname{div}(h) = 1$ or 2. $\square$

If X is of $\mathrm{K3}^{[n]}$ or Kum_n-type then its Lagrangian fibration $\pi : X \to B$ may have $\operatorname{div}(h) > 2$. See [Mar14] and [Wie18]. In such cases, π (and any of its deformation) would never admit any rational section.

3.3 Deformation invariance of the polarization type

Deformation invariance of the polarization type is first proved in [Wie16]. The goal of this section is to recover Wieneck's result using the language of sheaves and variation of Hodge structures.

Theorem 3.3.1 (Wieneck)**.** *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then the polarization type of π is invariant under deformations of π.*

The rest of this section is devoted to the proof of the theorem. To start our proof, let us recall from Proposition 3.1.9 that the abelian scheme P_0 is associated to the VHS $(R^1\pi_*\mathbb{Z})^\vee$ on B_0. The polarization λ corresponds to a polarization on the VHS $(R^1\pi_*\mathbb{Z})^\vee$. An injective morphism $(R^1\pi_*\mathbb{Z})^\vee \to R^1\pi_*\mathbb{Z}$ of VHS is induced, so we can consider its cokernel

$$0 \longrightarrow (R^1\pi_*\mathbb{Z})^\vee \longrightarrow R^1\pi_*\mathbb{Z} \longrightarrow \underline{K} \longrightarrow 0 \ . \tag{3.3.2}$$

Note that $\underline{K}$ is a local system of finite groups on the base B_0. In fact, it is the sheaf of analytic sections of the polarization scheme K.

The polarization type is thus modeled by a local system $\underline{K}$. We claim we can deform this local system $\underline{K}$. Let us thus consider a family of Lagrangian fibered hyper-Kähler manifolds $p : \mathcal{X} \xrightarrow{\pi} \mathcal{B} \xrightarrow{q} \Delta$ over an open disc Δ. Set $\mathcal{B}_0 \subset \mathcal{B}$ the smooth locus of π. For each $t \in \Delta$, we have a Lagrangian fibration $\pi : X_t \to B_t$ and its smooth locus $(B_t)_0$. The above process attaches a local system $\underline{K}_t$ on $(B_t)_0$.

Lemma 3.3.3. *There exists a local system $\underline{\mathcal{K}}$ on $\mathcal{B}_0$ parametrizing $\underline{K}_t$ on $(B_t)_0$ for every $t \in \Delta$.*

Proof. Let $\mathcal{X}_0 = \pi^{-1}(\mathcal{B}_0)$ be the preimage of $\mathcal{B}_0$ and let us temporarily call the restriction $\mathcal{X}_0 \to \mathcal{B}_0$ of π simply the same letter π. Therefore, throughout this lemma π is a smooth proper family of abelian varieties. Consider the local system $R^2\pi_*\mathbb{Z}$ on $\mathcal{B}_0$. Our first claim is $H^0(\mathcal{B}_0, R^2\pi_*\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Denoting by $j : \mathcal{B}_0 \to \mathcal{B}$ an open immersion, we claim

$$H^0(\mathcal{B}_0, R^2\pi_*\mathbb{Z}) = H^0(\Delta, q_*j_*R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$$

Notice that $q_*j_*R^2\pi_*\mathbb{Z}$ is a constructible sheaf on Δ, because $R^2\pi_*\mathbb{Z}$ is a local system on $\mathcal{B}_0$, its pushforward by j_* is a constructible sheaf on $\mathcal{B}$ (e.g., [KS90, Ex VIII.10]), and again its pushforward by q_* is a constructible sheaf. For each $t \in \Delta$, we have a Lagrangian fibered hyper-Kähler manifold $\pi : X_t \to B_t$ and we may apply our previous discussions in Proof of Theorem 3.1.1

$$H^0((B_t)_0, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}.$$

This proves every fiber of $q_*j_*R^2\pi_*\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. In this setting, we will abstractly prove the sheaf has $\mathbb{Z}$ global sections in Lemma 3.3.4. This proves the claim $H^0(\mathcal{B}_0, R^2\pi_*\mathbb{Z}) \cong \mathbb{Z}$.

We have a unique primitive morphism $(R^2\pi_*\mathbb{Z})^\vee \to \mathbb{Z}$ of local systems on $\mathcal{B}_0$. As π is a family of abelian varieties over $\mathcal{B}_0$, we have an isomorphism $R^2\pi_*\mathbb{Z} = \wedge^2 R^1\pi_*\mathbb{Z}$. This gives us a unique primitive morphism of local systems (in fact, a polarization of VHS by Proposition 3.1.5) $(R^1\pi_*\mathbb{Z})^\vee \otimes (R^1\pi_*\mathbb{Z})^\vee \to \mathbb{Z}$ over $\mathcal{B}_0$. This induces a morphism $(R^1\pi_*\mathbb{Z})^\vee \to R^1\pi_*\mathbb{Z}$ whose cokernel $\underline{\mathcal{K}}$ is a local system on $\mathcal{B}_0$, parametrizing $\underline{K}_t$ for each $t \in \Delta$. □

Lemma 3.3.4. *Let F be a constructible sheaf on a complex open disc Δ. If every fiber of F is isomorphic to $\mathbb{Z}$, then we have $H^0(\Delta, F) \cong \mathbb{Z}$.*

Proof. Since F is constructible, $F_{|U}$ is a local system on a completement U of a finite set of points $t_1, \cdots, t_k \in \Delta$. Let $U_i \subset U$ be a small punctured disc around t_i. The restriction $F_{|U_i}$ is determined by the representation

$$\rho_i : \mathbb{Z} \cong \pi_1(U_i) \to \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}.$$

We have only two possibilities $\rho_i(1) = \pm 1$ for each i. Suppose we have $\rho_i(1) = -1$ for some i. Consider the total space $f : \mathrm{Et}(F) \to \Delta$ of the entire constructible sheaf F (espace étalé). The map f is holomorphic and étale, i.e., a local isomorphism. The condition $\rho_i(1) = -1$ geometrically translates to the fact that $f^{-1}(U_i)$ consists of a single copy of U_i (the zero section) and infinite number of two-sheeted coverings of the punctured disc U_i. By the very assumption, the preimage $f^{-1}(t_i) = \{p_1, p_2, \cdots\}$ should be isomorphic to $\mathbb{Z}$. Since f is a local isomorphism, there should be an open disc neighborhood of each $p_i \in \mathrm{Et}(F)$. Along the two-sheeted coverings of U_i in $f^{-1}(U_i)$, this cannot happen. Therefore, the only possibility is that all p_i are the non-Hausdorff points filling in the unique punctured disc component in $f^{-1}(U_i)$ (i.e., the zero section). Hence we obtain at least $\mathbb{Z}$ global sections around the zero section and we are done.

The remaining case is when $\rho_i(1) = 1$ for all i. This means $F_{|U}$ is a constant sheaf $\mathbb{Z}$. The reader should be aware that this does not imply F is a constant sheaf $\mathbb{Z}$ on Δ. This can be again conveniently seen in the total space $f : \mathrm{Et}(F) \to \Delta$. Although f is a local homeomorphism, it is not a covering space unless $\mathrm{Et}(F)$ is Hausdorff. Indeed, the fibers $f^{-1}(t_i)$ can consist of non-Hausdorff points in $\mathrm{Et}(F)$ and this gives us a classification of such a constructible sheaf F. In any case, there are always $\mathbb{Z}$ global sections. $\square$

The desired Theorem 3.3.1 clearly follows from Lemma 3.3.3, because the polarization type was nothing but a single fiber of the local system $\underline{K}$.

Chapter 4

The H^2-trivial automorphisms

Introduction

An H^2*-trivial automorphism* of a hyper-Kähler manifold X is a biholomorphic map $f : X \to X$ whose f^*-action on $H^2(X, \mathbb{Z})$ is trivial. The group of all H^2-trivial automorphisms $\mathrm{Aut}^\circ(X)$ has been studied since the very beginning of the hyper-Kähler theory [Bea83a, Huy99]. Let us assume further X admits a Lagrangian fibration $\pi : X \to B$. We denote by $\mathrm{Aut}^\circ(X/B)$ the subgroup of $\mathrm{Aut}^\circ(X)$ consisting of H^2-trivial automorphisms respecting the Lagrangian fibration π. The goal of this section is to study this new group. It will play a crucial role in our construction of the dual Lagrangian fibration of hyper-Kähler manifolds.

The group $\mathrm{Aut}^\circ(X/B)$ is a finite abelian group and is invariant under deformations of π (Theorem 4.1.4). Our main conjecture 4.3.2 is that the group $\mathrm{Aut}^\circ(X/B)$ is big enough to contain the polarization scheme introduced in §3.2. We could not prove the conjecture in full generality, but will present some partial results in §4.3. These partial results will be used in Chapter 5 to verify the conjecture for all currently known deformation types of hyper-Kähler manifolds. We also present an explicit computation of $\mathrm{Aut}^\circ(X/B)$ for all known deformation types (Theorem 4.4.2).

Along the way to achieve the results above, we introduce two separate statements that can be of independent interest. The first is a canonical descent of the $\mathrm{Aut}^\circ(X)$-action on X to the base B of the Lagrangian fibration (Proposition 4.1.5). The second is a general method of constructing H^2-trivial automorphisms using Albanese fibrations in §4.2.

4.1 Deformation invariance of H^2-trivial automorphisms

Let X be a hyper-Kähler manifold. Beauville in his early paper [Bea83a] already defined the group of H^2*-trivial automorphisms*

$$\mathrm{Aut}^\circ(X) = \ker\big(\mathrm{Aut}(X) \to \mathrm{O}(H^2(X,\mathbb{Z}),\bar{q}), \quad f \mapsto (f^*)^{-1}\big). \tag{4.1.1}$$

Here $\mathrm{Aut}(X)$ is the group of biholomorphic automorphisms of X. Huybrechts [Huy99, Prop 9.1] together with Hassett–Tschinkel [HT13, Thm 2.1] proved that $\mathrm{Aut}^\circ(X)$ is a finite group and is invariant under deformations of X.

Theorem 4.1.2 (Huybrechts, Hassett–Tschinkel)**.** *Let X be a hyper-Kähler manifold. Then the group* $\mathrm{Aut}^\circ(X)$ *in* (4.1.1) *is a finite group invariant under deformations of X.*

Let us further assume X admits a Lagrangian fibration $\pi : X \to B$. We can then restrict our attention to H^2-trivial automorphisms that respect the Lagrangian fibration

$$\mathrm{Aut}^\circ(X/B) = \mathrm{Aut}(X/B) \cap \mathrm{Aut}^\circ(X). \tag{4.1.3}$$

This group is finite because it is a subgroup of $\mathrm{Aut}^\circ(X)$. The goal of this first section is to settle a similar result for $\mathrm{Aut}^\circ(X/B)$.

Theorem 4.1.4. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then the group* $\mathrm{Aut}^\circ(X/B)$ *in* (4.1.3) *is a finite abelian group invariant under deformations of π.*

The rest of this section will be devoted to the proof of this theorem. In fact, we will only prove the deformation invariance of $\mathrm{Aut}^\circ(X/B)$ in this section, and will prove it is abelian later in Proposition 4.3.8. The first main idea of the proof is to descend the $\mathrm{Aut}^\circ(X)$-action to the base B. This can be of independent interest, so let us also make it as a separate proposition.

Proposition 4.1.5. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then*

(i) *There exists a canonical* $\mathrm{Aut}^\circ(X)$*-action on B making π an* $\mathrm{Aut}^\circ(X)$*-equivariant morphism.*
(ii) *Under this descent,* $\mathrm{Aut}^\circ(X/B) = \ker\big(\mathrm{Aut}^\circ(X) \to \mathrm{Aut}(B)\big)$.

