Prismatic cohomology and p-adic homotopy theory

A Dissertation presented

by

Tobias Shin

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2022

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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This dissertation is an observation about the recent theory of prismatic cohomology developed by Bhatt and Scholze. In particular, by applying a functor of Mandell, we see that the étale comparison theorem in the prismatic theory reproduces the p-adic homotopy type for a smooth complex variety with good reduction mod p. Historically it was known, by work of Artin and Mazur, that the mod p reduction reproduces the ℓ-adic homotopy type for the complex variety, where ℓ is a prime not equal to p. This latter constraint is imposed to disallow étale coverings with degree equal to the characteristic. The observation of the thesis is that the p-adic homotopy type of the complex variety can still be recovered from an integral model over a p-complete ring of integers, using the prismatic ideas of Bhatt and Scholze, and the homotopical algebra ideas of Mandell.

Dedicated to all my haters

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Acknowledgements

I would like to first thank my advisor Dennis Sullivan. He taught me a lot about mathematics, and how to understand and appreciate it. He also taught me a lot about how to properly advise and teach students – possibly more than he knows. The true meaning of legacy is how we will be remembered by those closest to us, after all.

Second, I would like to thank the friends I made during my time at Stony Brook. They made learning mathematics a worthwhile endeavor; art only means something when you share it with someone.

In no particular order, there is my generation of students: Yoon-Joo Kim, Xujia Chen, Mohamed El Alami, Ying Hong Tham, Jae Ho Cho, Qianyu Chen, Dahye Cho, and Sasha Viktorova for being shameless Degenerates, as there is no shame because it is shameful; and Nathan Chen, Jordan Rainone, Ben Wu, Matt Dannenberg for partying as hard as we worked, and keeping the spirit of Stony Brook alive.

There are my elders, the older generation, the ones I learned from: Aleksandar Milivojevic and Frederik Benirschke, my Bagel Brothers; Lisandra Hernandez and Silvia Ghinassi, my older sisters; Jack Burkart, Matt Lam, and Paul Frigge, Kings; David Stapleton (include note why he is in this list); Michael Albanese, for worst banter exchanged; and Mu Zhao and John Sheridan for putting up with me barging into Fred's home and office respectively.

There are the zoomers, the younger generation, who made me feel old and reminded me how corrupt all of it really is: Jiahao Hu, Lisa Marquand, Sam Auyeung, Jin-Cheng Guu, Tim Alland, Yankl Mazor, Dan Brogan, Runjie Hu, Owen Mireles Briones, and Genya Zhukova.

I would also like to thank the staff at the Stony Brook math department: Christine, Diane, Lynne, Pat. Without them, this thesis would not even have been submitted.

A special shout-out to certain friends outside of the program, such as Jess Maghakian, Brandon Gong, Jennifer Pai, Frederick Vu, Taher Dahleh, the Moroccan crew, and the entire Fathi family. Thank you for reminding me there is more to life than pure mathematics.

Finally, I would like to thank my family: my mother, my father, and my sister, for supporting me throughout this journey. Your love and encouragement meant more than I can express. So I won't.

Introduction

A *homotopy type* is an equivalence class of a CW complex up to homotopy equivalence. Restricting to simply connected spaces (or spaces whose fundamental group acts trivially on higher homotopy groups), one can apply functorial algebraic constructions to the homotopy groups. The processes are best summarized in the following fiber product diagram, called the *arithmetic square*:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \widehat{\mathbb{Z}} \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \widehat{\mathbb{Z}} \end{array}$$

Here, the map $\mathbb{Z} \to \widehat{\mathbb{Z}}$ is the *profinite completion* of $\mathbb{Z}$, where $\widehat{\mathbb{Z}}$ is the inverse limit over all finite index subgroups of $\mathbb{Z}$. The ring $\widehat{\mathbb{Z}}$ is isomorphic to the direct product $\prod_p \mathbb{Z}_p$, indexed over all primes p, where $\mathbb{Z}_p$ denotes the p-adic integers. The vertical map $\mathbb{Z} \to \mathbb{Q}$ is *localization* at 0, or *rationalization*. The homology, cohomology, and homotopy groups of a simply connected homotopy type are all abelian groups, and so can all be profinitely completed or localized appropriately.

The work of Sullivan [25] constructs spaces whose homotopy invariants are precisely those obtained from the above algebraic processes applied to the homotopy invariants of a given space. For example, he constructs the *rationalization* $X_{\mathbb{Q}}$ of a space X, whose homology, cohomology, and homotopy groups are all rationalizations of those of the original space. Similarly, there is a notion of a *p-adic completion* X_p of a space X whose homotopy information is the p-adic completion of the homotopy information of X. The space X_p is constructed as the limit of an inverse system of homotopy types, indexed over maps from X into spaces with finitely many non-zero homotopy groups, all which have order powers of p. We shall refer to the homotopy type of X_p as the **p-adic homotopy type** of X.

The classical work of Artin and Mazur [2] defines the notion of an *étale homotopy type* for an algebraic variety. They construct the étale homotopy type by taking étale hypercoverings of the variety, yielding a simplicial object (up to simplicial homotopy), and passing to geometric realizations. The resulting object is an inverse system of homotopy types. By restricting to coverings with degree powers of p, one has a **p-adic étale homotopy type**. A natural question one may ask is:

Given a smooth, proper complex variety X with good reduction mod p, what is the relationship between the p-adic homotopy type of X and the étale homotopy type of the reduction $[X \bmod p]$?

Artin and Mazur in fact prove that if a smooth, proper complex variety X has good reduction mod p, then the inverse systems defining the ℓ-adic homotopy type of X and the ℓ-adic étale homotopy type of $[X \bmod p]$ are equivalent for $\ell \neq p$. In particular, the ℓ-adic homotopy type of a smooth proper complex variety with good reduction mod p can be recovered from its mod p reduction, where $\ell \neq p$.

The work of this dissertation is to show that this last constraint can be removed, and that the p-adic homotopy type is accessible from p-adic information. This is accomplished by utilizing a theorem of Mandell, which says that the p-adic homotopy type of a space is obtained from the E_∞-algebra structure of its mod p singular cochains, in conjunction with the recently constructed prismatic cohomology theory of Bhatt and Scholze [7]. In particular, the argument will still utilize characteristic p information, but in the form of tilts of local perfectoid coverings rather than the reduction mod p alone. That is, we will use the reduction mod p with the data of an integral model over a suitable p-adically complete ring of integers.

Prismatic cohomology is defined with reference to a base prism (A, I), such that the resulting cohomology groups are modules over A (see section 3.1 for preliminaries for prismatic cohomology). The notation $R\Gamma(X, \Delta_{X/A})$ will denote the cochains of the prismatic cohomology of X over the base prism (A, I). In particular, this paper will fix one particular prism for its setting. First, let $\mathbb{C}_p$ denote the p-completion of an algebraic closure of $\mathbb{Q}_p$ and let $\mathcal{O}_{\mathbb{C}_p}$ denote its ring of integers. We can then take the *tilt*, $\mathcal{O}^\flat_{\mathbb{C}_p}$, which is defined as the inverse limit of $(... \xrightarrow{\phi} \mathcal{O}_{\mathbb{C}_p}/p \xrightarrow{\phi} \mathcal{O}_{\mathbb{C}_p}/p)$ where ϕ denotes the Frobenius. Then we can take the Witt vectors of the tilt to obtain our δ-ring $A = A_{\text{inf}} := W(\mathcal{O}^\flat_{\mathbb{C}_p})$. This ring has a natural map $\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ and our prism is then the pair $(A_{\text{inf}}, \ker\, \theta)$.

The work of Bhatt, Morrow, and Scholze in [5] yields the following corollary of the étale comparison theorem in [4] (Lecture IX, Theorem 5.1).

Theorem 0.1. *(Bhatt-Morrow-Scholze) Let $\mathfrak{X}$ denote a smooth formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ and let X denote its fiber over $\mathbb{C}_p$. Then there is a quasi-isomorphism*

$$(\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\text{inf}}}) \otimes^L_{A_{\text{inf}}} \mathbb{C}^\flat_p)^{\phi=1} \simeq \mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{F}_p)$$

where the notation $(-)^{\phi=1}$ denotes the the homotopy fixed points of the lift of Frobenius ϕ on the prismatic complex over the prism $(A_{\text{inf}}\ , \ker(A_{\text{inf}} \xrightarrow{\theta} \mathcal{O}_{\mathbb{C}_p}))$ (i.e., the mapping co-cone of the chain map $\phi - Id$ in the derived category (so $Cone(\phi - Id)[-1]$).

In particular, identify $\mathbb{C}_p$ with $\mathbb{C}$ and let $X \subset \mathbb{P}^n$ be a smooth projective variety over $\mathbb{C}$ that arises as the generic fiber of a smooth, proper scheme $X_\mathbb{Z}$ over $\mathbb{Z}$, e.g., one whose defining homogeneous polynomials have coefficients in $\mathbb{Z}$ and whose Jacobian matrix has rank $n + 1 - \dim(X)$ modulo p. Then the above quasi-isomorphism holds after taking $\mathfrak{X}$ as the formal scheme whose underlying topological space $X_{\mathcal{O}/p}$ is the fiber of $X_\mathbb{Z}$ over $\mathcal{O}_{\mathbb{C}_p}/p$, and whose structure sheaf is the p-completion of the structure sheaf of $X_\mathcal{O}$.

In fact, the proof of the above theorem proves more: there are natural E_∞-$\mathbb{F}_p$-*algebra* structures on the cochains above, such that the quasi-isomorphism is a map of E_∞-$\mathbb{F}_p$-algebras. The natural E_∞-$\mathbb{F}_p$-algebra structures on the cochains arise via Godement resolutions. The theorem above then states one can recover the full E_∞-algebra structure on the étale $\mathbb{F}_p$-cochains of a smooth proper complex variety with good reduction mod p, by natural constructions applied to the prismatic complex over the prism $(A_{\mathrm{inf}}, \ker(\theta))$. On the other hand, we have the following theorem of Mandell [15] (see also [15], Remark 5.1).

Theorem 0.2. *(Mandell) Let $\mathfrak{H}$ denote the homotopy category of connected p-complete nilpotent spaces of finite p-type, and let $h\overline{\mathfrak{E}}$ denote the homotopy category of E_∞-$\overline{\mathbb{F}_p}$-algebras. The singular cochain functor $C^*_{\mathrm{sing}}(-, \overline{\mathbb{F}_p})$ induces a contravariant equivalence from $\mathfrak{H}$ to a full subcategory of $h\overline{\mathfrak{E}}$. The quasi-inverse on the subcategory is given by $\overline{\mathbb{U}}$, the right derived functor of the functor $A \mapsto \mathrm{Hom}_{\overline{\mathfrak{E}}}(A, C^*(\Delta[n], \overline{\mathbb{F}_p}))$. Moreover, there is the following adjunction*

$$[X, \overline{\mathbb{U}}A] \cong [A, C^*_{\mathrm{sing}}(X, \overline{\mathbb{F}_p})]$$

where $[-,-]$ denotes morphisms in the respective homotopy category. Moreover, for X connected and of finite p-type, the natural map $X \to \overline{\mathbb{U}}C^*_{\mathrm{sing}}(X, \overline{\mathbb{F}_p})$ via the adjunction is naturally isomorphic to p-completion in the sense of Bousfield-Kan.

There is also the analog of the above theorem ([15], Proposition A.2, A.3) for the singular cochain functor with coefficients in $\mathbb{F}_p$.

