Monopoles, Singularities and Hyperkähler Geometry

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Abstract of the Dissertation

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The main subjects of this thesis are monopoles. They are solutions to the Bogomolny equations on 3-manifolds, Calabi-Yau 3-folds and G_2-manifolds. Monopoles, conjecturally, can be used to define invariants of manifolds [21]. We prove the existence of non-trivial monopoles with Dirac singularities on rational homology 3-spheres, via a gluing construction. Furthermore, we will introduce some generalized Bogomolny equations in higher dimensions. The main difficulty in defining invariants of manifolds with special holonomy groups using these gauge-theoretic equations comes from the non-compactness of the moduli spaces of monopoles, which are governed by a first order differential operator, called the Fueter operator. The Fueter operator is a non-linear generalization of the Dirac operator over 3- and 4-manifolds, where the spinor bundle is replaced by a non-linear hyperkähler bundle. We prove partial compactness results by examining the different sources of non-compactness of the spaces of the Fueter sections and proving some of them, in fact, do not occur.

Donaldson proposed the possibility of studying G_2-manifolds from the viewpoint of coassociative fibrations and the adiabatic limit [14]. This approach is expected to be helpful in understanding the non-compactness problems. The adiabatic picture led Donaldson and Scaduto to conjecture the existence of certain associative submanifolds in G_2-manifolds with a coassociative $K3$-fibration near the adiabatic limit [20], which reduces to the question of the existence of certain asymptotically cylindrical special Lagrangians in certain Calabi-Yau 3-folds. We propose a strategy to prove this conjecture using the method of continuity and take several steps in that direction.

To Mom and Dad.

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Introduction

The study of the gauge-theoretic equations, emerging from theoretical-physics, and the discovery of invariants defined by 'counting' the solutions to these equations revolutionized the field of low-dimensional topology [19, 24]. Donaldson and Thomas proposed generalizing these invariants to higher-dimensional manifolds, in particular, manifolds with special holonomy groups [22]. Donaldson and Segal hinted at the idea of defining invariants of Calabi-Yau 3-folds and G_2-manifolds by counting monopoles on these manifolds [21]. In this writing, we address some of the key questions, specially relevant to the existence and compactness problems, in both low and higher dimensions.

Chapter 1: Singular Monopoles on 3-Manifolds

Monopoles on 3-manifolds appear as dimensional reduction of instantons on 4-manifolds. A monopole on a principal G-bundle $P \to M$ on an oriented Riemannian 3-manifold (M, g) is a pair (A, Φ) of a connection A on P and a section Φ of the adjoint bundle which satisfies the Bogomolny equation,

$$*F_A = d_A\Phi,$$

where $*$ denotes the Hodge star operator of the Riemannian metric g, F_A is the curvature 2-form of the connection A, and $d_A\Phi$ is the covariant derivative of Φ with respect to the connection A.

Every smooth monopole on a closed Riemannian 3-manifold, when G is a compact Lie group, satisfies a stronger condition,

$$*F_A = d_A\Phi = 0.$$

In order to get solutions to the Bogomolny equation which do not satisfy this stronger condition, one can consider monopoles with Dirac singularities at isolated points, where close to a singular point $p \in M$, we have

$$|\Phi| = \frac{k}{2r} + m + O(r),$$

where k is a positive integer, called the charge of the monopole at p, m is a constant, called the mass of the monopole at p, and r denotes the geodesic distance from the singular point p.

We prove the existence of non-trivial singular $SU(2)$-monopoles on rational homology 3-

spheres. This is partially motivated by the problem of the existence of monopoles in higher dimensions. This existence result is proven with the use of a gluing construction. The gluing constructions originate from the works of Taubes where he used them to construct irreducible smooth $SU(2)$-monopoles on $\mathbb{R}^3$ [90] and Yang-Mills instantons on non-self-dual 4-manifolds [90]. The gluing construction requires careful analysis of the linearized problem and the use of certain weighted Sobolev spaces, studied by Biquard [9], and in the monopole case by Foscolo [33, 30].

Chapter 2: Monopoles in higher dimensions

Donaldson and Segal proposed to define invariants of Calabi-Yau 3-folds and G_2-manifolds by 'counting' monopoles on these manifolds. Let $P \to Z$ be a principal G-bundle over a Calabi-Yau 3-fold (Z, g, ω, Ω). A pair (A, Φ) of a connection A on P and a section Φ of the adjoint bundle is called a Calabi-Yau monopole if it satisfies the Calabi-Yau-Bogomolny equations,

$$
\begin{aligned}
*(F_A \wedge \mathrm{Im}(\Omega)) &= d_A \Phi, \\
F_A \wedge \omega^2 &= 0.
\end{aligned}
$$

In this section, we consider their dimensional reduction to $U(1)$-bundles over hyperkähler 4-manifolds and derive Bogomolny equations on these 5-dimensional manifolds. We show they satisfy some similar formal properties as monopoles in other dimensions.

Moreover, we introduce the complexified gauge-theoretic equations on manifolds with special holonomy groups. Similar to the lower-dimensional cases, there are two ways to complexify these equations, which we call the Haydys type and the Kapustin-Witten type. We show the moduli spaces of solutions to the these Haydys type gauge-theoretic equations on manifolds with special holonomy groups can be understood as a Kähler manifold. Moreover, the solutions to the Kapustin-Witten type gauge-theoretic equations on manifolds with special holonomy groups satisfy certain vanishing properties.

Chapter 3: Fueter Sections and Monopoles

As conjectured by Donaldson and Segal, monopoles on Calabi-Yau 3-folds and G_2-manifolds are closely related to the calibrated submanifolds, more specifically, the special Lagrangians in the Calabi-Yau case and the coassociatives in the G_2 case [21]. This is similar to the Taubes' theorem, which relates the Seiberg-Witten and Gromov invariants of symplectic 4-manifolds [86]. A major role in this conjecture is played by a non-linear generalization of the Dirac operator, called the Fueter operator.

Fueter sections, which are harmonic spinors with respect to the Fueter operators on 3- and 4-manifolds, are also interesting independent of their relevance to the gauge theory in higher dimensions, and potentially, can be used to define invariants of 3- and 4-manifolds. Let (M, g) be an oriented Riemannian 3-manifold. Let $\pi : \mathfrak{X} \to M$ be a fiber bundle whose fibers are modeled on a hyperkähler manifold (X, g_X, I, J, K), with an isometric bundle identification

$$
\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X}),
$$

where STM is the unit tangent bundle of M and $\mathfrak{b}(\mathfrak{X})$ is the sphere bundle of the complex structures of the fibers of $\mathfrak{X}$. Let ∇ be the covariant derivative of a connection on this bundle. A section $f \in \Gamma(\mathfrak{X})$ is called a Fueter section if

$$\mathfrak{F}_{\nabla}(f) := I(\partial x_1)\nabla_{\partial x_1} f + I(\partial x_2)\nabla_{\partial x_2} f + I(\partial x_3)\nabla_{\partial x_3} f = 0,$$

where $(\partial x_1, \partial x_2, \partial x_3)$ is a local orthonormal frame on M. The operator $\mathfrak{F}_{\nabla}$ is called a Fueter operator.

The main difficulty in defining the monopole invariants in higher dimensions comes from the non-compactness problems. The non-compactness of these moduli spaces are governed by the Fueter operators. In our case, the fibers of the hyperkähler bundles are modeled on the moduli spaces of monopoles on $\mathbb{R}^3$.

We prove partial compactness results for Fueter sections of the monopole bundles, examining the different sources of non-compactness of the spaces of Fueter sections, and proving some of them, in fact, do not occur. The analysis follows the same line of thought as in the study of singularities of harmonic maps, which originates from the work of Schoen and Uhlenbeck [83].

Chapter 4: On the Donaldson-Scaduto Calibrated Submanifolds

Donaldson proposed the possibility of studying G_2-manifolds from the viewpoint of coassociative fibrations and the adiabatic limit, where the diameters of the fibers shrink to zero [20]. It is expected that this approach would be helpful in understanding the formation of singularities and the compactness problems. The adiabatic picture led Donaldson and Scaduto to conjecture the existence of certain associative submanifolds in G_2-manifolds with a coassociative $K3$-fibration near the adiabatic limit [20]. This problem can be reduced to the question of the existence of certain asymptotically cylindrical special Lagrangians in $X \times \mathbb{C}$, where X is a multi-Eguchi-Hanson hyperkähler manifold.

We propose a strategy to prove this conjecture using the method of continuity. We will show, under the hyperkähler deformation of X, the existence of these asymptotically cylindrical special Lagrangians is an open condition. Moreover, by considering a Hamiltonian action of $U(1)$ on $X \times \mathbb{C}$, we will reduce this existence problem to the question of existence of certain non-compact holomorphic curves with boundary in $\mathbb{R}^4$, with respect to a non-standard singular almost complex structure, where this existence problem is described by a singular Monge-Ampère equation.

Chapter 1

Singular Monopoles on 3-Manifolds

The theory of Yang-Mills connections and, in particular, instantons revolutionized the study of 4-manifolds [19, 15, 18, 68]. The Bogomolny monopoles appear as the dimensional reduction of instantons to 3-manifolds. Let (M, g) be an oriented Riemannian 3-manifold, $P \to M$ a principal G-bundle for a Lie group G, and $\mathfrak{g}_P$ the associated adjoint bundle. A pair (A, Φ) of a connection A on P and a section Φ of the adjoint bundle is called a monopole if it satisfies the Bogomolny equation,

$$*F_A = d_A \Phi,$$

where $*$ denotes the Hodge star operator of the Riemannian metric g, F_A is the curvature 2-form of the connection A, and $d_A\Phi$ is the covariant derivative of Φ with respect to the connection A.

The theory of monopoles on non-compact 3-manifolds is very rich and interesting. Jaffe and Taubes proved the existence of non-trivial $SU(2)$-monopoles on $\mathbb{R}^3$, using a gluing construction [49]. The gluing constructions, originating from the works of Taubes, have been used to construct solutions to various differential equations [90, 29, 85, 54, 26, 60, 48, 16, 95, 76, 30, 64]. From the gluing construction of monopoles, one can read the dimension of the moduli spaces of monopoles on $\mathbb{R}^3$. This can also be proven using a variation of the Atiyah-Singer index theorem, called the Callias index theorem, which is an index theorem for Dirac operators on non-compact odd-dimensional manifolds [4, 59]. Furthermore, there exists an explicit parametrization of the moduli spaces of monopoles on $\mathbb{R}^3$ in terms of rational maps, due to Donaldson [17, 47].

The moduli spaces of monopoles on $\mathbb{R}^3$ are ALF hyperkähler manifolds, which have been extensively studied, originating from the works of Atiyah and Hitchin [4]. Floer studied monopoles on asymptotically Euclidean 3-manifolds [28], more recently, Oliveira studied monopoles on asymptotically conical 3-manifolds and stated that there exists a $(4k-1)$-dimensional family of non-trivial irreducible smooth $SU(2)$-monopoles on any asymptotically conical 3-manifold (M, g) with $b^2(M) = 0$ [76]. It is proven by Kottke that the expected dimension of the moduli space of monopoles on an asymptotically conical 3-manifold, whose ends are asymptotic to a cone on Σ, is $4k + \frac{1}{2}b^1(\Sigma) - b^0(\Sigma)$ [59].

The theory of monopoles on compact 3-manifolds is quite different from the ones on non-compact manifolds. When the structure group G is compact, every smooth monopole on a closed oriented Riemannian 3-manifold satisfies a stronger condition,

$$*F_A = d_A\Phi = 0,$$

and therefore, A is a flat connection and Φ is a covariantly constant section. These monopoles are sometimes referred to as trivial monopoles.

There is another class of monopoles on compact 3-manifolds, which is quite interesting. These monopoles are smooth on the complement of finitely many points with prescribed Dirac singularities at these points. Pauly studied the deformation of these singular monopoles with the structure group $SU(2)$ [78], and using the Atiyah-Singer index theorem and exploiting a theorem of Kronheimer [61] — which states that close to the points with Dirac singularities, monopoles up to gauge, can be understood as smooth S^1-invariant instantons on a 4-dimensional space — proved that the expected dimension of the moduli space of singular monopoles with charge $k \in \mathbb{N}$ on a compact Riemannian 3-manifold (M, g) is equal to $4k$. However, this argument does not imply that the moduli spaces are non-empty.

In this chapter, we prove the existence of $SU(2)$-monopoles with Dirac singularities on rational homology 3-spheres. The proof is based on a gluing construction. Furthermore, this construction gives a geometric interpretation to Pauly's dimensional formula for the moduli spaces of singular monopoles on rational homology 3-spheres. The strategy of the proof follows the gluing argument of Taubes [49, 90], Foscolo's proof of the existence of singular monopoles on $\mathbb{R}^2 \times S^1$ [30], and Oliveira's work on smooth monopoles on asymptotically conical 3-manifolds [76].

Proposition 1 (Existence of Singular Monopoles)**.** *Let (M, g) be an oriented rational homology 3-sphere equipped with a Riemannian metric g. For any $k \in \mathbb{N}$, there exists a non-trivial irreducible $SU(2)$-monopole with Dirac singularities with charge k on a principal $SU(2)$-bundle $P \to M$.*

Outline of This Chapter Section 1.1 contains background material on monopoles. In Section 1.2.1, we study the basic properties of Dirac monopoles on closed 3-manifolds. In section 1.2.2 we present the construction of Dirac monopoles on rational homology 3-spheres. In sections 1.2.3 and 1.2.4, we construct an approximate irreducible $SU(2)$-monopole. In section 1.2.5, we set up the framework to solve the Bogomolny equation to find a genuine $SU(2)$-monopole near the constructed approximate monopole. In section 1.2.7.1, we solve the linearized Bogomolny equation near the points where the BPS-monopoles are located. In section 1.2.7.4, we solve the longitudinal part of the linear equation away from the glued BPS-monopoles and in section 1.2.7.8 the transverse part. Finally in section 1.2.8, we consider the quadratic terms, solve the Bogomolny equation and complete the proof of the main existence theorem.

1.1 Dimensional Reductions and Singularities

In this section, we review the basics of the theory of monopoles, their relations to instantons on 4-manifolds, and singular solutions to these equations. For more detailed account consult with [49], [4] and [33].

The monopoles, i.e., solutions to the Bogomolny equation, are the central topic of this thesis.

Definition 1 (The Bogomolny Monopole)**.** *Let* (M, g) *be an oriented Riemannian 3-manifold. Let* G *be a compact Lie group. Let* $P \to M$ *be a principal* G*-bundle and* $\mathfrak{g}_P$ *the associated adjoint bundle. Let* A *be a connection on* P *and* Φ *a section of* $\mathfrak{g}_P$*. The Bogomolny equation for a pair* (A, Φ) *is*

$$*F_A = d_A \Phi, \tag{1.1.1}$$

where $*$ *is the Hodge star operator on the* $\mathfrak{g}_P$*-valued differential forms on* M*, defined using the Riemannian metric* g *and the orientation on* M*.*

A pair (A, Φ) *which satisfies equation 1.1.1 is called a monopole.*

The Bogomolny equation on 3-manifolds is closely related to the anti-self-duality equation on 4-manifolds.

Definition 2 (Anti-Self-Dual Instanton)**.** *Let* (N, h) *be an oriented Riemannian 4-manifold. Let* G *be a compact Lie group. Let* $P \to N$ *be a principal* G*-bundle. A connection* A *on* P *is called an anti-self-dual instanton if*

$$F_A^+ := F_A + *F_A = 0, \tag{1.1.2}$$

where $*$ *is the Hodge star operator on* N *induced by the Riemannian metric* h *and the orientation on* N*.*

The Bogomolny equation on 3-manifolds can be understood as a dimensional reduction of anti-self-duality equation on 4-manifolds.

Lemma 1 (Dimensional Reduction of Instantons)**.** *Let* (M, g) *be an oriented Riemannian 3-manifold with volume form* vol_g*. Let* $P \to M$ *be a principal* G*-bundle for a Lie group* G*. Let* $X = M \times \mathbb{R}$ *— or* $X = M \times S^1$ *— equipped with the product Riemannian metric* $h = g + dt^2$*, where* t *denotes the coordinate on the* $\mathbb{R}$*-factor. Let* $vol_h = dt \wedge \pi^*(vol_g)$ *be the volume form on* X*, where* $\pi : X \to M$ *is the projection map onto* M*. Any connection* $\mathbb{A}$ *on the pull-back bundle* $\pi^* P \to X$*, invariant under the translation in the* $\mathbb{R}$*-direction, can be written as*

$$\mathbb{A} = \pi^* A + (\pi^* \Phi) dt,$$

for a connection A *on* P *and a section* Φ *of the adjoint bundle* $\mathfrak{g}_P$*.*

The pair (A, Φ) *is a monopole if and only if* $\mathbb{A}$ *is an anti-self dual instanton.*

Proof. The curvature of $\mathbb{A}$ is given by

$$F_{\mathbb{A}} = \pi^* F_A - dt \wedge \pi^*(d_A \Phi),$$

and therefore,

$$F_{\mathbb{A}}^+ = (\pi^* F_A)^+ - (dt \wedge \pi^*(d_A \Phi))^+.$$

We denote the Hodge star operators on M and X by $*_3$ and $*_4$, respectively. For any 1-form $\alpha \in \Omega^1(M)$, we have

$$dt \wedge \pi^*\alpha = *_4\pi^*(*_3\alpha).$$

This equation extends to 1-forms on M with values in any vector bundle, and therefore,

$$F_{\mathbb{A}}^+ = (\pi^*F_A)^+ - (*_4\pi^*(*_3 d_A\Phi))^+ = (\pi^*F_A)^+ - (\pi^*(*_3 d_A\Phi))^+ = (\pi^*(F_A - *_3 d_A\Phi))^+.$$

This shows if the pair (A, Φ) is a monopole, then $\mathbb{A}$ is an anti-self dual instanton.

Conversely, suppose $\mathbb{A}$ is an anti-self-dual connection,

$$F_{\mathbb{A}}^+ = (\pi^*(F_A - *_3 d_A\Phi))^+ = 0.$$

For any 2-form $\beta \in \Omega^2(M)$ — and therefore, for any 2-form on M with values in any vector bundle — we have

$$(\pi^*(\beta))^+ = 0 \Rightarrow \beta = 0.$$

To see this, note that on any sufficiently small open neighbourhood in M, with local coordinates (x, y, z), if $\beta = \beta_z dxdy + \beta_x dydz + \beta_y dzdx$, we have

$$0 = (\pi^*(\beta))^+ = \frac{1}{2}((\pi^*\beta_z)(dxdy + dtdz) + (\pi^*\beta_x)(dydz + dtdx) + (\pi^*\beta_y)(dzdx + dtdy)),$$

hence, $\pi^*\beta_x = \pi^*\beta_y = \pi^*\beta_z = 0$, which implies $\beta = 0$ on every coordinate ball, and therefore, everywhere on M. □

Although sometimes it is useful to think about monopoles as 3-dimensional counterparts of instantons, there are important differences between them too.

Lemma 2 (Trivial Monopole)**.** *Let (M, g) be a closed, oriented, Riemannian 3-manifold and G a compact Lie group. Any smooth monopole (A, Φ) on a principal G-bundle $P \to M$ satisfies the following equations,*

$$*F_A = d_A\Phi = 0. \tag{1.1.3}$$

A pair (A, Φ) satisfying 1.1.3 is sometimes referred to as a trivial monopole.

Proof. The Bogomolny equation implies

$$\Delta_A\Phi = d_A^* d_A\Phi = *d_A F_A = 0,$$

where the last equality follows from the Bianchi identity. Therefore, Φ is a harmonic section with respect to the connection A. Since G is a compact Lie group, there is an adjoint-invariant inner product on its Lie algebra $\mathfrak{g}$, and therefore, on the adjoint bundle $\mathfrak{g}_P$, denoted by $\langle -, - \rangle$. With respect to this inner product, we have the following pointwise equations,

$$0 = \langle \Delta_A \Phi, \Phi \rangle = \langle d_A^* d_A \Phi, \Phi \rangle = \langle d_A \Phi, d_A \Phi \rangle = |d_A \Phi|^2,$$

and therefore, $*F_A = d_A \Phi = 0$. □

The following corollary is an immediate consequence of Lemma 1 and Lemma 2.

Corollary 1. *Let (M, g) be a closed Riemannian 3-manifold. Let $X = M \times \mathbb{R}$ — or $X = M \times S^1$ — be the product Riemannian manifold and $\mathbb{A}$ an instanton on $\pi^* P \to X$, invariant under the $\mathbb{R}$-translation, with the same assumptions as in Lemma 1. Then $\mathbb{A}$ is a flat connection,*

$$F_{\mathbb{A}} = 0.$$

Lemma 2 suggests that on closed 3-manifolds, in order to have monopoles which do not satisfy the stronger equations 1.1.3, we should consider monopoles with singularities. One approach is to consider monopoles with Dirac singularities. These monopoles fit well with the idea of understanding monopoles as a dimensional reduction of instantons.

In this direction, Kronheimer introduced a second way to see monopoles as dimensional reduction of instantons [61], which incorporates the notion of Dirac singularity. Roughly speaking, monopoles on $B^3 \subset \mathbb{R}^3$ with Dirac singularity at the origin are dimensional reduction of $U(1)$-invariant instantons on $B^4 \subset \mathbb{R}^4 = \mathbb{C}^2$, as we will explain below.

Let $\pi : B^4 \setminus \{0\} \to B^3 \setminus \{0\}$ be the principal $U(1)$-bundle, where the $U(1)$-action on $B^4 \setminus \{0\}$ is given by

$$e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{-i\theta} z_2),$$

for all $\theta \in U(1)$, and the bundle map π is the radial extension of the Hopf map $S^3 \to S^2$. We can extend the map π to get a smooth map $\pi : B^4 \to B^3$ with $\pi^{-1}(0) = 0$. Let (x_1, x_2, x_3, x_4) be the coordinates on B^4. Let $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$. Let (u_1, u_2, u_3) be coordinates on B^3. The map π is given by the following formulas,

$$u_1 = 2\,\mathrm{Re}(z_1 z_2) = 2(x_1 x_3 - x_2 x_4), \tag{1.1.4}$$

$$u_2 = 2\,\mathrm{Im}(z_1 z_2) = 2(x_1 x_4 + x_2 x_3), \tag{1.1.5}$$

$$u_3 = |z_1|^2 - |z_2|^2 = x_1^2 + x_2^2 - x_3^2 - x_4^2. \tag{1.1.6}$$

Let $P \to B^3 \setminus \{0\}$ be a principal G-bundle and let $\pi^* P \to B^4 \setminus \{0\}$ be the pull-back bundle. Any $U(1)$-invariant connection $\mathbb{A}$ on $\pi^* P$, in a $U(1)$-invariant gauge, can be written as

$$\mathbb{A} = \pi^* A + (\pi^* \Phi) \xi, \tag{1.1.7}$$

for a connection A on P, a section Φ of the adjoint bundle $\mathfrak{g}_P$, and

$$\xi = 2(-x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4),$$

which is a $U(1)$-invariant 1-form on B^4 dual to the vector field generated by the infinitesimal action of the Lie algebra $\mathfrak{u}(1) \cong \mathbb{R}$ on B^4.

We equip B^3 and B^4 with the Euclidean metrics and volume forms $vol_{B^3} = du_1du_2du_3$ and $vol_{B^4} = -dx_1dx_2dx_3dx_4$, respectively. Then

$$F_{\mathbb{A}} = \pi^*F_A + (\pi^*\Phi)d\xi - \xi \wedge \pi^*(d_A\Phi) \Rightarrow (F_{\mathbb{A}})^+ = (\pi^*F_A)^+ - (\xi \wedge \pi^*(d_A\Phi))^+,$$

since $d\xi = 4(dx_1dx_2 + dx_3dx_4)$ is an anti-self dual 2-form. A similar argument as the one we saw in Lemma 1 proves the following.

Lemma 3 (Dimensional Reduction of $U(1)$-Invariant Instantons)**.** *A pair (A, Φ) on $P \to B^3 \backslash \{0\}$ is a monopole if and only if the connection $\mathbb{A}$ on $\pi^*P \to B^4 \setminus \{0\}$, given in 1.1.7, is an anti-self-dual instanton.*

This lemma can be generalized in two ways. First, following Kronheimer, one can replace the Euclidean metric on $\mathbb{R}^4$ by a Taub-NUT metric, and second, one can replace the Euclidean metric on B^3 with an arbitrary one, which is done by Pauly.

Lemma 4 (Kronheimer [61])**.** *Let (X, g_X, I, J, K) be a Taub-NUT space given by the Gibbons-Hawking Ansatz*[1]*, defined by positive harmonic map $V : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}$ and a connection 1-form on θ on $\pi : X \to \mathbb{R}^3 \setminus \{0\}$. Let $P \to \mathbb{R}^3 \setminus \{0\}$ be a principal G-bundle. A $U(1)$-invariant connection $\mathbb{A}$ on the pull-back bundle $\pi^*P \to X$ can be written as $\mathbb{A} = \pi^*(A) - \pi^*(\frac{\Phi}{V})\theta$, for a pair (A, Φ) of a connection A on $P \to \mathbb{R}^3 \setminus \{0\}$ and a section Φ of the adjoint bundle. Furthermore, $\mathbb{A}$ is an instanton on (X, g_X) if and only if (A, Φ) is a monopole with respect to the Euclidean metric on $\mathbb{R}^3 \setminus \{0\}$.*

An important feature of Kronheimer's theorem is that the anti-self-dual connection $\mathbb{A}$ and the bundle it is defined on — in a suitable $U(1)$-invariant gauge — extend smoothly over the $\pi^{-1}(0)$, even if the monopole itself cannot be extended over $0 \in \mathbb{R}^3$, which is indeed the case for monopoles with Dirac singularities. This follows from Uhlenbeck's removable singularity theorem for Yang-Mills connections; in particular, instantons [94].

Definition/Lemma 5 (Dirac Singularity)**.** *Let $P \to B^3 \setminus \{0\}$ be a principal $SU(2)$-bundle. Let (A, Φ) be a monopole with singularity at $0 \in B^3$ on $P \to B^3 \setminus \{0\}$. Let $\mathbb{A}$ be the instanton on $\pi^*P \to B^4$ defined in 1.1.7. Suppose*

$$\frac{1}{2}\int_{B^4} |F_{\mathbb{A}}|^2 vol_{B^4} < \infty.$$

Then close to the singular point, we have

$$|\Phi| = \frac{k}{2r} + m + O(r), \tag{1.1.8}$$

where the norm is defined with respect to the adjoint-invariant inner product on the adjoint bundle $\mathfrak{g}_P$, $k \in \mathbb{N}$ is a positive integer, called the charge of the monopole at the singular point, and m is a constant, called the mass of the monopole at the singular point. The pair (A, Φ) is called a monopole with a Dirac singularity.

[1]For detailed description of the Gibbons-Hawking Ansatz see Section 4.1.1.3.

When $G = U(1)$, we can define the signed charge $k \in \mathbb{Z}\setminus\{0\}$, as opposed to the non-Abelian case where $k > 0$. This is because in the Abelian case, close to the singular point, one can think about the section as a real-valued function,

$$\Phi = -\frac{k}{2r} + m + O(r). \tag{1.1.9}$$

Note that unlike 1.1.8, the left-hand-side is the section itself and not its norm. Here we used the identification $\mathfrak{u}(1) \cong \mathbb{R}$, in order to have $\mathbb{R}$-valued Higgs fields and connections, rather than $i\mathbb{R}$-valued.

As proven by Pauly [78], the local correspondence between singular monopoles on $B^3 \setminus \{0\}$ with Euclidean metric and anti-self-dual connections on $B^4 \setminus \{0\}$ with Euclidean or Taub-NUT metric can be generalized to the case where the metric on B^3 is an arbitrary one. He proved if B^3 is equipped with a Riemannian metric g^3, one can recover Theorem 4 by replacing the Taub-NUT metric with

$$g^4 = V\pi^*(g^3) + V^{-1}\theta_0^2,$$

which is not necessarily a hayperkähler metric anymore. Here, again close to a singular point the Higgs field of the monopole is of the form 1.1.8, and in the Abelian case 1.1.9, where r denotes the geodesic distance from the singular point.

Although this local picture is very convenient, because of topological reasons it does not extend to a global picture over smooth manifolds. Let $\{p_1, \ldots, p_n\}$ be n points on a Riemannian 3-manifold (M, g). Pauly showed there is no Riemannian 4-manifold (X, h) with a $U(1)$-action that is free everywhere on X except n fixed points $\{q_1, \ldots, q_n\}$, a smooth projection $\pi : X \to M$, which is a principal $U(1)$-bundle on $X \setminus \{q_1, \ldots, q_n\} \to M \setminus \{p_1, \ldots, p_n\}$ with $\pi(q_i) = p_i$, and a smooth $U(1)$-invariant 1-form $\xi \in \Omega^1(X)$ such that (A, Φ) is a monopole on (M, g) if and only if $\mathbb{A} = \pi^*A + (\pi^*\Phi)\xi$ is an anti-self-dual instanton on (X, h) [78].

In the following section, we will prove an existence theorem for these monopoles with Dirac singularities on rational homology 3-spheres.

1.2 The Gluing Construction

The main aim of this chapter is to construct irreducible $SU(2)$-monopoles with Dirac singularities on compact Riemannian 3-manifolds (M, g) with $H^2(M, \mathbb{Q}) = 0$. In fact, this gluing construction produces open subsets of the moduli spaces of monopoles with Dirac singularities. The Proposition 1 follows from the following.

Theorem 1 (Gluing Construction). *Let (M, g) be an oriented rational homology 3-sphere equipped with a Riemannian metric g. Let $S_p = \{p_1, \ldots, p_n\}$ and $S_q = \{q_1, \ldots, q_k\}$ be two sets of disjoint points in M. Let $k_1, \ldots, k_n$ be n negative integers, where $k + \sum_{i=1}^n k_i = 0$. Then there exists an irreducible $SU(2)$-monopole (A, Φ) with Dirac singularities with charge $|k_i|$ at p_i for all $i \in \{1, \ldots, n\}$ on a principal $SU(2)$-bundle $P \to M \setminus S_p$ such that*

$$(A, \Phi) = (A_0, \Phi_0) + (a, \varphi),$$

where (A_0, Φ_0) is equal to a scaled BPS-monopole on a small neighbourhood $B_{\varepsilon_j}(q_j)$ of each point q_j for $j \in \{1, \ldots, k\}$ and is equal to the lift of a $U(1)$-Dirac monopole with charge k_i at p_i for $i \in \{1, \ldots, n\}$ on $M \setminus \cup_{j=1}^{k} B_{2\varepsilon_j}(q_j)$. Moreover, the pair $(a, \varphi) \in W^{1,2}_{\alpha_1,\alpha_2}$ for suitable values of α_1 and α_2, where $W^{1,2}_{\alpha_1,\alpha_2}$ is a weighted Sobolev space, defined in Definition 8.

The proof of Theorem 1 is based on a gluing construction.

- The first step is to produce an Abelian Dirac monopole on (M, g) with some singular points p_i with negative charges and some singular points q_j with charge $+1$ such that the total charge of the monopole is zero.
- The second step is to smooth out the singularities with charge $+1$. The smoothing process is carried over by gluing model SU(2)-monopoles — called the scaled BPS-monopoles — to the singular points with charge $+1$ and leaving out the rest of the singular points not smoothed-out. In fact, it would be impossible to smooth out all of the singularities, as we know there is no smooth monopole on closed 3-manifolds with a non-flat connection. The gluing construction at each singular point depends on the choice of framing, which can be described by an element of $U(1)$.
- The third step is the deformation. The resulting configuration from the previous step is an approximate monopole and it does not necessarily satisfy the Bogomolny equation, but in a suitable norm, it is close to a solution and should be deformed into a genuine monopole.

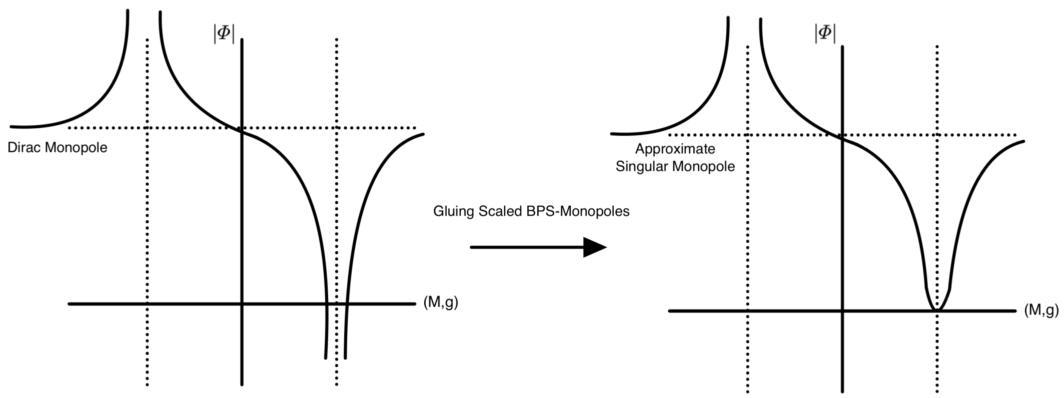

Figure 1.1: Gluing Construction

Remark 1. *Let $p_1, \ldots, p_n$ be n points in M. Let $k_1, \ldots, k_n$ be n negative integers. The gluing construction for $SU(2)$-monopoles with Dirac singularities at the points p_i with charges $|k_i|$ depends on $4k$-parameters, where $k = -\sum_{i=1}^{n} k_i$. This is equal to the expected dimension of the moduli space of singular $SU(2)$-monopoles with charge k, as computed by Pauly. $3k$ of this number is accounted by the position of the highly concentrated BPS-monopoles, $k-1$ of this* *number by choices of the framings at these points*[2]*, and 1 degree of freedom in changing the average mass of the monopole.*

1.2.1 Dirac Monopoles on Closed 3-Manifolds

In this section, we study the local model of Dirac monopoles close to the singular points, as a preparatory part to the next section, where we will present a construction of these monopoles.

Let (A_D, Φ_D) be a $U(1)$-monopole with a Dirac singularity at $p \in M$ with signed charge $k \in \mathbb{Z} \setminus \{0\}$, defined on a small neighbourhood of p in M. As mentioned earlier, a Dirac monopole is a monopole with isolated singularities on a bundle with structure group $U(1)$. Close to a singular point p, the Higgs field Φ_D has the following form,

$$\Phi_D = -\frac{k}{2r} + m + O(r), \tag{1.2.1}$$

where r denotes the geodesic distance from p and k is the signed charge at p.

The Bianchi identity shows that the curvature 2-form of a $U(1)$-connection is closed. Furthermore, from the Chern-Weil theory we know that the 2-form

$$\frac{F_{A_D}}{2\pi} = \frac{*d\Phi_D}{2\pi},$$

presents $c_1(L)$, the first Chern class of a line bundle L where the monopole is defined on. Restricting the bundle to a sufficiently small punctured neighbourhood of a singular point p with charge k, the line bundle $L_{|_{B_\varepsilon(p)\setminus\{p\}}} \to B_\varepsilon(p) \setminus \{p\}$ is isomorphic to H_p^k, where H_p is the Hopf line bundle centered at p, with the first Chern number c_1

$$c_1 = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\partial B_\varepsilon(p)} *d\Phi_D = \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\partial B_\varepsilon(p)} (\frac{k}{2\varepsilon^2} + O(1)) vol_{\partial B_\varepsilon(p)} = k.$$

The model connection A of a Dirac monopole on $\mathbb{R}^3$ close to a singular point $0 \in \mathbb{R}^3$ with charge k is an $SO(3)$-invariant connection defined on the line bundle $H_0^k \to \mathbb{R}^3 \setminus \{0\}$. Let $S_0^2(1)$ be the unit 2-sphere centred at the origin in $\mathbb{R}^3$. We can cover $S_0^2(1)$ by U^+ and U^-, where $U^+ = S_0^2(1) \setminus \{(0,0,-1)\}$ and $U^- = S_0^2(1) \setminus \{(0,0,1)\}$. In spherical coordinates (ρ, θ, φ), the connection A on U^+ and U^- is given by the following 1-forms,

$$A_{|_{U^-}} = k\frac{(1-\cos(\varphi))}{2} d\theta, \qquad A_{|_{U^+}} = k\frac{(-1-\cos(\varphi))}{2} d\theta,$$

with the transition function $e^{ik\theta}$. Note that on $U^- \cap U^+$

$$A_{|_{U^-}} - A_{|_{U^+}} = kd\theta.$$

We extend the connection radially to $H_0^k \to \mathbb{R}^3 \setminus \{0\}$ to get A.

[2]There are k points which we can fix the frames there; however, 1 parameter vanishes after taking the action of the gauge group into account.

Using geodesic normal coordinates, we can define a diffeomorphism

$$\eta : B_\varepsilon(0) \subset \mathbb{R}^3 \to B_\varepsilon(p) \subset M,$$

between a small neighbourhood of the origin in $\mathbb{R}^3$ and a small neighbourhood of a point $p \in M$. Furthermore, by choosing a bundle isomorphism, covering η, we can identify the bundles above these open neighbourhoods and pull back the connection A_D to a punctured neighbourhood of the origin in $\mathbb{R}^3$.

Lemma 6. *The connection of the Dirac monopole with charge k, denoted by A_D, close to a singular point $p \in M$, up to a gauge transformation, can be written as the following,*

$$\eta^* A_D = A + a, \quad \text{with} \quad |a| = O(r), \tag{1.2.2}$$

where the gauge transformation — which is just addition by an exact 1-form — corresponds to tensoring H_p^k by a flat line bundle.

Proof. The pair $(\eta^* A_D, \eta^* \varPhi_D)$ is not necessarily a monopole with respect to the Euclidean metric on $B_\varepsilon(0) \subset \mathbb{R}^3$; however, it is a monopole with respect to the pull-back metric $\eta^* g$, and therefore, $\eta^* \varPhi_D = -\frac{k}{2r} + m + O(r)$, where r is the geodesic distance from the origin with respect to $\eta^* g$.

A is the connection of a monopole with a Higgs field $\varPhi = -\frac{k}{2r_0} + m_0 + O(r_0)$, where r_0 is the distance to the origin with respect to the Euclidean metric, and therefore,

$$| *_0 \, d(\eta^* A_D - A)|_{g_0} = |d(\eta^* \varPhi_D - \varPhi)|_{g_0} = |d(\frac{k}{2r} - \frac{k}{2r_0})|_{g_0} + O(1).$$

Moreover,

$$|r - r_0| = \max\{R_{i,j,k,l}\} O(r_0^3) + O(r_0^4),$$

where $R_{i,j,k,l}$ is the Riemann curvature tensor of $\eta^* g$, and therefore,

$$|d(\eta^* A_D - A)|_{\eta^* g} = O(1),$$

which shows in a suitable gauge, $|a|_{\eta^* g} = O(r)$. $\square$

1.2.2 Construction of Dirac Monopoles

In this section, we construct a Dirac monopole $(A_D, \varPhi_D)$ with prescribed charges and singularities on a rational homology 3-sphere (M, g).

Theorem 2 (Existence of Dirac Monopoles)**.** *Let (M, g) be an oriented rational homology 3-sphere equipped with a Riemannian metric g. Let $p_1, \ldots, p_n$ be n distinct points in M with non-zero integer-valued charges $k_1, \ldots, k_n$, respectively, where $\sum_{i=1}^n k_i = 0$. Then there exists a monopole $(A_D, \varPhi_D)$ with Dirac singularities with charge k_i at p_i on a principal $U(1)$-bundle $P \to M \setminus \{p_1, \ldots, p_n\}$. This monopole, up to gauge transformations and adding a constant to the Higgs field, is unique.*

Proof. From the monopole equation it can be seen that on the complement of the singular points we have $\Delta\Phi = 0$[3], and therefore, Φ is a harmonic section of the adjoint bundle on $M \setminus \{p_1, \ldots, p_n\}$. A Dirac monopole singular at the points $p_1, \ldots, p_n$ with corresponding signed charges $k_1, \ldots, k_n$ is a solution to the equation

$$\Delta\Phi_D = \sum_{i=1}^{n} k_i \delta_{p_i}, \tag{1.2.3}$$

on a compact Riemannian 3-manifold (M, g), in the sense of currents, where δ_{p_i} is the Dirac delta function centered at the point p_i. The Dirac delta function can also be understood as a map $\delta_{p_i} : C^\infty(M) \to \mathbb{R}$, defined by $\delta_{p_i}(f) = f(p_i)$, or as a 3-dimensional cohomology satisfying the equation $\int_M f\delta_{p_i} = f(p_i)$ for any smooth function f. By a slight abuse of notation, we would denote any of them by δ_{p_i}.

The equation 1.2.3 has a solution if and only if

$$\sum_{i=1}^{n} k_i = 0. \tag{1.2.4}$$

This can be seen as a generalization of the well-known fact that on a closed, oriented, Riemannian manifold (M, g), the equation $\Delta f = h$, for a smooth function h, has a solution if and only if $\int_M hvol_g = 0$, to the case where h is not a smooth function but a distribution.

To see this, let $\Phi_i(x) = k_i G_{p_i}(x)$, where G_{p_i} is the Green's function based at the point p_i, which means it satisfies the equation $\Delta G_{p_i}(x) = \delta_{p_i}(x)$, defined on $2\varepsilon_i$-neighbourhood of p_i, for a sufficiently small ε_i [7, Theorem 4.17]. On a sufficiently small neighbourhood of p_i, $G_{p_i}(x) = -\frac{1}{2r_i} + m_i + O(r_i)$, where r_i denotes the geodesic distance from the point p_i. On a small neighbourhood of p_i, Φ_i is a solution to the equation $\Delta\Phi_i = k_i\delta_{p_i}$. Let $\Phi_D = \sum_{i=1}^{n} \xi_i\Phi_i + \phi$, where ξ_i is a cut-off function based at p_i, which is supported on $B_{2\varepsilon_i}(p_i)$ and equal to 1 on $B_{\varepsilon_i}(p_i)$. It can be arranged to have $\int_M \sum_{i=1}^{n} \Delta(\xi_i\Phi_i)vol_g = 0$, with suitable choices of the cut-off functions when $\sum_{i=1}^{n} k_i = 0$. The equation 1.2.3 for ϕ becomes

$$\Delta\phi = \sum_{i=1}^{n}(k_i\delta_{p_i} - \Delta(\xi_i\Phi_i)). \tag{1.2.5}$$

The point of this equation is that the right-hand-side is smooth everywhere on M. Note that in a neighbourhood of the points p_i, the right-hand-side is identically zero. Moreover,

$$\int_M \sum_{i=1}^{n}(k_i\delta_{p_i} - \Delta(\xi_i\Phi_i))vol_g = 0,$$

and therefore, there is a smooth ϕ satisfying the equation 1.2.5.

As mentioned before, another way to think about the Dirac delta function is in terms of differential forms. Let δ_{p_i} be a 3-form representative of the Poincaré dual of the 0-cycle $\{p_i\}$.

[3] In this writing, we use the convention $\Delta = (d + d^*)^2$.

The equation $\Delta\widetilde{\Phi}_D = \sum_i k_i\delta_{p_i}$ has a solution if and only if $\delta = \sum_i k_i\delta_{p_i}$ is an element of the orthogonal complement of harmonic 3-forms $\mathcal{H}^3$. From the Hodge decomposition theorem we have,

$$\Omega^3(M) = d\Omega^2(M) \oplus \mathcal{H}^3.$$

On a closed oriented Riemannian 3-manifold, $\mathcal{H}^3$ is 1-dimensional, generated by the volume form vol_g of the Riemannian metric g — note that the volume form of g is parallel and harmonic. On the other hand

$$\langle \delta, vol_g\rangle = \sum_{i=1}^{n} k_i \int_M \delta_{p_i} \wedge *vol_g = \sum_{i=1}^{n} k_i = 0,$$

and therefore, the equation $\Delta\widetilde{\Phi}_D = \sum_i k_i\delta_{p_i}$ has a solution. We can define the Higgs field of the Dirac monopole by $\Phi_D := *\widetilde{\Phi}_D$.

Furthermore, the solution to this equation is unique up to addition by a constant. For any two solutions Φ_D and Φ'_D of the equation 1.2.3, we have

$$\Delta(\Phi_D - \Phi'_D) = \Delta\Phi_D - \Delta\Phi'_D = \sum_{i=1}^{n} k_i\delta_{p_i} - \sum_{i=1}^{n} k_i\delta_{p_i} = 0,$$

and therefore, $\Phi_D - \Phi'_D$ is a harmonic function on the closed manifold M; hence, it is constant.

Also, note that the assumption on the total charge being zero is necessary. For any Φ with $\Delta\Phi = \sum_{i=1}^{n} k_i\delta_{p_i}$, we have

$$\begin{aligned}\int_{M\setminus\{p_1,\dots,p_n\}} \Delta\Phi vol_g &= \lim_{\varepsilon\to 0}\int_{M\setminus\cup_{i=1}^n B_\varepsilon(p_i)} \Delta\Phi vol_g = \lim_{\varepsilon\to 0}\int_{M\setminus\cup_{i=1}^n B_\varepsilon(p_i)} d^*d\Phi vol_g \\ &= \lim_{\varepsilon\to 0}\int_{M\setminus\cup_{i=1}^n B_\varepsilon(p_i)} d*d\Phi = \lim_{\varepsilon\to 0}\int_{\cup_{i=1}^n \partial B_\varepsilon(p_i)} *d\Phi,\end{aligned}$$

which is zero since $\Delta\Phi = 0$ on $M\setminus\{p_1,\dots,p_n\}$. On the other hand $\Phi = -\frac{k_i}{2r_i} + m_i + O(r_i)$ close to the point p_i, and therefore,

$$*d\Phi = \frac{k_i}{2r_i^2}\iota_{\partial r_i} vol_g + O(1),$$

hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\cup_{i=1}^n \partial B_\varepsilon(p_i)} *d\Phi &= \lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon(p_i)} \left(\frac{k_i}{2r_i^2}\iota_{\partial r_i} vol_g + O(1)\right) = \lim_{\varepsilon\to 0}\sum_{i=1}^{n}\int_{\partial B_\varepsilon(p_i)} \frac{k_i}{2r_i^2} vol_{\partial B_\varepsilon(p_i)} \\ &= 2\pi\sum_{i=1}^{n} k_i,\end{aligned}$$

and therefore, $k := \sum_{i=1}^{n} k_i = 0$. This is in contrast with the non-compact case, where some of the charges can run into infinity.

Note that in the construction of Φ_D, we have not used the assumption about the homology groups of M, and the argument works on any closed oriented Riemannian 3-manifold (M, g). In fact, it works on any closed, oriented, n-dimensional Riemannian manifold. The homology assumption is related to the existence and uniqueness of the connection of a Dirac monopole. We start with the uniqueness problem.

Suppose there exists a connection A_D on a line bundle $\pi : L \to M \setminus \{p_1, \ldots, p_n\}$ such that $F_{A_D} = *d\Phi_D$. For any other connection 1-form A'_D satisfying this equation, since $H^1_{dR}(M \setminus S) = 0$ and $d(A_D - A'_D) = 0$, we have $(A_D - A'_D) = \pi^*(df)$ for a smooth function $f \in C^\infty(M \setminus S)$. This implies that if such a connection exists, it is determined by the Higgs field up to addition by an exact form, which corresponds to tensoring the line bundle which the connection A_D is defined on by a flat line bundle.

Now we focus on the existence problem for such a connection. Let F_D be a 2-form defined by $F_D := *d\Phi_D$. We should determine when we can realize this 2-form as the curvature 2-form of a connection A_D on a principal $U(1)$-bundle on $M \setminus \{p_1, \ldots, p_n\}$. This would be the case if the 2-form F_D has integer periods in $H^2(M, \mathbb{R})$. Recall the following lemma from the Chern-Weil theory.

Lemma 7. *For any integral closed 2-form F on a manifold X, there is a line bundle $L \to X$, unique up to isomorphism, with a connection 1-form A with curvature 2-form F.*

In our case, note that $\frac{1}{2\pi} F_D$ is a closed 2-form on $M \setminus \{p_1, \ldots, p_n\}$, and therefore, we can consider the corresponding cohomology class $[\frac{1}{2\pi} F_D] \in H^2(M \setminus \{p_1, \ldots, p_n\}, \mathbb{R})$. We need to show that the cohomology class $[\frac{1}{2\pi} F_D]$ vanishes in $H^2(M \setminus \{p_1, \ldots, p_n\}, \mathbb{R}/\mathbb{Z})$. However, since M is a rational homology 3-sphere, $H_1(M, \mathbb{Z})$ is finite and $H_2(M, \mathbb{Z}) = 0$, and therefore, $H_2(M \setminus \{p_1, \ldots, p_n\}, \mathbb{Z})$ is generated by 2-spheres $\partial B_\varepsilon(p_i)$. We have

$$\frac{1}{2\pi} \int_{\partial B_\varepsilon(p_i)} F_D = \frac{1}{2\pi} \int_{\partial B_\varepsilon(p_i)} *d\Phi_D = k_i \equiv 0 \text{ modulo } \mathbb{Z},$$

and therefore, by Lemma 7, there is a principal $U(1)$-bundle $L \to M \setminus \{p_1, \ldots, p_n\}$ and a connection 1-form A_D on L, where F_D is the curvature of A_D and Φ_D is a section of the adjoint bundle. □

1.2.2.1 Mass of Monopoles

On non-compact manifolds with ends of suitable types, for instance asymptotically conical ones, the mass of a monopole (A, Φ) with a sufficiently fast decaying curvature can be defined at the ends of the manifold as the limit of the Higgs field $\lim_{r \to \infty} |\Phi|$, where r denotes the geodesic distance from a fixed point $x_0 \in M$. Similarly, we defined the mass of the monopole at a singular point to be the constant m appearing in formula 1.1.8.

Let the vector $\overrightarrow{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ denote the masses of the monopole at the singular points $p_1, \ldots, p_n$ on a closed manifold (M, g). For any Dirac monopole (A_D, Φ_D) with vector mass $\overrightarrow{m}$, the monopole $(A_D, \Phi_D + c)$ has the vector mass $\overrightarrow{m} + c = (m_1 + c, \ldots, m_n + c)$. In the asymptotically conical case, one can find a monopole with fixed charges and fixed masses at the ends of the manifold; however, recall that in the case of compact manifolds, the Dirac monopole, up to addition by a constant, is given by fixing the charges at the singular points, and we do not have much freedom in the choices of the masses of the monopole at the singular points. We can only add a constant to the mass vector, and therefore, we can only fix the position of the singular points, the charges at $(n-1)$ of them and the mass in one of the singular points.

We define the average mass by $\overline{m} = \frac{m_1 + \ldots + m_n}{n}$. The relative mass m'_i at each singular point p_i is defined by $m_i = \overline{m} + m'_i$. The relative mass at each point p_i is a function of the charges and the locations of the singular points, and as one moves these points around, these relative masses change. Here, for our gluing construction to work, we would make the average mass sufficiently large by adding a constant.

In the study of the moduli spaces of monopoles on $\mathbb{R}^3$, one can assume a normalizing condition, to have mass 1 at infinity, since there is a natural identification between the moduli spaces of monopoles with different masses. However, this is not true for the moduli spaces of monopoles over other 3-manifolds, and there is no natural identification between the moduli spaces of monopoles with different masses.

1.2.2.2 Lifting the Dirac Monopole

For carrying on our gluing construction, we should lift the Dirac monopole we constructed to an $SU(2)$-bundle, so we can glue the scaled $SU(2)$ BPS-monopoles to this lifted background Dirac monopole. Consider the rank 2 vector bundle $L \oplus L^{-1} \to M \setminus \{p_1, \ldots, p_n\}$. Let ∇_{A_D} be the covariant derivative of A_D on L. This induces a covariant derivative on $L \oplus L^{-1}$, namely $\nabla_{A_D} \oplus (-\nabla_{A_D})$.

Moreover, close to a singular point p_i, the adjoint bundle of the corresponding $SU(2)$-bundle can be decomposed as $\underline{\mathbb{R}} \oplus H_{p_i}^{k_i}$, with the induced Higgs field $\Phi_D \oplus 0$. Close to a singular point p_i with charge k_i, the rank 2 bundle is isomorphic to $H_{p_i}^{k_i} \oplus H_{p_i}^{-k_i}$.

We can fix a basis for $\mathfrak{su}(2)$,

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Suppose (A_D, Φ_D) is a $U(1)$-Dirac monopole, defined on a $U(1)$-bundle $P_{U(1)}$ with associated line bundle L. The induced $SU(2)$-monopole is $(A_D\sigma_3, \Phi_D\sigma_3)$, which for simplicity and by an abuse of notation we still denote this monopole by (A_D, Φ_D).

The other main ingredients of the gluing construction are the scaled BPS-monopoles on $\mathbb{R}^3$. In the following section, we study the model monopoles which we glue to the background Dirac monopole.

1.2.3 BPS-Monopoles on $\mathbb{R}^3$

In this section, we introduce the BPS-monopoles on $\mathbb{R}^3$ and recall a basic lemma about their asymptotic behaviour.

Let $P \to \mathbb{R}^3$ be a principal $SU(2)$-bundle. Let A be a connection on P and Φ a section of the adjoint bundle. For pairs (A, Φ) with suitable asymptotic decay, we can define the Yang-Mills-Higgs action functional,

$$\mathcal{YMH}(A, \Phi) := \frac{1}{2} \int_{\mathbb{R}^3} (|F_A|^2 + |d_A \Phi|^2) dx dy dz,$$

where the norms are defined with respect to the adjoint-invariant inner product on the adjoint bundle.

The critical points of Yang-Mills-Higgs action functional are the solutions to the following equations,

$$\begin{aligned} d_A^* F_A &= - * [\Phi, d_A \Phi], \\ d_A^* d_A \Phi &= 0. \end{aligned}$$

Monopoles satisfy these equations, in fact, they are the minimizers of this action functional. Monopoles with finite Yang-Mills-Higgs energy satisfy the following decay conditions,

$$|F_A| = |d_A \Phi| = O(r^{-2}), \qquad |\Phi| \to m, \quad \text{as} \quad r \to \infty, \tag{1.2.6}$$

where the constant $m = \lim_{|x| \to \infty} |\Phi(x)|$ is the mass of the monopole at infinity.

By a scaling, one can change the mass of a given monopole on $\mathbb{R}^3$. If (A, Φ) is a monopole on $\mathbb{R}^3$ with mass m at infinity, then

$$(A^\lambda, \Phi^\lambda)(x) := (A, \lambda \Phi)(\lambda x),$$

is also a monopole on $\mathbb{R}^3$ with mass λm. This shows there are natural identifications between the moduli spaces of monopoles with different positive masses, and therefore, one can assume a normalizing condition, and let $m = 1$. More generally, we have the following lemma.

Lemma 8. *Let (M, g) be a complete, oriented, Riemannian 3-manifold. Let*

$$exp_p^\lambda : T_p M \to M, \qquad exp_p^\lambda(v) := \exp_p(\lambda v),$$

be the exponential map based at a point $p \in M$, defined for any $\lambda \in \mathbb{R}^+$.

Let (A, Φ) be a monopole on a principal G-bundle $P \to M$. Then the pair $(A^\lambda, \Phi^\lambda) := ((exp_p^\lambda)^* A, \lambda (exp_p^\lambda)^* \Phi)$ is a monopole on the pull-back bundle $(exp_p^\lambda)^* P \to T_p M$ with respect to the Riemannian metric $g_\lambda := \lambda^{-2} (exp_p^\lambda)^* g$.

Proof. This lemma can be seen as an instance of the conformal invariance of the anti-self-duality equation on 4-dimensional manifolds, but can also be seen directly,

$$F_{A^\lambda} = d\left((exp_p^\lambda)^* A\right) + \frac{1}{2}[(exp_p^\lambda)^* A \wedge (exp_p^\lambda)^* A] = (exp_p^\lambda)^*(dA + \frac{1}{2}[A \wedge A]) = (exp_p^\lambda)^* F_A.$$

Furthermore,

$$d_{A^\lambda}\Phi^\lambda = d\left(\lambda(exp_p^\lambda)^*\Phi\right) + [(exp_p^\lambda)^*A, \lambda(exp_p^\lambda)^*\Phi] = \lambda(exp_p^\lambda)^*(d\Phi + [A,\Phi]) = \lambda(exp_p^\lambda)^* d_A\Phi.$$

Moreover, we have $exp_p^\lambda = Id \circ \exp_p \circ m_\lambda$, where $m_\lambda : T_pM \to T_pM$ is defined by $m_\lambda(v) = \lambda v$.

$$\begin{aligned}(T_pM, g_3 := g_\lambda = \lambda^{-2} m_\lambda^* \circ exp_p^* g) &\xrightarrow{Id} (T_pM, g_2 := m_\lambda^* \circ exp_p^* g) \\ &\xrightarrow{m_\lambda} (T_pM, g_1 := exp_p^* g) \xrightarrow{exp_p} (M, g_0 := g).\end{aligned}$$

The following maps are isometries,

$$\begin{aligned}&m_\lambda : (T_pM, g_2 = m_\lambda^* \circ exp_p^* g) \to (T_pM, g_1 = exp_p^* g), \\ &exp_p : (T_pM, g_1 = exp_p^* g) \to (M, g_0 = g),\end{aligned}$$

and therefore, we have $m_\lambda^* \circ *_{g_1} = *_{g_2} \circ m_\lambda^*$ and $exp_p^* \circ *_{g_0} = *_{g_1} \circ exp_p^*$.

The identity map

$$Id : (T_pM, g_3 = g_\lambda = \lambda^{-2} m_\lambda^* \circ exp_p^* g) \to (T_pM, g_2 = m_\lambda^* \circ exp_p^* g)$$

is not an isometry; however, it is a conformal map.

More generally, recall that on an oriented n-dimensional Riemannian manifold, under the conformal change of the metric $\tilde{g} := e^{2f} g$, the Hodge star operator on p-forms changes according to the formula $*_{\tilde{g}} = e^{(n-2p)f} *_g$. In our case $n = 3$, $p = 2$ and $f = -\ln(\lambda)$, and therefore, $*_{g_3} \circ Id^* = \lambda Id^* \circ *_{g_2}$. We have

$$*_{g_\lambda} \circ (exp_p^\lambda)^* = *_{g_\lambda} \circ exp_p^* \circ m_\lambda^* \circ Id^* = \lambda(exp_p^\lambda)^* \circ *_g,$$

and therefore,

$$*_{g_\lambda} F_{A^\lambda} = *_{g_\lambda}(exp_p^\lambda)^*(F_A) = \lambda(exp_p^\lambda)^* *_g (F_A) = \lambda(exp_p^\lambda)^* d_A\Phi = d_{A^\lambda}\Phi^\lambda.$$

□

On $\mathbb{R}^3$, the Euclidean metric is invariant under this transformation, $g_\lambda = g$, and therefore, for every monopole (A, Φ) with mass m, we have a monopole $(A_\lambda, \Phi_\lambda)$ with mass $\lim_{|x|\to\infty} |\Phi_\lambda(x)| = \lambda m$. This scaling plays an important role in the gluing construction.

In the case $G = SU(2)$, to any pair (A, Φ) on an $SU(2)$-bundle on $\mathbb{R}^3$, which is not necessarily a monopole, with the decay conditions 1.2.6 and mass m, one can assign an integer charge, defined by

$$k := \lim_{R\to\infty} \frac{1}{4\pi m} \int_{S_R(0)} \langle \Phi, F_A \rangle.$$

Note that this notion of charge is different from but closely related to the notion of charge at a singularity. To differentiate these two, we might call this one the charge of the monopole and the one we defined earlier the charge of the monopole at a singularity.

A very important problem in the theory of monopoles, also related to the gluing constructions, is to understand the moduli space of monopoles on $\mathbb{R}^3$ with charge k. The seminal work of Taubes shows that for any charge k, there are $SU(2)$-monopoles with charge k on $\mathbb{R}^3$. Atiyah and Hitchin showed that the moduli space of centered $SU(2)$-monopoles with charge k is a $(4k-4)$-dimensional smooth hyperkähler manifold [4]. In all of these constructions, the BPS-monopole plays a crucial role. The BPS-monopole is an explicit charge +1 solution to the Bogomolny equation which was discovered by Prasad and Sommerfield [80].

Definition/Lemma 9. *There is a unique $SU(2)$-monopole on $\mathbb{R}^3$, centred at the origin with mass 1 at infinity and charge $+1$, called the BPS-monopole. Denoting this monopole by (A_{BPS}, Φ_{BPS}), we have*

$$A_{BPS}(x) = (\frac{1}{\sinh(r)} - \frac{1}{r})(n \times \sigma) \cdot dx, \qquad \Phi_{BPS}(x) = (\frac{1}{\tanh(r)} - \frac{1}{r}) n \cdot \sigma,$$

where $r = |x|$, $n = \frac{x}{r}$, $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \otimes \mathfrak{su}(2)$, and

$$\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

where $\cdot$ and $\times$ are formal inner and cross product on vectors with three components.

Although the BPS-monopole looks singular at $\{0\}$, it extends smoothly to the origin. We have $\Phi_{BPS}(0) = 0$; moreover, this is the only zero of the Higgs field. Furthermore, $|\Phi(x)| < 1$ for all $x \in \mathbb{R}^3$ and $\lim_{|x| \to \infty} |\Phi(x)| \to 1$.

1.2.4 Approximate Solutions

In this section, we construct an approximate $SU(2)$-monopole (A_0, Φ_0) on an $SU(2)$-bundle $P \to M$, with prescribed Dirac singularities at some isolated points p_i for $i \in \{1, \dots, n\}$. The pair (A_0, Φ_0) would not be a genuine monopole, but an approximate one.

We take the lifted Dirac monopoles we constricted in the Section 1.2.2 as the background monopole. The monopoles we are gluing to these Dirac monopoles are defined by scalings of the BPS-monopole on $\mathbb{R}^3$.

The idea of constructing an irreducible singular $SU(2)$-monopole with singular points $p_1, \dots, p_n$ with corresponding charges $k_1, \dots, k_n$, where $k_i \in \mathbb{N}$, is to start with an Abelian Dirac monopole with singularities at the points p_i with negative charges k'_i such that $k'_i = -k_i$, and some other well-separated singular points $S_q = \{q_1, \dots, q_k\}$ all with signed charge $+1$, such that

$$\text{Total charge } := k + \sum_{i=1}^{n} k'_i = 0.$$

Since the total charge is zero, there exists a reducible Dirac monopole (A_D, Φ_D) on an $SU(2)$-bundle with these prescribed singularities. We will glue the scaled BPS-monopoles with charge $+1$ to this background Dirac monopole at the points q_j with charge $+1$ at the singularities. The scaling is necessary since close to the singular points q_j, $|\Phi_D(x)| \to \infty$ as $dist(x, q_j) \to 0$, and therefore, near the singular points $|\Phi_D|$ is quite large, which implies the Higgs field of the $SU(2)$-monopole which we are gluing to the background monopole close to the point q_j should be large too. We change the mass of the BPS-monopole by a suitable scaling.

The Higgs field of the BPS-monopole (A_{BPS}, Φ_{BPS}) is non-zero on $\mathbb{R}^3 \setminus \{0\}$, and therefore, it induces a decomposition of the adjoint bundle $\underline{\mathbb{R}} \oplus L$, where $\underline{\mathbb{R}}$ is the sub-bundle generated by the image of the Higgs field Φ_{BPS} and L is the orthogonal sub-bundle in the adjoint bundle. Corresponding to this decomposition any section of the adjoint-bundle or any adjoint-bundle-valued tensor f supported away from $0 \in \mathbb{R}^3$ can be written as $f = f^L + f^T$, where f^L and f^T are called the longitude and the transverse components, respectively. The following key lemma follows from the work of Jaffe and Taubes [49, Section IV.1], also Lemma 2.13 in [30], which is fundamental in the gluing construction.

Lemma 10. *Let (a, φ) be a pair of a connection denoted by a on the $SU(2)$-bundle $P \to \mathbb{R}^3 \setminus \{0\}$ and a section φ of the adjoint bundle, with finite Yang-Mills-Higgs energy, with charge k, and mass 1 at infinity, which does not necessarily satisfy the Bogomolny equation. Then we have*

$$|\Phi_D^L - \varphi^L| = O(r^\nu), \quad |\Phi_D^T - \varphi^T| = O(e^{-r}), \quad |A_D - a| = O(e^{-r}),$$

for some $\nu < 0$.

Moreover, for the BPS-monopole with charge $+1$ and mass 1 centered at the origin, we have

$$|\Phi_D^L - \Phi_{BPS}^L| = O(e^{-r}), \quad |\Phi_D^T - \Phi_{BPS}^T| = O(e^{-r}), \quad |A_D - A_{BPS}| = O(e^{-r}),$$

and therefore,

$$|\Phi_D - \Phi_{BPS}| = O(e^{-r}), \quad |A_D - A_{BPS}| = O(e^{-r}).$$

From the Lemma 8, recall that if (A, Φ) is a monopole on $\mathbb{R}^3$ with mass m at infinity, then $(A^\lambda, \Phi^\lambda)(x) := (A, \lambda\Phi)(\lambda x)$ is also a monopole on $\mathbb{R}^3$ with the same charge, but with the mass λm. The scaling by $\lambda > 1$ not only makes the mass larger, but also makes the convergence faster. We have

$$|\Phi_{BPS}^\lambda - \Phi_D^\lambda| = O(\lambda e^{-\lambda r}), \qquad |A_{BPS}^\lambda - A_D^\lambda| = O(\lambda e^{-\lambda r}). \tag{1.2.7}$$

The result of scaling Dirac monopoles on $\mathbb{R}^3$ is quite simple. The connection of the Dirac monopole is radially invariant, $A_D^\lambda = A_D$. Furthermore, for the Higgs field of the Dirac monopole we have

$$\Phi_D^\lambda(x) = \lambda \Phi_D(\lambda x) = \lambda(1 - \frac{2k}{\lambda r}) = \lambda - \frac{2k}{r} = \Phi_D(x) + (\lambda - 1).$$

Therefore, this scaling just adds the constant $(\lambda - 1)$ to the Higgs field of the Dirac monopole,

$$(A_D^\lambda, \Phi_D^\lambda)(x) = (A_D, \Phi_D + (\lambda - 1))(x).$$

Remark 2. *As we increase the average mass of the $SU(2)$-monopole we get closer to the boundary of the moduli space of monopoles. The Dirac monopoles can be understood as a part of the boundary — or corner — of the moduli space of monopoles, so the strategy, similar to the other gluing constructions, is to start from a boundary point of the moduli space and then move — deform — towards inside.*

By adding a positive large constant, if necessary, we can assume that the local description of the Higgs field of the monopole, close to each singular point p_i or q_j, has the form

$$\Phi_D = -\frac{k_i}{2r_i} + m_i + O(r_i),$$

with $m_i > 0$. Close to the singular points with positive charges, the Higgs field goes to negative infinity; however, for a fixed positive ε_0, we can increase the mass of the Higgs field such that on $\overline{M \setminus \cup_j B_{\varepsilon_0}(q_j)}$ we have $\Phi_D \geq \overline{m}/2$, simply because $\overline{M \setminus (\cup_i B_{\varepsilon_0}(p_i) \cup_j B_{\varepsilon_0}(q_j))}$ is compact and the Higgs field goes to plus infinitiy at the points p_i. Similar to the non-compact case, as observed by Oliveira [76], a more relevant inequality would be of the type where ε_0 depends on the average mass, as in the following lemma.

Lemma 11. *By increasing the mass of (A_D, Φ_D), if necessary, on $K(\varepsilon_0) := \overline{M \setminus \cup_j B_{\varepsilon_0}(q_j)}$ we have $\Phi_D \geq \overline{m}/2$, where $\varepsilon_0 = \sqrt{2/\overline{m}}$.*

Proof. Let $\varepsilon > 0$ be a sufficiently small positive number such that on ε-neighbourhood of singular points p_i or q_j, $\Phi_D = -k_i/2r_i + m_i + O(r_i)$ for a positive m_i . By making ε smaller, if necessary, we can assume $\Phi_D + 1 > -k_i/2r_i + m_i$. Furthermore, by adding a constant to the Higgs field, on $\overline{M \setminus \cup_j B_\varepsilon(q_j)}$, we would have $\Phi_D \geq \overline{m}/2$.

Now we need to show the same holds for $\varepsilon_0 < r_i < \varepsilon$. It is enough to show that on this region we have $-1/2r_i + m_i - 1 \geq \overline{m}/2$ or, equivalently, $\overline{m}/2 + m'_i \geq 1/2r_i + 1$. By adding a constant to the Higgs field, we assume $\overline{m} \geq 2$ and $\overline{m} \geq 2m'_i$ for all i, and therefore, it is enough to have

$$r_i \geq \frac{1}{2\overline{m} - 2}. \tag{1.2.8}$$

This holds if we let $\varepsilon_0 = \sqrt{\frac{2}{\overline{m}}}$, which is larger than $\frac{1}{2\overline{m}-2}$ when $\overline{m}$ is sufficiently large. □

The singular points p_i can be arbitrarily close to each other, but for fixed masses, the construction breaks down as one moves the singular points q_j very close to each other or to the points p_i. However, if we allow the average mass to increase, these points can be arbitrarily close. For the gluing construction to work we increase the average mass such that

$$\min_{i,j}\{dist_{i\neq j}(q_i, q_j), dist_{i,j}(q_i, p_j)\} \geq \varepsilon_0 = \sqrt{\frac{2}{\overline{m}}}.$$

The approximate monopole we are constructing is equal to the Dirac monopole on $M \setminus \cup_j B_{2\varepsilon_j}(q_j)$ with a large mass, and equal to the pull-back of an appropriately scaled BPS-monopole on each $B_{\varepsilon_j}(q_j)$, where the scaling factor $\lambda_j = m_j$ is the mass of the Dirac monopole at q_j. This can be done after identifying a small neighbourhood of q_j with a neighbourhood of the origin in $\mathbb{R}^3$ and the bundles above them.

Fix a diffeomorphism between $2\varepsilon_j$-neighbourhood of the singular point q_j and a neighbourhood of origin in $\mathbb{R}^3$ using the geodesic normal coordinates,

$$\eta_j : B_{2\varepsilon_j}(q_j) \subset M \to \mathbb{R}^3.$$

Moreover, we can fix an identification between the associated vector bundles above these neighbourhoods covering η_j, which by an abuse of notation, we also denote this bundle map, called the framing, by η_j,

$$\eta_j : (\underline{\mathbb{R}} \oplus H_{q_j})_{B_{2\varepsilon_j}(q_j)\setminus\{q_j\}} \to \underline{\mathfrak{su}(2)}_{\mathbb{R}^3\setminus\{0\}}.$$

Using these identifications we can pull back the scaled BPS-monopoles to the $2\varepsilon_j$-neighbourhood of q_j. Although the Dirac monopole is not defined at the point q_j, the pull-back of the BPS-monopole and the bundle it is defined on extend smoothly over q_j. Different identifications can result in different pairs on $M \setminus S_p$, even up to gauge. Up to isomorphism there is a $U(1)$-freedom in the choice of the framing for each q_j, and assuming we have k such points we would get k parameters for the choices of framings — up to gauge $(k-1)$ parameters.