The second main idea is to use the language of sheaves and representability of them by complex manifolds. Before getting into the proofs of Theorem 4.1.4 and 4.1.5, let us change our problems into the sheaf language.

4.1.1 Sheaf of automorphisms

Suppose we have a family of hyper-Kähler manifolds $p : \mathcal{X} \to S$ over a complex manifold S. Let $U \subset S$ be an analytic open subset and denote by $p : \mathcal{X}_U = p^{-1}(U) \to U$ the restricted family over U. We define the sheaf of H^2-trivial automorphism groups $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ on S by

$$\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}(U) = \{f : \mathcal{X}_U \to \mathcal{X}_U : U\text{-automorphism such that } f^* : R^2p_*\mathbb{Z} \to R^2p_*\mathbb{Z} \text{ is the identity}\}.$$

By Theorem 4.1.2, this sheaf is a local system of finite groups, parametrizing $\mathrm{Aut}^\circ(X_t)$ for $t \in S$. Similarly, given a family of Lagrangian fibered hyper-Kähler manifolds, we can define a sheaf parametrizing $\mathrm{Aut}^\circ(X_t/B_t)$:

Definition 4.1.6. Given a family of Lagrangian fibered hyper-Kähler manifolds $p : \mathcal{X} \xrightarrow{\pi} \mathcal{B} \xrightarrow{q} S$, we define a sheaf of groups $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}$ on S by

$$\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S}(U) = \{f : \mathcal{X}_U \to \mathcal{X}_U : \mathcal{B}_U\text{-automorphism such that } f^* : R^2p_*\mathbb{Z} \to R^2p_*\mathbb{Z} \text{ is the identity}\}.$$

Equivalently, we may define $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S} = q_*\underline{\mathrm{Aut}}_{\mathcal{X}/\mathcal{B}} \cap \underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$.

Consider the automorphism sheaf $\underline{\mathrm{Aut}}_{\mathcal{B}/S}$ of the $\mathbb{P}^n$-bundle $\mathcal{B} \to S$. It is the sheaf of analytic local sections of the $\mathrm{PGL}_{n+1}(\mathbb{C})$-group scheme $\mathrm{Aut}_{\mathcal{B}/S} \to S$. Proposition 4.1.5 is essentially the main idea of the theorem, but will not be enough to be used in the proof of the theorem. We will need to state the following stronger version of it. Both Theorem 4.1.4 and Proposition 4.1.5 will be a consequence of this stronger version.

Proposition 4.1.7. *Let $\mathcal{X} \to \mathcal{B} \to S$ be a family of Lagrangian fibered hyper-Kähler manifolds over an open ball S.*

(i) *There exists a canonical $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$-action on $\mathcal{B}$ making $\pi : \mathcal{X} \to \mathcal{B}$ an $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$-equivariant morphism over S.*

(ii) *Under this descent, $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/\mathcal{B}/S} = \ker\left(\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S} \to \underline{\mathrm{Aut}}_{\mathcal{B}/S}\right)$.*

4.1.2 G-linearized line bundles

Let us briefly recall the notion of a G-linearizability of a line bundle on a complex manifold. For simplicity we only consider finite group actions. Our references are [Bri18, §3], [Dol03, §7] and [MFK94], but we need to take some additional care since these references only consider the algebraic setting.

Let G be an arbitrary finite group and $\mathcal{X}$ a complex manifold equipped with a holomorphic G-action. A *G-linearized line bundle* on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ together with a collection of isomorphisms $\Phi_g : g^*\mathcal{L} \to \mathcal{L}$ for each $g \in G$, satisfying the condition $\Phi_{gg'} = \Phi_{g'} \circ g'^*\Phi_g$ for all $g, g' \in G$. A *G-invariant line bundle* on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ such that $g^*\mathcal{L} \cong \mathcal{L}$ for each $g \in G$ (without any condition). We denote by $\mathrm{Pic}^G(\mathcal{X})$ and $\mathrm{Pic}(\mathcal{X})^G$ the groups of G-linearized line bundles and G-invariant line bundles on $\mathcal{X}$ up to isomorphisms, respectively. The second group is precisely the G-invariant subgroup of $\mathrm{Pic}(\mathcal{X})$.

There is a forgetful homomorphism $\mathrm{Pic}^G(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})^G$, which is neither injective nor surjective in general. To understand the obstruction to its surjectivity, one considers an exact sequence of abelian groups ([Dol03, Rmk 7.2] or [Bri18, Prop 3.4.5])

$$\mathrm{Pic}^G(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X})^G \longrightarrow H^2(G, \Gamma)\ , \qquad \Gamma = H^0(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}).$$

Both Dolgachev and Brion's discussions are for algebraic varieties, but their proofs can be adapted to our analytic setting as well. With this exact sequence in hand, we have:

Lemma 4.1.8. *Every G-invariant line bundle $\mathcal{H}$ on $\mathcal{X}$ is G-linerizable up to a suitable tensor power.*

Proof. It is a general fact in the theory of group cohomology (for finite groups) that all the higher degree cohomologies $H^{\geq 1}(G, \Gamma)$ are $|G|$-torsion for any G-module Γ (e.g., [Ser79, Cor VIII.1]). Hence by the previous lemma, the $|G|$-th tensor $\mathcal{H}^{\otimes |G|}$ vanishes in $H^2(G, \Gamma)$ and hence comes from $\mathrm{Pic}^G(\mathcal{X})$. □

For us, the importance of the G-linearizability of a line bundle comes from the induced G-action on the higher direct images of a linearized line bundle. If $\mathcal{L}$ is a G-linearized line bundle on $\mathcal{X}$ and $p : \mathcal{X} \to S$ is a G-invariant holomorphic map, then we have a contravariant G-action on all the higher direct image sheaves

$$g^* : R^k p_* \mathcal{L} \to R^k p_* \mathcal{L}, \qquad (g \circ g')^* = g'^* \circ g^*.$$

Now assume further $\mathcal{L}$ is globally generated over S and $p_*\mathcal{L}$ is a vector bundle on S. Then we have a G-action on $\mathbb{P}_S(p_*\mathcal{L})$ making the holomorphic map $\mathcal{X} \to \mathbb{P}_S(p_*\mathcal{L})$ G-equivariant over S. See [MFK94, Prop 1.7].

4.1.3 Proof of the theorem

This subsection presents the proof of Proposition 4.1.7 and Theorem 4.1.4. To start, recall the sheaf of H^2-trivial automorphisms $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ is a local system on S. Both of the statements are local on the base S, so we may assume S is an open ball. Then $\underline{\mathrm{Aut}}^\circ_{\mathcal{X}/S}$ becomes a constant sheaf, so we may consider it as an abstract finite group

$$G = \mathrm{Aut}^\circ(X)$$

acting on $\mathcal{X} \to S$ fiberwise (where X is any fixed single fiber).

The following lemma proves every line bundle on $\mathcal{X}$ is G-invariant.

Lemma 4.1.9. *G acts trivially on* $\mathrm{Pic}(\mathcal{X})$.

Proof. We first claim G acts trivially on $H^2(\mathcal{X}, \mathbb{Z})$. Apply the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q p_* \mathbb{Z}) \quad \Rightarrow \quad H^{p+q}(\mathcal{X}, \mathbb{Z}).$$

Noticing that $R^0 p_* \mathbb{Z} = \mathbb{Z}$, $R^1 p_* \mathbb{Z} = 0$, and S is an open ball, we obtain an isomorphism $H^2(\mathcal{X}, \mathbb{Z}) \cong H^0(S, R^2 p_* \mathbb{Z})$. This isomorphism respects the G-action as the Leray spectral sequence is functorial. Now G acts on $H^2(X_t, \mathbb{Z})$ trivially for any fiber X_t, so G acts on $R^2 p_* \mathbb{Z}$ trivially and the claim follows.

It is enough to prove the first Chern class map $\mathrm{Pic}(\mathcal{X}) \to H^2(\mathcal{X}, \mathbb{Z})$ is injective. This homomorphism is induced by the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}^* \to 0$, so it suffices to prove $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. Again, use the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q p_* \mathcal{O}_{\mathcal{X}}) \quad \Rightarrow \quad H^{p+q}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}).$$

This time, we have $R^0 p_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$ and $R^1 p_* \mathcal{O}_{\mathcal{X}} = 0$. This implies $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$. □

Consider the line bundle $\mathcal{H} = \pi^* \mathcal{O}_{\mathcal{B}/S}(1)$ on $\mathcal{X}$. Since $\mathrm{Pic}(\mathcal{X})$ is G-invariant, we can apply Lemma 4.1.8 to $\mathcal{H}$ and conclude $\mathcal{H}^{\otimes m}$ is G-linearizable for some positive integer m. As a result, we have a G-equivariant morphism $\pi_m : \mathcal{X} \to \mathcal{B}_m$ where $\mathcal{B}_m = \mathbb{P}_S\big(p_* \mathcal{H}^{\otimes m}\big)$ is the dual of the complete linear system associated to $\mathcal{H}^{\otimes m}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, "\pi"] \arrow[drr, bend left, "\pi_m"] \arrow[dd, "p"'] & & \\
& \mathcal{B} \arrow[r, hook] \arrow[dl, "q"] & \mathcal{B}_m \arrow[dll, bend left, "q_m"] \\
S & &
\end{tikzcd} \quad . \tag{4.1.10}$$

Lemma 4.1.11. *The m-th relative Veronese embedding $\mathcal{B} \hookrightarrow \mathcal{B}_m$ makes the diagram* (4.1.10) *commute.*

Proof. Notice that the morphism $\pi : \mathcal{X} \to \mathcal{B}$ associated to $\mathcal{H}$ has connected fibers. Hence, for any $t \in S$, the fiber $\pi : X_t \to B_t$ becomes the Iitaka fibration of the line bundle H_t

(e.g., [Laz04, §2.1.B]). This in particular implies that any morphism $\pi_m : X_t \to (B_m)_t$ associated to $H_t^{\otimes m}$ factors through the Iitaka fibration π. The factorization is precisely the Stein factoriazation of π_m and $B_t \hookrightarrow (B_m)_t$ is the m-th Veronese embedding. In other words, the m-th relative Veronese embedding makes the diagram commute. □

Proof of Proposition 4.1.7. For the first item, we need to prove π is G-invariant. The morphism π_m in the above diagram was G-equivariant by construction. This implies π is G-equivariant since π_m is factorized into $\mathcal{X} \to \mathcal{B} \hookrightarrow \mathcal{B}_m$. The second item is an immediate consequence of the first item. □

Proof of Theorem 4.1.4. The sheaves $\underline{\mathrm{Aut}}^{\circ}_{\mathcal{X}/S}$ and $\underline{\mathrm{Aut}}_{\mathcal{B}/S}$ are represented by

$$\mathrm{Aut}^{\circ}_{\mathcal{X}/S} \cong \bigsqcup_{f \in \mathrm{Aut}^{\circ}(X)} S, \qquad \mathrm{Aut}_{\mathcal{B}/S} \cong \mathrm{PGL}_{n+1}(\mathbb{C}) \times S.$$

Hence the desired sheaf $\underline{\mathrm{Aut}}^{\circ}_{\mathcal{X}/\mathcal{B}/S}$ is representable by $\ker(\alpha : \mathrm{Aut}^{\circ}_{\mathcal{X}/S} \to \mathrm{Aut}_{\mathcal{B}/S})$ by Proposition 4.1.7. To prove the kernel is a constant subgroup scheme, it is enough to show the following: let S' be a connected component of $\mathrm{Aut}^{\circ}_{\mathcal{X}/S}$. Consider the restriction of α followed by the projection

$$\beta : S' \to \mathrm{PGL}_{n+1}(\mathbb{C}).$$

Then we claim that either $\beta(S') = \{\mathrm{id}\}$ or $\beta(S') \not\ni \mathrm{id}$. Notice that the image $\beta(S')$ consists of $|G|$-torsion matrices in $\mathrm{PGL}_{n+1}(\mathbb{C})$. Since the set of $|G|$-torsion matrices is a disjoint union of $\mathrm{PGL}_{n+1}(\mathbb{C})$-adjoint orbits (classified by eigenvalues), the connected set $\beta(S')$ has to lie in a single orbit. The adjoint orbit containing the identity matrix is a singleton set $\{\mathrm{id}\}$. Hence the claim follows. □

4.2 H^2-trivial automorphisms via Albanese fibrations

This section discusses an explicit construction of certain H^2-trivial automorphisms of geometric origin. The main result of this section will be Proposition 4.2.7, but to state it we need to introduce some relevant settings.