Theorem 0.3. *(Mandell) Let $h\mathfrak{E}$ denote the homotopy category of E_∞-$\mathbb{F}_p$-algebras. The right derived functor $\mathbb{U}$ of the functor $A \mapsto \mathrm{Hom}_{\mathfrak{E}}(A, C^*(\Delta[n], \mathbb{F}_p))$ from the category of E_∞-$\mathbb{F}_p$-algebras to simplicial sets is right adjoint to the right derived functor of the singular cochain functor $C^*_{\mathrm{sing}}(-, \mathbb{F}_p)$, such that there is a natural isomorphism $LX \to \mathbb{U}C^*_{\mathrm{sing}}(X, \mathbb{F}_p)$ in the homotopy category for X connected, p-complete, nilpotent, and of finite p-type, where LX denotes the free loop space of X.*

In other words, taking coefficients in $\mathbb{F}_p$ recovers the free loop space of the p-adic homotopy type. An immediate consequence of the theorems above is the following main theorem of the thesis:

Theorem 0.4. *Let X be as in the hypotheses of Theorem 0.1. Assume further that X is nilpotent and finite p-type. Then $\mathbb{U}(\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\mathrm{inf}}}) \otimes^L_{A_{\mathrm{inf}}} \mathbb{C}_p^\flat)^{\phi=1}$ is the free loop space of the Bousfield-Kan p-completion of the complex variety X. Similarly, $\overline{\mathbb{U}}((\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\mathrm{inf}}}) \otimes^L_{A_{\mathrm{inf}}} \mathbb{C}_p^\flat)^{\phi=1} \otimes^L_{\mathbb{F}_p} \overline{\mathbb{F}_p})$ is the Sullivan p-completion of the complex variety X.*

Recall that for simply connected CW-complexes of finite type, the Bousfield-Kan p-completion and Sullivan p-completion coincide. So as a corollary we have

Corollary 0.1. *Let X be as in the hypotheses of Theorem 0.1. Assume X is simply connected and finite p-type. Then $\mathbb{U}(\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\mathrm{inf}}}) \otimes^L_{A_{\mathrm{inf}}} \mathbb{C}_p^\flat)^{\phi=1}$ is the free loop space of the p-completion of X, and $\overline{\mathbb{U}}((\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\mathrm{inf}}}) \otimes^L_{A_{\mathrm{inf}}} \mathbb{C}_p^\flat)^{\phi=1} \otimes^L_{\mathbb{F}_p} \overline{\mathbb{F}_p})$ is the p-completion of X.*

That is, for a smooth variety over $\mathbb{C}$ with good reduction mod p, natural constructions applied to its prismatic complex over A_{inf} yield its p-adic homotopy type (in the senses above).

Notation and conventions. All rings are commutative with unit. All varieties over $\mathbb{C}$ are assumed connected in the analytic topology. We use k to denote the ground field, usually $\mathbb{F}_p$ or $\overline{\mathbb{F}_p}$. All E_∞-k-algebras are chain complexes of k-modules with an action of a fixed

E_∞-operad $\mathcal{E}_k$ in Ch(k-mod), where $\mathcal{E}_k$ has a fixed map of operads $\mathcal{E}_k \to \mathcal{Z}_k$ to the Eilenberg-Zilber operad $\mathcal{Z}_k$. We refer the proof that such a map of operads always exists to [14]. Often we will omit the subscript k if it is clear in context. By a *quasi-isomorphism of E_∞-algebras*, we mean a morphism of E_∞-algebras that induces a quasi-isomorphism on the underlying chain complexes. A complex variety with *good reduction mod p* is a variety that admits an integral model over $\mathrm{Spec}(\mathbb{Z})$ such that the reduction mod p is regular.

Given a site $\mathcal{C}$, we write $\mathrm{AbSh}(\mathcal{C})$ for the category of abelian sheaves on $\mathcal{C}$. We write $\mathrm{Sh}(\mathcal{C})$ and $\mathrm{PSh}(\mathcal{C})$ for the category of sheaves and presheaves of sets respectively on $\mathcal{C}$, and $\mathrm{Sh}(\mathcal{C}, D)$ for sheaves with values in a category D. **Ab** denotes the category of abelian groups, and $D\mathrm{AbSh}(\mathcal{C})$ and $D\mathbf{Ab}$ denote the respective derived categories. If C is some category, then ΔC denotes the category of cosimplicial objects of C. We avoid any ∞-categorical language because the author does not know it.

The author would like to thank Joana Cirici and Dan Petersen for helpful comments about E_∞-algebras. He would also like to thank Frederick Vu for helping him learn prismatic cohomology from the ground up.

1 E_∞-algebras and homotopy types

In this section we review some preliminaries regarding operads and E_∞-algebras, following Kriz and May [13].

1.1 Operads and algebras

We fix a ground field k. In this paper, k is $\mathbb{F}_p$ or an algebraic closure $\overline{\mathbb{F}_p}$. All tensor products are over k.

Definition 1.1. *An* ***operad*** $\mathcal{O}$ *consists of a sequence $\mathcal{O}(n)$, $n \geq 0$ of cochain complexes over k, with a unit map $\eta : k \to \mathcal{O}(1)$, a right action of the symmetric group Σ_n on $\mathcal{O}(n)$ for each n, and maps (of cochain complexes)*

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes ... \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + ... + n_k)$$

for $k \geq 1$ and $n_j \geq 0$. The maps γ must satisfy the following diagrams.

a) (Associativity) We have the following commutative diagram, where $\sum_s j_s = j$ and $\sum_t i_t = i$, with $g_s = j_1 + ... + j_s$ and $h_s = i_{g_{s-1}+1} + ... + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{array}{ccc}
\mathcal{O}(k) \otimes (\bigotimes_{s=1}^k \mathcal{O}(j_s)) \otimes (\bigotimes_{r=1}^j \mathcal{O}(i_r)) & \xrightarrow{\gamma \otimes Id} & \mathcal{O}(j) \otimes (\bigotimes_{r=1}^j \mathcal{O}(i_r)) \\
 & & \big\downarrow \gamma \\
\big\downarrow \text{shuffle} & & \mathcal{O}(i) \\
 & & \big\uparrow \gamma \\
\mathcal{O}(k) \otimes (\bigotimes_{s=1}^k \mathcal{O}(j_s) \otimes (\bigotimes_{q=1}^{j_s} \mathcal{O}(i_{g_s - 1 + q}))) & \xrightarrow{Id \otimes (\otimes_s \gamma)} & \mathcal{O}(k) \otimes (\bigotimes_{s=1}^k \mathcal{O}(h_s))
\end{array}$$

b) (Unital) The following unit diagrams commute:

$$\begin{array}{ccc} \mathcal{O}(n) \otimes k^{\otimes n} & \xrightarrow{\simeq} & \mathcal{O}(n) \\ \downarrow{\scriptstyle Id\otimes\eta^n} & \nearrow{\scriptstyle \gamma} & \\ \mathcal{O}(n) \otimes \mathcal{O}(1)^{\otimes n} & & \end{array}$$

and

$$\begin{array}{ccc} k \otimes \mathcal{O}(j) & \xrightarrow{\simeq} & \mathcal{O}(j) \\ \downarrow{\scriptstyle \eta\otimes Id} & \nearrow{\scriptstyle \gamma} & \\ \mathcal{O}(1) \otimes \mathcal{O}(j) & & \end{array}$$

c) (Equivariance) The following diagrams commute, where $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$, *and* $\sigma(j_1, ..., j_k) \in \Sigma_k$ *permutes* k *blocks of letters as* σ *permutes* k *letters, and* $\tau_1 \oplus ... \oplus \tau_k \in \Sigma_k$ *is the block sum:*

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{O}(k) \otimes \mathcal{O}(j_{\sigma(1)}) \otimes ... \otimes \mathcal{O}(j_{\sigma(k)}) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \mathcal{O}(j) & \xrightarrow{\sigma(j_{\sigma(1)},...,j_{\sigma(k)})} & \mathcal{O}(j) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) & \xrightarrow{Id\otimes\tau_1\otimes...\otimes\tau_k} & \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \mathcal{O}(j) & \xrightarrow{\tau_1\oplus...\oplus\tau_k} & \mathcal{O}(j). \end{array}$$

The complexes $\mathcal{O}(n)$ parameterize n-ary operations, which take as input n values and produces one output, via the γ maps.

We now define what operads act upon.

Definition 1.2. *Let* $\mathcal{O}$ *be an operad. An* ***algebra*** A *over an operad is a cochain complex with maps*

$$\theta : \mathcal{O}(n) \otimes A^{\otimes n} \to A$$

for every $n \geq 0$ *that are associative, unital, and equivariant in the following sense.*

a) (Associativity) The following associativity diagrams commute, where $\sum j_s = j$

$$\begin{array}{ccc} \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes ... \otimes \mathcal{O}(j_k) \otimes A^{\otimes j} & \xrightarrow{\gamma\otimes Id} & \mathcal{O}(j) \otimes A^{\otimes j} \\ & & \downarrow{\scriptstyle\theta} \\ \downarrow{\scriptstyle \text{shuffle}} & & A \\ & & \uparrow{\scriptstyle\theta} \\ \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes A^{\otimes j_1} ... \otimes \mathcal{O}(j_k) \otimes A^{\otimes j_k} & \xrightarrow{Id\otimes\theta^k} & \mathcal{O}(k) \otimes A^{\otimes k} \end{array}$$

b) (Unital) The following unit diagram commutes:

$$\begin{array}{ccc} k \otimes A & \xrightarrow{\simeq} & A \\ \big\downarrow{\scriptstyle \eta\otimes Id} & \nearrow_{\theta} & \\ \mathcal{O}(1) \otimes A & & \end{array}$$

c) (Equivariance) The following diagram commutes, for $\sigma \in \Sigma_j$:

$$\begin{array}{ccc} \mathcal{O}(j) \otimes A^{\otimes j} & \xrightarrow{\sigma\otimes\sigma^{-1}} & \mathcal{O}(j) \otimes A^{\otimes j} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & A & \end{array} .$$

A *morphism* of $\mathcal{O}$-algebras is a morphism of complexes such that the operadic action commutes. A *quasi-isomorphism* of $\mathcal{O}$-algebras is a morphism of $\mathcal{O}$-algebras that is a quasi-isomorphism of the underlying complexes.

Definition 1.3. *An operad $\mathcal{O}$ is* ***unital*** *if $\mathcal{O}(0) = k$. If $\mathcal{O}$ is unital, then there are augmentation maps $\epsilon = \gamma : \mathcal{O}(n) \otimes \mathcal{O}(0)^n \to \mathcal{O}(0) = k$. We say a unital operad $\mathcal{O}$ is* ***acyclic*** *if the augmentations are quasi-isomorphisms for every n. If the unital operad $\mathcal{O}$ is acyclic and Σ_n acts freely on $\mathcal{O}(n)$ for every n, we say $\mathcal{O}$ is an E_∞**-operad**. An E_∞**-algebra** is any algebra over an E_∞-operad.*

Example 1.1. *Consider the singular chains functor $\Lambda : \Delta \to$ Ch(k-mod) that sends a simplex $[n]$ to the normalized cochain complex $C^*(\Delta^n, k)$. We define Λ^j to be the functor that sends $[n]$ to the j-fold tensor power $C^*(\Delta^n, k)^{\otimes j}$. Let $Hom(\Lambda^j, \Lambda)$ denote the set of natural transformations between these two functors. We can define an operad as follows: let $\mathcal{Z}(j) = \mathrm{Hom}(\Lambda^j, \Lambda)$ for every j. This is the* ***Eilenberg-Zilber operad*** *[11]. By the acyclic carrier methods of Eilenberg and Zilber, this operad is acyclic, but the actions of the symmetric group are not free. However, one can show there always exists an E_∞-operad $\mathcal{E} \to \mathcal{Z}$ (see [14] section 4) so that any $\mathcal{Z}$-algebra can be considered an E_∞-algebra with respect to the operad $\mathcal{E}$.*

Example 1.2. *An explicit example of an E_∞-operad, pointed out to the author by John Morgan, is that of the* ***Barratt-Eccles operad*** *$\mathcal{E}$. Here, for each j, the chain complex $\mathcal{E}(j)$ is generated in degree d by $(d+1)$-tuples $(w_0, ..., w_d)$ where each w_i is an element of the symmetric group Σ_j. The symmetric group acts freely by definition, and this operad admits a map of operads to the Eilenberg-Zilber operad above. For more details see Berger and Fresse [3].*

For more examples of algebras over operads, see section 2.2. Given two E_∞-operads $\mathcal{E}$ and $\mathcal{E}'$, the homotopy categories of algebras over each operad respectively are equivalent. Thus, for the entirety of this paper, when we say E_∞-algebra, we only make reference to one fixed E_∞-operad (with fixed map $\mathcal{E} \to \mathcal{Z}$ to the Eilenberg-Zilber operad) unless otherwise specified.