Suppose for each q_j, a framing η_j is fixed. Now we can pull back the bundles and the scaled BPS-monopoles to $B_{2\varepsilon_j}(q_j)$. We denote these local pairs by

$$(\eta_j^*(A_{BPS}^{\lambda_j}), \eta_j^*(\Phi_{BPS}^{\lambda_j})).$$

Using cut-off functions we can glue these local monopoles to the background monopole. Suppose ξ_j is a cut-off function supported around q_j such that

$$\xi_j = \begin{cases} 1 & \text{on} \quad B_{\varepsilon_j}(q_j) \\ 0 & \text{on} \quad M \setminus B_{2\varepsilon_j}(q_j), \end{cases}$$

and

$$\xi_0 = \begin{cases} 1 & \text{on} \quad M \setminus \cup_j B_{2\varepsilon_j}(q_j) \\ 0 & \text{on} \quad \cup_j B_{\varepsilon_j}(q_j), \end{cases}$$

where on $\varepsilon_j \leq r_j \leq 2\varepsilon_j$ we have $\xi_0 + \xi_j = 1$ for each $j \in \{1, \dots, k\}$, and

$$|\nabla \xi_j| \leq 2\varepsilon_j^{-1} \quad \text{and} \quad |\nabla \xi_0| \leq 2 \max_{j\in\{1,\dots,k\}} \{\varepsilon_j^{-1}\}.$$

The approximate monopole has the form

$$(A_0, \Phi_0) = \xi_0(A_D, \Phi_D) + \sum_{j=1}^{k} \xi_j(\eta_j^*(A_{BPS}^{\lambda_j}), \eta_j^*(\Phi_{BPS}^{\lambda_j})).$$

Note that the assumption $\xi_0 + \xi_j = 1$ assures A_0 is a connection.

1.2.4.1 Pointwise Approximation of the Error

This pair (A_0, Φ_0) is an approximate solution and does not necessarily satisfy the Bogomolny equation. We define the error term by

$$e_0 = *F_{A_0} - d_{A_0}\Phi_0.$$

In this section, we estimate the error term e_0 in different regions on M.

e_0 is zero on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$, since on this region the approximate monopole is equal to the Dirac monopole. The error term is non-zero on $\cup_j B_{2\varepsilon_j}(q_j)$. It is correct that on each $B_{\varepsilon_j}(q_j)$ the approximate monopole is equal to the pull-back of the scaled BPS-monopole, but we only know that the scaled BPS-monopole is a monopole with respect to the Euclidean metric and not with respect to the arbitrary Riemannian metric g on M. The error term e_0 is also non-zero on the necks $\cup_j (B_{2\varepsilon_j}(q_j) \setminus B_{\varepsilon_j}(q_j))$, both because g is not necessarily flat, and also because of the use of the cut-off functions.

Lemma 12. *Let $\varepsilon_j = \lambda_j^{-\frac{1}{2}}$. Then when the average mass is sufficiently large, we have the following pointwise error estimate,*

$$(e_0)_{|B_{2\varepsilon_j}(q_j)} = O(1).$$

Proof. We denote the error coming from the manifold not being flat around q_j by

$$e_j^{BPS} := \left(*F_{\eta_j^*(A_{BPS}^{\lambda_j})} - d_{\eta_j^*(A_{BPS}^{\lambda_j})} \eta_j^*(\Phi_{BPS}^{\lambda_j}) \right)_{|B_{2\varepsilon_j}(q_j)}.$$

This error would vanish if the metric is flat on small neighbourhoods of the points q_j; however, it is not true in the general case. On each $B_{2\varepsilon_j}(q_j)$, the size of the error depends on how much the metric g is different from the Euclidean metric. For the comparison between the metric g on $B_{2\varepsilon_j}(q_j)$ and the Euclidean metric, we should first pull back the Euclidean metric g_0 to $B_{2\varepsilon_j}(q_j)$ using the same map that we used to pull back the scaled BPS-monopole to $B_{2\varepsilon_j}(q_j)$. We denote the Euclidean metric pulled back to $B_{2\varepsilon_j}(q_j)$ and its Hodge star operator by g_0 and $*_0$, respectively.

$$\begin{aligned}
(e_j^{BPS})_{|B_{2\varepsilon_j}(q_j)} &= *F_{\eta_j^*(A_{BPS}^{\lambda_j})} - d_{\eta_j^*(A_{BPS}^{\lambda_j})} \eta_j^*(\Phi_{BPS}^{\lambda_j}) \\
&= (*_0 F_{\eta_j^*(A_{BPS}^{\lambda_j})} - d_{\eta_j^*(A_{BPS}^{\lambda_j})} \eta_j^*(\Phi_{BPS}^{\lambda_j})) + (*F_{\eta_j^*(A_{BPS}^{\lambda_j})} - *_0 F_{\eta_j^*(A_{BPS}^{\lambda_j})}) \\
&= *F_{\eta_j^*(A_{BPS}^{\lambda_j})} - *_0 F_{\eta_j^*(A_{BPS}^{\lambda_j})} = (* - *_0) F_{\eta_j^*(A_{BPS}^{\lambda_j})}.
\end{aligned}$$

Note that

$$*_0 F_{\eta_j^*(A_{BPS}^{\lambda_j})} - d_{\eta_j^*(A_{BPS}^{\lambda_j})} \eta_j^*(\Phi_{BPS}^{\lambda_j}) = 0,$$

since $(A_{BPS}^{\lambda_j}, \Phi_{BPS}^{\lambda_j})$ is a monopole with respect to the Euclidean metric g_0.

For each j, for sufficiently small ε_j, fix a local geodesic normal coordinate system on $B_{2\varepsilon_j}(q_j)$, denoted by (x_1, x_2, x_3). We can think about the components of the Riemannian metric $g_{k,l}$ in this coordinate system as real-valued functions defined on this neighbourhood of q_j. We can write down the Taylor series expansion of these functions around the origin, which here corresponds to q_j,

$$\begin{aligned} g_{k,l}(x) &= \delta_k^l + \frac{1}{3}\sum_{m,n} R_{klmn} x_m x_n + O(|x|^3), \\ g^{k,l}(x) &= \delta_k^l - \frac{1}{3}\sum_{m,n} R_{klmn} x_m x_n + O(|x|^3), \end{aligned}$$

where R_{klmn} is the $(4,0)$-Riemann curvature tensor. Furthermore,

$$vol_g(x) = \left(1 - \frac{1}{6}\sum_{m,n} R_{m,n} x_m x_n + O(|x|^3)\right) dx_1 dx_2 dx_3,$$

where $R_{m,n}$ denotes the Ricci curvature tensor.

For any 2-form $\beta \in \Omega^2(M)$, which in the given coordinates system $\beta = \sum_{\text{cyclic } i,j,k} \beta_i dx_j dx_k$, we have

$$|\beta|_g^2 = \sum_{k,l} \beta_k \beta_l g^{k,l} = |\beta|_{g_0}^2 - \frac{1}{3}\sum_{k,l,m,n} R_{klmn} \beta_k \beta_l x_m x_n + O(|x|^3).$$

Similarly, if $\beta \in \Omega^2(M, V)$ is a V-valued 2-form for a vector bundle V equipped with a fiber-wise inner product $\langle -, - \rangle$,

$$|\beta|_g^2 = \sum_{i,j} \langle \beta_k, \beta_l \rangle g^{k,l} = |\beta|_{g_0}^2 - \frac{1}{3}\sum_{k,l,m,n} R_{klmn} \langle \beta_k, \beta_l \rangle x_m x_n + O(|x|^3).$$

On the other hand

$$\langle \beta \wedge *\beta \rangle = |\beta|_g^2 vol_g,$$

where $\langle - \wedge - \rangle$ is wedge product on the real differential form parts and inner product on V-valued parts, and therefore,

$$\langle \beta \wedge *\beta \rangle = \left(|\beta|_{g_0}^2 - \sum_{m,n} (\frac{1}{3} \sum_{k,l} R_{klmn} \langle \beta_k, \beta_l \rangle + \frac{1}{6} |\beta|_{g_0}^2 R_{m,n}) x_m x_n + O(|x|^3) \right) dx_1 dx_2 dx_3$$

$$\Rightarrow \beta \wedge (* - *_0)\beta = \left(- \sum_{m,n} (\frac{1}{3} \sum_{k,l} R_{klmn} \langle \beta_k, \beta_l \rangle + \frac{1}{6} |\beta|_{g_0}^2 R_{m,n}) x_m x_n + O(|x|^3) \right) dx_1 dx_2 dx_3$$

$$\Rightarrow (* - *_0)\beta = - \sum_{k} (\frac{1}{3} \sum_{l,m,n} R_{klmn} \beta_l x_m x_n + \frac{1}{6} \sum_{l,m,n} \beta_l g^{k,l} R_{m,n} x_m x_n) dx_k + O(|x|^3),$$

and therefore, pointwise and with respect to the metric g,

$$|(* - *_0)\beta|_g \leq C |\beta|_g |x|^2,$$

where the constant C depends only on the curvature tensor of (M, g). Going back to the curvature 2-form $F_{\eta_j^*(A_{BPS}^{\lambda_j})}$ on $B_{2\varepsilon_j}(q_j)$, the computations above show

$$|(*_0 - *) F_{\eta_j^*(A_{BPS}^{\lambda_j})}|_g \leq C |F_{\eta_j^*(A_{BPS}^{\lambda_j})}|_g |x|^2.$$

Following [49, Section IV.1], and using the same notations as in the Definition 9,

$$(d_{A_{BPS}} \Phi_{BPS})^L = (\frac{1}{\sinh^2(|x|)} - \frac{1}{|x|^2})(n \cdot \sigma) n \cdot dx,$$

$$(d_{A_{BPS}} \Phi_{BPS})^T = (\frac{1}{|x|} - \frac{1}{\tanh(|x|)})(\frac{1}{\sinh(|x|)})(\sigma - (n \cdot \sigma)\sigma) \cdot dx.$$

Although $(d_{A_{BPS}} \Phi_{BPS})^L$ and $(d_{A_{BPS}} \Phi_{BPS})^T$ look singular at the origin, they extend smoothly to the origin. In fact, the maximum of both of these components are achieved at the origin,

$$|(d_{A_{BPS}} \Phi_{BPS})^L|_{g_0} \leq \frac{1}{3}, \quad |(d_{A_{BPS}} \Phi_{BPS})^T|_{g_0} \leq \frac{1}{3}.$$

Furthermore,

$$|d_{A_{BPS}^{\lambda_j}} \Phi_{BPS}^{\lambda_j}|_{g_0} \leq \frac{C}{\lambda_j^{-2} + |x|^2},$$

for a constant $C > 0$.

For any 2-form β, with values in any vector bundle, we have

$$|\beta|_g^2 - |\beta|_{g_0}^2 = \sum_{k,l} \langle \beta_k, \beta_l \rangle (g^{k,l} - g_0^{k,l}) = -\frac{1}{3} \sum_{k,l,m,n} \langle \beta_k, \beta_l \rangle R_{klmn} x_m x_n + O(|x|^3)$$
$$\leq CR|\beta|_{g_0}^2 |x|^2,$$

where $C > 0$ is a constant and R is the maximum of the Riemann curvature tensor of g. Therefore,

$$|\beta|_g^2 \leq |\beta|_{g_0}^2 + CR|\beta|_{g_0}^2 |x|^2,$$

Let $\beta = (*_0 - *)F_{\eta_j^*(A_{BPS}^{\lambda_j})}$

$$|(*_0 - *)F_{\eta_j^*(A_{BPS}^{\lambda_j})}|_g^2 \leq C \left(\frac{|x|^4}{(\lambda_j^{-2} + |x|^2)^2} + R \frac{|x|^6}{(\lambda_j^{-2} + |x|^2)^2} \right),$$

for a constant C, when λ_j is sufficiently large.

These sum up to

$$(e_j^{BPS})_{|B_{2\varepsilon_j}(q_j)} \leq C', \tag{1.2.9}$$

for a positive constant C'.

On the neck $B_{2\varepsilon_j}(q_j) \setminus B_{\varepsilon_j}(q_j)$, the cut-off function is another source of error. On this region, we have

$$\begin{aligned} e_0 = & \sum_{j=1}^{k} \Big((*F_{\eta_j^*(A_{BPS}^{\lambda_j})} - d_{\eta_j^*(A_{BPS}^{\lambda_j})} \eta_j^*(\Phi_{BPS}^{\lambda_j}))_{|B_{\varepsilon_j}(q_j)} \\ & + \xi_0 (*d_{\eta_j^*(A_{BPS}^{\lambda_j})} (A_D^{\lambda_j} - \eta_j^*(A_{BPS}^{\lambda_j})) - d_{\eta_j^*(A_{BPS}^{\lambda_j})} (\Phi_D^{\lambda_j} - \eta_j^*(\Phi_{BPS}^{\lambda_j})) \\ & + * (d\xi_0 \wedge (A_D^{\lambda_j} - \eta_j^*(A_{BPS}^{\lambda_j}))) - d\xi_0 (\Phi_D^{\lambda_j} - \eta_j^*(\Phi_{BPS}^{\lambda_j})) \\ & + (\xi_0 (A_D^{\lambda_j} - \eta_j^*(A_{BPS}^{\lambda_j})))^2 - \xi_0^2 [A_D^{\lambda} - \eta_j^*(A_{BPS}^{\lambda}), \Phi_D^{\lambda} - \eta_j^*(\Phi_{BPS}^{\lambda})] \Big), \end{aligned}$$

where $*$ is the Hodge star of g on M.

First consider the case where the Riemannian metric g is flat, as we were to glue a scaled BPS-monopole to a scaled Dirac monopole on $\mathbb{R}^3$. Then following Lemma 10, we would have

$$|\Phi_{BPS}^{\lambda_j} - \Phi_D^{\lambda_j}| = O(\lambda_j e^{-\lambda_j r_j}), \quad |A_{BPS}^{\lambda_j} - A_D^{\lambda_j}| = O(\lambda_j e^{-\lambda_j r_j}).$$

and therefore, on this region

$$
\begin{aligned}
|\xi_0(*d_{\eta_j^*(A_{BPS}^{\lambda_j})}(A_D^{\lambda_j}-\eta_j^*(A_{BPS}^{\lambda_j}))| &\leq c_1(\lambda_j^2 e^{-\lambda_j r_j}),\\
|d_{\eta_j^*(A_{BPS}^{\lambda_j})}(\Phi_D^{\lambda_j}-\eta_j^*(\Phi_{BPS}^{\lambda_j}))| &\leq c_2(\lambda_j^2 e^{-\lambda_j r_j}),\\
|(d\xi_0\wedge(A_D^{\lambda_j}-\eta_j^*(A_{BPS}^{\lambda_j})))| &\leq c_3(\frac{\lambda_j}{\varepsilon_j}e^{-\lambda_j r_j}),\\
|d\xi_0(\Phi_D^{\lambda_j}-\eta_j^*(\Phi_{BPS}^{\lambda_j}))| &\leq c_4(\frac{\lambda_j}{\varepsilon_j}e^{-\lambda_j r_j}),\\
|(\xi_0(A_D^{\lambda_j}-\eta_j^*(A_{BPS}^{\lambda_j})))^2| &\leq c_5(\lambda_j^2 e^{-2\lambda_j r_j}),\\
|\xi_0^2[A_D^{\lambda}-\eta_j^*(A_{BPS}^{\lambda}),\Phi_D^{\lambda}-\eta_j^*(\Phi_{BPS}^{\lambda})]| &\leq c_6(\lambda_j^2 e^{-2\lambda_j r_j}),
\end{aligned}
$$

for constants $c_1,\dots,c_6$, independent of ε_j and λ_j.

Here λ_j and ε_j should be understood as a very large and a very small number, respectively. As we increase λ_j, we can make ε_j smaller. Although there is no unique choice for these parameters for the gluing construction to work, sometimes there are choices which minimize the error of the approximate solution.

Let's let $\varepsilon_j=\lambda_j^l$ for some $-1<l<0$. For l outside of this interval the errors listed above can be large. The appropriate value for l depends on the functional spaces we choose to work with. For sufficiently large λ_j, the leading term of the bounds for the error e_j^{neck}, up to a constant, would be $\lambda_j^2 e^{-\lambda_j{}^{l+1}}$. These errors are exponentially small and favorable.

However, the case over arbitrary Riemannian 3-manifolds is different, since the Green's function on a neighbourhood of a point q_j is not necessarily equal to $-\frac{1}{2r_j}+m_j$, but potentially there are higher order terms, and therefore,

$$
|\Phi_{BPS}^{\lambda_j}-\Phi_D^{\lambda_j}|=|\Phi_{BPS}^{\lambda_j}-(-\frac{k_j}{2r_j}+m_j)|+|(-\frac{k_j}{2r_j}+m_j)-\Phi_D^{\lambda_j}|=O(\lambda_j e^{-\lambda_j r_j})+O(r_j),
$$

and similarly,

$$
|A_{BPS}^{\lambda_j}-A_D^{\lambda_j}|=O(\lambda_j e^{-\lambda_j r_j})+O(r_j).
$$

and therefore, for sufficiently large λ_j,

$$
|\Phi_{BPS}^{\lambda_j}-\Phi_D^{\lambda_j}|=O(r_j),\quad |A_{BPS}^{\lambda_j}-A_D^{\lambda_j}|=O(r_j).
$$

hence, for $l = -\frac{1}{2}$,

$$
\begin{aligned}
|\xi_0(*d_{\eta_j^*(A^{\lambda_j}_{BPS})}(A^{\lambda_j}_D - \eta_j^*(A^{\lambda_j}_{BPS}))| &\leq c_1, \\
|d_{\eta_j^*(A^{\lambda_j}_{BPS})}(\Phi^{\lambda_j}_D - \eta_j^*(\Phi^{\lambda_j}_{BPS}))| &\leq c_2, \\
|(d\xi_0 \wedge (A^{\lambda_j}_D - \eta_j^*(A^{\lambda_j}_{BPS})))| &\leq c_3(\frac{r_j}{\varepsilon_0}), \\
|d\xi_0(\Phi^{\lambda_j}_D - \eta_j^*(\Phi^{\lambda_j}_{BPS}))| &\leq c_4(\frac{r_j}{\varepsilon_0}), \\
|(\xi_0(A^{\lambda_j}_D - \eta_j^*(A^{\lambda_j}_{BPS})))^2| &\leq c_5(r_j^2), \\
|\xi_0^2[A^{\lambda}_D - \eta_j^*(A^{\lambda}_{BPS}), \Phi^{\lambda}_D - \eta_j^*(\Phi^{\lambda}_{BPS})]| &\leq c_6(r_j^2),
\end{aligned}
$$

where the constants $c_1, \ldots, c_6$ are independent of ε_j and λ_j, and only depend on the geometry of (M, g). We denote this error on the neck containing the terms $A^{\lambda_j}_D - \eta_j^*(A^{\lambda_j}_{BPS})$ and $\Phi^{\lambda_j}_D - \eta_j^*(\Phi^{\lambda_j}_{BPS})$ by e_j^{neck}. We get

$$(e_0)_{|B_{2\varepsilon_j}(q_j) \setminus B_{\varepsilon_j}(q_j)} = O(1).$$

$\square$

1.2.5 Solving the Equation

The goal is to show there is a solution to the Bogomolny equation near the constructed approximate monopole (A_0, Φ_0). In other words, we are looking for a small (a, φ) — small in a suitable norm — such that $(A_0 + a, \Phi_0 + \varphi)$ is a genuine monopole. In this section, we set up the equations for (a, φ) and state the strategy to solve these equations.

We can write the equation for the pair (a, φ),

$$
\begin{aligned}
&*F(A_0 + a) - d_{A_0+a}(\Phi_0 + \varphi) = 0 \Rightarrow \\
&\qquad (*F_{A_0} - d_{A_0}\Phi_0) + (*d_{A_0}a - d_{A_0}\varphi - [a, \Phi_0]) + (*\frac{[a \wedge a]}{2} - [a, \varphi]) = 0.
\end{aligned}
$$

Let $d_2^{(A_0,\Phi_0)} : \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P) \to \Omega^1(M \setminus S_p, \mathfrak{g}_P)$ be the operator that appeared in the linearization of the Bogomolny equation at (A_0, Φ_0),

$$d_2^{(A_0,\Phi_0)}(a, \varphi) = *d_{A_0}a - d_{A_0}\varphi - [a, \Phi_0], \tag{1.2.10}$$

where $S_p = \{p_1, \ldots, p_n\}$.

Although $d_2^{(A_0,\Phi_0)}(a, \varphi)$ depends on the pair (A_0, Φ_0), whenever there is no fear of confusion we drop the subscript (A_0, Φ_0) and denote it by d_2.

Let $Q(a, \varphi) : \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P) \to \Omega^1(M \setminus S_p, \mathfrak{g}_P)$ be the operator defined by the quadratic part,

$$Q(a,\varphi) = *\frac{[a \wedge a]}{2} - [a,\varphi].$$

Equation 1.1.1 can be written as

$$(d_2 + Q)(a,\varphi) = -e_0. \tag{1.2.11}$$

The Bogomolny equation is invariant under the action of the gauge group, and therefore, not elliptic. In fact, it is elliptic modulo the action of the gauge group. The linearization of the gauge group action is given by

$$\begin{aligned}
&d_1^{(A_0,\Phi_0)} : \Omega^0(M \setminus S_p, \mathfrak{g}_P) \to \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P),\\
&d_1^{(A_0,\Phi_0)}\xi = (-d_{A_0}\xi, -[\Phi_0, \xi]).
\end{aligned}$$

Similar to d_2, this operator also depends on the pair (A_0, Φ_0), but we drop this subscript when there is no fear of confusion and denote it by d_1.

The gauge fixing equation $d_1^*(a,\varphi) = 0$ describes a local slice of the action of the gauge group at (A_0, Φ_0), where

$$d_1^*(a,\varphi) = -d_{A_0}^* a - [\Phi_0, \varphi],$$

is the formal adjoint of d_1 with respect to the L^2-inner product. Let

$$D_{(A_0,\Phi_0)} : \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P) \to \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P),$$

be the elliptic operator defined by

$$D := D_{(A_0,\Phi_0)} = d_2 \oplus d_1^*.$$

This can be used to define an elliptic equation. Instead of $d_2(a,\varphi) = f$, we can consider the equation

$$D(a,\varphi) = (f, 0). \tag{1.2.12}$$

Two important properties of 1.2.12: it is elliptic, and, for any small f, any solution of $d_2(a,\varphi) = f$ can be gauged into a solution of 1.2.12.

These operators fit into a sequence

$$\Omega^0(M \setminus S_p, \mathfrak{g}_P) \xrightarrow{d_1} \Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P) \xrightarrow{d_2} \Omega^1(M \setminus S_p, \mathfrak{g}_P). \tag{1.2.13}$$

Note that $d_2 \circ d_1\xi = *[(d_{A_0}\Phi_0 - *F_{A_0}) \wedge \xi]$, and therefore, $d_2 \circ d_1 = 0$ when (A_0, Φ_0) is a monopole. In fact, 1.2.13 is an elliptic complex when (A_0, Φ_0) is a monopole.

The formal adjoint of d_2 with respect to the L^2-inner product is given by

$$d_2^* : \Omega^1(M \setminus S_p) \to \Omega^1(M \setminus S_p) \oplus \Omega^0(M \setminus S_p), \qquad d_2^* u = (*d_{A_0} u + [u, \Phi_0], -d_{A_0}^* u).$$

We look for solutions to the equation 1.2.11, which are of the form $(a, \varphi) = d_2^* u$, and therefore, the equation 1.2.11 can be written as

$$(d_2 d_2^* + Q d_2^*) u = -e_0. \tag{1.2.14}$$

This equation is elliptic. In fact, $d_2 d_2^*$ has the same symbol as the Laplacian on $\mathfrak{su}(2)$-valued 1-forms. A key step in solving this equation would be solving the linear equation

$$d_2 d_2^* u = f. \tag{1.2.15}$$

The method for solving this linear equation can be summarized into 4 steps:

- solving the linearized equation $d_2 d_2^* u = f$ on $B_{3\varepsilon_j}(q_j)$ via a variational method;
- solving the linearized equation $d_2 d_2^* u = f$ on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$ via a variational method;
- solving the linearized equation $d_2 \xi = f$ on $M \setminus S_p$ via an iteration method;
- solving the Bogomolny equation on $M \setminus S_p$ using a fixed point theorem.

1.2.6 Analytic Preliminaries

In this section, we review the necessary background material to solve the linearized Bogomolny equation.

We start with the monopole Weitzenböck formulas. These formulas follow from the standard Weitzenböck formula for a connection on a vector bundle.

Lemma 13 (The Monopole Weitzenöck Formulas [28])**.** *Let (A_0, Φ_0) be a pair of a connection and a Higgs field on a principal bundle $P \to M$ where (M, g) is an oriented Riemannian 3-manifold — and (A_0, Φ_0) is not necessarily a monopole. Let $e_0 = *F_{A_0} - d_{A_0}\Phi_0$. Let $ad^2(\Phi_0)\xi = [\Phi_0, [\Phi_0, \xi]]$ and $u \in \Omega^1(M, \mathfrak{g}_P)$. The Monopole Weitzenböck formulas are*

$$d_2 d_2^* u = \nabla_{A_0}^* \nabla_{A_0} u - ad(\Phi_0)^2(u) + Ric(u) + *[e_0 \wedge u], \tag{1.2.16}$$

$$DD^*(a, \varphi) = \nabla_{A_0}^* \nabla_{A_0}(a, \varphi) - ad(\Phi_0)^2(a, \varphi) + Ric(a, \varphi) + *[e_0 \wedge (a, \varphi)], \tag{1.2.17}$$

$$D^* D(a, \varphi) = DD^*(a, \varphi) + 2\langle d_{A_0}\Phi_0, (a, \varphi)\rangle. \tag{1.2.18}$$

Variations of the Poincaré inequality are essential in the analysis of the linear problem. The standard Poincaré inequality $\|u\|_{L^p(U)} \le C \|\nabla u\|_{L^p(U)}$ is stated for compactly supported functions $u \in W^{1,p}(U)$ where $U \subset \mathbb{R}^n$ is a bounded domain and C is a positive constant. This inequality is also valid when U is a ball in a Riemmanian manifold (M, g) with a positive constant C which depends on the geometry of U. A variation of this inequality also holds for the compactly supported functions on $\mathbb{R}^n$, when $n \geq 2$.

Lemma 14 (The Gagliardo-Nirenberg-Sobolev Inequality [43]). *Let $n \geq 2$ and $1 \leq p < n$. Let p^* be the Sobolev conjugate of p; i.e., p^* satisfies $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

for a constant C which depends on n and p and for all compactly supported functions $u \in C_c^1(\mathbb{R}^n)$.

Moreover, this inequality holds when $\mathbb{R}^n$ is equipped with a metric g which is asymptotically Euclidean rather than Euclidean, for a positive constant C_g.

The space of smooth compactly supported functions on $\mathbb{R}^n$ is dense in $W^{1,p}(\mathbb{R}^n)$, and therefore, the following lemma is immediate.

Corollary 2 ($W^{1,p}$ Gagliardo-Nirenberg-Sobolev Inequality). *Let $n \geq 2$, $1 \leq p < n$ and p^* the Sobolev conjugate of p. Then*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)},$$

for a constant $C = C_{n,p}$ and any $u \in W^{1,p}(\mathbb{R}^n)$.

Furthermore, this inequality extends to asymptotically Euclidean spaces.

Now we turn to weighted Poincaré inequalities. A one dimensional version of this inequality states that if $f : \mathbb{R} \to \mathbb{R}$ is a non-negative function, $F(x) = \int_0^x f(t)dt$, and $p > 1$, then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx.$$

This inequality, which is called the Hardy's inequality, was first proved by Hardy; however, the constant $(\frac{p}{p-1})^p$ in this inequality, which is sharp, was later discovered by Landau. For a proof consult with the beautiful book 'Inequalities' written by Hardy, Littlewood and Pólya [39, Section 9.8].

Lewis proved a higher-dimensional version of this inequality [63], which — a special case of that — is stated below.

Lemma 15. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be in $W^{2,2}(\mathbb{R}^n)$. Then for all $u \in C_c^1(\mathbb{R}^n)$ we have*

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta g(x)||u(x)|^2 vol_{\mathbb{R}^n} &\leq 2\int_{\mathbb{R}^n} |\nabla g(x)||u(x)||\nabla u(x)| vol_{\mathbb{R}^n} \\ &\leq 4\int_{\mathbb{R}^n} |\Delta g(x)|^{-1}|\nabla g(x)|^2|\nabla u(x)|^2 vol_{\mathbb{R}^n}.\end{aligned}$$

In particular, when $n \geq 2$,

$$\int_{\mathbb{R}^n} |x|^{\beta-2}|u(x)|^2 vol_{\mathbb{R}^n} \leq \frac{4}{(\beta-2+n)^2}\int_{\mathbb{R}^n} |x|^\beta|\nabla u(x)|^2 vol_{\mathbb{R}^n},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Using Kato's inequality, we can extend these results about real-valued functions on $\mathbb{R}^n$ to $\mathfrak{su}(2)$–valued forms and their covariant derivatives.

Lemma 16. *For all* $\alpha \neq -1$ *and compactly supported section* $u \in C_0^\infty(\mathbb{R}^3, \mathfrak{su}(2))$*, we have*

$$\int_{\mathbb{R}^3} |x|^{-2\alpha-3}|u|^2 vol_{\mathbb{R}^3} \leq \frac{1}{(\alpha+1)^2}\int_{\mathbb{R}^3} |x|^{-2\alpha-1}|\nabla_A u|^2 vol_{\mathbb{R}^3}. \tag{1.2.19}$$

In fact, we will use a different version of this inequality. For a $\lambda \geq 0$, let

$$w(x) = \begin{cases} \sqrt{\lambda^{-2}+|x|^2}, & |x| \leq \frac{1}{2} \\ 1, & |x| \geq 1. \end{cases} \tag{1.2.20}$$

Corollary 3 ([30])**.** *For all* $\alpha \neq -1$ *and* $u \in C_0^\infty(\mathbb{R}^3, \mathfrak{su}(2))$ *we have*

$$\int_{\mathbb{R}^3} w^{-2\alpha-3}|u|^2 vol_{\mathbb{R}^3} \leq \frac{1}{(\alpha+1)^2}\int_{\mathbb{R}^3} w^{-2\alpha-1}|\nabla_A u|^2 vol_{\mathbb{R}^3}. \tag{1.2.21}$$

Moreover, this inequality is valid on asymptotically Euclidean manifolds, for a different positive constant C_α*, depending on* α *and the geometry of the manifold.*

1.2.7 The Linear Equation

We break solving the linear equation into three parts. In the first part we analyze, and later solve, the linear equation close to the points q_j. In the second part we analyze and solve the linear equation away from the points q_j, and finally, in the third part we solve the linear equation on the whole manifold.

In the following section, we start analyzing the linear equation on a small neighbourhood of the points q_j.

1.2.7.1 The Linear Equation over $B_{3\varepsilon_j}(q_j)$

In this section, we set up the framework for studying the linear equation $d_2 d_2^* u = f$ on $B_{3\varepsilon_j}(q_j)$, and transfer it to $\mathbb{R}^3$.

Suppose f is supported on $B_{3\varepsilon_j}(q_j)$. We can localize the problem on this region. When the metric is flat on a small neighborhood of the point q_j, it is convenient to consider $B_{3\varepsilon_j}(q_j)$ as a subset of $\mathbb{R}^3$. This reduces the problem to solving the equation $d_2 d_2^* u = f$ on $\mathbb{R}^3$.

More generally, when the metric around q_j is an arbitrary one, it is still useful to consider $B_{3\varepsilon_j}(q_j)$ as a subset of $\mathbb{R}^3$, but with a non-standard metric. Using a geodesic normal coordinate, we fix a diffeomorphism

$$\mu : B_{3\varepsilon_j}(q_j) \subset M \to U \subset \mathbb{R}^3.$$

We can pull back the Riemannian metric on $B_{3\varepsilon_j}(q_j)$ via μ^{-1} to $U \subset \mathbb{R}^3$. Furthermore, one can extend this metric defined on U to a Riemannian metric defined on the whole $\mathbb{R}^3$ such that it is flat outside of a slightly larger open subset V with geodesic radius $4\varepsilon_j$ containing U. To prevent any confusion, we denote $\mathbb{R}^3$ with this non-standard metric with $\mathfrak{R}^3$; however, note that the metric need not to be product. By an abuse of notation, we denote both metrics on M, and also the induced metric on $\mathfrak{R}^3$ by g. The case where the metric is flat has been investigated in the study of periodic singular monopoles by Foscolo [30].

Working with an arbitrary metric introduces two potential sources of difficulty. One is related to the error of the approximate solution, since the BPS-monopole is not a genuine monopole with respect to the arbitrary metric, as we observed earlier. The other difficulty is related to the Ricci terms in the monopole Wietzenböck formulas in Lemma 13, which appear in the estimations in the linear problem.

Let $q \in S_q = \{q_1, \ldots, q_k\}$. Let (A_0, Φ_0) be the approximate monopole defined on $B_{3\varepsilon}(q)$ by gluing the pull-back of the scaled BPS-monopole on $\mathbb{R}^3$ to the scaled Dirac monopole, both with mass λ and centered at the origin, using a cut-off function ξ as before,

$$(A_0, \Phi_0)(x) = \begin{cases} (\eta^*(A^\lambda_{BPS}), \eta^*(\Phi^\lambda_{BPS}))(x) & r \leq \lambda^{-\frac{1}{2}}, \\ (A^\lambda_D, \Phi^\lambda_D)(x) & r \geq 2\lambda^{-\frac{1}{2}}, \end{cases} \tag{1.2.22}$$

where r denotes the geodesic distance to the origin and $|\nabla\xi| \leq \lambda^{\frac{1}{2}}$.

Using the diffeomorphism μ^{-1} one can pull back the pair (A_0, Φ_0) to $U \subset \mathfrak{R}^3$. We can extend this pair $((\mu^{-1})^*(A_0), (\mu^{-1})^*\Phi_0)$ to a pair on $\mathfrak{R}^3$, by gluing it to the standard Dirac monopole scaled by the factor λ on $\mathfrak{R}^3 \setminus V = \mathbb{R}^3 \setminus V$, using a cut-off function $\tilde{\xi}$ such that $|\nabla\tilde{\xi}| \leq \lambda^{\frac{1}{2}}$. By an abuse of notation, we still denote this pair on $\mathfrak{R}^3$ by (A_0, Φ_0).

The first step is to set up the suitable Sobolev spaces for the linear problem. In order to solve 1.2.15 on $\mathfrak{R}^3$, naively, one might let $d_2 d_2^* : W^{2,2}(\Omega^1(\mathfrak{R}^3)) \to L^2(\Omega^1(\mathfrak{R}^3))$; however, this is not a suitable choice, and one needs to use the weighted Sobolev spaces.

1.2.7.2 Function Spaces on $\mathfrak{R}^3$

In this section, we introduce the appropriate weighted Sobolev spaces to study the linear operator $d_2 d_2^*$ on $\mathfrak{su}(2)$-valued 1-forms on $\mathfrak{R}^3$, for a pair (A_0, Φ_0) with a large mass. These weighted spaces have been investigated by Biquard [9, 10], and used in the case of monopoles by Foscolo [31, 30], also related to the weighted norms in [1].

The weights used in the definition of our function spaces are designed for the linear operator $d_2 d_2^*$ to have a bounded right-inverse. The motivation for the specific choices of the weights, follow from the observation that the terms F_{A_0} and $d_{A_0}\Phi_0$ appearing in the monopole Weitzenböck formula of $d_2 d_2^*$ blow up as the scaling factor $\lambda \to \infty$ and $r \to 0$.

Definition 3. *Let*

$$w(x) = \begin{cases} \sqrt{\lambda^{-2} + r^2}, & r \leq \frac{1}{2} \\ 1, & r \geq 1, \end{cases}$$

where $r : \mathfrak{R}^3 \to \mathbb{R}^{\geq 0}$ denotes the geodesic distance from the origin.

Let (A_0, Φ_0) be the approximate monopole we constructed earlier. Let $\alpha \in \mathbb{R}$. For all smooth compactly supported $\mathfrak{su}(2)$-valued differential forms $u \in \Omega^\bullet(\mathfrak{R}^3, \mathfrak{su}(2))$, let

$$\|u\|_{L^2_\alpha(\mathfrak{R}^3)} = \|w^{-\alpha-\frac{3}{2}} u\|_{L^2(\mathfrak{R}^3)},$$
$$\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} = \|u\|^2_{L^2_\alpha(\mathfrak{R}^3)} + \|\nabla_{A_0} u\|^2_{L^2_{\alpha-1}(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2_{\alpha-1}(\mathfrak{R}^3)}.$$

Let the spaces $L^2_\alpha(\mathfrak{R}^3)$ and $W^{1,2}_\alpha(\mathfrak{R}^3)$ be the completion of $C_0^\infty(\mathfrak{R}^3)$ with respect to these norms. Furthermore,

$$\|u\|^2_{W^{2,2}_\alpha(\mathfrak{R}^3)} := \|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} + \|\nabla_{A_0}(d_2^* u)\|^2_{L^2_{\alpha-2}(\mathfrak{R}^3)} + \|[\Phi_0, d_2^* u]\|^2_{L^2_{\alpha-2}(\mathfrak{R}^3)}.$$

More generally, for any $p > 1$, we can define the norm

$$\|u\|_{L^p_\alpha(\mathfrak{R}^3)} = \|w^{-\alpha-\frac{3}{p}} u\|_{L^p(\mathfrak{R}^3)},$$

and $L^p_\alpha(\mathfrak{R}^3)$ as the completion of $C_0^\infty(\mathfrak{R}^3)$ with respect to this norm.

Remark 3. *The integer 3 which appears in the exponent of w in the definition of the weighted norms reflects the dimension of the base manifold. In fact, for sections of bundles over $\mathbb{R}^n$, we can define*

$$\|u\|_{L^p_\alpha(\mathbb{R}^n)} = \|w^{-\alpha-\frac{n}{p}} u\|_{L^p(\mathbb{R}^n)}.$$

These spaces satisfy similar properties as the ordinary Sobolev spaces.

Lemma 17. *The weighted Sobolev spaces enjoy the following properties:*

- *Let $k \in \{0, 1, 2\}$. Let $u \in W^{k,p}_{\alpha,loc}(\mathfrak{R}^3)$. Suppose $W^{k,p}_\alpha(\mathfrak{R}^3)$-norm of u converges. Then $u \in W^{k,p}_\alpha(\mathfrak{R}^3)$.*
- *For any $u \in W^{2,2}_\alpha(\mathfrak{R}^3)$, we have $d_2 d_2^* u \in L^2_{\alpha-2}(\mathfrak{R}^3)$, when λ is sufficiently large.*
- *$C_c^\infty(\mathfrak{R}^3)$ is dense in $W^{1,2}_\alpha(\mathfrak{R}^3)$, and therefore, the Corollary 3 holds for the elements of $W^{1,2}_\alpha(\mathfrak{R}^3)$.*

The proofs are straightforward.

1.2.7.3 Solving the Linear Equation on $\mathfrak{R}^3$

The main theorem of this section is the following. In the case where the Riemannian metric g is the Euclidean metric, this is proposition 5.8. in [30].

Theorem 3. *Let $d_2 d_2^* : W^{2,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to L^2_{\alpha-2}(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$. For all $-\frac{1}{2} \le \alpha < 0$ there exist $\delta > 0$ such that if $\|we_0(A_0\Phi_0)\|_{L^3(\mathfrak{R}^3)} < \delta$, then $d_2 d_2^*$ is invertible. For $\delta > 0$*

sufficiently small, there exists $C > 0$ such that for all $f \in L^2_{\alpha-2}(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$, there exists a unique solution $u \in W^{2,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$ to $d_2 d_2^* u = f$ with

$$\|u\|_{W^{2,2}_\alpha(\mathfrak{R}^3)} \leq C\|f\|_{L^2_{\alpha-2}(\mathfrak{R}^3)},$$

where the constant C is independent of λ, which appears in the definition of the approximate monopole (A_0, Φ_0).

The proof is based on a direct variational method. We present the proof of the Theorem 3 in 10 steps, presented in a sequence of lemmas. The line of proof follows [30]. Since $C_c^\infty(\mathfrak{R}^3)$ is dense in $W^{2,2}_\alpha$, we only need to prove the theorem when f is a smooth compactly supported $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$.

Before stating the proof, we want to assume a normalizing condition for the Riemannian metric g on M. Note that (A, Φ) is a monopole on (M, g) if and only if $(A, \frac{1}{c}\Phi)$ is a monopole on $(M, c^2 g)$ for any positive constant c, and therefore, there is a one-to-one correspondence between monopoles on (M, g) and monopoles on $(M, c^2 g)$. Therefore, without loss of generality, by multiplying the metric g by a sufficiently small positive constant number c^2, we can assume $\sup_{x\in M} |Ric(x)| < \frac{1}{100}$, and therefore, $\sup_{x\in\mathfrak{R}^3} |Ric(x)| < \frac{1}{100}$

Lemma 18 (Step 1)**.** *Suppose f is a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. Let*

$$E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}, \qquad E(u) := \frac{1}{2}\int_{\mathfrak{R}^3} |d_2^* u|^2 vol_g - \langle u, f\rangle_{L^2(\mathfrak{R}^3)}. \tag{1.2.23}$$

The functional $E(u)$ is convergent when $u \in W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$.

Proof. We first show $\|d_2^* u\|^2_{L^2(\mathfrak{R}^3)}$ is finite. The key fact in proving this is the monopole Weitzenböck formula.

$$\|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} = \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} + \langle u, Ric(u)\rangle_{L^2(\mathfrak{R}^3)} + \langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)}. \tag{1.2.24}$$

Note that since $u \in W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$, the asymptotic terms do not appear in the formula above.

The first two terms on the right hand side of 1.2.24 are finite, since $1 \leq w^{-\alpha-\frac{1}{2}}$ when $-\frac{1}{2} \leq \alpha$, and therefore,

$$\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} \leq \|w^{-\alpha-\frac{1}{2}}\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} = \|\nabla_{A_0} u\|^2_{L^2_{\alpha-1}(\mathfrak{R}^3)} \leq \|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} < \infty,$$
$$\|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} \leq \|w^{-\alpha-\frac{1}{2}}[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} = \|[\Phi_0, u]\|^2_{L^2_{\alpha-1}(\mathfrak{R}^3)} \leq \|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} < \infty.$$

Regarding the Ricci term we have

$$\langle u, Ric(u)\rangle_{L^2(\mathfrak{R}^3)} \leq \sup_{\mathfrak{R}^3} |Ric|\|u\|^2_{L^2(\mathfrak{R}^3)} \leq \sup_{\mathfrak{R}^3} |Ric|\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} < \infty.$$

$\mathfrak{R}^3$ is flat outside of a compact subset, and therefore, $\sup_{\mathfrak{R}^3}|Ric|$ is finite.

As for the error term in 1.2.24, by applying the Hölder's inequality twice, we get

$$\begin{aligned}
&|\langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)}| \\
&\quad = |\langle *[we_0 \wedge u], w^{-1}u\rangle_{L^2(\mathfrak{R}^3)}| \leq \|[we_0 \wedge u]\|_{L^2(\mathfrak{R}^3)}\|w^{-1}u\|_{L^2(\mathfrak{R}^3)} \\
&\quad \leq 2\|(we)^2\|^{\frac{1}{2}}_{L^{\frac{3}{2}}(\mathfrak{R}^3)}\|u^2\|^{\frac{1}{2}}_{L^3(\mathfrak{R}^3)}\|w^{-1}u\|_{L^2(\mathfrak{R}^3)} = 2\|we\|_{L^3(\mathfrak{R}^3)}\|u\|_{L^6(\mathfrak{R}^3)}\|w^{-1}u\|_{L^2(\mathfrak{R}^3)} \\
&\quad \leq 2\delta\|u\|_{L^6(\mathfrak{R}^3)}\|w^{-1}u\|_{L^2(\mathfrak{R}^3)}.
\end{aligned}$$

Regarding the term $\|u\|_{L^6(\mathfrak{R}^3)}$, by the Sobolev inequality we have $\|u\|_{L^6(\mathfrak{R}^3)} \leq C_{Sob}\|u\|_{W^{1,2}(\mathfrak{R}^3)}$ — where C_{Sob} is independent of λ — and since $0 < w \leq 1$ on $\mathfrak{R}^3 \setminus \{0\}$,

$$\begin{aligned}
\|u\|^2_{W^{1,2}(\mathfrak{R}^3)} &= \|u\|^2_{L^2(\mathfrak{R}^3)} + \|\nabla_{A_0}u\|^2_{L^2(\mathfrak{R}^3)} \leq \|w^{-\alpha-\frac{3}{2}}u\|^2_{L^2(\mathfrak{R}^3)} + \|w^{-\alpha-\frac{1}{2}}\nabla_{A_0}u\|^2_{L^2(\mathfrak{R}^3)} \\
&\leq \|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} < \infty.
\end{aligned}$$

Regarding the term $\|w^{-1}u\|_{L^2(\mathfrak{R}^3)}$, following the Corollary 3, we have

$$\begin{aligned}
\|w^{-1}u\|_{L^2(\mathfrak{R}^3)} &\leq C\|\nabla_{A_0}u\|_{L^2(\mathfrak{R}^3)} \leq C\|w^{-\alpha-\frac{1}{2}}\nabla_{A_0}u\|_{L^2(\mathfrak{R}^3)} = C\|\nabla_{A_0}u\|_{L^2_{\alpha-1}(\mathfrak{R}^3)} \\
&\leq C\|u\|_{W^{1,2}_\alpha} < \infty,
\end{aligned}$$

for a constant C which depends on α and the metric on $\mathfrak{R}^3$, and therefore,

$$|\langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)}| \leq 2\delta C C_{Sob}\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} \leq C_1\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)},$$

for a constant $C_1 = 2\delta C C_{Sob}$, and therefore,

$$\begin{aligned}
\|d_2^*u\|^2_{L^2(\mathfrak{R}^3)} &\leq \|\nabla_{A_0}u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} + C_1\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} + \sup_{\mathfrak{R}^3}|Ric|\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} \\
&\leq (2 + C_1 + \sup_{\mathfrak{R}^3}|Ric|)\|u\|^2_{W^{1,2}_\alpha(\mathfrak{R}^3)} < \infty.
\end{aligned}$$

After observing that $\|d_2^*u\|^2_{L^2(\mathfrak{R}^3)}$ is convergent, we should prove the same for $\langle u, f\rangle_{L^2(\mathfrak{R}^3)}$. By the Cauchy–Schwarz inequality we have

$$\langle u, f\rangle_{L^2(\mathfrak{R}^3)} \leq \|u\|_{L^2(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} \leq \|u\|_{L^2_\alpha(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} \leq \|u\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} < \infty,$$

and therefore, we have a well-defined action functional $E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}$. □

Lemma 19 (Step 2)**.** *Let f be a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. The functional $E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}$ is continuous.*

Proof. We should show the following functions are continuous,

$$\|d_2^* -\|^2_{L^2(\mathfrak{R}^3)} : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}, \qquad \langle -, f\rangle_{L^2(\mathfrak{R}^3)} : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}.$$

To prove $\|d_2^* - \|^2_{L^2(\mathfrak{R}^3)}$ is continuous we should show

$$\|d_2^* u_i\|^2_{L^2(\mathfrak{R}^3)} \to \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)}, \quad \text{when} \quad u_i \to u \quad \text{in} \quad W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))). \tag{1.2.25}$$

By the monopole Weitzenböck formula we know

$$\begin{aligned} \|d_2^* u_i\|^2_{L^2(\mathfrak{R}^3)} = \|\nabla_{A_0} u_i\|^2_{L^2(\mathfrak{R}^3)} + \|[\varPhi_0, u_i]\|^2_{L^2(\mathfrak{R}^3)} &+ \langle Ric(u), u\rangle_{L^2(\mathbb{R}^3)} \\ &+ \langle *[e_0 \wedge u_i], u_i\rangle_{L^2(\mathfrak{R}^3)}. \end{aligned}$$

It follows directly from the definition of $W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$ that

$$\|\nabla_{A_0} u_i\|^2_{L^2(\mathfrak{R}^3)} + \|[\varPhi_0, u_i]\|^2_{L^2(\mathfrak{R}^3)} \to \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\varPhi_0, u]\|^2_{L^2(\mathfrak{R}^3)},$$

when $u_i \to u$ in $W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$.

Regarding the Ricci term,

$$\begin{aligned} &|\langle Ric(u), u\rangle_{L^2(\mathbb{R}^3)} - \langle Ric(u_i), u_i\rangle_{L^2(\mathbb{R}^3)}| \\ &\quad = |\langle wRic(u), w^{-1}u\rangle_{L^2(\mathbb{R}^3)} - \langle wRic(u_i), w^{-1}u_i\rangle_{L^2(\mathbb{R}^3)}| \\ &\quad \le |\langle wRic(u - u_i), w^{-1}u\rangle_{L^2(\mathbb{R}^3)}| + |\langle wRic(u_i), w^{-1}(u - u_i)\rangle_{L^2(\mathbb{R}^3)}| \\ &\quad \le \|wRic\|_{L^3}\|u - u_i\|_{L^6}\|w^{-1}u\|_{L^2} + \|wRic\|_{L^3}\|u_i\|_{L^6}\|w^{-1}(u - u_i)\|_{L^2} \\ &\quad \le \|wRic\|_{L^3}\|u - u_i\|_{L^6}\|w^{-1}u\|_{L^2} + \|wRic\|_{L^3}(\|u\|_{L^6} + 1)\|w^{-1}(u - u_i)\|_{L^2} \\ &\quad \le C\left(\|wRic\|_{L^3}\|u - u_i\|_{W^{1,2}}\|\nabla_{A_0}u\|_{L^2} + \|wRic\|_{L^3}(\|\nabla_{A_0}u\|_{L^2} + 1)\|u - u_i\|_{W^{1,2}}\right), \end{aligned}$$

which goes to zero as $i \to \infty$.

Furthermore, when i is sufficiently large,

$$\begin{aligned} &|(|\langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)}| - |\langle *[e_0 \wedge u_i], u_i\rangle_{L^2(\mathfrak{R}^3)}|)| \\ &\quad = |(|\langle *[we_0 \wedge u], w^{-1}u\rangle_{L^2(\mathfrak{R}^3)}| - |\langle *[we_0 \wedge u_i], w^{-1}u_i\rangle_{L^2(\mathfrak{R}^3)}|)| \\ &\quad \le |\langle *[we_0 \wedge u], w^{-1}(u - u_i)\rangle_{L^2(\mathfrak{R}^3)}| + |\langle *[we_0 \wedge (u - u_i)], w^{-1}u_i\rangle_{L^2(\mathfrak{R}^3)}| \\ &\quad \le \|[we_0 \wedge u]\|_{L^2(\mathfrak{R}^3)}\|w^{-1}(u - u_i)\|_{L^2(\mathfrak{R}^3)} + \|[we_0 \wedge (u - u_i)]\|_{L^2(\mathfrak{R}^3)}\|w^{-1}u_i\|_{L^2(\mathfrak{R}^3)} \\ &\quad \le 2\delta\|u\|_{L^6(\mathfrak{R}^3)}\|w^{-1}(u - u_i)\|_{L^2(\mathfrak{R}^3)} + 2\delta\|u - u_i\|_{L^6(\mathfrak{R}^3)}\|w^{-1}u_i\|_{L^2(\mathfrak{R}^3)} \\ &\quad \le 2\delta\|u\|_{W^{1,2}(\mathfrak{R}^3)}\|w^{-1}(u - u_i)\|_{L^2(\mathfrak{R}^3)} + 2\delta\|u - u_i\|_{W^{1,2}(\mathfrak{R}^3)}\|w^{-1}u_i\|_{L^2(\mathfrak{R}^3)} \\ &\quad \le 2\delta\|u\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}\|u - u_i\|_{W^{1,2}_\alpha(\mathfrak{R}^3)} + 2\delta\|u - u_i\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}\|u_i\|_{W^{1,2}_\alpha(\mathfrak{R}^3)} \\ &\quad \le 2\delta\|u\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}\|u - u_i\|_{W^{1,2}_\alpha(\mathfrak{R}^3)} + 2\delta\|u - u_i\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}(\|u\|_{W^{1,2}_\alpha(\mathfrak{R}^3)} + 1), \end{aligned}$$

which converges to 0 as $i \to \infty$, and therefore, 1.2.25 follows.

In order to prove $\langle -, f\rangle_{L^2(\mathfrak{R}^3)} : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}$ is continuous, note that this map is linear, and since $W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$ is a Hilbert space, continuity is equivalent to being bounded.

$$\langle u, f\rangle_{L^2(\mathfrak{R}^3)} \leq \|u\|_{L^2(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} \leq \|u\|_{W^{1,2}_\alpha(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)},$$

and therefore, the operator norm

$$\sup\{\langle u, f\rangle_{L^2(\mathfrak{R}^3)} \mid u \in W^{1,2}_\alpha, \|u\|_{W^{1,2}_\alpha} = 1\} \leq \|f\|_{L^2(\mathfrak{R}^3)} < \infty,$$

which proves the linear map is bounded and continuous. □

Lemma 20 (Step 3)**.** *Let f be a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. $E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}$ is Gateaux-differentiable.*

Proof. A direct computation shows

$$d_uE(v) = \int_{\mathfrak{R}^3} \langle d_2^* u, d_2^* v\rangle_{L^2} vol_g - \langle v, f\rangle_{L^2(\mathfrak{R}^3)}.$$

Note that for any $u, v \in W^{1,2}_\alpha$,

$$\int_{\mathfrak{R}^3} \langle d_2^* u, d_2^* v\rangle_{L^2} vol_g \leq \|d_2^* u\|_{L^2(\mathfrak{R}^3)}\|d_2^* v\|_{L^2(\mathfrak{R}^3)} < \infty,$$

and

$$\langle v, f\rangle_{L^2(\mathfrak{R}^3)} \leq \|v\|_{L^2(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} < \infty.$$

□

Lemma 21 (Step 4)**.** *Suppose f is a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. There exists $\delta > 0$ such that if $\|we_0\|_{L^3(\mathfrak{R}^3)} < \delta$, then the functional*

$$E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2))) \to \mathbb{R}, \qquad E(u) := \frac{1}{2}\int_{\mathfrak{R}^3} |d_2^* u|^2 vol_g - \langle u, f\rangle_{L^2(\mathfrak{R}^3)}, \tag{1.2.26}$$

is strictly convex.

Proof. Since $\langle u, f\rangle_{L^2(\mathfrak{R}^3)}$ is linear in u, we only need to show $\|d_2^* u\|^2_{L^2(\mathfrak{R}^3)}$ is strictly convex. Let $u, v \in W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3, \mathfrak{su}(2)))$ where $u_1 \neq u_2$. Let $t \in (0, 1)$. We should prove

$$\|d_2^*(tu_1 + (1-t)u_2)\|^2_{L^2(\mathfrak{R}^3)} < t\|d_2^* u_1\|^2_{L^2(\mathfrak{R}^3)} + (1-t)\|d_2^* u_2\|^2_{L^2(\mathfrak{R}^3)}.$$

In fact, we only need to check this for $t = \frac{1}{2}$. The strict inequality

$$\|d_2^*(\frac{u_1 + u_2}{2})\|^2_{L^2(\mathfrak{R}^3)} < \frac{1}{2}\|d_2^* u_1\|^2_{L^2(\mathfrak{R}^3)} + \frac{1}{2}\|d_2^* u_2\|^2_{L^2(\mathfrak{R}^3)},$$

is equivalent to $\|d_2^*(u_1-u_2)\|^2_{L^2(\mathfrak{R}^3)} > 0$. Let $u = u_1 - u_2$. We should show

$$u \in W^{1,2}_\alpha, \quad d_2^* u = 0 \quad \Rightarrow \quad u = 0. \tag{1.2.27}$$

The property 1.2.27 implies $\|d_2^* - \|_{L^2}$ is a norm on $W^{1,2}_\alpha$. The key fact to prove this is the Gagliardo–Nirenberg–Sobolev inequality. Suppose $\|d_2^* u\|_{L^2(\mathfrak{R}^3)} = 0$,

$$\begin{aligned} 0 = \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} = \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} &+ \langle Ric(u), u\rangle_{L^2(\mathfrak{R}^3)} \\ &+ \langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)}. \end{aligned}$$

Using Corollary 3, we have

$$\begin{aligned} \langle u, Ric(u)\rangle_{L^2(\mathfrak{R}^3)} &= \langle w^{-1}u, wRic(u)\rangle_{L^2(\mathfrak{R}^3)} \le \|w^{-1}u\|_{L^2(\mathfrak{R}^3)} \|wRic(u)\|_{L^2(\mathfrak{R}^3)} \\ &\le 5\|\nabla_{A_0} u\|_{L^2(\mathfrak{R}^3)} \|wRic\|_{L^3} \|u\|_{L^6(\mathfrak{R}^3)} \le \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)}, \end{aligned}$$

for ε small enough such that $\|wRic\|_{L^3} < \frac{1}{20C_{Sob}}$, which can be arranged when λ is sufficiently large, since $Ric(x) = 0$ outside of $B_0(4\varepsilon)$. Moreover, in the inequality

$$\|w^{-1}u\|_{L^2(\mathfrak{R}^3)} \le C_{g,\alpha} \|\nabla_{A_0} u\|_{L^2(\mathfrak{R}^3)}.$$

by taking $\varepsilon > 0$ small enough, we can take $C_{g,\alpha} < 5$. In fact, when the metric g on $\mathfrak{R}^3$ is flat, this constant is $1/(1+\alpha)^2$. In our case, the metric on $\mathfrak{R}^3$ coincide with the flat metric outside of a small ball $B_{4\varepsilon}(0)$. By taking the $\varepsilon > 0$ sufficiently small, we can take the constant $C_{g,\alpha}$ to be sufficiently close to $1/(\alpha+1)^2$, and therefore, less than 5.

Furthermore,

$$\begin{aligned} |\langle *[we_0 \wedge u], w^{-1}u\rangle_{L^2(\mathfrak{R}^3)}| &\le 2\|we\|_{L^3(\mathfrak{R}^3)} \|u\|_{L^6(\mathfrak{R}^3)} \|w^{-1}u\|_{L^2(\mathfrak{R}^3)} \\ &\le 2C\|we\|_{L^3(\mathfrak{R}^3)} \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)}. \end{aligned}$$

Pick $\delta < \frac{1}{8C}$,

$$\begin{aligned} 0 &= \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} \\ &= \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} + \langle Ric(u), u\rangle_{L^2(\mathfrak{R}^3)} + \langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)} \\ &\ge \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} - \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} - \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} \\ &= \frac{1}{2}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)}, \end{aligned}$$

and therefore, $\nabla_{A_0} u = 0 = [\Phi_0, u]$. Therefore, u is a covariantly constant section in $W^{1,2}_\alpha$, hence, $u = 0$. $\square$

Lemma 22 (Step 5). *Let f be a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. The action functional $E : W^{1,2}_\alpha(\Omega^1(\mathfrak{R}^3)) \to \mathbb{R}$ has a unique minimizer, and* *therefore, $d_2 d_2^* u = f$ has a unique solution.*

Proof. The proof of this lemma is based on the following fact.

Let $E : W \to \mathbb{R}$ be a convex, continuous and real Gateaux-differentiable functional defined on a real reflexive Banach space W such that $d_u E(u) > 0$, for any $u \in W$ with $\|u\|_W \geq R > 0$, for a positive constant R. Then there exists an interior point u_0 of $\{u \in W \mid \|u\|_W < R\}$ which is the unique minimizer of E and $d_{u_0} E = 0$.

Let $W = W_\alpha^{1,2}(\mathfrak{R}^3)$ but equipped with the norm $\|d^* - \|_{L^2(\mathfrak{R}^3)}$ — it follows from the proof of Lemma 21 that this is a norm. Moreover, similar to the Sobolev space $W^{1,2}$, it is straightforward to see $W_\alpha^{1,2}(\mathfrak{R}^3)$ is reflexive too.

We should show there is a constant $R > 0$ such that we have $d_u E(u) > 0$ for any $u \in W$ with $\|u\|_W \geq R > 0$. As in the proof of Lemma 21,

$$\begin{aligned}
\|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} &= \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} + \langle wRic(u), w^{-1}u\rangle_{L^2(\mathfrak{R}^3)} + \langle *[e_0 \wedge u], u\rangle_{L^2(\mathfrak{R}^3)} \\
&\geq \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} - \sup_{x \in \mathfrak{R}^3} Ric(x) \|w^{-1}u\|^2_{L^2(\mathfrak{R}^3)} - \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} \\
&\geq \|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} - \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} - \frac{1}{4}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} \\
&= \frac{1}{2}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)} + \|[\Phi_0, u]\|^2_{L^2(\mathfrak{R}^3)} \geq \frac{1}{2}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)},
\end{aligned}$$

and therefore,

$$\|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} \geq \frac{1}{2}\|\nabla_{A_0} u\|^2_{L^2(\mathfrak{R}^3)}.$$

This shows

$$\begin{aligned}
d_u E(u) &= \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} - \langle u, f\rangle_{L^2(\mathfrak{R}^3)} = \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} - \langle w^{-1}u, wf\rangle_{L^2(\mathfrak{R}^3)} \\
&\geq \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} - \|w^{-1}u\|_{L^2(\mathfrak{R}^3)}\|wf\|_{L^2(\mathfrak{R}^3)} \\
&\geq \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} - C\|\nabla_{A_0} u\|_{L^2(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} \\
&\geq \|d_2^* u\|^2_{L^2(\mathfrak{R}^3)} - \sqrt{2}C\|d_2^* u\|_{L^2(\mathfrak{R}^3)}\|f\|_{L^2(\mathfrak{R}^3)} \\
&= \|d_2^* u\|_{L^2(\mathfrak{R}^3)}(\|d_2^* u\|_{L^2(\mathfrak{R}^3)} - \sqrt{2}C\|f\|_{L^2(\mathfrak{R}^3)}).
\end{aligned}$$

Let $R = 1 + \sqrt{2}C\|f\|_{L^2(\mathfrak{R}^3)}$, and therefore, $\|d_2^* u\|_{L^2(\mathfrak{R}^3)} > R$ implies

$$\|d_2^* u\|_{L^2(\mathfrak{R}^3)} - \sqrt{2}C\|f\|_{L^2(\mathfrak{R}^3)} > 1,$$

hence

$$d_u E(u) > \|d_2^* u\|_{L^2(\mathfrak{R}^3)} > R > 0,$$

and therefore, E has a unique minimizer in $W^{1,2}_{\alpha}(\mathfrak{R}^3)$, inside

$$\{u \in W^{1,2}_{\alpha}(\mathfrak{R}^3) \mid \|d_2^* u\|_{L^2(\mathfrak{R}^3)} < 1 + \sqrt{2}C\|f\|_{L^2(\mathfrak{R}^3)}\}.$$

□

Φ_0 is non-zero outside of a large ball $V \subset \mathfrak{R}^3$, and therefore, it induces a decomposition of the adjoint bundle to the longitudinal and transverse parts. Let u^L and u^T denote the longitudinal and transverse components, respectively. Note that since the metric g is flat on $\mathfrak{R}^3 \setminus V$, we have $\mathfrak{R}^3 \setminus V = \mathbb{R}^3 \setminus V$

Lemma 23 (Step 6)**.** *Let u be the unique solution of Lemma 22, and u^L be its longitudinal component with respect to the decomposition of the bundle over $\mathbb{R}^3 \setminus V$ induced by Φ_0. We have*

$$|u^L(x)| \leq \frac{C}{|x|},$$

for a constant C.

Proof. Since f is compactly supported, if necessary we can enlarge V such that $f_{|_{\mathbb{R}^3 \setminus V}} = 0$, and therefore, u^L satisfies the equation

$$\Delta u^L = 0,$$

on $\mathbb{R}^3 \setminus V$, thus, u^L is a harmonic real-valued 1-form outside of a compact subset of $\mathbb{R}^3$. Let $u^L = u_1^L dx_1 + u_2^L dx_2 + u_3^L dx_3$ on $\mathbb{R}^3 \setminus V$. With respect to the Euclidean metric, we have

$$\Delta u^L = (\Delta u_1^L)dx_1 + (\Delta u_2^L)dx_2 + (\Delta u_3^L)dx_3,$$

and therefore, the 1-form u^L is harmonic if and only if its coefficients are harmonic. This shows

$$\Delta u_1^L = \Delta u_2^L = \Delta u_3^L = 0,$$

on $\mathbb{R}^3 \setminus V$, and therefore, they are harmonic functions on the complement of a compact subset of $\mathbb{R}^3$. Functions of this type have been studied in [2], which we burrow the following fact from.

Let v be a subharmonic function defined over $\{x \in \mathbb{R}^n \mid |x| > R\}$ for a positive real number R. Then there exist a non-constant subharmonic function $s(x)$ defined over $\mathbb{R}^n$, a real number $r > R$, and a constant $c \leq 0$ such that

$$v(x) = s(x) + c|x|^{2-n} \quad \text{when} \quad |x| > r.$$

Letting $v(x) = u_i^L(x)$ for $i \in \{1, 2, 3\}$, $n = 3$, and R large enough such that $V \subset B_0(R)$, we get $u_i^L(x) = s_i(x) + \frac{c_i}{|x|}$ when $|x| > r$ for some $r > R$ and constants c_i. Furthermore, a similar statement holds for superharmonic functions, and therefore, in our case $s_i(x)$ is harmonic over entire $\mathbb{R}^3$.

$u_i^L(x)$ is a harmonic section in $W^{1,2}(\mathbb{R}^3 \setminus V)$, and therefore, $\lim_{|x|\to\infty} u_i^L(x) = 0$, hence, $\lim_{|x|\to\infty} s_i(x) = 0$. This shows $s_i(x)$ is a bounded harmonic function on $\mathbb{R}^3$; thus $s_i \equiv 0$. This implies $u_i^L(x) = \frac{c_i}{|x|}$, and therefore,

$$u^L = \sum_{i=1}^{3} \frac{c_i}{|x|} dx_i,$$

on $\mathbb{R}^3 \setminus V$, which proves the lemma. $\square$

Lemma 24 (Step 7). *$u^T = O(e^{-r})$ as $r \to \infty$.*

Proof. The argument is completely similar to the step 3 of the proof of lemma 7.10 in [32]. $\square$

Lemma 25 (Step 8). *Let f be a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. There exist sufficiently small $\delta > 0$ and $\varepsilon_0 > 0$ such that if $\|we_0(A_0\Phi_0)\|_{L^3(\mathfrak{R}^3)} < \delta$ and $\varepsilon < \varepsilon_0$, then the unique solution $u(x)$ of Lemma 22 satisfies*

$$\|u\|_{W^{1,2}_\alpha} \leq C\|f\|_{L^2_{\alpha-2}}, \tag{1.2.28}$$

for a constant C independent of ε.

Proof. If $\|u\|_{W^{1,2}_\alpha} = 0$, then 1.2.28 is trivial. Suppose $\|u\|_{W^{1,2}_\alpha} \neq 0$. Then 1.2.28 is equivalent to

$$\|u\|^2_{W^{1,2}_\alpha} \leq C\|u\|_{W^{1,2}_\alpha}\|f\|_{L^2_{\alpha-2}}. \tag{1.2.29}$$

By the integration by parts we have

$$\begin{aligned}
\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} &\geq \|f\|_{L^2_{\alpha-2}}\|u\|_{L^2_\alpha} = \|w^{-\alpha+\frac{1}{2}}f\|_{L^2}\|w^{-\alpha-\frac{3}{2}}u\|_{L^2} \geq \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle f, u\rangle vol_g \\
&= \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle d_2 d_2^* u, u\rangle vol_g = \int_{\mathfrak{R}^3} w^{-2\alpha-1}(|\nabla_{A_0}u|^2 + |[\Phi_0, u]|^2) vol_g \\
&+ \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0 \wedge u], u\rangle vol_g - \alpha(1+2\alpha)\|u\|^2_{L^2_\alpha}.
\end{aligned}$$

The previous two lemmas show $u(x) = O(|x|^{-1})$ as $|x| \to \infty$, and therefore,

$$d_2^* u(x) u(x) = O(|x|^{-3}),$$

hence, the asymptotic terms do not appear in the integration by parts.

Let

$$\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} = a\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} + b\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} \text{ where } a+b=1 \text{ and } a, b > 0.$$

Using the Corollary 3,

$$\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} = a\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} + b\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} \geq \frac{a}{C_{g,\alpha}}\|u\|^2_{L^2_\alpha} + b\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}}.$$

Let $a/C_{g,\alpha} = b$. We get

$$a = \frac{C_{g,\alpha}}{C_{g,\alpha}+1}, \qquad b = \frac{1}{C_{g,\alpha}+1}.$$

For instance, when g is flat and $\mathfrak{R}^3 = \mathbb{R}^3$, we have $C_{g,\alpha} = 1/(\alpha+1)^2$, and therefore,

$$a = \frac{1}{(\alpha+1)^2+1}, \qquad b = \frac{(\alpha+1)^2}{(\alpha+1)^2+1}.$$

Note that $C_{g,\alpha} \to 1/(\alpha+1)^2$ as $\varepsilon \to 0$.