Throughout the section, we stick to the following setting. Let M be a *projective* symplectic manifold, not necessarily irreducible.[1] By the Beauville–Bogomolov decomposition theorem, M must admit a finite étale covering $X \times T \to M$ where X is a finite product of projective hyper-Kähler manifolds and T is an abelian variety. This is called a *split covering* of M. In fact, Beauville in [Bea83a, §3] also considered minimal such a covering.

[1] The projectiveness of M will be only used in Theorem 4.2.2.

Theorem-Definition 4.2.1 (Beauville). *Let M be a compact symplectic manifold, not necessarily irreducible. A* minimal split covering *of M is a split covering $\Phi : X \times T \to M$ such that any split covering $\Phi' : X' \times T' \to M$ factors through Φ. Then the minimal split covering of M always exists and is unique up to non-unique isomorphisms. Moreover, the minimal split covering is always finite Galois.*

Meanwhile, Kawamata [Kaw85, Thm 8.3] proved the following theorem.

Theorem 4.2.2 (Kawamata). *Let M be a K-trivial smooth projective variety. Then its Albanese morphism $\mathrm{Alb} : M \to \mathrm{Alb}(M)$ is an étale fiber bundle. More precisely, there exists an isogeny $\phi : T \to \mathrm{Alb}(M)$ of abelian varieties such that the base change of Alb becomes a trivial fiber bundle over T:*

$$\begin{array}{ccc} X \times T & \xrightarrow{\Phi} & M \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle \mathrm{Alb}} \\ T & \xrightarrow{\phi} & \mathrm{Alb}(M) \end{array} . \tag{4.2.3}$$

If we apply Kawamata's theorem to a projective symplectic manifold M then $\Phi : X \times T \to M$ in the theorem becomes a split covering of M. Combining the results of Beauville and Kawamata, we obtain:

Proposition 4.2.4. *Let M be a projective holomorphic symplectic manifold and $\mathrm{Alb} : M \to \mathrm{Alb}(M)$ its Albanese morphism, an étale fiber bundle by Kawamata. Assume $X = \mathrm{Alb}^{-1}(0)$ is a projective hyper-Kähler manifold. Then there exists a unique isogeny $\phi : T \to \mathrm{Alb}(M)$ such that the morphism Φ in the fiber diagram* (4.2.3) *becomes the minimal split covering of Beauville.*

Proof. Use Theorem 4.2.2 to construct an isogeny $\phi' : T' \to \mathrm{Alb}(M)$ trivializing the Albanese map as in (4.2.3). Since ϕ' is a finite Galois covering, Φ' is also a finite Galois covering with $\mathrm{Gal}(\Phi') \cong \mathrm{Gal}(\phi')$. The first lemma in [Bea83a, §3] claims $\mathrm{Aut}(X \times T') = \mathrm{Aut}(X) \times \mathrm{Aut}(T')$. Hence the $\mathrm{Gal}(\Phi')$-action on $X \times T'$ is by (f, a) where f and a are automorphisms on X and T, respectively. The isomorphism $\mathrm{Gal}(\Phi') \to \mathrm{Gal}(\phi')$ is by the second projection $(f, a) \mapsto a$. Since $\mathrm{Gal}(\phi')$ is the kernel of the isogeny ϕ', the automorphisms a must be translations of T'.

Now consider the homomorphism $\mathrm{Gal}(\Phi') \to \mathrm{Aut}(X)$ by $(f, a) \mapsto f$. Set H by the kernel of it; it consists of elements of the form (id_X, a). Under the isomorphism $\mathrm{Gal}(\Phi') \cong \mathrm{Gal}(\phi')$, we can consider it as a subgroup of $\mathrm{Gal}(\phi')$, so there exists a Galois covering $T' \to T = T'/H$ corresponding to it. Let $\phi : T \to \mathrm{Alb}(M)$ be the morphism factorizing ϕ'. We have a cartesian diagram

$$\begin{array}{ccccc} X \times T' & \longrightarrow & X \times T & \xrightarrow{\Phi} & M \\ \downarrow^{\mathrm{pr}_2} & & \downarrow^{\mathrm{pr}_2} & & \downarrow^{\mathrm{Alb}} \\ T' & \longrightarrow & T & \xrightarrow{\phi} & \mathrm{Alb}(M) \end{array} .$$

By construction, $\mathrm{Gal}(\Phi)$ consists of automorphisms (f, a) with no (id_X, a) (i.e., the $\mathrm{Gal}(\phi)$-action on X is effective). But this means Φ is precisely Beauville's minimal split covering [Bea83a, §3]. The uniqueness of ϕ follows from the uniqueness of the minimal split covering. □

The proposition in particular proves that the minimal split covering can be always realized by an isogeny $\phi : T \to \mathrm{Alb}(M)$ and the base change (4.2.3).

Definition 4.2.5. We call $\phi : T \to \mathrm{Alb}(M)$ in Proposition 4.2.4 the *minimal isogeny* trivializing the Albanese morphism $\mathrm{Alb} : M \to \mathrm{Alb}(M)$. It is unique up to non-unique isomorphisms.

In fact, the proof of Proposition 4.2.4 is saying more about an arbitrary isogny ϕ'.

Corollary 4.2.6. *Notation as in Proposition 4.2.4. Let $\phi' : T' \to \mathrm{Alb}(M)$ be any isogeny trivializing the Albanese morphism. Then*

(i) *ϕ' factors though the minimal isogeny ϕ.*
(ii) *There exists a canonical* $\mathrm{Gal}(\phi')$*-action on X.*
(iii) *The isogeny ϕ' is minimal if and only if the* $\mathrm{Gal}(\phi')$*-action on X is effective.*

Proof. All of these can be directly deduced from the proof of Proposition 4.2.4. Recall $\mathrm{Gal}(\Phi') \to \mathrm{Gal}(\phi')$, $(f, a) \mapsto a$ is an isomorphism. Therefore, $f = f_a$ is uniquely determined by a, and this defines $\mathrm{Gal}(\phi') \to \mathrm{Aut}(X)$, $a \mapsto f_a$. □

Now we can state the main result of this section. The ideas here were already contained in [Bea83a, Bea83b].

Proposition 4.2.7. *Notation as in Proposition 4.2.4 and 4.2.6. Then* $\mathrm{Gal}(\phi')$ *acts on X by H^2-trivial automorphisms. That is, we have a canonical homomorphism*

$$\mathrm{Gal}(\phi') \to \mathrm{Aut}^{\circ}(X),$$

which is injective if and only if ϕ' is minimal.

Proof. By Corollary 4.2.6, we may assume $\phi' = \phi$ is minimal and $\mathrm{Gal}(\phi) \subset \mathrm{Aut}(X)$. The content of the proposition is that it is further a subgroup of $\mathrm{Aut}^\circ(X)$.

Consider the diagram (4.2.3). Our first step is to equip T-actions on all the four spaces to make the diagram T-equivariant. Equip a T-action on T by translation, and on $X \times T$ only on the second factor again by translation. The T-action on $\mathrm{Alb}(M)$ is by translation via the morphism ϕ: if $a \in T$ and $z \in \mathrm{Alb}(M)$ then we define $a.z = z + \phi(a)$.

To equip a T-action on M, we claim the T-action on $X \times T$ descends to M via Φ. The descent works if the $\mathrm{Gal}(\Phi)$-action on $X \times T$ commutes with the T-action. Recall from the discussions in Proposition 4.2.4 that $\mathrm{Gal}(\Phi)$ acts on $X \times T$ by (f, a) where f is an automorphism of X and a is a translation of T. Now let $b \in T$ and $(x, t) \in X \times T$. Then we have a sequence of identities

$$b.\Big((f,a).(x,t)\Big) = (f(x), t + a + b) = (f,a).\Big(b.(x,t)\Big).$$

This proves the T-action and $\mathrm{Gal}(\Phi)$-action commutes, yielding the descent T-action on M. The conclusion is that Alb becomes automatically T-equivariant (and hence the diagram (4.2.3) becomes T-equivariant).

By definition, the stabilizer the T-action on $\mathrm{Alb}(M)$ is precisely $\ker \phi = \mathrm{Gal}(\phi)$. Since the Albanese map $\mathrm{Alb} : M \to \mathrm{Alb}(M)$ is T-equivariant, this induces a $\mathrm{Gal}(\phi)$-action on the fiber $\mathrm{Alb}^{-1}(0) = X$. One easily shows this coincides with our previous $\mathrm{Gal}(\phi)$-action on X. Now notice that any T-action on M is isotopic to the identity map because T is path connected. In particular, T acts on M trivially at the level of cohomology $H^*(M, \mathbb{Q})$. The embedding $X \subset M$ is $\mathrm{Gal}(\phi)$-equivariant, so we have a $\mathrm{Gal}(\phi)$-equivariant restriction homomorphism

$$H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q}).$$

Hence it suffices to prove this restriction homomorphism is surjective.

Now the question became topological. Deform the complex structure of the hyper-Kähler manifold X very generally so that $H^2(X, \mathbb{Q})$ becomes a simple $\mathbb{Q}$-Hodge structure (we will have to lose the projectiveness of X). The complex structure of M can be correspondingly chosen in a way that the finite covering map $\Phi : X \times T \to M$ becomes holomorphic. Therefore, the Hodge structure morphism $H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is either 0 or surjective. We only need to rule out the former possibility.

To prove it is nonzero, consider any global holomorphic symplectic form σ on M. Pulling it back to $X \times T$ gives a global holomorphic symplectic form on $X \times T$. But $H^{2,0}(X \times T) = H^{2,0}(X) \oplus H^{2,0}(T)$ by Künneth. If σ was 0 in the $H^{2,0}(X)$-component then this would mean

σ doesn't contain any 2-forms along the tangent direction of X, violating σ is a symplectic form. Hence $\sigma_{|X}$ cannot be 0. The claim follows. $\square$

Remark 4.2.8. The inclusion $\mathrm{Gal}(\phi) \subset \mathrm{Aut}^\circ(X)$ for the minimal isogeny ϕ in Proposition 4.2.7 may be a strict inclusion. For example, we will later see that when X if of Kum_n-type then

$$\mathrm{Gal}(\phi) \cong (\mathbb{Z}/n+1)^{\oplus 4}, \qquad \mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}.$$

4.3 Relative automorphism scheme and H^2-trivial automorphisms

The goal of this section is to relate the group $\mathrm{Aut}^\circ(X/B)$ and the polarization scheme K introduced in §3.2. We will see in Proposition 4.3.8 that every H^2-trivial automorphism $f \in \mathrm{Aut}^\circ(X/B)$ defines a global section of the abelian scheme $P_0 \to B_0$ in Theorem 3.1.1. If we consider $\mathrm{Aut}^\circ(X/B)$ as a constant group scheme over B_0, this means we have a closed immersion of group schemes

$$\mathrm{Aut}^\circ(X/B) \hookrightarrow P_0. \tag{4.3.1}$$

We expect the image of this map will contain the polarization scheme.