1.2 Homotopy types

Given a homotopy type X, one can take the p-completion of its homotopy groups. There is a construction (due to Sullivan) that builds a homotopy type X_p called the **p-completion of** X. The homotopy groups of X_p are precisely the p-completions of the homotopy groups of X. We say two spaces X and Y have the same p-adic homotopy type if there is a homotopy equivalence of their p-completions. Notice immediately then that X and X_p have the same p-adic homotopy type. In this section, we briefly review the notion of a p-completion (in the sense of Sullivan) and prove some elementary lemmas on base-changing for operads. For a field k, we let $\mathcal{E}_k$ denote the E_∞-operad for complexes of k-modules. We present the definition of the p-completion of a space in the sense of Sullivan below [25].

Definition 1.4. *Let X be a homotopy type. Consider the category $\{f\}_X$ where objects are maps*

$$X \xrightarrow{f} F$$

where F is connected with finitely many nonzero homotopy groups $\pi_i(F)$, all of which are finite groups of order a power of p for all i, and whose morphisms are given by homotopy commutative diagrams

$$\begin{array}{ccc} & & F \\ & \nearrow & \downarrow \\ X & & \\ & \searrow & \downarrow \\ & & F'. \end{array}$$

Let the notation $[X, F]$ denote the set of (based) homotopy classes of maps from X to F. For a given F with finitely many non-zero homotopy groups, all of which are finite groups, the set of homotopy classes of maps from a finite complex X (i.e., a homotopy type with finitely many attaching maps) is finite by basic obstruction theory. It is proved in [25] that the category above is actually cofiltered, and that one can take an inverse limit over all objects of the category.

Moreover, for such an F, we have the equivalence $[X, F] \xrightarrow{\simeq} \varprojlim_{X_\alpha} [X_\alpha, F]$, where X_α is a finite subcomplex of X. As each $[X_\alpha, F]$ is finite, this gives the set $[X, F]$ a topology as an inverse limit of finite discrete sets. The space $[X, F]$ is compact, Hausdorff, and totally disconnected.

Finally, since the category $\{f\}_X$ is cofiltered, Sullivan defines a homotopy functor

$$X_p(Y) := \varprojlim_{\{f\}_X} [Y, F]$$

from the homotopy category of spaces to the category of compact, Hausdorff, totally disconnected spaces. He then shows this functor is representable by a space X_p by appealing to the theory of Brownian compact functors.

Definition 1.5. *The **p-completion (in the sense of Sullivan)** of a homotopy type X is the (homotopy class of) map*

$$X \to X_p$$

corresponding to $\prod_{\{f\}}(X \xrightarrow{f} F)$, *i.e., the map from* X *to the inverse limit over all* $X \xrightarrow{f} F$. *The space* X_p *is the* ***p-adic homotopy type of*** X.

It is proven in [25] and the 1970 MIT notes that for a simply connected homotopy type X with finitely generated homotopy groups, we have that $\pi_*(X_p) \simeq (\pi_*(X))_p$ and $H^*(X,\mathbb{Z})_p \simeq H^*(X,\mathbb{Z}_p) \simeq H^*(X_p,\mathbb{Z}_p)$. Moreover, $H^*(X,\mathbb{Z}/p) \simeq H^*(X_p,\mathbb{Z}/p)$. The above results generalize to the case of "simple" fundamental group (i.e., where the fundamental group acts trivially on higher homotopy groups). In this case, we can say the p-adic homotopy type of a space X is simply a space X_p whose homotopy and integral cohomology are p-completions of those of X.

We restate below Mandell's main theorem 0.2 that we will utilize. The *homotopy category* of E_∞-algebras is obtained by forcing quasi-isomorphisms to be isomorphisms (following Quillen).

Theorem 1.1. *(Mandell) Let* $\mathfrak{H}$ *denote the homotopy category of connected p-complete nilpotent spaces of finite p-type, and let* $h\overline{\mathfrak{E}}$ *denote the homotopy category of* E_∞-$\overline{\mathbb{F}_p}$*-algebras. The singular cochain functor* $C^*_{\text{sing}}(-,\overline{\mathbb{F}_p})$ *induces a contravariant equivalence from* $\mathfrak{H}$ *to a full subcategory of* $h\overline{\mathfrak{E}}$. *The quasi-inverse on the subcategory is given by* $\overline{\mathbb{U}}$, *the right derived functor of the functor* $A \mapsto \operatorname{Hom}_{\overline{\mathfrak{E}}}(A, C^*(\Delta[n],\overline{\mathbb{F}_p}))$. *Moreover, there is the following adjunction*

$$[X, \overline{\mathbb{U}}A] \cong [A, C^*_{\text{sing}}(X,\overline{\mathbb{F}_p})]$$

where $[-,-]$ *denotes morphisms in the respective homotopy category. Moreover, for* X *connected and of finite p-type, the natural map* $X \to \overline{\mathbb{U}}C^*_{\text{sing}}(X,\overline{\mathbb{F}_p})$ *via the adjunction is naturally isomorphic to p-completion in the sense of Bousfield-Kan.*

The above theorem is about the p-adic homotopy type in the sense of *Bousfield-Kan*. We will not define this notion here; for its definition, see the book by Bousfield and Kan [8]. We will instead work with the p-adic homotopy type in the sense of Sullivan [25] throughout this paper. Below we will review some preliminary lemmas about base-change for operads, and show how Mandell's theorem relates the two notions of p-adic homotopy types, justifying why we work with the Sullivan type than the Bousfield-Kan type.

We state a lemma by Mandell, whose proof we refer to [15], in the paragraph preceding Proposition A.8.

Lemma 1.1. *There is a map of operads* $\mathcal{E}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to \mathcal{E}_{\overline{\mathbb{F}_p}}$. *By changing the operad* $\mathcal{E}_{\overline{\mathbb{F}_p}}$ *if necessary, this map is an isomorphism. This defines a functor from* E_∞-$\mathbb{F}_p$*-algebras to* E_∞-$\overline{\mathbb{F}_p}$*-algebras that sends* A *to* $A \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$.

Now we study the extended scalars $C^*_{\text{sing}}(X,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. This is an E_∞-$\overline{\mathbb{F}_p}$-algebra by the above lemma. However, there is another E_∞-$\overline{\mathbb{F}_p}$-algebra structure as described in the lemma below; we will prove that these two structures agree. See Appendix B in [15] for details.

Lemma 1.2. $C^*_{\text{sing}}(X,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ *is an algebra over* $\mathcal{E}_{\overline{\mathbb{F}_p}}$. *There is a natural map of* $\mathcal{E}_{\overline{\mathbb{F}_p}}$*-algebras* $C^*_{\text{sing}}(X,\mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to C^*_{\text{sing}}(X,\overline{\mathbb{F}_p})$.

Proof. Let $\{X_\alpha\}$ denote the inverse system of levelwise finite quotients of X; that is, X_α is a quotient of X as a simplicial set, and in each simplicial degree n, the set of n simplices is finite. We give $C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ the structure of an E_∞-$\overline{\mathbb{F}_p}$-algebra via the natural isomorphism $C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \cong \text{colim}_\alpha\, C^*(X_\alpha, \overline{\mathbb{F}_p})$. The inverse system of maps $X \to X_\alpha$ induces a map of E_∞-$\overline{\mathbb{F}_p}$-algebras $C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to C^*_{\text{sing}}(X, \overline{\mathbb{F}_p})$. □

The following argument was communicated to the author by Mandell.

Lemma 1.3. *The E_∞-$\overline{\mathbb{F}_p}$-algebra structure on $C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ from Lemma 1.1 and the E_∞-$\overline{\mathbb{F}_p}$-algebra structure from Lemma 1.2 agree.*

Proof. The maps $X \to X_\alpha$ induce a map of E_∞-$\overline{\mathbb{F}_p}$-algebras in the sense of Lemma 1.1

$$C^*_{\text{sing}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}.$$

There is a natural map $C^*_{\text{sing}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to C^*_{\text{sing}}(X_\alpha, \overline{\mathbb{F}_p})$ of E_∞-$\overline{\mathbb{F}_p}$-algebras, which is an isomorphism since X_α is degreewise finite; that is, the lemma holds for X_α by finiteness. Thus we obtain a system of maps of E_∞-$\overline{\mathbb{F}_p}$-algebras

$$C^*_{\text{sing}}(X_\alpha, \overline{\mathbb{F}_p}) \to C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}.$$

This system is filtered, so the colimit over α of $C^*(X_\alpha, \overline{\mathbb{F}_p})$ in the category of E_∞-$\overline{\mathbb{F}_p}$-algebras is the same colimit on the underlying differential graded modules over $\overline{\mathbb{F}_p}$. By universal property of colimits in E_∞-$\overline{\mathbb{F}_p}$-algebras, we obtain an E_∞-$\overline{\mathbb{F}_p}$-algebra map

$$\text{colim}_\alpha C^*_{\text{sing}}(X_\alpha, \overline{\mathbb{F}_p}) \to C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$$

which is an isomorphism on the underlying differential graded modules over $\overline{\mathbb{F}_p}$, and so is an isomorphism of E_∞-$\overline{\mathbb{F}_p}$-algebras. The E_∞-$\overline{\mathbb{F}_p}$-algebra structure on the colimit is in the sense of Lemma 1.2. □

The following theorem of Mandell ([15], Theorem B.1) relates the above discussion to the p-profinite completion of the variety X in the sense of Sullivan ([25], Section 3).

Theorem 1.2. *For any connected simplicial set X, the composite map*

$$X \to \overline{\mathbb{U}} C^*_{\text{sing}}(X, \overline{\mathbb{F}_p}) \to \overline{\mathbb{U}}(C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p})$$

is Sullivan p-completion.

The first map is from the unit of the adjunction in Theorem 0.2. By Lemma 1.2, we have a map $C^*_{\text{sing}}(X, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p} \to C^*_{\text{sing}}(X, \overline{\mathbb{F}_p})$. This induces the second map above, from the Bousfield-Kan completion to the Sullivan completion. If we assume X is simply connected, the second map above is an isomorphism. Thus, our notion of p-adic homotopy type in this paper will be in the sense of the Sullivan p-completion unless otherwise specified, though the distinction will not play a large part.