For these specific choices for a and b, we get

$$\|\nabla_{A_0} u\|^2_{L^2_{\alpha-1}} \geq \frac{1}{C_{g,\alpha}+1}\|u\|^2_{W^{1,2}_\alpha},$$

and therefore,

$$\begin{aligned}
\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} &\geq \frac{1}{C_{g,\alpha}+1}\|u\|^2_{W^{1,2}_\alpha} - |\alpha(1+2\alpha)|\|u\|^2_{L^2_\alpha} \\
&\quad + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0 \wedge u], u\rangle vol_g \\
&\geq \left(\frac{1}{C_{g,\alpha}+1} - |\alpha(1+2\alpha)|\right)\|u\|^2_{W^{1,2}_\alpha} \\
&\quad + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0 \wedge u], u\rangle vol_g.
\end{aligned}$$

As $\varepsilon \to 0$,

$$\frac{1}{C_{g,\alpha}+1} + \alpha(1+2\alpha) \to b_\alpha := \frac{(\alpha+1)^2}{(\alpha+1)^2+1} + \alpha(1+2\alpha). \tag{1.2.30}$$

Let $\varepsilon_1 > 0$ be sufficiently small such that for any $0 < \varepsilon < \varepsilon_1$,

$$|\left(\frac{1}{C_{g,\alpha}+1} + \alpha(1+2\alpha)\right) - b_\alpha| \leq \frac{1}{1000}.$$

The equation $b_\alpha = 0$ has no solutions. In fact, $b_\alpha > 0.18 > 0$. Moreover, for any $\varepsilon \in (0, \varepsilon_1)$, the solutions to the equation $\frac{1}{C_{g,\alpha}+1} + \alpha(1+2\alpha) > 0$. Moreover, for all $\alpha \in [-\frac{1}{2}, 0)$ and $0 < \varepsilon < \varepsilon_1$,

$$\left(\frac{1}{C_{g,\alpha}+1} + \alpha(1+2\alpha)\right) > \frac{1}{10},$$

and therefore,

$$\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} \geq \frac{1}{10}\|u\|^2_{W^{1,2}_\alpha} + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0 \wedge u], u\rangle vol_g.$$

As mentioned, by scaling the metric on M and without loss of generality, we can assume $sup_{x\in M}|Ric(x)| \leq \frac{1}{100}$, and therefore, by Corollary 3,

$$\int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g \leq \frac{1}{100}\|u\|^2_{L^2_{\alpha-1}} \leq \frac{1}{100}\|u\|^2_{L^2_\alpha} \leq \frac{C_{g,\alpha}}{100}\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} \leq \frac{C_{g,\alpha}}{100}\|u\|^2_{W^{1,2}_\alpha}.$$

hence,

$$\begin{aligned}\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} &\geq \frac{1}{10}\|u\|^2_{W^{1,2}_\alpha} + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle Ric(u), u\rangle vol_g + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g \\ &\geq (\frac{1}{10} - \frac{C_{g,\alpha}}{100})\|u\|^2_{W^{1,2}_\alpha} + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g.\end{aligned}$$

Moreover, $1/(\alpha+1)^2 \leq 4$ for $\alpha \in [-\frac{1}{2}, 0)$. Let $\varepsilon_2 > 0$ be sufficiently small such that

$$|C_{g,\alpha} - \frac{1}{(\alpha+1)^2}| \leq \frac{1}{10},$$

and therefore, $C_{g,\alpha} \leq \frac{9}{2}$. This implies

$$\begin{aligned}\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} &\geq (\frac{1}{10} - \frac{C_{g,\alpha}}{100})\|u\|^2_{W^{1,2}_\alpha} + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g \\ &\geq \frac{1}{20}\|u\|^2_{W^{1,2}_\alpha} + \int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g.\end{aligned}$$

Regarding the error term,

$$\begin{aligned}\int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g &\leq \|we_0\|_{L^3}\|u\|_{L^2_\alpha}\|w^{-\alpha-\frac{1}{2}}u\|_{L^6} \\ &\leq \|we_0\|_{L^3}\|u\|_{W^{1,2}_\alpha}\|w^{-\alpha-\frac{1}{2}}u\|_{L^6}.\end{aligned}$$

By the Sobolev inequality

$$\|w^{-\alpha-\frac{1}{2}}u\|_{L^6} \leq C_{Sob}\|w^{-\alpha-\frac{1}{2}}u\|_{W^{1,2}} \leq C' C_{Sob}\|u\|_{W^{1,2}_\alpha},$$

for a uniform constant C', and therefore,

$$\int_{\mathfrak{R}^3} w^{-2\alpha-1}\langle *[e_0\wedge u], u\rangle vol_g \leq \delta C' C_{Sob}\|u\|^2_{W^{1,2}_\alpha}.$$

Taking δ small enough such that $\delta < \frac{1}{100C'C_{Sob}}$, we get

$$\|f\|_{L^2_{\alpha-2}}\|u\|_{W^{1,2}_\alpha} \geq \frac{1}{25}\|u\|^2_{W^{1,2}_\alpha},$$

and therefore,

$$\|u\|_{L^2_\alpha} \leq C\|f\|_{L^2_{\alpha-2}}.$$

□

We are progressing towards proving

$$\|u\|_{W^{2,2}_\alpha} \leq C\|f\|_{L^2_{\alpha-2}}.$$

The next lemma is a necessary estimation in this direction.

Lemma 26 (Step 9)**.** *Let (A_0, Φ_0) be the constructed approximate monopole. Then we have the following pointwise approximation*

$$|F_{A_0}| = |d_{A_0}\Phi_0| \leq \frac{C}{\lambda^{-2} + r^2}.$$

Proof. The proof in the case $\mathfrak{R}^3 = \mathbb{R}^3$ can be found in [30, Proposition 4.14]. The essential point is that this approximation holds everywhere on $\mathbb{R}^3$, where the pair (A_0, Φ_0) is equal to the scaled BPS-monopole, where it is equal to the scaled Dirac monopole, and also over the region in between.

Let g and g_0 denote the Riemannian metrics on $\mathfrak{R}^3$ and $\mathbb{R}^3$, respectively. Recall that for any 2-form β with valued in any vector bundle, we have

$$\begin{aligned} |\beta|_g - |\beta|_{g_0} &= \sum_{k,l} \langle \beta_k, \beta_l \rangle (g^{k,l} - g_0^{k,l}) = -\frac{1}{3} \sum_{k,l,m,n} \langle \beta_k, \beta_l \rangle R_{klmn} \beta_k \beta_l x_m x_n + O(|x|^3) \\ &\leq C_1 R |\beta|^2_{g_0} |x|^2, \end{aligned}$$

where $C_1 > 0$ is a constant and R is the maximum of the Riemann curvature tensor of g. Therefore,

$$|\beta|_g \leq |\beta|_{g_0} + C_1 R |\beta|^2_{g_0} |x|^2,$$

Let $\beta = F_{\eta_j^*(A^{\lambda_j}_{BPS})}$,

$$|(*_0 - *)F_{\eta_j^*(A^{\lambda_j}_{BPS})}|_g \leq C_2(\frac{1}{\lambda_j^{-2} + |x|^2} + R\frac{|x|^2}{\lambda_j^{-2} + |x|^2}) \leq C_2(\frac{1 + R\varepsilon^2}{\lambda_j^{-2} + |x|^2}) \leq \frac{C}{\lambda_j^{-2} + |x|^2},$$

for a constant C, when λ_j is sufficiently large. □

The following lemma is the last step of proving Theorem 3.

Lemma 27 (Step 10)**.** *Let f be a smooth, compactly supported, $\mathfrak{su}(2)$-valued 1-form on $\mathfrak{R}^3$. Let $\alpha \in [-\frac{1}{2}, 0)$. There exists a sufficiently small $\delta > 0$ and ε_0 such that if $\|we_0(A_0\Phi_0)\|_{L^3(\mathfrak{R}^3)} < \delta$*

and $\varepsilon < \varepsilon_0$, then the unique solution $u(x)$ of Lemma 22 is an element of $W^{2,2}_\alpha$ and satisfies

$$\|u\|_{W^{2,2}_\alpha} \leq C\|f\|_{L^2_{\alpha-2}},$$

for a constant C independent of λ.

Proof. We should find a uniform bound on

$$\|\nabla_{A_0}(d_2^* u)\|_{L^2_{\alpha-2}} + \|[\Phi_0, d_2^* u]\|_{L^2_{\alpha-2}} = \|w^{-\alpha+\frac{1}{2}}\nabla_{A_0}(d_2^* u)\|^2_{L^2} + \|w^{-\alpha+\frac{1}{2}}[\Phi_0, d_2^* u]\|^2_{L^2},$$

in terms of $\|f\|_{L^2_{\alpha-2}}$.

Let $(a,\phi) = D^*(u,0) = d_2^* u$. By the monopole Weitzenböck formula we have

$$\begin{aligned} D^*D(d_2^*u) &= DD^*(d_2^*u) + 2\langle d_{A_0}\Phi_0, d_2^*u\rangle \\ &= \nabla_A^*\nabla_A(d_2^*u) - ad(\Phi)^2(d_2^*u) + \langle Ric(x), d_2^*u\rangle + *[e_0 \wedge d_2^*u] + 2\langle d_{A_0}\Phi_0, d_2^*u\rangle, \end{aligned}$$

By multiplying the formula by $w^{-2\alpha+1}d_2^*u$ and integrating over $\mathfrak{R}^3$, we get

$$\begin{aligned} &\|\nabla_A(d_2^*u)\|^2_{L^2_{\alpha-2}} + \|[\Phi,(d_2^*u)]\|^2_{L^2_{\alpha-2}} \\ &\qquad \leq \|D(d_2^*u)\|^2_{L^2_{\alpha-2}} + \sup_{x\in\mathfrak{R}^3}|Ric(x)|\|d_2^*u\|^2_{L^2_{\alpha-2}} \\ &\qquad + \int_{\mathfrak{R}^3} w^{-2\alpha+1}(|e_0| + 2|d_{A_0}\Phi_0|)|d_2^*u|^2 vol_g \\ &\qquad + (-2\alpha+1)\int_{\mathfrak{R}^3} w^{-2\alpha}|\nabla w||d_2^*u|(|\nabla_{A_0}(d_2^*u)| + |D(d_2^*u)|)vol_g, \end{aligned} \tag{1.2.31}$$

where the last integral is the asymptotic term of the Stokes' theorem.

We start by bounding $\|D(d_2^*u)\|^2_{L^2_{\alpha-2}}$. First note that

$$D(d_2^*u) = DD^*(u,0) = (f, *[e_0 \wedge *u]).$$

By multiplying this formula by $w^{-\alpha+\frac{1}{2}}$ and taking the L^2-norm over $\mathfrak{R}^3$, we get

$$\begin{aligned}
\|D(d_2^*u)\|^2_{L^2_{\alpha-2}} &= \|f\|^2_{L^2_{\alpha-2}} + \|[we_0 \wedge *w^{-\alpha-\frac{1}{2}}u]\|^2_{L^2} \\
&\leq \|f\|^2_{L^2_{\alpha-2}} + \|we\|^2_{L^3}\|w^{-\alpha-\frac{1}{2}}u\|^2_{L^6} \\
&\leq \|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\|w^{-\alpha-\frac{1}{2}}u\|^2_{W^{1,2}} \\
&= \|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\left(\|w^{-\alpha-\frac{1}{2}}u\|^2_{L^2} + \|\nabla_{A_0}(w^{-\alpha-\frac{1}{2}}u)\|^2_{L^2}\right) \\
&\leq \|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\left(\|u\|^2_{L^2_\alpha} + \|\nabla_{A_0}(w^{-\alpha-\frac{1}{2}}u)\|^2_{L^2}\right) \\
&\leq C'\|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\|\nabla_{A_0}(w^{-\alpha-\frac{1}{2}}u)\|^2_{L^2} \\
&\leq C'\|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\left((-\alpha-\frac{1}{2})^2\|w^{-\alpha-\frac{3}{2}}|\nabla w|u\|^2_{L^2} + \|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}}\right) \\
&\leq C''\|f\|^2_{L^2_{\alpha-2}} + \delta^2 C_{Sob}\|\nabla_{A_0}u\|^2_{L^2_{\alpha-1}} \\
&\leq C\|f\|^2_{L^2_{\alpha-2}},
\end{aligned}$$

for positive constants C_{Sob}, C', and C''.

Regarding the term

$$\sup_{x\in\mathfrak{R}^3}|Ric(x)|\|d_2^*u\|^2_{L^2_{\alpha-2}} \leq \sup_{x\in\mathfrak{R}^3}|Ric(x)|\|d_2^*u\|^2_{L^2_{\alpha-1}},$$

the Ricci curvature is bounded and $\|d_2^*u\|^2_{L^2_{\alpha-1}}$ can be bounded uniformly by $\|f\|^2_{L^2_{\alpha-2}}$, as we observed in the proof of Lemma 25.

Regarding the error term, using Lemma 12,

$$\int_{\mathfrak{R}^3} w^{-2\alpha+1}|e_0||d_2^*u|^2 vol_g \leq c\|d_2^*u\|^2_{L^2_{\alpha-2}} \leq c\|d_2^*u\|^2_{L^2_{\alpha-1}} \leq C\|f\|^2_{L^2_{\alpha-2}},$$

for positive constants c and C.

Regarding the term,

$$\int_{\mathfrak{R}^3} w^{-2\alpha+1}|d_{A_0}\Phi_0||d_2^*u|^2 vol_g,$$

recall that following Lemma 26, the term $|d_{A_0}\Phi_0|$ can be estimated,

$$|d_{A_0}\Phi_0| \leq \frac{c}{\lambda^{-2}+r^2} \to w^2|d_{A_0}\Phi_0| \leq \frac{cw^2}{\lambda^{-2}+r^2} \leq C',$$

for a uniform constant C', and therefore,

$$\int_{\mathfrak{R}^3} w^{-2\alpha+1}|d_{A_0}\Phi_0||d_2^*u|^2 vol_g \leq C' \int_{\mathfrak{R}^3} w^{-2\alpha-1}|d_2^*u|^2 vol_g$$
$$= C'\|d_2^*u\|^2_{L^2_{\alpha-1}} \leq C''\|f\|^2_{L^2_{\alpha-2}}.$$

Therefore,

$$\|u\|_{W^{2,2}_\alpha} \leq C\|f\|_{L^2_{\alpha-2}},$$

for a uniform constant C. $\square$

A key assumption in the Theorem 3 is that the error estimate $\|we_0(A_0\Phi_0)\|_{L^3(\mathfrak{R}^3)} < \delta$ for a sufficiently small δ. The following theorem states $\|we_0(A_0\Phi_0)\|_{L^3(\mathfrak{R}^3)}$ can be made as small as necessary by increasing the masses λ_j.

Theorem 4. *For any $\delta > 0$, there exists a sufficiently large $\lambda_j = \varepsilon_j^{-2} > 0$ such that the monopole (A_0, Φ_0) defined in 1.2.22 with the parameters λ_j, satisfies*

$$\|we_0(A_0, \Phi_0)\|_{L^3(\mathfrak{R}^3)} < \delta. \tag{1.2.32}$$

Proof. For each $j \in \{1, \ldots, k\}$, from the Lemma 12, we have

$$(e_0^{BPS})_{|B_{3\varepsilon_j}(q_j)} \leq C.$$

and therefore,

$$\|we_0^{BPS}\|^3_{L^3(B_{3\varepsilon_j}(q_j))} = \int_{B_{3\varepsilon_j}(q_j)} |we_0^{BPS}|^3 vol_g \leq C_1 \int_{B_{3\varepsilon_j}(q_j)} |w|^3 vol_g$$
$$= C_1 \int_{B_{3\varepsilon_j}(q_j)} (\lambda_j^{-2} + |x|^2)^{\frac{3}{2}} vol_g \leq C_2 \varepsilon_j^3 (\lambda_j^{-2} + \varepsilon_j^2)^{\frac{3}{2}}.$$

for a positive uniform constants C_1 and C_2, and therefore, it can be made as small as necessary. $\square$

1.2.7.4 The Linear Equation over $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$

In this section, we study the linearized equation $d_2 d_2^* u = f$ on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$, away from the points where the scaled BPS-monopoles are located, and set the stage for solving this linearized equation.

On this region the pair (A_0, Φ_0) is a reducible monopole with a non-zero Higgs field Φ_0, and therefore, it induces a decomposition of the adjoint bundle as $\mathfrak{g}_P = \underline{\mathbb{R}} \oplus L$, where $\underline{\mathbb{R}}$ is the sub-bundle generated by the image of Φ_0 and L is the orthogonal sub-bundle. Corresponding to the bundle decomposition $\mathfrak{g}_P = \underline{\mathbb{R}} \oplus L$, a section or a $\mathfrak{g}_P$-valued tensor f supported on this region can be written as $f = (f^L, f^T)$.

This bundle decomposition is preserved by d_2, d_2^*, and $d_2 d_2^*$. Hence the equation $d_2 d_2^* u = f$ on this region reduces to two equations for u^L and u^T. The equation for u^L is given by

$$\Delta u^L = f^L, \tag{1.2.33}$$

and the equation for u^T is

$$d_2 d_2^* u^T = f^T.$$

In the following section we will introduce the appropriate function spaces to solve these equations.

1.2.7.5 Function Spaces on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$

In this section, we set the stage to study the linearized equation over $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. We start by defining the suitable weighted Sobolev spaces on $\mathfrak{su}(2)$-valued differential forms on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. These spaces can be used to solve the problem away from the points q_j, following [9, 30].

Let δ_{p_i} be the injectivity radius at p_i. For each point $p_i \in S_p$, let

$$w_i(x) = \begin{cases} r_i, & r_i \leq \delta_{p_i} \\ 1, & r_i \geq 1, \end{cases} \tag{1.2.34}$$

where r_i is the geodesic distance from p_i.

Definition 4. *Let $U = M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$ and $U_{ext} = M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup_i B_{2\varepsilon_i}(p_i))$. Let $\alpha \in \mathbb{R}$. For all smooth compactly supported $\mathfrak{su}(2)$-valued differential forms $u \in \Omega^\bullet(U, \mathfrak{su}(2))$, let*

$$\|u\|^2_{L^2_\alpha(U)} = \|u\|^2_{L^2(U_{ext})} + \sum_{i=1}^n \|u^T\|^2_{L^2(B_{2\varepsilon_i}(p_i))} + \sum_{i=1}^n \|w_i^{-\alpha-\frac{3}{2}} u^L\|^2_{L^2(B_{2\varepsilon_i}(p_i))}.$$

Furthermore,

$$\|u\|^2_{W^{1,2}_\alpha(U)} = \|u\|^2_{L^2_\alpha(U)} + \|\nabla_{A_0} u\|^2_{L^2_{\alpha-1}(U)}.$$

Moreover,

$$\|u\|^2_{W^{2,2}_\alpha(U)} = \|u\|^2_{W^{1,2}_\alpha(U)} + \|\nabla_{A_0}(d_2^* u)\|^2_{L^2_{\alpha-2}(U)}.$$

The spaces $W^{k,2}_\alpha(U)$ are defined as the completion of $C_0^\infty(U)$ with respect to the corresponding norms for $k \in \{0, 1, 2\}$. Furthermore, one can define similar norms and weighted Sobolev spaces $W^{k,p}_\alpha(U)$ for any $p \geq 2$ and $k \in \{0, 1, 2\}$.

1.2.7.6 The Longitudinal Component and the Lockhart-McOwen Theory

In this section, we study the weighted Sobolev spaces of the sections of the longitudinal component, and set the necessary background to solve $\Delta u^T = f^T$. The main goal of this section is to show that these weighted Sobolev spaces on the longitudinal component are suitable for studying elliptic operators, more specifically, the Laplacian. These spaces are closely related to the Lockhart-McOwen Sobolev spaces on asymptotically cylindrical manifolds [65].

The following example gives a good picture of the real-valued sections in these weighted Sobolev spaces.

Example 1. *Let (M, g) be a closed, Riemannian, n-dimensional manifold. Let $p \in M$. Let δ_p be the injectivity radius at p and $r : M \to \mathbb{R}$ a smooth function such that*

$$r(x) = \begin{cases} \textit{geodesic distance from } p & \textit{on} \quad B_\delta(p), \\ 1 & \textit{on} \quad M \setminus B_{2\delta}(p_i). \end{cases}$$

Then $r^\delta \in L^p_\alpha(M)$ if and only if $\delta > \alpha$.

To understand the longitudinal part of these weighted Sobolev spaces, using a conformal mapping, one can transform them into the weighted Sobolev spaces over asymptotically cylindrical manifolds. The punctured ball $B_\varepsilon(0) \setminus \{0\} \subset \mathbb{R}^3$ can be identified with the cylinder $(-\log(\varepsilon), +\infty) \times S^2$, using a map $L_0 : B_\varepsilon(0) \setminus \{0\} \to (-\log(\varepsilon), +\infty) \times S^2$, defined by

$$(t, \theta, \varphi) := L_0(r, \theta, \varphi) = (-\log(r), \theta, \varphi), \tag{1.2.35}$$

where (r, θ, φ) denotes the spherical coordinates on $B_\varepsilon(0) \subset \mathbb{R}^3$.

Equip the punctured ball with the flat metric $g_0 = dx^2 + dy^2 + dz^2$, which in spherical coordinates can be written as $g_0 = dr^2 + r^2 g_{S^2} = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$, and the cylinder with the standard product metric $g_{Cyl} = dt^2 + g_{S^2}$. The map L_0 takes the flat metric on the punctured ball to $e^{-2t}(dt^2 + g_{S^2})$ on $(-\log(\varepsilon), +\infty) \times S^2$, which is conformally equivalent to the cylindrical metric $g_{cyl} = dt^2 + g_{S^2}$.

The Riemannian metric g on each ball $B_{\varepsilon_i}(p_i)$ using the exponential map and in geodesics normal coordinates can be written as $g = dr^2 + \psi(r, \theta) g_{S^2}$, where ψ is a smooth positive function such that $\lim_{r\to 0} \psi(r, \theta) \to 1$. Let $\mu(t, \theta) = r^{-2}\psi(r, \theta) = e^{-2t}\psi(e^{-t}, \theta)$. One can define the diffeomorphism $L_i : B_{\varepsilon_i}(p_i) \to (-\log(\varepsilon_i), +\infty) \times \mathbb{R}$, similar to 1.2.35, that takes the metric g to $e^{-2t}(dt^2 + \mu(t.\theta) g_{S^2})$, which is conformally equivalent to

$$\tilde{g} = dt^2 + \mu(t, \theta) g_{S^2},$$

where $\mu(t, \theta) \to 1$ as $t \to \infty$. The metric $\tilde{g}$ is asymptotically cylindrical

Definition 5 (Asymptotically Cylindrical Manifold)**.** *Let (X, g_X) be a non-compact, n-dimensional, Riemannian manifold, and $X_0 \subset X$ a compact submanifold with boundary. Let (Σ, g_Σ) be a closed $(n-1)$-dimensional manifold with l connected components, $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_l$, with a* *diffeomorphism*

$$\Psi : X \setminus X_0 \to (1, \infty) \times \Sigma. \tag{1.2.36}$$

Let $(1, \infty) \times \Sigma$ *be equipped with the cylindrical Riemannian metric* $g_{Cyl} = dt^2 + g_\Sigma$*, where* t *is the coordinates on* $(1, \infty)$*.*

(X, g_X) *is called an asymptotically cylindrical manifold with rate* $\beta < 0$ *if*

$$|\nabla^j_{g_{Cyl}}(\Psi_*(g_X) - g_{Cyl})| = O(e^{\beta t}), \quad \forall j \in \{0, 1, \ldots\},$$

where $\nabla_{g_{Cyl}}$ *is the Levi-Civita connection of the Riemannian metric* g_{Cyl}*.*

By gluing the maps L_i to the identity map on $M \setminus \cup_i B_{2\varepsilon_i}(p_i)$ and extend it smoothly to the necks $\cup_i(B_{2\varepsilon_i}(p_i) \setminus B_{\varepsilon_i}(p_i))$, we get a diffeomorphism

$$L : M \setminus S_p \to M_{Cyl} := (M \setminus \cup_{i=1}^n B_{\varepsilon_i}(p_i)) \bigcup (\cup_i(-log(\varepsilon_i), +\infty) \times S^2),$$

where M_{Cyl} is equipped with an asymptotically cylindrical metric. Furthermore, L takes the vector bundle of differential forms on M to asymptotically translation-invariant asymptotically cylindrical bundles over M_{Cyl}.

Definition 6 (Asymptotically Cylindrical Vector Bundle)**.** *Let* $E_{Cyl} \to (1, \infty) \times \Sigma$ *be a vector bundle invariant under translations in the* $(1, \infty)$*-direction equipped with a translation-invariant fiber metric* h_{Cyl} *and a translation-invariant connection* $\nabla_{E_{Cyl}}$ *compatible with* h_{Cyl}*. Let* $E \to X$ *be a vector bundle over asymptotically cylindrical Riemannian manifold* (X, g_X)*, where* E *is equipped with fiber metric* h *and a connection* ∇_E *compatible with* h*. The triple* (E, h, ∇_E) *is an asymptotically cylindrical vector bundle, asymptotic to* $(E_{Cyl}, h_{Cyl}, \nabla_{E_{Cyl}})$ *with rate* $\beta < 0$*, if there exists a bundle identification*

$$\tilde{\Psi} : E_{|_{X \setminus X_0}} \to E_{Cyl}|_{(1,\infty)\times\Sigma},$$

covering Ψ *in 1.2.36, such that*

$$\tilde{\Psi}_*(h) = h_{Cyl} + O(e^{\beta t}), \quad \tilde{\Psi}_*(\nabla_E) = \nabla_{E_{Cyl}} + O(e^{\beta t}).$$

In order to have the Fredholm property for the elliptic differential operators like Laplacian or $d + d^*$ on asymptotically cylindrical manifolds, one should use suitable classes of Banach spaces as domain and co-domain, as introduced by Lockhart and McOwen [65].

Definition 7 (Lockhart-McOwen Sobolev Spaces)**.** *Let* (X, g_X) *be an* n*-dimensional asymptotically cylindrical Riemannian manifold. Let* $\rho : X \to \mathbb{R}$ *be a smooth function such that on* $X \setminus X_0$ *it agrees with the geodesic distance from a point* $x_0 \in X$*. Let* $(E, h_E, \nabla_E) \to X$ *be an asymptotically cylindrical bundle. Let* $p \geq 1$*,* $k \geq 0$*, and* $\beta \in \mathbb{R}$*. For any smooth compactly* *supported section $u \in \Gamma(E)$, let*

$$\|u\|^p_{W^{k,p}_{Cyl,\beta}(E)} = \sum_{j=0}^{k} \int_X |e^{-\beta\rho}\nabla^j_E u|^p_{h_E} vol_{g_X}.$$

Let $W^{k,p}_{Cyl,\beta}(X,E)$ denote the completion of $C^\infty_0(X,E)$ with respect to this norm.

These weighted Sobolev spaces over asymptotically cylindrical manifolds are closely related to the weighted spaces defined in Definition 8.

Lemma 28. *Let L_0 be the map defined in 1.2.35. Let f be a section of a vector bundle above $B_\varepsilon(0) \setminus \{0\} \subset \mathbb{R}^3$. We have*

$$f \in W^{k,p}_\alpha(B_\varepsilon(0)) \Longleftrightarrow (L_0^{-1})^* f \in W^{k,p}_{Cyl,-\alpha}((-\log(\varepsilon), +\infty) \times S^2).$$

Moreover,

$$\|f\|_{W^{k,p}_\alpha(B_\varepsilon(0))} = \|(L_0^{-1})^* f\|_{W^{k,p}_{Cyl,-\alpha}((-\log(\varepsilon),+\infty)\times S^2)},$$

and therefore, $\|f\|_{W^{k,p}_\alpha(M\setminus S_p)}$ and $\|(L_0^{-1})^* f\|_{W^{k,p}_{Cyl,-\alpha}(M_{Cyl})}$ are equivalent norms.

Proof. Let L^p_{Cyl} and $W^{k,p}_{Cyl}$ denote the ordinary Sobolev spaces on the cylinder with respect to its cylindrical Riemannian metric g_{Cyl}.

Let $k = 0$. By a change of variable we have

$$\|f\|_{L^p_\alpha(B_\varepsilon(0))} = \|e^{\alpha t}(L_0^{-1})^* f\|_{L^p_{Cyl}((-\log(\varepsilon),+\infty)\times S^2)} = \|(L_0^{-1})^* f\|_{L^p_{Cyl,-\alpha}((-\log(\varepsilon),+\infty)\times S^2)}.$$

Let $k > 0$. Let ∇ be the Levi-Civita connection on $B_\varepsilon(0)$ with respect to the standard euclidean metric and ∇_{Cyl} be the Levi-Civita connection on $(-\log(\varepsilon), +\infty) \times S^2$ with respect to the cylindrical metric. We have the pointwise equality

$$|e^{-jt}\nabla^j f| = |\nabla^j_{Cyl}(L_0^{-1})^* f|,$$

and therefore,

$$\begin{aligned}\|r^{-\alpha-\frac{3}{2}+j}\nabla^j f\|_{L^p(B_\varepsilon(0))} &= \|e^{\alpha t}\nabla^j_{Cyl}(L_0^{-1})^* f\|_{L^p_{Cyl}((-\log(\varepsilon),+\infty)\times S^2)} \\ &= \|\nabla^j_{Cyl}(L_0^{-1})^* f\|_{L^p_{Cyl,-\alpha}((-\log(\varepsilon),+\infty)\times S^2)}.\end{aligned}$$

By summing over j,

$$\|f\|_{W^{k,p}_\alpha(B_\varepsilon(0))} = \|e^{\alpha t}(L_0^{-1})^* f\|_{W^{k,p}_{Cyl}((-\log(\varepsilon),+\infty)\times S^2)} = \|(L_0^{-1})^* f\|_{W^{k,p}_{Cyl,-\alpha}((-\log(\varepsilon),+\infty)\times S^2)}.$$

□

One can also transform the Laplacian and $d + d^*$ to the cylindrical space, on the more familiar weighted spaces of Lockhart and McOwen. Let d^* and Δ denote the corresponding differential operators over $B_{\varepsilon_i}(p_i)$, and d^*_{Cyl} and Δ_{Cyl} denote the corresponding differential operators over the cylinder $(-\log(\varepsilon), +\infty) \times S^2$, with respect to the cylindrical metric.

Lemma 29. *Let*

$$\begin{aligned}
&d^* : \Omega^*(B_{\varepsilon_i}(p_i)) \to \Omega^*(B_{\varepsilon_i}(p_i)),\\
&d^*_{Cyl} : \Omega^*((-\log(\varepsilon), +\infty) \times S^2) \to \Omega^*((-\log(\varepsilon), +\infty) \times S^2).
\end{aligned}$$

Let $\alpha \in \Omega^1(B_{\varepsilon_i}(p_i))$. We have

$$d^*\alpha = e^{2t} \left(- *_{Cyl} \left(dt \wedge *_{Cyl}(L_0^{-1}\alpha)\right) + d^*_{Cyl}(L_0^{-1}\alpha)\right).$$

Let $\beta \in \Omega^2(B_{\varepsilon_i}(p_i))$. We have

$$d^*\beta = e^{2t} \left(*_{Cyl}(dt \wedge *_{Cyl}(L_0^{-1}\beta)) + d^*_{Cyl}(L_0^{-1}\beta)\right).$$

Let $\Delta : \Omega^1(B_{\varepsilon_i}(p_i)) \to \Omega^1(B_{\varepsilon_i}(p_i))$. We have

$$\Delta\alpha = e^{2t} \left(\Delta_{Cyl}(L_0^{-1}\alpha) + F(\alpha)\right),$$

where $F : \Omega^1(B_{\varepsilon_i}(p_i)) \to \Omega^1(B_{\varepsilon_i}(p_i))$ is a 1st order differential operator, given by

$$\begin{aligned}
F(\alpha) = -2dt \wedge (*_{Cyl}(dt \wedge *_{Cyl} L_0^{-1*}(\alpha))) - d(*_{Cyl}(dt \wedge *_{Cyl} L_0^{-1*}(\alpha)))\\
+ *_{Cyl} \left(dt \wedge *_{Cyl} L_0^{-1*}(d\alpha)\right).
\end{aligned}$$

Proof. Let $*^k$ and $*^k_{Cyl}$ be the Hodge star operator on differential k-forms on $B_\varepsilon(0) \setminus \{0\}$ and $(-\log(\varepsilon), +\infty) \times S^2$, respectively. These two Hodge star operators are related by the following formulas,

$$\begin{aligned}
*^0 f &= -e^{-3t} *^0_{Cyl} (L_0^{-1})^* f, && \text{for all } f \in \Omega^0(B_\varepsilon(0) \setminus \{0\}),\\
*^1 \alpha &= -e^{-t} *^1_{Cyl} (L_0^{-1})^* \alpha, && \text{for all } \alpha \in \Omega^1(B_\varepsilon(0) \setminus \{0\}),\\
*^2 \beta &= -e^{t} *^2_{Cyl} (L_0^{-1})^* \beta, && \text{for all } \beta \in \Omega^2(B_\varepsilon(0) \setminus \{0\}),\\
*^3 \gamma &= -e^{3t} *^3_{Cyl} (L_0^{-1})^* \gamma, && \text{for all } \gamma \in \Omega^3(B_\varepsilon(0) \setminus \{0\}).
\end{aligned}$$

Let $(d^*)_k$ and $(d^*_{Cyl})_k$ denote the formal adjoint of d on differential k-forms on $B_\varepsilon(0) \setminus \{0\}$ and $(-\log(\varepsilon), +\infty) \times S^2$, respectively.

Let $\alpha \in \Omega^1(B_{\varepsilon_i}(p_i))$. Then

$$\begin{aligned}
(d^*)_1\alpha = *^3 d *^1 \alpha &= *^3 d \left(-e^{-t} *^1_{Cyl} L_0^{-1*}(\alpha)\right)\\
&= e^{-t} *^3 \left(dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha) - d(*^1_{Cyl} L_0^{-1*}(\alpha)\right)\\
&= e^{2t} \left(- *^3_{Cyl} \left(dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha)\right) + *^3_{Cyl}(d(*^1_{Cyl} L_0^{-1*}(\alpha)))\right)\\
&= e^{2t} \left(- *^3_{Cyl} \left(dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha)\right) + (d^*_{Cyl})_1 L_0^{-1*}(\alpha)\right).
\end{aligned}$$

Let $\beta \in \Omega^2(B_{\varepsilon_i}(p_i))$. Then

$$\begin{aligned}(d^*)_2\beta &= *^2 d *^2 \beta = *^2 d\left(-e^t *^2_{Cyl} (L_0^{-1})^*\beta\right)\\ &= -e^t *^2 \left(dt \wedge *^2_{Cyl} L_0^{-1*}(\beta) + d(*^2_{Cyl} L_0^{-1*}(\alpha)\right)\\ &= e^{2t}\left(*^2_{Cyl}(dt \wedge *^2_{Cyl} L_0^{-1*}(\beta)) + *^2_{Cyl}(d(*^2_{Cyl} L_0^{-1*}(\alpha))\right)\\ &= e^{2t}\left(*^2_{Cyl}(dt \wedge *^2_{Cyl} L_0^{-1*}(\beta)) + (d^*_{Cyl})_2(L_0^{-1*}(\beta))\right).\end{aligned}$$

Let $\Delta : \Omega^1(B_{\varepsilon_i}(p_i)) \to \Omega^1(B_{\varepsilon_i}(p_i))$. Then

$$\begin{aligned}\Delta\alpha = d(d^*)_1\alpha + (d^*)_2 d\alpha &= d\left(e^{2t}\left(- *^3_{Cyl} (dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha)) + (d^*_{Cyl})_1 L_0^{-1*}(\alpha)\right)\right)\\ &+ e^{2t}\left(*^2_{Cyl}(dt \wedge *^2_{Cyl} L_0^{-1*}(d\alpha)) + (d^*_{Cyl})_2(L_0^{-1*}(d\alpha))\right)\\ &= e^{2t}(2dt \wedge (- *^3_{Cyl} (dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha)))\\ &- d(*^3_{Cyl}(dt \wedge *^1_{Cyl} L_0^{-1*}(\alpha))) + *^2_{Cyl}(dt \wedge *^2_{Cyl} L_0^{-1*}(d\alpha)))\\ &+ e^{2t}\Delta_{Cyl}\alpha = e^{2t}(F(\alpha) + \Delta_{Cyl}\alpha),\end{aligned}$$

where $F : \Omega^1(B_{\varepsilon_i}(p_i)) \to \Omega^1(B_{\varepsilon_i}(p_i))$ is a 1st order differential operator given by

$$\begin{aligned}F(\alpha) = -2dt \wedge (*_{Cyl}(dt \wedge *_{Cyl} L_0^{-1*}(\alpha))) &- d(*_{Cyl}(dt \wedge *_{Cyl} L_0^{-1*}(\alpha)))\\ &+ *_{Cyl} (dt \wedge *_{Cyl} L_0^{-1*}(d\alpha)).\end{aligned}$$

□

The key property of these weighted Sobolev spaces is described in the following lemma.

Lemma 30. *On real-valued 1-forms,* $\Delta : W^{2,2}_\alpha(\Omega^1(M\backslash S_p)) \to L^2_{\alpha-2}(\Omega^1(M\backslash S_p))$ *is Fredholm, for* α *outside of a discrete subset of* $\mathbb{R}$*, denoted by* $D(\Delta)$*.*

The lemma follows from Corollary 3.2.13, Lemma 3.3.4, Lemma 3.3.6 in [33].

Lemma 31. *Let* $\Delta : W^{2,2}_\alpha(\Omega^1(M\backslash S_p)) \to L^2_{\alpha-2}(\Omega^1(M\backslash S_p))$*, where* M *is a rational homology 3-sphere. Let* $\alpha \notin D(\Delta)$*. Then*

$$\ker\Delta = \{0\}.$$

The proof is similar to the lemma 4.4 in [11] and lemma 3.4.4 in [33].

Proof. To show Δ is Fredholm, one can construct an inverse for this operator, by taking the inverse of the Laplacian away from the singular points $M \setminus \cup_i B_{\varepsilon_i}(p_i)$ — which can be done since $H^1(M \setminus \cup_i B_{\varepsilon_i}(p_i), \mathbb{R}) = 0$ — and glue it to the inverse of the Laplacian on the weighted Sobolev spaces defined on the neighbourhood of the singular points. These local inverses close to the singular points can be constructed by solving the Dirichlet problem, similar to the proposition 3.3.11 in [33]. □

1.2.7.7 Solving the Longitudinal Part over $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$

In this section, we solve the linear equation over $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. In the Theorem 5, we solve the longitudinal component of the linear problem, and in the Theorem 6, the transverse one.

Theorem 5. *Let $-\frac{1}{2} \leq \alpha < 0$ such that $\alpha, -\alpha - 1$ are not in $D(\Delta)$. Let $f^L \in L^2_{\alpha-2}(M \setminus S_p)$. Then there exists a solution to the equation $\Delta u^L = f^L$, where $u^L \in W^{2,2}_{\alpha}(\Omega^1(M \setminus S_p))$, and*

$$\|u^L\|_{W^{2,2}_{\alpha}(M\setminus S_p)} \leq C\|f^L\|_{L^2_{\alpha-2}(M\setminus S_p)},$$

for a constant C, independent of λ.

Proof. Pick $\alpha \in [-\frac{1}{2}, 0) \setminus D(\Delta)$. Therefore, $\Delta : W^{2,2}_{\alpha}(\Omega^1(M \setminus S_p)) \to L^2_{\alpha-2}(\Omega^1(M \setminus S_p))$ is Fredholm. We have

$$L^2_{\alpha-2} = \Delta(W^{2,2}_{\alpha}) \oplus \ker \Delta_{W^{2,2}_{-\alpha-1}}.$$

On the other hand,

$$\ker \Delta_{W^{2,2}_{-\alpha-1}} \cong H^1(M \setminus S_p, \mathbb{R}) = 0,$$

and therefore, $\Delta : W^{2,2}_{\alpha}(\Omega^1(M \setminus S_p)) \to L^2_{\alpha-2}(\Omega^1(M \setminus S_p))$ is surjective. Furthermore, since it is a Fredholm operator, there is a constant C such that

$$\|u^L\|_{W^{2,2}_{\alpha}} \leq C\|\Delta u^L\|_{L^2_{\alpha-2}}.$$

□

1.2.7.8 The Transverse Component

In this section, we solve the transverse part of the linear equation, away from the points where the scaled BPS-monopoles are located. Recall that on the transverse component our norms agree with the standard Sobolev norms $W^{k,p}$ on the transverse component on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. For all smooth compactly supported L-valued differential forms $u^T \in \Omega^\bullet(U, \mathfrak{su}(2))$, we have

$$\begin{aligned}
\|u^T\|^2_{L^2_{\alpha}(U)} &= \|u^T\|^2_{L^2(U)}, \\
\|u^T\|^2_{W^{1,2}_{\alpha}(U)} &= \|u^T\|^2_{W^{1,2}(U)} := \|u^T\|^2_{L^2(U)} + \|\nabla_{A_0} u^T\|^2_{L^2(U)}, \\
\|u^T\|^2_{W^{2,2}_{\alpha}(U)} &= \|u^T\|^2_{W^{2,2}(U)} := \|u^T\|^2_{W^{1,2}(U)} + \|\nabla_{A_0}(d_2^* u^T)\|^2_{L^2(U)}.
\end{aligned}$$

The main theorem of this section is the following. The essential assumption is that the monopole has a very large mass.

Theorem 6. *Let $U = M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. Let $d_2 d_2^* : W^{2,2}(\Omega^1(U, L)) \to L^2(\Omega^1(U, L))$. Let $f^T \in L^2(\Omega^1(U; L))$ such that $f^T_{|\cup_j B_{2\varepsilon_j}(q_j)} = 0$. For a sufficiently large average mass $\overline{m}$,* *there exists a unique solution $\xi^T \in W^{1,2}(\Omega^1(U,L))$ to $d_2\xi^T = f^T$ of the form $\xi^T = d_2^* u^T$, and*

$$\|u^T\|_{W^{2,2}(U)} \leq C\|f^T\|_{L^2(U)},$$

where the constant C is independent of λ, which appears in the definition of the approximate monopole (A_0, Φ_0).

The proof of this theorem is presented in a series of lemmas. Since the set of smooth compactly supported 1-forms $\Omega^1_c(U,L)$ is dense in the subset of $L^2(\Omega^1(U,L))$ where the 1-forms vanish on $\cup_j B_{2\varepsilon_j}(q_j)$, we only to prove the lemma for the case where f^T is a smooth compactly supported element in $\Omega^1_c(U,L)$.

Lemma 32 (Step 1)**.** *Suppose f^T is a smooth compactly supported L-valued 1-form on U. Let*

$$\begin{aligned}
&E : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R},\\
&E(u^T) := \frac{1}{2}\int_U |d_2^* u^T|^2 vol_g - \langle u^T, f^T\rangle_{L^2(U)},
\end{aligned}$$

Then $E(u^T)$ is convergent for all $u^T \in W^{1,2}(\Omega^1(U,L))$.

Proof. First we show $\|d_2^* u^T\|^2_{L^2(U)}$ is finite. Again, the key fact in proving this is the monopole Weitzenböck formula. Note that on this region $e_0 = 0$.

$$\begin{aligned}
\|d_2^* u^T\|^2_{L^2(U)} &= \|\nabla_{A_0} u^T\|^2_{L^2(U)} + \|[\Phi_0, u^T]\|^2_{L^2(U)} + \langle u^T, Ric(u^T)\rangle_{L^2(U)}\\
&\leq (1 + 2\|\Phi_0\|^2_{L^2(U)} + \sup_x |Ric(x)|)\|u^T\|^2_{W^{1,2}(U)} < \infty.
\end{aligned}$$

Note that Φ_0 is continuous and bounded on U_{ext}, and therefore, it has finite L^2-norm on U_{ext}. Moreover, on each $B_{2\varepsilon_i}(p_i)$,

$$\|\Phi_0\|^2_{L^2(B_{2\varepsilon_j}(p_j))} \leq C_1 \int_{B_{2\varepsilon_i}(p_i)} \frac{1}{r^2} vol_g \leq C_2\varepsilon_i,$$

and therefore, $\|\Phi_0\|^2_{L^2(U)} < \infty$.

By the Cauchy–Schwarz inequality we have

$$\langle u^T, f^T\rangle_{L^2(U)} \leq \|u^T\|_{L^2(U)}\|f^T\|_{L^2(U)} \leq \|u^T\|_{W^{1,2}(U)}\|f^T\|_{L^2(U)} < \infty,$$

and therefore, the action functional $E(u^T)$ is convergent for all $u^T \in \Omega^1(U,L)$. □

Lemma 33 (Step 2)**.** *The functional $E : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}$ is continuous.*

Proof. We should show the following functions are continuous,

$$\|d_2^* - \|^2_{L^2(U)} : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}, \qquad \langle -, f^T\rangle_{L^2(U)} : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}.$$

To prove $\|d_2^* - \|^2_{L^2(U)}$ is continuous we should show

$$\|d_2^* u_i^T\|^2_{L^2(U)} \to \|d_2^* u^T\|^2_{L^2(U)}, \quad \text{when} \quad u_i^T \to u^T \quad \text{in} \quad W^{1,2}(\Omega^1(U,L)).$$

By the Weitzenböck formula we know

$$\begin{aligned}
\|d_2^*(u^T - u_i^T)\|^2_{L^2(U)} &= \|\nabla_{A_0}(u^T - u_i^T)\|^2_{L^2(U)} + \|[\Phi_0, (u^T - u_i^T)]\|^2_{L^2(U)} \\
&+ \langle Ric((u^T - u_i^T)), (u^T - u_i^T)\rangle_{L^2(U)} \\
&\le \|(u^T - u_i^T)\|^2_{W^{1,2}(U)} + \|[\Phi_0, (u^T - u_i^T)]\|^2_{L^2(U)} \\
&+ \sup_x |Ric(x)| \|u^T - u_i^T\|_{L^2(U)},
\end{aligned}$$

which goes to zero as $i \to \infty$.

In order to prove $\langle -, f^T\rangle_{L^2(U)} : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}$ is continuous, note that this map is linear, and since $W^{1,2}(\Omega^1(U,L))$ is a Hilbert space, continuity is equivalent to being bounded.

$$\langle u^T, f^T\rangle_{L^2(U)} \le \|u^T\|_{L^2(U)} \|f^T\|_{L^2(U)} \le \|u^T\|_{W^{1,2}(U)} \|f^T\|_{L^2(U)},$$

and therefore, the operator norm

$$\sup\{\langle u^T, f^T\rangle_{L^2(U)} \mid u^T \in W^{1,2}, \|u^T\|_{W^{1,2}} = 1\} \le \|f\|_{L^2(U)} < \infty,$$

which proves the linear map is bounded and continuous. □

Lemma 34 (Step 3)**.** *$E : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}$ is Gateaux-differentiable.*

Proof. A direct computation shows

$$d_u^T E(v^T) = \int_U \langle d_2^* u^T, d_2^* v^T\rangle_{L^2} vol_g - \langle v^T, f^T\rangle_{L^2(U)}.$$

Note that for any $u^T, v^T \in W^{1,2}$,

$$\int_U \langle d_2^* u^T, d_2^* v^T\rangle_{L^2} vol_g \le \|d_2^* u^T\|_{L^2(U)} \|d_2^* v^T\|_{L^2(U)} < \infty,$$

and

$$\langle v^T, f^T\rangle_{L^2(U)} \le \|v^T\|_{L^2(U)} \|f^T\|_{L^2(U)} < \infty.$$

□

Lemma 35 (Step 4)**.** *The functional*

$$E : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}, \qquad E(u^T) := \frac{1}{2}\int_U |d_2^* u^T|^2 vol_g - \langle u^T, f^T\rangle_{L^2(U)}.$$

is strictly convex, when $\overline{m}$ is sufficiently large.

Proof. $\langle u^T, f^T\rangle_{L^2(U)}$ is linear in u and we only need to show $\|d_2^* u^T\|^2_{L^2(U)}$ is strictly convex. This reduces to showing that $\|d_2^*(u_1^T - u_2^T)\|^2_{L^2(U)} > 0$ when $u_1^T, u_2^T \in W^{1,2}(\Omega^1(U,L))$ and $u_1^T \neq u_2^T$. Let $u^T = u_1^T - u_2^T$. We should show

$$u^T \in W^{1,2}(\Omega^1(U,L)), \quad d_2^* u^T = 0 \quad \Rightarrow \quad u^T = 0.$$

In fact, this shows $\|d_2^* - \|_{L^2}$ is a norm on $W^{1,2}(\Omega^1(U,L))$.

$$\begin{aligned} 0 = \|d_2^* u\|^2_{L^2(\Re^3)} &\geq \|\nabla_{A_0} u^T\|^2_{L^2(U)} + (\frac{\overline{m}^2}{4} - \sup_x |Ric(x)|)\|u^T\|^2_{L^2(U)} \\ &\geq \|\nabla_{A_0} u^T\|^2_{L^2(U)} + \|u^T\|^2_{L^2(U)}, \end{aligned}$$

when $\overline{m}$ is large enough such that

$$\frac{\overline{m}^2}{4} - \sup_x |Ric(x)| \geq 1,$$

and therefore, $u^T = 0$. □

Lemma 36 (Step 5). *$E : W^{1,2}(\Omega^1(U,L)) \to \mathbb{R}$ has a unique minimizer, when $\overline{m}$ is sufficiently large, and therefore, $d_2 d_2^* u^T = f^T$ has a unique solution.*

Proof. In order to prove this, we use the following fact.

Let $E : W \to \mathbb{R}$ be a convex, continuous, real-Gateaux-differentiable functional defined on a real reflexive Banach space W such that $d_u E(u) > 0$ for any $u \in W$ with $\|u\|_W \geq R > 0$. Then there exists an interior point u_0 of $\{u \in W \mid \|u\|_W < R\}$ which is the unique minimizer of E and $d_{u_0} E = 0$.

Let $W = W^{1,2}(\Omega^1(U,L))$ equipped with the norm L^2 defined by $\| - \|_{L^2(U)}$. Then

$$\begin{aligned} \|d_2^* u^T\|^2_{L^2(U)} &= \|\nabla_{A_0} u^T\|^2_{L^2(U)} + \|[\Phi_0, u^T]\|^2_{L^2(U)} + \langle Ric(u^T), u^T\rangle_{L^2(U)} \\ &\geq \|\nabla_{A_0} u^T\|^2_{L^2(U)} + (\frac{\overline{m}^2}{4} - \sup_x |Ric(x)|)\|u^T\|^2_{L^2(U)} \\ &\geq \|\nabla_{A_0} u^T\|^2_{L^2(U)} + \|u^T\|^2_{L^2(U)}, \end{aligned}$$

when

$$\frac{\overline{m}^2}{4} - \sup_x |Ric(x)| \geq 1.$$

Let $R = 1 + \|f^T\|_{L^2(U)}$. Then $\|u^T\|_{L^2(U)} > 1 + \|f^T\|_{L^2(U)}$. Therefore,

$$\begin{aligned} d_u^T E(u^T) &= \|d_2^* u^T\|_{L^2(U)}^2 - \langle u^T, f^T \rangle_{L^2(U)} \\ &\geq \|\nabla_{A_0} u^T\|_{L^2(U)}^2 + \|u^T\|_{L^2(U)}^2 - \|u^T\|_{L^2(U)} \|f^T\|_{L^2(U)} \\ &= \|\nabla_{A_0} u^T\|_{L^2(U)}^2 + \|u^T\|_{L^2(U)} (\|u^T\|_{L^2(U)} - \|f^T\|_{L^2(U)}) \\ &\geq \|\nabla_{A_0} u^T\|_{L^2(U)}^2 + \|u^T\|_{L^2(U)} > R > 0, \end{aligned}$$

when

$$\|u^T\|_{L^2(U)} > R = 1 + \|f^T\|_{L^2(U)} \quad \text{and} \quad \overline{m} > \sqrt{1 + 4 \sup_x |Ric(x)|}.$$

Therefore, E has a unique minimizer u^T where $\|u^T\|_{L^2(U)} < 1 + \|f^T\|_{L^2(U)}$. $\square$

Lemma 37 (Step 6)**.** *For $\overline{m}$ sufficiently large, the unique solution u^T of the previous lemma is in $W^{1,2}(U, L)$, and satisfies*

$$\|u^T\|_{W^{1,2}(U)} \leq C \|f\|_{L^2(U)},$$

for a uniform constant C.

Proof. We have

$$\|f^T\|_{L^2}^2 = \|d_2 d_2^* u^T\|_{L^2}^2 \geq \|\nabla_A u^T\|_{L^2}^2 + (\frac{\overline{m}^2}{4} - \sup |Ric_x|) \|u^T\|_{L^2}^2,$$

assuming $\overline{m}$ is sufficiently large, we get

$$\|f^T\|_{L^2}^2 \geq C \|u^T\|_{W^{1,2}}^2,$$

for a positive constant C. $\square$

Lemma 38 (Step 7)**.** *Suppose $\overline{m}$ is sufficiently large. Then the unique solution u^T of Lemma 36 is in $W^{2,2}(U, L)$, and satisfies*

$$\|u^T\|_{W^{2,2}(U)} \leq C \|f^T\|_{L^2(U)},$$

for a positive constant C.

Proof. We should find a uniform bound on $\|\nabla_{A_0} d_2^* u^T\|_{L^2}$ in terms of $\|f^T\|_{L^2}$. Let $(a^T, \varphi^T) = D^*(u^T, 0) = d_2^* u^T$ in the Weitzenböck formula for DD^*. By multiplying this formula by $d_2^* u^T$, and integrating over U, we get

$$\begin{aligned} \|\nabla_A (d_2^* u^T)\|_{L^2}^2 &\leq \|\nabla_A (d_2^* u^T)\|_{L^2}^2 + \|[\Phi, (d_2^* u^T)]\|_{L^2}^2 \\ &\leq \|D(d_2^* u^T)\|_{L^2}^2 + \sup_x |Ric(x)| \|d_2^* u^T\|_{L^2}^2. \end{aligned} \tag{1.2.37}$$

We start by bounding $\|D(d_2^* u^T)\|_{L^2}^2$. Recall that

$$D(d_2^* u^T) = DD^*(u^T, 0) = (f^T, *[e_0 \wedge *u^T]) = (f^T, 0).$$

By taking the L^2-norm over U, we get

$$\|D(d_2^* u^T)\|_{L^2}^2 = \|f^T\|_{L^2}^2,$$

The term $\|d_2^* u^T\|_{L^2}^2$ can be bounded uniformly by $\|f^T\|_{L^2}^2$, and therefore,

$$\|\nabla_{A_0}(d_2^* u^T)\|_{L^2(U)}^2 \leq \|D(d_2^* u^T)\|_{L^2}^2 + \sup_x |Ric(x)| \|d_2^* u^T\|_{L^2}^2 \leq (1 + \sup_x |Ric(x)|)\|f^T\|_{L^2}^2.$$

hence,

$$\|u^T\|_{W^{2,2}(U)} \leq C\|f^T\|_{L^2(U)},$$

for a uniform constant C. □

1.2.7.9 The Function Spaces over $M \setminus S_p$

We start by setting up the suitable function spaces over $M \setminus S_p$ by gluing the weighted function spaces defined over $\mathfrak{R}^3$ and the function spaces defined for the longitudinal component and transverse component over U. Then we use these spaces to solve the linearized problem on $M \setminus S_p$.

Let δ_{q_j} and δ_{p_i} be the injectivity radius at the points $q_j \in S_q$ and $p_i \in S_p$, respectively. For each $q_j \in S_q$, let

$$w_j(x) = \begin{cases} \sqrt{\lambda_j^{-2} + r_j^2}, & r_j \leq \delta_{q_j} \\ 1, & r_j \geq 1, \end{cases} \tag{1.2.38}$$

and for each point $p_i \in S_p$, let

$$w_i(x) = \begin{cases} r_i, & r_i \leq \delta_{p_i} \\ 1, & r_i \geq 1, \end{cases} \tag{1.2.39}$$

where r_j and r_i are the geodesic distance from the points q_i and p_i, respectively.

Definition 8. *Let $U_{ext} = M \setminus (\cup_i B_{2\varepsilon_i}(p_i) \cup_j B_{2\varepsilon_j}(q_j))$. For any smooth compactly supported differential form $u \in \Omega_c(M \setminus S, \mathfrak{su}(2))$, let*

$$\|u\|_{L^2_{\alpha_1,\alpha_2}(M \setminus S_p)}^2 = \|u\|_{L^2(U_{ext})}^2 + \sum_{j=1}^{k} \|w_j^{-\alpha_1 - \frac{3}{2}} u\|_{L^2(B_{2\varepsilon_j}(q_j))}^2 + \sum_{i=1}^{n} \|u^T\|_{L^2(B_{2\varepsilon_i}(p_i))}^2 + \sum_{i=1}^{n} \|w_i^{-\alpha_2 - \frac{3}{2}} u^L\|_{L^2(B_{2\varepsilon_i}(p_i))}^2.$$

Furthermore,

$$\|u\|^2_{W^{1,2}_{\alpha_1,\alpha_2}(M\setminus S_p)} = \|u\|^2_{L^2_{\alpha_1,\alpha_2}(M\setminus S_p)} + \|\nabla_{A_0}u\|^2_{L^2_{\alpha_1-1,\alpha_2-1}(M\setminus S_p)} + \sum_{j=1}^{k} \|[\Phi_0, u]\|^2_{L^2_{\alpha_2-1}(B_{2\varepsilon_j}(q_j))}.$$

Moreover,

$$\|u\|^2_{W^{2,2}_{\alpha_1,\alpha_2}(M\setminus S_p)} = \|u\|^2_{W^{1,2}_{\alpha_1,\alpha_2}(M\setminus S)} + \|\nabla_{A_0}(d_2^*u)\|^2_{W^{1,2}_{\alpha_1-2,\alpha_2-2}(M\setminus S_p)} + \sum_{j=1}^{k} \|[\Phi_0, d_2^*u]\|^2_{W^{1,2}_{\alpha_2-2}(B_{2\varepsilon_j}(q_j))}.$$

The spaces $W^{k,2}_{\alpha_1,\alpha_2}$ are defined as the completion of C_0^∞ with respect to the corresponding norms, for $k \in \{0, 1, 2\}$. Furthermore, one can define similar norms and weighted Sobolev spaces $W^{k,p}_{\alpha_1,\alpha_2}$ for any $p \geq 2$.

1.2.7.10 Solving the Linear Equation over $M \setminus S_p$ and the Fixed Point Theorem

In this section, we solve the linear equation for an $\mathfrak{su}(2)$-valued 1-form f on $M \setminus S_p$. This can be done by patching Theorems 3, 5 and 6. The idea is writing $f = \chi_0 f + \sum_{j=1}^{k} \chi_j f$, where χ_j is a cut-off function which is equal to 1 on $B_{2\varepsilon_j}(q_j)$, $\chi_j = 0$ on $M \setminus B_{3\varepsilon_j}(q_j)$, and $\chi_0 + \chi_j = 1$ for all $j \in \{1, \ldots, k\}$. Each equation $d_2 d_2^* u_j = \chi_j f$ can be solved using Theorem 3, and the equation $d_2 d_2^* u_0 = \chi_0 f$, using Theorems 5 and 6.

Then we would glue these solutions to get an approximate solution for the linear equation on $M \setminus S_p$. Using an iteration argument one can see the linear equation has a solution. However, in order to be able to use an iteration argument, we will need more subtle cut-off functions. This is the 3-dimensional version of Lemma 7.2.10 in [19], which is about cut-off functions on 4-dimensional manifolds.

Lemma 39. *There is a constant K and for any N and λ there is a smooth function $\beta = \beta_{N,\lambda}$ on $\mathbb{R}^3$ with $\beta(x) = 1$ where $|x| \leq N^{-1}\lambda^{\frac{1}{2}}$ and $\beta(x) = 0$ where $|x| \geq N\lambda^{\frac{1}{2}}$ such that*

$$\|\nabla\beta\|_{L^3} \leq \frac{K}{(\log(N))^{\frac{2}{3}}}.$$

Proof. This lemma is clearer in cylindrical coordinates. We can transfer the problem to a cylindrical space since the L^p-norm on $\Omega^1(M)$ is conformally invariant if and only if $p = \dim(M)$. Therefore, L^3 is conformally invariant on 1-forms on $\mathbb{R}^3$. Let $L : \mathbb{R}^3 \setminus \{0\} \cong (-\infty, +\infty) \times S^2$, be the identification defined by

$$L(r, \theta, \varphi) = (-\log(r), \theta, \varphi),$$

where (r, θ, φ) denotes the spherical coordinates on $\mathbb{R}^3$.

Let (t,θ,φ) denotes the cylindrical coordinates on $(-\infty,+\infty)\times S^2$. We are looking for a cut-off function

$$\widetilde{\beta}:\mathbb{R}\times S^2\to\mathbb{R},$$

such that

$$\begin{aligned}
&\widetilde{\beta}(t,\theta,\varphi)=0, \quad \text{when} \quad t<-\frac{1}{2}\log(\lambda)-\log(N),\\
&\widetilde{\beta}(t,\theta,\varphi)=1, \quad \text{when} \quad t>-\frac{1}{2}\log(\lambda)+\log(N),\\
&\|\nabla\widetilde{\beta}\|_{L^3}\leq\frac{K}{\log(N)}.
\end{aligned}$$

Moreover, we ask $\widetilde{\beta}$ to be only a function of t,

$$\widetilde{\beta}(t,\theta_1,\varphi_1)=\widetilde{\beta}(t,\theta_2,\varphi_2),$$

for all $(\theta_1,\varphi_1),(\theta_2,\varphi_2)\in S^2$.

One can take $\widetilde{\beta}$ to be a smooth function where

$$\begin{aligned}
&\widetilde{\beta}(t,\theta,\varphi)=0, \quad \text{when} \quad t<-\frac{1}{2}\log(\lambda)-\log(N),\\
&\widetilde{\beta}(t,\theta,\varphi)=1, \quad \text{when} \quad t>-\frac{1}{2}\log(\lambda)+\log(N),\\
&\partial_t\widetilde{\beta}\approx-\frac{1}{2\log(N)}, \quad \text{when} \quad -\frac{1}{2}\log(\lambda)-\log(N)<t<-\frac{1}{2}\log(\lambda)+\log(N).
\end{aligned}$$

Then, we have

$$\|\nabla\widetilde{\beta}\|_{L^3}=\|\partial_t\widetilde{\beta}\|_{L^3}\leq\frac{K}{(\log(N))^{\frac{2}{3}}},$$

for a constant K.

$\square$

Theorem 7. *Let*

$$d_2:W^{1,2}_{\alpha_1-1,\alpha_2-1}(\Omega^1(M\setminus S_p,\mathfrak{g}_P)\oplus\Omega^0(M\setminus S_p,\mathfrak{g}_P))\to L^2_{\alpha_1-2,\alpha_2-2}(\Omega^1(M\setminus S_p,\mathfrak{g}_P)).$$

For any $\alpha_1\in[-\frac{1}{2},0)$ and α_2 outside of a discrete subset, and for sufficiently large $\overline{m}$, the linear equation $d_2\xi=f$ has a unique solution $\xi\in W^{1,2}_{\alpha_1-1,\alpha_2-1}$, for each $f\in L^2_{\alpha_1-2,\alpha_2-2}$. Moreover,

$$\|\xi\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}(M\setminus S_p)}\leq\|f\|_{L^2_{\alpha_1-2,\alpha_2-2}(M\setminus S_p)}.$$

Proof. Let χ_j be the cut-off function, centered at q_j,

$$\chi_j = \begin{cases} 1 & \text{if} \quad dist(x, q_j) \leq 2\varepsilon_j, \\ 0 & \text{if} \quad dist(x, q_j) \geq 3\varepsilon_j, \end{cases}$$

χ_j for $j \in \{1, \ldots, k\}$ such that $|\nabla \chi_j| \leq \frac{2}{\varepsilon_j}$.

Let χ_0 be the cut-off function, centered away from the points $\{q_1, \ldots, q_k\}$,

$$\chi_0 = \begin{cases} 1 & \text{if} \quad dist(x, q_j) \geq 3\varepsilon_j \quad \text{for all } j \in \{1, \ldots, k\} \\ 0 & \text{if} \quad dist(x, q_j) \leq 2\varepsilon_j \quad \text{for some } j \in \{1, \ldots, k\}, \end{cases}$$

such that $\chi_0 + \chi_j = 1$, and $|\nabla \chi_0| \leq \max_j \{2/\varepsilon_j\}$.

Let $f_j = \chi_j f$. For each $j \in \{1, \ldots, k\}$, f_j is supported close to a point q_j, and as explained before, we can localize the problem on $B_{3\varepsilon_j}(q_j)$, and transfer it to $\mathfrak{R}^3$. Note that

$$\|f_j\|_{L^2_{\alpha_1 - 2}} \leq \|f\|_{L^2_{\alpha_1 - 2, \alpha_2 - 2}} < \infty,$$

and therefore, using theorem 3, there are $u_j \in W^{2,2}_{\alpha}$ such that on $B_{3\varepsilon_j}(q_j)$ we have $d_2 d_2^* u_j = f_j$, where

$$\|u_j\|_{W^{2,2}_{\alpha_1}(\mathfrak{R}^3)} \leq C \|f_j\|_{L^2_{\alpha_1 - 2, \alpha_2 - 2}(\mathfrak{R}^3)} \leq C \|f\|_{L^2_{\alpha_1 - 2, \alpha_2 - 2}(M \setminus S_p)}.$$

Let $f_0 = \chi_0 f$. The section f_0 is supported on $M \setminus (\cup_{j=1} B_{2\varepsilon_j}(q_j) \cup S_p)$, and is in $L^2_{\alpha_2 - 2}$. Moreover, it vanishes on $\partial(\cup_j B_{2\varepsilon_j}(q_j))$. Using theorems 5 and 6, there is $u_0 \in W^{2,2}_{\alpha_2}(U)$ such that

$$d_2 d_2^* u_0 = f_0, \qquad \|u_0\|_{W^{2,2}_{\alpha_2}(U)} \leq C \|f_0\|_{L^2_{\alpha_2 - 2}(U)}.$$

Let β_j be the cut-off function introduced in Lemma 39, centered at q_j,

$$\begin{aligned} &\beta_j(x) = 1, \quad \text{if} \quad |x| < N^{-1} \lambda^{\frac{1}{2}}, \\ &\beta_j(x) = 0, \quad \text{if} \quad |x| > N \lambda^{\frac{1}{2}}, \\ &\|\nabla \beta_j\|_{L^3} \leq \frac{K}{(\log(N))^{\frac{2}{3}}}, \end{aligned}$$

for all $j \in \{1, \ldots, k\}$, for a constant $K > 0$, and any $N > 0$.

Let β_0 the cut-off function supported away from the points q_j such that $\beta_0 + \beta_j = 1$. Using these cut-off functions we can transfer the solutions back to M and glue them together. Let

$$F : L^2_{\alpha_1 - 2, \alpha_2 - 2}(\Omega^1(M \setminus S_p, \mathfrak{g}_P)) \to W^{1,2}_{\alpha_1 - 1, \alpha_2 - 1}(\Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P)),$$

be the map defined by

$$F(f) = \sum_{j=0}^{k} \beta_j d_2^* u_j.$$

Note that $F(f) \in W^{1,2}_{\alpha_1-1,\alpha_2-1}$. In fact,

$$\begin{aligned}\|F(f)\|_{L^2_{\alpha_1-1,\alpha_2-1}} &= \|\sum_{j=0}^{k} \beta_j d_2^* u_j\|_{L^2_{\alpha_1-1,\alpha_2-1}} \leq \sum_{j=0}^{k} \|d_2^* u_j\|_{L^2_{\alpha_1-1,\alpha_2-1}} \leq C \sum_{j=0}^{k} \|u_j\|_{W^{1,2}_{\alpha_1,\alpha_2}} \\ &\leq C' \sum_{j=0}^{k} \|f_j\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq (k+1)C'\|f\|_{L^2_{\alpha_1-2,\alpha_2-2}}.\end{aligned}$$

Note that $\beta_j f = \beta_j \xi_j f = \beta_j f_j$, when $\varepsilon_j > 0$ is sufficiently small. Moreover,

$$\begin{aligned}\|\nabla_{A_0} F(f)\|_{L^2_{\alpha_1-2,\alpha_2-2}} &= \|\sum_{j=0}^{k} \nabla_{A_0}(\beta_j d_2^* u_j)\|_{L^2_{\alpha_1-2,\alpha_2-2}} \\ &\leq \sum_{j=0}^{k} (\|\nabla\beta_j \cdot d_2^* u_j\|_{L^2_{\alpha_1-2,\alpha_2-2}} + \|\beta_j \nabla_{A_0}(d_2^* u_j)\|_{L^2_{\alpha_1-2,\alpha_2-2}}) \\ &\leq \sum_{j=0}^{k} (\|\nabla\beta_j\|_{L^3} \|d_2^* u_j\|_{L^6_{\alpha_1-2,\alpha_2-2}} + \|\nabla_{A_0}(d_2^* u_j)\|_{L^2_{\alpha_1-2,\alpha_2-2}}) \\ &\leq \sum_{j=0}^{k} \left(\frac{K}{(\log(N))^{\frac{2}{3}}} + 1 \right) \|d_2^* u_j\|_{W^{1,2}_{\alpha_1-2,\alpha_2-2}} \\ &\leq C \sum_{j=0}^{k} \left(\frac{K}{(\log(N))^{\frac{2}{3}}} + 1 \right) \|f_j\|_{L^2_{\alpha_1-2,\alpha_2-2}} \\ &\leq C(k+1) \left(\frac{K}{(\log(N))^{\frac{2}{3}}} + 1 \right) \|f\|_{L^2_{\alpha_1-2,\alpha_2-2}},\end{aligned}$$

where in the second inequality we have used

$$\|fg\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq \|f\|_{L^3} \|g\|_{L^6_{\alpha_1-2,\alpha_2-2}},$$

which follows from the Hölder's inequality. Also in the third inequality we have used

$$\|h\|_{L^6_{\alpha_1-2,\alpha_2-2}} \leq \|h\|_{W^{1,2}_{\alpha_1-2,\alpha_2-2}},$$

which follows from the Sobolev inequality.

Moreover, we have

$$\|[\Phi, F(f)]\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq \sum_{j=0}^{k} \|[\Phi, d_2^* u_j]\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq \sum_{j=0}^{k} \|d_2^* u_j\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}$$
$$\leq C \sum_{j=0}^{k} \|f_j\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq C(k+1)\|f\|_{L^2_{\alpha_1-2,\alpha_2-2}},$$

and therefore,

$$\|F(f)\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}} \leq C'\|f\|_{L^2_{\alpha_1-2,\alpha_2-2}},$$

for a positive constant C'.

The map F is close to being a right-inverse of d_2.

$$\|f - d_2F(f)\|_{L^2_{\alpha_1-2,\alpha_2-2}} = \|\sum_{j=0}^{k} \nabla\beta_j \cdot d_2^* u_j\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq \sum_{j=0}^{k} \|\nabla\beta_j\|_{L^3}\|d_2^* u_j\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}$$
$$\leq \frac{C}{(\log(N))^{\frac{2}{3}}}\|f\|_{L^2_{\alpha_1-2,\alpha_2-2}}.$$

This shows, the map $(Id - d_2 \circ F)$ is a contraction on $L^2_{\alpha_1-2,\alpha_2-2}$, when N is sufficiently large, and therefore, by the method of iteration, we get a map which is the right-inverse of d_2.

□

The next step is to solve the full non-linear Bogomolny equation.

1.2.8 The Quadratic Term and the Fixed Point Theorem

In this section, we complete the construction of a family of solutions to the Bogomolny equation by the use of a fixed point theorem. In the previous section, we saw that there is a solution to the linearized equation. The remaining part of the equation is quadratic, which we will consider in this section.

Let

$$Q : W^{1,2}_{\alpha_1-1,\alpha_2-1}\left(\Omega^1(M \setminus S_p, \mathfrak{g}_P) \oplus \Omega^0(M \setminus S_p, \mathfrak{g}_P)\right) \to L^2_{\alpha_1-2,\alpha_2-2}\left(\Omega^1(M \setminus S_p, \mathfrak{g}_P)\right),$$

be the map defined by the quadratic part of the Bogomolny equation, given by

$$Q(a,\varphi) = *\frac{[a \wedge a]}{2} - [a,\varphi].$$

The equation 1.1.1 can be written as

$$(d_2 + Q)(a,\varphi) = -e_0.$$

Following Theorem 7, let d_2^{-1} be a right-inverse of d_2.

Let $f = d_2(a, \varphi)$. Let ξ be a solution to the equation $d_2\xi = f$, and therefore, $\xi = (a, \varphi)$. Let

$$q(f) := Q \circ d_2^{-1}(f),$$

and therefore, we have

$$f + q(f) = -e_0.$$

The proof of the existence of the solution to the Bogomolny equation is based on the following lemma from [19].