Conjecture 4.3.2. *The polarization scheme K is contained in the image of* (4.3.1)*.*

Note that this conjecture would in particular imply

(i) the polarization scheme K is a constant group scheme; and
(ii) the polarization scheme K is extendable to the entire base B.

Although we were not able to prove this conjecture in its full generality, we will later in Theorem 5.2.1 prove this conjecture for all known deformation types of hyper-Kähler manifolds.

Remark 4.3.3. In the first version of our arXiv paper [Kim21], we have incorrectly claimed that the polarization scheme K would be precisely the image of (4.3.1). This turns out to be not always the case, as there is a counterexample when $\pi : X \to B$ is a Lagrangian fibered Kum_3-type hyper-Kähler manifold with $\mathrm{div}(h) = 2$. In such a case, we can compute $\mathrm{Aut}^\circ(X/B) \cong (\mathbb{Z}/2)^{\oplus 5}$ and K is a constant group scheme $(\mathbb{Z}/2)^{\oplus 4}$. See Theorem 4.4.2. This counterexample was pointed out to us by Salvatore Floccari.

The main results of this section are the following two propositions, which will be later used in the proof of Theorem 5.2.1. To state the results, we need the following new definition. Given any positive integer a, we define

$$K_a = \ker\big(a\lambda : P_0 \to \check{P}_0\big). \tag{4.3.4}$$

All K_a are finite étale commutative group schemes over B_0. The original polarization scheme K is K_1.

Proposition 4.3.5. *Let $\pi : X \to B$ and $\pi' : X' \to B'$ be two deformation equivalent Lagrangian fibered hyper-Kähler manifolds. Let a be any positive integer. Then* (4.3.1) *factors through*

$$\mathrm{Aut}^{\circ}(X/B) \hookrightarrow K_a \tag{4.3.6}$$

if and only if the same holds for π'.

Proposition 4.3.7. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold and $(d_1, \cdots, d_n)$ its polarization type. Assume we have an equality $c_X = d_1 \cdots d_n$. Then* (4.3.6) *holds for $a = \mathrm{div}(h)$.*

The first proposition says that for a fixed integer a, the property (4.3.6) is deformation invariant on π. The second proposition gives an upper bound of such an integer a making (4.3.6) holds. The reason for taking $a = \mathrm{div}(h)$ is because the image of the restriction map $H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is precisely $\mathrm{div}(h)$ times the primitive polarization (Proposition 3.2.6). We also recall that the condition $c_X = d_1 \cdots d_n$ is satisfied for all known deformation types of hyper-Kähler manifolds by Theorem 3.2.3.

The rest of this section will be devoted to the proof of Proposition 4.3.5 and 4.3.7.

Proposition 4.3.8. *Every H^2-trivial automorphism in $\mathrm{Aut}^{\circ}(X/B)$ defines a global section of $P_0 \to B_0$. That is, we have a closed immersion of group schemes*

$$\mathrm{Aut}^{\circ}(X/B) \hookrightarrow P_0.$$

Proof. We first claim $\mathrm{Aut}^{\circ}(X/B)$ acts on $\pi : X_0 \to B_0$ by fiberwise translation automorphisms. Consider the quotient $\bar{X} = X/\mathrm{Aut}^{\circ}(X/B)$ with a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "p"] \arrow[dd, "\pi"'] & \\ & \bar{X} \arrow[dl, "\bar{\pi}"] \\ B & \end{tikzcd} .$$

We first claim p is étale on general fibers over B. Let $S \subset X$ be the ramified locus of p. The quotient p is symplectic so it is quasi-étale, i.e., S has codimension ≥ 2. Let $b \in B$ be a general point, so that the fibers $F = X_b$ and $\bar{F} = \bar{X}_b$ are both smooth. Observe the ramification locus of $p : F \to \bar{F}$ is precisely $S \cap F$, which is of codimension ≥ 2 since b is general. The purity of the branch locus theorem forces $p : F \to \bar{F}$ to be étale.

Now we have a finite étale quotient $p : F \to \bar{F} = F/\operatorname{Aut}^\circ(X/B)$ between smooth projective varieties. Its Galois group $\operatorname{Aut}^\circ(X/B)$ acts on F by fixed point free automorphisms. Since F is an abelian variety, this means $\operatorname{Aut}^\circ(X/B)$ acts on F by translations. The conclusion is that on a general fiber of π, the group $\operatorname{Aut}^\circ(X/B)$ acts by translation.

Recall from Proposition 3.1.9 that P_0 is the abelian scheme representing the translation automorphism sheaf of $\pi : X_0 \to B_0$. This means $\operatorname{Aut}^\circ(X/B)$ defines a rational section of $\nu : P_0 \to B_0$. By Proposition 3.1.12, the rational section must be defined over the entire B_0 and becomes an honest section. □

An immediate byproduct of this proposition is that $\operatorname{Aut}^\circ(X/B)$ is an abelian group, completing the promised proof of Theorem 4.1.4. Let us prove an elementary lemma before we get into the proof of the desired propositions.

Lemma 4.3.9. *Let $\mathcal{P}_0 \to \mathcal{B}_0$ be an abelian scheme over a complex manifold $\mathcal{B}_0$ and $a\lambda : \mathcal{P}_0 \to \check{\mathcal{P}}_0$ a polarization with $\mathcal{K}_a = \ker(a\lambda)$. Assume there exists a torsion section $f : \mathcal{B}_0 \to \mathcal{P}_0$. If $f(\mathcal{B}_0) \cap \mathcal{K}_a \neq \varnothing$ then $f(\mathcal{B}_0) \subset \mathcal{K}_a$.*

Proof. The statement is topological and local on the base $\mathcal{B}_0$, so we may assume $\mathcal{B}_0$ is a complex open ball S and $\mathcal{P}_0 \to \mathcal{B}_0$ is homeomorphic to the topological constant group scheme $(\mathbb{R}/\mathbb{Z})^{2n} \times S \to S$. In this setting, the kernel $\mathcal{K}_a$ is a constant subgroup scheme and the *torsion* section f is a constant section. Hence $f(S) \cap \mathcal{K}_a \neq \varnothing$ if and only if $f(S) \subset \mathcal{K}_a$. □

Proof of Proposition 4.3.5. Consider a one-parameter family of Lagrangian fibered hyper-Kähler manifolds $\mathcal{X} \to \mathcal{B} \to \Delta$ over a complex disc Δ. By Lemma 3.3.3, there exists a notion of a family of abelian schemes $\mathcal{P}_0 \to \mathcal{B}_0$ and a family of finite étale group schemes $\mathcal{K}_a \subset \mathcal{P}_0$. Proposition 4.3.8 proves we have a closed immersion $\operatorname{Aut}^\circ(X/B) \hookrightarrow P_0$ for a single fiber. In fact, the argument applies to the entire family and produces $\operatorname{Aut}^\circ(X/B)$ global sections of $\mathcal{P}_0 \to \mathcal{B}_0$, or equivalently an embedding

$$\operatorname{Aut}^\circ(X/B) \hookrightarrow \mathcal{P}_0.$$

Since $\operatorname{Aut}^\circ(X/B)$ is finite, the global sections are torsion. Suppose we had $\operatorname{Aut}^\circ(X/B) \hookrightarrow K_a$ for the original Lagrangian fibration over $0 \in \Delta$. Then this forces $\operatorname{Aut}^\circ(X/B) \hookrightarrow \mathcal{K}_a$ over the entire Δ by Lemma 4.3.9. The claim follows. □

Proof of Proposition 4.3.7. Recall from Proposition 3.2.6 that the restriction map $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ has a rank 1 image generated by the class $a\theta$, where $a = \operatorname{div}(h)$ and θ is the primitive ample class corresponding to our polarization $\lambda : F \to \check{F}$. The preimage of $a\theta \in H^2(F,\mathbb{Z})$ under this restriction homomorphism is precisely $S = \{x \in H^2(X,\mathbb{Z}) : q(x,h) = a\}$. By the previous Proposition 4.3.5, the claim is invariant under deformations of π. We may thus deform π and assume $\operatorname{Pic}(X) \cap S \neq \varnothing$. In other words, we may assume the composition $\operatorname{Pic}(X) \subset H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ is generated by $a\theta$.

The assertion $\operatorname{Aut}^\circ(X/B) \hookrightarrow K_a = \ker(a\lambda)$ is equivalent to $a\lambda(\operatorname{Aut}^\circ(X/B)) = 0$. The latter equality may be verified fiberwise, so we may concentrate on a single fiber $F = \nu^{-1}(b) = \pi^{-1}(b)$. Let L be any line bundle on X such that its image under $\operatorname{Pic}(X) \to H^2(F,\mathbb{Z})$ is $a\theta$. This means the polariztion $a\lambda$ can be described as

$$a\lambda : F \to \check{F}, \qquad t_x \mapsto [t_x^*(L_{|F}) \otimes L_{|F}^{-1}].$$

If we assume $t_x = f_{|F}$ is from a global H^2-trivial automorphism $f \in \operatorname{Aut}^\circ(X/B)$, then we have a sequence of identities

$$t_x^*(L_{|F}) = (f_{|F})^*(L_{|F}) = (f^*L)_{|F} \cong L_{|F},$$

where the last isomorphism follows from the fact f acts on $\operatorname{Pic}(X) \subset H^2(X,\mathbb{Z})$ trivially. This proves $a\lambda$ sends $\operatorname{Aut}^\circ(X/B)$ to 0 and the claim follows. □

4.4 Computing H^2-trivial automorphisms for known deformation types

The following computations are obtained by [Bea83a], [BNWS13] and [MW17]. Together with the deformation invariance by [HT13] it computes the group $\operatorname{Aut}^\circ(X)$ for all currently known deformation types of hyper-Kähler manifolds.

Theorem 4.4.1 (Beauville, Boissière–Nieper-Wißkirchen–Sarti, Mongardi–Wandel)**.** *Let X be a hyper-Kähler manifold. Then*

$$\operatorname{Aut}^\circ(X) \cong \begin{cases} \{\mathrm{id}\} & \textit{if } X \textit{ is of } K3^{[n]} \textit{ or OG10-type,} \\ \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4} & \textit{if } X \textit{ is of } Kum_n\textit{-type,} \\ (\mathbb{Z}/2)^{\oplus 8} & \textit{if } X \textit{ is of OG6-type.} \end{cases}$$

The main result of this section is to prove the following analogue of this theorem.