We very briefly and informally mention the Artin-Mazur construction of an *étale homotopy type* of an algebraic variety X over any field k. For a given variety, one can take a covering by étale opens $U \to X$ and take iterated fiber products: pairwise, then by triples, etc. This yields a simplicial set (scheme), called the *Čech nerve*, just as in the ordinary topological setting. To define the étale homotopy type, Artin and Mazur take a *hypercovering* by étale opens, which is roughly the Čech nerve construction, but modified as follows. Instead of taking the iterated fiber product at each step, one takes a finer étale open covering of the iterated fiber product. For example, instead of taking pairwise intersections $U \cap V$, one takes an étale open covering of the intersection $U \cap V$. One takes triple intersections in this new covering. For each triple intersection in the new covering, one takes *another* new covering by étale opens, and we reiterate the process. This yields a simplicial scheme, called a **hypercovering**.

Artin and Mazur then define a notion of simplicial homotopy for hypercoverings, and prove that the homotopy category of hypercoverings is cofiltered (so one can form inverse systems). The **étale homotopy type** of X is the inverse system of homotopy types, given by the geometric realizations of all hypercoverings of X (i.e., it is a pro-homotopy type). Artin and Mazur prove that the étale homotopy type is in fact **pro-finite**, i.e., it is isomorphic to the inverse system of homotopy types whose homotopy groups are all finite.

For a fixed p, a p-homotopy type is a space whose homotopy groups are finite groups with order powers of p. There is an evident inclusion of categories of p-homotopy types into all homotopy types, and likewise for the respective pro- categories. The adjoint to the inclusion functor is called **p-completion**. We can then consider the étale homotopy type up to isomorphism after p-completion. We call this equivalence class the **p-adic étale homotopy type**. The construction of this adjoint functor follows from using functoriality of p-completion for ordinary groups, applied to inverse system diagrams of groups. Thus the p-adic étale homotopy type is an inverse system of homotopy types, each of whose homotopy groups are all finite p-groups.

Artin and Mazur prove in [2] the following two statements: 1) for a complex variety X, the limit of its p-adic étale homotopy type is isomorphic to its p-adic homotopy type. 2) For a complex variety X with good reduction mod p, the limit of the ℓ-adic étale homotopy type of $[X \bmod p]$ is isomorphic to the ℓ-adic homotopy type of X, for $\ell \neq p$.

For the rest of this thesis, we shall no longer discuss ℓ.

2 Étale cochains and E_∞-Artin comparison

In this mainly expository section, we review some classical theorems regarding the E_∞-algebra structure on étale cochains. We emphasize there are no original theorems proven in this section; many of the arguments can be found in, for example, [9], [19], and [20], with the main conceptual ideas originating from Godement [10]. The aim of this section is to cover the following well known theorem.

Theorem 2.1. *Let X be a smooth complex variety. There is a quasi-isomorphism of E_∞-algebras between the singular $\mathbb{F}_p$-cochains $C^*_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{F}_p)$ and the étale $\mathbb{F}_p$-cochains $\mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{F}_p)$.*

Here is an outline of the section: first, we compare the étale site of X over $\mathbb{C}$ with the site of analytic open sets on its underlying complex manifold $X(\mathbb{C})$, by passing to the site of local homeomorphisms mapping to $X(\mathbb{C})$. We analyze the site of local homeomorphisms and show it has enough points. We then use the Godement resolution on all three sites to obtain E_∞-algebras on their sheaf cohomologies; the quasi-isomorphism of the underlying complexes is omitted.

We have the following theorem from SGA IV (XII-4) [1]; we provide a translation of part of the proof, with some details provided using lemmas from the Stacks Project [24].

Theorem 2.2. *Let X be a smooth complex variety. There is a zig-zag of sites*

$$\begin{array}{ccccc} & & X_{\mathrm{cl}} & & \\ & \overset{\delta}{\swarrow} & & \overset{\varepsilon}{\searrow} & \\ X(\mathbb{C}) & & & & X_{\text{ét}} \end{array}$$

where $X(\mathbb{C})$ denotes the site of analytic open sets on the underlying complex manifold of X, X_{cl} denotes the site of local homeomorphisms $U \to X(\mathbb{C})$, and $X_{\text{ét}}$ denotes the étale site of X.

Proof. An object of X_{cl} is a continous map of topological spaces $f : U \to X(\mathbb{C})$ such that for every point $x \in U$, there is a neighborhood U_x such that the restriction of f to U_x is a homeomorphism onto an open neighborhood around $f(x)$; that is, f is a local homeomorphism. Since inclusions of open sets in $X(\mathbb{C})$ are local homeomorphisms, we obtain a morphism of sites $\delta : X_{\mathrm{cl}} \to X(\mathbb{C})$ by the continuous functor $U \mapsto (U \hookrightarrow X(\mathbb{C}))$; this continuous functor is the inclusion of categories $X(\mathbb{C}) \subset X_{\mathrm{cl}}$.

On the other hand, let $f : X' \to X$ be étale. Then the induced map on the underlying smooth manifolds $f(\mathbb{C}) : X'(\mathbb{C}) \to X(\mathbb{C})$ is a local isomorphism, by the Jacobian criterion and implicit function theorem. The functor $X' \mapsto X'(\mathbb{C})$ then induces the morphism of sites $\varepsilon : X_{\mathrm{cl}} \to X_{\text{ét}}$. □ □

Lemma 2.1. *The functor δ_* sends surjective maps of sheaves of sets to surjective maps of sheaves of sets. Moreover, it is an equivalence of the associated topoi, and reflects injections and surjections (i.e., $\delta_* f$ is an injection (resp. surjection) implies f is an injection (resp. surjection)).*

Proof. For each local homeomorphism $f : U \to X(\mathbb{C})$, for each $x \in U$, there is a neighborhood homeomorphic to an open neighborhood around $f(x)$ in $X(\mathbb{C})$. Thus, we can cover U by open sets that are homeomorphic to open sets in $X(\mathbb{C})$; that is, there exists a family of open sets $\{U_i \hookrightarrow X(\mathbb{C})\}$ such that we have a commutative diagram of local homeomorphisms

$$\begin{array}{ccc} U_i & \hookrightarrow & U \\ & \searrow \qquad \swarrow & \\ & X(\mathbb{C}) & \end{array}$$

and where U is the union of the images of the maps from U_i. The above geometric argument immediately implies that the hypotheses of ([24], Tag 04D5, Lemma 7.41.2) are satisfied, and so δ_* sends surjective maps of sheaves to surjective maps of sheaves. Similarly, the hypotheses of ([24], Tag 04D5, Lemma 7.41.4) are also satisfied, so δ_* reflects injections and surjections.

Lastly, to show δ_* is an equivalence of topoi, notice that the inclusion functor $X(\mathbb{C}) \hookrightarrow X_{\text{cl}}$ is cocontinuous, and that the hypotheses of ([24], Tag 039Z, Lemma 7.29.1) are likewise satisfied by the above geometric argument. The morphism of topoi $g : \text{Sh}(X(\mathbb{C})) \to \text{Sh}(X_{\text{cl}})$ associated to the inclusion as a cocontinuous functor is then an equivalence, with the adjunction mappings $g^{-1}g_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to g_*g^{-1}\mathcal{G}$ being isomorphisms. It follows from the definition of induced morphism of topoi from a cocontinuous functor that $\delta_* = g^{-1}$; in fact, since g^{-1} is left adjoint to g_*, this shows δ_* is right exact as well. □

Remark. The functor δ^{-1} is *not* the same as the induced functor g_* in the argument above; the map of topoi (δ^{-1}, δ_*) is induced from the inclusion as a *continuous* functor, whereas the map of topoi (g^{-1}, g_*) is induced from the inclusion as a *cocontinuous* functor.

The above theorem and lemma say that one can replace $X(\mathbb{C})$ with X_{cl} for calculation of usual sheaf cohomology. Moreover there is the following property of the morphism ε, again explained in SGA IV (XII-4).

Theorem 2.3. *Let X be smooth over $\mathbb{C}$. There is an equivalence of categories given by the quasi-inverse functors ε_* and ε^* between the category of locally constant constructible torsion sheaves on $X_{\text{ét}}$, and the category of locally constant finite fiber torsion sheaves on X_{cl}.*

The proof of the above theorem is what ultimately gives the desired quasi-isomorphism between sheaf cohomologies

$$\mathrm{R}\Gamma(X_{\text{ét}}, \mathbb{F}_p) \xrightarrow{\simeq} \mathrm{R}\Gamma(X_{\text{cl}}, \mathbb{F}_p) \xleftarrow{\simeq} \mathrm{R}\Gamma(X(\mathbb{C}), \mathbb{F}_p)$$

where the first isomorphism is from ε_* in the theorem above, and the second is by δ_* from Theorem 2.2. The quasi-isomorphism is really the difficult part of Theorem 2.1; the rest of this section is to show that the above zig-zag of maps are maps of E_∞-$\mathbb{F}_p$-algebras. First, we show how we obtain induced maps on cohomology from morphisms of sites:

Lemma 2.2. *Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of sites. The following diagram commutes:*

$$\begin{array}{ccc} D\text{AbSh}(\mathcal{C}) & \xrightarrow{\mathrm{R}f_*} & D\text{AbSh}(\mathcal{C}') \\ & \searrow^{\mathrm{R}\Gamma} & \downarrow \mathrm{R}\Gamma \\ & & D\mathbf{Ab} \end{array}$$

In particular, we have $\mathrm{R}\Gamma(\mathcal{C}', \mathrm{R}f_*\mathcal{F}) \simeq \mathrm{R}\Gamma(\mathcal{C}, \mathcal{F})$ for any $\mathcal{F}$ in $\text{Sh}(\mathcal{C})$.

Proof. This follows from the following commutative diagram:

$$\begin{array}{ccc} \text{AbSh}(\mathcal{C}) & \xrightarrow{f_*} & \text{AbSh}(\mathcal{C}') \\ & \searrow^{\Gamma} & \downarrow \Gamma \\ & & \mathbf{Ab} \end{array}$$

We have that f_* is right adjoint to an exact functor f^{-1}. Hence, we have the following string of natural isomorphisms:

$$\begin{aligned}\Gamma(\mathcal{C}', f_*\mathcal{F}) &:= \mathrm{Hom}_{\mathrm{AbSh}(\mathcal{C}')}(\underline{\mathbb{Z}}_{\mathcal{C}'}, f_*\mathcal{F})\\ &\cong \mathrm{Hom}_{\mathrm{AbSh}(\mathcal{C})}(f^{-1}(\underline{\mathbb{Z}}_{\mathcal{C}'}), \mathcal{F})\\ &\cong \mathrm{Hom}_{\mathrm{AbSh}(\mathcal{C})}(\underline{\mathbb{Z}}_{\mathcal{C}}, \mathcal{F})\\ &=: \Gamma(\mathcal{C}, \mathcal{F}).\end{aligned}$$

Here, $\underline{\mathbb{Z}}$ denotes the constant sheaf that takes values in $\mathbb{Z}$: the category $\mathrm{AbSh}(\mathcal{C})$ has a unique morphism of topoi (p_*, p^{-1}) to **Ab**, where $\underline{\mathbb{Z}}_{\mathcal{C}} = p^{-1}\mathbb{Z}$. The second isomorphism holds by uniqueness of this morphism (see SGA IV (IV-4.3) [1]). All functors in the above diagram are left exact but not right exact. Since f_* preserves injective objects ([24], Tag 015Z), it follows by ([24], Tag 015L, Lemma 13.22.1) that the natural morphism $\mathrm{R}(\Gamma \circ f_*) \to \mathrm{R}\Gamma \circ \mathrm{R}f_*$ is an isomorphism. □

2.1 Godement resolutions

Let us briefly recall the Godement construction [10], considered in a general categorical context. Here, we use the language of Rodríguez-Gonález and Roig [21]. The Godement construction will be where all of our E_∞-algebra structures arise, using a theorem of Hinich and Schechtman.