Lemma 40 (Donaldson-Kronheimer)**.** *Let B be a Banach space and $q : B \to B$ a smooth map such that for all $f, f' \in B$,*

$$\|q(f) - q(f')\| \leq K(\|f\| + \|f'\|)\|f - f'\|, \tag{1.2.40}$$

for a constant K. Then, if $\|e\| \leq \frac{1}{10K}$ there is a unique solution f to the equation

$$f + q(f) = e,$$

where $\|f\| \leq 2\|e\|$.

Following this lemma, let the Banach space $B = L^2_{\alpha_1-2,\alpha_2-2}(\Omega^1(M \setminus S_p, \mathfrak{g}_P))$, and let $e = -e_0$. We should show the assumptions of Lemma 40 holds in our case.

Lemma 41. *The error estimate $\|e_0\|_{L^2_{\alpha_1-2,\alpha_2-2}}$ can be made sufficiently small.*

Proof. Note that $e_0 = 0$ on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. On each $B_{2\varepsilon_j}(q_j)$, following Lemma 12,

$$\|e_0\|_{L^2_{\alpha_1-2}(B_{2\varepsilon_j}(q_j))} = \int_{B_{2\varepsilon_j}(q_j)} w_j^{-2\alpha+1}|e_0|^2 vol_g \leq C \int_{B_{2\varepsilon_j}(q_j)} w_j^{-2\alpha+1} vol_g \leq C\varepsilon_j^4,$$

for a constant $C > 0$, and therefore, it can be made as small as necessary. $\square$

To complete the proof, we should show $q(f) = Q \circ d_2^{-1}(f)$ satisfies 1.2.40 for some K, which is the content of the following lemma.

Lemma 42. *There exists a constant K such that*

$$\|q(f) - q(f')\|_{L^2_{\alpha_1-2,\alpha_2-2}} \leq K\|f + f'\|_{L^2_{\alpha_1-2,\alpha_2-2}}\|f - f'\|_{L^2_{\alpha_1-2,\alpha_2-2}}.$$

Let

$$\tilde{q}(f, f') := \frac{1}{2}(q(f + f') - q(f) - q(f')).$$

The proof of Lemma 42 is based on the following lemma.

Lemma 43. *The map*

$$\tilde{q}: W^{1,2}_{\alpha_1-1,\alpha_2-1}(\Omega^1(M\setminus S_p))\times W^{1,2}_{\alpha_1-1,\alpha_2-1}(\Omega^1(M\setminus S_p))\to L^2_{\alpha_1-2,\alpha_2-2}(\Omega^1(M\setminus S_p)),$$

is continuous. Moreover

$$\|\tilde{q}(f,f')\|_{L^2_{\alpha_1-2,\alpha_2-2}}\leq C\|f\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}\|f'\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}, \tag{1.2.41}$$

for a positive constant C.

Proof of Lemma 42 using 43. We have

$$\begin{aligned}\|q(f)-q(f')\|_{L^2_{\alpha_1-2,\alpha_2-2}}&=\|\tilde{q}(f+f',f-f')\|_{L^2_{\alpha_1-2,\alpha_2-2}}\\&\leq C\|f+f'\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}\|f-f'\|_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}.\end{aligned}$$

□

Proof of Lemma 43. One only need to show

$$\|q(f)\|^2_{L^2_{\alpha_1-2,\alpha_2-2}}\leq C\|f\|^2_{W^{1,2}_{\alpha_1-1,\alpha_2-1}}.$$

The Lemma 43 can be localized over different regions of $M\setminus S_p$. Suppose f is supported on $B_{2\varepsilon_j}(q_j)$ for some $j\in\{1,\ldots,k\}$. Similar to the study of the linearized equation on this region, we can transform the problem to $\mathfrak{R}^3$. We should show

$$\|q(f)\|^2_{L^2_{\alpha_1-2}(\mathfrak{R}^3)}\leq C\|f\|^2_{W^{1,2}_{\alpha_1-1}(\mathfrak{R}^3)}.$$

By the Hölder's inequality, when λ_j is sufficiently large,

$$\begin{aligned}\|q(f)\|^2_{L^2_{\alpha_1-2}(\mathfrak{R}^3)}&=\int_{\mathfrak{R}^3}w_j^{-2\alpha_1+1}|q(f)|^2vol_g\leq C_1\varepsilon_j^{2\alpha_1}\int_{\mathfrak{R}^3}w_j^{-4\alpha_1+1}|f|^4vol_g\\&\leq C_1\varepsilon_j^{2\alpha_1}\|w_j^{-\alpha_1-\frac{1}{2}}f\|_{L^2}\|w_j^{-\alpha_1+\frac{1}{2}}f\|^3_{L^6},\end{aligned}$$

for a positive constant C_1.

By the Sobolev inequality we have

$$\|w_j^{-\alpha_1+\frac{1}{2}}f\|_{L^6}\leq C_{Sob}\|w_j^{-\alpha_1+\frac{1}{2}}f\|_{W^{1,2}}\leq C_2\|f\|_{W^{1,2}_{\alpha_1-2}}\leq C_2\|f\|_{W^{1,2}_{\alpha_1-1}},$$

for a positive constant C_2, and therefore,

$$\|q(f)\|^2_{L^2_{\alpha_1-2}(\mathfrak{R}^3)}\leq C_1C_2\varepsilon_j^{2\alpha_1}\|f\|^4_{W^{1,2}_{\alpha_1-1}},$$

hence on this region, 1.2.41 holds with

$$C = C_1 C_2 \varepsilon_j^{2\alpha_1}.$$

Second, suppose f is supported on $U = M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup S_p)$. Let $f = f^L + f^T$.

$$q(f) = \tilde{q}(f, f) = \tilde{q}(f^L + f^T, f^L + f^T) = \tilde{q}(f^L, f^L) + \tilde{q}(f^L, f^T) + \tilde{q}(f^T, f^L) + \tilde{q}(f^T, f^T).$$

We have

$$\tilde{q}(f^L, f^L) = q(f^L) = Q \circ d_2^{-1}(f^L).$$

The linear maps d_2 and d_2^{-1}, preserve the bundle decomposition induced by a Higgs fields Φ_0, and therefore, $(a_1^T, \varphi_1^T) = d_2^{-1}(f^L)$ is a section of longitudinal part. However, the Lie bracket vanishes when restricted to the longitudinal sub-bundle,

$$Q \circ d_2^{-1}(f^L) = 0,$$

and therefore,

$$q(f) = (\tilde{q}(f^L, f^T) + \tilde{q}(f^T, f^L)) + \tilde{q}(f^T, f^T),$$

where $(\tilde{q}(f^L, f^T) + \tilde{q}(f^T, f^L))$ is the transverse component and $\tilde{q}(f^T, f^T)$ is the longitudinal one.

For the transverse component 1.2.41 becomes

$$\|\tilde{q}(f^L, f^T)\|_{L^2(U)} \leq C \|f^L\|_{W^{1,2}_{\alpha_2 - 1}(U)} \|f^T\|_{W^{1,2}(U)}. \tag{1.2.42}$$

By the Hölder's inequality, we have

$$\begin{aligned} \|\tilde{q}(f^L, f^T)\|_{L^2(U)} &\leq C \|f^L\|_{L^3} \|f^T\|_{L^6} \leq C \|w^{-\alpha_2 - \frac{1}{2}} f^L\|_{L^3} \|f^T\|_{W^{1,2}} \\ &\leq C \|w^{-\alpha_2 - \frac{1}{2}} f^L\|_{W^{1,2}} \|f^T\|_{W^{1,2}} \leq C \|f^L\|_{W^{1,2}_{\alpha_2 - 1}} \|f^T\|_{W^{1,2}}, \end{aligned}$$

where C is a uniform constant.

For the longitudinal component 1.2.41 becomes

$$\|q(f^T)\|_{L^2_{\alpha_2 - 2}(U)} \leq C \|f^T\|^2_{W^{1,2}(U)}. \tag{1.2.43}$$

By the Hölder's and Sobolev inequalities we have

$$\begin{aligned} \|q(f^T)\|_{L^2_{\alpha_2 - 2}(U)} &= \|w^{-\alpha_2 + \frac{1}{2}} q(f^T)\|_{L^2(U)} \leq C \|w^{-\alpha_2 + \frac{1}{2}} f^T\|_{L^3} \|f^T\|_{L^6} \\ &\leq C \|f^T\|_{L^3} \|f^T\|_{L^6} \leq C \|f^T\|^2_{W^{1,2}}, \end{aligned}$$

for a uniform constant C.

□

This completes the gluing construction of irreducible $SU(2)$-monopoles with Dirac singularities on rational homology 3-spheres.

1.3 Monopoles on Asymptotically Cylindrical 3-Manifolds

In this section, we address the problem of the existence of monopoles, both smooth and singular, on asymptotically cylindrical 3-manifolds. However, since the problem is very similar to the compact case, we will be very brief.

Let (M, g) be an asymptotically cylindrical 3-manifold. Let (Σ, g_Σ) be the cross-section of the 3-manifold at infinity, where $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_l$ and $l = b_0(\Sigma)$. Let $t_i : M \to \mathbb{R}$ be a radius function on the ith end of the manifold.

There are two interesting classes of monopoles to consider on asymptotically cylindrical 3-manifolds.

- **Singular Monopoles.** Monopoles with Dirac singularities and asymptotic conditions

$$\lim_{t_i \to \infty} |d_A \Phi| = 0, \qquad \lim_{t_i \to \infty} |\Phi| = M_i, \tag{1.3.1}$$

  where $M_i > 0$ is a constant, called the mass of the monopole on the ith end.

- **Smooth Monopoles.** Smooth monopoles (without singularities) such that on each end of the manifold

$$\lim_{t_i \to \infty} |\Phi| = \frac{K_i}{\mathrm{Vol}(\Sigma_i)} t_i + M_i + O(t_i^{-1}), \tag{1.3.2}$$

  where $K_i \in \mathbb{N}$ and $M_i > 0$ are called the charge of the monopole and the mass of the monopole on the ith end, respectively.

A natural condition to consider is to ask the monopole to have finite Yang-Mills-Higgs energy. This would imply

$$F_A \to 0, \qquad d_A \Phi \to 0, \qquad |\Phi| \to M_i, \text{ as } t_i \to \infty. \tag{1.3.3}$$

This assumption implies the asymptotic conditions 1.3.1. Similar to the case of monopoles on closed 3-manifolds, smooth monopoles on asymptotically cylindrical 3-manifolds with the asymptotic condition 1.3.3 satisfy stronger conditions.

Lemma 44. *Let (M, g) be a asymptotically cylindrical Riemannian 3-manifold. Let G be a compact Lie group. Any smooth monopole (A, Φ) on a principal G-bundle $P \to M$ with the asymptotic condition 1.3.3 satisfies the following equations,*

$$*F_A = d_A \Phi = 0. \tag{1.3.4}$$

Proof. Let $t : M \to (0, \infty)$ denote the distance from a fixed point $x \in M$. Let $M_T = t^{-1}(0, T]$ for any $T \in \mathbb{R}^+$. The Bogomolny equation implies

$$\Delta_A \Phi = d_A^* d_A \Phi = *d_A F_A = 0, \tag{1.3.5}$$

and therefore,

$$0 = \langle \Delta_A \Phi, \Phi \rangle_{M_T} = \langle d_A^* d_A \Phi, \Phi \rangle_{M_T} = \int_{M_T} |d_A \Phi|^2 vol_g - \int_{\partial M_T} (*d_A \Phi)\Phi.$$

As we take limit $T \to \infty$, the integrand $(*d_A\Phi)\Phi \to 0$, and the area of the slice ∂M_T converges to a constant, and therefore,

$$0 = \langle \Delta_A \Phi, \Phi \rangle_M = \|d_A \Phi\|^2_{L^2(M)},$$

thus, $*F_A = d_A\Phi = 0$. □

Note that with the asymptotic conditions 1.3.1 the total charge of a singular monopole at the Dirac singularities vanishes.

Lemma 45. *Let (M, g) be a asymptotically cylindrical Riemannian 3-manifold. Let (A, Φ) be a monopole with Dirac singularites on a principal $U(1)$-bundle $P \to M \setminus \{p_1, \ldots, p_n\}$ with the asymptotic condition*

$$\lim_{t_i \to \infty} |\Phi(x)| = M_i + O(t_i^{-1}), \tag{1.3.6}$$

and Dirac singularities with charge $k_i \in \mathbb{N}$ at p_i for $i \in \{1, \ldots, n\}$. Then

$$\sum_{i=1}^{n} k_i = 0. \tag{1.3.7}$$

Proof. Let $t : M \to \mathbb{R}$ denote the distance from a fixed point $x_0 \in M$. Let $M_T = t^{-1}[0, T]$ for any $T \in \mathbb{R}^+$. The Bogomolny equation implies

$$\Delta_A \Phi = d_A^* d_A \Phi = *d_A F_A = 0, \tag{1.3.8}$$

on $M \setminus \{p_1, \ldots, p_n\}$, and therefore, by the Stokes' theorem and for any $T > 0$, we have

$$0 = \int_{M_T \setminus \cup_i \{p_i\}} (\Delta_A \Phi) vol_g = \sum_{i=1}^{n} \int_{\partial B_\varepsilon(p_i)} *d\Phi - \int_{t^{-1}(T)} *d\Phi = 2\pi \sum_{i=1}^{n} k_i - \int_{t^{-1}(T)} *d\Phi.$$

On the other hand, the asymptotic condition 1.3.6 on an asymptotically cylindrical manifold implies

$$\lim_{T \to \infty} \int_{t^{-1}(T)} *d\Phi = 0,$$

and therefore, $\sum_{i=1}^{n} k_i = 0$. □

In fact, as mentioned earlier we can consider a more generalized asymptotic conditions 1.3.2. The asymptotic conditions 1.3.3 imply $K_i = 0$. With the asymptotic conditions 1.3.2 some of the charges can run into infinity. In this case, the sum of the charges at the singular points and at the ends of the manifold vanishes, with the similar proof as the lemma above.

Singular Monopoles on Asymptotically Cylindrical 3-Manifolds. Let (M, g) be an oriented asymptotically cylindrical Riemannian 3-manifold with $b_2(M) = 0$. For any $k \in \mathbb{N}$, there exists an irreducible monopole with Dirac singularities with charge k and asymptotic conditions 1.3.1 on a principal $SU(2)$-bundle $P \to M \setminus \{p_1, \ldots, p_n\}$.

The proof of this statement follows the same line of thought as the existence theorem of singular monopoles on closed 3-manifolds. Since the proof is similar to the closed case, we only sketch the proof here without going into the details.

- The first step is to produce an Abelian Dirac monopole with singularities on (M, g), where
$$\lim_{t_i(x)\to\infty} |\Phi(x)| = M_i + O(t_i^{-1}), \tag{1.3.9}$$
with Dirac singularities at points $p_1, \ldots, p_n$ with integer charges $k_1, \ldots, k_n$ such that $\sum_{i=1}^{n} k_i = 0$ such that some of them are of charge $+1$. The proof is similar to the construction of the Dirac monopoles on closed 3-manifolds, as we saw in section 1.2.2.

- The second step is to smooth out some of the singularities with charge $+1$ by gluing scaled BPS $SU(2)$-monopoles and leave out the rest of the singular points not smoothed-out to construct an approximate singular monopole similar to what we did in section 1.2.4 for approximate monopoles over closed 3-manifolds.

- The third step is the deformation. The resulting configuration from the step two is an approximate monopole and it does not necessarily satisfy the Bogomolny equation, but in a suitable norm, it is close to a solution and should be deformed into a genuine monopole. To solve the linear problem, we should combine the weighted Sobolev spaces we defined earlier with the Lockhart-McOwen Sobolev spaces on asymptotically cylindrical manifolds.

Smooth Monopoles on Asymptotically Cylindrical 3-Manifolds. Let (M, g) be an oriented asymptotically cylindrical Riemannian 3-manifold with $b_2(M) = 0$. For any $k \in \mathbb{N}$, there exists a smooth irreducible monopole with charge k and asymptotic conditions 1.3.2 on a principal $SU(2)$-bundle $P \to M$.

The proof is similar to the previous gluing constructions we studied in this chapter. In order to prove the existence of smooth monopoles,

- The first step is to produce an Abelian Dirac monopole with singularities on (M, g), where
$$\lim_{t_i(x)\to\infty} |\Phi| = \frac{K_i}{Vol(\Sigma_i)} t_i + M_i + O(t_i^{-1}), \tag{1.3.10}$$
with Dirac singularities at points $p_1, \ldots, p_n$ with charge $+1$ such that
$$n - \sum_{i=1}^{b_0(\Sigma)} K_i = 0.$$

- The second step is to smooth out all of the singular points $p_1, \dots, p_n$ with charge $+1$ by gluing scaled BPS $SU(2)$-monopoles and construct an approximate monopole.
- The third step is deforming the approximate monopole to a genuine one.

1.3.1 Monopoles and Connected Sum

A relevant problem to our gluing argument is the construction of monopoles on connected sum manifolds. Let (M_1, g_1) and (M_2, g_2) be two compact oriented Riemannian 3-manifolds with $b_2(M_i) = 0$ for $i \in \{1, 2\}$. Let $P_1 \to M_1$ and $P_2 \to M_2$ be two principal $SU(2)$-bundles. Let (A_i, Φ_i) be a singular $SU(2)$-monopole on P_i for $i \in \{1, 2\}$. In this section, very briefly, we address the problem of constructing a monopole on the connected sum $M_1 \# M_2$ using the monopoles (A_1, Φ_1) and (A_2, Φ_2).

Suppose (A_1, Φ_1) and (A_2, Φ_2) are two monopoles with Dirac singularities and large mass, similar to the ones we constructed in this chapter. Suppose $p_i \in M_i$ is a point which is a Dirac singularity of (A_i, Φ_i) for $i \in \{1, 2\}$. Moreover, assume these monopoles have a common signed charge k at these points. By adding a constant, let's assume the mass of (A_1, Φ_1) at p_1 is equal to the mass of (A_2, Φ_2) at p_2, where this common value is denoted by m. One can construct a monopole (A, Φ) on a principal $SU(2)$-bundle $P \to M_1 \# M_2$ which, up to gauge, is close to (A_i, Φ_i) on $P_i \to M_i$ for $i \in \{1, 2\}$.

Suppose an identification of a small ball around p_1 with a small ball around p_2 is fixed,

$$f : B_\varepsilon(p_1) \to B_\varepsilon(p_2),$$

such that $f(p_1) = p_2$.

Moreover, suppose a bundle identification above these open balls is chosen such that it covers $f : B_\varepsilon(p_1) \to B_\varepsilon(p_2)$. This can be used to construct a connected sum bundle $P \to M_1 \# M_2$. On this small open ball, we have

$$|(A_1, \Phi_1) - (A_2, \Phi_2)| = O(me^{-mr}),$$

where r denotes the geodesic distance from the point $p_1 = p_2$.

We can define the approximate connected sum monopole by letting

$$(A, \Phi) = \xi_1(A_1, \Phi_1) + \xi_2(A_2, \Phi_2),$$

where ξ_1 and ξ_2 are the functions where

$$\xi_i = \begin{cases} 1 & \text{on} \quad M \setminus B_\varepsilon(p_i) \\ \frac{1}{2} & \text{on} \quad B_{\varepsilon/2}(p_i), \end{cases}$$

and $\xi_1 + \xi_2 = 1$.

This is an approximate connected sum monopole with Dirac singularities on $M_1 \# M_2$. Let $e_0 = *F_A - d_A \Phi$. By increasing the mass m, we can make e_0 sufficiently small and satisfy the assumption of our gluing method, which we used to construct monopoles on rational homology

3-spheres, and therefore, this approximate singular monopole can be deformed to a genuine singular monopole on the connected sum space.

The picture is clearer over asymptotically cylindrical manifolds. Let (A_i, Φ_i) be a singular $SU(2)$-monopole on an asymptotically cylindrical 3-manifold (M_i, g_i) for $i \in \{1, 2\}$ with the asymptotic conditions 1.3.1, with the common mass m at infinity. One can follow the same line of thought to construct a monopole on the manifold with long neck constructed by gluing M_1 and M_2. In this case, rather than increasing the mass m, one can extend the neck of the base manifold to decrease the error term.

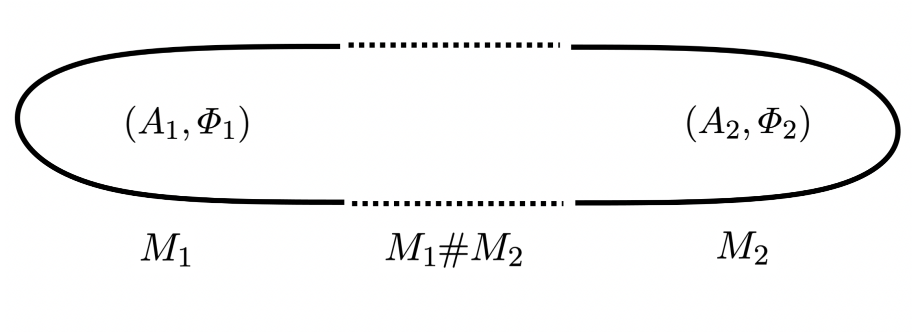

Figure 1.2: Connected Sum Monopole

1.4 Monopoles with Singularities Along Knots

One can study solutions to the Bogomolny equation on 3-manifolds with other types of singularities. One interesting example has been studied by Sun [84], where he investigates monopoles on $\mathbb{R}^3$ with singularity along a knot $K \subset \mathbb{R}^3$. In this section, we investigate a framework to generalize this idea to study monopoles with knot singularity on closed 3-manifolds.

Sun studied pairs (A_K, Φ_K), defined on a principal $SU(2)$-bundle $P \to \mathbb{R}^3 \setminus K$, which satisfy the Bogomolny equation, and the connection A has monodromy γ around K. Suppose an arc length parametrization of the knot K is fixed, which by an abuse of notation, it is denoted by $K : [0, l] \to \mathbb{R}^3$, where $l = \text{length}(K)$. Let N_K be the normal bundle of the knot K in $\mathbb{R}^3$ and $e_1, e_2 : [0, l] \to N_K$ an orthonormal frame of N_K. The points on a tubular neighbourhood of the knot K can be parametrized by

$$(s, x, y) \to K(s) + xe_1(s) + ye_2(s) \in \mathbb{R}^3. \tag{1.4.1}$$

On a tubular neighbourhood of the knot K, the model singular solution (A, Φ) is given by

$$(A_K, \Phi_K) = (\gamma\sigma d\theta, m\sigma), \tag{1.4.2}$$

where $\sigma \in \mathfrak{su}(2)$, $d\theta$ is only defined on an ε-neighbourhood of the knot for sufficiently small ε by the equation $e^{i\theta} = x + iy$, and m is a constant.

By the monopoles with singularity along a knot K, we refer to monopoles defined on the complement of K, which on a small neighbourhood of K, they agree with the model solution

1.4.2 up to higher order terms in r, where r denotes the geodesic distance from K.

Sun extends this monopole to the entire $\mathbb{R}^3 \setminus K$ with different charges by gluing some BPS-monopoles to (A_K, Φ_K). Moreover, he shows a neighbourhood of a solution to the Bogomolny equations on $\mathbb{R}^3$ with this prescribed knot singularity in the moduli space of these monopoles has an analytical structure.

Similar to Sun's work, one would hope to study solutions to the Bogomolny equation on other Riemannian 3-manifolds with singularity along a knot, or more generally, a link. An interesting case is when this 3-manifold is closed. However, a monopole with a knot singularity on a closed 3-manifold has a vanishing curvature.

Theorem 8. *Let (M, g) be a closed, oriented, Riemannian 3-manifold. Let G be a compact Lie group. Let (A_K, Φ_K) be a monopole on a principal G-bundle $P \to M \setminus K$ with the model knot singularity 1.4.2 along a knot $K \subset M$. Then on $M \setminus K$ the pair (A_K, Φ_K) satisfies the stronger equations*

$$*F_{A_K} = d_{A_K}\Phi_K = 0.$$

Proof. On the complement of the knot $\Delta_{A_K}\Phi_K = 0$. We have

$$\lim_{r\to 0} |\Phi_K| = m,$$

where m is a positive constant and r is the geodesic distance from the knot, and therefore,

$$0 = \langle \Delta_{A_K}\Phi_K, \Phi_K\rangle_{L^2(M\setminus N_K(\varepsilon))} = \|d_{A_K}\Phi_K\|^2_{L^2(M\setminus B_\varepsilon(K))} - \int_{\partial B_K(\varepsilon)} *(d_{A_K}\Phi_K)\Phi_K,$$

where $\partial N_K(\varepsilon)$ is the boundary of the ε-tubular neighbourhood of the knot.

However, as $\varepsilon \to 0$, both the area of $\partial B_K(\varepsilon)$ and $*(d_{A_K}\Phi_K)\Phi_K$ converge to zero, and therefore, by taking limit and letting $\varepsilon \to 0$, we have

$$0 = \langle \Delta_{A_K}\Phi_K, \Phi_K\rangle_{L^2(M\setminus K)} = \|d_{A_K}\Phi_K\|^2_{L^2(M\setminus K)},$$

hence, $*F_{A_K} = d_{A_K}\Phi_K = 0$. □

This shows monopoles with knot singularity on closed 3-manifolds are essentially just flat connections with singularity along a knot. Flat connections on knot complements can be used to study knots and links. Using the $SU(2)$-flat connections on 3-manifolds with knot singularities, Kronheimer and Mrowka defined an instanton Floer homology for knots in 3-manifolds [62]. In order to get monopoles on closed 3-manifolds with knot singularities and with non-flat connections, one can consider monopoles with mixed singularities.

Definition 9 (Monopoles with Mixed Singularities)**.** *Let (M, g) be an oriented, closed, Riemannian 3-manifold. Let $p_1, \ldots, p_n$ be n distinct points in M with corresponding positive charges $k_1, \ldots, k_n$. Let $K \subset M$ be a knot. Let G be a Lie group. Let $P \to M \setminus (K \cup \{p_1, \ldots, p_n\})$ be a principal G-bundle. Let A be a connection on P and Φ a section of the adjoint bundle. We call a pair (A, Φ) a monopole with mixed singularities if*

- (A, Φ) *satisfies the Bogomolny equation on* $M \setminus (K \cup \{p_1, \ldots, p_n\})$.
- (A, Φ) *has a Dirac singularity with charge* k_i *at* p_i.
- (A, Φ) *has a knot singularity along* K.

We expect to have monopoles with mixed singularities on closed 3-manifolds.

Conjecture 1 (Existence of Monopoles with Mixed Singularities)**.** *Let* (M, g) *be a closed, oriented, Riemannian 3-manifold with* $b^2(M) = 0$. *Let* $p_1, \ldots, p_n$ *be* n *points in* M *with corresponding charges* $k_1, \ldots, k_n$. *Let* $K \subset M$ *be a knot in* M. *We expect to have a irreducible non-trivial monopole on a principal* $SU(2)$*-bundle* $P \to M \setminus (K \cup \{p_1, \ldots, p_n\})$ *with Dirac singularities with charge* k_i *at* p_i *and knot singularity along* K.

We expect it would be possible to solve this conjecture using a gluing construction. For simplicity suppose there exists a small open ball $B \subset M$ where the Riemannian metric g restricted to B is flat and $K \subset B$. The analysis of the Bogomolny equation around the knot, is similar to the Euclidean case, which is carried over by Sun. Let (A_K, Φ_K) be the model solution with singularity along K defined on $B \subset M$,

$$(A_K, \Phi_K) = (\gamma\sigma d\theta, m\sigma).$$

Let $p_1, \ldots, p_n$ be well-separated points in M away from the knot K with corresponding negative integers charges $-k_1, \ldots, -k_n$ and another set of points $q_1, \ldots, q_k$ with charges equal to $+1$ such that $k - \sum_{i=1}^{n} k_i = 0$.

Let (A_D, Φ_D) be a monopole with Dirac singularities at the points p_i with corresponding charges k_i. Moreover, by adding a constant if necessary, we can assume the average mass of this Dirac monopole on the small ball B is a sufficiently large number m. Let (A_j, Φ_j) be the appropriately scaled BPS-monopoles transformed to a neighbourhood of the points q_j, similar to what we did in the case of singular monopoles on closed 3-manifolds.

We can define the approximate monopole with mixed singularities by

$$(A_0, \Phi_0) = \xi_0(A_D, \Phi_D) + \sum_{j=1}^{k} \xi_j(A_j, \Phi_j) + \xi_K(A_K, \Phi_K), \tag{1.4.3}$$

for suitable cut-off functions ξ_0, ξ_K and ξ_j for $j \in \{1, \ldots, k\}$, where

$$\xi_0 + \xi_j = 1, \qquad \xi_0 + \xi_K = 1.$$

This pair has the prescribed singularity on a neighbourhood of the knot and the singular points, and it is a monopole on $M \setminus (\cup_j B_{2\varepsilon_j}(q_j) \cup B_{2\varepsilon}(K) \cup S_p)$ where $S_p = \{p_1, \ldots, p_n\}$. However, similar to the previous gluing constructions we studied, this pair is not a monopole on a neighbourhood of the points $q_1, \ldots, q_k$, and more generally, when the metric on B is not flat, it is also not necessarily a monopole on the neighbourhood of the knot K.

The next step is to deform this approximate monopole with mixed singularities to a genuine one. To do this, we should define the appropriate Sobolev spaces to set up the deformation problem. We expect this would be possible by combining the function spaces we defined in the previous sections, in the study of monopoles with Dirac singularities on closed 3-manifolds and the function spaces Sun defined to study the deformation of monopoles on $\mathbb{R}^3$ with knot singularities.

Chapter 2

Monopoles in Higher Dimensions

An interesting feature of the Bogomolny equation is that it can be generalized to certain higher-dimensional spaces. The most interesting examples appear on Calabi-Yau 3-folds and G_2-manifolds. It is proposed by Donaldson and Segal that one can define invariants of non-compact Calabi-Yau 3-folds and G_2-manifolds with suitable ends, by a count of monopoles [21].

In this chapter, we study these higher-dimensional Bogomolny equations and show their relevance to the 3-dimensional monopoles and Fueter operators which are non-linear generalizations of the Dirac operator. Furthermore, we introduce a monopole equation on $U(1)$-bundles over 4-dimensional hyperkähler manifolds. Moreover, we introduce the complexification of gauge-theoretic equations on manifolds with special holonomy group, and study their basic properties.

2.1 Preliminaries: Gauge Theory on Manifolds with Special Holonomy Groups

In this section, we recall the basic definitions and results about the geometry of manifolds with special holonomy groups, more specifically, Calabi-Yau 3-folds, G_2- and $Spin(7)$-manifolds. Moreover, we review the instanton and the Bogomolny equations on these manifolds. Our review is quite brief. For a more detailed account on manifolds with special holonomy groups you can consult with the book by Dominic Joyce [53]. This book mainly focuses on the construction of Riemannian metrics with exceptional holonomy groups on compact manifolds. For more detailed introduction to gauge theory on manifolds with special holonomy groups, you can see [22, 21, 98, 77].

2.1.1 Manifolds with Special Holonomy Groups

The classification of manifolds based on their holonomy groups has a long history, which goes back to the works of Élie Cartan. In the case of symmetric Riemannian manifolds, the holonomy groups can be classified completely using the theory of Lie groups, as done by Cartan in 1925 [13]. In the case of non-symmetric spaces, 30 years later, the celebrate theorem of Berger illustrated the situation.

Theorem 9 (Berger [8])**.** *Suppose (M, g) is a simply-connected, irreducible, non-symmetric, n-dimensional Riemannian manifold. Then exactly one of the following seven cases holds.*

(i) ***Generic.*** *Hol$(g) = SO(n)$.*

(ii) ***Kähler.*** *$n = 2m$ with $m \geq 2$ and Hol$(g) = U(m) \subset SO(2m)$.*

(iii) ***Calabi-Yau.*** *$n = 2m$ with $m \geq 2$ and Hol$(g) = SU(m) \subset SO(2m)$.*

(iv) ***Hyperkähler.*** *$n = 4m$ with $m \geq 2$ and Hol$(g) = Sp(m) \subset SO(4m)$.*

(v) ***Quaternionic Kähler.*** *$n = 4m$ with $m \geq 2$ and Hol$(g) = Sp(m)Sp(1) \subset SO(4m)$.*

(vi) ***G_2.*** *$n = 7$ and Hol$(g) = G_2 \subset SO(7)$.*

(vii) ***Spin(7).*** *$n = 8$ and Hol$(g) = Spin(7) \subset SO(8)$.*

It was recognized by Hitchin that all manifolds with covariantly constant spinors appear in the list of Berger. More specifically, they are Calabi-Yau, hyperkähler, 7-dimensional G_2- or 8-dimensional $Spin(7)$-manifolds [44].

Manifolds with special holonomy groups also play an important role in String Theory and M-Theory. As observed by Candelas, Horowitz, Strominger and Witten, in the 10-dimensional spacetime of String Theory, Calabi–Yau manifolds are the spaces that satisfy the requirement for being the six unseen spatial dimensions. You can find more about this in the expository account by S.T. Yau [100]. In M-theory, where the spacetime is 11-dimensional, G_2-manifolds can be the hidden 7-dimensional space [6].

2.1.1.1 Calabi-Yau Manifolds

We start with the definition of Calabi–Yau manifold.

Definition 10 (Calabi-Yau n-Fold)**.** *A Calabi–Yau n-fold is a quadruple (M, g, ω, Ω) where (M, g, ω) is a $2n$-dimensional Kähler manifold with the compatible integrable almost complex structure J and with a nonzero $(n, 0)$-form Ω on M, called the holomorphic volume form, which satisfies*

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} (\frac{i}{2})^n \Omega \wedge \overline{\Omega}.$$

There are a distinguished class of real n-dimensional submanifolds of Calabi-Yau n-folds, called special Lagrangians. They are calibrated with respect to the real n-form $\mathrm{Re}(\Omega)$, in the sense of Harvey and Lawson [40]. They are important in String Theory, and also play a key role in the mathematical theory of Mirror Symmetry [35]. Moreover, they are closely related to the gauge theory on Calabi-Yau manifolds. We will discuss calibrated submanifolds, and more specially, special Lagrangians in much more detail in Chapter 4.

Definition 11 (Calibration)**.** *Let (M, g) be an n-dimensional Riemannian manifold. Let $\alpha \in \Omega^k(M)$ be a differential k-form for some $0 \leq k \leq n$. The k-form α is called a calibration on M if*

- *α is closed: $d\alpha = 0$;*
- *for any $x \in M$ and any oriented k-dimensional subspace $V \subset T_xM$, we have*

$$\alpha_{|_V} = \lambda vol_V \quad \textit{with} \quad \lambda \leq 1,$$

where vol_V is the volume form of V defined with respect to the restriction of the Riemannian metric g to V.

A k-dimensional submanifold $N \subset M$ is called a calibrated submanifold with respect to the calibration α, if

$$\alpha_{T_xN} = vol_{T_xN},$$

for all $x \in N$.

Definition 12 (Special Lagrangian)**.** *Let (M, g, ω, Ω) be a Calabi-Yau n-fold. $\mathrm{Re}(\Omega)$ is a calibration, and calibrated submanifolds with respect to $\mathrm{Re}(\Omega)$ are called special Lagrangians.*

The following lemma explains why these submanifolds are called special Lagrangians.

Lemma 46 (Special Lagrangian Equations [40])**.** *Let L be a real n-dimensional submanifold of a Calabi-Yau n-fold (M, g, ω, Ω). Then L admits an orientation making it a special Lagrangian submanifold if and only if*

$$\omega_{|_L} = 0, \quad \textit{and} \quad \mathrm{Im}(\Omega)_{|_L} = 0.$$

The condition $\omega_{|_L} = 0$ states L is a Lagrangian submanifold and $\mathrm{Im}(\Omega)_{|_L} = 0$ asserts it is a special one.

2.1.1.2 G_2-Manifolds

G_2-manifolds provide another class of manifolds which one can study a Bogomolny type equation on. What makes dimension 7 stand out is that it is the only dimension other than 3, where it can be equipped with a cross product. Moreover, on $\mathbb{R}^7$, similar to $\mathbb{R}^3$, one can use the cross product and the Riemannian metric to define a triple product.

Definition 13 (Associative Form)**.** *On $\mathbb{R}^7$ one can use the cross and inner products to define a map*

$$\phi_0 : \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R} : (u, v, w) \to g(u \times v, w).$$

ϕ_0 is an alternating 3–form, called the associative form, which can be written as

$$\phi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$, and $e^1, \ldots, e^7$ is the dual basis for $e_1, \ldots, e_7$.

The exceptional group G_2 can be defined as the stabilizer of this 3-form.

Definition/Lemma 47 (Exceptional Group G_2). *G_2 is the subgroup of $GL(7,\mathbb{R})$ that preserves ϕ_0,*

$$G_2 := \{g \in GL(7,\mathbb{R}) \mid g^*\phi_0 = \phi_0\}.$$

G_2 is a compact, connected, simply connected, 14-dimensional, simple Lie group. Moreover, $G_2 \subset SO(7)$.

ϕ_0 is non-degenerate in the following sense.

Definition 14 (Non-Degenerate 3-Form). *A 3–form ϕ on a 7–dimensional vector space V is called non-degenerate or positive if for each $v \in V \setminus \{0\}$ the induced 2–form $i_v\phi$ on $V/\langle v\rangle$ is a symplectic form.*

A 3-form $\phi \in \Omega^3(M)$ is called a G_2-structure if at each point $x \in M$ it looks like the model ϕ_0 on $\mathbb{R}^7$.

Definition 15 (G_2-Structure). *Let M be a smooth 7-dimensional manifold. A G_2-structure on M is a non-degenerate 3-form $\phi \in \Omega^3(M)$ such that at every $p \in M$ there exists a linear isomorphism $T_pM \cong \mathbb{R}^7$ with respect to which $\phi \in \Lambda^3(T_p^*M)$ corresponds to $\phi_0 \in \Lambda^3(\mathbb{R}^7)^*$. The 3-form ϕ induces a Riemannian metric and an orientation on M such that*

$$\iota_u\phi \wedge \iota_u\phi \wedge \phi = -6g_\phi(u,v)vol_\phi.$$

We can define the coassociative 4-form by

$$\psi = *_{g_\phi}\phi.$$

We suppress the subscript ϕ of the Riemannian metric, volume form and the Hodge star operator, when there is no fear of confusion.

The following theorem shows there are many 7-dimensional manifolds which admit a G_2-structure.

Lemma 48 (Existence of G_2-Structure [34]). *A smooth 7-dimensional manifold M admits a G_2-structure if and only if M is both orientable and spinnable, and therefore, if and only if the first and second Stiefel-Whitney classes vanish.*

A G_2-structure ϕ on M induces a Riemannian metric g, which uniquely determines a Levi-Civita connection ∇ on M. Using this connection one can define the torsion of the G_2-structure.

Definition 16 (G_2-Manifold). *A G_2-manifold (M,ϕ) is a 7-dimensional manifold with a positive G_2-structure ϕ such that the torsion of the G_2-structure $T(\phi) := \nabla_{g_\phi}\phi = 0$; i.e., ϕ is parallel. By a theorem of Fernández and Gray [27], a G_2-structure is torsion-free if and only if $d\phi = 0 = d\psi$. In this case, $Hol(g_\phi) \subseteq G_2 \subset SO(7)$.*

On a 7-dimensional manifold M with a G_2-structure ϕ, there are orthogonal decompositions of the bundles Λ^kT^*M into irreducible representations of G_2, which induces decomposition of the spaces of differential k-forms.

Lemma 49. *Let V be a 7-dimensional vector space equipped with a non-degenerate 3-form ϕ. Then $\Lambda^* V^*$ splits into irreducible representations of G_2 as follows,*

$$\Lambda^1(V) = \Lambda^1_7(V), \qquad \Lambda^2(V) = \Lambda^2_7(V) \oplus \Lambda^2_{14}(V),$$
$$\Lambda^3(V) = \Lambda^3_1(V) \oplus \Lambda^3_7(V) \oplus \Lambda^3_{27}(V), \qquad \Lambda^4(V) = \Lambda^4_1(V) \oplus \Lambda^4_7(V) \oplus \Lambda^4_{27}(V),$$
$$\Lambda^5(V) = \Lambda^5_7(V) \oplus \Lambda^5_{14}(V), \qquad \Lambda^6(V) = \Lambda^6_7(V),$$

where the indices denote the rank of the bundles and

$$\begin{aligned}
\Lambda^2_7(V) &= \{\beta \in \Lambda^2(V) \mid *(\phi \wedge \beta) = -2\beta\} = \{*(\alpha \wedge \psi) \mid \alpha \in \Lambda^1(V)\} \\
&= \{\iota_u \phi \mid u \in \Gamma(V)\} \cong \Lambda^1_7(V), \\
\Lambda^2_{14}(V) &= \{\beta \in \Lambda^2(V) \mid \beta \wedge \psi = 0\} = \{\beta \in \Lambda^2(V) \mid *(\phi \wedge \beta) = \beta\} \cong \mathfrak{g}_2, \\
\Lambda^3_1(V) &= \langle \phi \rangle, \\
\Lambda^3_7(V) &= \{*(\alpha \wedge \phi) \mid \alpha \in \Lambda^1(V)\} = \{\iota_u \psi \mid u \in \Gamma(V)\} \cong \Lambda^1_7(V), \\
\Lambda^3_{27}(V) &= \{\gamma \in \Lambda^3(V) \mid \gamma \wedge \phi = 0 = \gamma \wedge \psi\},
\end{aligned}$$

and

$$\Lambda^k_d(V) \cong \Lambda^{7-k}_d(V), \qquad \forall k \in \{1, 2, 3\}.$$

We denote the corresponding projection maps by $\pi_d : \Lambda^k(V) \to \Lambda^k_d(V)$.

A proof can be found in [81, Theorem 8.5].

Definition 17 (Associative and Coassociative Submanifolds)**.** *Let (M, ϕ) be a 7-dimensional manifold with a positive G_2-structure ϕ. The associative form ϕ is a calibration, and the 3-dimensional submanifolds calibrated with respect to ϕ are called associative. Moreover, the coassociative form $\psi = *\phi$ is also a calibration on M, and 4-dimensional submanifolds calibrated with respect to ψ are called coassociative.*

2.1.1.3 *Spin*(7)-Manifolds

We start with the linear model.

Definition 18 (Admissible 4-Form)**.** *A 4-form Ω on an 8-dimensional vector space V is called admissible if there exists a basis of V in which Ω is identified with the 4-form Ω_0 on $\mathbb{R}^8$, given by*

$$\begin{aligned}
\Omega_0 = {} & e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^1 \wedge e^2 \wedge e^7 \wedge e^8 + e^3 \wedge e^4 \wedge e^5 \wedge e^6 \\
& + e^3 \wedge e^4 \wedge e^7 \wedge e^8 + e^1 \wedge e^3 \wedge e^5 \wedge e^7 - e^1 \wedge e^3 \wedge e^6 \wedge e^8 \\
& - e^2 \wedge e^4 \wedge e^5 \wedge e^7 + e^2 \wedge e^4 \wedge e^6 \wedge e^8 - e^1 \wedge e^4 \wedge e^5 \wedge e^8 \\
& - e^1 \wedge e^4 \wedge e^6 \wedge e^7 - e^2 \wedge e^3 \wedge e^5 \wedge e^8 - e^2 \wedge e^3 \wedge e^6 \wedge e^7 \\
& + e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^5 \wedge e^6 \wedge e^7 \wedge e^8.
\end{aligned}$$

Definition/Lemma 50 (Exceptional Group $Spin(7)$)**.** *$Spin(7)$ is the subgroup of $GL(8,\mathbb{R})$ that preserves Ω_0,*

$$Spin(7) = \{g \in GL(8,\mathbb{R}) \mid g^*\Omega_0 = \Omega_0\}.$$

$Spin(7)$ is a compact, connected, simply connected, 21-dimensional, simple Lie group. Moreover, $Spin(7) \subset SO(8)$.

Definition 19 (Almost $Spin(7)$-Manifold)**.** *A $Spin(7)$-structure on a smooth 8-dimensional manifold M is an admissible 4-form $\Omega \in \Omega^4(M)$. The pair (M,Ω) is called an almost $Spin(7)$-manifold.*

Definition 20 ($Spin(7)$-manifold)**.** *Let (M,Ω) be an almost $Spin(7)$-manifold. (M,Ω) is called a $Spin(7)$-manifold if the Spin(7)-structure is torsion-free,*

$$T(\Omega) := \nabla_{g_\Omega}\Omega = 0,$$

where ∇_{g_Ω} is the Levi-Civita connection of the Riemannian metric induced by the $Spin(7)$-structure Ω.

Definition 21 (Cayley Submanifold)**.** *Let (M,Ω) be a $Spin(7)$-manifold. Ω is a calibration on M. A 4-dimensional submanifold calibrated with respect to Ω is called a Cayley submanifold.*

2.1.2 Donaldson-Thomas Program

Donaldson and Thomas, and later Donaldson and Segal, proposed studying manifolds with special holonomy groups from the viewpoint of gauge theory. They proposed defining numerical, or more ambitiously, homological invariants of manifolds with special holonomy groups by a count of instantons, monopoles, and Calibrated submanifolds. In this section, we briefly review the definition of instantons and monopoles on these manifolds.

2.1.2.1 *Spin*(7)-Instantons

We start with the $Spin(7)$-instantons on $Spin(7)$-manifolds.

Definition 22 ($Spin(7)$-Instanton)**.** *Let G be a compact Lie group. Let $P \to M$ be a principal G-bundle above a $Spin(7)$-manifold (M,Ω). A connection A on this bundle is called a $Spin(7)$-instanton if it satisfies the following equation,*

$$*(F_A \wedge \Omega) = -F_A.$$

The following theorem of Walpuski gives a good picture of $Spin(7)$-instantons on certain closed $Spin(7)$-manifolds.

Theorem 10 (Walpuski [96])**.** *Let (M,Ω) be a compact, irreducible, $Spin(7)$–manifold. Let $Q \subset M$ be a Cayley submanifold, diffeomorphic to the smooth 4-manifold underlying any $K3$-surface and with self-intersection number zero, whose induced metric is sufficiently close* *to a hyperkähler metric. Moreover, suppose that the induced connection on the normal bundle $\nu(Q) \to Q$ is almost flat. Then there exists a 5–dimensional family of $Spin(7)$–instantons on an $SU(2)$–bundle $P \to M$ with $c_2(P) = PD[Q]$. Furthermore, let $Q_1, \ldots, Q_k$ be k disjoint Cayley submanifolds as above. Then there exists a $(8k-2)$-dimensional family of $Spin(7)$-instantons on an $SU(2)$–bundle $P \to M$ with $c_2(P) = \sum_{i=1}^{k} PD[Q_i]$.*

Dimensional reduction of $Spin(7)$-instantons leads to a theory of monopole on G_2-manifolds.

2.1.2.2 G_2-Monopoles

Here is the definition of G_2-monopole.

Definition 23 (G_2-Monopole)**.** *Let (M, ϕ) be a G_2-manifold and $\psi = *\phi$, where $*$ is the Hodge star of g_ϕ. A pair (A, Φ) of a connection A on a principal G-bundle $P \to M$ and a section Φ of the adjoint bundle is called a G_2-monopole if it satisfies the G_2-Bogomolny equation,*

$$*(F_A \wedge \psi) = d_A \Phi.$$

Example 2. *Let $M = \mathbb{R}^7$ equipped with the standard Euclidean metric $g_0 = \sum_{i=1}^{7} dx_i^2$, the volume form $vol_0 = dx_1 \wedge \ldots \wedge dx_7$, and the standard G_2-structure on $\mathbb{R}^7$, given by*

$$\begin{aligned}\phi_0 &= e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356}, \\ \psi_0 &= e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}.\end{aligned}$$

Let $\pi : \mathbb{R}^7 = \mathbb{R}^4 \times \mathbb{R}^3 \to \mathbb{R}^3$ be the projection on the last three components. Let $P \to \mathbb{R}^3$ be a principal G-bundle and (A, Φ) a monopole on P. The pair $(\pi^* A, \pi^* \Phi)$ is a G_2-monopole on $\pi^* P \to \mathbb{R}^7$ since

$$*_7(F_{(\pi^* A)} \wedge \psi_0) = *_7(\pi^*(F_A) \wedge \psi_0) = *_7(\pi^*(*_3 d_A \Phi) \wedge \psi_0),$$

where $*_7$ and $*_3$ are the Hodge star of the Euclidean metric on $\mathbb{R}^7$ and $\mathbb{R}^3$, respectively. More generally, for any 2-form $\beta \in \Omega^2(\mathbb{R}^3; V)$ with values in any vector bundle $V \to \mathbb{R}^7$, we have

$$*_7(\pi^*(\beta) \wedge \psi_0) = \pi^*(*_3 \beta),$$

and therefore, $*_7(F_{(\pi^* A)} \wedge \psi_0) = *_7(\pi^*(*_3 d_A \Phi) \wedge \psi_0) = \pi^*(d_A \Phi) = d_{\pi^*(A)} \pi^*(\phi)$.

G_2-monopoles can be understood as a dimensional reduction of $Spin(7)$-instantons, similar to the 3-dimensional monopoles, which are dimensional reduction of anti-self-dual instantons on 4-manifolds. The proof is also similar to the low-dimensional case.

Lemma 51 (Dimensional Reduction of $Spin(7)$-Instantons)**.** *Let (M, ϕ) be a G_2-manifold. Let $N = M \times \mathbb{R}$ be an 8-dimensional manifold with the $Spin(7)$-structure Ω, given by*

$$\Omega = dt \wedge \pi^*(\phi) + \pi^*(\psi),$$

where $\pi : N \to M$ is the obvious projection map and t is the coordinate on the $\mathbb{R}$-factor.

Let $P \to M$ be a principal G-bundle. Any t-translation-invariant connection $\mathbb{A}$ on $\pi^*P \to N$, in a translation-invariant gauge, can be written as

$$\mathbb{A} = \pi^*(A) + dt \wedge \pi^*(\Phi),$$

for a connection A on $P \to M$ and a section Φ on the adjoint bundle.

The connection $\mathbb{A}$ is an $Spin(7)$-instanton if and only if the pair (A, Φ) is a G_2-monopole.

G_2-monopoles on closed G_2-manifolds satisfy a stronger condition.

Lemma 52. *Let (M, ϕ) be a closed G_2-manifold. Any smooth G_2-monopole (A, Φ) on a principal G-bundle, for a compact Lie group G, satisfies the equations*

$$*(F_A \wedge \psi) = d_A\Phi = 0. \tag{2.1.1}$$

Proof. The proof is similar to the 3-dimensional case 2. The G_2-Bogomolny equation implies

$$\Delta_A\Phi = d_A^* d_A\Phi = *d_A(F_A \wedge \psi) = *(d_AF_A \wedge \psi + F_A \wedge d\psi) = 0,$$

where the last equality follows from the Bianchi identity and the fact that ψ is closed. Therefore, Φ is a harmonic section with respect to the connection A. Since G is a compact Lie group, there is an adjoint-invariant inner product on its Lie algebra $\mathfrak{g}$, and therefore, on the adjoint bundle $\mathfrak{g}_P$, denoted by $\langle -, - \rangle$. With respect to this inner product, we have the following pointwise equations,

$$0 = \langle \Delta_A\Phi, \Phi \rangle = \langle d_A^* d_A\Phi, \Phi \rangle = \langle d_A\Phi, d_A\Phi \rangle = |d_A\Phi|^2,$$

and therefore, $*(F_A \wedge \psi) = d_A\Phi = 0$. □

This lemma motivates the definition of G_2-instantons.

2.1.2.3 G_2-Instantons

Here is the definition of G_2-instanton.

Definition 24 (G_2-Instanton)**.** *A connection A on a principal G-bundle $P \to M$ over a G_2-manifold (M, ϕ) is called a G_2-instanton if*

$$F_A \wedge \psi = 0,$$

where $\psi = *\phi$ or, equivalently,

$$*(F_A \wedge \phi) = -F_A.$$

Example 3. *Let $\pi : \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}^4$ be the projection on the last 4-components. The pullback of an anti-self-dual instanton over $\mathbb{R}^4$ to $\mathbb{R}^7$ is a G_2–instanton. The proof is similar to Example 2.*

One can go one dimension lower.

2.1.2.4 Calabi-Yau Monopoles

We start with the definition of complex monopole.

Definition 25 (Complex Monopole)**.** *Let (M, g, ω, Ω) be a compact Calabi-Yau 3-fold. Let G be a compact Lie group. Let $P_G \to M$ and $P_{G_{\mathbb{C}}}$ be a principal G-bundle and its complexification, respectively. Let g_{P_G} and $g_{P_{G_{\mathbb{C}}}}$ be the adjoint bundle of P_G and its complexification, respectively. Let A be a connection on P_G and $\Phi = \Phi_1 + i\Phi_2$ a section of the complexified adjoint bundle $g_{P_{G_{\mathbb{C}}}}$. The pair (A, Φ) is called a complex monopole if it satisfies the complex Bogomolny equations,*

$$*(F_A \wedge \Omega) = 2\partial_A \Phi, \qquad \Lambda F_A = 2i[\Phi, \overline{\Phi}],$$

where $\Lambda\beta = *(\beta \wedge \omega^2)$ for any $\beta \in \Omega^2(M, \mathbb{C})$ and $*$ is the complex linear extension of the Hodge star operator.

An interesting class of complex monopoles appears when $\Phi_2 = 0$.

Definition 26 (Calabi-Yau Monopole)**.** *Let (M, g, ω, Ω) be a Calabi-Yau 3-fold. Let $P \to M$ be a principal G-bundle, where G is a compact Lie group. A pair (A, Φ) of a connection A on P and a section Φ of the adjoint bundle is called a Calabi-Yau monopole if it satisfies the Calabi-Yau Bogomolny equations,*

$$*(F_A \wedge \Omega_1) = d_A \Phi,$$
$$\Lambda F_A = 0,$$

where $\Omega = \Omega_1 + i\Omega_2$ and $\Lambda\beta = *(\beta \wedge \omega^2)$ for any 2-form β.

Example 4. *Let $M = \mathbb{C}^3 = \mathbb{R}^6$ with $z_j = x_j + iy_j$ for $j \in \{1, 2, 3\}$, the standard Euclidean metric g_0, the volume form $vol_0 = dz_1 \wedge dz_2 \wedge dz_3$, the holomorphic volume form $\Omega = dz_1 dz_2 dz_3$ with the real part*

$$\Omega_1 = dx_1 dx_2 dx_3 - dy_1 dy_2 dx_3 - dy_1 dx_2 dy_3 - dx_1 dy_2 dy_3,$$

and the symplectic form $\omega = dx_1 dy_2 + dx_2 dy_2 + dx_3 dy_3$.

Let $\pi : \mathbb{R}^6 \to \mathbb{R}^3_y$ be the projection

$$(z_1, z_2, z_3) \to (y_1, y_2, y_3).$$

Note that the 3-plane

$$\mathbb{R}^3_y = \{(0, y_1, 0, y_2, 0, y_3) \mid y_1, y_2, y_3 \in \mathbb{R}\},$$

is a special Lagrangian subspace,

$$\omega_{|_{\mathbb{R}^3_y}} = \Omega_{1|_{\mathbb{R}^3_y}} = 0.$$

Let $P \to \mathbb{R}^3_y$ be a principal G-bundle and (A, Φ) a monopole on P. The pair $(\pi^* A, \pi^* \Phi)$ is a Calabi-Yau monopole on $\pi^* P \to \mathbb{R}^6$ since

$$*_6(F_{(\pi^* A)} \wedge \Omega_1) = *_6(\pi^*(F_A) \wedge \Omega_1) = *_6(\pi^*(*_3 d_A \Phi) \wedge \Omega_1),$$

where $*_6$ and $*_3$ are the Hodge star of the Euclidean metric on $\mathbb{R}^6$ and $\mathbb{R}^3$, respectively. More generally, for any 2-form $\beta \in \Omega^2(\mathbb{R}^3; V)$ with values in any vector bundle $V \to \mathbb{R}^3$, we have

$$\pi^*(\beta) \wedge \Omega_1 = *_6 \pi^*(*_3 \beta),$$

and therefore, $F_{(\pi^* A)} \wedge \Omega_1 = *_6 \pi^*(d_A \Phi) = *_6 d_{\pi^*(A)} \pi^*(\phi)$. Furthermore, it is straightforward to see $\Lambda F_{\pi^*(A)} = \Lambda \pi^*(F_A) = 0$.

Similar to the 3-dimensional and G_2 cases, smooth Calabi-Yau monopoles on closed Calabi-Yau-manifolds satisfy a stronger equations.

Lemma 53. *Let (M, g, ω, Ω) be a closed Calabi-Yau 3-fold. Any smooth Calabi-Yau monopole (A, Φ) on a principal G-bundle, for a compact Lie group G, satisfies the stronger equations*

$$F_A \wedge \Omega_1 = * d_A \Phi = 0. \tag{2.1.2}$$

The proof is similar to the G_2 case, which we saw in Lemma 52.

Calabi-Yau monopoles can be understood as dimensional reduction of G_2-instantons.

Lemma 54. *Let (Z, g, ω, Ω) be a Calabi-Yau 3-fold. Let $M = Z \times \mathbb{R}$ be the 7-dimensional manifold equipped with the G_2-structure ϕ, given by*

$$\phi = dt \wedge \omega + \mathrm{Re}(\Omega),$$

where t denotes the coordinate on the $\mathbb{R}$-factor.

Let $P \to Z$ be a principal G-bundle. Any t-translation invariant connection $\mathbb{A}$ on $\pi^* P \to M$, in a translation invariant gauge, can be written as

$$\mathbb{A} = \pi^*(A) + dt \wedge \pi^*(\Phi),$$

for a connection A on $P \to Z$ and a section Φ on the adjoint bundle.

The connection $\mathbb{A}$ is a G_2-instanton if and only if the pair (A, Φ) is a Calabi-Yau monopole.

The proof is similar to the lower-dimensional case 1.

One can consider the dimensional reduction of Calabi-Yau monopoles to 5-dimensional spaces.

2.2 5-Dimensional Monopoles on U(1)-bundles over Hyperkähler Manifolds

Here is the definition of 5-dimensional monopole.

Definition 27 (5-Dimensional Monopole)**.** *Let (X, g_X, I, J, K) be a hyperkähler 4-manifold with the triple of kähler structures ω_1, ω_2 and ω_3. Let $(M, g_M, \alpha, \beta, \theta_1, \theta_2)$ be the 5-dimensional manifold, given as a $U(1)$-bundle above X equipped with the geometric structure given by differential forms $\theta_1 \in \Omega^3(M)$, $\theta_2 \in \Omega^2(M)$, $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$, where*

$$\theta_2 = \omega_2, \qquad \beta = \omega_1. \tag{2.2.1}$$

and in a local trivialization,

$$\theta_1 = dt \wedge \omega_3, \qquad \alpha = dt,$$

for a local coordinate t on the fibers.

Let G be a compact Lie group. Let $P \to M$ be a principal G-bundle. The triple (A, ψ, Φ) is called a 5-dimensional monopole if

$$*(F_A \wedge \theta_1) = [\psi, \Phi], \tag{2.2.2}$$

$$*(F_A \wedge \theta_2 - d_A\psi \wedge \theta_1) = d_A\Phi, \tag{2.2.3}$$

$$F_A \wedge \alpha \wedge \beta = -d_A\psi \wedge \beta^2. \tag{2.2.4}$$

5-dimensional monopoles can be seen as dimensional reduction of Calabi-Yau monopoles. In the following example, we consider the linear case.

Example 5 (5-Dimensional Monopoles on $\mathbb{R}^5$)**.** *Let $M = \mathbb{R}^5$ equipped with the Euclidean metric g_0. Let $\theta_1 \in \Omega^3(\mathbb{R}^5)$, $\theta_2 \in \Omega^2(\mathbb{R}^5)$, $\alpha \in \Omega^1(\mathbb{R}^5)$ and $\beta \in \Omega^2(\mathbb{R}^5)$ be differential forms, given by*

$$\begin{aligned} \theta_1 &= -dx_1 \wedge dx_3 \wedge dx_4 - dx_1 \wedge dx_2 \wedge dx_5, & \theta_2 &= dx_2 \wedge dx_4 - dx_3 \wedge dx_5, \\ \alpha &= dx_1, & \beta &= dx_2 \wedge dx_3 + dx_4 \wedge dx_5. \end{aligned}$$

Let $\mathbb{C}^3 = \mathbb{R}^6 = \mathbb{R}_t \times \mathbb{R}^5$ equipped with the standard Calabi-Yau structure. Let $\pi : \mathbb{R}_t \times \mathbb{R}^5 \to \mathbb{R}^5$ be the obvious projection map. Let $P \to \mathbb{R}^5$ be a principal G-bundle and $\pi^*P \to \mathbb{R}^6$ the pull-back bundle. Any t-translation-invariant pair $(\widetilde{A}, \widetilde{\Phi})$ on π^*P can be written as

$$\widetilde{A} = \pi^*A + (\pi^*\psi)dt \quad \text{and} \quad \widetilde{\Phi} = \pi^*\Phi,$$

for a connection A on P and sections ψ and Φ of the adjoint bundle.

The pair $(\widetilde{A}, \widetilde{\Phi})$ is a Calabi-Yau monopole if and only if (A, ψ, Φ) is a 5-dimensional monopole.

To see this, note that

$$\Omega_1 = \theta_1 + dt \wedge \theta_2,$$

where Ω_1 is the real part of the holomorphic volume form on $\mathbb{C}^3$, and therefore,

$$\begin{aligned} F_{\widetilde{A}} \wedge \Omega_1 &= (\pi^*(F_A) + \pi^*(d_A\psi) \wedge dt) \wedge (\theta_1 + \theta_2 dt) \\ &= \pi^*(F_A) \wedge \theta_1 + (\pi^*(F_A) \wedge \theta_2 - \pi^*(d_A\psi) \wedge \theta_1) \wedge dt, \end{aligned}$$

moreover,

$$*_6 d_{\widetilde{A}}\widetilde{\Phi} = \pi^*(*_5 d_A \Phi) \wedge dt + \pi^*(*_5[\psi, \Phi]),$$

where $*_6$ *and* $*_5$ *are the Hodge star operators on* $\mathbb{R}^6$ *and* $\mathbb{R}^5$*, respectively.*

By equating the terms containing dt *we can see that the equation* $*d_{\widetilde{A}}\widetilde{\Phi} = F_{\widetilde{A}} \wedge \Omega_1$ *is equivalent to the following equations on* $P \to \mathbb{R}^5$*,*

$$F_A \wedge \theta_1 = *_5[\psi, \Phi], \quad \textit{and} \quad F_A \wedge \theta_2 - d_A\psi \wedge \theta_1 = *_5 d_A \Phi.$$

Moreover, notice

$$\omega = dt \wedge \alpha + \beta,$$

and therefore, $\Lambda F_{\widetilde{A}} = 0$ *is equivalent to*

$$F_A \wedge \alpha \wedge \beta + d_A\psi \wedge \beta^2 = 0, \qquad F_A \wedge \beta^2 = 0.$$

More generally, we have the following lemma.

Lemma 55. *Let* M *be a* $U(1)$*-bundle over a hyperkähler manifold* (X, g_X, I, J, K)*. Let* $Z = \mathbb{R}_t \times M$ *be a cylindrical Calabi-Yau manifold. Let* $\pi : Z \to M$ *be the projection map. Let* $P \to M$ *be a principal* G*-bundle, and* $\pi^*P \to \mathbb{R}^6$ *the pull-back bundle. The pair* $(\widetilde{A}, \widetilde{\Phi})$ *is a translation-invariant Calabi-Yau monopole if and only if* (A, ψ, Φ) *is a 5-dimensional monopole, where*

$$\widetilde{A} = \pi^*A + (\pi^*\psi)dt \quad \textit{and} \quad \widetilde{\Phi} = \pi^*\Phi.$$

The proof is similar to the linear case, we studied earlier.

Anti-self-dual instantons give rise to some examples of 5-dimensional monopoles.

Example 6. *Let* $\mathbb{R}^5 = \mathbb{R}_t \times \mathbb{R}^4_{(x_1,x_2,x_3,x_4)}$*. Let* $\pi : \mathbb{R}^5 \to \mathbb{R}^4_{(x_1,x_2,x_3,x_4)}$ *be the projection map. Let* $P \to \mathbb{R}^4$ *be a principal* G*-bundle. Let* A *be an anti-self-dual connection on* P*. The triple* $(\pi^*(A), 0, 0)$ *is a 5-dimensional monopole on* $\pi^*P \to \mathbb{R}^5$*.*

To see this, note that

$$\begin{aligned} &\theta_1 = dt \wedge (dx_1 \wedge dx_4 + dx_2 \wedge dx_3) = dt \wedge \omega_3, && \theta_2 = dx_1 \wedge dx_3 + dx_4 \wedge dx_2 = \omega_2, \\ &\alpha = dt, && \beta = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 = \omega_1, \end{aligned}$$

where $(\omega_1, \omega_2, \omega_3)$ *are self-dual 2-forms, forming a hyperkähler structure on* $\mathbb{R}^4$*, and therefore,*

$$F_{\pi^*(A)} \wedge \theta_1 = 0, \quad F_{\pi^*(A)} \wedge \theta_2 = 0, \quad F_{\pi^*(A)} \wedge \alpha \wedge \beta = 0.$$

We finish this section with a question and a conjecture.

Question 56. *Is there any* 5*-dimensional monopole on* $\mathbb{R}^5$ *which is not pull-back of an instanton on* $\mathbb{R}^4$*?*

The positive answer to this question suggests that there is an interesting theory of monopoles on — potentially singular — $U(1)$-bundles over hyperkähler 4-manifolds. Negative answer to this question would imply that all translation-invariant Calabi-Yau monopoles on $\mathbb{R}^6$ come from the instantons on $\mathbb{R}^4$. We expect the answer to the question 56 to be positive.

An interesting case occurs when in the 5-dimensional Bogomolny equations we have $\psi = 0$.

Definition 28 (Pure 5-Dimensional Monopole)**.** *Let (X, g_X, I, J, K) be a hyperkähler 4-manifold. Let $(M, g_M, \alpha, \beta, \theta_1, \theta_2)$ be a $U(1)$-bundle above X. Let $P \to M$ be a principal G-bundle. The triple (A, Φ) is called a pure 5-dimensional monopole if*

$$F_A \wedge \theta_1 = 0, \qquad F_A \wedge \alpha \wedge \beta = 0, \qquad *(F_A \wedge \theta_2) = d_A \Phi.$$

One can consider the case where $\Phi = \psi = 0$, to get an instanton theory on these 5-manifolds.

Definition 29 (5-Dimensional Instanton)**.** *Let (X, g_X, I, J, K) be a hyperkähler 4-manifold. Let $(M, g_M, \alpha, \beta, \theta_1, \theta_2)$ be a $U(1)$-bundle above X. Let $P \to M$ be a principal G-bundle. A connection A on P is called a 5-dimensional instanton if*

$$F_A \wedge \theta_1 = 0, \qquad F_A \wedge \alpha \wedge \beta = 0, \qquad F_A \wedge \theta_2 = 0.$$

Example 7 (5-Dimensional Instanton)**.** *Let (X, g_X, I, J, K) be a hyperkähler 4-manifold. Let M be a $U(1)$-bundle above X. Let $P \to X$ be a principal G-bundle. Let A be an anti-self-dual connection on P. Then $\pi^* A$ is a 5-dimensional instanton on $\pi^* P \to M$.*

Conjecture 2. *There are 5-dimensional instantons on M which are not pull-back of any instanton on X.*

2.3 Singular Monopoles in Higher Dimensions

As we saw in Lemmas 52 and 53, G_2- and Calabi-Yau monopoles on closed manifolds satisfy stronger conditions. Similar to the 3-dimensional case, in order to have monopoles in higher dimensions which do not satisfy these conditions, one should allow singularities. In this direction, Oliveira proposed the study of monopoles with singularities along certain calibrated submanifolds [75]. In the Calabi-Yau case, one could study monopoles with Dirac singularities along special Lagrangians.

Definition 30 (Singular Calabi-Yau Monopole [75])**.** *Let (M, g, ω, Ω) be a Calabi-Yau 3-fold. Let $L = L_1 \cup \ldots \cup L_k$, where$L_i$ are disjoint, compact, connected, embedded special Lagrangian submanifolds. A pair (A, Φ) is called a Calabi-Yau monopole with Dirac singularity along L if*

- *It satisfies the Calabi-Yau-Bogomolny equations on $M \setminus L$.*
- *For each $i \in \{1, \ldots, k\}$ there exists $k_i \in \mathbb{Z}^+$ such that*

$$\lim_{r_i \to 0} r_i |\Phi| = \lim_{r_i \to 0} r_i^2 |F_A| = \frac{k_i}{2},$$

where r_i is the geodesic distance from L_i, defined in a small tubular neighbourhood of L_i. k_i is called the charge of the monopole along L_i. In particular, when the structure group $G = U(1)$, one can define the signed charge $k_i \in \mathbb{Z}$, where

$$-2 \lim_{r_i \to 0} r_i \Phi = k_i.$$

Similarly, one can consider G_2-monopoles with Dirac singularity along certain codimension 3, calibrated submanifolds.

Definition 31 (Singular G_2-Monopole [75])**.** *Let (M, ϕ) be a closed G_2-manifold. Let $N = N_1 \cup \ldots \cup N_k$, where$N_i$ are disjoint, compact, connected, embedded coassociative submanifolds. A pair (A, Φ) is called a G_2-monopole with Dirac singularity along N if*

- *It satisfies the G_2-Bogomolny equation on $M \setminus N$.*
- *For each $i \in \{1, \ldots, k\}$ there exists $k_i \in \mathbb{Z}^+$ such that*

$$2 \lim_{r_i \to 0} r_i |\Phi| = 2 \lim_{r_i \to 0} r_i^2 |F_A| = k_i$$

where r_i is the geodesic distance from N_i in a small tubular neighbourhood of N_i.

k_i is called the charge of the monopole along N_i. In particular, when the structure group $G = U(1)$, one can define the signed charge $k_i \in \mathbb{Z}$, where

$$-2 \lim_{r_i \to 0} r_i \Phi = k_i.$$

There is a 5-dimensional version of this theory.

Lemma 57. *Let (X, g_X, I, J, K) be a closed hyperkähler 4-manifold. Let $(M, g_M, \alpha, \beta, \theta_1, \theta_2)$ be a $U(1)$-bundle above X. Let G be a compact Lie group. Let $P \to M$ be a principal G-bundle. Let (A, Φ) be a pure 5-dimensional monopole on P. Then $d_A \Phi = 0$ and A is a 5-dimensional instanton.*

Proof. The pull-back of the 5-dimensional monopole to the Calabi-Yau 3-fold $M \times S^1$ is a Calabi-Yau monopole, and Calabi-Yau monopoles on the closed Calabi-Yau manifolds satisfy the following equations,

$$F_A \wedge (\theta_1 + dt \wedge \theta_2) = 0, \qquad d_A \Phi = 0.$$

These equations imply

$$F_A \wedge \theta_2 = 0, \qquad F_A \wedge \theta_1 = 0, \qquad d_A \Phi = 0.$$

□

Similar to the Calabi-Yau and G_2 cases, one can define 5-dimensional monopoles with Dirac singularities on $U(1)$-bundles over a 4-dimensional hyperkähler manifolds. The Dirac singularities of monopoles appear along certain calibrated submanifolds with codimension 3.