Theorem 4.4.2. *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Then*

(i) $$\mathrm{Aut}^\circ(X/B) \cong \begin{cases} \{\mathrm{id}\} & \textit{if } X \textit{ is of } K3^{[n]} \textit{ or OG10-type,} \\ (\mathbb{Z}/2)^{\oplus 4} & \textit{if } X \textit{ is of OG6-type.} \end{cases}$$

(ii) *Assume X is of Kum_n-type and $(1, \cdots, 1, d_1, d_2)$ is the polarization type of π in Theorem 3.2.3. Then*

$$\mathrm{Aut}^\circ(X/B) \cong \begin{cases} (\mathbb{Z}/2)^{\oplus 5} & \textit{if } n = 3 \textit{ and the polarization type is } (1,2,2), \\ (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \textit{otherwise.} \end{cases}$$

Since $\mathrm{Aut}^\circ(X)$ is already trivial for $\mathrm{K3}^{[n]}$ and OG10-types, our theorem is obvious for the first two cases. Recall we have proved Theorem 4.1.4 that $\mathrm{Aut}^\circ(X/B)$ is invariant under deformations of π. By [MR21], every Lagrangian fibration of an OG6-type hyper-Kähler manifold is deformation equivalent to each other, and by [MW17, §5], $\mathrm{Aut}^\circ(X/B) = (\mathbb{Z}/2)^{\oplus 4}$ for at least one Lagrangian fibration of an OG6-type hyper-Kähler manifold. This proves the theorem for the OG6-case. Therefore, the only remaining case in the theorem is the Kum_n-type.

By [Wie18, §6.28], every Lagrangian fibration of a Kum_n-type hyper-Kähler manifold is deformation equivalent to the construction in Example 1.3.14. Therefore, we only need to prove the following more concrete result.

Proposition 4.4.3. *Let $\pi : X \to B$ be a Lagrangian fibered Kum_n-type hyper-Kähler manifold in Example 1.3.14, and let (d_1, d_2) be the polarization type of the ample class l on A. Then*

$$\mathrm{Aut}^\circ(X/B) \cong \begin{cases} (\mathbb{Z}/2)^{\oplus 5} & \textit{if } n = 3 \textit{ and } d_1 = d_2 = 2, \\ (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \textit{otherwise.} \end{cases}$$

Our proof of this proposition is by explicit computations. In fact, the computation can be carried out further and calculates the polarization scheme K as well. This is the second main result of this section.

Proposition 4.4.4. *Let $\pi : X \to B$ be a Lagrangian fibered Kum_n-type hyper-Kähler manifold in Example 1.3.14.*

(i) *If $n = 3$ and $d_1 = d_2 = 2$, then K is an order 2 subgroup of $\mathrm{Aut}^\circ(X/B)$ via (4.3.1).*
(ii) *In other cases, $K = \mathrm{Aut}^\circ(X/B)$ via (4.3.1).*

The proof of these propositions are quite lengthy. In this article, we omit the proof of Proposition 4.4.4 and highlight only the outline of the proof of Proposition 4.4.3. We refer to the original paper [Kim21] for full details.

4.4.1 Describing the H^2-trivial automorphisms of Kum_n-type moduli constructions

Recall from Theorem 4.4.1 that any Kum_n-type hyper-Kähler manifold has $\mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}$. The goal of this subsection is to explicitly describe such automorphisms for Example 1.3.5. Lagrangian fibrations play no role in this subsection.

In Example 1.3.5 and 1.3.14, we have identified Pic^l_A with $\check{A}$. Strictly speaking, this is possible only after choosing a reference point of Pic^l_A. We choose a *symmetric* ample line bundle $[L_0] \in \mathrm{Pic}^l_A$.[2] By the general theory of abelian varieties, there exists a dual ample line bundle $\check{L}_0$ on $\check{A}$. The ample line bundles L_0 and $\check{L}_0$ induce polarization isogenies

$$\varphi : A \to \check{A}, \qquad \check{\varphi} : \check{A} \to A,$$

making their compositions the multiplication endomorphisms

$$[n+1] : A \xrightarrow{\varphi} \check{A} \xrightarrow{\check{\varphi}} A, \qquad [n+1] : \check{A} \xrightarrow{\check{\varphi}} A \xrightarrow{\varphi} \check{A}. \tag{4.4.5}$$

Since L_0 has a polarization type (d_1, d_2), the dual line bundle $\check{L}_0$ also has a polarization type (d_1, d_2) (see [BL04, Prop 14.4.1]). In particular, we have an isomorphism

$$\ker \check{\varphi} \cong (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}. \tag{4.4.6}$$

A closed point x on A defines a translation automorphism by x. Our notation for the translation automorphism is

$$t_x : A \to A, \qquad y \mapsto y + x.$$

A closed point ξ on $\check{A}$ represents a numerically trivial line bundle on A. Considering ξ both as a closed point on $\check{A}$ and a line bundle on S can possibly lead to a confusion. Thus, we will write

$$L_\xi : \text{ numerically trivial line bundle on } A \text{ corresponding to } \xi \in \check{A}.$$

With these notation in mind, we can explicitly realize the $\mathrm{Aut}^\circ(X)$-action for the moduli of sheaves construction X.

Proposition 4.4.7. *Let X be a Kum_n-type hyper-Kähler manifold in Example 1.3.5. Then*

[2] There are precisely 16 symmetric line bundles in Pic^l_A by [BL04, Lem 4.6.2]. It will be crucial we have chosen L_0 to be symmetric when we describe the $\mathbb{Z}/2$-part of $\mathrm{Aut}^\circ(X)$ explicitly.

(i) *We have an isomorphism*

$$\mathrm{Aut}^\circ(X) = \{\pm 1\} \ltimes \{(x,\xi) \in A[n+1] \times \check{A}[n+1] : \ \varphi(x) = 0, \quad \check{\varphi}(\xi) = sx\}.$$

(ii) *With the above identification, the* $\mathrm{Aut}^\circ(X)$*-action on* X *is defined by*

$$(1,x,\xi).[E] = [t_x^* E \otimes L_\xi], \qquad (-1,x,\xi).[E] = [t_x^*([-1]^*E) \otimes L_\xi],$$

where $[-1] : A \to A$ *is the multiplication by* -1 *automorphism on* A*.*

The rest of this subsection is devoted to sketching the proof of Proposition 4.4.7. We omit all the technical details of its proof. To start, we note that Yoshioka has already computed an explicit trivialization of Albanese morphism $\mathrm{Alb} : M \to A \times \check{A}$. Yoshioka's trivialization is obtained by the base change $[n+1] : A \times \check{A} \to A \times \check{A}$, which is a degree $(n+1)^8$ isogeny. This turns out to be not a minimal isogeny trivializing the Ablanese morphism in the sense of §4.2. The minimal isogeny is precisely

$$\phi : A \times \check{A} \to A \times \check{A}, \qquad (x,\xi) \mapsto (sx - \check{\varphi}(\xi),\ \varphi(x)). \tag{4.4.8}$$

Proposition 4.4.9. *The base change* (4.4.8) *is the minimal isogeny trivializing the Albanese morphism* $\mathrm{Alb} : M \to A \times \check{A}$ *in the sense of Definition 4.2.5.*

By Proposition 4.2.7, we have a canonical, effective and H^2-trivial $\mathrm{Gal}(\phi)$-action on X. The Galois group $\mathrm{Gal}(\phi)$ is captured by the kernel of ϕ, so we have

$$\mathrm{Gal}(\phi) = \{(x,\xi) \in A[n+1] \times \check{A}[n+1] : \ \varphi(x) = 0,\ \check{\varphi}(\xi) = sx\}.$$

This explains the isomorphism in Proposition 4.4.7. We can also compute $\mathrm{Gal}(\phi)$ more explicitly.

Lemma 4.4.10. $\mathrm{Gal}(\phi) \cong (\mathbb{Z}/n+1)^{\oplus 4}$.

This will describe the $\mathrm{Gal}(\phi) \cong (\mathbb{Z}/n+1)^{\oplus 4}$-action on X acting trivially on H^2. Since $\mathrm{Aut}^\circ(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n+1)^{\oplus 4}$, we still need an additional $\mathbb{Z}/2$-action to describe. Fortunately, this is not hard to guess. Define an involution ι on M by

$$\iota([E]) = [[-1]^*E].$$

A Riemann–Roch computation tells us the subvariety $X \subset M$ is indeed invariant under ι (here we need the fact L_0 is symmetric). It remains to prove ι acts on the second cohomology of X as the identity. We have already proved in Proposition 4.2.7 that $H^2(M,\mathbb{Q}) \to H^2(X,\mathbb{Q})$

is surjective. Hence it will be enough to prove ι acts on $H^2(M,\mathbb{Q})$ as the identity. This follows because ι is induced from the automorphism $[-1]$ on A, $[-1]$ acts on $H^2(A,\mathbb{Q})$ trivially and finally the Hodge structure $H^2(M,\mathbb{Q})$ is obtained by a tensor construction of $H^2(A,\mathbb{Q})$ by [B2̈0]. This exhausts the entire $\mathrm{Aut}^\circ(X)$-action description on X and hence completes the proof of Proposition 4.4.7.

4.4.2 Automorphisms respecting the Lagrangian fibration

With Proposition 4.4.7 at hand, we can complete the proof of Proposition 4.4.3. Any H^2-trivial automorphism is of the form

$$f = (1, x, \xi) \quad \text{or} \quad (-1, x, \xi), \qquad x \in A[n+1], \quad \xi \in \check{A}[n+1].$$

Let us first consider the automorphisms of the form $f = (1, x, \xi)$. It acts on X by

$$f.[E] = [t_x^* E \otimes L_\xi].$$

Recall $\pi : X \to B$ is the Fitting support map $\pi([E]) = [\operatorname{Supp} E]$. Hence f respects π if and only if $\operatorname{Supp} E = \operatorname{Supp} E - x$ for all $[E] \in X$. This is equivalent to $x = 0$, so we obtain precisely

$$\{(1, 0, \xi) : \xi \in \check{A}[n+1],\ \check{\varphi}(\xi) = 0\} \cong \ker \check{\varphi} \quad \subset \mathrm{Aut}^\circ(X/B).$$

Note also that $\ker \check{\varphi} \cong (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}$ by (4.4.6).

We now consider the automorphisms of the form $f = (-1, x, \xi)$. It acts on X by

$$f.[E] = [t_x^*([-1]^* E) \otimes P_\xi].$$

Therefore, f respects π if and only if $\operatorname{Supp} E = [-1]^* \operatorname{Supp} E - x$ for all $[E] \in X$. In other words, we have $D = [-1]^* D - x$ for all $D \in |L_0|$. Fix any $\frac{1}{2}x \in A$ with $2 \cdot \left(\frac{1}{2}x\right) = x$. Then this condition is equivalent to the fact that every divisor in the complete linear system $|t^*_{-\frac{1}{2}x} L_0|$ is a symmetric divisor. The following lemma completes the proof of Proposition 4.4.3. We again refer to [Kim21] for the proof of the lemma.

Lemma 4.4.11. *Let A be an abelian surface and L_0 a symmetric ample line bundle on it. Then every divisor in the complete linear system $|L_0|$ is symmetric if and only if L_0 is isomorphic to $\Theta^{\otimes 2}$ for a principal symmetric ample divisor Θ on A.*

Chapter 5

Constructing the dual Lagrangian fibration

Introduction

Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. The Strominger–Yau–Zaslow conjecture predicts an existence of the notion of its *dual Lagrangian fibration*. The dual Lagrangian fibration should be a new holomorphic map $\check{\pi} : \check{X} \to B$ and expected to satisfy the following two conditions:

(i) $\check{\pi}$ is a *Lagrangian fibration* of an orbifold $\check{X}$.

(ii) If $\pi : X_0 \to B_0$ is the associated torus fibration, then $\check{\pi}$ is a compactification of its *dual torus fibration* $\check{\pi} : \check{X}_0 \to B_0$.