Definition 2.1. *Let $\mathcal{C}$ be a site. A* ***point of the site*** *$\mathcal{C}$ is a pair of adjoint functors $x = (x^*, x_*)$*

$$\mathrm{Sh}(\mathcal{C}) \underset{x_*}{\overset{x^*}{\rightleftarrows}} \mathrm{Set},$$

$$\mathrm{Hom}_{\mathrm{Set}}(x^*\mathcal{F}, E) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{F}, x_*E)$$

where x^* commutes with finite limits. The site $\mathcal{C}$ ***has enough points*** *if there exists a set S of points x such that a morphism f in $\mathrm{Sh}(\mathcal{C})$ is an isomorphism if and only if x^*f is a bijection for all $x \in S$.*

The right adjoint x_* assigns to each set E the *skyscraper sheaf* x_*E at the point x. The left adjoint x^* assigns to each sheaf $\mathcal{F}$ the *stalk* $\mathcal{F}_x := x^*\mathcal{F}$ at the point x.

Example 2.1. *Consider the site $X(\mathbb{C})$ of (analytic) open subsets of the complex manifold $X(\mathbb{C})$. Take S to be the underlying set of $X(\mathbb{C})$. The geometric points $x \in S$ determine points of the site by taking x^* to be the stalk functor and x_* to be the skyscraper sheaf functor. Moreover, a map of sheaves is an isomorphism if and only if it is so on the stalks. Thus the site $X(\mathbb{C})$ has enough points.*

Example 2.2. *It is a classical fact that the étale site $X_{\text{ét}}$ has enough points; this follows from, for example, Deligne's criterion ([1], VI Appendix, Proposition 9.0). Isomorphisms of sheaves on the étale site can be checked stalk-wise at geometric points $\overline{x}$, where a geometric point $\overline{x}$ (for a given point x of the underlying topological space X) is the spectrum of a separably closed field $k(\overline{x})$ containing $k(x)$, with the map $\overline{x} \to X$ induced by the inclusion of* *the residue fields. It suffices to choose, for each point x of the underlying topological space X, a separably closed field that contains the residue field of x (see the lecture notes of Milne [17], by the argument of Lemma 7.4 therein and Remark 7.7). This gives the set of points for the étale site. We caution that the collection of* ***all*** *geometric points does not form a set, but rather a proper class.*

Lemma 2.3. *The site X_{cl} has enough points.*

Proof. Recall we have an equivalence of topoi, with the equivalence induced by an inclusion of categories, with induced functor $\delta_* : \text{Sh}(X_{\text{cl}}) \to \text{Sh}(X(\mathbb{C}))$ with quasi-inverse g_*. In fact, δ_* is left adjoint to g_*. By the example above, we know $X(\mathbb{C})$ has enough points. For each point $x = (x^*, x_*)$ of $X(\mathbb{C})$, define $y = (y^*, y_*) := (x^* \circ \delta_*, g_* \circ x_*)$. We check adjointness:

$$\begin{aligned}
\text{Hom}_{\text{Set}}(y^*\mathcal{F}, E) &= \text{Hom}_{\text{Set}}(x^* \circ \delta_*\mathcal{F}, E) \\
&\cong \text{Hom}_{\text{Sh}(X(\mathbb{C}))}(\delta_*\mathcal{F}, x_*E) \\
&\cong \text{Hom}_{\text{Sh}(X_{\text{cl}})}(\mathcal{F}, g_* \circ x_*E) \\
&= \text{Hom}_{\text{Sh}(X_{\text{cl}})}(\mathcal{F}, y_*E)
\end{aligned}$$

where all isomorphisms follow from the adjunction pairs (x^*, x_*) and (δ_*, g_*). Moreover, since x^* and δ_* are both exact, their composition y^* also commutes with finite limits.

The above argument shows that y as defined is a point. We take as our set of points S a set with cardinality at most the cardinality of the underlying set of the manifold $X(\mathbb{C})$, via $(x^*, x_*) \mapsto (x^* \circ \delta_*, g_* \circ x_*)$. We now show we can check isomorphisms of sheaves stalk-wise.

Assume f is an isomorphism in $\text{Sh}(X_{\text{cl}})$. By exactness of δ_*, we have that $\delta_* f$ is an isomorphism in $\text{Sh}(X(\mathbb{C}))$. Thus, $x^*\delta_* f$ is a bijection for all points x. Therefore, $y^* f = x^*\delta_* f$ is a bijection for all y.

Conversely, suppose $y^* f$ is a bijection for all y, i.e., that $x^*\delta_* f$ is a bijection for all x. Then, $\delta_* f$ is an isomorphism in $\text{Sh}(X(\mathbb{C}))$. However, by lemma 2.1, δ_* reflects injections and surjections. Thus f is an isomorphism. □

We are now ready to discuss the Godement resolution, following [21]. In fact, we have the following proposition ([21], Proposition 3.3.1):

Proposition 2.1. *Let D be a category closed under products and filtered colimits, and let x be a point of the site $\mathcal{C}$. Then there is an adjoint pair (x^*, x_*) of functors*

$$\text{Sh}(\mathcal{C}, D) \underset{x_*}{\overset{x^*}{\rightleftarrows}} D,$$

$$\text{Hom}_D(x^*\mathcal{F}, E) \cong \text{Hom}_{\text{Sh}(\mathcal{C},D)}(\mathcal{F}, x_*E)$$

Rodríguez-Gonález and Roig prove the above proposition by noting that we have explicit formulas for x^*, using filtered colimits, and x_*, using products. We refer the formulas to their paper ([21], 3.3).

Definition 2.2. *Let $\mathcal{C}$ be a site with enough points, with S as its set of points. Let D be a category closed under products and filtered colimits. By abuse of notation, let S denote the collection considered as a discrete category, with only identity morphisms. There is a pair (p^*, p_*) of adjoint functors*

$$\mathrm{Sh}(\mathcal{C}, D) \underset{p_*}{\overset{p^*}{\rightleftarrows}} D^S,$$

$$\mathrm{Hom}_{D^S}(p^*\mathcal{F}, E) \cong \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C},D)}(\mathcal{F}, p_*E)$$

defined, for $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, D)$ and $E = (E_x)_{x\in S} \in D^S$, by $p^*\mathcal{F} := (\mathcal{F}_x)_{x\in S}$ and $p_*E := \prod_{x\in S} x_*(E_x)$.

Definition 2.3. *Let $\mathcal{C}$ be a site with enough points and D a symmetric monoidal category closed under products and filtered colimits. The* ***Godement functor*** *is a functor $G : \mathrm{Sh}(\mathcal{C}, D) \to \mathrm{Sh}(\mathcal{C}, D)$ defined as $G = p_*p^*$ for the adjoint pair (p^*, p_*) defined in definition 2.2. The* ***cosimplicial Godement functor*** *is a functor $G^\bullet : \mathrm{Sh}(\mathcal{C}, D) \to \Delta\mathrm{Sh}(\mathcal{C}, D)$ into cosimplicial sheaves, where the i-th term is given by the i-th iterate G^i.*

For the purposes of this paper, we will mainly be concerned with $D = \mathrm{Ch}(k\text{-mod})$ the category of cochain complexes of k-modules. In this situation, we have the following classical theorem, proven for far more general D by Rodríguez-Gonález and Roig (see [20] Proposition 3.4.5, Corollary 3.4.6, Proposition 3.4.7) though definitely known to Godement [10] albeit without modern language. See also Chataur and Cirici ([9] Definition 2.5).

Theorem 2.4. *Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of sites induced by a continuous functor, where $\mathcal{C}$ and $\mathcal{C}'$ have enough points. Let $D = \mathrm{Ch}(k\text{-mod})$. Then the functors $G^\bullet$, f_*, and Γ are lax symmetric monoidal.*

Remark. The symmetric monoidal structure on $\mathrm{Sh}(\mathcal{C}, D)$ is by sheafifying the presheaf whose values are tensor products object-wise. That is, given $\mathcal{F}$ and $\mathcal{G}$, we can define a presheaf by $U \mapsto \mathcal{F}(U) \otimes_k \mathcal{G}(U)$, and then sheafify.

We now use some definitions following Chataur and Cirici ([9], Definition 2.4), and Mandell ([14], Proposition 5.2). See also ([24], Tag 019H) for general Dold-Kan considerations.

Definition 2.4. *The* ***normalized complex functor*** *is the functor $N : \Delta\mathrm{Ch}(k\text{-mod}) \to \mathrm{Ch}(k\text{-mod})$ given as the composition of the cosimplicial degree-wise normalization functor and the totalization functor. Normalization of a cosimplicial object means forming a chain complex by taking the intersections of kernels of the cosimplicial maps, then restricting the alternating sum of cosimplicial maps to the intersections. The* ***associated complex functor*** *is the functor $s : \Delta\mathrm{Ch}(k\text{-mod}) \to \mathrm{Ch}(k\text{-mod})$ given by taking alternating sums of the cosimplicial maps to obtain an associated double complex, then applying totalization.*

It is well known that for every cosimplicial complex $C^\bullet$, the complexes $NC^\bullet$ and $sC^\bullet$ are homotopy equivalent ([26], Lemma 8.3.7, Theorem 8.3.8).

Definition 2.5. *Let $D = \mathrm{Ch}(k\text{-mod})$. The* ***normalized Godement resolution*** *is the functor $NG^\bullet : \mathrm{Sh}(\mathcal{C}, D) \to \mathrm{Sh}(\mathcal{C}, D)$ given as object-wise composition of the normalized complex functor and the cosimplicial Godement functor.*

Definition 2.6. *Let* $D = \mathrm{Ch}(k\text{-mod})$*. The* ***Godement resolution*** *is the functor* $sG^\bullet : \mathrm{Sh}(\mathcal{C}, D) \to \mathrm{Sh}(\mathcal{C}, D)$ *given as object-wise composition of the associated complex functor and the cosimplicial Godement functor.*

The normalized cochain functor is monoidal but *not* symmetric monoidal; instead we have the following theorem of Hinich and Schechtman [11] (see also Chataur and Cirici [9], Proposition 2.2, and Mandell [14], Theorem 5.5).

Theorem 2.5. *(Hinich-Schechtman) Let* $A^\bullet$ *be a cosimplicial* $\mathcal{P}$*-algebra for an arbitrary operad* $\mathcal{P}$ *in* $\mathrm{Ch}(k\text{-mod})$*. Then* $NA^\bullet$ *is a* $(\mathcal{P} \otimes_k \mathcal{Z})$*-algebra, for the Eilenberg-Zilber operad* $\mathcal{Z}$*, functorial in* $A^\bullet$ *and* $\mathcal{P}$*.*

2.2 Sheaves of commutative DGAs

This subsection will discuss how to obtain an E_∞-algebra structure on the cochains of a sheaf of commutative DGAs (CDGAs). An outline is as follows: the above theorem of Hinich and Schechtman says by normalizing and totalizing a cosimplicial CDGA, the resulting complex is an E_∞-algebra. The Godement resolution precisely takes a sheaf of CDGAs to cosimplicial sheaves of CDGAs, and then normalizes and totalizes. Taking global sections then yields an E_∞-algebra structure on the sheaf cochains.

We introduce two necessary definitions from [20] and [21] before proceeding. By the remark on the symmetric monoidal structure on $\mathrm{Sh}(\mathcal{C}, D)$ via the sheafification functor, the following is well defined. See ([20], Remark 3.4.3).