Definition 32 (Singular 5-Dimensional Monopole)**.** *Let M be a $U(1)$-bundle over a hyperkähler manifold (X, g_X, I, J, K). Let $N = N_1 \cup \ldots \cup N_k$, where$N_i$ are disjoint, compact, connected, embedded surfaces satisfying the equations*

$$\beta_{|_{N_i}} = \theta_2|_{N_i} = 0. \tag{2.3.1}$$

A pair (A, ψ, Φ) is called a 5-dimensional monopole with Dirac singularity along N if

- *It satisfies the 5-dimensional-Bogomolny equations on $M \setminus N$.*
- *For each $i \in \{1, \ldots, k\}$ there exists $k_i \in \mathbb{Z}^+$ such that*

$$2 \lim_{r_i \to 0} r_i |\Phi| = \lim_{r_i \to 0} r_i^2 (|F_A| + |d_A \psi|) = k_i$$

where r_i is the geodesic distance from N_i in a small tubular neighbourhood of N_i.

Note that, in the product case $M = X \times U(1)$, the holomorphic curves $\Sigma \times \{x\} \subset X \times \{x\}$,

$$\omega_{1|_L} = \omega_{2|_L} = 0,$$

satisfy the equations 2.3.1.

One hopes to prove the existence of singular monopoles on closed manifolds in higher dimensions, similar to what we did in Chapter 1 over 3-manifolds. Under suitable topological assumptions, Oliveira proved the existence of these singular G_2-monopoles when the structure group $G = U(1)$.

Lemma 58 (Oliveira [75])**.** *Suppose (X, ϕ) is a closed G_2-manifold with full holonomy group G_2 and $N = N_1 \cup \ldots \cup N_n$ a union of disjoint, compact, connected, embedded, coassociative submanifolds. For every $\alpha \in H^2(X \setminus N, \mathbb{Z})$, there exists a line bundle $L \to X \setminus N$ with $c_1(L) = \alpha$ and a Dirac G_2-monopole (A, Φ) on $L \to X \setminus N$. Moreover, the charge of (A, Φ) along N_i, $k_i = ev_i(\alpha)$, where $ev_i : H^2(X \setminus N, \mathbb{Z}) \to \mathbb{Z}$ is defined by integrating a 2-form over any fiber of the unit 2-sphere bundle of N_i in $X \setminus N$.*

Furthermore, Oliveira found examples of non-Abelian singular G_2-monopoles on the Bryant-Salamon G_2-manifolds; however, a systematic proof of the existence of irreducible non-Abelian G_2-monopoles with Dirac singularities is still missing.

2.4 Adiabatic Limits and Fueter Sections

In this section, we see how monopoles in higher dimensions, in the adiabatic limit, are related to the 3-dimensional monopoles on $\mathbb{R}^3$. Moreover, we will show how a non-linear generalization of the Dirac operator, called the Fueter operator, appears in the study of these monopoles. This section mainly serves as a motivating section to the next chapter, which we will study Fueter sections on hyperkähler bundles.

We consider the product case, where the picture is clearer.

2.4.1 Monopoles in Higher Dimensions and Fueter Sections

Let (Y, ϕ) be non-compact asymptotically conical G_2-manifold. Let $P \to Y$ be a principal $SU(2)$-bundle. To any pair (A, Φ) of a connection A on P and section Φ of the adjoint bundle, one can assign an the intermediate energy, as defined by Oliveira in [77],

$$\mathcal{E}(A, \Phi) = \int_Y (|F_A \wedge \psi|^2 + |d_A\Phi|^2) vol_g,$$

where g is the G_2-metric and $\psi = *_g \phi$.

As mentioned by Donaldson and Segal [21], and proved by Fadel, Nagy and Oliveira [25], to any G_2-monopole (A, Φ) on (Y, ϕ) with finite intermediate energy, one can assign a mass at infinity, which is defined by

$$m := \lim_{|x| \to \infty} |\Phi(x)|.$$

Let $\{(A_i, \Phi_i)\}_{i=1}^{\infty}$ be a sequence of G_2-monopoles on $P \to Y$ with finite intermediate energy and masses $\{m_i\}_{i=1}^{\infty}$ where $m_i \to \infty$ as $i \to \infty$. It is conjectured by Donaldson and Segal that such sequence of G_2-monopoles with large mass will concentrate along coassociatives $N = N_1 \cup \ldots \cup N_n \subset Y$.

At each point $t \in N$, one can decompose the G_2-Bogomolny equation into two equations. Suppose t_0, t_1, t_2, t_3 denote local coordinates on N around $t \in N$ and x_1, x_2, x_3 the coordinates on the normal direction. As mentioned in [21], the Bogomolny equations schematically can be written as

$$\nabla_{\underline{x}}\Phi = F_{\underline{xx}} + F_{\underline{tt}}, \qquad \nabla_{\underline{t}}\Phi = F_{\underline{tx}}. \tag{2.4.1}$$

We call the first equation in 2.4.1, the transverse Bogomolny equation, and the second one, the mixed Bogomolny equation. At each $t \in N$, the leading terms of the G_2-Bogomolny equation, for pairs with large mass, in the direction normal to N at $t \in N$, define a 3-dimensional Bogomolny equation on $(T_tN)^{\perp} \cong \mathbb{R}^3$. Moreover, the mixed G_2-Bogomolny equation is given by a non-linear Dirac operator.

Let's consider the linear model. Let $Y = \mathbb{R}^4_t \times \mathbb{R}^3_x$. Let $\omega_1, \omega_2, \omega_3$ be the anti-self-dual 2-forms on $\mathbb{R}^4_t$, given by

$$\omega_i = dt_0 \wedge dt_i + dt_j \wedge dt_k,$$

for a cyclic permutation of i, j, k.

The G_2-structure ϕ is given by

$$\phi = \sum_{i=1}^{3} \omega_i \wedge dx_i - dx_1 \wedge dx_2 \wedge dx_3,$$

with the induced G_2-metric $g = \sum_{i=0}^{4} dt_i^2 + \sum_{i=1}^{3} dx_i^2$ and the orientation given by

$$vol_g = -dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3 \wedge dx_1 \wedge dx_2 \wedge dx_3.$$

The coassociative 4-form is given by

$$\psi = *\phi = \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k + dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3.$$

Note that the submanifold $N = \{t_0\} \times \mathbb{R}_x^3 \subset Y$ for any $t_0 \in \mathbb{R}_x^3$ is a coassociative submanifold, since we have $\phi_{|_N} = 0$.

For each $\varepsilon > 0$, let $T_\varepsilon : \mathbb{R}_t^4 \times \mathbb{R}_x^3 \to \mathbb{R}_t^4 \times \mathbb{R}_x^3$ be the diffeomorphism, given by

$$T_\varepsilon(t_0, t_1, t_2, t_3, x_1, x_2, x_3) = (\varepsilon^{-1}t_0, \varepsilon^{-1}t_1, \varepsilon^{-1}t_2, \varepsilon^{-1}t_3, x_1, x_2, x_3).$$

For any $\varepsilon > 0$, we can define the G_2-structure, given by

$$\begin{aligned}
\phi_\varepsilon &= T_\varepsilon^* \phi = \varepsilon^{-2} \sum_{i=1}^{3} \omega_i \wedge dx_i - dx_1 \wedge dx_2 \wedge dx_3, \\
\psi_\varepsilon &= T_\varepsilon^* \psi = \varepsilon^{-2} \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k + \varepsilon^{-4} dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3, \\
g_\varepsilon &= T_\varepsilon^* g = \varepsilon^{-2} \sum_{i=0}^{3} dt_i^2 + \sum_{i=1}^{3} dx_i^2, \\
vol_{g_\varepsilon} &= T_\varepsilon^* vol_g = -\varepsilon^{-4} dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3 \wedge dx_1 \wedge dx_2 \wedge dx_3.
\end{aligned}$$

For any $\varepsilon > 0$, the 3-form ϕ_ε is a positive 3-form. However, the 3-form ϕ_0, defined by letting $\varepsilon = 0$, is not a positive form, but we still can define the G_2-Bogomolny equation with respect to this structure. This can be thought of as an adiabatic limit for the G_2-Bogomolny equation, as in [21]. By setting $\varepsilon = 0$, we get the following.

Theorem 11. *The adiabatic limit of the G_2-Bogomolny equation in the direction transverse to N at each $t \in N$ is the 3-dimensional Bogomolny equation on $(T_t N)^\perp \cong \mathbb{R}^3$.*

Proof. We start by considering the linear case $Y = \mathbb{R}_t^4 \times \mathbb{R}_x^3$ equipped with the coassociative 4-form $\psi = \psi_{2,2} + \psi_{4,0}$ where

$$\psi_{2,2} = \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k, \qquad \psi_{4,0} = dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3.$$

Let $A = A_t + A_x$, where $A_t = \sum_{i=0}^{3} B_i dt_i$ and $A_x = \sum_{j=1}^{3} A_j dx_j$. Schematically, let

$F_A = F_{t,t} + F_{x,x} + F_{t,x}$, where

$$F_{t,t} = \sum_{0\le i<j\le 3} (\partial_{t_j} B_i - \partial_{t_i} B_j) dt_j \wedge dt_i,$$
$$F_{x,x} = \sum_{1\le i<j\le 3} (\partial_{x_j} A_i - \partial_{x_i} A_j) dx_j \wedge dx_i,$$
$$F_{t,x} = \sum_{\substack{0\le i\le 3\\ 1\le j\le 3}} (\partial_{t_i} A_j - \partial_{x_j} B_i) dt_i \wedge dx_j.$$

Therefore

$$F_A \wedge \psi_\varepsilon = (\varepsilon^{-4} F_{x,x} \wedge dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3 + \varepsilon^{-2} F_{t,t} \wedge \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k) + (\varepsilon^{-2} F_{t,x} \wedge \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k).$$

Moreover,

$$d_A \Phi = d_{A_t}\Phi + d_{A_x}\Phi = \sum_{i=0}^{3} (\partial_{t_i}\Phi + [B_i, \Phi]) dt_i + \sum_{j=1}^{3} (\partial_{x_j}\Phi + [A_j, \Phi]) dx_j.$$

Let $*_{g_\varepsilon}$ and denote the Hodge star operator with respect to the metric g_ε.

$$*_{g_\varepsilon} d_A \Phi = \varepsilon^{-2} \sum_{i=0}^{3} (\partial_{t_i}\Phi + [B_i, \Phi]) \widehat{dt}_i + \varepsilon^{-4} \sum_{j=1}^{3} (\partial_{x_j}\Phi + [A_j, \Phi]) \widehat{dx}_j,$$

where $\widehat{dt}_1 = dt_2 dt_3 dx_0 \ldots dx_3$ and $\widehat{dx}_1 = dt_1 dt_2 dt_3 dx_0 dx_2 dx_3$, and the rest are defined with cyclic permutations.

For any $\varepsilon > 0$, by multiplying the G_2-Bogomolny equation for ϕ_ε by ε^4 we get

$$\begin{aligned}(F_{x,x} \wedge dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3 + \varepsilon^2 F_{t,t} \wedge \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k)& \\ + (\varepsilon^2 F_{t,x} \wedge \sum_{i=1}^{3} \omega_i \wedge dx_j \wedge dx_k)& \\ = \varepsilon^2 \sum_{i=0}^{3} (\partial_{t_i}\Phi + [B_i, \Phi]) \widehat{dt}_i + \sum_{j=1}^{3} (\partial_{x_j}\Phi + [A_j, \Phi]) \widehat{dx}_j.&\end{aligned}$$

Let's let $\varepsilon = 0$. We get

$$F_{x,x} \wedge dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3 = \sum_{i=1}^{3} (\partial_{x_j} \Phi + [A_j, \Phi]) \widehat{dx_j},$$

and therefore, by letting the coefficients of $dt_0 \wedge dt_1 \wedge dt_2 \wedge dt_3$ equal on the both sides we get the standard 3-dimensional Bogomolny equation on $(T_x N)^{\perp} \cong \mathbb{R}^3$.

The proof in the more general setting is similar, essentially because at each $t \in N$, in a suitable coordinates $\psi = \psi_{2,2} + \psi_{4,0} + O(|x|)$, where

$$\psi_{2,2} := \sum_{i=1}^{3} \omega_i \wedge dt_j \wedge dt_k, \qquad \psi_{4,0} := dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3,$$

for coordinates t_0, t_1, t_2, t_3 on N around $t \in N$ and x_1, x_2, x_3 in the normal direction. $\square$

In the non-linear case, where there is no identification between monopoles with different masses, the affect of scaling the metric the way we observed above is similar to changing the mass of the monopole, and rather than sending $\varepsilon \to 0$, one can send mass $m \to \infty$, and still G_2-monopoles will concentrate along coassociatives, and in the transverse directions at each point along the coassociative submanifolds, a G_2-monopole with a large mass is approximately a 3-dimensional monopole.

Now we consider the mixed Bogomolny equations. Let (Y, ϕ) be an asymptotically conical G_2-manifold. Let $N \subset Y$ be a coassociative submanifold. Let (A, Φ) be a G_2-monopole on $P \to Y$ with a large mass. Considering the dominant part of the G_2-Bogomolny equation in the transverse direction, at each point $t \in N$ we have an approximate 3-dimensional monopole on $(T_t N)^{\perp}$, and therefore, in the adiabatic limit, there is a 4-dimensional family of 3-dimensional monopoles, parametrized by the points of N. This can be understood as a section of a bundle above N, with fibers modeled on the moduli spaces of centered monopoles on $\mathbb{R}^3$. The mixed G_2-Bogomolny equation asserts that this section satisfies a non-linear Dirac operator on a hyperkähler bundle.

Again for simplicity, consider the linear case $Y = \mathbb{T}^4 \times \mathbb{R}^4$ with the G_2-structure ϕ_ε. Let $N = \mathbb{T}^4 \times \{x_0\}$ for a $x_0 \in \mathbb{R}^4$. Let $f : \mathbb{T}^4 \times \{x_0\} \to M_k$, where M_k denotes the moduli space of centered k-monopoles on $\mathbb{R}^3$[4]. Let $f(x) = (A_f(x), \Phi_f(x))$. Moreover, for any k, we have

$$\Lambda^k(\mathbb{T}^4 \times \mathbb{R}^4) = \oplus_{p+q=k} (\Lambda^p \mathbb{T}^4 \otimes \Lambda^q \mathbb{R}^4).$$

Corresponding to this decomposition, for any k-form α we can write $\alpha = \sum_{p+q=k} \alpha_{p,q}$. Then the G_2-Bogomolny equation reads as

$$0 = (d_{A_f} \Phi_f)_{1,0} - *(F^{1,1}_{A_f} \wedge \psi_{2,2}) = d\Phi_f - \sum_{i=1}^{3} \mathcal{I}(\partial_{t_i}) d(A_f)_i,$$

[4] For detailed description of the moduli spaces of monopoles on $\mathbb{R}^3$ see Section 3.4.1.

where the expression on the right-hand side is a non-linear Dirac operator applied to f. In the non-linear case, the map f is a section of the centered monopole bundle on an oriented Riemannian 4-manifold N, studied in more details in the next chapter.

Over the Calabi-Yau manifolds, the Calabi-Yau monopoles with large mass concentrate along special Lagrangians. Similar to the G_2 case, the Calabi-Yau monopole equation, in the adiabatic limit, reduces to the 3-dimensional Bogomolny equation in the transverse directions to the special Lagrangian. The mixed equations read as a Fueter equation on 3-manifolds. These equations will be studied in more detail in the next chapter.

2.5 Complex Gauge Theory in Higher Dimensions

We can complexify gauge theories, and they turn out to be quite interesting. For instance, it is conjectured by Witten that a certain complexification of the instantons can be used to give a gauge-theoretic definition of the Jones polynomial and Khovanov homology of knots and links in 3-manifolds [99, 70]. There exists a different complexification of instantons, introduced by Haydys [41]. Same thing can be done for the Bogomolny equation on 3-manifolds. These complex Bogomolny equations have been introduced and studies by Nagy and Oliveira [73, 74]. In this section, we do the same for gauge-theoretic equations defined on manifolds with special holonomy groups.

2.5.1 Preliminaries

We start by introducing the basic framework of the complexified gauge theories.

2.5.1.1 Complexification

The first step of defining these complexified gauge theories is to complexify the Lie groups and Lie algebras.

Definition 33 (Complex Lie Group)**.** *A complex Lie group G is a Lie group and a complex manifold where the group multiplication $m : G \times G \to G$ and the inverse map $i : G \to G$, given by*

$$m(g,h) = gh, \qquad\qquad i(g) = g^{-1},$$

are holomorphic. There is a natural induced complex structure on the Lie algebra $\mathfrak{g}$ of a complex Lie group, with a complex bilinear Lie bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Definition 34 (Complexification of Lie Groups: Definition by the Universal Property)**.** *A complex Lie group $G_{\mathbb{C}}$ is a complexification of a Lie group G if there exists a continuous injective group homomorphism $\iota : G \to G_{\mathbb{C}}$ such that for any complex Lie group H and any continuous group homomorphism $f : G \to H$ there is a unique unique holomorphic homomorphism $f_{\mathbb{C}} : G_{\mathbb{C}} \to H$,* *such the following diagram is commutative,*

$$\begin{tikzcd} G \arrow[r, "\iota"] \arrow[dr, "f"'] & G_{\mathbb{C}} \arrow[d, "f_{\mathbb{C}}"] \\ & H \end{tikzcd}$$

Remark 4. *In the definition above, although it is asked for the existence of $f_{\mathbb{C}}$ for any continuous group homomorphism f, it should noted that any continuous group homomorphism between Lie groups is, in fact, (real) analytic.*

Example 8. *$GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$ and $PSL(n,\mathbb{C})$ are complex Lie groups for any $n \in \mathbb{N}$. In fact,*

$$(U(n))_{\mathbb{C}} \cong GL(n,\mathbb{C}), \qquad (SU(n))_{\mathbb{C}} \cong SL(n,\mathbb{C}), \qquad (PU(n))_{\mathbb{C}} \cong PSL(n,\mathbb{C}).$$

Theorem 12 (Existence and Uniqueness)**.** *Every compact Lie group admits a complexification. Furthermore, the complexification is unique up to isomorphism.*

A proof of this theorem can be found in [38, Section 7.1].

Remark 5. *There are non-compact Lie groups which do not admit a complexification.*

One can also complexify a Lie algebra simply by letting $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. Lie algebra of the complexification of a Lie group is the same as complexification of the Lie algebra of the Lie group.

One important difference between the complex Lie groups and the non-complex ones in example 8 is that the non-complex ones are compact, whereas the complex ones are non-compact.

Theorem 13. *Every connected complex Lie group, which is not a complex torus, is non-compact.*

The main difficulty for studying gauge theory, when the structure group is non-compact, is the compactness problems. For instance, the Uhlenbeck compactness theorem applies to connections on bundles with compact structure group. There are different forms of generalizations of Uhlenbeck compactness theorem to the principal $SL(2,\mathbb{C})$- and $PSL(2,\mathbb{C})$-bundles, due to Taubes [88, 91, 87].

The key fact about compact Lie groups, which is essential in the Uhlenbeck compactness theorem, is that the Lie algebra of a compact Lie group admits an adjoint invariant inner product, which here it means the Yang-Mills functional is invariant under the action of gauge group. However, that is not necessarily the case for non-compact Lie groups, including the examples appearing in 8.

The next step is to complexify bundles.

Definition 35 (Complexification of Bundles)**.** *Let $P_G \to M$ be a principal G-bundle for a Lie group G, with complexification $G_{\mathbb{C}}$. This bundle induces a principal $G_{\mathbb{C}}$-bundle $P_{G_{\mathbb{C}}} = P_G \times_G G_{\mathbb{C}}$ over M, which is called the complexified principal bundle. The associated adjoint bundle of $P_{G_{\mathbb{C}}}$ is the complexification of the adjoint bundle of P_G; i.e., $\mathfrak{g}_P \otimes \mathbb{C} = \mathfrak{g}_P \oplus i\mathfrak{g}_P$.*

This sets the stage to study the complex gauge theories.

2.5.2 Complex Gauge Theories in Low Dimensions

In this section, we briefly review the complex gauge theories on 3- and 4-manifolds, including complex instantons and monopoles. There are two ways to complexify the gauge-theoretic equations, we call one of the Haydys type and the other of the Kapustin-Witten type.

2.5.2.1 Complex Flat Connections

Haydys studied the moduli space of stable flat $PSL(2,\mathbb{C})$-connections on Riemannian 3-manifolds. These connections are related to the non-compactness phenomenon in the study of the higher rank Seiberg-Witten equations.

Definition 36. *Let $P_{G_{\mathbb{C}}}$ be a complexified principal bundle. A connection $\mathbb{A}$ on this bundle can be written as $A_1 + iA_2$, where A_1 is a connection on a principal G-bundle P_G and A_2 is a 1-forms with values in the adjoint bundle $\mathfrak{g}_P$.*

$$F_{\mathbb{A}} = (F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + i d_{A_1} A_2.$$

$\mathbb{A}$ is a flat connection if and only if

$$F_{A_1} - \frac{1}{2}[A_2 \wedge A_2] = 0, \qquad d_{A_1} A_2 = 0. \tag{2.5.1}$$

These equations are under-determined. One can get an elliptic system of equation by introducing a new condition. A flat $PSL(2,\mathbb{C})$ connection $\mathbb{A} = A_1 + iA_2$ is called stable if

$$d^*_{A_1} A_2 = 0. \tag{2.5.2}$$

A section Φ of the adjoint bundle $\mathfrak{g}_{P_{G_{\mathbb{C}}}}$ can be written as $\Phi = \Phi_1 + i\Phi_2$, and

$$d_{\mathbb{A}}\Phi = (d_{A_1}\Phi_1 - [A_2, \Phi_2]) + i(d_{A_1}\Phi_2 + [A_2, \Phi_1]),$$

and therefore, Φ is parallel with respect to $\mathbb{A}$ if

$$d_{A_1}\Phi_1 - [A_2, \Phi_2] = 0, \qquad d_{A_1}\Phi_2 + [A_2, \Phi_1] = 0. \tag{2.5.3}$$

The point about the equations 2.5.1 and 2.5.3 is that they are written on a bundles with a real structure group.

2.5.2.2 Complex Instantons

In order to complexify the monopole and instanton equations, both in low- and also higher dimensions, one should extend the Hodge star operator to complex-valued differential forms. This basically boils down to deciding what the constant c should be in the equation $*i = ci*$.

$$*(1) = *(-i^2) = ci * (-i) = -c^2 i^2 * (1) = c^2 * (1),$$

and therefore, $c = 1$ or -1. The first case, $c = 1$ corresponds to the linear extension of the Hodge star operator to the complex-valued differential forms. We denote this operator by $*_{\mathbb{C}}$. The second case, $c = -1$, corresponds to the conjugate linear (anti-linear) extension of the Hodge star operator. We denote this operator by $\overline{*}_{\mathbb{C}}$.

$$*_{\mathbb{C}} \text{ and } \overline{*}_{\mathbb{C}} : \Omega^k(M, \mathfrak{g}_{P_{G_{\mathbb{C}}}}) \to \Omega^{n-k}(M, \mathfrak{g}_{P_{G_{\mathbb{C}}}}).$$

Haydys introduced complex instantons with respect to the complex linear Hodge star operator [41], Kapustin and Witten introduced complex instantons with respect to the conjugate linear Hodge star [58], and Nagy and Oliveira studied the dimensional reduction of both of these equations on 3-manifolds [73, 74]. We review these works and then generalize them to the higher-dimensional gauge theories.

Definition 37 (Haydys Instanton)**.** *Let $P_{\mathcal{G}_{\mathbb{C}}}$ be a complexified principal bundle over an oriented Riemannian 4-manifold (M, g). Let $\mathbb{A}$ be a connection on $P_{\mathcal{G}_{\mathbb{C}}}$. $\mathbb{A}$ is called a complex anti-self-dual connection if it is anti-self-dual with respect to the complex linear Hodge star operator $*_{\mathbb{C}}$,*

$$*_{\mathbb{C}} F_{\mathbb{A}} = -F_{\mathbb{A}}.$$

The complex anti-self-duality equation for $\mathbb{A} = A_1 + iA_2$ reduces to the real equations,

$$(F(A_1) - \frac{1}{2}[A_2 \wedge A_2])^+ = 0, \qquad (d_{A_1} A_2)^+ = 0. \tag{2.5.4}$$

The equations 2.5.4 with the symmetry breaking condition $d^*_{A_1} A_2 = 0$, modulo real gauge group $\mathcal{G}$, form an elliptic system of equations,

$$(F(A_1) - \frac{1}{2}[A_2 \wedge A_2])^+ = 0, \qquad (d_{A_1} A_2)^+ = 0, \qquad d^*_{A_1} A_2 = 0. \tag{2.5.5}$$

The solutions to these equations are called Haydys instantons.

The complex equation $*_{\mathbb{C}} F_{\mathbb{A}} + F_{\mathbb{A}} = 0$ is invariant under the action of the complex gauge group, $\mathcal{G}_{\mathcal{C}} = Aut(P_{G_{\mathcal{C}}})$, and therefore, the real equations 2.5.4, although they are written on a G-bundle, they are invariant under $\mathcal{G}_{\mathcal{C}}$. These equations, even modulo the real gauge group $\mathcal{G}$, are underdetermined and not elliptic. One hopes to get an elliptic system, modulo $\mathcal{G}$, by adding an extra equation. This equation should be invariant under the action of $\mathcal{G}$, but not $\mathcal{G}_{\mathbb{C}}$. The equation $d^*_{A_1} A_2 = 0$ is such. This is a gauge symmetry breaking, similar to the stability condition in the study of complex flat connections.

The space of complex connections is an infinite-dimensional flat Kähler manifold with a Hamiltonian action of the real gauge group $\mathcal{G}$. The equation $d^*_{A_1} A_2 = 0$ describes the zero-set of the moment map.

Theorem 14 (Haydys [41])**.** *Let (M, g) be a closed oriented Riemannian 4-dimensional manifold. Denote the moduli space of Haydys instantons on $P_{G_{\mathbb{C}}}$ and the moduli space of (real) instantons* *on P_G by $\mathcal{M}_{\mathbb{C}}$ and $\mathcal{M}$, respectively. We have*

$$virdim\mathcal{M}_{\mathbb{C}} = 2virdim\mathcal{M}.$$

Assuming $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ are smooth manifolds of expected dimensions, $\mathcal{M}_{\mathbb{C}}$ is Kähler. Furthermore, suppose (M, J_M, ω_M) is a 4-dimensional Kähler manifold. Then $\mathcal{M}_{\mathbb{C}}$ is hyperkähler and $\mathcal{M}$ is a complex Lagrangian submanifold of $\mathcal{M}_{\mathbb{C}}$.

Complex instantons, with respect to conjugate linear Hodge star operator, have quite different behaviours. The prototype of these equations are the Kapustin-Witten equations.

Definition 38 (Kapustin-Witten Instanton)**.** *Let (M, g) be an oriented Riemannian 4-manifold. Let $P_{\mathcal{G}_{\mathbb{C}}} \to M$ be a complexified principal bundle. Let $\mathbb{A}$ be a connection on $P_{\mathcal{G}_{\mathbb{C}}}$. We call $\mathbb{A}$ a complex conjugate anti-self-dual instanton if it satisfies the following equation,*

$$\overline{*}_{\mathbb{C}} F_{\mathbb{A}} = -F_{\mathbb{A}},$$

where $\overline{*}_{\mathbb{C}}$ is the complex conjugate extension of the Hodge star operator.

For $\mathfrak{g}_P$-valued pair (A_1, A_2) where $\mathbb{A} = A_1 + iA_2$, this equation reduces to

$$(F(A_1) - \frac{1}{2}[A_2 \wedge A_2])^+ = 0, \qquad (d_{A_1} A_2)^- = 0. \tag{2.5.6}$$

A connection $\mathbb{A} = A_1 + iA_2$ is called a stable complex conjugate anti-self-dual instanton if

$$(F(A_1) - \frac{1}{2}[A_2 \wedge A_2])^+ = 0, \qquad (d_{A_1} A_2)^- = 0, \qquad d^*_{A_1} A_2 = 0. \tag{2.5.7}$$

One interesting feature of these equations is that they fit into a larger family of equations. The Kapustin-Witten equations, with phase $e^{i\theta}$, are

$$\begin{aligned}
(\cos(\theta)(F(A_1) - \frac{1}{2}[A_2 \wedge A_2]) - \sin(\theta) d_{A_1} A_2)^+ &= 0, \\
(\sin(\theta)(F(A_1) - \frac{1}{2}[A_2 \wedge A_2]) + \cos(\theta) d_{A_1} A_2)^- &= 0, \\
d^*_{A_1} A_2 &= 0.
\end{aligned}$$

The Kapustin-Witten connections are the Kapustin-Witten instantons with $\theta = \frac{\pi}{4}$,

$$F(A_1) - \frac{1}{2}[A_2 \wedge A_2] = *d_{A_1} A_2, \qquad d^*_{A_1} A_2 = 0.$$

Smooth Kapustin-Witten instantons on closed 4-manifolds satisfy a stronger condition. We have the following theorem, from Gagliardo and Uhlenbeck [36].

Theorem 15 (The Vanishing Theorem)**.** *The Kapustin-Witten instantons with $\theta \neq 0, \frac{\pi}{2}$, on compact 4-manifolds, potentially with boundary, are complex flat connections,*

$$F(A_1) - \frac{1}{2}[A_2 \wedge A_2] = *d_{A_1} A_2 = 0, \qquad d^*_{A_1} A_2 = 0.$$

2.5.2.3 Complexification of Monopoles

Nagy and Oliveira introduced complex monopoles on oriented Riemannian 3-manifolds [73, 73], as dimensional reduction of Haydys and Kapustin-Witten instantons. We start with the definitions.

Definition 39 (Haydys Monopole [73]). *Let $P_{\mathcal{G}_{\mathbb{C}}}$ be a complexified principal bundle over an oriented Riemannian 3-manifold (M, g). Let $(\mathbb{A}, \Upsilon)$ be a pair of a connection $\mathbb{A}$ on $P_{\mathcal{G}_{\mathbb{C}}}$ and a section Υ of the complexified adjoint bundle. Let $*_{\mathbb{C}}$ be the complex Hodge star operator. The complex Bogomolny equation for a pair $(\mathbb{A}, \Upsilon)$ is*

$$*_{\mathbb{C}} F_{\mathbb{A}} = d_{\mathbb{A}} \Upsilon.$$

Let $\mathbb{A} = A_1 + iA_2$ and $\Upsilon = \Phi_1 + i\Phi$. The complex Bogomolny equation for A_1, A_2, Φ_1 and Φ_2 reduces to

$$*(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) - d_{A_1}\Phi_2 + [A_2, \Phi_2] = 0, \tag{2.5.8}$$

$$*d_{A_1}A_2 - d_{A_1}\Phi_2 - [A_2, \Phi_1] = 0. \tag{2.5.9}$$

The complex Bogomolny equations are invariant under the action of the complex gauge group $\mathcal{G}_{\mathbb{C}}$. This shows the complex Bogomolny equations for quadruplet $(A_1, A_2, \Phi_1, \Phi_2)$, even modulo real gauge group $\mathcal{G}$, are not elliptic. The equations can be made elliptic by introducing a new equation

$$d^*_{A_1}A_2 + [\Phi_1, \Phi_2] = 0. \tag{2.5.10}$$

The quadruplets satisfying the complex Bogomolny equations and 2.5.10 are called Haydys monopoles.

These equations can be understood as dimensional reduction of Haydys instantons.

Lemma 59. *Let (M, g) be an oriented Riemannian 3-manifold. Let $N = M \times \mathbb{R}_t$, or $N = M \times S^1_t$, equipped with the product metric $h = g + dt^2$ where t is the variable in the $\mathbb{R}$-direction. Let $\pi : N \to M$ be the projection map. Let $P_G \to M$ be a principal G-bundle and $\pi^* P_G \to N$ the pull-back bundle. A pair of translation-invariant connections $(\widetilde{A}_1, \widetilde{A}_2)$, in a translation-invariant gauge, can be written as*

$$\widetilde{A}_i = \pi^*(A_i) + \pi^*(\Phi_i)dt,$$

for a connection A_i on P_G, a section Φ_i of the adjoint bundle, and $i \in \{1, 2\}$.

A pair (A_1, A_2) is a Haydys instanton if and only if the quadruplet $(A_1, A_2, \Phi_1, \Phi_2)$ is a Haydys monopole.

The proof is similar to the proof of Lemma 1.

Assuming maximal symmetry breaking, Nagy and Oliveira showed that the moduli space of finite energy Haydys monopoles on $\mathbb{R}^3$ is a hyperkähler manifold in 3-different ways and contains the ordinary Bogomolny moduli space as a complex Lagrangian submanifold.

Theorem 16 (Nagy-Oliveira [73]). *Let $M = \mathbb{R}^3$ equipped with the Euclidean metric g_0. Let G be a compact Lie group. Let $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ the moduli spaces of Bogomolny and Haydys monopoles. $\mathcal{M}$ embedds naturally into $\mathcal{M}_{\mathbb{C}}$ by letting $A_2 = 0 = \Phi_2$.*

- *There are finite energy Haydys monopoles that are not Bogomolny monopoles.*
- *$\mathcal{M}_{\mathbb{C}}$ carries an $Cl(4)$-structure, which means there are 3 different hyperkähler structures, denoted by $(I_1, I_2, I_3), (J_1, J_2, J_3), and (K_1, K_2, K_3)$, all compatible with the L^2-metric on the moduli space. Furthermore, $I_1 = J_1 = K_1$ and I_2, J_2 and K_2 pairwise anti-commute. Moreover, $e_1 = I_1, e_2 = I_2, e_3 = J_2$, and $e_4 = K_4$ are algebraically independent, anti-commuting complex structures, and therefore, giving the tangent bundle the structure of a $Cl(4)$-module.*
- *$\mathcal{M}$ is a complex Lagrangian submanifold of $\mathcal{M}_{\mathbb{C}}$ with respect to the either of the 3 hyperkähler structures. Moreover, it is complex with respect to complex structures I_2, J_2 and K_2. Furthermore, it is Lagrangian with respect to the Kähler structures induced by the other complex structures.*

Similar to the case of instantons, one can extend the Hodge star to the complex-valued forms by letting $\overline{*}_{\mathbb{C}} i = -i \overline{*}_{\mathbb{C}}$. These monopoles are called the Kapustin-Witten monopoles.

Definition 40 (Kapustin-Witten Monopole [74]). *Let $P_{\mathcal{G}_{\mathbb{C}}}$ be a complexified principal bundle over an oriented Riemannian 3-manifold (M, g). Let $(\mathbb{A}, \Upsilon)$ be a pair of a connection $\mathbb{A}$ on $P_{\mathcal{G}_{\mathbb{C}}}$ and a section Υ of the complexified adjoint bundle. Let $*_{\mathbb{C}}$ be the complex conjugate Hodge star operator. The complex conjugate Bogomolny equation for a pair $(\mathbb{A}, \Upsilon)$ is*

$$\overline{*}_{\mathbb{C}} F_{\mathbb{A}} = d_{\mathbb{A}} \Upsilon.$$

Let $\mathbb{A} = A_1 + iA_2$ and $\Upsilon = \Phi_1 + i\Phi$. The complex conjugate Bogomolny equation for A_1, A_2, Φ_1 and Φ_2 reduces to

$$*(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) - d_{A_1}\Phi_2 + [A_2, \Phi_2] = 0, \tag{2.5.11}$$

$$*d_{A_1} A_2 + d_{A_1}\Phi_2 + [A_2, \Phi_1] = 0. \tag{2.5.12}$$

The complex conjugate Bogomolny equations are invariant under the action of the complex gauge group $\mathcal{G}_{\mathbb{C}}$. This shows the complex conjugate Bogomolny equations for quadruplets $(A_1, A_2, \Phi_1, \Phi_2)$, even modulo real gauge group $\mathcal{G}$, are not elliptic. The equations can be made elliptic by introducing the equation

$$d^*_{A_1} A_2 + [\Phi_1, \Phi_2] = 0. \tag{2.5.13}$$

The quadruplet satisfying the complex conjugate Bogomolny equations and 2.5.13 are called Kapustin-Witten monopoles.

Nagy and Oliveira showed the Kapustin-Witten monopoles satisfy a vanishing theorem.

Theorem 17 (Nagy-Oliveira [74]). *Let $M = \mathbb{R}^3$ equipped with the Euclidean metric g_0. Let $(A_1, A_2, \Phi_1, \Phi_2)$ be a Kapustin-Witten monopole on $P_{SU(2)} \to \mathbb{R}^3$ with finite energy. Then (A_1, Φ_1) is a finite energy monopole and $(A_2, \Phi_2) = (0, 0)$.*

In the following section, we will see a higher-dimensional generalizations of this theory.

2.5.3 Complex Gauge Theory and Manifolds with Special Holonomy Groups

In this section, we introduce the complexified gauge-theoretic equations, instantons and monopoles, over manifolds with special holonomy groups.

Similar to the low dimensional case, one can consider the complex linear extension of the Hodge star operator or the complex conjugate one. We call the equations defined with respect to $*_{\mathbb{C}}$ of the Haydys type, and the equations defined with respect to $\overline{*}_{\mathbb{C}}$ of the Kapustin-Witten type.

We start by complexifying the $Spin(7)$-instanton equation on $Spin(7)$-manifolds.

2.5.3.1 Complex *Spin*(7)-Instantons

In this section, we introduce the stable complex $Spin(7)$-instantons, and show they form a Kähler space, similar to the lower-dimensional case.

Definition 41 (Stable Complex $Spin(7)$-Instanton). *Let (M, Ω) be a $Spin(7)$-manifold. Let $P_{\mathcal{G}_{\mathbb{C}}} \to M$ be a complexified principal bundle. A connection $\mathbb{A}$ on $P_{\mathcal{G}_{\mathbb{C}}}$ is called a complex $Spin(7)$-instanton if satisfies the $Spin(7)$-instanton equation with respect to the complex linear $*_{\mathbb{C}}$,*

$$*_{\mathbb{C}}(F_{\mathbb{A}} \wedge \Omega) = -F_{\mathbb{A}}.$$

The complex $Spin(7)$-instanton equations for $\mathbb{A} = A_1 + iA_2$ reduces to the real equations,

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \Omega \right) + F_{A_1} - \frac{1}{2}[A_2 \wedge A_2] = 0, \tag{2.5.14}$$

$$* \left((d_{A_1} A_2) \wedge \Omega \right) + d_{A_1} A_2 = 0, \tag{2.5.15}$$

where $*$ is the Hodge star operator associated to the $Spin(7)$-metric on M.

We call solutions to the complex $Spin(7)$-instanton equations with the symmetry breaking condition

$$d^*_{A_1} A_2 = 0, \tag{2.5.16}$$

the stable complex $Spin(7)$-instantons.

Lemma 60. *The complex $Spin(7)$-instanton equations with the symmetry breaking condition 2.5.16 form an elliptic system. Moreover,*

$$virdim \mathcal{M}_{\mathbb{C}} = 2 virdim \mathcal{M},$$

where $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ are the moduli space of $Spin(7)$-instantons and moduli space of stable complex $Spin(7)$-instantons, respectively.

Proof. The linearized equations at $\mathbb{A} = A_1 + iA_2$ are given by the operator

$$L_{\mathbb{A}} = (L^1_{(A_1,A_2)}, L^2_{(A_1,A_2)}, L^3_{(A_1,A_2)}) : \Omega^1(M, \mathfrak{g}_P) \times \Omega^1(M, \mathfrak{g}_P) \\ \to \Omega^2_7(M, \mathfrak{g}_P) \times \Omega^2_7(M, \mathfrak{g}_P) \times \Omega^0(M, \mathfrak{g}_P),$$

where $L^1_{(A_1,A_2)}, L^2_{(A_1,A_2)}$ and $L^3_{(A_1,A_2)}$ are

$$\begin{aligned} L^1_{(A_1,A_2)}(a_1, a_2) &= *((d_{A_1}a_1 - [A_2 \wedge a_2]) \wedge \Omega) + d_{A_1}a_1 - [A_2 \wedge a_2], \\ L^2_{(A_1,A_2)}(a_1, a_2) &= *((d_{A_1}a_2 + [a_1 \wedge A_2]) \wedge \Omega) + d_{A_1}a_2 + [a_1 \wedge A_2], \\ L^3_{(A_1,A_2)}(a_1, a_2) &= d^*_{A_1}a_2 + *[a_1 \wedge *A_2]. \end{aligned}$$

Note the image of the linear maps $L^1_{(A_1,A_2)}$ and $L^1_{(A_1,A_2)}$ are subset of $\Omega^2_7(M, \mathfrak{g}_P)$. Let

$$d_{(A_1,A_2)} : \Omega^0(M, \mathfrak{g}_P) \to \Omega^1(M, \mathfrak{g}_P) \times \Omega^1(M, \mathfrak{g}_P), \qquad d_{(A_1,A_2)}(\xi) = (-d_{A_1}\xi, -d_{A_2}\xi).$$

We get the following deformation complex,

$$0 \to \Omega^0(M, \mathfrak{g}_P) \xrightarrow{d_{(A_1,A_2)}} \Omega^1(M, \mathfrak{g}_P) \times \Omega^1(M, \mathfrak{g}_P) \\ \xrightarrow{L_{(A_1,A_2)}} \Omega^2_7(M, \mathfrak{g}_P) \times \Omega^2_7(M, \mathfrak{g}_P) \times \Omega^0(M, \mathfrak{g}_P) \to 0.$$

Note that

$$rank(\Omega^0(M, \mathfrak{g}_P)) + rank(\Omega^2_7(M, \mathfrak{g}_P) \times \Omega^2_7(M, \mathfrak{g}_P) \times \Omega^0(M, \mathfrak{g}_P)) = 16 \\ = rank(\Omega^1(M, \mathfrak{g}_P) \times \Omega^1(M, \mathfrak{g}_P)),$$

and the operator

$$D_{(A_1,A_2)} := L_{(A_1,A_2)} \oplus d^*_{(A_1,A_2)} : \Omega^1(M) \times \Omega^1(M) \\ \to \Omega^2_7(M) \times \Omega^2_7(M) \times \Omega^0(M) \times \Omega^0(M).$$

is elliptic.

Moreover,

$$T_{\mathbb{A}}\mathcal{M}_{\mathbb{C}} = ker D_{(A_1,A_2)}.$$

Note that by setting $A_2 = 0$ we get a natural embedding of $\mathcal{M}$ into $\mathcal{M}_{\mathbb{C}}$. Moreover,

$$D_{(A_1,0)}(a_1, a_2) = (*((da_1) \wedge \Omega) + da_1, *((da_1) \wedge \Omega) + da_2, 2d^*a_1, d^*a_2),$$

and therefore

$$\begin{aligned} kerD_{(A_1,0)} = \{(a_1,a_2) \mid &* ((da_1)\wedge\Omega) + da_1 + d^*a_1 = 0 \\ &\text{and } * ((da_2)\wedge\Omega) + da_2 + d^*a_2 = 0\} \cong T_{A_1}\mathcal{M} \oplus T_{A_1}\mathcal{M}, \end{aligned}$$

which implies $virdim\mathcal{M}_{\mathbb{C}} = 2virdim\mathcal{M}$ at $(A_1, 0)$. □

The following lemma is similar to the case of Haydys instantons in dimension four.

Theorem 18. *Suppose $\mathcal{M}$ and $\mathcal{M}_{\mathbb{C}}$ are smooth manifolds of expected dimensions. $\mathcal{M}_{\mathbb{C}}$ is Kähler.*

Proof. The space of complex connections $\mathcal{A}_{\mathbb{C}}$ form a flat infinite-dimensional Kähler manifold, with almost complex structure given by

$$\begin{aligned} &I(a_1 + ia_2) = -a_2 + ia_1, \\ &g(a_1 + ia_2, b_1 + ib_2) = \int_M tr(a_1 \wedge *b_1 + a_2 \wedge *b_2), \\ &\omega(a_1 + ia_2, b_1 + ib_2) = \int_M tr(a_1 \wedge *b_2 - a_2 \wedge *b_1). \end{aligned}$$

The action of the real gauge group is Hamiltonian with the moment map $\mu : \mathcal{A}_{\mathbb{C}} \to Lie(\mathcal{G}) \otimes \mathbb{R}^*$, given by

$$\mu(A_1 + iA_2) = d^*_{A_1} A_2.$$

We have,

$$\begin{aligned} \mathcal{M}_{\mathbb{C}} = &\text{ Stable complex } Spin(7)\text{-instantons } /\mathcal{G} \\ &= (\text{Complex } Spin(7)\text{-instantons } /\mathcal{G}) \cap (\mu^{-1}(0)/\mathcal{G}). \end{aligned}$$

Assuming transversality, $\mu^{-1}(0)/\mathcal{G}$ is a Kähler manifold with the virtue of infinite-dimensional Kähler reduction. Furthermore, we have

$$\begin{aligned} L^1_{(A_1,A_2)}(J(a_1,a_2)) &= L^1_{(A_1,A_2)}(-a_2,a_1) = * ((-d_{A_1}a_2 - [A_2 \wedge a_1]) \wedge \Omega) - d_{A_1}a_2 - [A_2 \wedge a_1] \\ &= -L^2_{(A_1,A_2)}(a_1,a_2). \\ L^2_{(A_1,A_2)}(J(a_1,a_2)) &= L^2_{(A_1,A_2)}(-a_2,a_1) = * ((d_{A_1}a_1 - [a_2 \wedge A_2]) \wedge \Omega) + d_{A_1}a_1 - [a_2 \wedge A_2] \\ &= L^1_{(A_1,A_2)}(a_1,a_2). \end{aligned}$$

This shows the map $L^1_{(A_1,A_2)} \oplus L^2_{(A_1,A_2)}$ is J-holomorphic, and therefore, the space complex $Spin(7)$-instantons is a complex subvariety of $\mathcal{A}_{\mathbb{C}}$, thus it is Kähler. Furthermore, $\mathcal{G}$ preserves the Kähler structure, hence assuming transversal intersection, $\mathcal{M}_{\mathbb{C}}$ is Kähler. □

Similar statements hold for complex G_2-instantons on G_2-manifolds. We mention them very briefly.

2.5.3.2 Complex G_2-Instantons

Definition 42 (Stable Complex G_2-Instanton)**.** *Let (M, ϕ) be a G_2-manifold. Let $\mathbb{A}$ be a connection on a complexified principal bundle $P_{\mathcal{G}_{\mathbb{C}}} \to M$. We call a connection $\mathbb{A}$ on $P_{\mathcal{G}_{\mathbb{C}}}$ a complex G_2-instanton if*

$$*_{\mathbb{C}}(F_{\mathbb{A}} \wedge \phi) = -F_{\mathbb{A}},$$

where $*_{\mathbb{C}}$ is the complex linear extension of the Hodge star operator.

For $\mathfrak{g}$-valued pair (A_1, A_2), this equation reduces to

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \phi \right) = -F_{A_1} + \frac{1}{2}[A_2 \wedge A_2], \tag{2.5.17}$$

$$* \left((d_{A_1} A_2) \wedge \phi \right) = -d_{A_1} A_2. \tag{2.5.18}$$

A connection $\mathbb{A} = A_1 + iA_2$, is called a stable complex G_2-instanton if

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \phi \right) = -F_{A_1} + \frac{1}{2}[A_2 \wedge A_2], \tag{2.5.19}$$

$$* \left((d_{A_1} A_2) \wedge \phi \right) = -d_{A_1} A_2, \tag{2.5.20}$$

$$d^*_{A_1} A_2 = 0. \tag{2.5.21}$$

Theorem 19. *The stable complex G_2-instanton equations, modulo the real gauge group $\mathcal{G}$, form a an elliptic system of equations. Furthermore, assuming the transversality,* $\dim \mathcal{M}^7_{\mathbb{C}} = 2 \dim \mathcal{M}^7$, *where $\mathcal{M}^7_{\mathbb{C}}$ and $\mathcal{M}^7$ are the moduli spaces of stable complex G_2-instantons and G_2-instantons, respectively. $\mathcal{M}^7_{\mathbb{C}}$ has a Kähler structure and $\mathcal{M}^7$ embeds into $\mathcal{M}^7_{\mathbb{C}}$ as a Lagrangian submanifold.*

The proof is similar to the $Spin(7)$ case.

2.5.3.3 Kapustin-Witten $Spin(7)$ and G_2-Instantons

In this section, we consider the complex $Spin(7)$ and G_2-instantons with respect to the complex conjugate Hodge star.

Definition 43 (Kapustin-Witten $Spin(7)$-Instanton)**.** *Let (M, Ω) be a $Spin(7)$-manifold. Let $P_{\mathcal{G}_{\mathbb{C}}} \to M$ be a complexified principal bundle. A connection $\mathbb{A}$ on $P_{\mathcal{G}_{\mathbb{C}}}$ is called a complex conjugate $Spin(7)$-instanton if*

$$\overline{*}_{\mathbb{C}}(F_A \wedge \Omega) = -F_A,$$

where $\overline{*}_{\mathbb{C}}$ is the complex conjugate Hodge star operator.

The complex conjugate $Spin(7)$-instanton equation for $\mathbb{A} = A_1 + iA_2$ reduces to

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \Omega \right) = -F_{A_1} + \frac{1}{2}[A_2 \wedge A_2], \tag{2.5.22}$$

$$* \left((d_{A_1} A_2) \wedge \Omega \right) = d_{A_1} A_2. \tag{2.5.23}$$

We call the solutions to these equations the stable complex-conjugate $Spin(7)$-instantons if they also satisfy the symmetry breaking condition,

$$d^*_{A_1} A_2 = 0. \tag{2.5.24}$$

These equations, fit into a larger class of equations,

$$\begin{aligned}
* \left((\cos(\theta)(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) - \sin(\theta)(d_{A_1} A_2)) \wedge \Omega \right) \\
&= -\cos(\theta)(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + \sin(\theta)(d_{A_1} A_2), \\
* \left((\sin(\theta)(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + \cos(\theta)(d_{A_1} A_2)) \wedge \Omega \right) \\
&= \sin(\theta)(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + \cos(\theta)(d_{A_1} A_2), \\
d^*_{A_1} A_2 &= 0.
\end{aligned}$$

We call solutions to the equations above, when $\theta = \frac{\pi}{4}$, the Kapustin-Witten $Spin(7)$-instantons,

$$\begin{aligned}
& * \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2] \right) - (d_{A_1} A_2) \wedge \Omega) = -(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2), \\
& * \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2) \wedge \Omega \right) = (F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2), \\
& d^*_{A_1} A_2 = 0.
\end{aligned}$$

Similarly one can define the Kapustin-Witten G_2-instantons, which on a complexified principal bundle $P_{G_{\mathbb{C}}} \to N$, over a G_2-manifolds (N, ϕ) are given by

$$\begin{aligned}
& * \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) - (d_{A_1} A_2) \wedge \phi \right) = -(F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2), \\
& * \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2) \wedge \phi \right) = (F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) + (d_{A_1} A_2), \\
& d^*_{A_1} A_2 = 0.
\end{aligned}$$

We have the following conjecture.

Conjecture 3 (Vanishing Properties)**.** *The $Spin(7)$-Kapustin-Witten-instantons with $\theta \neq 0, \frac{\pi}{2}$, on compact $Spin(7)$-manifolds, potentially with boundary, satisfy*

$$(F(A_1) - \frac{1}{2}[A_2 \wedge A_2] - *d_{A_1} A_2) \wedge \Omega = 0, \qquad d^*_{A_1} A_2 = 0.$$

We expect the similar statement holds in the G_2-case too.

2.5.3.4 Complex Monopoles on Manifolds with Special Holonomy

Definition 44 (Complex G_2-Monopole)**.** *Let (M, ϕ) be a G_2-manifold with a complexified principal bundle $P_{G_{\mathbb{C}}} \to M$. Let $(\mathbb{A}, \Upsilon)$ be a connection $\mathbb{A} = A_1 + iA_2$ on $P_{G_{\mathbb{C}}}$ and $\Upsilon = \Phi_1 + i\Phi_2$ a section of the complexified adjoint bundle. We call $(\mathbb{A}, \Upsilon)$ a complex G_2-monopole if it satisfies the G_2-Bogomolny equation with respect to the complex linear Hodge star operator, which for a quadruple $(A_1, A_2, \Phi_1, \Phi_2)$ reduces to*

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \psi \right) = d_{A_1}\Phi_1 - [A_2, \Phi_2], \tag{2.5.25}$$

$$* \left((d_{A_1} A_2) \wedge \psi \right) = d_{A_1}\Phi_2 + [A_2, \Phi_1]. \tag{2.5.26}$$

Similar to the case of complex instantons, these equations, even modulo the real gauge group, are not elliptic. We impose the symmetry breaking equation

$$d^*_{A_1} A_2 - [\Phi_1, \Phi_2] = 0.$$

We call quadruples $(A_1, A_2, \Phi_1, \Phi_2)$ satisfying the complex G_2-monopole equations with the symmetry breaking equation the stable complex G_2-monopoles.

Theorem 20. *The complex G_2-monopoles are dimensional reduction of complex $Spin(7)$-instantons. The complex G_2-monopole equations,modulo the real gauge group $\mathcal{G}$, are elliptic. Furthermore, assuming the transversality, $\dim \mathcal{M}^7_{\mathbb{C}} = 2 \dim \mathcal{M}^7$, where $\mathcal{M}^7_{\mathbb{C}}$ and $\mathcal{M}^7$ are the moduli spaces of stable complex G_2-monopoles and G_2-monopoles, respectively. $\mathcal{M}^7_{\mathbb{C}}$ has a Kähler structure and $\mathcal{M}^7$ embeds into $\mathcal{M}^7_{\mathbb{C}}$ as a complex Lagrangian submanifold.*

The proof is similar to the case of $Spin(7)$-instantons.

Definition 45 (Kapustin-Witten G_2-monopole)**.** *Let (M, ϕ) be a G_2-manifold with a complexified principal bundle $P_{G_{\mathbb{C}}} \to M$. Let $(\mathbb{A}, \Upsilon)$ be a connection $\mathbb{A} = A_1 + iA_2$ on $P_{G_{\mathbb{C}}}$ and $\Upsilon = \Phi_1 + i\Phi_2$ a section of the complexified adjoint bundle. We call $(\mathbb{A}, \Upsilon)$ a complex conjugate G_2-monopole if it satisfies G_2-Bogomolny equation with respect to the conjugate complex Hodge star operator, which for the quadruple $(A_1, A_2, \Phi_1, \Phi_2)$ reduces to*

$$* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \psi \right) = d_{A_1}\Phi_1 - [A_2, \Phi_2], \tag{2.5.27}$$

$$* \left((d_{A_1} A_2) \wedge \psi \right) = -d_{A_1}\Phi_2 - [A_2, \Phi_1]. \tag{2.5.28}$$

We have the symmetry breaking equation

$$d^*_{A_1} A_2 - [\Phi_1, \Phi_2] = 0.$$

We call quadruples $(A_1, A_2, \Phi_1, \Phi_2)$ satisfying the complex conjugate G_2-monopole equations with the symmetry breaking condition the Kapustin-Witten G_2-monopoles.

Theorem 21. *The Kapustin-Witten G_2-monopoles are dimensional reduction of Kapustin-Witten $Spin(7)$-instantons. Moreover, the Kapustin-Witten G_2-monopole equations form an elliptic system.*

The proof is similar to the case of $Spin(7)$-instantons.

On closed G_2-manifolds, we expect the Kapustin-Witten G_2-monopoles satisfy stronger conditions.

Conjecture 4. *Suppose* $(A_1, A_2, \varPhi_1, \varPhi_2)$ *is a Kapustin-Witten* G_2*-monopole quadruple on a closed* G_2*-manifold* (M, ϕ)*. Then we have*

$$
\begin{aligned}
&* \left((F_{A_1} - \frac{1}{2}[A_2 \wedge A_2]) \wedge \psi \right) = d_{A_1}\varPhi_1 - [A_2, \varPhi_2] = 0, \\
&* ((d_{A_1} A_2) \wedge \psi) = d_{A_1}\varPhi_2 - [A_2, \varPhi_1] = 0.
\end{aligned}
$$

Chapter 3

Fueter Sections and Monopoles

The Fueter operator is a non-linear generalization of the Dirac operator defined over orientable Riemannian 3- and 4-manifolds, where the spinor bundle is replaced by a hyperkähler bundle with fibers modeled on a non-linear hyperkähler manifold. In this section, we study the regularity and singularities of Fueter sections, i.e., harmonic spinors with respect to this non-linear Dirac operator, and show how singularities of these sections are related to the existence of certain minimal spheres in the hyperkähler manifolds. Using this, we will prove a compactness theorem for the spaces of Fueter sections of the bundles whose fibers are modeled on the moduli spaces of monopoles on $\mathbb{R}^3$.

3.1 Preliminaries

We start by defining the Fueter sections on 3-manifolds.

3.1.1 Fueter Sections on 3-manifolds

Let (M, g) be an oriented Riemannian 3-manifold. Let $\pi : \mathfrak{X} \to M$ be a fiber bundle with fibers modeled on a (compact or non-compact) hyperkähler manifold (X, g_X, I, J, K) with an isometric bundle identification

$$\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X}),$$

where STM is the unit tangent bundle of M and $\mathfrak{b}(\mathfrak{X})$ is the sphere bundle of the complex structures of the fibers of $\mathfrak{X}$. We call $\pi : \mathfrak{X} \to M$ a hyperkähler bundle.

Definition 46 (3D Fueter Section)**.** *Let $\pi : \mathfrak{X} \to M$ be a hyperkähler bundle. Let ∇ be a connection on this bundle. A section $f \in \Gamma(\mathfrak{X})$ is called a Fueter section if*

$$\mathfrak{F}_\nabla(f) := I(\partial x_1)\nabla_{\partial x_1} f + I(\partial x_2)\nabla_{\partial x_2} f + I(\partial x_3)\nabla_{\partial x_3} f = 0 \in \Omega^0(M, f^*V\mathfrak{X}),$$

where $V\mathfrak{X} := ker(d\pi) : T\mathfrak{X} \to TM$ is the vertical bundle and $(\partial x_1, \partial x_2, \partial x_3)$ is a local orthonormal frame on M. The operator $\mathfrak{F}_\nabla$ is called a Fueter operator.

Lemma 61. *The Fueter operators on 3-manifolds are well-defined.*

Proof. One can see this by a direct computation. Let $(\partial x_1, \partial x_2, \partial x_3)$ and $(\partial y_1, \partial y_2, \partial y_3)$ be two oriented orthogonal basis at a point $x \in M$, related by an orthogonal matrix $A \in SO(T_xM)$; i.e., $\partial x_i = \sum_{j=1}^{3} A_{i,j}\partial y_j$. Then

$$\sum\nolimits_{i=1}^{3} I(\partial x_i) \cdot df(\partial x_i) = \sum\nolimits_{i=1}^{3} I(\sum_{j=1}^{3} A_{i,j}\partial y_i) \cdot df(\sum_{k=1}^{3} A_{i,k}\partial y_i) =$$

$$\sum\nolimits_{j,k=1}^{3} \sum_{i=1}^{3} (A_{i,j}A_{i,k}) I(\partial y_j) \cdot df(\partial y_k) = \sum\nolimits_{i=1}^{3} I(\partial y_i) \cdot df(\partial y_i).$$

It can also be seen by expressing the Fueter operator in a coordinate-free way. The isometric bundle identification $\mathcal{I}$ and the Riemannian metric g induce a bundle homomorphism, which by an abuse of notation, we still denote it by $\mathcal{I}$,

$$\mathcal{I} : T^*M \otimes V\mathfrak{X} \to V\mathfrak{X}.$$

This map is a generalization of the Clifford multiplication of the spinor bundles to the non-linear case. The Fueter operator can be written as

$$\mathfrak{F}_\nabla(f) = \mathcal{I}(\nabla(f)),$$

which is independent of the chosen local frame on M. ☐

Example 9. If the bundle $\mathfrak{X}$ is trivial, $\mathfrak{X} = X \times M$, with trivial connection $\nabla = d$, we can think of a Fueter section of this bundle as a map $f : M \to X$ such that

$$\sum_{i=1}^{3} I(\partial x_i)\partial x_i f = 0.$$

We call f a Fueter map.

Example 10. *The Fueter operator is a non-linear generalization of Dirac operator. Let $\mathfrak{s}$ be a spin structure on M and $\mathfrak{X}$ be the corresponding spinor bundle. Let Δ be the induced Levi-Civita connection and $\mathcal{I}$ the Clifford multiplication. The corresponding Fueter operator is the Dirac operator associated with the spin structure $\mathfrak{s}$. This non-linear generalization has been introduced in [89] and [3].*

There is a 4-dimensional version of this theory.

3.1.2 Fueter Sections on 4-manifolds

Let V be a 4-dimensional vector space equipped with an inner product. Using the metric, we can identify 2-forms on V with skew-symmetric endomorphisms of V. Let $\iota : \Lambda^+V^* \to \mathfrak{so}(V)$. Let (N, h) be an oriented Riemannian 4-manifold. Let $\pi : \mathfrak{X} \to N$ be a fiber bundle, with fibers modeled on a hyperkähler manifold (X, g_X, I, J, K), together with a fixed isometric bundle identification

$$\mathcal{J} : S\Lambda^+(T^*N) \to \mathfrak{b}(\mathfrak{X}),$$

where $S\Lambda^+(T^*N)$ is the unit length sphere bundle of the self-dual 2-forms on N and $\mathfrak{b}(\mathfrak{X})$ is the sphere bundle of the complex structures of the fibers of $\mathfrak{X}$.

Definition 47 (4D Fueter Section). *A section $f \in \Gamma(\mathfrak{X})$ is called a Fueter section if*

$$\widetilde{\mathfrak{F}}_\nabla(f) := \nabla f - \sum_{i=1}^{3} \mathcal{J}(\omega_i) \circ (\nabla f) \circ \iota(\omega_i) = 0 \in \Gamma(f^* Hom(\pi^* TN, V\mathfrak{X})),$$

for some local orthonormal frame $(\omega_1, \omega_2, \omega_3)$ of $\Lambda^+(T^*N)$ and a connection ∇ on $\mathfrak{X}$. Similar to the 3-dimensional case, it is independent of the chosen orthonormal frame.

Lemma 62. *Fueter operators on 4-manifolds are well-defined.*

Proof. Similar to the 3-dimensional case, one can see this by a direct computation. Let $(\partial x_1, \partial x_2, \partial x_3, \partial x_4)$ and $(\partial y_1, \partial y_2, \partial y_3, \partial y_4)$ be two oriented orthogonal basis at a point $x \in N$, related by an orthogonal matrix $A \in SO(T_x N)$. Let $\omega_i = dx_1 \wedge dx_i + dx_j \wedge dx_k$ and $\widetilde{\omega}_i = dy_1 \wedge dy_i + dy_j \wedge dy_k$, where $\{i, j, k\} = \{1, 2, 3\}$ are cyclic. Since the matrix A preserves the metric, it induces a linear map $A^* \in SO(\Lambda^+_x N)$. We have

$$\sum_{i=1}^{3} \mathcal{J}(\omega_i) \circ (\nabla f) \circ \iota(\omega_i) = \sum_{i=1}^{3} \mathcal{J}(\sum_{j=1}^{3} A^*_{i,j} \widetilde{\omega}_j) \circ (\nabla f) \circ \iota(\sum_{k=1}^{3} A^*_{i,k} \widetilde{\omega}_k) =$$
$$\sum_{j,k=1}^{3} \sum_{i=1}^{3} A^*_{i,j} A^*_{i,k} \mathcal{J}(\widetilde{\omega}_j) \circ (\nabla f) \circ \iota(\widetilde{\omega}_k) = \sum_{j=1}^{3} \mathcal{J}(\widetilde{\omega}_j) \circ (\nabla f) \circ \iota(\widetilde{\omega}_j).$$

□

Remark 6. *Let $N = \mathbb{R}_t \times M$ for some Riemannian 3-manifold (M, g) equipped with a Riemannian metric $h = dt^2 + g$, where t denotes the coordinates on $\mathbb{R}$. Let $\pi : N \to M$ be the obvious projection map. Let $\mathfrak{X} \to M$ be a hyperkähler fiber bundle and $\pi^*\mathfrak{X} \to N$ the pull-back bundle. Moreover, suppose the identification $\mathcal{J} : S\Lambda^+(T^*N) \to \mathfrak{b}(\mathfrak{X})$ is induced by an identification $\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X})$, and the connection on $\pi^*\mathfrak{X}$ is the pullback of a connection on $\mathfrak{X}$. Then the 4-dimensional Fueter equation can be written as*

$$\partial_t u - \mathfrak{F} u = 0,$$

with $\mathfrak{F}$ denoting the 3-dimensional Fueter operator and ∂_t the derivative in the $\mathbb{R}$-direction. This 4-dimensional Fueter operator appears in [46], in order to define the differentials of a hyperkähler Floer theory.

3.1.3 Group Actions on Hyperkähler Manifolds

With motivations coming from higher-dimensional gauge theory, generalized Seiberg-Witten theory and a new — potentially — invariant of 3- and 4-manifolds, an interesting case to consider is when we have a hyperkähler bundle with a certain G-action on the fibers, which is defined below.

Note that on hyperkähler manifolds there exists an S^2-family of complex and Kähler structures. For any $\zeta = \zeta_1 i + \zeta_1 j + \zeta_1 k \in S^2 \subset \mathbb{R}^3$, we have corresponding complex structure and Kähler form, defined by

$$I_\zeta = \zeta_1 I + \zeta_1 J + \zeta_1 K, \qquad \omega_\zeta = \zeta_1 \omega_1 + \zeta_1 \omega_2 + \zeta_1 \omega_3.$$

Let $\omega \in \mathfrak{sp}(1)^* \otimes \Omega^2(X)$ be the hyperkähler form, defined by

$$\langle \omega, \zeta \rangle = \omega_\zeta \quad \text{for any} \quad \zeta = \zeta_1 i + \zeta_2 j + \zeta_3 k \in \mathfrak{sp}(1) \cong \mathrm{Im}(\mathbb{H}).$$

A certain type of group actions, which appears naturally in the study of hyperkähler manifolds, is the permuting hyperkähler action.

Definition 48 (Permuting Hyperkähler Action)**.** *Let (X, g_X, I, J, K) be a hyperkähler manifold with an $SO(3)$-action. This actions is called permuting if*

- *$SO(3)$ acts isometrically.*
- *The induced action of $SO(3)$ on the 2-sphere of complex structures — or, equivalently, on Kähler structures — is the standard action of $SO(3)$ on $S^2 \subset \mathbb{R}^3$.*

Similarly, an isometric action of $SU(2)$ on M is called permuting, if the induced action on the 2-sphere of complex structures is the standard action of $SU(2)$ factoring through $SO(3)$ on $S^2 \subset \mathbb{R}^3$.

Example 11. *$SO(3)$ acts on $\mathbb{R}^3$, and therefore, on the moduli spaces of solution to the Bogomolny equation on $\mathbb{R}^3$. These actions are permuting.*

There is a strong obstruction for a hyperkähler manifold to have a permuting hyperkähler action.

Lemma 63. *[12, Proposition 1.1] Let (M, g, I, J, K) be a hyperkähler manifold with a permuting $SO(3)$ action. Then every symplectic form ω_ζ in the two-sphere family of symplectic structures is exact, and therefore, M is non-compact.*

Permuting actions can be used to construct hyperkähler bundles.

3.1.4 Hyperkähler Bundles

In this section, we define the main bundles where our Fueter sections are defined on.

Definition 49 (Permuting Hyperkähler $SO(3)$-Bundle over a 3-Manifold)**.** *Let (M, g) be an oriented Riemannian 3-manifold with the oriented orthonormal frame bundle $Fr_{SO(3)} \to M$. Let (X, g_X, I, J, K) be a hyperkähler manifold with a permuting $SO(3)$-action. The permuting hyperkähler $SO(3)$-bundle $\mathfrak{X} \to M$ is defined by*

$$\mathfrak{X} := Fr_{SO(3)} \times_{SO(3)} X \xrightarrow{\pi} M.$$

Example 12. *In particular, when $X = Mon_k^\circ$ is the moduli space of centered k-monopoles with the natural induced $SO(3)$-action, we get a permuting hyperkähler bundle, called the monopole bundle, denoted by $\mathfrak{Mon}_k^\circ \to M$.*

An interesting aspect of permuting hyperkähler bundles is that, unlike the general case, these bundles come with a natural identification $\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X})$ and the induced Levi-Civita connection, which can be used to define the Fueter operator.

Definition 50 (Permuting Hyperkähler $SU(2)$-Bundle over a 3-Manifold)**.** *Let (M, g) be an oriented Riemannian 3-manifold with a spin structure and a corresponding $SU(2)$-bundle $Fr_{SU(2)} \to M$. Let (X, g_X, I, J, K) be a hyperkähler manifold with a permuting $SU(2)$-action. Let $\mathfrak{X} \to M$ be the permuting hyperkähler $SU(2)$-bundle, defined by*

$$\mathfrak{X} := Fr_{SU(2)} \times_{SU(2)} X \xrightarrow{\pi} M.$$

Similar to the $SO(3)$ case, there is a natural isomorphism between the unit sphere bundle and the complex structures on X, and we can use the Levi-Citiva connection to define the Fueter operator.

Definition 51 (Auxiliary Hyperkähler Bundle over a 3-Manifold)**.** *Let (M, g) be an oriented Riemannian 3-manifold with the oriented orthonormal frame bundle $Fr_{SO(3)} \to M$. Let (X, g_X, I, J, K) be a hyperkähler manifold with a permuting $SO(3)$-action. Furthermore, suppose G is a Lie group with a hyperkähler action on X, and therefore, X is equipped with an action of $G \times SO(3)$. Moreover, let $Q \to M$ be a principal G-bundle. The auxiliary $G \times SO(3)$-hyperkähler bundle $\mathfrak{X} \to M$ is defined by*

$$\mathfrak{X} := (Q \times Fr) \times_{G \times SO(3)} X \xrightarrow{\pi} M.$$

Similarly we can define an auxiliary $G \times SU(2)$-hyperkähler bundle by replacing $Fr_{SO(3)}$ by a $SU(2)$-bundle corresponding to a spin structure and replacing X with a hyperkähler manifold with a permuting $SU(2)$-action.