The precise meaning of the two emphasized words in the conditions are not defined, so the conjecture in general should be interpreted liberally. The goal of this chapter is to realize this conjecture when X is of known deformation type of hyper-Kähler manifolds. In particular, we will provide a precise definition of the dual torus fibration in Definition 5.1.2, and prove $\check{X}$ is a hyper-Kähler orbifold admitting $\check{\pi}$ as a Lagrangian fibration.

The technical main ingredient of our result is the proof of Conjecture 4.3.2 for known types of hyper-Kähler manifolds. The dual Lagrangian fibration will be defined by a global quotient

$$\check{\pi} : \check{X} \to B \qquad \text{for} \quad \check{X} = X/K,$$

where K is a subgroup of $\mathrm{Aut}^\circ(X/B)$ that corresponds to the polarization scheme via (4.3.1). The resulting hyper-Kähler orbifold $\check{X}$ has the same universal deformation behavior to X.

Its Beauville–Bogomolov form $\bar{q}_{\check{X}}$ is the same as the original $\bar{q}_X$ over $\mathbb{Q}$ but may be different over $\mathbb{Z}$. Its Fujiki constant $c_{\check{X}}$ is precisely $1/c_X$.

5.1 The dual torus fibration

Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold. Recall that we called the restriction $\pi : X_0 \to B_0$ of the Lagrangian fibration a *torus fibration* of π. It is a family of abelian varieties of dimension n, a torsor under an abelian scheme $\nu : P_0 \to B_0$ by Theorem 3.1.1.

Guided by the Strominger–Yau–Zaslow (SYZ) conjecture [SYZ96], we expecte there should be a certain notion of the *dual* of a Lagrangian fibration $\pi : X \to B$; this should be a new morphism $\check{\pi} : \check{X} \to B$ from a compact Calabi–Yau orbifold $\check{X}$, and $\check{\pi}$ should be a Lagrangian fibration in a certain sense. It is tempting to expect that the conjectural $\check{X}$ should be also hyper-Kähler and $\check{\pi}$ is Lagrangian with respect to this hyper-Kähler structure.

The SYZ conjecture further predicts that the dual Lagrangian fibration $\check{\pi}$ needs to be a compactification of a "dual torus fibration" of the original torus fibration $\pi : X_0 \to B_0$. It is often tacitly assumed in the literature that the dual torus fibration should be the relative Picard scheme $\mathrm{Pic}^0_{X_0/B_0} \to B_0$. We claim this should *not* be the case, and the main point of this section is to propose an alternative definition of the dual torus fibration.

Remark 5.1.1. Here is one reason why we believe the relative Picard scheme construction $\mathrm{Pic}^0_{X_0/B_0} \to B_0$ may not be a correct definition: it always admits a section, the identity section. Note that the section issue does not arise in the symplectic geometry context as any family of tori admits a C^∞ section.[1] In contrast, existence of a holomorphic (rational) section is at least a codimension 1 condition in moduli as we have seen in §1.2.1. If one believes the conjectural dual Lagrangian fibration $\check{\pi} : \check{X} \to B$ behaves like a Lagrangian fibered hyper-Kähler orbifold, then $\check{\pi}$ should not admit a rational section once we deform the original Lagrangian fibration π very generally.

The following is the main definition of this section. Recall from Definition 3.2.1 that we have a notion of a polarization scheme K, a finite étale commutative group scheme over B_0. Since X_0 is a P_0-torsor and K is a subgroup scheme of P_0, we conclude there is a K-action on X_0 over B_0.

[1] This is because torsors are classified by H^1 of the group sheaf that they are acted by. Any C^∞ sheaf on B_0 has the vanishing higher cohomology groups because they are fine sheaves.

Definition 5.1.2. Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold and $\pi : X_0 \to B_0$ its associated torus fibration. The *dual torus fibration* of π is

$$\check{\pi} : \check{X}_0 \to B_0 \qquad \text{for} \quad \check{X}_0 = X_0/K,$$

where $K \to B_0$ is the polarization scheme of π in §3.2.

The following two propositions justify our definition of the dual torus fibration.

Proposition 5.1.3. *Let $\check{\pi} : \check{X}_0 \to B_0$ be the dual torus fibration of π. Then*

(i) *$\check{X}_0$ is a torsor under $\check{P}_0$.*

(ii) *For any $b \in B_0$, the fibers $\pi^{-1}(b)$ and $\check{\pi}^{-1}(b)$ are dual abelian varieties.*

(iii) *The abelian group homomorphism*

$$H^1(B_0, P_0) \to H^1(B_0, \check{P}_0)$$

induced by the polarization $\lambda : P_0 \to \check{P}_0$ sends $[X_0]$ to $[\check{X}_0]$.

Proof. The P_0-action on X_0 over B_0 uniquely descends to a $\check{P}_0$-action on $\check{X}_0$; notice an isomorphism $\check{P}_0 \cong P_0/K$ and quotient out everything by K. The local trivialization of the P_0-action on X_0 gives the local trivialization of the $\check{P}_0$-action on $\check{X}_0$. This proves $\check{X}_0$ is a $\check{P}_0$-torsor.

Write $F = \pi^{-1}(b)$ the fiber of π at b. Since X_0 is a P_0-torsor, they are fiberwise isomorphic and hence $\nu^{-1}(b) \cong F$. Similarly the fibers of $\check{\pi}$ and $\check{\nu}$ are isomorphic. But the fiber $\check{\nu}^{-1}(b)$ is isomorphic to $\check{F}$ because $\check{\nu}$ is the dual abelian scheme of ν. The second item follows. The third item is clear from the Čech cohomology description of the torsors. $\square$

Proposition 5.1.4. *If $\pi : X \to B$ admits at least one rational section then the dual torus fibration $\check{\pi} : \check{X}_0 \to B_0$ is isomorphic to $\mathrm{Pic}^0_{X_0/B_0} \to B_0$.*

Proof. If π has a rational section then the torus fibration $\pi : X_0 \to B_0$ has a section by Proposition 1.2.8. This implies X_0 is a trivial P_0-torsor, and hence $\check{X}_0$ is a trivial $\check{P}_0$-torsor by Proposition 5.1.3. Thus $\check{X}_0 \cong \check{P}_0 \cong \mathrm{Pic}^0_{X_0/B_0}$ by Proposition 3.1.9. $\square$

5.2 The dual Lagrangian fibration for known deformation types

This section is the main section of the entire article. We begin with a technical main result of this article.

Theorem 5.2.1. *Conjecture 4.3.2 holds if X is of $K3^{[n]}$, Kum_n, OG10 or OG6-type. That is, the polarization scheme is a constant group scheme contained in the image of* (4.3.1).

Proof. Recall from Theorem 3.2.3 and 4.4.2 that we have an explicit computation of the group $\mathrm{Aut}^\circ(X/B)$ and the polarization type. When X is of $\mathrm{K3}^{[n]}$ or OG10-type, both the polarization scheme K and $\mathrm{Aut}^\circ(X/B)$ are trivial, so the claim is obvious. When X is of OG6-type, lattice theory forces $\mathrm{div}(h) = 1$ as shown in [MR21, Lem 7.1]. Therefore, Proposition 4.3.7 applies and we get an inclusion $\mathrm{Aut}^\circ(X/B) \hookrightarrow K$. Notice every fiber of K is abstractly the same group as $\mathrm{Aut}^\circ(X/B)$. This means the inclusion is forced to be an equality fiberwise, whence the global equality $\mathrm{Aut}^\circ(X/B) = K$.

Assume X is of Kum_n-type. By [Wie18], the given Lagrangian fibration $\pi : X \to B$ is deformation equivalent to the moduli of torsion sheaves construction $\pi' : X' \to B'$ in Example 1.3.14 and its deformation type is determined by the value $d_1 = \mathrm{div}(h)$. If $n \neq 3$ or $d_1 \neq 2$, then we are in the non-exceptional case of Proposition 4.4.4, so $\mathrm{Aut}^\circ(X'/B') = K'$. Use Proposition 4.3.5 to deform this to $\mathrm{Aut}^\circ(X/B) \hookrightarrow K$. The inclusion has to be an equality fiberwise by comparing their sizes, so again we obtain a global equality $\mathrm{Aut}^\circ(X/B) = K$.

It remains to consider one exceptional case when $n = 3$ and $d_1 = 2$. In this case, on the one hand from Proposition 4.3.7 we have an inclusion $\mathrm{Aut}^\circ(X'/B') \hookrightarrow K'_2$, and on the other hand from Proposition 4.4.4, we have $K' \hookrightarrow \mathrm{Aut}^\circ(X'/B')$. We have a sequence of inclusions of group schemes

$$K' \hookrightarrow \mathrm{Aut}^\circ(X'/B') \hookrightarrow K'_2.$$

The latter inclusion deforms to an inclusion $\mathrm{Aut}^\circ(X/B) \hookrightarrow K_2$ for the original Lagrangian fibration π by Proposition 4.3.5. But note that $K \subset K_2$ can be characterized as a topological group scheme as $K = 2K_2$. Hence the former inclusion is also preserved by deformations and yields an inclusion $K \hookrightarrow \mathrm{Aut}^\circ(X/B)$, proving our desired claim. □

Remark 5.2.2. We have in fact proved $K = \mathrm{Aut}^\circ(X/B)$ for most of the cases, except for a single case when X is of Kum_3-type and π has a polarization type $(1, 2, 2)$. In this case, $K \subset \mathrm{Aut}^\circ(X/B)$ is an order 2 subgroup. When $\pi : X \to B$ is a moduli of torsion sheaves construction in Example 1.3.14, then K can be characterized by the translation automorphisms as we have seen in the explicit computations in §4.4. The exceptional case is pointed out to us by Salvatore Floccari.

A direct consequence of this theorem is a promised compactification of the dual torus fibration $\check{\pi} : \check{X}_0 \to B_0$ into $\check{\pi} : \check{X} \to B$.

Theorem 5.2.3 (Main theorem)**.** *Let $\pi : X \to B$ be a Lagrangian fibered hyper-Kähler manifold of $K3^{[n]}$, Kum_n, OG10 or OG6-type. Let K be the polarization scheme, considered as a subgroup of* $\mathrm{Aut}^\circ(X/B)$ *by Theorem 5.2.1. Then*

$$\check{X} = X/K \qquad \textit{for} \quad \check{\pi} : \check{X} \to B$$

defines a compactification of the dual torus fibration $\check{\pi} : \check{X}_0 \to B_0$ in Definition 5.1.2.

Proof. The dual torus fibration is defined by $\check{X}_0 = X_0/K$. For known deformation types, $K \subset \mathrm{Aut}^\circ(X/B)$ is a constant group scheme and acts on the entire $\pi : X \to B$ by Theorem 5.2.1. Therefore, the quotient $\check{X} = X/K$ makes sense and compactifies the dual torus fibration $\check{X}_0 = X_0/K$. □

Remark 5.2.4. After all, our construction of the dual Lagrangian fibration is very simple. The reader may notice the only important ingredient we used is Conjecture 4.3.2, after setting up the relevant notions correctly. However, we would like to emphasize Conjecture 4.3.2 is nontrivial. For example, the conjecture depends on the primitiveness of the polarization λ; if we consider twice a primitive polarization then its kernel $K_2 = \ker(2\lambda : P_0 \to \check{P}_0)$ will certainly *not* be contained in $\mathrm{Aut}^\circ(X/B)$. Note also that one does not expect such a simple quotient construction works for (special Lagrangian fibrations of) Calabi–Yau manifolds.