Definition 2.7. *Let* $D = \mathrm{Ch}(k\text{-mod})$*. A* ***sheaf of operads*** $\mathcal{P}$ *in* D *is an operad in* $\mathrm{Sh}(\mathcal{C}, D)$*.*

Definition 2.8. *Let* $D = \mathrm{Ch}(k\text{-mod})$ *and let* $\mathcal{P}$ *be a sheaf of operads in* D *on a site* $\mathcal{C}$*. A sheaf of cochain complexes* $\mathcal{F} \in \mathrm{Sh}(\mathcal{C}, D)$ *is a* ***sheaf of*** $\mathcal{P}$***-algebras*** *if it is a* $\mathcal{P}$*-algebra. Equivalently, for each object* U *in* $\mathcal{C}$*, there are the usual structure morphisms for an algebra over an operad* $\mathcal{P}(n)(U) \otimes_k \mathcal{F}^{\otimes n}(U) \to \mathcal{F}(U)$*.*

Example 2.3. *Let* $\mathcal{O}$ *be a sheaf of* k*-algebras. We can view this as a sheaf of trivial commutative DGAs, which are concentrated in degree 0. Then* $\mathcal{O}$ *is a sheaf of* $Comm$*-algebras, where* $Comm$ *denotes the commutative operad, which has* $Comm(n) = k$ *in degree 0 for all* n*, with trivial symmetric group actions. Note that* $Comm$*-algebras are equivalent to commutative DGAs (CDGAs) ([13] Example 2.2).*

Example 2.4. *Recall that the Eilenberg-Zilber operad is acyclic ([11] Theorem 2.3, [14] Proposition 5.4), and so admits a map of operads* $\mathcal{Z} \to Comm$*. There is also the fixed map of operads* $\mathcal{E} \to \mathcal{Z}$*, where* $\mathcal{E}$ *is our fixed* E_∞*-operad. Thus a sheaf* $\mathcal{O}$ *of* k*-algebras is a sheaf of* E_∞*-algebras.*

Remark. Recall that the symmetric monoidal structure on $\Delta\mathrm{Sh}(\mathcal{C}, D)$ is induced degree-wise by the symmetric monoidal structure on $\mathrm{Sh}(\mathcal{C}, D)$, so the definitions above make sense for cosimplicial sheaves as well.

Remark. The equivalence above in definition 2.8 follows from the existence of the sheafification functor for presheaves with values in $D = \text{Ch}(k\text{-mod})$. Part of the work of Rodríguez-Gonález and Roig in [20] concerns the conditions on the coefficient category D for which a sheafification functor is available. For the purposes of this paper, we will actually mainly be concerned with the underlying presheaf structure than the sheaf structure per se.

Lemma 2.4. *The category of Comm-algebras, equivalent to the category of commutative differential graded k-algebras (CDGAs), is closed under limits and colimits.*

Proof. Limits and colimits of cochain complexes are computed degree-wise on the underlying k-modules, with the differential defined by universal property. A CDGA is precisely a cochain complex $C^\bullet$ with a chain map $a : C^\bullet \otimes_k C^\bullet \to C^\bullet$ unital and associative in the appropriate sense, and satisfying graded commutativity. We sketch a proof of the colimit case in this paragraph and omit the limit case, which is identical. For colimits in CDGAs, we take the colimits of the underlying cochain complexes. Let $D^\bullet$ denote the colimit of a system of CDGAs $C_i^\bullet \xrightarrow{f_{ji}} C_j^\bullet$. We define the algebra map by

$$D^\bullet \otimes_k D^\bullet = \text{colim} C_i^\bullet \otimes_k \text{colim} C_j^\bullet \cong \text{colim}(C_i^\bullet \otimes_k C_i^\bullet) \xrightarrow{\text{colim } a_i} \text{colim} C_i^\bullet = D^\bullet$$

where the center natural isomorphism follows from invoking the universal property on colim $(C_i^\bullet \otimes_k C_i^\bullet)$ for the system $\{C_i^\bullet \otimes_k C_i^\bullet\}$, via the maps $C_i^\bullet \otimes_k C_i^\bullet \to \text{colim} C_i^\bullet \otimes_k \text{colim} C_j^\bullet$. For the inverse, one fixes an index i and defines maps $C_i^\bullet \otimes_k C_j^\bullet \xrightarrow{f_{ji} \otimes \text{id}} C_j^\bullet \otimes_k C_j^\bullet \to \text{colim } C_i^\bullet \otimes C_i^\bullet$ if $i \leq j$, and $C_i^\bullet \otimes_k C_j^\bullet \xrightarrow{\text{id} \otimes f_{ij}} C_i^\bullet \otimes_k C_i^\bullet \to \text{colim } C_i^\bullet \otimes C_i^\bullet$ if $i > j$; one then uses the universal property twice, and commutativity of all diagrams involved and uniqueness of the maps arising from the universal property verify the isomorphism. Graded commutativity and the Leibniz formula follow from all the terms in the equations being colimits of elements and colimits of maps respectively, and functoriality of colim.

□

Lemma 2.5. *Let $\mathcal{F}$ be a sheaf of Comm-algebras and let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism of sites with enough points induced by a continuous functor $u : \mathcal{C}' \to \mathcal{C}$. Then $G^\bullet \mathcal{F}$ is a cosimplicial presheaf of Comm-algebras. If $\mathcal{F}$ is a sheaf of $\mathcal{P}$-algebras for a fixed operad $\mathcal{P}$, then $f_* \mathcal{F}$ is a presheaf of $\mathcal{P}$-algebras.*

Proof. Recall that $G^0 \mathcal{F} = p_* p^* \mathcal{F}$. By definition, G^0 is a product of filtered colimits of *Comm*-algebras, and so is a presheaf of *Comm*-algebras by Lemma 2.4. By definition, $f_* \mathcal{F}(U) = \mathcal{F}(u(U))$ for objects U in $\mathcal{C}$; since $\mathcal{F}$ is a presheaf of $\mathcal{P}$-algebras, it follows that $f_* \mathcal{F}$ is a presheaf of $\mathcal{P}$-algebras. □

Remark. By Theorem 2.4 and results of [20], we obtain an induced functor, which we also denote by $G^\bullet$, that sends sheaves of operads $\mathcal{P}$ in $\text{Sh}(\mathcal{C}, D)$ to cosimplicial sheaves of operads $G^\bullet \mathcal{P}$, which is by definition the same as a cosimplicial operad in $\text{Sh}(\mathcal{C}, D)$. Similarly, we obtain an induced functor $G^\bullet$ that sends sheaves $\mathcal{F}$ of $\mathcal{P}$-algebras to cosimplicial sheaves $G^\bullet \mathcal{F}$ which are $G^\bullet \mathcal{P}$-algebras, i.e., algebras over the cosimplicial operad $G^\bullet \mathcal{P}$.

An immediate consequence of Theorem 2.4 and Theorem 2.5 is then the following corollary.

Corollary 2.1. *Let $D = \mathrm{Ch}(k\text{-mod})$ and let $\mathcal{C}$ be a site with enough points with a terminal object X. Let $\mathcal{F}$ be a sheaf of Comm-algebras. Then $NG^\bullet(\mathcal{F})$ is a presheaf of $\mathcal{Z}$-algebras. In particular, $\mathrm{R}\Gamma(\mathcal{C}, \mathcal{F})$ is an $\mathcal{Z}$-algebra.*

Proof. By Theorem 2.4 and Lemma 2.5, $G^\bullet\mathcal{F}$ is a cosimplicial presheaf of $Comm$-algebras. Thus on each object U of $\mathcal{C}$, we have that $NG^\bullet(\mathcal{F})(U)$ is the normalized complex of a cosimplicial $Comm$-algebra. By Theorem 2.5, $NG^\bullet(\mathcal{F})(U)$ is a $(Comm \otimes_k \mathcal{Z})$-algebra for each object U. Since $Comm \otimes_k \mathcal{Z} = \mathcal{Z}$, we have that this is a presheaf of $\mathcal{Z}$-algebras. Finally, we have $\mathrm{R}\Gamma(\mathcal{C}, \mathcal{F}) = NG^\bullet(\mathcal{F})(X)$ since the Godement resolution is a resolution by flasque sheaves, which are acyclic. □

We need just one more theorem, due to Mandell ([14], Theorem 5.8). See also Chataur and Cirici ([9], Definition 2.4).

Theorem 2.6. *There is a cosimplicial normalization functor N that sends cosimplicial E_∞-algebras to E_∞-algebras, that agrees with the normalized cochain functor N on the underlying cochain complexes, and such that, for a constant cosimplicial E_∞-algebra $A^\bullet$, the isomorphism of cochain complexes $A^0 \cong N(A^\bullet)$ is a morphism of E_∞-algebras.*

Lemma 2.6. *Let $\mathcal{O}$ be a sheaf of k-algebras. The cosimplicial Godement functor has an augmentation $\mathcal{O} \to G^\bullet\mathcal{O}$ which is a map of presheaves of k-algebras in degree 0. The existence of the augmentation is equivalent to the existence of a map of cosimplicial presheaves of k-algebras $\mathcal{O}^\bullet \to G^\bullet\mathcal{O}$. The composition $\mathcal{O} \xrightarrow{\cong} N\mathcal{O}^\bullet \to NG^\bullet\mathcal{O}$ is a map of presheaves of E_∞-algebras.*

Proof. That an augmentation of a cosimplicial object is equivalent to a map from the constant cosimplicial object is by composing the augmentation map with the cosimplicial maps; see ([24] Tag 018F Lemma 14.20.2). The map $\mathcal{O}^\bullet \to G^\bullet\mathcal{O}$ of cosimplicial presheaves of k-algebras is then a map of cosimplicial presheaves of $Comm$-algebras, and so E_∞-algebras. Applying Theorem 2.6 gives the last statement. □

The following proposition summarizes the above section in one clean statement.

Proposition 2.2. *Let $\mathcal{F}$ be a sheaf of commutative k-algebras (or more generally of commutative DGAs) on a site $\mathcal{C}$ with enough points. Then $\Gamma(\mathcal{C}, NG^\bullet\mathcal{F})$ is an E_∞-k-algebra, where $NG^\bullet\mathcal{F}$ is the normalization of the cosimplicial Godement resolution. Thus the object $\mathrm{R}\Gamma(\mathcal{C}, \mathcal{F})$ representing sheaf cohomology has an E_∞-algebra structure.*

2.3 E_∞-Artin comparison

We are now ready to prove the statement that the classical Artin comparison theorem is a quasi-isomorphism of E_∞-algebras, with the corresponding E_∞-algebra structures arising from Godement resolutions.