Similar constructions exist over Riemannian 4-manifolds. Let (N, h) be an oriented Riemannian 4-manifold. The Hodge star of the Riemannian metric $* : \Lambda^2(T^*N) \to \Lambda^2(T^*N)$ induces a bundle decomposition

$$\Lambda^2(T^*N) = \Lambda^2_+(T^*N) \oplus \Lambda^2_-(T^*N).$$

The self dual and anti-self dual 2-forms form vector bundles over N with structure group $SO(3)$, with the following associated frame bundles

$$Fr^{\pm}_{SO(3)} \to N.$$

Definition 52 (Permuting Hyperkähler $SO(3)$-Bundles over 4-Manifolds)**.** *Let (N, h) be an oriented Riemannian 4-manifold with frame bundles $Fr^{\pm}_{SO(3)} \to N$. Let (X, g, I, J, K) be a hyperkähler manifold with a permuting $SO(3)$-action. The permuting hyperkähler $SO(3)$-bundles $\mathfrak{X}^{\pm} \to N$ are defined by*

$$\mathfrak{X}^{\pm} := Fr^{\pm}_{SO(3)} \times_{SO(3)} X \xrightarrow{\pi} N.$$

In the following section, we study the basic analysis of the Fueter sections.

3.2 The Fredholm Theory

A key property governing the analytical properties of Fueter sections is an energy identity.

3.2.1 The Energy Identity

In the Floer theory introduced by Hohloch, Noetzel and Salamon [46], the crucial role is played by the hypersymplectic action functional, defined by

$$\mathcal{A} : Map(M, X) \to \mathbb{R}, \qquad \mathcal{A}(f) := -\int_M \sum_{i=1}^{3} \alpha_i \wedge f^*\omega_i,$$

where (M, g) is a quotient of S^3 by a finite subgroup of $SU(2)$, $(\alpha_1, \alpha_2, \alpha_3)$ is a global frame of the cotangent bundle dual to the Reeb vector fields, and (X, g_X, I, J, K) is a hyperkähler manifold.

Here, the framing is fixed globally on the manifold, and therefore, there is no question of well-definedness. However, the same action functional can be defined in the gauge-theoretic setting and is well-defined.

Lemma 64. *Let $\mathfrak{X} \to M$ be a hyperkähler bundle over an oriented Riemannian 3-manifold (M, g) with an isometric bundle identification $\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X})$ and a connection on $\mathfrak{X} \to M$ with covariant derivative ∇. The gauged hypersymplectic action functional, defined by*

$$\mathcal{A} : \Gamma(\mathfrak{X}) \to \mathbb{R}, \qquad \mathcal{A}(f) := -\int_M \sum_{i=1}^{3} dx_i \wedge f^*\omega_{\mathcal{I}(\partial x_i)},$$

— where $(\partial x_1, \partial x_2, \partial x_3)$ is a local oriented orthonormal frame, (dx_1, dx_2, dx_3) is the dual frame is well-defined, where $f^*\omega_{\mathcal{I}(\partial x_i)}$ is defined with the use of the connection.

Note that $\omega_{\mathcal{I}(\partial x_i)}$ is not a differential form on $\mathfrak{X}$. Let $x \in M$. Let $B_x(\varepsilon)$ be a sufficiently small open neighbourhood of x in M, with a fixed unit vector field ∂x_i. $\omega_{\mathcal{I}(\partial x_i)}$ defines a fiber-wise 2-form on fibers above points in $B_x(\varepsilon)$, and $f^*\omega_{\mathcal{I}(\partial x_i)}$ a 2-form on $B_x(\varepsilon)$. However, $\sum_{i=1}^3 dx_i \wedge f^*\omega_{\mathcal{I}(\partial x_i)}$ is a global well-defined 2-form on M, as we will see below.

Proof. Similar to the proof of well-definedness of the Fueter operator, let $(\partial x_1, \partial x_2, \partial x_3)$ and $(\partial y_1, \partial y_2, \partial y_3)$ be two local orthogonal frames at a point $x \in M$, related by a matrix $A \in SO(3)$; i.e., $\partial x_i = \sum_{j=1}^3 A_{i,j} \partial y_j$, and therefore, $dx_i = \sum_{j=1}^3 A_{j,i} dy_j$. Then

$$
\begin{aligned}
-\sum_{i=1}^3 dx_i \wedge f^*\omega_{\mathcal{I}(\partial x_i)} &= -\sum_{i=1}^3 (\sum_{j=1}^3 A_{j,i} dy_j) \wedge f^*\omega_{\mathcal{I}(\sum_{k=1}^3 A_{i,k}\partial y_k)} \\
&= -\sum_{j,k=1}^3 \sum_{i=1}^3 A_{j,i} A_{i,k} dy_j \wedge f^*\omega_{\mathcal{I}(\partial y_k)} \\
&= -\sum_{j}^3 dy_j \wedge f^*\omega_{\mathcal{I}(\partial y_j)}.
\end{aligned}
$$

□

Hohloch, Noetzel and Salamon's energy identity holds in the gauge-theoretic setting too, with a similar proof.

Lemma 65. *Let (M, g) be an oriented Riemannian 3-manifold. We have the following energy identity,*

$$\int_M |\nabla f|^2 vol_M = \int_M |\mathfrak{F}(f)|^2 vol_M - \int_M \sum_{i=1}^3 dx_i \wedge f^*\omega_{\mathcal{I}(\partial x_i)}. \tag{3.2.1}$$

Proof. Let $(\partial x_1, \partial x_2, \partial x_3)$ be a local orthonormal frame,

$$
\begin{aligned}
\int_M |\nabla f|^2 vol_g - \int_M |\mathfrak{F}(f)|^2 vol_g &= \int_M \sum_{i=1}^3 |\nabla f(\partial x_i)|^2 vol_g - \int_M |\mathfrak{F}(f)|^2 vol_g \\
&= -\int_M \langle \mathcal{I}(\partial x_3)\nabla f(\partial x_1), \nabla f(\partial x_2)\rangle + \langle (\mathcal{I}(\partial x_1)\nabla f(\partial x_2), \nabla f(\partial x_3)\rangle \\
&\qquad + \langle (\mathcal{I}(\partial x_2)\nabla f(\partial x_3), \nabla f(\partial x_1)\rangle) vol_g \\
&= -\int_M (f^*\omega_{\mathcal{I}(\partial x_1)}(\partial_2, \partial_3) + f^*\omega_{\mathcal{I}(\partial x_2)}(\partial_3, \partial_1) + f^*\omega_{\mathcal{I}(\partial x_3)}(\partial_1, \partial_2)) vol_g \\
&= -\int_M (dx_1 \wedge f^*\omega_{\mathcal{I}(\partial x_1)} + dx_2 \wedge f^*\omega_{\mathcal{I}(\partial x_2)} + dx_3 \wedge f^*\omega_{\mathcal{I}(\partial x_3)})(\partial x_1, \partial x_2, \partial x_3) vol_g \\
&= -\int_M (dx_1 \wedge f^*\omega_{\mathcal{I}(\partial x_1)} + dx_2 \wedge f^*\omega_{\mathcal{I}(\partial x_2)} + dx_3 \wedge f^*\omega_{\mathcal{I}(\partial x_3)}) = \mathcal{A}(f).
\end{aligned}
$$

□

There exists a 4-dimensional version of the action functional defined above, which we call hyperkähler action functional.

Lemma 66. *Let $\mathfrak{X} \to M$ be a hyperkähler bundle over an oriented Riemannian 4-manifold (N, h) with an isometric bundle identification $\mathcal{J} : S\Lambda^+(T^*N) \to \mathfrak{b}(\mathfrak{X})$ and a connection on $\mathfrak{X} \to N$ with covariant derivative ∇. The gauged hyperkähler action functional, defined by*

$$\mathcal{A} : \Gamma(\mathfrak{X}) \to \mathbb{R}, \qquad \mathcal{A}(f) := -\int_N \sum_i \Omega_i \wedge f^*(\omega_{\mathcal{J}(\Omega_i)}),$$

— where $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$ is a local oriented orthonormal frame for $\Lambda^+(N)$ corresponding to a local frame $(\partial x_1, \partial x_2, \partial x_3)$ and $\Omega_i = dx_1 \wedge dx_i + dx_j \wedge dx_k$ — is well-defined.

The proof is similar to the 3-dimensional case.

There exists a 4-dimensional version of the energy identity 67. This is the gauged version of proposition 2.7. in [42].

Lemma 67. *Let (N, h) be an oriented Riemannian 4-manifold. We have the following energy identity,*

$$\int_N |\nabla f|^2 vol_N = \int_N |\tilde{\mathfrak{F}}(f)|^2 vol_N - \int_N \sum_i \Omega_i \wedge f^*(\omega_{\mathcal{J}(\Omega_i)}), \tag{3.2.2}$$

for some local orthonormal frame $(\Omega_1, \Omega_2, \Omega_3)$ of $\Lambda^+(T^*N)$.

The proof is essentially the same as proposition 2.7. in [42].

There is a major different between the hypersymplectic — and hyperkähler — action functional for maps and sections in the gauge-theoretic setting. For maps, in the hypercontact setting, the energy identity 67 shows that the energy of a Fueter map is a topological invariant, which only depends on the homotopy class of the map. As mentioned in [97] it is rarely the case in the gauge-theoretic setting. We can let

$$-\sum_{i=1}^{3} dx_i \wedge f^*\omega_{\mathcal{I}(\partial x_i)} = f^*(\Lambda),$$

for a 3-form $\Lambda \in \Omega^3(\mathfrak{X})$, which is not necessarily closed. If it is the case that this form is closed we would get

$$\int_M |\nabla f|^2 vol_g = \int_M |\mathfrak{F}(f)|^2 vol_g - 2\langle [M], f^*\Lambda \rangle. \tag{3.2.3}$$

Note that if Λ is closed and f is a Fueter map, then 3.2.3 gives a bound on $\|\nabla f\|^2_{L^2}$, which is crucial in the study of compactness problems of Fueter sections. However, this might not be true in general for Fueter sections.

3.2.2 The Elliptic Theory

The Fueter operators are non-linear generalizations of Dirac operators. One crucial property of the Dirac operators is that they are elliptic. The Fueter operators are non-linear elliptic operators. A non-linear differential operator is elliptic at a section f if its linearization at f is elliptic in the usual sense.

Lemma 68. *The 3-dimensional Fueter operator is elliptic.*

Proof. Let $f \in \Gamma(\mathfrak{X})$. Let $\mathfrak{L}_{\mathfrak{f}}$ denote the linearization of the Fueter operator along the section f. The operator $\mathfrak{L}_{\mathfrak{f}} : W^{1,2}(M, f^*V\mathfrak{X}) \to L^2(M, f^*V\mathfrak{X})$ is given by

$$\mathfrak{L}_{\mathfrak{f}}\xi = \sum_{i=1}^{3} \mathcal{I}(\partial x_i)\nabla_{\partial x_i}\xi,$$

which shows $\mathrm{Symbol}(\mathfrak{L}_{\mathfrak{f}})_x(h, v) = i\mathcal{I}(v)(h)$ for all $x \in M, v \in T_xM$ and $h \in f^*V\mathfrak{X}_x$, and therefore, $\mathfrak{F}$ is an Elliptic operator. $\square$

The similar statement is correct in the 4-dimensional case.

Lemma 69. *The 4-dimensional Fueter operator is elliptic.*

The proof is similar to the proof of the previous lemma.

The regularity theory of the Fueter sections is very similar to the case of Fueter maps, studied in [46], with almost an identical proof.

Lemma 70. *If $p > 3$ every Fueter section over a 3-dimensional manifold which belongs to $W^{1,p}$ is smooth. If $p > 4$ every Fueter section over a 4-dimensional manifold which belongs to $W^{1,p}$ is smooth.*

Recall that elliptic operators on compact manifolds are Fredholm.

Corollary 4. *If the base manifold is compact then the Fueter operator is Fredholm.*

Since the Fueter operators are Fredholm, we can consider their Fredholm index. Note that the index of a non-linear elliptic operator is the index of its linearization at each section $f \in \Gamma(\mathfrak{X})$.

Lemma 71. *Let $\mathfrak{X} \to M$ be a permuting hyperkähler bundle over a closed oriented Riemannian 3-manifold (M, g). The linearization of Fueter operators over 3-manifolds at each section $f \in \Gamma(\mathfrak{X})$ is self-adjoint.*

Proof. Let $\xi_1, \xi_2 \in \Omega^0(M, f^*V\mathfrak{X})$,

$$\begin{aligned}\langle \mathfrak{L}_f(\xi_1), \xi_2\rangle_{L^2} &= \sum_{i=1}^{3}\langle \mathcal{I}(\partial x_i)\nabla_{\partial x_i}\xi_1, \xi_2\rangle_{L^2} = -\sum_{i=1}^{3}\langle \nabla_{\partial x_i}\xi_1, \mathcal{I}(\partial x_i)\xi_2\rangle_{L^2} \\ &= \sum_{i=1}^{3}\langle \xi_1, \nabla_{\partial x_i}\mathcal{I}(\partial x_i)\xi_2\rangle_{L^2} = \sum_{i=1}^{3}\langle \xi_1, \mathcal{I}(\partial x_i)\nabla_{\partial x_i}\xi_2\rangle_{L^2} = \langle \xi_1, \mathfrak{L}_f(\xi_2)\rangle_{L^2}.\end{aligned}$$

Note that in the case of permuting hyperkähler bundles, there is a natural isometric identification $\mathcal{I}$, which is covariantly constant. □

Elliptic operators over closed odd-dimensional manifolds have index zero.

Corollary 5. *Let $\mathfrak{X} \to M$ be a permuting hyperkähler bundle over a closed oriented Riemannian 3-manifold (M, g). Then*

$$Ind(\mathfrak{F}) = 0.$$

In this setting, since we are dealing with index zero Fredholm operators, one can hope to count solutions to the equation $\mathfrak{F}(f) = 0$.

Definition 53. *Let $\mathfrak{X} \to M$ be a permuting hyperkähler bundle, with fibers modeled on a hyperkähler manifold (X, g_X, I, J, K), over a closed oriented Riemannian 3-manifold (M, g). Let $\mathfrak{Fuet}(\mathfrak{X})$ be the space of Fueter sections,*

$$\mathfrak{Fuet}(\mathfrak{X}) = \{f \in \Gamma(\mathfrak{X}) \mid \mathfrak{F}(f) = 0\}.$$

Optimistically, we assume for a generic Riemannian metric g on M — or potentially larger set of perturbations — $\mathfrak{Fuet}(\mathfrak{X})$ is a zero-dimensional manifold. If $\mathfrak{Fuet}(\mathfrak{X})$ is compact — or can be compactified — we can count the number of Fueter sections.

$$n(M, \mathfrak{X}) := \#\mathfrak{Fuet}(\mathfrak{X}).$$

This count can be understood as a $\mathbb{Z}_2$-count, or more ambitiously one can hope to define a signed count,

$$n(M, \mathfrak{X}) := \sum_{f \in \mathfrak{Fuet}(\mathfrak{X})} sign(f).$$

This number a priori depends on the Riemannian metric g — although we drop it from the notation — and might not be necessarily an invariant of the 3-manifold M.

Let $\mathfrak{X} = \mathfrak{Mon}_k^\circ$, be the moduli space of centered k-monopoles on $\mathbb{R}^3$. Ignoring technical difficulties to define these counts, let

$$\mathfrak{mon}_k(M) := n(M, \mathfrak{Mon}_k^\circ) = \sum_{f \in \mathfrak{Fuet}(\mathfrak{Mon}_k^\circ)} sign(f). \tag{3.2.4}$$

We call $\mathfrak{mon}_k(M)$ the kth monopole number of (M, g).

Conjecture 5. *For a generic g, $\mathfrak{mon}_k(M)$ is a topological invariant of the 3-manifold M. More ambitiously, one can think of $\mathfrak{Fuet}(\mathfrak{Mon}_k^\circ)$ as the generators of a Floer theory — where the differential is defined by counting the solutions to the 4-dimensional Fueter operator on $\mathbb{R} \times M$.*

The main difficulty in defining these invariants come from the compactness problems.

3.3 Compactness

In this section, we study the compactness properties of Fueter sections over 3- and 4-dimensional manifolds. We start with the case where the hyperkähler manifold X is compact or more generally, the case where the image of Fueter sections fall inside a fixed compact subset of the bundle $\mathfrak{X}$. The following lemma is the gauged version of Theorem 3.2 in [46] with a similar proof.

Theorem 22. *Let $\mathfrak{X} \to M$ be a hyperkähler bundle over a closed oriented Riemannian 3-manifold (M, g). Let $p > 3$. Let $\{f_i\}_{i=1}^{\infty}$ be sequence of Fueter sections of $\mathfrak{X}$ such that $\cup_{i=1}^{\infty} f_i(M) \subset C$, where C is a compact subset of $\mathfrak{X}$. Furthermore, suppose $sup_i\{\|\nabla f_i\|_{L^p}\} < \infty$. The sequence $\{f_i\}_{i=1}^{\infty}$ has a subsequent which converges in $C^{\infty}_{loc}(M)$.*

Let $\mathfrak{X} \to N$ be a hyperkähler bundle over an oriented Riemannian 4-manifold (N, h). Let $p > 4$. Let $\{f_i\}_{i=1}^{\infty}$ be sequence of Fueter sections of $\mathfrak{X}$ such that $\cup_{i=1}^{\infty} f_i(N) \subset C$, where C is a compact subset of $\mathfrak{X}$. Furthermore, suppose $sup_i\{\|\nabla f_i\|_{L^p}\} < \infty$. The sequence $\{f_i\}_{i=1}^{\infty}$ has a subsequent which converges in $C^{\infty}_{loc}(N)$.

There are different assumption in this theorem which result in the compactness, but they are rather strong conditions. If we remove the assumption about the norm of the derivative of the sections, the sections might blow up, bubble and form singularities. Also, if we remove the assumption that the images of the sections stay in a compact subset, we can have a sequence of Fueter section which their image diverge to infinity.

The first step in the direction of removing these assumptions is taken by Walpuski [97]. Walpuski proves a sequence of Fueter sections $\{f_i\}_{i=1}^{\infty}$ of a hyperkähler bundle $\mathfrak{X}$ with compact fibers over a closed oriented Riemannian 3-manifold (M, g), with bounded energy, after passing to a subsequence, converges outside of a closed rectifiable subset $S \subset M$, called the blow-up locus, with Hausdorff dimension $dim(S) \leq 1$. Furthermore, Walpuski shows the non-compactness along S has two different sources. First, bubbling of holomorphic spheres. This non-compactness has a codimension two nature. Second, codimension three non-removable singularities.

The following example demonstrates the emergence of the bubbling phenomenon .

Example 13 ([97])**.** *Let (X, g_X, I, J, K) be a hyperkähler manifold such that for some complex structure I_ζ in the S^2-family of complex structures on X, there is a non-trivial I_ζ-holomorphic 2-sphere in X,*

$$f : S^2 \to X \quad \textit{such that} \quad \overline{\partial}_{I_\zeta} f = 0.$$

Let $M = S^3$ equipped with the standard round metric. Furthermore, let $\mathfrak{X} = S^3 \times X \to S^3$ be the trivial hyperkähler bundle above S^3 equipped with the trivial connection d. Let $\pi : S^3 \to S^2$ be the Hopf map and $\bar{\cdot} : S^2 \to S^2$ the complex conjugation map. Since the bundle is trivial, the sections can be identified with maps from S^3 to X. Let

$$u = f \circ \bar{\cdot} \circ \pi : S^3 \to X,$$

u is Fueter.

Let $s_\lambda : \mathbb{C} \to \mathbb{C}$ be the scaling map, defined by $s_\lambda(z) = \lambda z$ for all $\lambda \in \mathbb{R}^+$. This map extends to the one point compactification of $\mathbb{C}$, i.e., $\mathbb{C}P^1 \cong S^2$, by letting $s_\lambda(\infty) = \infty$. By an abuse of *notation, we still denote this map by* $s_\lambda : S^2 \to S^2$. *We can define a 1-parameter family of Fueter maps*

$$u_\lambda = f \circ \underset{\cdot}{-} \circ s_\lambda \circ \pi : S^3 \to X.$$

Sending $\lambda \downarrow 0$, *the family of Fueter maps* u_λ *blows up along* $\pi^{-1}(\infty) \cong S^1$ *and converges to the constant map on the complement of this circle. Note that,the energy of these maps, i.e.,* $\mathcal{E}(u_\lambda) = \frac{1}{2}\int_{S^3} |\nabla u_\lambda|^2$ *is independent of* λ *and bounded.*

The following theorem is due to Walpuski. In the original statement X is assumed to be compact; however, with no difficulty one can replace this assumption with a slightly weaker one, where X is not necessarily compact but

$$\cup_{i=1}^{\infty} f_i(M) \subset C,$$

where C is a compact subset of $\mathfrak{X}$.

Theorem 23 (Walpuski)**.** *Let* (M, g) *be a closed oriented Riemannian 3-manifold. Let* $\mathfrak{X} \to M$ *be a hyperkähler bundle with fibers modeled on a hyperkähler manifold* (X, g_X, I, J, K). *Suppose the bundle is equipped with a connection* ∇ *and an isomorphism* $\mathcal{I} : STM \to \mathfrak{b}(\mathfrak{X})$. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a sequence of Fueter sections on this bundle such that* $f_i(M) \subset C$, *where* C *is a compact subset of* $\mathfrak{X}$ *for every* $i \in \mathbb{N}$. *Moreover, let* c *be a positive constant such that*

$$\mathcal{E}(f_i) := \frac{1}{2}\int_M |\nabla f_i|^2 \leq c.$$

Then after passing to a subsequence the following holds:

- ***Convergence away from the blow-up locus.*** *There exists a closed subset* S *with finite* 1*-dimensional Hausdorff measure* $\mathcal{H}^1(S) < \infty$ *and a Fueter section* $f \in \Omega^0(M \setminus S, \mathfrak{X})$ *such that* $f_i|_{M \setminus S}$ *converges to* f *in* $C^\infty_{loc}(M)$.
- ***Decomposition of the blow-up locus.*** *There exist a constant* $\varepsilon_0 > 0$ *and an upper semi-continuous function* $\Theta : S \to [\varepsilon_0, \infty)$ *such that the sequence of measures* $\mu_i := |\nabla f_i|^2 \mathcal{H}^3$ *converges weakly to* $\mu = |\nabla f|^2 \mathcal{H}^3 + \Theta \mathcal{H}^1|_S$.

  S *decomposes as*

  $$S = \Gamma \cup sing(f),$$

  with

  $$\Gamma := supp(\Theta \mathcal{H}^1_S)$$

  and

  $$sing(f) := \{x \in M : \limsup_{r \to 0} \frac{1}{r} \int_{B_r(x)} |\nabla f|^2 > 0\},$$

  Γ *is* $\mathcal{H}^1$*-rectifiable and* $\mathcal{H}^1(sing(f)) = 0$.

- ***Bubbling-off of holomorphic spheres*** *For each smooth point $x \in \Gamma$ there exists a non-trivial $-I(v)$-holomorphic sphere*

$$f_x : S^2 \to \mathfrak{X}_x,$$

with v the unit tangent vector in $T_x\Gamma$.

Four dimensional version of this theorem is as following:

Theorem 24 (Walpuski)**.** *Let (N, h) be an oriented Riemannian 4-manifold. Let $\mathfrak{X} \to M$ be a hyperkähler bundle with fibers modeled on a hyperkähler manifold (X, g_X, I, J, K) over N. Suppose the bundle is equipped with a connection ∇, an isomorphism $\mathcal{I} : S\Lambda^+(T^*N) \to \mathfrak{b}(\mathfrak{X})$ and the standard identification $\iota : \Lambda^+(T^*N) \to \mathfrak{so}(TN)$. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of (4-dimensional) Fueter sections on this bundle such that $f_i(M) \subset C$, where C is a compact subset of $\mathfrak{X}$, for every $i \in \mathbb{N}$. Moreover, let c be a positive constant such that*

$$\mathcal{E}(f_i) := \frac{1}{2}\int_M |\nabla f_i|^2 \leq c.$$

Then after passing to a subsequence the following holds:

- ***Convergence away from the blow-up locus.*** *There exists a closed subset S which $\mathcal{H}^2(S) < \infty$ and a Fueter section $f \in \Omega^0(M \setminus S, \mathfrak{X})$ such that $f_i|_{M\setminus S}$ converges to f in $C^{\infty}_{loc}(N)$.*
- ***Decomposition of the blow-up locus.*** *There exist a constant $\varepsilon_0 > 0$ and an upper semi-continuous function $\Theta : S \to [\varepsilon_0, \infty)$ such that the sequence of measures $\mu_i := |\nabla f_i|^2\mathcal{H}^4$ converges weakly to $\mu = |\nabla f|^2\mathcal{H}^4 + \Theta\mathcal{H}^2|_S$.*

  S decomposes as

$$S = \Gamma \cup sing(f),$$

  with

$$\Gamma := supp(\Theta\mathcal{H}^2_S)$$

  and

$$sing(f) := \{x \in M : limsup_{r\to 0}\frac{1}{r^2}\int_{B_r(x)} |\nabla f|^2 > 0\},$$

  Γ is $\mathcal{H}^2$-rectifiable and $\mathcal{H}^2(sing(f)) = 0$.
- ***Bubbling-off of holomorphic spheres.*** *For each smooth point $x \in \Gamma$ there exists a non-trivial $-I(\alpha)$-holomorphic sphere*

$$f_x : S^2 \to \mathfrak{X}_x,$$

  which α is unit self-dual 2–form on T_xM.

In the direction of defining the monopole invariant $\mathfrak{mon}_k(M)$, and also in the direction of addressing the Donaldson-Segal conjecture, we investigate the compactness properties, in the case where X is the moduli space of centered k-monopoles.

3.4 Fueter Sections of Monopole Bundles

In this section, we mainly consider the case where the hyperkähler manifold X is a moduli space of centered k-monopoles. We start by reviewing the basic facts about the geometry of the moduli spaces of monopoles on $\mathbb{R}^3$ with the structure group $SU(2)$. Our treatment is brief. For more detailed account consult with [4].

3.4.1 Geometry of the Moduli Spaces of Monopoles on $\mathbb{R}^3$

In this subsection we review the basic results about the geometry of the moduli spaces of monopoles on $\mathbb{R}^3$.

Let $P \to \mathbb{R}^3$ be a principal $SU(2)$-bundle over $\mathbb{R}^3$. Let (A, Φ) be a pair of a connection A on P and a section Φ of the adjoint bundle. The Yang-Mills-Higgs action functional of this pair is defined by

$$\mathcal{YMH}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|F_A|^2 + |d_A \Phi|^2) dxdydz.$$

Here the norms are taken with respect to the adjoint-invariant inner product on $\mathfrak{su}(2)$.

Let $\mathcal{A}$ denote the space of connections on P and $\Gamma(\mathfrak{g}_P)$ the space of sections of the adjoint bundle $\mathfrak{g}_P$. Let $\mathcal{G} = Aut(P_G)$ be the space of bounded gauge transformations. $\mathcal{G}$ acts on $\mathcal{A} \times \Gamma(\mathfrak{g}_P)$ by

$$g \cdot (A, \Phi) = (gAg^{-1} - dgg^{-1}, g\Phi g^{-1}).$$

The Yang-Mills-Higgs action functional is invariant under the action of the gauge group, and therefore, it induces an action functional on the space of equivalent classes of pairs (A, Φ).

Let (A, Φ) be a pair with finite energy. To any such pair we can assign mass and charge. The Yang-Mills-Higgs functional can be rewritten as

$$\mathcal{YMH}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} | * F_A - d_A \Phi|^2 dxdydz + \int_{\mathbb{R}^3} d\langle \Phi, F_A \rangle.$$

For any pair with finite energy we have $|\Phi(x)| \to m$ as $|x| \to \infty$, for a non-negative constant m. We let $m = 1$ as a normalizing assumption.

Let $R >> 0$ be a sufficiently large number such that $B_R(0)$ contains all the zeros of Φ. We would have a map from a large sphere in $\mathbb{R}^3$ to the unit 2-sphere in $\mathfrak{su}(2)$,

$$\frac{\Phi}{|\Phi|} : S^2_R(0) \to S^2_1(0) \subset \mathfrak{su}(2).$$

For a pair (A, Φ) with finite energy we have

$$-k = \deg(\frac{\Phi}{|\Phi|}) = -\frac{1}{4\pi} \lim_{R \to \infty} \int_{S_0^2(R)} \langle \Phi, F_A \rangle \in \mathbb{Z}.$$

In this section, we are concerned with the moduli spaces of $SU(2)$-monopoles on $\mathbb{R}^3$ with charge $k \in \mathbb{Z}$.

Definition 54 (Moduli Spaces of Monopoles on $\mathbb{R}^3$)**.** *Let $P \to \mathbb{R}^3$ be a principal $SU(2)$-bundle. Let $k \in \mathbb{Z}$. The moduli space of k monopoles, denoted by N_k, is defined by*

$$N_k = \{(A, \Phi) \mid *F_A = d_A\Phi,\ \mathcal{YMH}(A, \Phi) < \infty,\ |\Phi| \to 1,\ \textit{charge}(A, \Phi) = k\}/\mathcal{G}.$$

For the most purposes it is better to enlarge the moduli spaces of k-monopoles by a circle phase factor, to get moduli spaces of framed monopoles.

Definition/Lemma 72 (Moduli Spaces of Framed Monopoles on $\mathbb{R}^3$ [4])**.** *A pair (A, φ) with finite energy is framed in a direction — say — x_1 if*

$$\lim_{x_1 \to \infty} \varphi(x_1, 0, 0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

A gauge transformation $g \in \mathcal{G}$ is called x_1-diagonal if $\lim_{x_1 \to \infty} g(x_1, 0, 0)$ is diagonal. These gauge transformations preserve the x_1-framed monopoles. Furthermore, every monopole can be transformed to an x_1-framed monopole using a suitable gauge transformation, and therefore,

$$\frac{\textit{Solutions to Bogomolnyi equation}}{\textit{gauge transformations}} \cong \frac{x_1 - \textit{framed solutions to Bogomolnyi equation}}{x_1 - \textit{diagonal gauge transformations}}.$$

An x_1-framed gauge transformation $g \in \mathcal{G}$ is defined to be an x_1-diagonal gauge transformation where

$$\lim_{x_1 \to \infty} g(x_1, 0, 0) = 1.$$

The moduli space of framed k-monopoles over $\mathbb{R}^3$, denoted by M_k, is defined by

$$M_k = \frac{x_1 - \textit{framed k-monopoles}}{x_1 - \textit{framed gauge transformations}}.$$

M_k is a $U(1)$-bundle over N_k, which can be seen by noticing

$$U(1) \to \frac{x_1 - \textit{framed k-monopoles}}{x_1 - \textit{framed gauge transformations}} \to \frac{x_1 - \textit{framed k-monopoles}}{x_1 - \textit{diagonal gauge transformations}}.$$

The Bogomolny equation is invariant under the gauge transformations. Moreover, the translation group of $\mathbb{R}^3$ acts naturally on the moduli spaces of monopoles and framed monopoles on $\mathbb{R}^3$.

Definition 55 (Moduli Spaces of Centered Monopoles on $\mathbb{R}^3$)**.** *The moduli spaces of centered k-monopoles on $\mathbb{R}^3$ are defined by*

$$Mon_k^\circ = \frac{N_k}{\mathbb{R}^3}.$$

The deformation theory of these spaces has been studied by Atiyah and Hitchin [4]. In particular, we have the following lemma.

Lemma 73. *The moduli spaces of monopoles, framed monopoles and centered monopoles are smooth finite dimensional manifolds with dimensions $4|k|-1$, $4|k|$ and $4|k|-4$, respectively.*

There is a natural hyperkähler structure on the moduli spaces of framed and centered monopoles on $\mathbb{R}^3$.

3.4.2 Hyperkähler Structure on the Moduli Spaces of Monopoles

In this section, we study the hyperkähler structure on the moduli spaces of monopoles on $\mathbb{R}^3$. There are different ways to see the hyperkähler structure on the moduli spaces of monopoles. Here we review two closely related approaches.

- By studying the linearized equations.
- By the virtue of the hyperkähler reduction.

3.4.2.1 The Linear Theory

Let $\mathcal{C}_k \subset \mathcal{A} \times \Gamma(\mathfrak{g}_P)$ be the space of smooth pairs (A, Φ) on a principal $SU(2)$-bundle on $\mathbb{R}^3$ defined by

$$\mathcal{C}_k = \{(A,\Phi) \mid \mathcal{YMH}(A,\Phi) < \infty,\ \text{charge}(A,\Phi) = k,\ \lim_{|x|\to\infty} |\Phi| = 1\}.$$

The linearization of the action of the gauge group at on the tangent space of $\mathcal{C}_k$ at (A, Φ) is given by

$$d_1 : \Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \to \Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \oplus \Omega^1(\mathbb{R}^3, \mathfrak{su}(2)), \qquad d_1 g = -([\Phi, g], d_A g).$$

Let $\mathcal{G}$ be the gauge group, more specifically, defined by

$$\mathcal{G} = \{g \in Aut(P) \mid g \text{ is bounded},\ (d_1 g)g^{-1} \in L^2(\mathbb{R}^3)\},$$

and $\mathcal{G}_0$ the space of framed gauge transformations.

The linearization of the Bogomolny equation at (A, Φ) is given by

$$d_2 : \Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \oplus \Omega^1(\mathbb{R}^3, \mathfrak{su}(2)) \to \Omega^1(\mathbb{R}^3, \mathfrak{su}(2)), \qquad d_2(a,\psi) = *d_A a - d_A \psi + [\Phi, a].$$

The equation $d_1^*(a,\psi) = 0$ is a gauge fixing condition. d_1^* and d_2 can be coupled to produce an elliptic operator, as we saw in Chapter 1,

$$D = d_1^* + d_2 : \Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \oplus \Omega^1(\mathbb{R}^3, \mathfrak{su}(2)) \to \Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \oplus \Omega^1(\mathbb{R}^3, \mathfrak{su}(2)).$$

The tangent space of the space of monopoles at (A,φ) can be identified with the $\ker D$. Quaternions act on $\Omega^0(\mathbb{R}^3, \mathfrak{su}(2)) \oplus \Omega^1(\mathbb{R}^3, \mathfrak{su}(2))$ induced by the identification

$$(a_1 dx_1 + a_2 dx_2 + a_3 dx_3, \psi) \to \psi + a_1 I + a_2 J + a_3 K.$$

Moreover, the linearized Bogomolny equations are invariant under the $\mathbb{H}$-action. This induces an action of $\mathbb{H}$ on the moduli spaces of solutions.

With the suitable boundary conditions, where the infinitesimal deformations are L^2-integrable, the L^2-inner product restricted to the $\ker D$ defines a Riemannian metric on the moduli spaces of monopoles. It can be seen this metric together with the almost complex structures introduced by the action of $\mathbb{H}$ forms a hyperkähler manifold on the moduli spaces of monopoles. The details can be found in the chapter 4 of [4].

3.4.2.2 The Hyperkähler Reduction

The second approach is to understand the moduli spaces of monopoles as a hyperkähler quotient. The hyperkähler quotient construction first arose in the context of mathematical physics, and more specifically, in supersymmetry [45]. One can consider two cases, the finite-dimensional construction, and the infinite-dimensional construction. In the infinite-dimensional case, one might have an infinite dimensional hyperkähler manifold with an infinite-dimensional hyperkähler action on it; however, the quotient space might be a finite-dimensional manifold. Among the most interesting cases are the quotient construction of the hyperkähler metrics on the moduli spaces of instantons and monopoles.

Definition 56 (Tri-Hamiltonian Action)**.** *A smooth action of a Lie group G on a hyperkähler manifold (X, g_X, I, J, K) is called a tri-Hamiltonian action, if*

- *G acts isometrically; i.e., for any $h \in G$ we have $L_h^* g_X = g_X$, where $L_h : X \to X$ is the action of the element $h \in G$ on X.*
- *G preserves the hyperkähler structure; i.e., for any $h \in G$, we have $L_h^*\omega = \omega$. The induced hyperkähler G-action on TX commutes with the complex structures; i.e., $L_h \circ I_\zeta = I_\zeta \circ L_h$ for any $h \in G$. In particular, the hyperkähler action is symplectic with respect to ω_1, ω_2 and ω_3.*
- *It admits a hyperkähler moment map, which is a smooth map $\mu : X \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ such that $d\langle \mu, \xi \rangle = -\iota_{X_\xi}\omega$ for any $\xi \in \mathfrak{g}$, where X_ξ is the vector field on X generated by the infinitesimal action of ξ.*
- *μ is G-equivariant, where G acts on $\mathfrak{g}$ by the co-adjoint action and trivially on $\mathfrak{sp}(1)^*$.*

In particular, this action is Hamiltonian with respect to ω_1, ω_2 and ω_3 with moment maps $\mu_1 = \langle \mu, i\rangle$, $\mu_2 = \langle \mu, j\rangle$ and $\mu_3 = \langle \mu, k\rangle$, respectively.

The tri-Hamiltonian actions can be used to construct new hyperkähler manifolds, through a process called hyperkähler reduction.

Definition 57 (Hyperkähler Reduction [45])**.** *Let (X, g_X, I, J, K) be a hyperkähler manifold with a tri-Hamiltonian action of a Lie group G and a moment map $\mu : M \to \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$. Let $\xi = \xi_1 i + \xi_2 j + \xi_3 k \in \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*$ be a regular value of μ. Then $\mu^{-1}(\zeta)/G$ is a hyperkähler manifold, called a hyperkähler reduction.*

One can realize the moduli spaces of framed and centered monopoles as hyperkähler quotients. Let P be the trivial principal G-bundle over $\mathbb{R}^3$ with the adjoint vector bundle $\mathfrak{g}_P$. Let

$$V = \mathcal{A} \times \Omega^0(\mathbb{R}^3, \mathfrak{g}_P).$$

The space V is a flat infinite-dimensional affine space. This space can be equipped with a quaternionic structure. The tangent space at each $(A, \Phi) \in V$ is given by

$$T_{(A,\Phi)}M = \Omega^1(\mathbb{R}^3, \mathfrak{g}_P) \times \Omega^0(\mathbb{R}^3, \mathfrak{g}_P).$$

The quaternionic structure can be seen by the correspondence we saw earlier,

$$(a_1 dx_1 + a_2 dx_2 + a_3 dx_3, \psi) \to \psi + a_1 I + a_2 J + a_3 K.$$

The action of the gauge group $\mathcal{G}$ preserves this quaternionic structure. Moreover, it is a tri-hamiltonian action with a moment map $\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathcal{G}^* \cong \Omega^1(\mathbb{R}^3, \mathfrak{g}_P)$, given by

$$\begin{aligned}
\mu_1(A, \Phi) &= [\Phi, A_1] + [A_2, A_3], \\
\mu_2(A, \Phi) &= [\Phi, A_2] + [A_3, A_1], \\
\mu_3(A, \Phi) &= [\Phi, A_3] + [A_1, A_2],
\end{aligned}$$

where $A = \sum_{i=1}^3 A_i dx_i$.

The components of the moment maps are given by the components of the Bogomolny equation, and therefore, the hyperkähler reduction $\cap_i \mu_i^{-1}(0)/\mathcal{G}$, defines the moduli space of monopoles, which appears as the union of the disjoint connected components of the moduli spaces of monopoles with different charges.

More examples of this construction can be found in [45].

3.5 The Fueter Sections of the Moduli Spaces of Monopoles and the Compactness Problems

In this section, we prove a partial compactness theorem for Fueter sections of monopole bundles.

The theorems of Walpuski show that the non-compactness of Fueter sections of a bundle with compact fibers has two sources, Bubbling-off of holomorphic spheres in the fibers of $\mathfrak{X}$, and the formation of non-removable singularities. Non-compactness of the moduli spaces of monopoles introduces a new source of non-compactness, divergence to infinity. In this section, we prove a partial compactness theorem for the sequences of Fueter sections of the monopole bundles when they do not go to infinity.

3.5.1 Bubbling

Depending on the geometry of X, one might be able to rule out some sources of the non-compactness and potentially reduce the dimension of the singular set. For instance, the bubbling does not occur if X does not contain any holomorphic sphere, and this can reduce the dimension of the singular set.

An important case appears when there is a permuting $SO(3)$-action on the fibers, for instance the case of the moduli spaces of monopoles on $\mathbb{R}^3$. The existence of the permuting $SO(3)$-action rules out the existence of bubbles.

Theorem 25. *Let (M, g) be an oriented closed Riemannian 3-manifold. Let $\mathfrak{X} \to M$ be a hyperkähler bundle with fibers modeled on a hyperkähler manifold (X, g_X, I, J, K) with a permuting $SO(3)$-action — or a permuting $SU(2)$-action. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of Fueter sections on this bundle such that $f_i(M) \subset C$ for every $i \in \mathbb{N}$, where C is a compact subset of $\mathfrak{X}$. Moreover, let c be a positive constant such that*

$$\mathcal{E}(f_i) := \frac{1}{2}\int_M |\nabla f_i|^2 \leq c.$$

Then after passing to a subsequence the following holds:

- ***Convergence away from the blow-up locus.*** *There exists a closed rectifiable subset S with Hausdorff dimension strictly less than 1, and a Fueter section $f \in \Omega^0(M \setminus S, \mathfrak{X})$ such that $f_i|_{M\setminus S}$ converges to f in $C^{\infty}_{loc}(M)$.*

- ***Description of the blow-up locus.*** *There exist a constant $\varepsilon_0 > 0$ and an upper semi-continuous function $\Theta : S \to [\varepsilon_0, \infty)$ such that the sequence of measures $\mu_i := |\nabla f_i|^2\mathcal{H}^3$ converges weakly to $\mu = |\nabla f|^2\mathcal{H}^3$, where*

$$S = sing(f) = \{x \in M : \limsup_{r\to 0} \frac{1}{r}\int_{B_r(x)} |\nabla f|^2 > 0\}.$$

  Let $\Gamma_{smooth} \subset \Gamma$ denote the set of smooth points of Γ — which the tangent space can be defined there — then

$$\Gamma_{smooth} = \varnothing.$$

In particular, assuming the singular locus is smooth, the bubbling locus vanishes, $\Gamma = \varnothing. Assumingthatthesingularlocusisanembeddedgraphin$M$, thesingularlocuswouldreducetoasetofisolated$

Proof. Using Walpuski's theorem, the proof reduces to showing $\Gamma_{\text{smooth}} = \varnothing$. Recall that for each smooth point $x \in \Gamma$, there exists a non-trivial holomorphic sphere in $\mathfrak{X}_x$, with respect to the holomorphic structure on X_x associated to the tangent direction $T_x\Gamma$; however, hyperkähler manifolds with a permuting $SO(3)$-action do not contain any holomorphic sphere.

To see this recall that every symplectic form ω_ζ in the two-sphere family of symplectic structures of hyperkähler manifolds with a permuting $SO(3)$ action is exact 63. Moreover, there is no J-holomorphic sphere in an exact symplectic manifold (X, ω) for any J compatible with ω. Note that if $f : S^2 \to X$ is a non-trivial J-holomorphic sphere,

$$\int_{S^2} f^*\omega = \int_{S^2} \omega(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})dx_1dx_2 = \int_{S^2} \omega(-J\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_2})dx_1dx_2$$
$$= \int_{S^2} \omega(\frac{\partial f}{\partial x_2}, J\frac{\partial f}{\partial x_2})dx_1dx_2 = \int_{S^2} \|\frac{\partial f}{\partial x_2}\|^2 dx_1dx_2 > 0,$$

for a local orthonormal frame $(\partial x_1, \partial x_2)$.

But on the other hand since ω is exact $\omega = d\theta_\zeta$,

$$\int_{S^2} f^*\omega = \int_{S^2} f^*d\theta = 0$$

which the last equality follows from the Stokes' theorem. This is a contradiction; hence Γ is a rectifiable set with no smooth point, and therefore, $\Gamma_{\text{smooth}} = \varnothing$. This shows the singular set is a closed subset of M, with Hausdorff dimension less than 1. □

The 4-dimensional version of this theorem also follows similarly.

Theorem 26. *Let (N, h) be a closed oriented Riemannian 4-manifold. Let $\mathfrak{X} \to N$ be a hyperkähler bundle with fibers modeled on a hyperkähler manifold (X, g_X, I, J, K) with a permuting $SO(3)$-action — or a permuting $SU(2)$-action. Let $\{f_i\}_{i=1}^\infty$ be a sequence of Fueter sections on this bundle such that $f_i(N) \subset C$ for every $i \in \mathbb{N}$, where C is a compact subset of $\mathfrak{X}$. Moreover, let c be a positive constant such that*

$$\mathcal{E}(f_i) := \frac{1}{2}\int_N |\nabla f_i|^2 \leq c.$$

Then after passing to a subsequence the following holds:

- ***Convergence away from the blow-up locus.*** *There exists a closed subset S with Hausdorff dimension strictly less than 2, and a Fueter section $f \in \Omega^0(N \setminus S, \mathfrak{X})$ such that $f_i|_{N\setminus S}$ converges to f in $C^\infty_{loc}(N)$.*
- ***Description of the blow-up locus.*** *There exist a constant $\varepsilon_0 > 0$ and an upper semi-continuous function $\Theta : S \to [\varepsilon_0, \infty)$ such that the sequence of measures $\mu_i := |\nabla f_i|^2\mathcal{H}^4$ converges weakly to $\mu = |\nabla f|^2\mathcal{H}^4$, where*

$$S = sing(f) = \{x \in N : limsup_{r\to 0}\frac{1}{r^2}\int_{B_r(x)} |\nabla f|^2 > 0\}.$$

In other words $\Gamma_{\textit{smooth}} = \varnothing$.

The proof is similar to the proof of the previous theorem.

We have the following immediate corollary.

Corollary 6. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a sequence of Fueter sections on a bundle* $\mathfrak{X}$ *over a closed oriented Riemannian 3- or 4-manifold, with fibers isomorphic to the moduli space of centered monopoles* $\mathfrak{Mon}_k^0$*, or the moduli space of monopoles framed monopoles* M_k*, where* $f_i(N) \subset C$ *for every* $i \in \mathbb{N}$*, where* C *is a compact subset of* $\mathfrak{X}$*, and with an energy bound*

$$\mathcal{E}(f_i) := \frac{1}{2}\int_N |\nabla f_i|^2 \leq c.$$

Then assuming the bubbling locus is an immersed submanifold, we get $\Gamma_{\textit{smooth}} = \varnothing$*, and therefore, the Hausdorff dimension of the blow-up locus on 3-manifolds is zero, and on 4-manifolds is at most one.*

To completely rule-out the bubbling phenomenon, one should show the bubbling locus can't be very irregular. For instance, in the 3-dimensional case, one can hope to show the bubbling locus is a finite graph inside the 3-manifold. A difficulty in showing this is that potential existence of a bubbling locus which is a graph with infinitely many edges, where the length of a sequence the edges converges to zero, such that the total length of the edges of the graph is still finite. However, ruling out the existence of such a bubbling locus seems crucial for proving the compactness theorem, but We will not follow this direction any further in this thesis.

3.5.2 Non-Removable Singularities of Fueter Sections

In this section, we study the non-removable singularities of the Fueter sections. We will see the tangent map of a Fueter section at a singular point satisfies a certain Cauchy-Riemann type condition. Using this property, and under certain analytic assumptions, we will rule out the existence of non-removable singular points for Fueter sections on certain bundles, in particular, the monopole bundles, and we will prove a partial compactness theorem.

3.5.2.1 Non-Removable Singularities and Tri-Holomorphic-Maps

The tri-holomorphic maps are in the heart of the study of the singularities of Fueter sections. In fact, they appear as the tangent map of the Fueter sections at the non-removable singular points.

Definition 58 (Tri-Holomorphic Map)**.** *Let* (X, g_X, I, J, K) *be a hyperkähler manifold. Let* $\mathcal{I} : S^2 \to \mathfrak{b}(\mathfrak{X})$ *be an identification between* S^2 *and the 2-sphere of the complex structures on* X*. A map* $f : S^2 \to X$ *is called a* $\mathcal{I}$*-tri-holomorphic map if*

$$\partial_1\phi + I(x)\partial_2\phi = 0,$$

where ∂_1, ∂_2 *is a local orthogonal frame at each point* $x \in S^2$*.*

This means at each point $x \in S^2$, the map $d_x f$ is complex linear with respect to the complex structure on X associated to the point $x \in S^2$. Moreover, note that the definition is independent of the chosen local orthogonal frame ∂_1, ∂_2.

In the study of the singularities of the energy minimizing maps, the tangent maps play a crucial role. Let $f : (M^m, g) \to (N^n, h)$ be an energy minimizing map with a singular point $y \in M_1$. Schoen and Uhlenbeck proved that for every sequence r_i of positive numbers converging to zero, there exists a subsequence of the maps

$$x \to f(y + r_i x),$$

which converges weakly to a map

$$\varphi_y : \mathbb{R}^m \setminus \{0\} \to N.$$

Moreover, they showed this map φ_y, which is not necessarily unique and potentially depends on the choice of r_i and the chosen subsequence, is radially invariant. This map is called the tangent map.

The following theorem shows the relationship between the tri-holomorphic maps, the non-removable singularities, and the tangent maps.

Theorem 27. *Let (M, g) be a closed oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^{\infty}$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X} \to M$, where $f_i(M) \subset C$ for a compact subset $C \subset \mathfrak{X}$. Let f be the section where $f_i \to f$ in $C^{\infty}_{loc}(M \setminus S)$ for a singular set S. Suppose f has a radially invariant tangent map at $y \in M$. Let*

$$\varphi_y : S^2 \setminus S \to \mathfrak{X}_y,$$

be the restriction of the radially invariant tangent map of f at y to the unit sphere $S^2 \subset \mathbb{R}^3$. φ_y is a $-\mathcal{I}$-tri-holomorphic map. In other words,

$$\partial_1 \varphi_y - I(x) \partial_2 \varphi = 0.$$

Proof. The limiting section f satisfies the Fueter equation at each point $y \in M$ where f is smooth, and therefore, the re-scaled sections are Fueter too. By taking limit one can see that every tangent map φ_y satisfies the Fueter equation.

At a point $x \in S^2 \setminus S$, we choose an orthonormal frame $(\partial_r, \partial_1, \partial_2)$ for $\mathbb{R}^3$ where (∂_1, ∂_2) is a local frame for S^2. We get

$$I(\partial_r)\partial_r \phi + I(\partial_1)\partial_1 \phi + I(\partial_2)\partial_2 \phi = 0.$$

On the other hand, assuming the tangent map is radially invariant we get $\partial_r \varphi = 0$, and therefore,

$$I(\partial_1)\partial_1 \varphi + I(\partial_2)\partial_2 \varphi = 0.$$

By multiplying this equation by $-I(\partial_1)$ we get

$$\partial_1\phi - I(\partial_r)\partial_2\phi = 0.$$

Moreover, we have $I(\partial_r) = I(x)$ at each point $x \in S^2 \setminus S$, and therefore,

$$\partial_1\phi - I(x)\partial_2\phi = 0.$$

□

Remark 7. *This is similar to equation for J-holomorphic curves; however, here the complex structure on the target manifold is not fixed. The complex structure depends on the point x in the domain, In fact, at each point $x \in S^2$, the map ϕ is a $-I(x)$-holomorphic map — and not $I(x)$-holomorphic. It is true that for any almost complex structure $I(x)$, its negative, $-I(x)$ is also a complex structure, but interestingly enough, this negative sign can affect the behaviour of these maps extensively.*

3.5.2.2 The Almost Monotonicity Formula

The behaviour of Fueter sections near the singular points is controlled by an important property of Fueter sections, called the almost monotonicity, due to Walpuski [97]. This is quite similar to almost monotonicity inequalities for harmonic maps.

Lemma 74 (Walpuski)**.** *Let f be a smooth Fueter section over an oriented Riemannian 3-manifold (M, g). Let $x \in M$ and $\rho := d(x, .)$. Take $0 < s < r < r_0$, where r_0 is injectivity radius of (M, g) at x. We have*

$$\frac{e^{cr}}{r}\int_{B_r(x)} |\nabla f|^2 - \frac{e^{cs}}{s}\int_{B_s(x)} |\nabla f|^2 \geq \int_{B_r(x)\setminus B_s(x)} \frac{1}{\rho}|\nabla_r f|^2 - c(r^2 - s^2),$$

where c is a constant depending the geometry and independent of the section f. In particular, if the base manifold is flat and the bundle is trivial with the trivial connection,

$$\frac{1}{r}\int_{B_r(x)} |df|^2 - \frac{1}{s}\int_{B_s(x)} |df|^2 = 2\int_{B_r(x)\setminus B_s(x)} \frac{1}{\rho}|\partial_r f|^2.$$

Similarly one can prove the almost monotonicity formula for Feuter sections on 4-manifolds.

Lemma 75. *Let f be a smooth Fueter section over an oriented Riemannian 4-manifold (N, h). Then for every $x \in M$ and $0 < s < r < r_0$, where r_0 is injectivity radius of (N, h), and $\rho := d(x, .)$, we have*

$$\frac{e^{cr}}{r^2}\int_{B_r(x)} |\nabla f|^2 - \frac{e^{cs}}{s^2}\int_{B_s(x)} |\nabla f|^2 \geq \int_{B_r(x)\setminus B_s(x)} \frac{1}{\rho^2}|\nabla_r f|^2 - c(r^2 - s^2)$$

for a constant c. In the case that the base manifold is flat and the bundle is trivial with trivial *connection,*

$$\frac{1}{r^2}\int_{B_r(x)}|df|^2 - \frac{1}{s^2}\int_{B_s(x)}|df|^2 = 2\int_{B_r(x)\setminus B_s(x)}\frac{1}{\rho^2}|\partial_r f|^2.$$

To rule out the existence of non-removable singularities for the limiting Fueter sections, one should extend the almost monotonicity formula, which holds for smooth Fueter sections, to the limiting Fueter sections, which are potentially singular.

Assumption 1. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\mathfrak{X} \to M$ be a hyperkähler bundle, where the fibers are modeled on a hyperkähler manifold (X, g_X, I, J, K) which does not contain any holomorphic sphere. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of Fueter section on this bundle, where $f_i \to f \in \Omega^0(M \setminus S, \mathfrak{X})$. We assume the almost monotonicity formula holds for f*

$$\frac{e^{cr}}{r}\int_{B_r(x)}|\nabla f|^2 - \frac{e^{cs}}{s}\int_{B_s(x)}|\nabla f|^2 \geq \int_{B_r(x)\setminus B_s(x)}\frac{1}{\rho}|\nabla_r f|^2 - c(r^2 - s^2),$$

where c is a constant.

This assumption will be used to show the existence of the tangent maps for the limiting Fueter sections at the singular points.

3.5.2.3 Fueter Sections and the Tangent Maps

An important technique in the study the singular points of harmonic maps over Riemannian manifolds is blowing-up the harmonic map at the singular point, and produce tangent maps. The same technique can be used in the study of Fueter sections. In this section, we see how the almost monotonicity formula can be used to produce the tangent maps of Fueter sections at the singular points.

Theorem 28. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^{\infty}$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X} \to M$, where $f_i(M) \subset C$ for a compact subset $C \subset \mathfrak{X}$. Let f be the section where $f_i \to f$ in $C^{\infty}_{loc}(M \setminus S)$ for a singular set S. Let $y \in M$. Let*

$$f_{y,s_j} : B_0(r_y) \subset \mathbb{R}^3 \to \mathfrak{X}, \qquad f_{y,s_j}(x) = f(y + s_j x),$$

be a sequence of re-scaled sections, where $s_j \to 0$. Then, assuming 1, the sequence $\{f_{y,s_j}\}_{j=1}^{\infty}$ converges to a map,

$$\varphi_y : \mathbb{R}^3 \setminus S_y \to \mathfrak{X}_y, \tag{3.5.1}$$

for a closed rectifiable subset $S_y \subset \mathbb{R}^3$ with Hausdorff dimension 1. We call

$$\varphi_y : \mathbb{R}^3 \setminus S_y \to \mathfrak{X}_y, \tag{3.5.2}$$

a tangent map of the section f at the point $y \in M$.

Proof. Let $y \in M$, and r_y the injectivity radius at y. Let $f_{y,s} : B_0(r_y) \subset M \to \mathfrak{X}$ be the scaled map given by $f_{y,s}(x) = f(y + sx)$. The expression $y + sx$ is defined using the exponential map of the Riemannian metric g. Moreover, one can pull back the bundle and the connection over the domain of $f_{y,s}$.

The Fueter property of f implies that $f_{y,s}$ is also a Fueter section,

$$\mathfrak{F}(f_{y,s})(x) = r\mathfrak{F}(f)(y + sx) = 0.$$

Let $s_j \to 0$ be a positive increasing sequence. Let $r < r_y/s_j$. Each $f_{y,s}$ is a Fueter section, and therefore, by the assumption 1,

$$\frac{1}{r}\int_{B_0(r)} |\nabla f_{y,s_j}|^2 = \frac{1}{rs_j}\int_{B_0(rs_j)} |\nabla f|^2 \le \frac{1}{r_y}\int_{B_0(r_y)} |\nabla f|^2 < \infty, \tag{3.5.3}$$

when s_i — and thus, c — is sufficiently small, and therefore,

$$\mathcal{E}(f_{y,s_j}) = \frac{1}{2}\int_{B_0(r)} |\nabla f_{y,s_j}|^2 < C_{\mathcal{E}}, \tag{3.5.4}$$

for a constant $C_{\mathcal{E}}$.

Moreover, for each $j \in \{1, 2, \ldots\}$, we have

$$f_{y,s_j}(M) \subset f(M) \subset C, \tag{3.5.5}$$

which is a compact subset of $\mathfrak{X}$.

The conditions 3.5.4 and 3.5.5 allow up to apply the Theorem 23 to the sequence $\{f_{y,s_j}\}_{j=1}^{\infty}$. Therefore, there exists a closed rectifiable subset $S_y \subset \mathbb{R}^3$ with finite 1-dimensional Hausdorff measure and a Fueter map

$$\varphi_y : \mathbb{R}^3 \setminus S_y \to \mathfrak{X}_y, \tag{3.5.6}$$

such that $f_{y,s_j} \to \varphi_y$ in $C^{\infty}_{loc}(\mathbb{R}^3 \setminus S_y)$. □

Definition 59 (The Density Function)**.** *Let f be a section of a bundle over an oriented Riemannian 3-manifold (M, g). The density function $\Theta_f : M \to \mathbb{R}; \ge 0 \cup \{+\infty\}$ is defined by*

$$\Theta_f(y) = \lim_{\rho\to 0} \frac{1}{\rho}\int_{B_\rho(y)} |\nabla f|^2.$$

In the case of 4-manifolds, let f be a section of a bundle over an oriented Riemannian 4-manifold (N, h). The density function $\Theta_f : N \to \mathbb{R}$ is defined by

$$\Theta_f(y) = \lim_{\rho\to 0} \frac{1}{\rho^2}\int_{B_\rho(y)} |\nabla f|^2.$$

In both 3- and 4-dimensional cases, if y is a smooth point, then $\Theta_f(y) = 0$. From Walpuski's theorem, recall that $0 < \Theta_f(y) < \infty$ implies y is a non-removable singularity. Moreover, if

$\Theta_{f_i}(y) \to \infty$, for a sequence $f_i \to f$ in $C^\infty(M \setminus \{y\})$, then y is a bubbling point.

Lemma 76. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^\infty$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X} \to M$. Let f be the map where $f_i \to f$ in $C^\infty_{loc}(M \setminus S)$ for a singular set S. Then the density function $\Theta_f : M \to \mathbb{R}^{\geq 0}$ is upper semi-continuous.*

The proof is similar to the case of harmonic maps, since the proof only uses the almost monotonicity formula.

Proof. Let $y_i \to y$ in M. We should show

$$y_i \to y \quad \Rightarrow \quad \Theta_f(y) \geq \limsup_{i\to\infty} \Theta_f(y_i).$$

By letting $s \to 0$ in the almost monotonicity formula, for each y_i,

$$\frac{e^{cr}}{r}\int_{B_r(y_i)} |\nabla f|^2 - \Theta_f(y_i) = \int_{B_r(y_i)} \frac{1}{\rho}|\nabla_r f|^2,$$

and therefore,

$$\Theta_f(y_i) \leq \frac{e^{cr}}{r}\int_{B_r(y_i)} |\nabla f|^2.$$

Let $\varepsilon > 0$ be a sufficiently small number. Let $N > 0$ be sufficiently large such that for $i \geq N$, we have $B_r(y_i) \subset B_{r+\varepsilon}(y)$, and therefore,

$$\limsup_{i\to\infty} \Theta_f(y_i) \leq \frac{e^{cr}}{r}\int_{B_{r+\varepsilon}(y)} |\nabla f|^2.$$

Let $\varepsilon \to 0$,

$$\limsup_{i\to\infty} \Theta_f(y_i) \leq \frac{e^{cr}}{r}\int_{B_r(y)} |\nabla f|^2.$$

Let $r \to 0$, and therefore, $c \to 0$,

$$\limsup_{i\to\infty} \Theta_f(y_i) \leq \Theta_f(y).$$

□

Theorem 29. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^\infty$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X} \to M$ with bounded energy, where $f_i(M) \subset C$ for a compact subset $C \subset \mathfrak{X}$. Let f be the section where $f_i \to f$ in $C^\infty_{loc}(M \setminus S)$ for a singular set S. Suppose the almost monotonicity formula holds for f at y. Any tangent map of f at y,*

$$\varphi_y : \mathbb{R}^3 \setminus S_y \to \mathfrak{X}_y,$$

is radially invariant.

The proof is similar to the case of energy-minimizing maps.

Proof. The crucial fact is that $\frac{1}{r}\int_{B_r(y)}|\nabla\varphi_y|^2$, for $0<r<r_y$, is independent of r. To see this note that

$$\frac{1}{r}\int_{B_0(r)}|\nabla f_{y,s}|^2 = \frac{1}{rs}\int_{B_0(rs)}|\nabla f|^2 \le \frac{1}{r_y}\int_{B_0(r_y)}|\nabla f|^2 < \infty.$$

Let $s\to 0$,

$$\frac{1}{r}\int_{B_0(r)}|\nabla\varphi|^2 = \Theta_f(y),$$

where the right-hand side, and therefore the left-hand side, is independent of r. Thus,

$$\Theta_f(y) = \Theta_\varphi(0) = \frac{1}{r}\int_{B_r(0)}|\nabla\varphi|^2.$$

Inserting this into the monotonicity formula for $\varphi : \mathbb{R}^3\setminus S \to X_y$, we get

$$0 = \frac{1}{r}\int_{B_r(0)}|\nabla\varphi|^2 - \frac{1}{s}\int_{B_s(0)}|\nabla\varphi|^2 = \int_{B_r(x)\setminus B_s(x)}\frac{1}{\rho}|\nabla_r\varphi|^2,$$

and therefore, almost everywhere

$$\nabla_r\varphi = 0.$$

By integration along the rays originating from the $0\in\mathbb{R}^3$, we get

$$\varphi(\lambda x) = \varphi(x), \quad \text{for all } \lambda > 0.$$

□

This shows a non-constant tangent map φ has a cone singularity at the origin $0\in\mathbb{R}^3$. We can consider the restriction of the map φ to the unit sphere in $\mathbb{R}^3\setminus S$, by an abuse of notation we still denote this map with φ. We get

$$\varphi : S^2\setminus S \to X_y.$$

This map satisfies a very specific equation.

We get a stronger result when X does not contain any holomorphic sphere.

Corollary 7. *Let (M,g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^\infty$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X}\to M$ with bounded energy, where X does not contain any holomorphic sphere, and $f_i(M)\subset C$ for a compact subset $C\subset\mathfrak{X}$. Let f be the section where $f_i\to f$ in $C^\infty_{loc}(M\setminus S)$ for a singular set S. Let φ_y denote the tangent map of f at y. Then the singular set associated to φ_y is either empty or has one isolated non-removable* *singularity at the origin,*

$$\varphi_y : \mathbb{R}^3 \to \mathfrak{X}_y,$$

with a $-\mathcal{I}$-tri-holomorphic map

$$\varphi_{y|_{S^2}} : S^2 \to \mathfrak{X}_y.$$

Proof. Suppose there exists $0 \neq y \in S$. Since φ is radially invariant, all points, $\lambda y \in S$ for all $\lambda \in \mathbb{R}$. Therefore, we get a ray of singular points with Hausdorff dimension at least 1. However, since $\Gamma = \varnothing$, this is not possible. □

This gives a nice characterization of regular points, similar to the case of harmonic maps,

$$y \in reg(f) \Leftrightarrow \text{Every tangent map at } y \text{ is constant.}$$

The following compactness result is immediate.

Corollary 8. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^{\infty}$ a sequence of smooth Fueter sections of a hyperkähler bundle $\mathfrak{X} \to M$ with bounded energy, where X does not contain any holomorphic or $-\mathcal{I}$-tri-holomorphic sphere. Let $f_i(M) \subset C$ for a compact subset $C \subset \mathfrak{X}$. There exists a section f, which after passing to a subsequence, $f_i \to f$ in $C^{\infty}_{loc}(M)$.*

Holomorphic maps between kähler manifolds are harmonic. There is an analogous statement for I-triholomorphic maps.

Theorem 30. *$\mathcal{I}$-tr-holomorphic maps — and $-\mathcal{I}$-tr-holomorphic maps — are harmonic.*

Proof. For a point $x \in S^2$ and complex structure $I(x)$ on X, we have

$$\nabla df(X, I(x)Y) = \nabla_X df(I(x)Y) - df \nabla_X (I(x)Y),$$

but since at the point $x \in S^2$, f is $I(x)$-holomorphic,

$$\nabla df(X, jY) = I(x) \nabla df(X, Y),$$

where j is the complex structure on S^2. Moreover, since ∇df is a symmetric tensor we have

$$\nabla df(jX, jY) = -\nabla df(X, Y).$$

Let ∂x and $\partial y = j\partial x$ be a local orthonormal frame at $x \in S^2$. Let $\tau(f)$ denote the tension field of f at x,

$$\tau(f) = trace(\nabla df) = \nabla df(\partial x, \partial x) + \nabla df(\partial y, \partial y) = \nabla df(\partial x, \partial x) + \nabla df(j\partial x, \partial jx) = 0,$$

and therefore, f is harmonic. □

Theorem 31. *The image of a $\mathcal{I}$-tri-holomorphic sphere — and a $-\mathcal{I}$-tri-holomorphic sphere — inside a hyperkähler manifold X is a minimal sphere.*

Proof. Any harmonic map $f : \mathbb{C}P^1 \to X$ is conformal, and therefore, $\mathcal{I}$-tri-holomorphic spheres are conformal too. The image of such a harmonic, conformal map is a minimal surface. $\square$

This shows the existence of non-removable singularities in the singular locus of the limiting Fueter section implies the existence of minimal spheres in the target hyperkähler manifold. This motivates the study of minimal spheres in hyperkähler manifolds.

3.5.3 Minimal Spheres in Hyperkähler Manifolds

The previous section gives us a motivation to study minimal spheres in hyperkähler manifolds — especially in the moduli spaces of monopoles — which are not image of a holomorphic curve, since the $\mathcal{I}$-tri-holomorphic maps are not holomorphic. We collect some of the basics facts about these spheres here, which we will use in the next section, to rule out the existence of non-removable singularities in certain cases.

Lemma 77. *Let (X, g_X, I, J, K) be a hyperkähler manifold. Let $f : S^2 \to X$ be a $\mathcal{I}$-tri-holomorphic map. Furthermore, suppose $f : S^2 \to X$ is holomorphic with respect to a complex structure I_0 on X. Then f is a constant map.*

Proof. Let j be the complex structure on S^2. At each points $x \in S^2$ we have

$$d_x f + \mathcal{I}(x) \circ d_x f \circ j = 0, \qquad d_x f + I_0 \circ d_x f \circ j = 0,$$

and therefore,

$$(\mathcal{I}(x) - I_0) \circ df \circ j = 0 \in \Omega^0(S^2, f^*TX),$$

which implies $df = 0$ everywhere on S^2, and therefore, f is constant. $\square$

There is no closed minimal surface in the model hyperkähler manifolds $\mathbb{H}^k$. However, one can study closed minimal surfaces in $\mathbb{T}^{4k}$ equipped with the hyperkähler structure induced from $\mathbb{H}^k$. There is the following result of Micallef in this direction [72].

Theorem 32 (Micallef)**.** *Any area minimizing surface in a flat $\mathbb{T}^4$ is holomorphic with respect to a complex structure compatible with the given flat metric on $\mathbb{T}^4$.*

The following theorem follows from the theorem above and the Lemma 77.

Corollary 9. *There is no non-trivial $\mathcal{I}$-tri-holomorphic map $f : S^2 \to \mathbb{T}^4$.*

Remark 8. *The Corollary 9 is compatible with the compactness theorem of [46] for Fueter maps on a hyperkähler bundle where the fiberes on modeled on a hyperkähler manifold (X, g_X, I, J, K) where X is compact and g_X is flat, over an oriented Riemannian 3-manifold (M, g) which is a quotient of S^3 by a finite subgroup of $SU(2)$*

The case of $K3$ surfaces is different. Michallef and Wolfson, and later Foscolo showed there exist hyperkähler metrics on the $K3$-surfaces that admit a strictly stable minimal sphere which is not holomorphic with respect to any complex structure compatible with the metric. The counterexamples of Foscolo are the images of $\mathcal{I}$-tri-holomorphic maps.

Another family of manifolds to consider are the ones constructed by Gibbons-Hawking Ansatz. We will study these manifolds in more detail in Section 4.1.1.3. As we will see, all compact minimal spheres in the multi-Eguchi-Hanson and multi-Taub-NUT are holomorphic. This implies the following lemma.

Lemma 78. *Let (X, g_X, I, J, K) be a multi-Eguchi-Hanson or multi-Taub-NUT space. There is no non-trivial $\mathcal{I}$-tri-holomorphic map $f : S^2 \to X$.*

3.5.4 Minimal Surfaces in the Moduli Spaces of Monopoles

In this section, we study minimal surfaces in the moduli spaces of monopoles on $\mathbb{R}^3$, which potentially can be a source of non-compactness of the spaces of Fueter sections on monopole bundles. There are certain minimal spheres, called the axi-symmetric ones, in these moduli spaces; however, we will show, because of topological reasons, these spheres cannot be a source of non-compactness. Furthermore, we state a conjecture about the minimal surfaces in the moduli spaces of monopoles on $\mathbb{R}^3$ which, if true, will result in a compactness theorem.

One way of constructing minimal submanifolds in the moduli spaces of solutions to a gauge-theoretic equation is to consider the solutions which are invariant under some group action and have certain symmetries.

Example 14. *For any given direction in $\mathbb{R}^3$, we can consider the solutions invariant under rotation around this axis. These solutions are called axi-symmetric solutions. The space of axi-symmetric solutions with charge k to the Bogomolny equation is a minimal $\mathbb{RP}^2$ in the moduli space of k-monopoles, or a minimal sphere in the universal cover of this space. In fact, as we will see, this minimal sphere in the universal cover is image of a $-\mathcal{I}$-tri-holomorphic map.*

In the case charge $k = 2$, it is proven by Tsai and Wang that this minimal sphere is the only compact minimal surface inside the Atiyah-Hitchin space [93].

3.5.4.1 The Unique Minimal Sphere in the Atiyah-Hitchin Space

The moduli space of reduced 2-monopoles Mon_2^0 is the first non-trivial example of the moduli spaces of reduced monopoles. About this 4-dimensional manifold we have $\pi_1(\mathrm{Mon}_2^0) = \mathbb{Z}_2$. Let $\widetilde{\mathrm{Mon}_2^0}$ be the double cover of this space, called the Atiyh-Hitchin manifold. Any minimal surface $\Sigma \subset \mathrm{Mon}_2^0$ would lift to a minimal surface — potentially disconnected — $\widetilde{\Sigma} \subset \widetilde{\mathrm{Mon}_2^0}$.