The polarization scheme $K \subset \mathrm{Aut}^\circ(X/B)$ is computed for all known deformation types of hyper-Kähler manifolds X in Theorem 3.2.3. When X is of $\mathrm{K3}^{[n]}$ or OG10-type, the group is trivial and hence $\check{X} = X$. We can call them "self-dual". When X is of Kum_n or OG6-type, the group is nontrivial and hence $\check{X}$ cannot be *homeomorphic* to X (their local fundamental groups are different around the quotient singularities[2]). We call the corresponding $\check{X}$ the *dual Kummer variety* and *dual OG6*, respectively.

Since the quotient is taken by H^2-trivial automorphisms, the resulting space $\check{X}$ satisfies many pleasant properties. We collect some properties of $\check{\pi}$ and $\check{X}$ in the following propositions, but we will refer to [Kim21] for their proof. We also omit the definitions of primitive symplectic orbifolds and irreducible symplectic varieties. See, e.g., [BL18], [Sch20], [Men20] or [Kim21, §A] for their precise definitions.

Proposition 5.2.5. *Keep the notation in Theorem 5.2.3. Then*

(i) *$\check{X}$ is a compact primitive symplectic orbifold and also an irreducible symplectic variety. It is simply connected.*

[2]We have learned this fact from Mirko Mauri.

(ii) $\check{\pi} : \check{X} \to B$ *is a Lagrangian fibration of a possibly singular hyper-Kähler variety* $\check{X}$.

(iii) *The dual Kummer variety and dual OG6 have nonempty singular loci* $\check{X}_{\mathrm{sing}} \subset \check{X}$ *of codimension* ≥ 4. *We have* $\check{\pi}(\check{X}_{\mathrm{sing}}) \subset \Delta$.

Note that the dual Kummer variety and dual OG6 do not admit a symplectic resolution because their singular loci are of codimension ≥ 4. The following result says X and $\check{X}$ have identical deformation behaviors:

Proposition 5.2.6. *Keep the notation in Theorem 5.2.3. Denote by*

$$\mathcal{X} \to \mathrm{Def}(X), \qquad \mathcal{X}_H \to \mathcal{B} \to \mathrm{Def}(X, H)$$

the universal deformations of X *and* π, *respectively. Then*

(i) $\mathcal{X} / \mathrm{Aut}^\circ(X/B) \to \mathrm{Def}(X)$ *is the (locally trivial) universal deformation of* $\check{X}$.

(ii) $\mathcal{X}_H / \mathrm{Aut}^\circ(X/B) \to \mathcal{B} \to \mathrm{Def}(X, H)$ *is the (locally trivial) universal deformation of* $\check{\pi}$.

The following compare the cohomological properties of X and $\check{X}$.

Proposition 5.2.7. *Keep the notation in Theorem 5.2.3. Then*

(i) *The Beauville–Bogomolov quadratic space* $(H^2(\check{X}, \mathbb{Q}), \bar{q}_{\check{X}})$ *is isomorphic to* $(H^2(X, \mathbb{Q}), \bar{q}_X)$. *The Fujiki constant of* $\check{X}$ *is* $c_{\check{X}} = 1/c_X$.

(ii) *The LLV algebra* $\mathfrak{g}$ *of* X *and* $\check{X}$ *are isomorphic. The pullback* $H^*(\check{X}, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ *is an injective* $\mathfrak{g}$*-module homomorphism.*

(iii) $H^2(\check{X}, \mathbb{Q})$ *and* $H^2(X, \mathbb{Q})$ *are isomorphic as Hodge structures. The Mumford–Tate algebras of* $H^*(X, \mathbb{Q})$ *and* $H^*(\check{X}, \mathbb{Q})$ *are isomorphic.*

Appendix A

Representation theory backgrounds

Introduction

Representation theory of special orthogonal Lie algebras (algebraic groups) are of particular interest for the study of the cohomology of hyper-Kähler manifolds. We provide a summary of their basic representation theory facts here. We will be particularly interested in the theory over the rational number field $\mathbb{Q}$. Our main reference is [FH91], [Kir08] and [Mil17]. Throughout this section, we denote by

$$(V, q) : \text{nondegenerate quadratic space over } \mathbb{Q} \text{ of dimension } b, \qquad \mathfrak{g} = \mathfrak{so}(V, q).$$

A $\mathfrak{g}$-*module* is a finite dimensional $\mathbb{Q}$-vector space E equipped with a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(E)$. Every $\mathfrak{g}$-module in this article is assumed to be finite dimensional. We will often base change the Lie algebra to $\mathfrak{g}_{\mathbb{C}}$. A $\mathfrak{g}_{\mathbb{C}}$-*module* is a finite dimensional $\mathbb{C}$-vector space F equipped with a Lie algebra homomorphism $\mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(F)$.

A.1 Irreducible representations of special orthogonal Lie algebras

The goal of this section is to highlight some representation theory facts on the special orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(V, q)$. Since $\mathfrak{g}$ is a simple $\mathbb{Q}$-Lie algebra, any (finite dimensional) $\mathfrak{g}$-module is completely reducible. This means we can concentrate on the study of irreducible (=simple) $\mathfrak{g}$-modules. The study of irreducible $\mathfrak{g}$-modules depend on the parity of $b = \dim V$.

A.1.1 Type B

Assume $b = 2r+1 \geq 3$ is odd, so that $\mathfrak{g}$ is a type B_r simple Lie algebra.

Any $\mathfrak{g}$-module E induces a $\mathfrak{g}_{\mathbb{C}}$-module $E_{\mathbb{C}}$ after base change. The representation theory of $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}(b,\mathbb{C})$ is well-known to be classified by its highest weights. Let us briefly recall this. Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$. There exists a preferred choice of a basis $\{\varepsilon_1, \cdots, \varepsilon_r\}$ of $\mathfrak{h}_{\mathbb{R}}^{\vee}$ such that the simple roots become $\varepsilon_1 - \varepsilon_2, \cdots, \varepsilon_{r-1} - \varepsilon_r$ and ε_r. (With this basis, the weights associated to the standard $\mathfrak{g}_{\mathbb{C}}$-module $V_{\mathbb{C}}$ is $0, \pm\varepsilon_1, \cdots, \pm\varepsilon_r$.) Let $\Lambda, \Lambda^+ \subset \mathfrak{h}_{\mathbb{R}}^{\vee}$ be the lattice of weights and the monoid of dominant weights of $\mathfrak{g}_{\mathbb{C}}$, respectively. We can explicitly describe them in terms of the preferred basis

$$\begin{aligned} \Lambda &= \{\lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i : \lambda_i \in \tfrac{1}{2}\mathbb{Z}, \quad \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i,j\}, \\ \Lambda^+ &= \{\lambda \in \Lambda : \lambda_1 \geq \cdots \geq \lambda_r \geq 0\}. \end{aligned} \tag{A.1.1}$$

Any element $\lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i$ contained in Λ^+ is called a *dominant weight*. We will frequently denote it by $\lambda = (\lambda_1, \cdots, \lambda_r)$, and omit the zeros appearing at the end for simplicity. For example, $\lambda = (2,1)$ denotes a dominant weight $2\varepsilon_1 + \varepsilon_2$ for any $r \geq 2$. Given any dominant weight λ, there exists a unique irreducible $\mathfrak{g}_{\mathbb{C}}$-module $V_{\lambda,\mathbb{C}}$ with the "highest weight" λ and conversely, any irreducible $\mathfrak{g}_{\mathbb{C}}$-module is isomorphic to $V_{\lambda,\mathbb{C}}$ for some dominant weight λ. We call $V_{\lambda,\mathbb{C}}$ an irreducible $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ.

The classification of irreducible $\mathfrak{g}$-modules over $\mathbb{Q}$ is more complicated than the theory over $\mathbb{C}$. See [Mil17, §25.d] or the original paper [Tit71]. Luckily, we can use [DMOS82, Prop 3.1(a)] and deduce the following.

Proposition A.1.2. *Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant weight of $\mathfrak{g}$. If λ_i are integers, then there exists a unique irreducible $\mathfrak{g}$-module V_λ such that its base change over $\mathbb{C}$ is the irreducible $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ.*[1] *We call this V_λ an irreducible $\mathfrak{g}$-module of highest weight λ.*

If λ_i are half-integers but not integers, then we do not know when $V_{\lambda,\mathbb{C}}$ are defined over $\mathbb{Q}$. Over $\mathbb{R}$, this issue is discussed in [Del99].

A.1.2 Type D

Assume $b = 2r \geq 4$ is even, so that $\mathfrak{g}$ is a type D_r simple Lie algebra.

[1]In other words, V_λ is absolutely irreducible.

Let us first describe the representation theory of $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}(b, \mathbb{C})$. Fix a Cartan and a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. There is again a preferred choice of basis in this situation, making $\pm\varepsilon_1, \cdots, \pm\varepsilon_r$ the weights associated to the standard $\mathfrak{g}_{\mathbb{C}}$-module $V_{\mathbb{C}}$. The lattice of weights and monoid of dominant weights are explicitly described as

$$\Lambda = \{\lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i : \lambda_i \in \tfrac{1}{2}\mathbb{Z}, \quad \lambda_i - \lambda_j \in \mathbb{Z} \text{ for all } i, j\},$$

$$\Lambda^+ = \{\lambda \in \Lambda : \lambda_1 \geq \cdots \geq \lambda_{r-1} \geq |\lambda_r| \geq 0\}.$$

Again we write $(\lambda_1, \cdots, \lambda_r)$ for a dominant integral weight $\lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i$ and omit 0's at the end if possible. The irreducible $\mathfrak{g}_{\mathbb{C}}$-modules are classified by the dominant weights λ. A similar proposition on irreducible $\mathfrak{g}$-modules holds, but this time slightly more complicated.

Proposition A.1.3. *Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be a dominant weight with integer λ_i's.*

(i) *If $\lambda_r = 0$ then there exists a unique irreducible $\mathfrak{g}$-module V_λ such that its base change over $\mathbb{C}$ is the irreducible $\mathfrak{g}_{\mathbb{C}}$-module $V_{\lambda,\mathbb{C}}$ of highest weight λ.*

(ii) *Assume $\lambda_r \neq 0$ and denote $\lambda' = (\lambda_1, \cdots, \lambda_{r-1}, -\lambda_r)$. Then there are two possibilities:*

  (a) *Both $V_{\lambda,\mathbb{C}}$ and $V_{\lambda',\mathbb{C}}$ are defined over $\mathbb{Q}$.*

  (b) *There exists a unique irreducible $\mathfrak{g}$-module whose base change over $\mathbb{C}$ is $V_{\lambda,\mathbb{C}} \oplus V_{\lambda',\mathbb{C}}$.*

A.2 Restricting representations to Lie subalgebras

Let E be a $\mathfrak{g}$-module and g' any Lie subalgebra of $\mathfrak{g}$. A *restriction representation* (or *restriction of scalars*) of E to $\mathfrak{g}'$ is a $\mathfrak{g}'$-module structure on E defined by a composition

$$\mathfrak{g}' \hookrightarrow \mathfrak{g} \to \mathfrak{gl}(E).$$

We present two lemmas on restriction representations in this section. They are crucially used in our computation of the LLV structures in §2.2.

A.2.1 Two lemmas on restriction representations

Recall that we are assuming $\mathfrak{g} = \mathfrak{so}(V, q)$ for a nondegenerate quadratic space (V, q).