Lemma 2.7. *Let $\mathcal{O}$ and $\mathcal{O}'$ be sheaves of k-algebras on sites $\mathcal{C}$, $\mathcal{C}'$ respectively. Assume $\mathcal{C}$ and $\mathcal{C}'$ both have terminal objects. A morphism of ringed sites $(\mathcal{C}, \mathcal{O}) \xrightarrow{f} (\mathcal{C}', \mathcal{O}')$ induced by a continuous functor $\mathcal{C}' \to \mathcal{C}$ induces a map of E_∞-algebras $\mathrm{R}\Gamma(\mathcal{C}', \mathcal{O}') \to \mathrm{R}\Gamma(\mathcal{C}, \mathcal{O})$.*

Proof. A morphism of ringed sites by definition has the data of a map of sheaves of rings $\mathcal{O}' \to f_*\mathcal{O}$. Viewing $f_*\mathcal{O}$ as an object in the derived category concentrated in degree 0, there is a natural chain map $f_*\mathcal{O} \to \mathrm{R}f_*\mathcal{O}$. By functoriality of $\mathrm{RHom}_{\mathrm{PSh}(\mathcal{C}')}(*_{\mathcal{C}'}, -)$ we obtain maps

$$\mathrm{RHom}_{\mathrm{PSh}(\mathcal{C}')}(*_{\mathcal{C}'}, \mathcal{O}') \to \mathrm{RHom}_{\mathrm{PSh}(\mathcal{C}')}(*_{\mathcal{C}'}, f_*\mathcal{O}) \to \mathrm{RHom}_{\mathrm{PSh}(\mathcal{C}')}(*_{\mathcal{C}'}, \mathrm{R}f_*\mathcal{O}) \simeq \mathrm{RHom}_{\mathrm{PSh}(\mathcal{C})}(*_{\mathcal{C}}, \mathcal{O})$$

where the last isomorphism follows from Lemma 2.2. The composition of these maps gives our map $\mathrm{R}\Gamma(\mathcal{C}', \mathcal{O}') \to \mathrm{R}\Gamma(\mathcal{C}, \mathcal{O})$. The claim is that this naturally induced map is a map of E_∞-algebras. Note that Theorem 2.4 also implies that the last isomorphism from Lemma 2.2 is a quasi-isomorphism of E_∞-algebras, for sheaves of k-algebras.

First, note we already have maps of bounded below cochain complexes of sheaves of abelian groups $\mathcal{O}' \to f_*\mathcal{O} \to \mathrm{R}f_*\mathcal{O}$ where we view $\mathcal{O}'$ and $f_*\mathcal{O}$ as complexes concentrated in degree 0. The first map $\mathcal{O}' \to f_*\mathcal{O}$ is a map of presheaves of k-algebras, so of *Comm*-algebras, and so of E_∞-algebras. We have that $\mathrm{R}f_*\mathcal{O}$ is computed as $f_*NG^\bullet\mathcal{O}$, which is a presheaf of $\mathcal{Z}$-algebras, and so of E_∞-algebras, by Corollary 2.1 and Lemma 2.5. The second map is in fact the functor f_* applied to the map $\mathcal{O} \xrightarrow{\cong} N\mathcal{O}^\bullet \to NG^\bullet\mathcal{O}$ in Lemma 2.6; since this map is a map of presheaves of E_∞-algebras, applying the functor f_* yields a map of presheaves of E_∞-algebras. We then have the following commutative square

$$\begin{array}{ccc} \mathcal{O}' & \xrightarrow{\simeq} & NG^\bullet(\mathcal{O}') \\ \downarrow & & \downarrow * \\ \mathrm{R}f_*\mathcal{O} & \xrightarrow{\simeq} & NG^\bullet(\mathrm{R}f_*\mathcal{O}) \end{array}$$

by functoriality of the Godement resolution on the underlying complexes. The left hand vertical map is a map of presheaves of E_∞-algebras, by the above discussion. The right hand vertical map is then a map of presheaves of $(E_\infty \otimes_k \mathcal{Z})$-algebras, and so a map of presheaves of E_∞-algebras; this follows from the existence of maps of operads $\mathcal{E} \to \mathcal{E} \otimes_k \mathcal{E} \to \mathcal{E} \otimes_k \mathcal{Z}$. The map $\mathcal{E} \to \mathcal{E} \otimes_k \mathcal{E}$ exists using cofibrancy of an appropriate choice of E_∞-operad, or for example using the Barratt-Eccles operad, which is known to have this property [3]. Taking global sections gives us a map of E_∞-algebras

$$\mathrm{R}\Gamma(\mathcal{C}', \mathcal{O}') = \Gamma(\mathcal{C}', NG^\bullet(\mathcal{O}')) \xrightarrow{*} \Gamma(\mathcal{C}', NG^\bullet(\mathrm{R}f_*\mathcal{O})) = \mathrm{R}\Gamma(\mathcal{C}, \mathcal{O})$$

where the map $*$ is exactly the map described in the first paragraph above, by definition of derived global sections. □

Corollary 2.2. *(Artin comparison) Let X be a smooth complex variety. The zig-zag of sites in Theorem 2.2 gives a quasi-isomorphism of E_∞-algebras between $\mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{F}_p)$ and $\mathrm{R}\Gamma(X(\mathbb{C}), \mathbb{F}_p)$.*

Recall that the singular cochains of a space $C^*_{\mathrm{sing}}(X, k)$ have a natural E_∞-k-algebra structure, as an algebra over the Eilenberg-Zilber operad $\mathcal{Z}_k$; this structure arises from the failure of commutativity of the cup product on cochains with values in a general field k, and the Eilenberg-Zilber operad action is essentially given by the Alexander-Whitney maps ([16], Theorem 3.9). This E_∞-algebra structure can be computed again using the Godement resolution. We conclude with the following (again, very classical) statement neatly explained by Petersen in [19], who also assumes the space only be *cohomologically locally connected*.

Theorem 2.7. *Let X be a locally contractible, paracompact Hausdorff space. There is a quasi-isomorphism of E_∞-algebras between* $\mathrm{R}\Gamma(X, k)$ *and* $C^*_{\mathrm{sing}}(X, k)$.

Petersen's proof of the above follows the same strategy as the other proofs in this section, where one passes to the Godement resolution to see that one has a map of E_∞-algebras. Of course, one uses that the constant sheaf admits a resolution by the sheaf of singular cochains, and then applies Godement to both complexes.

3 Prismatic cohomology and p-adic homotopy theory

This section will cover preliminaries regarding prisms and prismatic cohomology. We will also briefly overview the étale comparison theorem of Bhatt and Scholze.

3.1 Prismatic cohomology

Prismatic cohomology arises as the sheaf cohomology of a sheaf of rings $\mathcal{O}_{\Delta}$ on an appropriate site. The underlying category of the site is a category of objects, called *prisms*, with morphisms to the variety of interest. Informally, a prism is a special type of characteristic 0 ring that has the data of a lift of Frobenius; they are pairs (A, I), where A is a δ-ring, i.e., a ring equipped with the data of a set map $\delta : A \to A$ such that the map $\phi(x) = x^p + p\delta(x)$ reduces mod p to the Frobenius endomorphism, and I is an invertible ideal. An illustrative example to keep in mind is $(\mathbb{Z}_p, (p))$ where $\mathbb{Z}_p$ denotes the p-adic integers, and $\delta(x) = \frac{x - x^p}{p}$, which is well defined as $\mathbb{Z}_p$ is p-torsion free. Morally, prisms are mixed characteristic thickenings of characteristic p varieties whose infinitesimal neighborhoods record a lift of the Frobenius endomorphism. We formally define them below, starting with the definition of a δ-ring.

Definition 3.1. *A **δ-ring** is a ring A equipped with a set map $\delta : A \to A$ such that $\delta(0) = \delta(1) = 0$, and satisfying the two identities:*

$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

and

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x + y)^p}{p}.$$

Given a δ, one can define a map $\phi : A \to A$ by $\phi(x) = x^p + p\delta(x)$. By the identities in the definition above, ϕ is a ring homomorphism. Moreover, ϕ reduces mod p to the Frobenius homomorphism. We call ϕ a *lift of Frobenius.* A *morphism* of δ-rings is a map of rings which commute with the respective δ maps.

We now briefly review the notion of derived completion.

Definition 3.2. *Let A be a ring with finitely generated ideal I. An A-complex $M \in D(A)$ is **derived I-complete** if for each $f \in I$, the natural map $M \to R\lim(M \otimes^L_{\mathbb{Z}[x]} Z[x]/(x^n))$ is a quasi-isomorphism, where we view M as a $\mathbb{Z}[x]$-module where x acts by multiplication by f.*

If an A-module M is classically I-complete, then it is derived I-complete.

Definition 3.3. *A* ***prism*** *is a pair (A, I) where A is a δ-ring and I defines a Cartier divisor on* $\mathrm{Spec}(A)$ *such that A is derived (p, I)-complete and $p \in I + \phi(I)A$.*

All prisms used seriously in this paper will be equipped with principal ideals $I = (d)$ for some choice of generator d, and will all be classically (p, I)-complete. We include the definitions as they are, for full generality. Morphisms of prisms $(A, I) \to (B, J)$ are maps of δ-rings that send the ideal I into J. In fact, it is a small lemma that any map of prisms forces $J = IB$.

Definition 3.4. *A prism is* ***perfect*** *if the lift of Frobenius ϕ is an isomorphism. A ring R is* ***(integral) perfectoid*** *if $R = A/I$ for a perfect prism (A, I).*

The above notion of perfectoid differs slightly from the notion of perfectoid in Fontaine's sense [22]. For the relation between the two notions, see Lemma 3.20 in [5]; in context for the étale comparison theorem later, the two notions will be compatible via this lemma. We use the notion of perfectoid above in this paper.

We now fix a base prism (A, I). We now assume all formal schemes are equipped with the p-adic topology. Recall that ordinary schemes in characteristic p can be considered p-adic formal schemes, since $p = 0$.

Definition 3.5. *Fix a smooth formal scheme $\mathfrak{X}$ over A/I. The* ***prismatic site*** *of $\mathfrak{X}$ is a category, where objects are diagrams*

$$\begin{array}{ccc} Spf(B/IB) & \longrightarrow & Spf(B) \\ \downarrow & & \\ \mathfrak{X} & & \downarrow \\ \downarrow & & \\ Spf(A/I) & \longrightarrow & Spf(A) \end{array}$$

where the pairs (B, IB) are prisms over (A, I). We equip the category with the flat topology. The structure sheaf $\mathcal{O}_{\mathbb{\Delta}}$ sends a prism (B, IB) to B.

There is a map of topoi $\nu : Sh(X/A)_{\mathbb{\Delta}} \to Sh(X_{\text{ét}})$ (see [7] Construction 4.4) that localizes prismatic cohomology on étale coverings, given by sending the above diagram to étale coverings $Spf(B/IB) \to \mathfrak{X}$. The *prismatic complex* $\mathbb{\Delta}_{\mathfrak{X}/A}$ is defined as $R\nu_* \mathcal{O}_{\mathbb{\Delta}} \in D(\mathfrak{X}_{\text{ét}})$. This is a complex of étale sheaves of A-modules, and in fact a commutative algebra object in the derived category. That is, it can be viewed as a presheaf of E_∞-algebras.

Definition 3.6. *We define* ***prismatic cohomology*** *$H_{\mathbb{\Delta}}(\mathfrak{X}/A)$ over the base prism (A, I) to be the hypercohomology of the complex of étale sheaves $\mathbb{\Delta}_{\mathfrak{X}/A}$.*

We finally end our review with one last definition.

Definition 3.7. *The* ***tilt*** *of a ring R is the inverse limit $\varprojlim_\phi R/p$ where ϕ denotes the Frobenius $x \mapsto x^p$.*

If (A, I) is a perfect prism, then there is a natural isomorphism $A \simeq W((A/I)^\flat)$. Thus, if $R = A/I$ is perfectoid with perfect prism (A, I) then its tilt $R^\flat$ is also perfectoid with perfect prism $(W(R^\flat), (p)) = (A, (p))$. Indeed, there is an equivalence of categories between the category of perfectoid rings, and the category of perfect prisms.

We are now ready to relate the result of Bhatt-Morrow-Scholze to the p-adic homotopy theory of the complex variety X. We simply apply $- \otimes^L_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ to the Bhatt-Morrow-Scholze result and apply Mandell's functor $\overline{\mathbb{U}}$. First we will briefly review the proof of Theorem 0.1, and then recall the basic change-of-scalars results we mentioned in section 1.2, whose proofs we will refer to Mandell [15], or Kriz and May [13]. Notice that since $\overline{\mathbb{F}_p}$ is a field extension over $\mathbb{F}_p$, it is faithfully flat. Thus, the functor $- \otimes^L_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ is equal to the functor $- \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ on the underlying complexes in the derived category.