As shown by Atiyah and Hitchin this space has a minimal 2-sphere at its core [4]. In this section, we review a theorem of Tsai and Wang which states this is the only minimal submanifold of the Atiyah-Hitchin space.

Theorem 33 ([93])**.** *Let Σ denote the minimal sphere of axi-symmetric solutions in the Atiyah-Hitchin space. Then*

- *Σ is the only compact minimal surface in $\widetilde{Mon_2^0}$.*

- *Σ is a calibrated, and therefore, it minimizes the area within its homology class.*
- *There is no compact, three-dimensional, minimal submanifold in $\widetilde{Mon_2^0}$.*

Sketch of the proof. A four dimensional manifold is hyperkähler if and only if it is anti self-dual Einstein. The group $SO(3)$ acts on $\mathbb{R}^3$, and therefore, on the moduli spaces of monopoles. This action preserves the L^2-metric of the moduli space, and therefore, the Riemannian metric of the Atiyah-Hitchin space. The Atiyah-Hitchin metric is an $SO(3)$-invariant anti self-dual Einstein metric. Because of this $SO(3)$ symmetry the partial differential equation for the metric reduces to an ordinary differential equation.

By a theorem of Gibbons and Pope such a metric can be written as

$$ds^2 = dr^2 + a^2(\sigma^1)^2 + b^2(\sigma^2)^2 + c^2(\sigma^3)^2.$$

The function $r : \widetilde{\mathrm{Mon}_2^0} \to \mathbb{R}^{\geq 0}$ can be understood as a radius function on the Atiyah-Hithin space. The minimal sphere Σ is the zero section, $\Sigma = r^{-1}(0)$. In fact, r measures the geodesic distance from this minimal sphere.

Consider the map $r^2 : \widetilde{\mathrm{Mon}_2^0} \to \mathbb{R}^{\geq 0}$. The restriction of this map to any minimal submanifold N is subharmonic,

$$\Delta_N(r^2_{|N}) = tr(Hess(r^2_{|N})) \geq 0,$$

and therefore, by the maximal principle, r^2 is constant on N. In particular, the Hessian of r^2, restricted to N, vanishes, which implies $r = 0$. $\square$

The following corollary is immediate.

Corollary 10. *The minimal $\mathbb{RP}^2$ of axi-symmetric solutions is the only minimal surface in Mon_2^0.*

3.5.4.2 Minimal Spheres in the Moduli Spaces of Monopoles with Higher Charges

In this section, we state a conjecture regarding the minimal submanifolds of the moduli spaces reduced k-monopoles, which generalizes the case of $k = 2$.

Theorem 34 (Atiyah-Hitchin [4])**.** *For every charge k, the moduli space of axi-symmetric solutions is a minimal $\mathbb{RP}^2$ in Mon_k^0. It will pull back to copies of minimal S^2 or $\mathbb{RP}^2$ in the universal cover, depending on whether k is even or odd.*

It is natural to ask the following question.

Question 79. *Is there any compact minimal submanifold in the moduli spaces of centered k-monopoles for $k > 2$?*

The main difficulty in answering this question is the absence of radius function r, unlike the case of charge 2-monopoles.

A slightly different approach to study these spheres stems from the twistorial description.

3.5.5 A Twistorial Description of $-\mathcal{I}$-Tri-Holomorphic Maps

A standard method to study hyperkähler manifolds is to use the twistororial description, introduced by Roger Penrose [79], and more relevant to our setup in [45]. As we mentioned earlier, if (X, g_X, I, J, K) is a hyperkähler manifold, then any $(aI + bJ + cK)$ is also a covariantly constant complex structure, where $a^2 + b^2 + c^2 = 1$.

The twistor space Z associated to a $4k$-dimensional hyperkähler manifold X is defined to be - topologically - the product manifold $X \times S^2$ with the almost complex structure

$$\mathcal{J}_1 = (aI + bJ + cK, j),$$

at the point $(x, a, b, c) \in X \times S^2$, where j is the standard complex structure on S^2. We call this complex structure, the standard complex structure of the twistor space. This complex structure is integrable.

Theorem 35 (Atiyah-Hitchin-Singer)**.** *$(Z, \mathcal{J}_1)$ is a complex manifold.*

Proof can be found in [45, 5]

However, there is a non-standard almost complex structure defined on Z, introduced by Eells and Salamon [23]. This almost complex structure on the twistor space is defined by

$$\mathcal{J}_2 = (-aI - bJ - cK, I_0),$$

at each point $(x, a, b, c) \in X \times S^2$. This almost complex structure is called the Eells-Salamon complex structure [23]. These two complex structures are quite different. For instance, we have the following theorem.

Theorem 36 (Eells-Salamon [23])**.** *The almost complex structure $\mathcal{J}_2$ is never integrable.*

As observed by Eells and Salamon, the twistor space equipped with this non-integrable complex structure is quite suitable for studying the harmonic maps into the hyperkähler manifold X. It turns out the Eells-Salamon twistor space $(Z, \mathcal{J}_2)$ gives a amenable description of $-\mathcal{I}$-holomorphic maps into X, by considering their Gauss lift to Z.

Definition 60. *Let (X, g_X, I, J, K) be a $4k$-dimensional hyperkähler manifold. Let $Z = S^2 \times X$ be the Twistor space. Let $g : S^2 \to X$. Then the map $\hat{g} : S^2 \to Z$ defined by*

$$\tilde{g}(x) := (x, g(x)),$$

is called the Gauss lift of g to the twistor space Z, which can be understood as a section of the bundle $Z \to S^2$, or simply the graph of g.

The following theorem shows the relevance of the Eells-Salamon almost complex structure to $-\mathcal{I}$-tri-holomorphic maps.

Theorem 37. *Let (X, g_X, I, J, K) be a hyperkähler manifold. The map $g : S^2 \to X$ is a $-\mathcal{I}$-tri-holomorphic map if and only if $\tilde{g}$ is a $\mathcal{J}_2$-holomorphic section of the twistor space Z.*

Proof. Let $y = (a, b, c) \in S^2 \subset \mathbb{R}^3$. The theorem follows from

$$2\overline{\partial}_{\mathcal{J}_2}\tilde{g} = d\tilde{g} + \mathcal{J}_2 \circ d\tilde{g} \circ j = (0, dg - \mathcal{I} \circ dg \circ j) = (0, 2\overline{\partial}_{-\mathcal{I}}g).$$

$\square$

Example 15. *The theorem above can be used to construct $\mathcal{J}_2$-holomorphic sections of $Z \to S^2$. Let $g : S^2 \to Mon_2^0$ be the $-\mathcal{I}$-holomorphic parametrization of the axi-symmetric solutions. $\tilde{g}$ is a $\mathcal{J}_2$-holomorphic section of the twistor space $Z = S^2 \times Mon_2^0 \to S^2$.*

The twistor space can be defined for any oriented Riemannian 4-manifold X. The Grassmann bundle of 2-planes $Z = \widetilde{Gr}_2(TX)$ over X is a vector bundle whose fiber at the point $x \in X$ is the space of real oriented two dimensional subspaces in T_xX, denoted by $\widetilde{Gr}_2(T_xX)$, which is the double cover of $Gr_2(T_xX)$. Note that $\pi_1(Gr_2(TX)) = \mathbb{Z}_2$.

The Hodge star operator $* : \Lambda^2TX \to \Lambda^2TX$ give rise to a decomposition

$$\Lambda^2TX = \Lambda^2_+TX \oplus \Lambda^2_-TX.$$

Let

$$S_\pm = S(\Lambda^2_\pm TX),$$

be the corresponding unit 2-sphere bundles over X. We have the following bundle isomorphism

$$\widetilde{Gr}_2(TX) \cong S_+ \times S_-.$$

The bundle S_- generalizes the twistor bundle Z we constructed above for hyperkähler 4-manifolds to general oriented Riemannian 4-manifolds. The almost complex structure $\mathcal{J}_2$ can be generalized to S_- [23]. We have projection maps

$$\pi_\pm : \widetilde{Gr}_2(TX) \to S_\pm.$$

For any map $g : S^2 \to X$, let $\tilde{g} : S^2 \to \widetilde{Gr}_2(TX)$ be the corresponding Gauss lift. We can define subsidiary Gauss lifts $\widetilde{g}_\pm : S^2 \to S_\pm$,

$$\widetilde{g}_\pm = \pi_\pm \circ \widetilde{g}.$$

Theorem 38. *[Eells-Salamon [23]] Let (X, g_X) be an oriented Riemannian 4-manifold. The correspondences*

$$g \to \widetilde{g}, \qquad g \to \widetilde{g},$$

are bijective correspondences between non-constant conformal harmonic maps $g : S^2 \to X$ and non-trivial $\mathcal{J}_2$-holomorphic spheres $\widetilde{g}_\pm : S^2 \to S_\pm$.

This theorem gives another argument for the following fact, which we can encountered before.

Corollary 11. *Let (X, g_X, I, J, K) be a $4k$-dimensional hyperkähler manifold. The image of a $-\mathcal{I}$-tri-holomorphic map $g : S^2 \to X$ is minimal sphere.*

Proof. Let $f : S^2 \to X$ be a $-\mathcal{I}$-tri-holomorphic map. Its Gauss lift $\tilde{g}$ is a $\mathcal{J}_2$-holomorphic section of $S_- \to S^2$, and by the theorem above, g itself is a conformal harmonic map, and therefore, its image in X is a minimal sphere. □

3.5.5.1 Non-Removable Singularities and the Topology of the Moduli Spaces of Monopoles

In order to completely rule out the existence of the non-removable singularities for a sequence of Fueter sections of a bundle with fiber Mon_2^0, we should show the minimal sphere of the axi-symmetric solutions, which is in fact an $\mathbb{RP}^2 \subset \mathrm{Mon}_2^0$, wouldn't appear as the image of a tangent map of a Fueter section at any point.

Theorem 39. *Let (M, g) be an oriented Riemannian 3-manifold. Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of smooth Fueter sections of a monopole bundle $\mathfrak{X} \to M$, with fibers modeled on Mon_k^0. Moreover, suppose $f_i \to f$ in $C^{\infty}_{loc}(M \setminus \{y\})$, where f has a non-removable singularity at $y \in M$. The minimal $\mathbb{RP}^2 \subset Mon_k^0$ of the axi-symmetric k-monopoles cannot appear as the image of a tangent map of the section f at the point y.*

Proof. Let $\{s_j\}_{j=1}^{\infty}$ be a sequence of positive numbers, where $s_j \to 0$. Let

$$f_{y,s_j} : B_0(r_y) \subset \mathbb{R}^3 \to \mathfrak{X}, \qquad f_{y,s_j}(x) = f(y + s_j x),$$

be a sequence of re-scalings of f. Then the sequence $\{f_{y,s_j}\}_{j=1}^{\infty}$, assuming the Assumption 1, converges to a radially invariant map φ_y, which can be restricted to $S^2 \subset \mathbb{R}^3$ to get

$$\varphi_y : \mathbb{R}^3 \setminus S_y \to \mathfrak{X}_y, \tag{3.5.7}$$

for a closed rectifiable subset $S_{i,y} \subset \mathbb{R}^3$ with Hausdorff dimension less than 1.

For each smooth Fueter section f_i, let

$$f_{i,y,s_j} : B_0(r_y) \subset \mathbb{R}^3 \to \mathfrak{X}, \qquad f_{i,y,s_j}(x) = f_i(y + s_j x),$$

be a sequence of re-scaled sections. Then for each i the sequence $\{f_{i,y,s_j}\}_{j=1}^{\infty}$ converges to a map $\varphi_{i,y}$, which is radially invariant, and therefore, we can define

$$\varphi_{i,y} : \mathbb{R}^3 \setminus S_{i,y} \to \mathfrak{X}_y. \tag{3.5.8}$$

for a closed rectifiable subset $S_{i,y} \subset \mathbb{R}^3$ with Hausdorff dimension less than 1.

For each j, the sequence $\{f_{i,y,s_j}\}_{i=1}^{\infty} \to f_{y,s_j}$ in $C^{\infty}_{loc}(M \setminus S_{y,s_j})$ for a closed rectifiable subset $S_{y,s_j} \subset M$ with Hausdorff dimension less than 1, and therefore,

$$\varphi_{i,y} \to \varphi_y, \quad \text{in} \quad C^{\infty}_{loc}(\mathbb{R}^3 \setminus S),$$

for a closed rectifiable subset $S \subset \mathbb{R}^3$ with Hausdorff dimension less than 1.

The maps $\varphi_{i,y}$ and φ_y are $-\mathcal{I}$-tri-holomorphic spheres in Mon_k^0. However, since each map f_i, and therefore, f_{i,y,s_j} is smooth and the image of $\varphi_{i,y}(B^3)$ is contractible in Mon_k^0. Since $\varphi_{i,y} \to \varphi_y$ in $C^\infty_{loc}(\mathbb{R}^3 \setminus S)$, the subset $\varphi_y(B^3)$ is also contractible in Mon_k^0. However, the minimal $\mathbb{RP}^2$ of axi-symmetric solutions is non-trivial in second homology, and therefore, it can't be the image of the map φ_y. $\square$

Corollary 12. *The minimal spheres of axi-symmetric solutions is not a source of non-compactness.*

In fact, we expect there would not be any minimal sphere in the moduli spaces of centered monopoles which can be a source of non-compactness.

Conjecture 6. *There is no minimal sphere Σ in the universal cover of the moduli space of centered k-monopoles which is trivial in the second homology.*

Conjecture 7. *Let (M, g) be an oriented Riemannian 3- or 4-manifold. Let $\{f_i\}_{i=1}^\infty$ be a sequence of smooth Fueter sections of a monopole hyperkähler bundle $\mathfrak{X} \to M$, with fibers modeled on the moduli space of centered k-monopoles on $\mathbb{R}^3$, such that $f_i(M) \subset C$ for a compact subset $C \subset \mathfrak{X}$, and with bounded energy $\mathcal{E}(f_i) < C_\varepsilon$, for a constant C_ε. Then there exists a Fueter section $f \in \Gamma(M, \mathfrak{X})$, such that $f_{i_j} \to f$ in $C^\infty(M)$, for a subsequence $\{f_{i_j}\}$.*

We finish this section with another conjecture. Throughout this section we assumed there is bound on the energy of the Fueter sections, since it is not a topological quantity in the case of Fueter sections. However, we expect that this assumption is not necessary.

Conjecture 8. *There exists a uniform bound on the energy of a sequence of Fueter sections of the monopole bundles, when their images are in a compact subset of the bundle.*

3.6 Divergence to Infinity

Another source of the non-compactness, which we mostly ignored in this writing, is the potential divergence of the Fueter sections to infinity. The moduli spaces of centered charge 2 $SU(2)$-monopoles is asymptotically $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$. We can start by considering the Fueter sections to $(\mathbb{R}^3 \times S^1)/\mathbb{Z}_2$ as the model space at infinity. Motivated by the study of monopoles on Calabi-Yau 3-folds, we would consider the case where the hyperkähler bundle is defined by

$$\mathfrak{X} = T^*M \times U(1) \to M,$$

where (M, g) is an oriented Riemannian 3-manifold, $\mathfrak{X}$ equipped with the connection obtained from both the Levi-Civita connection on T^*M and the trivial connection on the trivial $U(1)$-bundle above M.

Let $f = (a, \phi) \in \Gamma(\mathfrak{X})$ is Fueter section of this bundle, where a is a 1-form on M and ϕ a function $\phi : M \to U(1)$. The Fueter equation reads as

$$*da = d\phi,$$

which is a $U(1)$-Bogomolny equation on M.

As we have observed before, assuming (a, ϕ) is a smooth monopole, we have $\phi = c$ for a constant $c \in \mathbb{R}$ and a is a closed 1-form. In particular, in the case where M is a homology 3-sphere $a = df$, and therefore, up to gauge this equation has only a trivial solution.

The study of the singular cases, and the case of monopoles with higher charges is necessary for resolving the compactness problems completely; however, we will not follow this direction any further here.

Chapter 4

On the Donaldson-Scaduto Calibrated Submanifolds

Donaldson proposed the possibility of studying G_2-manifolds from the viewpoint of coassociative fibrations and the adiabatic limit, where the diameters of the fibers shrink to zero [14]. In particular, it is expected that this approach would be helpful in the study of the following two fundamental problems:

- determining when a compact oriented smooth 7-dimensional manifold admits an integrable G_2-structure;
- understanding the formation of singularities and corresponding failure of compactness in the counting problems in order to define invariants of G_2-manifolds, using instantons, monopoles, associative and coassociative submanifolds.

The adiabatic picture led Donaldson and Scaduto to conjecture the existence of certain associative submanifolds in G_2-manifolds with a coassociative $K3$-fibration near the adiabatic limit [20].

Conjecture 9 (Donaldson-Scaduto)**.** *Let X be the smooth 4-manifold underlying any complex $K3$-surface. Let α_1, α_2 and α_3 be -2 classes on X such that $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $H \subset H^2(X, \mathbb{R}) \cong \mathbb{R}^{3,19}$ be a maximal positive subspace $H \cong \mathbb{R}^3$. Let v_i be the projection of α_i to H. Then there is an associative submanifold $P \subset X \times \mathbb{R}^3$ with three ends asymptotic to cylinders $\Sigma_i \times \mathbb{R}^+ v_i$, where Σ_i is a 2-sphere representing α_i with respect to the complex structure defined by v_i. The associative submanifold P is unique up to translations in $\mathbb{R}^3$.*

As explained in [20], one can consider the non-compact manifold obtained as resolution of the A_2-singularity, which can be emebedded in the $K3$-manifold. This non-compact hyperkähler 4-manifold can be described using the Gibbons-Hawking Ansatz. This results in a similar conjecture involving the multi-Eguchi-Hanson spaces.

In this chapter, we propose a strategy to prove the Donaldson-Scaduto conjecture and take several steps in that direction.

4.1 The Preliminaries

In the this section, we set up the basics to explain, motivate and study this conjecture. We start by reviewing the basics of the geometry of calibrated submanifolds and their deformation. For more detailed introduction to the subject consult with [40, 71, 69, 56, 66].

4.1.1 Calibrated Geometry

A classical topic in Riemannian geometry is the study of minimal submanifolds in a given Riemannian manifold. For instance geodesics on Riemann surfaces, minimal surfaces in 3-dimensional Riemannian manifolds, and complex curves in Kähler manifolds are among the most well-studied ones. Information about minimal submanifolds of a Riemannian manifold can help us in understanding the geometry of the ambient manifold.

Motivated by complex submanifolds in Kähler manifolds, Harvey and Lawson introduced a special class of minimal submanifolds, called the calibrated submanifolds [40]. These minimal submanifolds are defined by a first order partial differential equation, which is easier to analyze that a general minimal submanifold, which is described by a second order partial differential equation.

Calibrated geometry is closely related to the theory of manifolds with special holonomy groups. In fact, there are natural classes of calibrations and calibrated submanifolds in manifold with special holonomy groups. Moreover, these submanifolds play an important role in the study of gauge theories over manifolds with special holonomy groups.

Definition 61 (Calibrated Submanifold)**.** *Let (M, g) be a Riemannian manifold. Let $\phi \in \Omega^k(M)$ be a closed k-form. ϕ is called a calibration on M if for every $x \in M$ and every oriented k-dimensional vector subspace $V \subset T_xM$ equipped with the induced Euclidean metric g_V, we have*

$$\phi_{|_V} \leq vol_V,$$

where vol_V is the induced volume form on V associated to g_V.

Let $N \subset M$ be an oriented k-dimensional submanifold. N is called a calibrated submanifold if for every $x \in N$

$$\phi_{|_{T_xN}} = vol_{T_xN}.$$

In other words, for each $x \in N$, T_xN is a calibrated k-plane with respect to ϕ.

Calibrated submanifolds are minimal. Recall that minimal submanifolds are critical points of the volume functional, and they are not necessarily volume minimizers. The minimal submanifolds can even be the maximizers of the volume functional; however, compact calibrated submanifold are the volume minimizers in their homology class.

The prototypical and motivating examples of calibrated submanifolds appear in Kähler manifolds.

Example 16. *Let (M, g, ω) be a Kähler manifold. $2k$-form $\omega^k/k!$ is a calibration on M and calibrated submanifolds with respect to this form are the complex k-dimensional submanifolds, and therefore, compact complex submanifolds of Kähler manifolds are volume minimizing in their homology class.*

4.1.1.1 Special Lagrangians

The relevance of calibrated geometry to the geometry of Calabi-Yau manifolds follows from the following.

Definition 62 (Special Lagrangian)**.** *Let (Z, g, ω, Ω) be a Calabi-Yau n-fold. Real-valued differential n-forms*

$$\operatorname{Re}(e^{i\theta}\Omega) = \cos\theta \operatorname{Re}(\Omega) - \sin\theta \operatorname{Im}(\Omega),$$

are calibrations on Z, where $e^{i\theta}$ is called the phase of the calibration. Let $L \subset Z$ be an n-dimensional oriented submanifold. L is called a special Lagrangian with phase $e^{i\theta}$ if L is calibrated with respect to the form $\operatorname{Re}(e^{i\theta}\Omega)$. *If we do not explicitly state the phase of the special Lagrangian, it is assumed that $\theta = 0$.*

There is a different characterization of the special Lagrangians which is quite useful, and also explains the etymology of the denomination of the special Lagrangian submanifolds.

Lemma 80. *Let (Z, g, ω, Ω) be a Calabi-Yau n-fold. Let $L \subset Z$ be an n-dimensional oriented submanifold. L is a special Lagrangian if and only if*

$$\omega_{|_L} = 0 \quad \textit{and} \quad \operatorname{Im}\Omega_{|_L} = 0. \tag{4.1.1}$$

The first condition asserts that L is a Lagrangian submanifold, and the second condition implies L is a special one.

We start by considering the linear model.

4.1.1.2 Special Lagrangians in $\mathbb{C}^2$

Let $Z = \mathbb{C}^n$ with the Calabi-Yau structure

$$g = |dz_1|^2 + \ldots + |dz_n|^2, \quad \omega = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + \ldots + dz_n \wedge d\overline{z}_n) \quad \text{and} \quad \Omega = dz_1 \wedge \ldots \wedge dz_n.$$

The Calabi-Yau structure on $\mathbb{C}^2$ in real coordinates (x_1, y_1, x_2, y_2) on $\mathbb{R}^4 = \mathbb{C}^2$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, takes the form

$$\begin{aligned} g =& dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2, \quad \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2, \\ \operatorname{Re}(\Omega) =& dx_1 \wedge dx_2 - dy_1 \wedge dy_2 \quad \text{and} \quad \operatorname{Im}(\Omega) = dx_1 \wedge dy_2 + dy_1 \wedge dx_2. \end{aligned}$$

A 2-dimensional submanifold $L \subset \mathbb{C}^2$ is special Lagrangian if and only if

$$(\omega - i\,\mathrm{Im}(\Omega))_{|_L} = 0.$$

Following Joyce [55], if we introduce the complex variables

$$w_1 = x_1 + ix_2, \quad w_2 = y_1 - iy_2,$$

we can see

$$\omega - i\,\mathrm{Im}\,\Omega = dw_1 \wedge dw_2,$$

and therefore, the condition of being special Lagrangian translates into

$$(dw_1 \wedge dw_2)_{|_L} = 0.$$

However, this is equivalent to L being holomorphic with respect to the integrable almost complex structure J, defined by

$$J(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, \quad J(\frac{\partial}{\partial y_1}) = -\frac{\partial}{\partial y_2}.$$

This is called a hyperkähler rotation. Note that $\mathbb{C}^2 = \mathbb{H}$ can be understood as a Calabi-Yau 2-fold and also as a hyperkähler 4-manifold. In fact, since the calibrated condition consists of equations in the linear algebra level, this observation extends to all hyperkähler 4-manifolds.

$\mathbb{C}^2$ can be considered as a special case of a more general family of hyperkähler 4-manifolds with a $U(1)$-symmetry, discovered by Gibbons and Hawking [37], which we also saw earlier in chapter 3.

4.1.1.3 The Gibbons–Hawking Ansatz

The Gibbons-Hawking Ansatz describes a system of coordinates on a family of hyperkähler 4-manifolds with a $U(1)$-symmetry, in terms of a positive harmonic function defined on an open subset of $\mathbb{R}^3$.

Definition 63 (The Gibbons–Hawking Ansatz)**.** *Let $U \subset \mathbb{R}^3$ be an open subset with coordinates u_1, u_2 and u_3. Let $p_1, \dots, p_n$ be n distinct points in U. Let $\pi : X \to U \setminus \{p_1, \dots, p_n\}$ be a principal $U(1)$-bundle. Let t be the coordinate along the fibers, normalized to have period 2π, with ∂t the corresponding vector field of the S^1-action, θ a connection 1-form on X such that $\theta(\partial t) = i$. Let $\theta_0 = \frac{\theta}{2\pi i}$. Let β be the curvature 2-form defined by $d\theta = \pi^*(\beta)$ and $V : U \to \mathbb{R}$ a positive harmonic real-valued function such that*

$$*dV = \frac{1}{2\pi i}\beta.$$

X can be equipped with a hyperkähler structure, given by the Riemannian metric

$$g_X = V \sum_{i=1}^{3} du_i^2 + V^{-1}\theta_0^2,$$

the Kähler forms $\omega_1, \omega_2, \omega_3$, and corresponding integrable almost complex structures I, J and K given by

$$\begin{aligned}
&\omega_1 = du_1 \wedge \theta_0 + V du_2 \wedge du_3, \quad \omega_2 = du_2 \wedge \theta_0 + V du_3 \wedge du_1,\\
&\omega_3 = du_3 \wedge \theta_0 + V du_1 \wedge du_2,\\
&I(du_2) = -du_3, \quad I(du_1) = -\frac{1}{V} d\theta_0, \quad J(du_3) = -du_1, \quad J(du_2) = -\frac{1}{V} d\theta_0,\\
&K(du_1) = -du_2, \quad K(du_3) = -\frac{1}{V} d\theta_0.
\end{aligned}$$

Furthermore, the $U(1)$-action is symplectic and Hamiltonian with respect to each symplectic form, with the moment maps u_1, u_2 and u_3. The map $\mu : X \to \mathbb{R}^3$ defined by $\mu = (u_1, u_2, u_3)$ is the hyperkähler moment map of the $U(1)$-action on X.

The Calabi-Yau structure on this space is given by the metrig g_X mentioned above, holomorphic volume form Ω and the kähler 2-form ω,

$$\Omega = \omega_1 - i\omega_2 = (\theta_0 + iV du_3) \wedge (du_1 - i du_2), \qquad \omega = \omega_3.$$

Note that

$$\Omega \wedge \overline{\Omega} = 2\omega_3^2.$$

Example 17 (Multi-Eguchi-Hanson Spaces)**.** *Let $U = \mathbb{R}^3 \setminus \{0\}$. Let $V : U \to \mathbb{R}$ be the function defined by*

$$V(x) = \frac{1}{4\pi|x|}.$$

Then $X = \mathbb{R}^4 \setminus \{0\}$ with the Euclidean metric g which can be extended over the origin $0 \in \mathbb{R}^4$ to get $\overline{X} = \mathbb{R}^4$.

Let $U = \mathbb{R}^3 \setminus \{p, q\}$, where $p \neq q$. Let $V : U \to \mathbb{R}$ be the function given by

$$V(x) = \frac{1}{4\pi|x-p|} + \frac{1}{4\pi|x-q|}.$$

This metric on X extends smoothly to the points $\pi^{-1}(p)$ and $\pi^{-1}(q)$. The resulting hyperkähler manifold $\overline{X}$ is called the Eguchi-Hanson space, which is in fact T^*S^2. The Eguchi–Hanson metric is ALE (asymptotically locally Euclidean), asymptotic to the flat metric on $\mathbb{R}^4/\mathbb{Z}_2$.

Let $U = \mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ for distinct points $p_1, \ldots, p_n$. Let $V : U \to \mathbb{R}$ be the function *defined by*

$$V(x) = \sum_{i=1}^{n} \frac{1}{4\pi|x - p_i|}.$$

This metric on X, similar to the previous cases, extends smoothly to the points $\pi^{-1}(p_i)$. The resulting hyperkähler manifold $\overline{X}$ is called a multi-Eguchi-Hanson space. A multi-Eguchi–Hanson metric is ALE, asymptotic to the Euclidean metric on $\mathbb{R}^4/\mathbb{Z}_n$.

Example 18 (Multi-Taub-NUT Spaces)**.** *Let $m > 0$. Let $U = \mathbb{R}^3 \setminus \{0\}$. Let $V : U \to \mathbb{R}$ be the function defined by*

$$V(x) = m + \frac{1}{4\pi|x|}.$$

The metric g on X extends smoothly over $\pi^{-1}(0)$. The resulting space is topologically $\mathbb{R}^4$, but with a different metric, called the Taub-NUT metric. The Taub-NUT metric is ALF (asymptotically locally flat), asymptotic to the product metric on $\mathbb{R}^3 \times S^1$.

Let $m > 0$. Let $U = \mathbb{R}^3 \setminus \{p_1, \dots, p_n\}$ for distinct points $p_1, \dots, p_n$. Let $V : U \to \mathbb{R}$ be the function defined by

$$V(x) = m + \sum_{i=1}^{n} \frac{1}{4\pi|x - p_i|}.$$

This metric on X extends smoothly over the isolated points $\pi^{-1}(p_i)$. The resulting hyperkähler manifold $\overline{X}$ is called a multi-Taub-NUT space. Multi-Taub–NUT metrics are ALF.

4.1.1.4 Special Lagrangians in the Gibbons-Hawking Spaces

We start by considering closed special Lagrangians in the Gibbons-Hawking spaces. By the maximal principle there is no closed minimal submanifold in $\mathbb{R}^n$, and therefore, there is no closed special Lagrangian in $\mathbb{R}^4$. A similar result holds for the Taub-NUT space. For a proof of the following lemma consult with [67].

Lemma 81. *There is no closed minimal submanifold in the Taub-NUT space.*

There are compact holomorphic spheres, and therefore, special Lagrangians, in the multi-Eguchi Hanson and the multi-Taub-NUT spaces.

Lemma 82. *We have the following,*

- *The zero section S^2 inside the Eguchi-Hanson space T^*S^2 is a minimal surface, a holomorphic sphere with respect to the complex structure specified by the direction of the vector connecting q to p in the Gibbons-Hawking construction.*

- *The 2-sphere $\pi^{-1}([p_i, p_j])$ in the multi-Eguchi-Hanson or the multi-Taub-NUT spaces, where $[p_i, p_j]$ is the line segment connecting p_i to p_j in $\mathbb{R}^3$, is a minimal sphere, holomorphic with respect to the complex structure specified by the direction of the vector connecting p_i to p_j.*

The proofs are straightforward.

One can consider the special Lagrangians in the Gibbons-Hawking spaces which are invariant under the $U(1)$-action. The following theorem shows the significance of these special Lagrangians.

Theorem 40 (Trinca [92]). *Let (X, g) be a multi-Eguchi-Hanson or a multi-Taub-NUT space with a harmonic function with two singular points. Then, compact minimal submanifolds are $U(1)$-invariant or are contained in the unique $U(1)$-invariant compact minimal surface.*

There are also complete non-compact $U(1)$-invariant special Lagrangians in the Gibbons-Hawking spaces.

Lemma 83. *$U(1)$-invariant special Lagrangian fibrations with singularities of the Gibbons-Hawking spaces, with respect to ω_1, are given by the following equations,*

$$u_1 = c_1,$$
$$u_2 = c_2,$$

for constants c_1 and c_2.

Proof. The $U(1)$-action on X is symplectic with respect to ω_1. In fact, it is Hamiltonian, with the moment map $\mu_1 : X \to \mathfrak{u}(1)^* \cong \mathbb{R}$,

$$\mu_1 = u_1.$$

The $U(1)$-invariant special Lagrangians are in the level sets of the moment map; i.e., for any $U(1)$-invariant special Lagrangian L, there exists $c_1 \in \mathbb{R}$ such that $L \subset \mu_1^{-1}(c_1)$,

$$\mu_1 = u_1 = c_1.$$

Let v be the vector field associated to the infinitesimal action of $1 \in \mathbb{R}$. L is a $U(1)$-invariant special submanifold, and therefore, $\iota_v \operatorname{Im}(\Omega_{|_L}) = 0$.

$$2\iota_v \operatorname{Im}(\Omega_{|_L}) = du_2 = 0,$$

and therefore, $u_2 = c_2$, for a constant c_2.

Sending each of these special Lagrangians to (c_1, c_2), we get a $U(1)$-invariant special Lagrangian fibration with singularities of a Gibbons-Hawking space over $\mathbb{R}^2$. □

In the Taub-NUT case, these special Lagrangians are asymptotically conical. If $(c_1, c_2) \neq (0, 0)$ they are topologically cylinder $S^1 \times \mathbb{R}$, and if $(c_1, c_2) = (0, 0)$, it is cone on S^1. More generally, in the multi-Taub-NUT case, these special Lagrangians are topologically cylinders with some meridians each to a point.

4.1.1.5 U(1)-Invariant Special Lagrangians in $\mathbb{C}^3$

Relevant to Donaldson-Scaduto's conjecture are $U(1)$-invariant special Lagrangian submanifolds in certain Calabi-Yau 3-folds. One can first consider the linear case. $U(1)$-invariant special Lagrangians in $\mathbb{C}^3$ have been studied by Joyce [50, 52, 51].

Let $U(1)$ act on $\mathbb{C}^3$ by

$$e^{i\theta} \dots (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \ \text{ for } e^{i\theta} \in U(1).$$

Let $L \subset \mathbb{C}^3$ be a $U(1)$-invariant special Lagrangian. Locally L can be written as

$$\begin{aligned} L = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid \ &\mathrm{Im}(z_3) = u(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)), \\ &\mathrm{Re}(z_1 z_2) = v(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)), |z_1|^2 - |z_2|^2 = 2a\}, \end{aligned}$$

where $a \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy a non-linear Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{\frac{1}{2}} \frac{\partial u}{\partial y}.$$

Joyce studied the existence, uniqueness and regularity properties of these special Lagrangians. The Donaldson-Scaduto special Lagrangians, which we will study in this chapter, can be understood as a non-linear generalization of this problem, where $\mathbb{C}^2$ in $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$ is replaced by a non-linear hyperkähler 4-manifold with a $U(1)$-action.

4.1.1.6 Deformation of Compact Special Lagrangian Submanifolds

Let L be a compact special Lagrangian submanifold of a Calabi-Yau n-fold M. McLean showed that the moduli space of nearby special Lagrangians is a smooth manifold with dimension $b^1(L)$ [71].

In McLean's theorem, the Calabi-Yau structure of the ambient Calabi-Yau manifold M is fixed. One can consider the case where the Calabi-Yau structure can change too. The detailed proof can be found in Marshal's thesis [69]. We sketch an almost identical but slightly different proof here, in the case $n = 3$, which is more suitable for our needs.

Theorem 41 (McLean [71, 69])**.** *Let M be a smooth $2n$-dimensional manifold and $(g(p), \omega(p), \Omega(p))$ a smooth family of Calabi-Yau structures on M, parameterised by $p \in \mathbb{R}^m$. Let $L \subset M$ be a closed submanifold which is special Lagrangian with respect to the Calabi-Yau structure $(g(0), \omega(0), \Omega(0))$ on M. Moreover, suppose $[\omega(p)_{|_L}] = 0 = [\Omega(p)_{|_L}]$ in $H^*_{dR}(L, \mathbb{R})$ for all $p \in \mathbb{R}^m$. Then there exist an open subset $W \subset \mathbb{R}^m$ containing 0, and a family of smooth manifolds $\mathcal{M}_p$ for all $p \in W$, each with dimension $b^1(L)$ such that for each $p \in W$, the smooth manifold $\mathcal{M}_p$ is the moduli space of smooth submanifolds near L which are special Lagrangian with respect to $(g(p), \omega(p), \Omega(p))$. Moreover, the parametrized moduli space*

$$\begin{aligned} \mathcal{M}_L = \cup_{p \in W} \mathcal{M}_{L,p} = \{(p, L_\xi) \mid p \in W \textit{ and } L_\xi \textit{ is a smooth special Lagrangian submanifold} \\ \textit{with respect to } (g(p), \omega(p), \Omega(p)) \textit{ near } L\}, \end{aligned}$$

is a smooth manifold with dimension $b^1(L) + m$*.*

For simplicity, here we only prove the case where the Calabi-Yau structure is fixed and $n = 3$.

Proof. Let $f : L \to M$ be an embedding of a compact special Lagrangian L. We have

$$f^* \operatorname{Im}(\Omega) = 0 = f^*\omega.$$

Let $\flat_g : TL \to T^*L$ be the bundle isomorphism between vector fields and 1-forms on L, defined using the pull-back of the Riemannian metric g to L. Let $\nu(L) \to L$ be the normal bundle of L in M. Let $\tilde{U}$ be a small tubular neighbourhoods of L in M, identified with a small neighbourhood of the zero section in the normal bundle, still denoted by $\tilde{U}$. Let

$$\tilde{U}^{k,p} = \{\xi \in W^{k,p}(\nu(L)) \mid \xi(x) \in \tilde{U} \text{ for all } x \in L\},$$

for $k \geq 0$ and $p > 1$.

Note that since L is a special Lagrangian we have the bundle isomorphisms

$$J : \nu(L) \to TL, \qquad \flat_g \circ J : \nu(L) \to T^*L.$$

Let $U = \flat_g \circ J(\tilde{U})$ and

$$U^{k,p} = \{\eta \in W^{k,p}(T^*L) \mid \eta(x) \in U \text{ for all } x \in L\},$$

for $k \geq 0$ and $p > 1$.

For any $\eta \in U$, let $f_\eta : L \to M$ be the deformation map defined by

$$f_\eta(x) = exp_{f(x)}((\flat_g \circ J)^{-1}(\eta_x)).$$

$f_\eta(L)$ is an embedding of L homotopic to $f(L)$, when η is sufficiently small. Moreover, any nearby submanifold homotopic to L can be presented in such a way.

Let $G : U \to \Omega^0(L) \oplus \Omega^2(L)$ be the map defined by

$$G(\eta) = *f_\eta^*(\Omega) + f_\eta^*(\omega).$$

$f_\eta(L)$ is a special Lagrangian if and only if $G(\eta) = 0$. In order to show that the zero set of G forms a manifold, we should use the inverse function theorem, and therefore, we should set up the map G between the suitable Banach spaces. Let

$$G : U^{k+1,p} \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)), \qquad G(\eta) = *f_\eta^*(\Omega) + f_\eta^*(\omega).$$

However, note that

$$d_0G : W^{k+1,p}(\Lambda^1(T^*L)) \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)),$$

is not elliptic, and therefore, not Fredholm. This can be seen by looking at the rank of the domain and target spaces, which are not equal,

$$rank(W^{k+1,p}(\Lambda^1(T^*L))) = 3, \qquad rank(W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L))) = 1 + 3 = 4.$$

To resolve this problem, let

$$F : U^{k+1,p} \oplus W^{k+1,p}(\Lambda^3(T^*L)) \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)),$$

be the map defined by

$$F(\eta, \chi) = G(\eta) + d^*\chi.$$

Note that if $F(\eta, \chi) = G(\eta) + d^*\chi = 0$, then

$$\|d^*\chi\|_{L^2}^2 = \langle d^*\chi, d^*\chi\rangle_{L^2} = -\langle G(\eta), d^*\chi\rangle_{L^2} = -\langle dG(\eta), \chi\rangle_{L^2} = -\langle df_\eta^*(\omega), \chi\rangle_{L^2} = 0.$$

This shows $F(\eta, \chi) = 0$ implies $d^*\chi = 0$, and therefore, $\chi = Cvol_{g_L}$ for a constant $C \in \mathbb{R}$, which shows

$$F^{-1}(0) = G^{-1}(0) \times H^3(L, \mathbb{R}).$$

The linearized map

$$d_{(0,0)}F : W^{k+1,p}(\Lambda^1(T^*L)) \oplus W^{k+1,p}(\Lambda^3(T^*L)) \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)),$$

is given by

$$d_{(0,0)}F(\eta, \chi) = (d + d^*)(\eta + \chi),$$

which is elliptic, and since L is compact, it is Fredholm.

Let

$$\mathcal{V} := (d + d^*)(W^{k+1,p}(\Lambda^1(T^*L)) \oplus W^{k+1,p}(\Lambda^3(T^*L))) \subset W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)).$$

By the Hodge decomposition theorem,

$$W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)) = \mathcal{V} \oplus \mathcal{H}^0(L) \oplus \mathcal{H}^2(L),$$

where $\mathcal{H}^0(L)$ and $\mathcal{H}^2(L)$ are constant functions and harmonic 2-forms on L, respectively. In fact, by the Hodge decomposition theorem, $\mathcal{V}$ is the Banach subspace of $W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L))$ which is L^2-orthogonal to $\mathcal{H}^0(L) \oplus \mathcal{H}^2(L)$.

The next step is to show Image$(F) \subset \mathcal{V}$, which is proposition 3.18 in [69]. To see this, note that $*f_\eta^*(\Omega)$ and $f_\eta^*(\omega)$ are co-closed and closed, respectively. Moreover, since the maps f and f_η are homotopic, we have

$$[f_\eta(\omega)] = [f(\omega)] = 0 \quad \text{and} \quad [f_\eta(\Omega)] = [f(\Omega)] = 0,$$

and therefore,

$$*f_\eta^*(\Omega) = d^*\alpha_1 \quad \text{and} \quad f_\eta^*(\omega) = d\alpha_2,$$

for some $\alpha_1, \alpha_2 \in \Omega^1(L)$. Moreover, since $d^*\alpha_1, d\alpha_2 \in W^{k+1,p}(\Lambda^1(T^*L))$, we can take

$$\alpha_1, \alpha_2 \in W^{k+1,p}(\Lambda^1(T^*L)).$$

Furthermore, we have

$$d_{(0,0)}F(W^{k+1,p}(\Lambda^1(T^*L)) \oplus W^{k+1,p}(\Lambda^3(T^*L))) = \mathcal{V}.$$

This means there is a solutions $\eta \in W^{k+1,p}(\Lambda^1(T^*L))$ to the equation

$$(d^* + d)\eta = d^*\alpha_1 + d\alpha_2,$$

for $\alpha_1, \alpha_2 \in W^{k+1,p}(\Lambda^1(T^*L))$, which is theorem 3.15 in [69].

Moreover,

$$(\eta, \chi) \in ker(d_{(0,0)}F) \iff d^*\eta = 0 \text{ and } d\eta + d^*\chi = 0.$$

Note that if $d\eta + d^*\chi = 0$,

$$\|d^*\chi\|_{L^2}^2 = \langle d^*\chi, d^*\chi\rangle_{L^2} = -\langle d\eta, d^*\chi\rangle_{L^2} = 0,$$

and therefore, $d\eta = d^*\eta = d^*\chi = 0$, which since L is compact, η and χ are harmonic 1-form and 3-form, respectively.

$$ker(d_{(0,0)}F) = \mathcal{H}^1(L) \oplus \mathcal{H}^3(L),$$

and therefore, $\dim ker(d_{(0,0)}F) = b^1(L) + b^3(L) = b^1(L) + 1$. Moreover, $T_0\mathcal{M}_L = ker(d_0G)$, thus $\dim T_0\mathcal{M}_L = b^1(L)$. □

This can be used to compute the index of the operator F.

Corollary 13. *Let (M, g, ω, Ω) be a Calabi-Yau 3-fold and $L \subset M$ a compact special Lagrangian submanifold. Let*

$$\begin{aligned} &F : U^{k+1,p} \oplus W^{k+1,p}(\Lambda^3(T^*L)) \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)), \\ &F(\eta, \chi) = *f_\eta^*(\Omega) + f_\eta^*(\omega) + d^*\chi. \end{aligned}$$

we have index$_{(0,0)}F = 0$.

Proof. We have the linearized map

$$d_{(0,0)}F : W^{k+1,p}(\Lambda^1(T^*L)) \oplus W^{k+1,p}(\Lambda^3(T^*L)) \to W^{k,p}(\Lambda^0(T^*L)) \oplus W^{k,p}(\Lambda^2(T^*L)).$$

The cokernel can be identified with the L^2-orthogonal complement of the image, and therefore,

$$\ker(d_{(0,0)}F) = \mathcal{H}^1 \oplus \mathcal{H}^3, \qquad \operatorname{coker}(d_{(0,0)}F) \equiv \mathcal{H}^0 \oplus \mathcal{H}^2.$$

Therefore, $\dim\ker(d_{(0,0)}F) = b^1(L) + b^3(L)$ and $\dim\operatorname{coker}(d_{(0,0)}F) = b^0(L) + b^2(L)$, and therefore, by the Poincaré duality,

$$\text{index } d_{(0,0)}F = (b^1(L) + b^3(L)) - (b^0(L) + b^2(L)) = 0.$$

□

4.2 Donaldson-Scaduto Calibrated Submanifolds

In this section, we give the definition of the asymptotically cylindrical calibrated submanifolds, which their existence has been conjectured by Donaldson and Scaduto.

4.2.0.1 Donaldson-Scaduto Special Lagrangians

There exists a Calabi-Yau version of the Donaldson-Scaduto conjecture for special Lagrangians in $X \times \mathbb{C}$ for a hyperkähler 4-manifold X. We start by setting up the basics to define the asymptotically cylindrical special Lagrangians, which appear in this version of the Donaldson-Scaduto conjecture.

Let (X, g_X, I, J, K) be a 4-dimensional hyperkähler manifold with Kähler structures ω_1, ω_2 and ω_3, corresponding to the complex structures I, J and K, respectively. Let $Z = X \times \mathbb{C} = X \times \mathbb{R}^2$ be the 6-dimensional manifold equipped with the Calabi-Yau-structure given by

$$\begin{aligned} g_Z &= g_X + dx \otimes dx + dy \otimes dy, \qquad \omega = \omega_3 + dx \wedge dy \\ \Omega &= (\omega_1 - i\omega_2) \wedge (dx + idy), \end{aligned} \tag{4.2.1}$$

where x and y denotes the coordinates on $\mathbb{R}^2$.

Suppose there is a non-trivial $U(1)$-action on X. We extend this action to Z by letting $U(1)$ act trivially on $\mathbb{R}^2$. In this section, we are interested in asymptotically cylindrical special submanifolds in Z which are invariant under this $U(1)$-action. Here, we assume (X, g_X, I, J, K) is given by the Gibbons-Hawking construction, for instance X can be a multi-Taub-NUT or a multi-Eguchi-Hanson space, with the harmonic function given by

$$V(u) = m + \sum_{i=1}^{n} \frac{1}{4\pi|x - p_i|},$$

for points $\{p_1, \dots, p_n\} \subset \mathbb{R}^3$, $n \geq 3$ and $m \geq 0$. For simplicity assume

$$\{p_1, \dots, p_n\} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3. \tag{4.2.2}$$

In this section, of special interest is the case $n = 3$. In this case, the condition 4.2.2 holds for a suitable choice of $\mathbb{R}^2$ inside $\mathbb{R}^3$.

Consider the n minimal 2-spheres $\Sigma_{i,i+1} = \pi^{-1}[p_i, p_{i+1}] \subset X$, which generate $H_2(X)$. The sphere $\Sigma_{i,j} = \pi^{-1}[p_i, p_j]$, $i \neq j$, is holomorphic with respect to the complex structure $aI + bJ + cK$ where

$$(a, b, c) = \frac{p_i - p_j}{|p_i - p_j|} \in S^2 \subset \mathbb{R}^3.$$

Let the vectors $v_{1,2}, v_{2,3}, \ldots, v_{n,1} \in \mathbb{R}^2$ be defined by

$$v_{i,j} = (p_i - p_j) \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3.$$

Let $v^c_{i,j}$ be the straight line defined by $v^c_{i,j} = c + \mathbb{R}v_{i,j}$ for any $c \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Let

$$L^c_{i,j} := \Sigma_{i,j} \times (v^c_{i,j}) \subset X \times \mathbb{R}^2,$$

for $i \neq j$.

Lemma 84. *The cylindrical submanifolds $L^c_{i,j} \subset Z$ are $U(1)$-invariant special Lagrangians in $X \times \mathbb{C}$ with respect to the Calabi-Yau structure defined in 4.2.1.*

Proof. Any set of the form $\pi^{-1}(A)$, including $\pi^{-1}[p_i, p_j]$, is invariant under the $U(1)$-action on X, and therefore, $L^c_{i,j}$ is invariant under the $U(1)$-action on Z.

First we show $L^c_{i,j}$ is a Lagrangian submanifold. Let $(z, t) \in \Sigma_{i,j} \times (\mathbb{R}v_{i,j})$. Let $w_{i,j}$ be the unique lift of the vector $p_i - p_j \in \mathbb{R}^3$ to the point $z \in X$ such that $d\pi_z(w_{i,j}) = p_i - p_j \in T_{\pi(z)}\mathbb{R}^3$.

$$T_{(z,t)}L^c_{i,j} = \langle \partial t, w_{i,j}, v_{i,j} \rangle.$$

Recall that $\omega = du_3 \wedge \theta_0 + V du_1 \wedge du_2 + dx \wedge dy$. We have $\omega(v_{i,j}, w_{i,j}) = 0$. Moreover,

$$\omega(\partial t, w_{i,j}) = -du_3(w_{i,j}) = 0, \quad \text{and} \quad \omega(\partial t, v_{i,j}) = -du_3(v_{i,j}) = 0,$$

since p_i and p_j are in $\mathbb{R}^2 \times \{0\}$, and therefore, $L^c_{i,j}$ is a Lagrangian submanifold.

Second we show $L^c_{i,j}$ is a special submanifold. We have

$$\mathrm{Im}(\Omega) = \frac{1}{2}\theta \wedge (-du_1 dy + du_2 dx) - \pi V du_3 \wedge (du_1 dx + du_2 dy).$$

Let $v_{i,j} = (a, b, 0)$, and therefore, $w_{i,j} = (u_1, u_2, u_3, \partial t) = (a, b, 0, 0)$,

$$\mathrm{Im}(\Omega)(\partial t, w_{i,j}, v_{i,j}) = \frac{1}{2}(-du_1(w_{i,j})dy(v_{i,j}) + du_1(w_{i,j})dy(v_{i,j})) = \frac{1}{2}(-ab + ab) = 0,$$

and therefore, it is a special Lagrangian. □

More generally, we are interested in the asymptotically cylindrical special Lagrangians in $X \times \mathbb{C}$. Asymptotically cylindrical special Lagrangians can be defined in asymptotically cylindrical Calabi-Yau manifolds; however, here we consider the case where the ambient Calabi-Yau manifold is cylindrical in certain directions. Our definition is different from the usual one, since in the common definition the ambient manifold is asymptotically cylindrical, for instance asymptotic to the Calabi-Yau 3-fold $K3 \times S^1 \times \mathbb{R}$, and the asymptotes of the special Lagrangian submanifolds are assumed to be in the same direction, whereas here different ends of the special Lagrangian can go towards infinity in different directions.

Definition 64 (Asymptotically Cylindrical Special Lagrangian in $X \times \mathbb{C}$)**.** *Let $(X, g_X, \omega_1, \omega_2, \omega_3)$ be a hyperkähler 4-manifold. Let (Z, g, ω, Ω) be the Calabi-Yau 3-fold where $Z = X \times \mathbb{R}^2_{x,y}$ and*

$$g_Z = g_X + dx \otimes dx + dy \otimes dy, \qquad \omega = \omega_3 + dx \wedge dy \tag{4.2.3}$$

$$\Omega = (\omega_1 - i\omega_2) \wedge (dx + idy). \tag{4.2.4}$$

Let l_i^+ be any half-line in $\mathbb{R}^2_{x,y}$. Let $\Sigma_i \subset X$ be a complex curve in X, holomorphic with respec to the complex structure on X specified by the unit vector in the direction of l_i^+. Let $L_i = \Sigma_i \times l_i^+ \subset X \times \mathbb{C}$ be the resulting cylindrical special Lagrangian in Z. Moreover, let $t_i : L_i \to \mathbb{R}^+$ be a radius function which measure the distance on L_i from a fixed point. Let $L_1, \ldots, L_n$ be a number of two by two disjoint special Lagrangians of the form described above.

A connected, complete special Lagrangian $L \subset Z$ is called asymptotically cylindrical special Lagrangian, asymptotic to $L_1, \ldots, L_n$ in directions $l_1^+, \ldots, l_n^+$ with decay rate $\beta_1, \ldots, \beta_n < 0$, if there exists a compact subset $L' \subset L$, normal vectors field $v_1, \ldots, v_n$ on $L_1, \ldots, L_n$ where $t_i > R$ for a sufficiently large $R > 0$ and all $i \in \{1, \ldots, n\}$, and a diffeomorphism

$$\varPhi : \big((L_1 \cap t_1^{-1}(R, +\infty)) \cup \ldots \cup (L_n \cap t_n^{-1}(R, +\infty))\big) \to L \setminus L',$$

such that the following diagram commutes,

$$\begin{tikzcd} \big((L_1 \cap t_1^{-1}(R, +\infty)) \cup \ldots \cup (L_n \cap t_n^{-1}(R, +\infty))\big) \arrow[r, "\varPhi"] \arrow[d, "\exp_{v_1} \oplus \ldots \oplus \exp_{v_n}"] & L \setminus L' \arrow[d] \\ X \times \mathbb{C} \arrow[r, "Id"] & X \times \mathbb{C} \end{tikzcd}$$

where for each i,

$$|\nabla^k v_i| = O(e^{\beta_i t}),$$

on $(L_i \cap t_i^{-1}(R, +\infty))$ for all $k \in \{0, 1, 2, \ldots\}$.

As we go towards infinity in different directions in $X \times \mathbb{C}$, we would see different geometries; however, we are only interested in the cylindrical ends of $X \times \mathbb{C}$. For any line $v_{i,j}^c \subset \mathbb{R}^2$, let $R_{v_{i,j}^c}$ be rectangular open neighbourhood of this line in $\mathbb{R}^2$. Let $X' \subset X$ be a compact subset of X. The ambient space for the asymptotically special Lagrangians we are interested in are union of finitely many subspaces of Z of the form $X' \times R_{v_{i,j}^c}$, which are cylindrical Calabi-Yau manifolds with boundary. Moreover, the special Lagrangians we will study are away from the boundary of these manifolds, and therefore, the boundary does not introduce any complications.

Let X be a multi-Eguchi-Hanson space, constructed over $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$, where p_1, p_2 and p_3 are not collinear. Let's let $c = (0,0) \in \mathbb{R}^2$. We would have three cylindrical special Lagrangians $L^0_{1,2}, L^0_{2,3}$ and $L^0_{3,1}$. Note that these special Lagrangians, two by two, intersect only in one point,

$$
\begin{aligned}
L^0_{1,2} \cap L^0_{2,3} &= (\pi^{-1}(p_2), c), \\
L^0_{2,3} \cap L^0_{3,1} &= (\pi^{-1}(p_3), c), \\
L^0_{3,1} \cap L^0_{1,2} &= (\pi^{-1}(p_1), c).
\end{aligned}
$$

The points p_1, p_2 and p_3 form a triangle in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, and there is a natural direction on each Lagrangian, given by the positive direction of $\mathbb{R}^+ \ldots (p_i - p_j)$.

In this situation, Donaldson-Scaduto conjecture asserts the following.

Conjecture 10 (Donaldson-Scaduto Special Lagrangian)**.** *Let X be a multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$, where p_1, p_2 and p_3 are not collinear. There exists an asymptotically cylindrical special Lagrangian $L \subset X \times \mathbb{C}$, homeomorphic to S^3 minus three 3-dimensional balls, where L has three ends L_1, L_2 and L_3 which in the positive directions are asymptotic to $L^0_{1,2}, L^0_{2,3}$ and $L^0_{3,1}$, respectively. We call L a Donaldson-Scaduto special Lagrangian 4.11.*

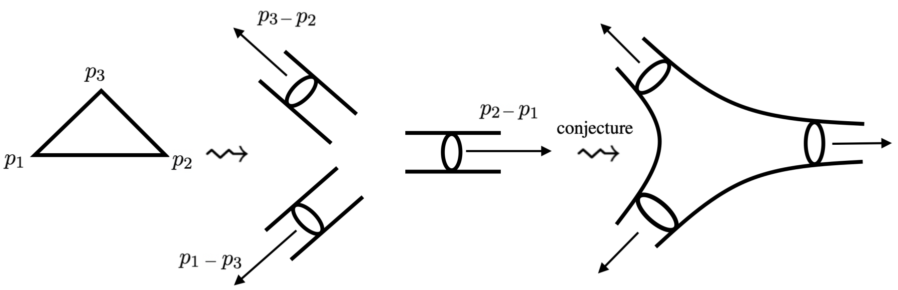

Figure 4.1: Donaldson-Scaduto Conjecture

In the conjecture, we set the vector $c = 0 \in \mathbb{R}^2$. More generally, one can consider the asymptotically cylindrical special Lagrangians $L \subset X \times \mathbb{C}$ with three ends L_1, L_2 and L_3 which in the positive directions are asymptotic to $L^{c_1}_{1,2}, L^{c_2}_{2,3}$ and $L^{c_3}_{3,1}$, respectively, for three vectors $c_1, c_2, c_3 \in \mathbb{R}^2$. The existence of the Donaldson-Scaduto special Lagrangian would imply c_1, c_2, c_3 must intersect in a point.

Lemma 85. *Let X be a multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$ where p_1, p_2 and p_3 are not collinear. Let $c_1, c_2, c_3 \in \mathbb{R}^2$. Let $L \subset X \times \mathbb{C}$ be an asymptotically cylindrical special Lagrangian with three ends L_1, L_2 and L_3 which in the positive directions are asymptotic to $L^{c_1}_{1,2}, L^{c_2}_{2,3}$ and $L^{c_3}_{3,1}$, respectively. Then three* *lines l_1, l_2 and l_3 defined by*

$$l_1 = \{t\frac{\overrightarrow{p_2 - p_1}}{|\overrightarrow{p_2 - p_1}|} + c_1 \in \mathbb{R}^2 \times \{0\} \mid t \in \mathbb{R}\},$$
$$l_2 = \{t\frac{\overrightarrow{p_3 - p_2}}{|\overrightarrow{p_3 - p_2}|} + c_2 \in \mathbb{R}^2 \times \{0\} \mid t \in \mathbb{R}\},$$
$$l_3 = \{t\frac{\overrightarrow{p_1 - p_3}}{|\overrightarrow{p_1 - p_3}|} + c_3 \in \mathbb{R}^2 \times \{0\} \mid t \in \mathbb{R}\},$$

are concurrent. In other words, they intersect at a point.

Proof. We have

$$\mathrm{Im}(\Omega) = \omega_1 \wedge dy - \omega_2 \wedge dx = d\lambda,$$

where $\lambda = -x\omega_2 + y\omega_1$.

Let $r : Z \to \mathbb{R}$ be a radius function, denoting the distance from a fixed point on L. Let $L_t = L \cap r^{-1}[0, t]$. Let $t >> 1$. The boundary of L_t consists of three 2-spheres, which we denote them by $\Sigma^t_{1,2}, \Sigma^t_{2,3}$ and $\Sigma^t_{3,1}$. Let $c_1 = (x_1, y_1), c_2 = (x_2, y_2), c_3 = (x_3, y_3)$. Let

$$\begin{aligned}
\frac{\overrightarrow{p_2 - p_1}}{|\overrightarrow{p_2 - p_1}|} &= \frac{1}{\sqrt{a_1^2 + b_1^2}}(a_1, b_1, 0),\\
\frac{\overrightarrow{p_3 - p_2}}{|\overrightarrow{p_3 - p_2}|} &= \frac{1}{\sqrt{a_2^2 + b_2^2}}(a_2, b_2, 0),\\
\frac{\overrightarrow{p_1 - p_3}}{|\overrightarrow{p_1 - p_3}|} &= \frac{1}{\sqrt{a_3^2 + b_3^2}}(a_3, b_3, 0).
\end{aligned}$$

We have

$$\begin{aligned}
v_{1,2} + v_{2,3} + v_{3,1} &= (a_1 + a_2 + a_3, b_1 + b_2 + b_3, 0)\\
&= (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_1) = (0, 0, 0).
\end{aligned}$$

By the Stokes theorem,

$$\int_{L^t} \mathrm{Im}(\Omega) = \int_{\partial L^t} \lambda = \int_{\Sigma^t_{1,2}} \lambda + \int_{\Sigma^t_{2,3}} \lambda + \int_{\Sigma^t_{3,1}} \lambda.$$

As $t \to +\infty$, we have

$$\begin{aligned}
&x \to a_1 t + x_1, \quad y \to b_1 t + y_1 \quad \text{along } L_1,\\
&x \to a_2 t + x_2, \quad y \to b_2 t + y_2 \quad \text{along } L_2,\\
&x \to a_3 t + x_3, \quad y \to b_3 t + y_3 \quad \text{along } L_3.
\end{aligned}$$

Therefore, for large t

$$
\begin{aligned}
0 = \int_{L^t} \mathrm{Im}(\Omega) &= \int_{\Sigma^t_{1,2}} (x\omega_2 - y\omega_1) + \int_{\Sigma^t_{2,3}} (x\omega_2 - y\omega_1) + \int_{\Sigma^t_{3,1}} (x\omega_2 - y\omega_1) \\
&\to \int_{\Sigma^t_{1,2}} ((x_1 + a_1 t)\omega_2 - (y_1 + b_1 t)\omega_1) + \int_{\Sigma^t_{2,3}} ((x_2 + a_2 t)\omega_2 - (y_2 t + b_2)\omega_1) \\
&+ \int_{\Sigma^t_{3,1}} ((x_3 + a_3 t)\omega_2 - (y_3 + b_3 t)\omega_1).
\end{aligned}
$$

The integral of symplectic forms over the holomorphic spheres in the multi-Eguchi-Hanson and the multi-Taub-NUT spaces can be computed explicitly. More generally, suppose (X, g_X, I, J, K) is a multi-Eguchi-Hanson or a multi-Taub-NUT space, constructed above $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$.

Let ω_ζ be a symplectic form in the S^2-family of symplectic structures on X. Let $\Sigma = \pi^{-1}[p_i, p_j]$. Then

$$\int_\Sigma \omega_\zeta = 2\pi \mathrm{length}_\zeta([p_i, p_j]),$$

where $\mathrm{length}_\zeta([p_i, p_j])$ denotes the signed length of the projection of $[p_i, p_j]$ onto the direction specified by x_ζ in $\mathbb{R}^2$. Applying that to our previous computations, we get

$$
\begin{aligned}
\int_{L^t} \mathrm{Im}(\Omega) \to &+ (x_1 + a_1 t)\mathrm{length}_2([p_2, p_1]) - (y_1 + b_1 t)\mathrm{length}_1([p_2, p_1]) \\
&+ (x_2 + a_2 t)\mathrm{length}_2([p_3, p_2]) - (y_2 + b_2 t)\mathrm{length}_1([p_3, p_2]) \\
&+ (x_3 + a_3 t)\mathrm{length}_2([p_1, p_3]) - (y_3 + b_3 t)\mathrm{length}_1([p_1, p_3]) \\
&= t(a_1 b_1 - b_1 a_1 + a_2 b_2 - b_2 a_2 + a_3 b_3 - b_3 a_3) \\
&+ (x_1 b_2 - y_1 a_1 + x_2 b_2 - y_2 a_2 + x_3 b_3 - y_3 a_3),
\end{aligned}
$$

as $t \to +\infty$, and therefore,

$$x_1 b_2 - y_1 a_1 + x_2 b_2 - y_2 a_2 + x_3 b_3 - y_3 a_3 = 0,$$

which is the equation of l_1, l_2 and l_3 being concurrent. $\square$

There exists a similar conjecture for associative submanifolds in G_2-manifolds.

4.2.0.2 Donaldson-Scaduto Associative Submanifolds

Let $(X, g_X, \omega_1, \omega_2, \omega_3)$ be a 4-dimensional hyperkähler manifold. Let $Y = X \times \mathbb{R}^3$ be the 7-dimensional manifold equipped with the integrable G_2-structure given by

$$\begin{aligned} g_Y &= V \sum_{i=1}^{3} du_i^2 + V^{-1}\theta_0^2 + dx \otimes dx + dy \otimes dy + dz \otimes dz, \\ \phi &= vol_{\mathbb{R}^3} - dx \wedge du_1 \wedge \theta_0 - V dx \wedge du_2 \wedge du_3 \\ &\quad - dy \wedge du_2 \wedge \theta_0 - V dy \wedge du_3 \wedge du_1 \\ &\quad - dz \wedge du_3 \wedge \theta_0 - V dz \wedge du_1 \wedge du_2, \end{aligned}$$

where x, y and z denotes the coordinates on $\mathbb{R}^3$.

Suppose there is a $U(1)$-action on X. We extend this action to Y by letting $U(1)$ act trivially on $\mathbb{R}^3$. In this section, we are interested in asymptotically cylindrical associative submanifolds in Y which are invariant under this $U(1)$-action. Moreover, similar to the previous section, suppose X is given via the Gibbons-Hawking construction, for instance X can be a multi-Taub-NUT or a multi-Eguchi-Hanson space, constructed over $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$, where $n \geq 3$ and $m \geq 0$.

Recall that the 2-spheres $\Sigma_{i,i+1} = \pi^{-1}[p_i, p_{i+1}] \subset X$ are minimal and holomorphic with respect to the complex structure $aI + bJ + cK$ where

$$(a, b, c) = \frac{p_i - p_j}{|p_i - p_j|} \in S^2 \subset \mathbb{R}^3.$$

Let the vectors $v_{1,2}, v_{2,3}, \ldots, v_{n,1} \in \mathbb{R}^3$ be defined by

$$v_{i,j} = (p_i - p_j) \in \mathbb{R}^3.$$

Let $v_{i,j}^c$ be the straight line defined by $v_{i,j}^c = c + \mathbb{R} \ldots v_{i,j}$ for any $c \in \mathbb{R}^3$. Let

$$N_{i,j}^c := \Sigma_{i,j} \times (v_{i,j}^c) \subset X \times \mathbb{R}^3.$$

Lemma 86. *The cylindrical submanifolds $N_{i,j}^c \subset Y$ are $U(1)$-invariant associative submanifolds.*

Proof. The proof of invariance under the $U(1)$-action is similar to the special Lagrangian case 84.

In the special case where the points $p_i \in \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, we have $N_{i,j}^c \subset X \times \mathbb{R}^2 \times \{0\} \subset Y$. As we saw in 84, $N_{i,j}^c$ is a special Lagrangian in $X \times \mathbb{R}^2 \times \{0\}$, and therefore, an associative submanifold in Y.

For a general configuration of points, let $w_{i,j}$ be the unique lift of the vector $p_i - p_j \in \mathbb{R}^3$ to the point $z \in X$. Let $v_{i,j} = (a, b, c)$ and $w_{i,j} = (a, b, c, 0)$,

$$T_{(z,t)} N_{i,j}^c = \langle \partial t, w_{i,j}, v_{i,j} \rangle.$$

Note that

$$\begin{aligned}\psi = *\phi = du_1 \wedge du_2 \wedge du_3 \wedge \theta - V dy \wedge dz \wedge du_2 \wedge du_3 - dy \wedge dz \wedge du_1 \wedge \theta \\ - V dz \wedge dx \wedge du_3 \wedge du_1 - dz \wedge dx \wedge du_2 \wedge \theta \\ - V dx \wedge dy \wedge du_1 \wedge du_2 - dx \wedge dy \wedge du_3 \wedge \theta.\end{aligned}$$

Recall that $N^c_{i,j}$ is associative if and only if at each point $p \in N^c_{i,j}$ and for each basis (e_1, e_2, e_3) of $T_p N^c_{i,j}$ we have $\psi(e_1, e_2, e_3, \ldots) = 0$. In fact, we only need to show $\psi(\partial t, v_{i,j}, w_{i,j}, v) = 0$ for any vector $v \in T_p Y$. We have

$$\begin{aligned}&\psi(\partial t, w_{i,j}, v_{i,j}, v) \\ &= (dy \wedge dz)(v_{i,j}, v) du_1(w_{i,j}) + (dz \wedge dx)(v_{i,j}, v) du_2(w_{i,j}) + (dx \wedge dy)(v_{i,j}, v) du_3(w_{i,j}) \\ &= (bdz(v) - dy(v)c)a + (cdx(v) - adz(v))b + (ady(v) - bdx(v))c = 0.\end{aligned}$$

□

More generally, we are interested in the asymptotically cylindrical associatives in $X \times \mathbb{R}^3$.

Conjecture 11 (Donaldson-Scaduto Associative Submanifold)**.** *Let X be a multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$, where no three of the points $p_1, \ldots, p_n$ are collinear. There exists an asymptotically cylindrical associative submanifold $N \subset X \times \mathbb{R}^3$, homeomorphic to S^3 minus n 3-dimensional balls which has n cylindrical ends $N_1, \ldots, N_{n-1}, N_n$ which in the positive directions are asymptotic to $N^0_{1,2}, \ldots, N^0_{n-1,n}, N^0_{n,1}$, respectively. We call N a Donaldson-Scaduto associative submanifold.*

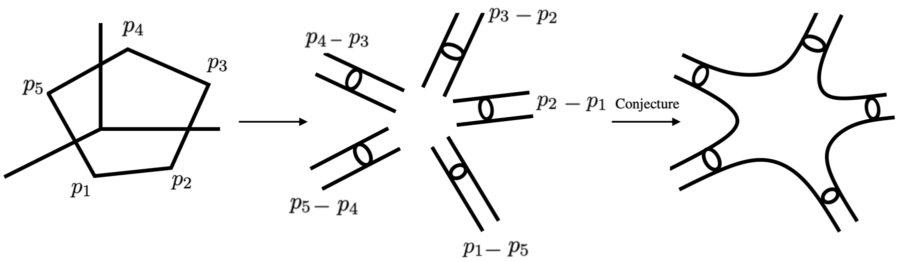

Figure 4.2: Donaldson-Scaduto Associative for 5 Points

4.3 The Method of Continuity and the Donaldson-Scaduto Conjecture

In this section, we propose a strategy to approach the Donaldson-Scaduto conjecture by the method of continuity. Moreover, we take first steps in that direction.

Let $P = \{p_1, \ldots, p_n\}$ be the set of distinct points in $\mathbb{R}^3$ where no three of them are collinear. Let $(X_P, g_P, I_P, J_P, K_P)$ be the multi-Eguchi-Hanson space, constructed via the Gibbons-

Hawking construction over $\mathbb{R}^3 \setminus P$. One can move the points $p_1, \dots, p_n$ in $\mathbb{R}^3$. As one moves these points around the hyperkähler manifold X_P, and therefore, the Calabi-Yau manifold $Z_P := X_P \times \mathbb{C}$ (or the G_2-manifold $Y_P := X_P \times \mathbb{R}^3$) changes too. In order to show the Donaldson-Scaduto conjecture holds for any of these manifolds for any generic configuration of points P, we propose to use the method of continuity.