For the first statement, we further assume the quadratic space (V, q) is decomposed into

$$(V, q) = (\bar{V}, \bar{q}) \oplus U,$$

where the latter space is the hyperbolic plane, a 2-dimensional quadratic space with a Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We call in this case (V, q) is a *Mukai completion* of $(\bar{V}, \bar{q})$. A formal argument shows there exists a semisimple element $h \in \mathfrak{g} = \mathfrak{so}(V,q)$ such that we have an h-eigenspace decomposition (see [KSV19])

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}, \qquad \mathfrak{g}_0 = \bar{\mathfrak{g}} \oplus \mathbb{Q}h, \quad \bar{\mathfrak{g}} = \mathfrak{so}(\bar{V}, \bar{q}), \quad \mathfrak{g}_{-2} \cong \mathfrak{g}_2 \cong \bar{V}. \tag{A.2.1}$$

We call this decomposition the *Mukai completion* of $\bar{\mathfrak{g}} = \mathfrak{so}(\bar{V}, \bar{q})$ into $\mathfrak{g} = \mathfrak{so}(V,q)$.

Lemma A.2.2. *Let $(\bar{V}, \bar{q})$ be a nondegenerate quadratic space, (V,q) its Mukai completion, $\mathfrak{g} = \mathfrak{so}(V,q)$ and $\mathfrak{g}_0 = \mathfrak{so}(\bar{V}, \bar{q}) \oplus \mathbb{Q}h$ as in* (A.2.1)*. Then any $\mathfrak{g}$-module E is determined by its restricted $\mathfrak{g}_0$-module structure (up to isomorphism).*

For the second statement, we assume there exists a nondegenerate quadratic subspace $(T,q) \subset (V,q)$ of dimension $b-1$. The orthogonal complement of T splits V into

$$(V,q) = (T,q) \oplus \langle a \rangle,$$

where $\langle a \rangle$ denotes a 1-dimensional quadratic space equipped with a 1×1 Gram matrix $a \in \mathbb{Q}^\times$.

Lemma A.2.3. *Let (V,q) be a nondegenerate quadratic space of dimension $b = 2r+1$, (T,q) its nondegenerate subspace of dimension $b-1 = 2r$, $\mathfrak{g} = \mathfrak{so}(V,q)$ and $\mathfrak{m} = \mathfrak{so}(T,q)$. Then*

(i) *Any $\mathfrak{g}$-module E is determined by its restricted $\mathfrak{m}$-module structure.*

(ii) *The formal $\mathfrak{m}$-character of E is the same as its formal $\mathfrak{g}$-character.*

We will recall the notion of the formal character in Definition A.2.4. We also note that Lemma A.2.3 does not hold when $b = 2r$. The reason will be evident after the proof of it.

A.2.2 The representation ring of special orthogonal Lie algebras

Let us temporarily assume $\mathfrak{g}$ is any reductive Lie algebra over $\mathbb{Q}$. Consider the categories $\mathrm{Rep}(\mathfrak{g})$ and $\mathrm{Rep}(\mathfrak{g}_\mathbb{C})$ of finite dimensional $\mathfrak{g}$-modules and $\mathfrak{g}_\mathbb{C}$-modules, respectively. They are both semisimple categories, i.e., every object in the category is completely reducible. Their Grothendieck rings $K(\mathfrak{g})$ and $K(\mathfrak{g}_\mathbb{C})$ are called the *representation ring* of $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$.

Any $\mathfrak{g}$-module E up to isomorphism is determined by its base change $E_\mathbb{C}$ as a $\mathfrak{g}_\mathbb{C}$-module. See, e.g., [Mil17, §25.d]. This has a following interpretation. Consider a base change functor

$$\mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{g}_\mathbb{C}), \qquad E \mapsto E_\mathbb{C}.$$

The induced ring homomorphism $K(\mathfrak{g}) \to K(\mathfrak{g}_\mathbb{C})$ is then injective.

The representation ring $K(\mathfrak{g}_\mathbb{C})$ of the orthogonal Lie algebra $\mathfrak{g}_\mathbb{C} \cong \mathfrak{so}(b, \mathbb{C})$ is completely understood. Let Λ be the weight lattice of $\mathfrak{g}_\mathbb{C}$ and $\mathbb{Z}\Lambda$ its group ring. We use the notation $e^\mu \in \mathbb{Z}\Lambda$ to represent $\mu \in \Lambda$ as an element in the group ring $\mathbb{Z}\Lambda$.

Definition A.2.4. Let F be any $\mathfrak{g}_\mathbb{C}$-module. Consider its weight decomposition $F = \bigoplus_{\mu\in\Lambda} F(\mu)$, where $F(\mu)$ indicates the weight μ subvector space of F. We define the *formal character map* of $\mathfrak{g}_\mathbb{C}$ by a ring homomorphism

$$\mathrm{ch} : K(\mathfrak{g}_\mathbb{C}) \to \mathbb{Z}\Lambda, \qquad [F] \mapsto \sum_{\mu\in\Lambda} \dim F(\mu) \cdot e^\mu.$$

Recall the notion of the Weyl group of $\mathfrak{g}_\mathbb{C}$, a group generated by all reflections s_α with respect to the simple roots α. The following result is standard.

Theorem A.2.5. *The formal character map* ch *is injective, and its image is precisely* $(\mathbb{Z}\Lambda)^W$*. That is,* $\mathrm{ch} : K(\mathfrak{g}_\mathbb{C}) \to (\mathbb{Z}\Lambda)^W$ *is a ring isomorphism.*

Let us now specialize our discussion to the case of our primary interest, $\mathfrak{g} = \mathfrak{so}(V, q)$ for a nondegenerate quadratic space (V, q) over $\mathbb{Q}$. Assume $b = 2r + 1$ is odd for $r \geq 1$. Recall we had a preferred choice of basis $\varepsilon_1, \cdots, \varepsilon_r$ of the dual Cartan algebra. Denote $x_i = e^{\varepsilon_i}$ for $i = 1, \cdots, r$. The description (A.1.1) of Λ translates to

$$\mathbb{Z}\Lambda = \mathbb{Z}[x_1^{\pm1}, \cdots, x_r^{\pm1}, (x_1 \cdots x_r)^{\frac{1}{2}}].$$

The Weyl group W_{2r+1} of the Lie algebra is isomorphic to $\mathfrak{S}_r \ltimes (\mathbb{Z}/2)^{\times r}$. It acts on the group ring $\mathbb{Z}[x_1^{\pm1}, \cdots, x_r^{\pm1}, (x_1 \cdots x_r)^{\frac{1}{2}}]$ as follows: $\mathfrak{S}_r$ acts as a permutation on $x_1, \cdots, x_r$, and the i-th factor $\mathbb{Z}/2$ acts as an involution $x_i \mapsto x_i^{-1}$. Theorem A.2.5 tells us an explicit description of $K(\mathfrak{g}_\mathbb{C})$ by an isomorphism

$$\mathrm{ch} : K(\mathfrak{g}_\mathbb{C}) \to \mathbb{Z}[x_1^{\pm1}, \cdots, x_r^{\pm1}, (x_1 \cdots x_r)^{\frac{1}{2}}]^{W_{2r+1}}. \tag{A.2.6}$$

Assume $b = 2r$ is even for $r \geq 2$. Again write $x_i = e^{\varepsilon_i}$, where $\varepsilon_1, \cdots, \varepsilon_r$ is a preferred basis. The group ring $\mathbb{Z}\Lambda$ has the same description as above. However, the Weyl group W_{2r} becomes smaller, an order 2 subgroup of $\mathfrak{S}_r \ltimes (\mathbb{Z}/2)^{\times r}$ consisting of elements with even number of 1's in $(\mathbb{Z}/2)^{\times r}$. It acts on the group ring $\mathbb{Z}\Lambda$ in the same way. Theorem A.2.5 gives us an isomorphism

$$\mathrm{ch} : K(\mathfrak{g}_\mathbb{C}) \to \mathbb{Z}[x_1^{\pm1}, \cdots, x_r^{\pm1}, (x_1 \cdots x_r)^{\frac{1}{2}}]^{W_{2r}}. \tag{A.2.7}$$

A.2.3 Proof of the lemmas

With the notion of the representation ring, Lemma A.2.2 and A.2.3 are equivalent to the following.

Lemma A.2.8. (i) *In the setting of Lemma A.2.2, the ring homomorphism* $\mathrm{Res}: K(\mathfrak{g}) \to K(\mathfrak{g}_0)$ *is injective.*

(ii) *In the setting of Lemma A.2.3, the ring homomorphism* $\mathrm{Res}: K(\mathfrak{g}) \to K(\mathfrak{m})$ *is injective. It becomes an honest inclusion after describing* $K(\mathfrak{g})$ *and* $K(\mathfrak{m})$ *using the formal character isomorphisms* (A.2.6) *and* (A.2.7).

Proof. Consider the commutative diagrams

$$\begin{array}{ccc} K(\mathfrak{g}) & \xrightarrow{\mathrm{Res}} & K(\mathfrak{g}_0) \\ \downarrow & & \downarrow \\ K(\mathfrak{g}_{\mathbb{C}}) & \xrightarrow{\mathrm{Res}} & K((\mathfrak{g}_0)_{\mathbb{C}}) \end{array}, \qquad \begin{array}{ccc} K(\mathfrak{g}) & \xrightarrow{\mathrm{Res}} & K(\mathfrak{m}) \\ \downarrow & & \downarrow \\ K(\mathfrak{g}_{\mathbb{C}}) & \xrightarrow{\mathrm{Res}} & K(\mathfrak{m}_{\mathbb{C}}) \end{array},$$

where all the vertical maps are base changes and the horizontal maps are restrictions. We have shown in the previous section that the base change map is injective. Therefore, it is enough to prove the lower maps $\mathrm{Res}: K(\mathfrak{g}_{\mathbb{C}}) \to K((\mathfrak{g}_0)_{\mathbb{C}})$ and $\mathrm{Res}: K(\mathfrak{g}_{\mathbb{C}}) \to K(\mathfrak{m}_{\mathbb{C}})$ are injective.

For the first restriction map, note that $(\mathfrak{g}_0)_{\mathbb{C}} = \bar{\mathfrak{g}}_{\mathbb{C}} \times \mathbb{C}h$. Hence its representation ring is $K((\mathfrak{g}_0)_{\mathbb{C}}) \cong K(\bar{\mathfrak{g}}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$. The formal character isomorphism gives us explicit descriptions of $K(\mathfrak{g}_{\mathbb{C}})$ and $K((\mathfrak{g}_0)_{\mathbb{C}})$. If $r = 2b+1$ is odd, then the restriction map becomes

$$\mathrm{Res}: \mathbb{Z}[x_1^{\pm 1}, \cdots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\frac{1}{2}}]^{W_{2r+1}} \to \mathbb{Z}[x_1^{\pm 1}, \cdots, x_{r-1}^{\pm 1}, (x_1 \cdots x_{r-1})^{\frac{1}{2}}, t^{\pm 1}]^{W_{2r-1}},$$

sending $\mathrm{Res}(x_i) = x_i$ for $i = 1, \cdots, r-1$ and $\mathrm{Res}(x_r) = t^2$.[2] This map is injective. Similar computation also works when r is even.

For the second restriction map, again use the formal character isomorphism to make the homomorphism into

$$\mathrm{Res}: \mathbb{Z}[x_1^{\pm 1}, \cdots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\frac{1}{2}}]^{W_{2r+1}} \to \mathbb{Z}[x_1^{\pm 1}, \cdots, x_r^{\pm 1}, (x_1 \cdots x_r)^{\frac{1}{2}}]^{W_{2r}}.$$

This is an honest inclusion because the Weyl group W_{2r} is an order 2 subgroup of the Weyl group W_{2r+1}. □

[2]To make the restriction map as desired we need to choose a Cartan subalgebra of $\mathfrak{g}$ precisely by the Cartan subalgebra of $\bar{\mathfrak{g}}$ together with h.

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