Theorem 0.1 can be obtained by first proving the following theorem ([7], Theorem 9.1) for affine opens of $\mathfrak{X}$, then gluing the result to pass from local to global. We note that the fraction field of $\mathcal{O}^\flat_{\mathbb{C}_p}$ is $\mathcal{O}^\flat_{\mathbb{C}_p}[1/d] = \mathbb{C}^\flat_p$ where d is the element that generates the kernel of the map $W(\mathcal{O}^\flat_{\mathbb{C}_p}) \to \mathcal{O}_{\mathbb{C}_p}$. Since $\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A})$ takes values in $A = A_{\mathrm{inf}}$-modules, tensoring with the fraction field is equivalent (locally) to inverting d and taking mod p.

Recall that there is a short exact sequence (the Artin-Schreier-Witt short exact sequence) of étale sheaves:

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{\phi - Id} \mathbb{G}_a \to 0.$$

There is also an analogous version for truncated Witt vectors, thought of as a group scheme with respect to addition:

$$0 \to \mathbb{Z}/p^n \to W_n \xrightarrow{\phi - Id} W_n \to 0$$

which will be used in the argument below. The point is that prismatic cohomology is acyclic (as a complex of sheaves) for perfectoid rings, and is given by the Witt vectors of their tilt in degree 0. Moreover, the étale theory of perfectoid rings is equivalent to their tilts.

Theorem 3.1. *(Bhatt-Scholze) Let $\mathfrak{X} = \mathrm{Spf}(S)$ be a formal affine scheme over a perfectoid ring R corresponding to a perfect prism $(A, (d))$. There is a canonical quasi-isomorphism of E_∞-algebras (with the E_∞-algebra structure induced by the Godement resolution for étale sheaves)*

$$\mathrm{R}\Gamma(\mathrm{Spec}(S[1/p], \mathbb{Z}/p^n) \simeq (\Delta_{S/A}[1/d]/p^n)^{\phi=1}$$

for each $n \geq 1$.

Very rough sketch of proof. Bhatt and Scholze first prove the two functors

$$\mathrm{Spf}(S) \mapsto \mathrm{R}\Gamma(\mathrm{Spec}(S[1/p], \mathbb{Z}/p^n)$$
$$\mathrm{Spf}(S) \mapsto (\Delta_{S/A}[1/d]/p^n)^{\phi=1}$$

are sheaves in the arc topology on affine formal schemes. Étale cohomology satisfies descent with respect to the arc topology ([6], Theorem 5.4). Given this, they assume S is perfectoid, using that affine perfectoids are a basis for the arc topology ([7], Lemma 8.8). Since S is perfectoid, they can then identify $(\mathbb{\Delta}_{S/A}[1/d]/p^n)^{\phi=1}$ with $\mathrm{R}\Gamma(\mathrm{Spec}(S^\flat[1/d]), \mathbb{Z}/p^n)$ where $S^\flat$ is the tilt of S, described as follows. As S is perfectoid, the complex of sheaves $\mathbb{\Delta}_{S/A}[1/d]/p^n$ is concentrated in degree 0 and given by $W(S^\flat)[1/d]/p^n$. By the Artin-Schreier-Witt sequence, taking Frobenius fixed points yields $\mathrm{R}\Gamma(\mathrm{Spec}(S^\flat[1/d]), \mathbb{Z}/p^n)$, after viewing the Witt vectors as group schemes (which is why we get cohomology on $S^\flat[1/d]$). Moreover, as S is perfectoid, the space $\mathrm{Spa}(S[1/p], S)$ is perfectoid in the sense of Fontaine, by Lemma 3.20 in [5].

By Huber's comparison theorem ([12], Corollary 3.2.2), there is an equivalence of étale sites between $\mathrm{Spec}(S[1/p])$ and the étale site of $\mathrm{Spa}(S[1/p], S)$, where étale in the latter sense is appropriately defined for adic spaces. The analogous equivalence holds for $S^\flat[1/d]$. Moreover, we have that the tilt of the perfectoid space $\mathrm{Spa}(S[1/p], S)$ is the perfectoid space $\mathrm{Spa}(S^\flat[1/d], S^\flat)$. By Scholze ([22], Theorem 1.11), there is an equivalence of étale sites induced by the tilting functor for perfectoid spaces. Thus we have an equivalence of étale sites between $S[1/p]$ and $S^\flat[1/d]$.

The equivalence of étale sites by Scholze is then the root of the E_∞-algebra comparison as in section 2, again using that the étale site has enough points, and passing these points through via the equivalences. $\square$

Remark. The above lemma is a comparison theorem on affine opens. To recover the full theorem as stated, one glues affines together in such a way that the isomorphisms are compatible. In the original context of [4] (Lecture IX, Theorem 5.1), the hypotheses of smooth and proper are used to impose a finiteness condition on the complexes involved. Then, under this finiteness condition, they utilize a semicontinuity theorem to obtain a dimension inequality between de Rham cohomology of the special fiber and étale cohomology of the generic fiber. In fact, the hypothesis of proper is unnecessary for our theorem.

3.2 Proof of Main Theorem

We are now ready to prove the main theorem 0.4. There is a slight nuance kindly pointed out to me by Mark De Cataldo: there are many non-canonical identifications of $\mathbb{C}$ with $\mathbb{C}_p$. For a complex variety $X = X_\mathbb{C}$ that admits a model over $\mathrm{Spec}(\mathbb{Z})$, to apply the work of Bhatt and Scholze we must consider the model over the ring of integers $\mathcal{O}_{\mathbb{C}_p}$. We then have a model $X_{\mathbb{C}_p}$ over $\mathbb{C}_p$. Fixing an identification of $\mathbb{C}$ with $\mathbb{C}_p$ yields $X_\mathbb{C}$ as the pullback of $X_{\mathbb{C}_p}$ under the field isomorphism.

Lemma 3.1. *Let $X = X_\mathbb{C}$ denote the smooth complex variety over $\mathbb{C}$ with an integral model over $\mathbb{Z}$, and let $Y = X_{\mathbb{C}_p}$ denote the model over $\mathbb{C}_p$. Choose a field isomorphism $\mathbb{C}_p \to \mathbb{C}$ so that we have the following fiber product diagram:*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}_p) \end{array}$$

We then have an equivalence of sites between $X_{\text{ét}}$ and $Y_{\text{ét}}$.

Proof. Using the map $X \to Y$ induced by the field isomorphism, one can pull back étale coverings over Y to étale coverings over X. Base-changing along this map produces a continuous functor between sites. By base-changing along the map induced by the inverse field isomorphism, one obtains the inverse continuous functor. To prove that the composition of base-changes with respect to the induced maps is naturally isomorphic to the identity, we present the following diagram:

$$\begin{array}{ccccc} (U \times_X Y) \times_Y X & \longrightarrow & U \times_X Y & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & X \end{array}$$

where each inner square is a pullback diagram and $U \to X$ is étale . This implies the larger square with the four corners is a pullback diagram. However, the composition of horizontal arrows on the bottom is the identity, since it is the composition of the maps induced by the field isomorphisms. Thus the top left corner is naturally isomorphic to the top right corner. $\square$

Corollary 3.1. *Fix a field isomorphism* $\mathbb{C} \to \mathbb{C}_p$. *We have a quasi-isomorphism of* E_∞-*algebras* $\mathrm{R}\Gamma(X_{\mathbb{C}_p}, \mathbb{F}_p) \simeq \mathrm{R}\Gamma(X_{\mathbb{C}}, \mathbb{F}_p)$.

Remark. Suppose X only admits a model over $\mathcal{O}_{\mathbb{C}_p}$ without admitting a model over $\mathrm{Spec}(\mathbb{Z})$. One can produce a complex variety by using any field isomorphism between $\mathbb{C}$ and $\mathbb{C}_p$. However, it is entirely possible that two different field isomorphisms could yield two non-isomorphic complex varieties. The two varieties produced would then be related by an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. However, there are explicit examples due to Serre [23], of complex varieties that differ by a Galois group action and whose fundamental groups are not isomorphic; these examples are not even homeomorphic. This checks out, as these latter automorphisms are not even continuous.

However, the argument above of Lemma 3.1 shows that if two complex varieties differ by a field automorphism, their étale sites are nonetheless equivalent. So they have isomorphic étale cohomology and étale fundamental groups (which are the profinite completions of their ordinary fundamental groups). Moreover, they have isomorphic p-adic (étale) homotopy types. Thus, the output of the main theorem being "the" p-adic homotopy type of a complex variety still makes sense.

We now prove the main theorem.

Proof of Theorem 0.4. Fix an identification of $\mathbb{C}$ with $\mathbb{C}_p$. The sheaf cohomology of any sheaf on the étale site of X inherits an E_∞-algebra structure by Godement considerations, as in section 2. By Bhatt-Morrow-Scholze 0.1 and Theorem 3.1, we have that

$$(\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\mathrm{inf}}}) \otimes^L_{A_{\mathrm{inf}}} \mathbb{C}_p^\flat)^{\phi=1} \simeq \mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{F}_p)$$

is a quasi-isomorphism of E_∞-$\mathbb{F}_p$-algebras. By the entirety of section 2, the right hand side is quasi-isomorphic to $\mathrm{R}\Gamma(X(\mathbb{C}), \mathbb{F}_p)$ as E_∞-$\mathbb{F}_p$-algebras. By Theorem 2.7, we again obtain a quasi-isomorphism of E_∞-$\mathbb{F}_p$-algebras with $C^*_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{F}_p)$. By Mandell 0.3, applying the functor $\mathbb{U}$ to the left hand side then yields the free loop space of the Bousfield-Kan p-completion of $X(\mathbb{C})$.

On the other hand, we can apply $- \otimes^L_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ to the equation above. Again by Mandell's Theorem 0.2 and Theorem 1.2, applying the functor $\overline{\mathbb{U}}$ to the resulting expression gives the Sullivan p-completion of $X(\mathbb{C})$. □

Remark. We have seen that applying Mandell's functors $\mathbb{U}$ and $\overline{\mathbb{U}}$ to the Bhatt-Morrow-Scholze result recovers the free loop space of the Bousfield-Kan p-completion, and the Sullivan p-completion respectively. One may wonder if the Bousfield-Kan p-completion itself is accessible, without the presence of the free loop space. The author suspects this is doable given the following: Mandell proves Theorem 0.3 by taking homotopy fixed points of the Frobenius on $A \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ where A is a cofibrant replacement of $C^*_{\text{sing}}(X, \mathbb{F}_p)$. Mandell shows this is weak equivalent (for X connected, p-complete, nilpotent, and of finite p-type) to the homotopy fixed points of the space X with the trivial action, which yields the free loop space. At the same time, the theorem of Bhatt-Morrow-Scholze recovers the étale $\mathbb{F}_p$-cochains by taking homotopy fixed points of the Frobenius on $(\mathrm{R}\Gamma(\mathfrak{X}, \Delta_{\mathfrak{X}/A_{\text{inf}}}) \otimes^L_{A_{\text{inf}}} \mathbb{C}_p^\flat)$. However, this latter object is enormous over $\overline{\mathbb{F}_p}$. The author suspects by passing to spectra and analyzing an appropriate Frobenius, as in [18] or [27], and using the relationship between prismatic cohomology and topological Hothschild homology, one could recover a clean statement regarding the Bousfield-Kan p-completion of the variety X over $\mathbb{C}$.

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