Let Δ be the set of n-tuples of points in $\mathbb{R}^3$, which contains three collinear points,

$$\Delta = \{(q_1, \dots, q_n) \in (\mathbb{R}^3)^n \mid q_i, q_j \text{ and } q_k \text{ are collinear for some distinct } i, j, k \in \{1, \dots, n\}\}.$$

$(\mathbb{R}^3)^n \setminus \Delta$ is the parameter space. Let $\Lambda \subset (\mathbb{R}^3)^n \setminus \Delta$ be the set n points P, where the Donaldson-Scaduto conjecture holds for Z_P. In order to apply the method of continuity one should prove three claims,

- Λ is an open subset of $(\mathbb{R}^3)^n \setminus \Delta$,
- Λ is a closed subset of $(\mathbb{R}^3)^n \setminus \Delta$,
- Λ is non-empty.

In the following section, we prove the first claim in the case special Lagrangians.

4.3.1 Λ is an open subset

In this section, we focus on the case of special Lagrangians. However, we consider the slightly more general case of special Lagrangians with n ends, rather that just 3.

Definition 65 (Donaldson-Scaduto Special Lagrangian)**.** *Let X be a multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus \{p_1, \dots, p_3\}$, where no three of these points are collinear. Let $\Sigma_{i,j} = \pi^{-1}[p_i, p_j]$, where $\pi : X \to \mathbb{R}^3$ is the bundle map used in the Gibbons-Hawking construction. Let $L^0_{i,j} = \Sigma_{i,j} \times (\mathbb{R} \dots (p_j - p_i))$. The asymptotically cylindrical special Lagrangian $L \subset X \times \mathbb{C}$, homeomorphic to S^3 minus n 3-dimensional balls, where L has n ends $L_1, \dots, L_n$ which in the positive directions are asymptotic to $L^0_{1,2}, \dots, L^0_{n,1}$, respectively, is called a Donaldson-Scaduto special Lagrangian.*

In order to show that the subset $\Lambda \subset (\mathbb{R}^3)^n \setminus \Delta$ is an open subset, we should prove that if there is a special Lagrangian submanifold $L_P \subset X_P \times \mathbb{C}$ which satisfies the Donaldson-Scaduto conjecture, then after sufficiently small deformation of the points $P = \{p_1, \dots, p_n\}$ to the points $P' = \{p'_1, \dots, p'_n\}$, it is still possible to find a special Lagrangian $L_{P'} \subset X_{P'} \times \mathbb{C}$ which satisfies the conjecture.

Theorem 42. *Let $P = \{p_1, \dots, p_n\}$ be the set of distinct points in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ where no three of them are collinear. Let $(X_P, g_P, I_P, J_P, K_P)$ be the multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus P$. The existence of the special Lagrangians in $X_P \times \mathbb{C}$ predicted by the Donaldson-Scaduto conjecture is an open condition with respect to the deformation of the points $P = \{p_1, \dots, p_n\}$.*

This theorem is based on the infinite-dimensional inverse function theorem. There are many similar theorems in the literature where one deforms the geometric structure of the background ambient space and then deforms the calibrated submanifold accordingly so it stays calibrated [69]. However, here one major difference is that when we moves the points $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, it is the manifold X_P which is changing — and therefore, $X_P \times \mathbb{C}$ — and not just its hyperkähler — or the Calabi-Yau — structure on the fixed ambient manifold.

4.3.1.1 Geometric Deformation of the Ambient Spaces

Let $(X_P, g_P, I_P, J_P, K_P)$ be a multi-Eguchi-Hanson or multi-Taub-NUT space constructed via the Gibbons-Hawking construction over $\mathbb{R}^3 \setminus P$ where $P = \{p_1, \dots, p_n\}$. If we move the position of the points $P = \{p_1, \dots, p_n\}$ to get a new set of disjoint points $P' = \{p'_1, \dots, p'_n\}$, the resulting hyperkähler manifold $X_{P'}$, as a hyperkähler manifold, is not necessarily isomorphic to X_P. In fact, X_P and $X_{P'}$ are not necessarily isometric. Therefore, as we move the points P, it is the Calabi-Yau manifold Z_P which is changing, and not only its Calabi-Yau structure.

Although X_P and $X_{P'}$ are not isometric as hyperkähler manifolds, they are diffeomorphic, and therefore, Z_P and $Z_{P'}$ are diffeomorphic. Using an appropriate diffeomorphism, one can pull-back the Calabi-Yau structure of $Z_{P'}$ to Z_P. This helps us to fix an ambient manifold $Z = Z_P$, and only change its Calabi-Yau structure as we are deforming the configuration of points P. The idea of studying the deformation problem of special Lagrangians in $Z_P = X_P \times \mathbb{C}$ while deforming P is to transform the problem to a fixed 6-dimensional manifold Z where it is the Calabi-Yau structure is changing. This is the topic of the rest of this section.

Fix a set of distinct points $P = \{p_1, \dots, p_n\}$ in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$ where no three of them are collinear as a reference set of points. Suppose they are ordered in a way that the open segments connecting the consecutive points $(p_1, p_2), \dots, (p_{n-1}, p_n), (p_n, p_1)$ are disjoint. We see an example below 4.3.1.1.

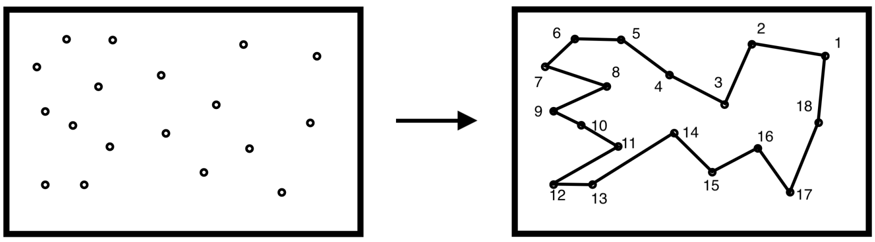

Figure 4.3: A Non-Intersecting Cycle.

The following lemma states we can always find such a non-intersecting cycle which connects all the points.

Lemma 87. *Let $n \geq 3$. Let $q_1, \dots, q_n$ be a set of n distinct points in $\mathbb{R}^2$ where no three of them are collinear. We can order them as $p_1, \dots, p_n$ where $\{p_1, \dots, p_n\} = \{q_1, \dots, q_n\}$, and the segments connecting the consecutive points $(p_1, p_2), \dots, (p_{n-1}, p_n), (p_n, p_1)$ are disjoint.*

Proof. Consider the convex hull of these points. Let p_1 be a point among $\{q_1, \ldots, q_n\}$ on the convex hull. By connecting p_1 to all the other points, we get $n-1$ oriented rays. Order them based on their slope in the counter clockwise order. We name these $n-1$ points such that these ordered rays are $p_1p_2, p_1p_3, \ldots, p_1p_n$. The cycle we look for is $p_1p_2 \ldots p_n$.

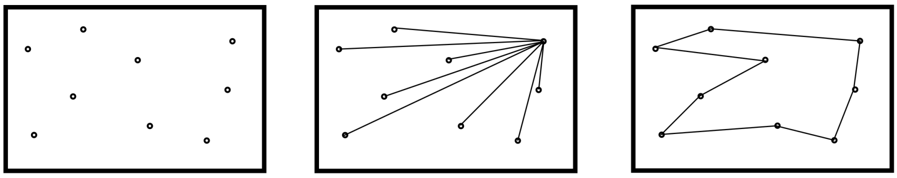

Figure 4.4: Finding a Non-Intersecting Cycle

□

Remark 9. *For a given set of n points, which no three of them are collinear, there can be different ways of ordering them to construct non-intersecting cycles. Each different order, and therefore, different cycle will result in a different asymptotic condition for the Donaldson-Scaduto conjecture, and therefore, conjecturally there would be different special Lagrangians assigned to a set of n points, as shown in figure 9.*

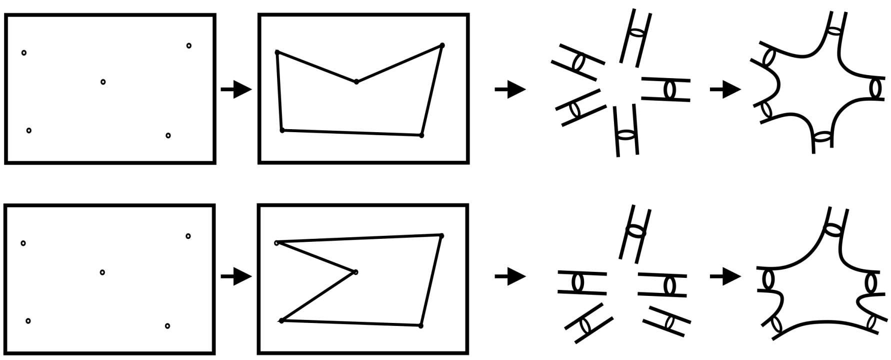

Figure 4.5: Two Different Special Lagrangians for the Same Set of Points

Let $B_\varepsilon(p_i) \subset \mathbb{R}^3$ be a sufficiently small open ball around p_i such that

$$B_{4\varepsilon}(p_i) \cap B_{4\varepsilon}(p_j) = \varnothing, \quad \text{for all distinct } i, j \in \{1, \ldots, n\}. \tag{4.3.1}$$

Let $P' = \{p'_1, \ldots, p'_n\}$ be another configuration of points where $p'_i \in (\mathbb{R}^2 \times \{0\}) \cap B_\varepsilon(p_i)$. Let the diffeomorphism $f_{p'} : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that on $\cup_{i=1}^n B_\varepsilon(p_i)$ it is defined by

$$f_{p'}(x) = x + (p'_i - p_i), \quad \text{when } x \in B_\varepsilon(p_i).$$

The assumption 4.3.1 assures that one can extend $f_{p'}$ to get a diffeomorphism on $\mathbb{R}^3$ such that $f(p_i) = p'_i$ and

- The map $f : B_\varepsilon(p_1) \times \ldots \times B_\varepsilon(p_n) \to \mathrm{Diff}(\mathbb{R}^3)$ defined by $f(p'_1, \ldots, p'_n) = f_{P'}$, where $P' = \{p'_1, \ldots, p'_n\}$, is smooth.
- $f_{P'}([p_i, p_{i+1}]) = [p'_i, p'_{i+1}]$.
- $f(p_1, \ldots, p_n) = f_P = Id : \mathbb{R}^3 \to \mathbb{R}^3$.

Note that this extension is not unique and there are many ways to define f; however, we can pick any of them which satisfies the conditions above.

Let $\tilde{f}_{P'} : X_P \to X_{P'}$ be any $U(1)$-equivariant diffeomorphism which covers $f_{p'_i} : \mathbb{R}^3 \to \mathbb{R}^3$. Locally, in a $U(1)$-invariant gauge, this can be written as

$$\tilde{f}_{P'_i}(u_1, u_2, u_3, t) = (f_{p'_i}(u_1, u_2, u_3), t). \tag{4.3.2}$$

Using this map we can pull back the hyperkähler structure of $X_{P'}$ to X_P. For any set of points $P' = \{p'_1, \ldots, p'_n\}$, we define the hyperkähler structure $(g_{P'}, I_{P'}, I_{P'}, I_{P'})$ on X_P by pulling-back the hyperkähler structure of $X_{P'}$ via $\tilde{f}_{P'_i}$.

The following lemma states a key property of these maps.

Lemma 88. *The map $\tilde{f}_{P'} : X_P \to X_{P'}$ induces a bi-holomorphism*

$$\tilde{f}_{P'|_{\Sigma_{i,i+1}}} : \Sigma_{i,i+1} \to \Sigma'_{i,i+1}.$$

Proof. First note that the map $\tilde{f}_{P'}$ takes $\Sigma_{i,i+1}$ to $\Sigma'_{i,i+1}$, since the map $f_{P'}$ takes $[p_i, p_{i+1}]$ to $[p'_i, p'_{i+1}]$, and therefore, the diffeomorphism $\tilde{f}_{P'}$ which covers $f_{P'}$ induces a diffeomorphism from the 2-sphere $\pi^{-1}([p_i, p_{i+1}]) = \Sigma_{i,i+1}$ to $\pi^{-1}([p'_i, p'_{i+1}]) = \Sigma'_{i,i+1}$.

For points x and $x' = f_{P'}(x)$ in $\Sigma_{i,i+1}$ and $\Sigma'_{i,i+1}$, respectively, we have

$$T_x\Sigma_{i,i+1} = \langle w_{i,i+1}, \partial t\rangle, \qquad T_{x'}\Sigma'_{i,i+1} = \langle w'_{i,i+1}, \partial t\rangle,$$

where $w_{i,i+1}$ and $w'_{i,i+1}$ are the vectors covering $v_{i,i+1}$ and $v'_{i,i+1}$, respectively.

The linearized map $d_x\tilde{f}_{P'}$ takes $w_{i,i+1}$ to a multiple of $w'_{i,i+1}$, and since it is equivariant it takes ∂t to a multiple of ∂t, and therefore, it is complex linear. □

The next step is to extend this diffeomorphism to the Calabi-Yau manifolds $Z_P = X_P \times \mathbb{C}$. One can simply define a diffeomorphism by taking the identity map on the $\mathbb{C}$ component; however, this is not suitable for the deformation problem, as we will explain below.

Let $l_{p_1,p_2}, l_{p_2,p_3}, \ldots, l_{p_{n-1},p_n}, l_{p_n,p_1} \subset \mathbb{R}^2$ be the half-lines through origin which are in the positive direction of the vectors $p_2 - p_1, p_3 - p_2, \ldots, p_n - p_{n-1}, p_1 - p_n$, respectively. Let the rectangles $\tilde{R}_{p_1,p_2}, \tilde{R}_{p_2,p_3}, \ldots, \tilde{R}_{p_{n-1},p_n}, \tilde{R}_{p_n,p_1} \subset \mathbb{R}^2 \cong \mathbb{C}$ be tubular neighbourhood of $l_{p_1,p_2}, l_{p_2,p_3}, \ldots, l_{p_{n-1},p_n}, l_{p_n,p_1}$, which are half-infinite rectangles with infinite length and ε-width for a sufficiently small ε.

Let $r : \mathbb{R}^2 \to [0, \infty)$ be the distance from the origin in $\mathbb{R}^2$. Let

$$R_{p_i,p_{i+1}} = \tilde{R}_{p_i,p_{i+1}} \cap r^{-1}(T, \infty),$$

for a sufficiently large $T >> 1$ such that the half-infinite rectangles $R_{p_i,p_{i+1}}$ are two-by-two disjoint.

Let $\theta(l_{p_i,p_{i+1}}, l_{p'_i,p'_{i+1}})$ be the angle from $l_{p_i,p_{i+1}}$ to $l_{p'_i,p'_{i+1}}$. Let $g_{P'} : \mathbb{R}^2 \to \mathbb{R}^2$ be the map which in polar coordinates on each $R_{p_i,p_{i+1}}$ is defined by

$$g_{P'}(\theta, r) = (\theta + \theta(l_{p_i,p_{i+1}}, l_{p'_i,p'_{i+1}}), r), \text{ for } (\theta, r) \in R_{p_i,p_{i+1}}.$$

The map $g_{P'}$ takes each half-rectangle $R_{p_i,p_{i+1}}$ to $R_{p'_i,p'_{i+1}}$ by simply rotating it. Moreover, it takes each half-line $l_{p_i,p_{i+1}} \cap r^{-1}(T, \infty)$ to $l_{p'_i,p'_{i+1}} \cap r^{-1}(T, \infty)$. We can extend $g_{P'}$ to get a diffeomorphism $g_{P'} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- The map $g : B_\varepsilon(p_1) \times \ldots \times B_\varepsilon(p_n) \to \mathrm{Diff}(\mathbb{R}^2)$ defined by $g(p'_1, \ldots, p'_n) = g_{P'}$, where $P' = \{p'_1, \ldots, p'_n\}$, is smooth.
- $g(p_1, \ldots, p_n) = g_P = Id : \mathbb{R}^2 \to \mathbb{R}^2$.

Let the diffeomorphism $h_{P'} : X_P \times \mathbb{C} \to X_{P'} \times \mathbb{C}$ be the map defined by

$$h_{P'}(x, z) = (\tilde{f}_{P'}(x), g_{P'}(z)).$$

We can pull back the Calabi-Yau structure $(g_{P'}, \Omega_{P'}, \omega_{P'})$ of $Z_{P'} = X_{P'} \times \mathbb{C}$ to $Z_P = X_P \times \mathbb{C}$ via the map $h_{P'}$. By an abuse of notation, we still denote this pulled-back Calabi-Yau structure on Z_P by $(g_{P'}, \Omega_{P'}, \omega_{P'})$.

With this understanding, for any set of distinct points $P' = \{p'_1, \ldots, p'_n\}$ we have a Calabi-Yau structure on the fixed smooth manifold $Z = Z_P$. Let W be the space parametrizing the Calabi-Yau structure on Z close to the reference Calabi-Yau structure $(g_P, \Omega_P, \omega_P)$. In the Calabi-Yau case — as opposed to the G_2 case — we only move the points p_i inside $B_\varepsilon(p_i) \subset \mathbb{R}^2 \times \{0\}$. Let $\pi' : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the first two coordinates. The parameter space is given by

$$Q = \pi'(B_\varepsilon(p_1)) \times \ldots \times \pi'(B_\varepsilon(p_n)),$$

and therefore, $\dim Q = 2n$.

The following lemma explains the reason behind the specific choice of the maps $g_{P'}$.

Lemma 89. *The map $h_{P'} : Z_P \to Z_{P'}$ induces a diffeomorphism between the special Lagrangians $L^0_{i,i+1} = \Sigma_{i,i+1} \times (v^0_{i,i+1})$ and $(L^0_{i,i+1})' = \Sigma'_{i,i+1} \times (v^0_{i,i+1})'$ in Z_p and $Z_{P'}$, respectively, sufficiently away from the origin,*

$$h_{P'}|_{L^0_{i,i+1} \cap r^{-1}[T,\infty)} : L^0_{i,i+1} \cap r^{-1}(T, \infty) \to (L^0_{i,i+1})' \cap r^{-1}(T, \infty),$$

for a sufficiently large T.

Proof. As we observed in the Lemma 88, $\tilde{f}_{P'}$ takes $\Sigma_{i,i+1}$ to $\Sigma'_{i,i+1}$. On the other hand, $g_{P'}$ takes the half-line $l_{p_i,p_{i+1}} \cap r^{-1}(T, \infty)$ to $l'_{p_i,p_{i+1}} \cap r^{-1}(T, \infty)$, and therefore, the diffeomorphism $h_{P'}$ takes $\Sigma_{i,i+1} \times (l_{p_i,p_{i+1}} \cap r^{-1}(T, \infty))$ to $\Sigma'_{i,i+1} \times (l'_{p_i,p_{i+1}} \cap r^{-1}(T, \infty))$. □

This means as we move the point P' in the parameter space Q and change the Calabi-Yau structure, these topologically cylindrical special Lagrangians, in the positive direction as induced by the vectors $p_{i+1} - p_i$ and outside of a compact subset, do not change. In particular, as we deform the Calabi-Yau structure, the asymptotic condition for the Donaldson-Scaduto special Lagrangians — which are asymptotic to these cylindrical special Lagrangians in the positive directions — do not change, and the deformation problem of these special Lagrangians fit into the framework of the deformation of the asymptotically cylindrical special Lagrangians with fixed asymptotic ends, which has been studied previously in the literature, for instance in [82].

The case of the deformation of the G_2-manifolds of the form $Y_P = X_P \times \mathbb{R}^3$ is quite similar. In the G_2 case the points $p_1, \ldots, p_n$ can move inside $\mathbb{R}^3$. Similar to the Calabi-Yau case, we can construct diffeomorphisms $\mathtt{g}_{P'} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathtt{g}_{P'}$ takes half-infinite cylindrical neighbourhood of each half-line $l_{p_i,p_{i+1}} \cap r^{-1}(T, \infty)$ to the half-infinite cylindrical neighbourhood $l'_{p_i,p_{i+1}} \cap r^{-1}(T, \infty)$ such that

- The map $\mathtt{g} : B_{\varepsilon_1}(p_1) \times \ldots \times B_{\varepsilon_n}(p_n) \to \text{Diff}(\mathbb{R}^3)$ defined by $\mathtt{g}(p'_1, \ldots, p'_n) = \mathtt{g}_{P'}$, where $P' = \{p'_1, \ldots, p'_n\}$, is smooth.
- $\mathtt{g}(p_1, \ldots, p_n) = \mathtt{g}_P = Id : \mathbb{R}^3 \to \mathbb{R}^3$.

The parameter space is given by

$$Q = B_\varepsilon(p_1) \times \ldots \times B_\varepsilon(p_n),$$

and therefore, $\dim Q = 3n$.

Let the diffeomorphism $\mathtt{h}_{P'} : X_P \times \mathbb{R}^3 \to X_{P'} \times \mathbb{R}^3$ defined by $\mathtt{h}_{P'}(x, w) = (\tilde{f}_{P'}(x), \mathtt{g}_{P'}(w))$ for any $x \in X_P$ and $w \in \mathbb{R}^3$.

Lemma 90. *The map $\mathtt{h}_{P'} : Y_P \to Y_{P'}$ induces a diffeomorphism between the associatives $N^0_{i,i+1} = \Sigma_{i,i+1} \times (v^0_{i,i+1})$ and $(N^0_{i,i+1})' = \Sigma'_{i,i+1} \times (v^0_{i,i+1})'$ in Y_P and $Y_{P'}$, respectively,*

$$\mathtt{h}_{P'}|_{N^0_{i,i+1} \cap r^{-1}[T,\infty)} : N^0_{i,i+1} \cap r^{-1}(T, \infty) \to (N^0_{i,i+1})' \cap r^{-1}(T, \infty),$$

for a sufficiently large T.

The proof is similar to 89.

4.3.1.2 Deformation of the Asymptotically Cylindrical Special Lagrangians in $X \times \mathbb{C}$

Let $P_0 = (p_1, \ldots, p_n)$ be an ordered set of n distinct points in $(\mathbb{R}^2 \times \{0\})^n \subset (\mathbb{R}^3)^n$, where no three of them are collinear. Let $B_\varepsilon(p_1), \ldots, B_\varepsilon(p_n) \subset \mathbb{R}^3$ be sufficiently small and disjoint open neighbourhoods of $p_1, \ldots, p_n$, respectively. Let $\pi' : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the first two components. Let Q be the parameter space, $Q = \pi'(B_\varepsilon(p_1)) \times \ldots \times \pi'(B_\varepsilon(p_n))$.

Let X be the smooth 4-manifold underlying a multi-Eguchi-Hanson or a multi-Taub-NUT hyperkähler manifold. Let $Z = X \times \mathbb{C}$. Let P denote a configuration of points in the parameter space Q. For each P, we have a hyperkähler structure on X, denoted by $(X, g_{X,P}, I_P, J_P, K_P)$, and therefore, a Calabi-Yau structure on Z, denoted by $(Z, g_{Z,P}, \omega_P, \Omega_P)$.

The open property of $\Lambda \subset (\mathbb{R}^2)^n \setminus \Delta$ follows from the following theorem, which is the main theorem of this subsection.

Theorem 43. *Let L_{P_0} be a Donaldson-Scaduto special Lagrangian in $Z = X \times \mathbb{C}$ with respect to the Calabi-Yau structure associated to $P_0 = (p_1, \ldots, p_n) \in W$ with a decay rate $\alpha < 0$, where $|\alpha|$ is sufficiently small. The moduli space $\mathcal{M}_{L_{P_0}}$ of asymptotically cylindrical special Lagrangian submanifolds in $(Z, g_{Z,P_0}, I_{P_0}, J_{P_0}, K_{P_0})$ with rate $\beta < 0$, where $\alpha < \beta < 0$, near and homotopic to L_{P_0} — which are necessarily of the type predicted by the Donaldson-Scaduto conjecture — is a smooth 0-dimensional manifold, and therefore, L_{P_0} is isolated. Moreover, the parametrized moduli space*

$$\widetilde{\mathcal{M}}_{L_{P_0}} = \{(P, L) \mid P \in Q \text{ and } L \text{ is an asymptotically cylindrical special Lagrangian with decay rate } \beta \text{ near and homotopic to } L_{P_0} \text{ with respect to the Calabi-Yau structure } (g_{Z,P}, \omega_P, \omega_P)\},$$

is a $2n$-dimensional smooth manifold, where $2n = \dim(Q)$. Furthermore, for any $P \in Q$ there exists a unique L_P where

$$(P, L_P) \in \widetilde{\mathcal{M}}_{L_{P_0}}.$$

The rest of this subsection is devoted to proving this theorem. We present the proof in a sequence of lemmas. The proof follows the same line of thought as in [82, 57, 57, 69].

Let L_0 denote the smooth 3-dimensional manifold underlying the special Lagrangian L_{P_0}. Let $f : L_0 \to Z$ be an embedding of the special Lagrangian L_{P_0} in Z. We have

$$f^* \operatorname{Im}(\Omega_{P_0}) = f^* \omega_{P_0} = 0.$$

Let $\flat_{g_0} : TL_0 \to T^*L_0$ be the bundle isomorphism between vector fields and 1-forms on L_0, defined using the pull-back of the Riemannian metric g_{Z,P_0} to L_0. Let $\nu(L_{P_0}) \to L_{P_0}$ be the normal bundle of L_{P_0} in Z. Let $\widetilde{U}$ be a small tubular neighbourhoods of L_{P_0} in Z with a fixed radius at different points of L_{P_0}, identified with a small neighbourhood of the zero section of the normal bundle, by an abuse of notation still denoted by $\widetilde{U}$. Let

$$\widetilde{U}^{k,p}_\alpha = \{\xi \in W^{k,p}_\alpha(\nu(L_{P_0})) \mid \xi(x) \in \widetilde{U} \text{ for all } x \in L_{P_0}\},$$

for $k \geq 0, p > 1$ and $\alpha < 0$.

Note that since L_{P_0} is a special Lagrangian we have the bundle isomorphisms

$$J_0 : \nu(L_{P_0}) \to TL_0, \qquad \flat_{g_0} \circ J_0 : \nu(L_{P_0}) \to T^*L_0.$$

Let $U = \flat_{g_0} \circ J_0(\widetilde{U})$ and

$$U^{k,p}_\alpha = \{\eta \in W^{k,p}_\alpha(T^*L_0) \mid \eta(x) \in U \text{ for all } x \in L_0\},$$

for $k \geq 0, p > 1$ and $\alpha < 0$.

For any $\eta \in U_\alpha^{k,p}$, let $f_\eta : L_{P_0} \to Z$ be the deformation map defined by

$$f_\eta(x) = exp_{f(x)}((\flat_{g_0} \circ J_0)^{-1}(\eta_x)).$$

$f_\eta(L_{P_0})$ is an embedding of L_0 homotopic to $L_{P_0} = f(L_0)$, when η is sufficiently small. Moreover, any nearby submanifold homotopic to L_{P_0} can be presented in this manner.

For a moment let's assume the Calabi-Yau structure on Z, associated to $P_0 \in Q$, is fixed. Let $G : U \to \Omega^0(L_0) \oplus \Omega^2(L_0)$ be the map defined by

$$G(\eta) = *_{P_0} f_\eta^*(\mathrm{Im}(\Omega_{P_0})) + f_\eta^*(\omega_{P_0}),$$

where $*_{P_0}$ is the Hodge star operator, defined using the pull-back of the Riemannian metric g_{Z,P_0} to L_0.

$f_\eta(L_0)$ is a special Lagrangian in $(Z, g_{Z,P_0}, \omega_{P_0}, \Omega_{P_0})$ if and only if $G(\eta) = 0$. In order to show that the zero set of G forms a smooth manifold, we should use the inverse function theorem, and therefore, we should set up the map G between the appropriate Banach spaces. Let

$$G_\alpha^{k+1,p} : U_\alpha^{k+1,p} \to W_\alpha^{k,p}(\Lambda(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)), \qquad G_\alpha^{k+1,p}(\eta) = *_{P_0} f_\eta^*(\Omega_{P_0}) + f_\eta^*(\omega_{P_0}).$$

However, note that similar to the compact case, the linearized map

$$d_0 G_\alpha^{k+1,p} : W_\alpha^{k+1,p}(\Lambda^0(T_0^*L_{P_0})) \to W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)),$$

is not elliptic, and therefore, not Fredholm. This can be seen by looking at the rank of the domain and the target space, which are not equal,

$$rank(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) = 3, \qquad rank(W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0))) = 1 + 3 = 4.$$

To resolve this problem, let

$$\widetilde{F}_\alpha^{k+1,p} : U_\alpha^{k+1,p} \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_0)) \to W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)),$$

defined by

$$\widetilde{F}_\alpha^{k+1,p}(\eta, \chi) = G_\alpha^{k+1,p}(\eta) + d_{P_0}^*\chi,$$

where $d_{P_0}^*\chi = - *_{P_0} d *_{P_0} \chi$.

Lemma 91. *The map $\widetilde{F}_\alpha^{k+1,p} : U_\alpha^{k+1,p} \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_0)) \to W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0))$, is elliptic, where the linearization is given by*

$$d_{(0,0)}\widetilde{F}_\alpha^{k+1,p} : W_\alpha^{k+1,p}(\Lambda^{odd}(T^*L_0)) \to W_\alpha^{k,p}(\Lambda^{even}(T^*L_0)), \qquad d_{(0,0)}\widetilde{F}_\alpha^{k+1,p}(\eta, \chi) = (d^* + d)(\eta + \chi).$$

The proof is similar to the compact case, 41.

More generally, we want to allow the Calabi-Yau structure of the ambient space to change.

Lemma 92. *Let*

$$F_\alpha^{k+1,p} : Q \times U_\alpha^{k+1,p} \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_0)) \to W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)),$$

be the map defined by

$$F_\alpha^{k+1,p}(P,\eta,\chi) = *_P f_\eta^*(\mathrm{Im}(\Omega_P)) + f_\eta^*(\omega_P) + d_P^*\chi.$$

where $*_P$ and d_P^* are defined with respect to the Riemannian metric $g_{Z,P}$, pulled-back to L_0. The map $F_\alpha^{k+1,p}$ is elliptic.

Let

$$\begin{aligned}(d_2)_{(P_0,0,0)}F_\alpha^{k+1,p} : W_\alpha^{k+1,p}&(\Lambda^1(T^*L_0)) \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_0)) \\ &\to W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)),\end{aligned}$$

denote the derivative of $F_\alpha^{k+1,p}$ at $(P_0,0,0)$ in the direction of $W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))\oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_0))$. We have

$$(d_2)_{(P_0,0,0)}F_\alpha^{k+1,p}(\eta,\chi) = (d_{g_P}^* + d)(\eta+\chi).$$

The lemma follows from 91.

Lemma 93. *For any $k > 0, p \geq 1$ and $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned}G_\alpha^{k+1,p}(U_\alpha^{k+1,p}) \subset d_{P_0}^*(W_\alpha^{k+1,p}&(\Lambda^1(T^*L_0)))\oplus d(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) \\ &\subset W_\alpha^{k,p}(\Lambda^0(T^*L_0)) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_0)).\end{aligned}$$

Proof. First note that $*f_\eta^*(\Omega_{P_0})$ and $f_\eta^*(\omega_{P_0})$ are co-closed and closed, respectively.

$$d_{P_0}^*(*f_\eta^*(\Omega_{P_0})) = *_{P_0} f_\eta^*(d\Omega_{P_0}) = 0, \qquad d(f_\eta^*(\omega_{P_0})) = f_\eta^*(d\omega_{P_0}) = 0.$$

Moreover, $f_\eta^*(\Omega_{P_0})$ and $f_\eta^*(\omega_{P_0})$ are exact. To see this note that f_η is homotopic to f, and therefore, it is enough to show $f^*(\Omega_{P_0})$ and $f^*(\omega_{P_0})$ are exact; however, since L_{P_0} is a special Lagrangian in $(Z, g_{Z,P_0}, \omega_{P_0}, \Omega_{P_0})$, we have $f^*(\Omega_{P_0}) = f^*(\omega_{P_0}) = 0$,

$$[f_\eta^*(\Omega_{P_0})] = [f^*(\Omega_{P_0})] = 0, \qquad [f_\eta^*(\omega_{P_0})] = [f^*(\omega_{P_0})] = 0,$$

and therefore,

$$*f_\eta^*(\Omega_{P_0}) = d^*\theta_1, \qquad f_\eta^*(\omega_{P_0}) = d\theta_2,$$

for 1-forms θ_1 and θ_2 on L_0. Moreover,

$$d^*\theta_1 = *f_\eta^*(\Omega_{P_0}) - *f_0^*(\Omega_{P_0}) = *d(\int_0^1 f_{s\eta}^*(\iota_{s\eta}\Omega_{P_0})ds),$$
$$d\theta_2 = f_\eta^*(\omega_{P_0}) - f_0^*(\omega_{P_0}) = d(\int_0^1 f_{s\eta}^*(\iota_{s\eta}\omega_{P_0})ds),$$

and we can take

$$\theta_1 = *\int_0^1 f_{s\eta}^*(\iota_{s\eta}\Omega_{P_0})ds, \qquad \theta_2 = \int_0^1 f_{s\eta}^*(\iota_{s\eta}\omega_{P_0})ds.$$

Therefore,

$$\theta_1 = O(e^{\alpha t}), \text{ and } \theta_2 = O(e^{\alpha t}),$$

which proves the lemma. □

This lemma implies

$$\begin{aligned}&\widetilde{F}_\alpha^{k+1,p}(U_\alpha^{k+1,p} \times W_\alpha^{k+1,p}(\Lambda^3(T^*L_0))) \\ &\quad \subset d_{P_0}^*(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) \oplus d(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) \oplus d_{P_0}^*(W_\alpha^{k+1,p}(\Lambda^3(T^*L_0))).\end{aligned}$$

Lemma 94. *For any $k > 0, p \geq 1$ and $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned}&F_\alpha^{k+1,p}(Q \times U_\alpha^{k+1,p}) \\ &\quad \subset d_{P_0}^*(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) \oplus \quad d(W_\alpha^{k+1,p}(\Lambda^1(T^*L_0))) \oplus d_{P_0}^*(W_\alpha^{k+1,p}(\Lambda^3(T^*L_0))).\end{aligned}$$

Proof. First note that $*f_\eta^*(\Omega_P)$ and $f_\eta^*(\omega_P)$ are co-closed and closed, respectively.

$$d_P^*(*f_\eta^*(\Omega_P)) = *_P f_\eta^*(d\Omega_P) = 0, \qquad d(f_\eta^*(\omega_P)) = f_\eta^*(d\omega_P) = 0.$$

Moreover, $f_\eta^*(\Omega_P)$ and $f_\eta^*(\omega_P)$ are exact, since f_η is homotopic to f, and therefore, we should only show $f^*(\Omega_P)$ and $f^*(\omega_P)$ are exact. This is true because of the types of deformations which we are considering. As we saw in the Lemma 85, when $c = 0$, $[f^*(\Omega_P)] = [f^*(\omega_P)] = 0$. The negative decay rate shows all deformations $f_\eta(L_0)$ still have $c = 0$. If we were considering the deformations where the ends were moving too, this assumption would not have been valid anymore. Therefore,

$$*_P f_\eta^*(\Omega_P) = d_{P_0}^*\theta_1, \qquad f_\eta^*(\omega_P) = d\theta_2,$$

for 1-forms θ_1 and θ_2 on L_0.

$$\theta_1 = O(e^{\alpha t}), \text{ and } \theta_2 = O(e^{\alpha t}),$$

which proves the lemma. $\square$

Let

$$\mathcal{C} := (d + d^*_{P_0})(W^{k+1,p}_\alpha(\Lambda^1(T^*L_0)) \oplus W^{k+1,p}_\alpha(\Lambda^3(T^*L_0))) \subset W^{k,p}_\alpha(\Lambda^0(T^*L_0)) \oplus W^{k,p}_\alpha(\Lambda^2(T^*L_0)).$$

Let $\mathcal{D}$ be the L^2-orthogonal complement of $\mathcal{C}$

$$Q \oplus W^{k+1,p}_\alpha(\Lambda^0(T^*L)) \oplus W^{k+1,p}_\alpha(\Lambda^2(T^*L)) = \mathcal{C} \oplus \mathcal{D}.$$

The following lemma characterizes $\mathcal{D}$.

Lemma 95. *We have $\mathcal{D} = \{(P, \theta) \mid P \in Q$ and $\theta \in \mathcal{D}_P\}$, where*

$$\mathcal{D}_P = \ker(d + d^*_P : W^{k+1,q}_{-\alpha}(\Lambda^{even}(L_0)) \to W^{k+1,q}_{-\alpha}(\Lambda^{odd}(L_0))),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. $\mathcal{D}_P$ can be identified with the co-kernel of $(d+d^*_P) : W^{k+1,p}_\alpha(\Lambda^{\mathrm{odd}}(L_0)) \to W^{k+1,p}_\alpha(\Lambda^{\mathrm{even}}(L_0))$, which itself can be identified with the kernel of $(d+d^*_P) : W^{k+1,q}_{-\alpha}(\Lambda^{\mathrm{even}}(L_0)) \to W^{k+1,q}_{-\alpha}(\Lambda^{\mathrm{odd}}(L_0))$. $\square$

Lemma 96. *We have*

$$F^{k+1,p}_\alpha(Q \times U^{k+1,p}_\alpha \times W^{k+1,p}_\alpha(\Lambda^3(T^*L_{P_0})))) \subset \mathcal{C} \subset W^{k,p}_\alpha(\Lambda^0(T^*L_{P_0})) \oplus W^{k,p}_\alpha(\Lambda^2(T^*L_{P_0})).$$

Proof. Let $(P, \eta, \chi) \in W \times U^{k+1,p}_\alpha \times W^{k+1,p}_\alpha(\Lambda^3(T^*L_{P_0})))$. We should show

$$\langle F^{k+1,p}_\alpha(P, \eta, \chi), \mathcal{D}_P \rangle_{L^2(L_0)} = 0.$$

Let $\theta = h + \beta \in \mathcal{D}_P$, for a constant 0-form h and a closed and co-closed 2-form β. For each $P \in Q$,

$$\begin{aligned}
\langle F^{k+1,p}_\alpha(P, \eta, \chi), h + \beta \rangle_{L^2(L_P)} &= \langle *_P f^*_\eta(\mathrm{Im}(\Omega_P)) + f^*_\eta(\omega_P) + d^*_P\chi, h + \beta \rangle_{L^2(L_P)} \\
&= \langle *_P f^*_\eta(\mathrm{Im}(\Omega_P)), h \rangle_{L^2(L_P)} + \langle f^*_\eta(\omega_P), \beta \rangle_{L^2(L_P)} + \langle d^*_P\chi, \beta \rangle_{L^2(L_P)} \\
&= \langle d^*_P\theta_1, h \rangle_{L^2(L_P)} + \langle d\theta_2, \beta \rangle_{L^2(L_P)} + \langle d^*_P\chi, \beta \rangle_{L^2(L_P)} \\
&= \langle \theta_1, dh \rangle_{L^2(L_P)} + \langle \theta_2, d^*_P\beta \rangle_{L^2(L_P)} + \langle \chi, d\beta \rangle_{L^2(L_P)} = 0.
\end{aligned}$$

Note that the boundary terms $\lim_{t\to\infty}\langle \theta_1, h \rangle_{L^2(L^t_P)}$ and $\lim_{t\to\infty}\langle \theta_2, *\beta \rangle_{L^2(L^t_P)}$ vanish, since θ_1 and θ_2 decay exponentially, and h and β are bounded. $\square$

Lemma 97. *For each $P \in Q$, let $G^{k+1,p}_{P,\alpha} : U^{k+1,p}_\alpha \to W^{k,p}_\alpha(\Lambda(T^*L_P)) \oplus W^{k,p}_\alpha(\Lambda^2(T^*L_P))$ be the map defined by*

$$G^{k+1,p}_{P,\alpha}(\eta) = *_P f^*_\eta(\Omega_P) + f^*_\eta(\omega_P),$$

and $\widetilde{F}^{k+1,p}_{P,\alpha} : U^{k+1,p}_\alpha \oplus W^{k+1,p}_\alpha(\Lambda^3(T^*L_P)) \to W^{k,p}_\alpha(\Lambda^0(T^*L_P)) \oplus W^{k,p}_\alpha(\Lambda^2(T^*L_P))$ *the map defined by*

$$\widetilde{F}^{k+1,p}_{P,\alpha}(\eta,\chi) = *_P f^*_\eta(\Omega_P) + f^*_\eta(\omega_P) + d^*_P\chi.$$

Then $(\widetilde{F}^{k+1,p}_{P,\alpha})^{-1}(0) = (G^{k+1,p}_{P,\alpha})^{-1}(0) \times \{0\}$*.*

Proof. It is clear that

$$(G^{k+1,p}_{P,\alpha})^{-1}(0) \times \{0\} \subset (\widetilde{F}^{k+1,p}_{P,\alpha})^{-1}(0).$$

Suppose $(\eta,\chi) \in (\widetilde{F}^{k+1,p}_{P,\alpha})^{-1}(0)$. Let $r : L_{P_0} \to \mathbb{R}$ be a radius function on L_P which measures distance from a fixed point $x_0 \in L_P$. Let $L^R_P = L_P \cap r^{-1}[0,R]$. Note that if $\widetilde{F}^{k+1,p}_{P,\alpha}(\eta,\chi) = G^{k+1,p}_{P,\alpha} + d^*\chi = 0$, then

$$\begin{aligned}
\|d^*_P\chi\|^2_{L^2(L^R_P)} &= \langle d^*_P\chi, d^*_P\chi\rangle_{L^2(L^R_P)} = -\langle G^{k+1,p}_{P,\alpha}(\eta), d^*_P\chi\rangle_{L^2(L^R_P)} \\
&= -\langle dG^{k+1,p}_{P,\alpha}(\eta), \chi\rangle_{L^2(L^R_P)} + \int_{\partial L^R_P} \langle G^{k+1,p}_{P,\alpha}(\eta), \chi\rangle \\
&= -\langle df^*_\eta(\omega), \chi\rangle_{L^2(L^R_P)} + \int_{\partial L^R_{P_0}} \langle G^{k+1,p}_\alpha(\eta), \chi\rangle = \int_{\partial L^R_P} \langle G^{k+1,p}_{P,\alpha}(\eta), \chi\rangle.
\end{aligned}$$

By taking limit $R \to \infty$ and since χ and $G^{k+1,p}_{P,\alpha}(\eta)$ are exponentially decaying, we get

$$\|d^*_P\chi\|^2_{L^2(L^R_P)} = 0,$$

and therefore, $d^*_P\chi = 0$, which implies $\chi = C vol_P$ for a constant $C \in \mathbb{R}$. However, since χ is in a certain Sobolev space, $\chi = 0$. $\square$

With this setup, for each $P \in Q$, the linearized map

$$d_{(0,0)}\widetilde{F}^{k+1,p}_{P,\alpha} : W^{k+1,p}_\alpha(\Lambda^1(T^*L_P)) \oplus W^{k+1,p}_\alpha(\Lambda^3(T^*L_P)) \to \mathcal{C}$$

is surjective.

Lemma 98. $T_{L_{P_0}}\widetilde{\mathcal{M}}_{L_{P_0}} \cong \ker d_{(P_0,0,0)}F^{k+1,p}_\alpha$*, and therefore,* $\dim \widetilde{\mathcal{M}}_{L_{P_0}} = 2n$*.*

Proof. Let $(P,\eta,\chi) \in \ker d_{(P_0,0,0)}F^{k+1,p}_\alpha$. We know $\chi = 0$ and $(d + d^*_{P_0})\eta = 0$. Let

$$\psi : \ker d_{(P_0,0,0)}F^{k+1,p}_\alpha \to Q \times H^1(L_{P_0},\mathbb{R}),$$

be the map given by

$$(P,\eta,\chi) \to (P,[\eta]).$$

The map $\psi : \ker d_{(P_0,0,0)}F_\alpha^{k+1,p} \to Q \times H^1(L_{P_0}, \mathbb{R})$ is injective. This follows from the fact that when $\alpha < 0$ is sufficiently small such that $[0, \alpha^2]$ does not include any eigenvalue of Laplacian on functions on L_P,

$$\ker(d + d^*)_\alpha^{k+1,p} = \ker(d + d^*)_\alpha^{k+1,2} \subset \ker(d + d^*)_0^{k+1,p}.$$

Suppose $(P_1, [\eta_1]) = (P_2, [\eta_2])$, and therefore, $P_1 = P_2$, and $[\eta] = 0$, where $\eta = \eta_1 - \eta_2$. We have $\eta = dk$ for some $k \in W_0^{k+2,p}(\Lambda^0(T^*L_P))$, thus

$$0 = d^*\eta = \Delta k,$$

and therefore, $k = 0$.

For each $P \in Q$, let $\psi_P(\eta, \chi) = \psi(P, \eta, \chi)$. We have image$(\psi_P) =$ image(ι_P), which is the natural inclusion $\iota_P : H_c^1(L_P) \to H^1(L_P)$, appearing in the following exact sequence,

$$\begin{aligned} 0 \to H^0(L_P) \to H^0(\partial\overline{L_P}) \to H_c^1(L_P) \xrightarrow{\iota} H^1(L_P) \to H^1(\partial\overline{L_P}) \to H_c^2(L_P) \\ \to H^2(L_P) \to H^2(\partial\overline{L_P}) \to H_c^3(L_P) \to 0. \end{aligned}$$

Note that $H^4(L_P, \mathbb{R}) \cong H_c^0(L_P, \mathbb{R}) = 0$, since L_0 is simply connected. The exact sequence shows

$$\dim \text{ image } \iota_P = b^2(L) - b^0(\partial\overline{L}) + b^0(L) = (n-1) - n + 1 = 0,$$

and therefore,

$$\dim \text{ image } \psi = 2n + 0 = 2n.$$

□

4.3.1.3 The Index Problem

The map

$$d_{(0,0)}F_\alpha^{k+1,p} : W_\alpha^{k+1,p}(\Lambda^1(T^*L_{P_0})) \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_{P_0})) \to \mathcal{C},$$

is surjective and has 0-dimensional kernel, and therefore,

$$\text{index } d_{(0,0)}F_\alpha^{k+1,p} = 0.$$

The map

$$\begin{aligned} d_{(0,0)}F_\alpha^{k+1,p} : W_\alpha^{k+1,p}(\Lambda^1(T^*L_{P_0})) \oplus W_\alpha^{k+1,p}(\Lambda^3(T^*L_{P_0})) \\ \to W_\alpha^{k,p}(\Lambda^0(T^*L_{P_0})) \oplus W_\alpha^{k,p}(\Lambda^2(T^*L_{P_0})), \end{aligned}$$

is Fredholm, has 0-dimensional kernel, and moreover,

$$\text{coker } d_{(0,0)}F_{\alpha}^{k+1,p} = \mathcal{D} = \ker(d + d^* : W_{-\alpha}^{k+1,q}(\Lambda^{\text{even}}) \to W_{-\alpha}^{k+1,q}(\Lambda^{\text{odd}})).$$

For α sufficiently small, $\ker(d+d^* : W_{-\alpha}^{k+1,q}(\Lambda^{\text{even}}) \to W_{-\alpha}^{k+1,q}(\Lambda^{\text{odd}}))$ consists the vector space of smooth closed and co-closed 0-forms and 2-forms, which consists only of constant functions, and therefore, is 1-dimensional. With this setup,

$$\text{index } d_{(0,0)}F_{\alpha}^{k+1,p} = -1.$$

4.4 Dimensional Reduction of the Donaldson-Scaduto Conjecture

It is expected that the Donaldson-Scaduto calibrated submanifolds are $U(1)$-invariant. This symmetry can be helpful in finding these conjectured submanifolds. More generally, in the presence of a suitable group action G on a manifold Z, the problem of finding a certain G-invariant submanifold L in the ambient space Z can be dimensionally reduced to the problem of finding some submanifold $L/G \subset Z/G$; however, the latter problem might be a singular one, and the study of singular spaces can be difficult. In the case of G-invariant Lagrangians in symplectic manifolds, if the action is Hamiltonian, one can do better and dimensionally reduce the problem to the symplectic quotient space.

In this section, by studying the problem in the level of the symplectic quotient, we will show that the Donaldson-Scaduto special Lagrangians in $X \times \mathbb{C}$ will correspond to certain non-compact $\underline{J}$-holomorphic curves with boundary in $\mathbb{R}^4$ with respect to a non-standard singular almost complex structure $\underline{J}$. Moreover, we will show that these $\underline{J}$-holomorphic curves are described by a real Monge-Ampère equation. This approach might be useful in proving that $\Lambda \subset (\mathbb{R}^3)^n \setminus \Delta$ is a closed subspace. Finally we address the non-generic case where the points p_1, p_2 and p_3, used in the Gibbons-Hawking construction of X, are collinear. This case can be understood as the special case used in the continuity method.

4.4.1 Dimensional Reduction of Special Lagrangians

The main theorem of this subsection is the following.

Theorem 44. *Let (X, g_X, I, J, K) be a 4-dimensional hyperkähler manifold given by the Gibbons-Hawking Ansatz, specified by a positive harmonic map $V : U \subset \mathbb{R}^3 \to \mathbb{R}$. There is a one-to-one correspondence between $U(1)$-invariant special Lagrangians in $Z = X \times \mathbb{C}$ and $\underline{J}$-holomorphic curves in the symplectic quotient for some non-integrable almost complex structure $\underline{J}$, given by*

$$\underline{J}(du_1) = \frac{1}{\sqrt{V}}dx_3, \qquad \underline{J}(du_2) = -\frac{1}{\sqrt{V}}dy_3. \tag{4.4.1}$$

where u_1, u_2, x_3, y_3 denote the standard Euclidean coordinates on $\mathbb{R}^4$.

In tha case $V = m + \sum_i \frac{1}{|x-p_i|}$ for $m \geq 0$, one can complete X by adding some isolated points to get a multi-Eguchi-Hanson or a multi-Taub-NUT space — still denoted by X — which correspondingly results in completing the symplectic quotient to get $\mathbb{R}^4$. Then the one-to-one correspondence extends to the $U(1)$-invariant special Lagrangians in $X \times \mathbb{C}$ and the non-compact $\underline{J}$-holomorphic curves with boundary in $\mathbb{R}^4$, with respect to a non-standard, singular, non-integrable almost complex structure $\underline{J}$.

Proof. Let $\pi : X \to \mathbb{R}^3$ be the bundle map which appears in the definition of the Gibbons-Hawking construction, $\pi(x) = (u_1(x), u_2(x), u_3(x))$. The $U(1)$-action on $X \times \mathbb{C}$, equipped with the symplectic form $\omega = \omega_3 + dx_3 \wedge dy_3$, is Hamiltonian with the moment map $\mu : X \times \mathbb{C} \to \mathbb{R}$, given by $\mu(x, z) = u_3(x)$, and therefore, any $U(1)$-invariant Lagrangian L will be in a level set of the moment map, $\mu_{|L} = c$ for a constant $c \in \mathbb{R}$. Let

$$Z_c := \mu^{-1}(c)/U(1) \quad \text{and} \quad \underline{L} := L/U(1) \subset Z_c.$$

On the reduction space u_3 is constant; moreover, as we take quotient by the $U(1)$-action, there is no t-coordinate. Therefore, the remaining variables are u_1, u_2, x_3, y_3. This is a global coordinate system on the symplectic quotient, which is $Z_c = \mathbb{R}^4_{u_1,u_2,x_3,y_3}$.

The symplectic structure ω on Z induces a symplectic structure $\underline{\omega}$ on the symplectic quotient Z_c, given by

$$\omega_{\text{red}} = V du_1 \wedge du_2 + dx_3 \wedge dy_3.$$

Note that $V : U \subset \mathbb{R}^3 \to \mathbb{R}$ induces a real-valued positive function on the symplectic quotient, which by an abuse of notation we still denote it with $V : Z_c \to \mathbb{R}$.

On the other hand the holomorphic volume form Ω induces a holomorphic volume Ω_{red} form on the reduction space, which is a holomorphic 2-form, given by

$$\Omega_{\text{red}} = (du_1 - i du_2) \wedge (dx_3 + i dy_3).$$

The holomorphic volume form Ω_{red} determines a complex structure I_{red} on Z_c, given by

$$I_{\text{red}}(du_1) = du_2, \qquad I_{\text{red}}(dx_3) = -dy_3.$$

The Riemannian metric g on Z induces a Riemannian metric g_{red} on Z_c, given by

$$g_{\text{red}} = V du_1 \otimes du_1 + V du_2 \otimes du_2 + dx_3 \otimes dx_3 + dy_3 \otimes dy_3.$$

The equations $\mathrm{Im}(\Omega)_{|L} = \omega_{|L} = 0$ imply $\mathrm{Im}(\Omega_{\text{red}})_{|\underline{L}} = \omega_{\text{red}|\underline{L}} = 0$, which are special Lagrangian type conditions for $\underline{L}$ in Z_c; however, note that $(Z_c, g_{\text{red}}, \omega_{\text{red}}, \Omega_{\text{red}})$ is not a Calabi-Yau manifold. Let $u = u_1 + iu_2$ and $z_3 = x_3 + iy_3$. We have

$$\omega_{\text{red}}^2 = -\frac{V}{4} du \wedge d\overline{u} \wedge dz_3 \wedge d\overline{z}_3, \qquad \Omega_{\text{red}} \wedge \overline{\Omega}_{\text{red}} = du \wedge d\overline{u} \wedge dz_3 \wedge d\overline{z}_3,$$

and as we can see, unlike Z, on the reduction space the differential forms ω_{red}^2 and $\Omega_{\text{red}} \wedge \overline{\Omega}_{\text{red}}$ do not agree up to a constant. It is more pleasant to work with structures which satisfy this formal relation. This can be resolved with the help of introducing a new singular symplectic form, denoted by $\underline{\omega}$, given by

$$\underline{\omega} = \sqrt{\frac{2}{V}}\omega_{\text{red}},$$

moreover, let $\underline{g} = g_{\text{red}}$ and $\underline{\Omega} = \Omega_{\text{red}}$. Then

$$2\underline{\omega}^2 = du \wedge d\overline{u} \wedge dz_3 \wedge d\overline{z}_3 = -\underline{\Omega} \wedge \overline{\underline{\Omega}},$$

and therefore, $(Z_c, \underline{g}, \underline{\omega}, \underline{\Omega})$ satisfies a formal property similar to Calabi-Yau manifolds. Note that this space is not a genuine singular Calabi-Yau manifold, since $d\underline{w} \neq 0$.

Let

$$\theta = \sqrt{2}\underline{\omega} + i\,\mathrm{Im}\,(\underline{\Omega}).$$

We have

$$\theta_{|\underline{L}} = 0.$$

In real coordinates

$$\sqrt{2}\underline{\omega} = 2\sqrt{V} du_1 \wedge du_2 + \frac{2}{\sqrt{V}} dx_3 \wedge dy_3, \quad \mathrm{Im}(\underline{\Omega}) = du_1 \wedge dy_3 - du_2 \wedge dx_3,$$

and therefore,

$$\begin{aligned}\theta &= (2\sqrt{V} du_1 \wedge du_2 + \frac{2}{\sqrt{V}} dx_3 \wedge dy_3) + i(du_1 \wedge dy_3 - du_2 \wedge dx_3) \\ &= 2(\sqrt{V} du_1 + i dx_3) \wedge (du_2 - \frac{i}{\sqrt{V}} dy_3).\end{aligned}$$

This means $\underline{L}$ is pseudo-holomorphic curve with respect to the almost complex structure $\underline{J}$, given by

$$\underline{J}(du_1) = \frac{1}{\sqrt{V}} dx_3, \quad \underline{J}(du_2) = -\frac{1}{\sqrt{V}} dy_3. \tag{4.4.2}$$

$\square$

Remark 10. *As mentioned in the proof of the lemma above, $(Z_c = \mathbb{R}^4, \underline{g}, \underline{\omega}, \underline{\Omega})$ is not a Calabi-Yau manifold. Let Z_c^{sing} denote the points in Z_c where V is singular. In the multi-Eguchi-Hanson and the multi-Taub-NUT cases, the singular set*

$$Z_c^{sing} := \{(u_1, u_2, x_3, y_3) \mid V(u_1, u_2, 0) = 0\} = \{p_1, p_2, p_3\} \times \mathbb{R}^2,$$

which is a 2-dimensional subset of $\mathbb{R}^4$ — of codimension 2. At each point $p = (u_1, u_2, x_3, y_3) \in$

Z_c^{sing}*, let*

$$T_{(u_1,u_2,x_3,y_3)}\mathbb{R}^4 = \langle \partial u_1(p), \partial u_2(p)\rangle \oplus \langle \partial x_3(p), \partial y_3(p)\rangle.$$

The metric $\underline{g}$ *blows up along* Z_c^{sing}*. More specifically,* $\underline{g}(p)|_{\langle \partial u_1(p), \partial u_2(p)\rangle}$ *blows up, and* $\underline{g}(p)|_{\langle \partial x_3(p), \partial y_3(p)\rangle}$ *is the standard Euclidean metric. The 2-form* $\underline{\omega}$ *is not closed; moreover, it is singular along* Z_c^{sing}*. In fact,* $\underline{\omega}(p)|_{\langle \partial u_1(p), \partial u_2(p)\rangle}$ *blows up, and* $\underline{\omega}(p)|_{\langle \partial x_3(p), \partial y_3(p)\rangle} = 0$*. Furthermore,* $\underline{\Omega}$ *and* $\underline{I}$ *are smooth, and in fact, the standard ones on* $\mathbb{C}^2$*. The almost complex structure* $\underline{J}$ *is also singular along* Z_c^{sing}*,*

$$\underline{J}_p(du_1) = 0, \qquad \underline{J}_p(du_2) = 0. \tag{4.4.3}$$

The cylindrical special Lagrangians $L^0_{i,j} \subset Z$ would map into bands $\underline{L}^0_{i,j}$ inside the symplectic reduction space,

$$\underline{L}^0_{i,j} = [p_i, p_j] \times ((p_i - p_j)\ldots\mathbb{R}) \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$

The Donaldson-Scaduto conjecture reduces to the existence of $\underline{J}$-holomorphic curves asymptotic to these bands.

Conjecture 12 (Dimensionally Reduced Donaldson-Scaduto Conjecture)**.** *Let* X *be a multi-Eguchi-Hanson space, constructed via the Gibbons-Hawking construction over* $\mathbb{R}^3 \setminus \{p_1, p_2, p_3\}$*, where* p_1, p_2 *and* p_3 *are not collinear. There exists* $\underline{L} \subset \mathbb{R}^4$ *a non-compact* $\underline{J}$*-holomorphic curve with non-compact boundary with three ends* $\underline{L}_1$*,* $\underline{L}_2$ *and* $\underline{L}_3$ *which in the positive directions are asymptotic to the* $\underline{J}$*-holomorphic bands* $\underline{L}^0_{1,2}$*,* $\underline{L}^0_{2,3}$ *and* $\underline{L}^0_{3,1}$*, respectively. We call* $\underline{L}$ *a Donaldson-Scaduto holomorphic curve.*

We see an schematic drawing of such $\underline{J}$*-holomorphic band.*

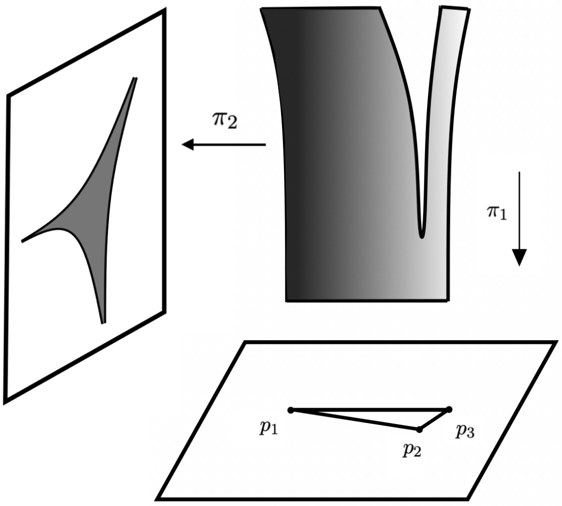

Figure 4.6: Dimensionally Reduced Donaldson-Scaduto Conjecture in $\mathbb{R}^2 \times \mathbb{R}^2$

This approach can be helpful in the direction of proving the closed property of the existence of the Donaldson-Scaduto special Lagrangians. In the dimensionally reduced picture, we would have a sequence of singular almost complex structures $\underline{J}_i$ and a corresponding sequence of $\underline{J}_i$-holomorphic curves with boundary $\underline{L}_i$, and one hopes to show as $\underline{J}_i \to \underline{J}_0$, we would get $\underline{L}_i \to \underline{L}_0$.

An important step in that direction would be to show there is no $\underline{J}$-holomorphic sphere in $\mathbb{R}^4$ for a $\underline{J}$ of the form mentioned in 4.4.1. It seems quite plausible; however, here we would not follow that any further.

4.4.2 Special Lagrangians and the Monge-Ampère Equation

In this section, we show that the existence of the Donaldson-Scaduto submanifolds are described by a Monge-Ampère equation.

Let $\pi_1 : \mathbb{R}^4_{(u_1,u_2,x_3,y_3)} \to \mathbb{R}^2_{(u_1,u_2)}$ and $\pi_2 : \mathbb{R}^4_{(u_1,u_2,x_3,y_3)} \to \mathbb{R}^2_{(x_3,y_3)}$ be the projection maps on the first two components and the last two components, respectively. The expected projections of a Donaldson-Scaduto $\underline{J}$-holomorphic curve on $\mathbb{R}^2_{(u_1,u_2)}$ and $\mathbb{R}^2_{(x_3,y_3)}$ are presented in the figure 4.11.

The triangle in $\mathbb{R}^2_{(u_1,u_2)}$ is the triangle we used in the Gibbons-Hawking construction in $\mathbb{R}^2 \times \{0\}$. The projection map π_1 projects each asymptotic band $\underline{L}^0_{i,j} = [p_i, p_j] \times ((p_i - p_j) \cdot \mathbb{R})$ to the segment $[p_i, p_j] \subset \mathbb{R}^2$. Moreover, the map π_1 sends the non-compact boundaries of each band $\underline{L}^0_{i,j}$ to a vertex of this triangle.

The projection map π_2 projects each asymptotic band $\underline{L}^0_{i,j} = [p_i, p_j] \times ((p_i - p_j) \cdot \mathbb{R})$ to the line $(p_i - p_j) \cdot \mathbb{R} \subset \mathbb{R}^2$. It is expected that the image of $\underline{L}$ under the projection map π_2 is an amoeba, which in three directions converges to the positive direction of the vectors $(p_1 - p_2) \cdot \mathbb{R}$, $(p_2 - p_3) \cdot \mathbb{R}$ and $(p_3 - p_1) \cdot \mathbb{R}$.

Conjecture 13. *Let T_{p_1,p_2,p_3} be the region in $\mathbb{R}^3_{(u_1,u_2)}$ bounded with the triangle with vertices p_1, p_2, p_3. Let $\underline{L}$ be the Donaldson-Scaduto $\underline{J}$-holomorphic curve. We expect*

$$\pi_1(\underline{L}) = T_{p_1,p_2,p_3}.$$

In other words, the projection of $\underline{L}$ would fill the triangle, and also does not go outside of this region.

Moreover, the image of $\pi_2(\partial(\underline{L}))$ would consists of three curves, bounding an amoeba, which is filled by $\pi_2(\underline{L})$.

Motivated by the figure 4.11, we would think about the Donaldson-Scaduto $\underline{J}$-holomorphic curve $\underline{L}$ as a graph of a map $F : A \subset \mathbb{R}^2_{(x_3,y_3)} \to T \subset \mathbb{R}^2_{(u_1,u_2)}$ or $G : T \subset \mathbb{R}^2_{(u_1,u_2)} \to A \subset \mathbb{R}^2_{(x_3,y_3)}$ for two sets A and T. In fact, Conjecture 13 states T is the triangle T_{p_1,p_2,p_3} and A is an amoeba. The condition $\theta_{|\underline{L}}$ would translate to a partial differential equation for F, which is investigated in the following theorem.

Theorem 45. *The existence of the Donaldson-Scaduto $\underline{J}$-holomorphic curves are described by a* *map*

$$\phi : A \subset \mathbb{R}^2_{(x_3,y_3)} \to \mathbb{R},$$

such that

$$\det\left(Hess(\phi)\right) = 1/V,$$

where $T = \pi_1(\underline{L})$ is a subset of $\mathbb{R}^2$, conjecturally T_{p_1,p_2,p_3}, as presented in the figure 4.4.2.

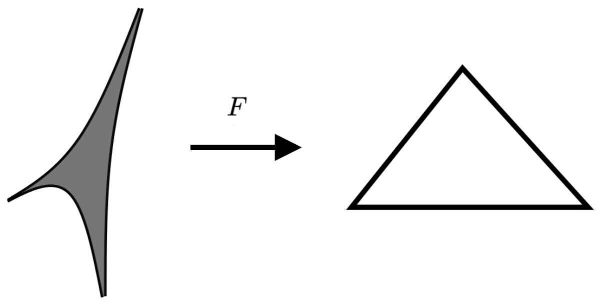

Figure 4.7: Map F describing $\underline{L}$

In this equation, ϕ is a function of x_3 and y_3, and $V : \mathbb{R}^2_{(u_1,u_2)} \to \mathbb{R}$ is a function of u_1 and u_2, where u_1 and u_2 are the derivatives of ϕ with respect to y_3 and x_3, respectively,

$$\partial_{y_3}\phi = u_1, \qquad \partial_{x_3}\phi = u_2.$$

Proof. Let

$$F = (F_1, F_2) : \mathbb{R}^2_{(x_3,y_3)} \to \mathbb{R}^2_{(u_2,u_1)},$$

and therefore,

$$T_{(x_3,y_3,u_1,u_2)}(Graph(F)) = \langle (1, 0, \partial_{x_3}F_1, \partial_{x_3}F_2), (0, 1, \partial_{y_3}F_1, \partial_{y_3}F_2)\rangle \subset T_{(x_3,y_3,u_1,u_2)}\mathbb{R}^4.$$

The graph of F is a Lagrangian if $\underline{\omega}_{|T(Graph(F))} = 0$. We only need to check

$$\begin{aligned}
\underline{\omega}((1, 0, \partial_{x_3}F_1, \partial_{x_3}F_2),&(0, 1, \partial_{y_3}F_1, \partial_{y_3}F_2)) \\
&= (V du_1 \wedge du_2 + dx_3 \wedge dy_3)((1, 0, \partial_{x_3}F_1, \partial_{x_3}F_2), (0, 1, \partial_{y_3}F_1, \partial_{y_3}F_2)) \\
&= V(\partial_{x_3}F_1\partial_{y_3}F_2 - \partial_{x_3}F_2\partial_{y_3}F_1) + 1 = V \det(dF) + 1 = 0,
\end{aligned}$$

and therefore,

$$\det(dF) = -1/V.$$

The special condition $\mathrm{Im}(\underline{\Omega})_{|T(Graph(F))} = 0$ can be written as

$$\begin{aligned}
&\mathrm{Im}(\underline{\Omega})((1,0,\partial_{x_3}F_1,\partial_{x_3}F_1),(0,1,\partial_{y_3}F_1,\partial_{y_3}F_2))\\
&\qquad = (du_1\wedge dy_3 + du_2\wedge dx_3)((1,0,\partial_{x_3}F_1,\partial_{x_3}F_2),(0,1,\partial_{y_3}F_1,\partial_{y_3}F_2))\\
&\qquad = \partial_{x_3}F_1 - \partial_{y_3}F_2 = 0,
\end{aligned}$$

and therefore,

$$\partial_{x_3}F_1 = \partial_{y_3}F_2.$$

The special Lagrangian equations for the graph of F are

$$\det(dF) = -1/\pi V, \qquad \partial_{x_3}F_1 = \partial_{y_3}F_2.$$

Let $\hat{F} := (\hat{F}_1,\hat{F}_2) = (F_2,F_1) : \mathbb{R}^2_{(x_3,y_3)} \to \mathbb{R}^2_{(u_1,u_2)}$. The equation $\partial_{x_3}\hat{F}_2 = \partial_{y_3}\hat{F}_2$ shows that one can define a function $\phi : \mathbb{R}^2_{(x_3,y_3)} \to \mathbb{R}$ such that

$$\partial_{y_3}\phi = \hat{F}_1, \qquad \partial_{x_3}\phi = \hat{F}_2,$$

and therefore, $det(dF) = -det(d\hat{F}) = det(Hess(\phi))$. By rewriting the special Lagrangian equations in terms of ϕ we get

$$\det(Hess(\phi)) = 1/V,$$

which is a real Monge–Ampère equation. □

One could consider the Donaldson-Scaduto $\underline{J}$-holomorphic curves as a graph of a map above the triangular region $T \subset \mathbb{R}^2_{(u_1,u_2)}$,

$$G = (G_1,G_2) : T \subset \mathbb{R}^2_{(u_1,u_2)} \to A \subset \mathbb{R}^2_{(x_3,y_3)}.$$

The same type of calculations as the one in the proof of Theorem 45 shows there exists a map

$$\psi : T \subset \mathbb{R}^2_{(u_1,u_2)} \to \mathbb{R},$$

such that

$$\det(Hess(\psi)) = V.$$

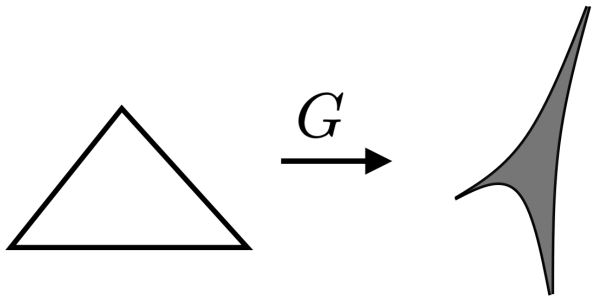

Figure 4.8: Map G describing $\underline{L}$

Assuming $T = T_{p_1,p_2,p_3}$, the boundary conditions of ψ are described by the following,

$$|d\psi(u_1, u_2)| = |G(u_1, u_2)| \to \infty \text{ as } x \to [p_1, p_2] \cup [p_2, p_3] \cup [p_3, p_1].$$

4.4.3 The Degenerate Case

In the study of the existence problem, in the analysis of the deformation of Donaldson-Scaduto special Lagrangians and also in writing down the Mongè Ampere equation describing the Donaldson-Scaduto $\underline{J}$-holomoprhic curves, we considered a generic case where the points p_1, p_2 and p_3 are not collinear. In this section, we consider the non-degenerate case, where p_1, p_2 and p_3 are collinear. The material in this section, are rather speculative.

The triangle T_{p_1,p_2,p_3}, when p_1, p_2 and p_3 are collinear, would be degenerate and is just a line segment. Without loss of generality, let's assume p_1, p_2 and p_3 lie on the x-axis and p_2 lies on the segment $[p_3, p_1]$. Let e_1 be the vector in the positive direction of the x-axis. We would have two parallel special Lagrangians in $X \times \mathbb{C}$, namely,

$$L^0_{1,2} = \pi^{-1}[p_1, p_2] \times \langle e_1 \rangle, \qquad L^0_{2,3} = \pi^{-1}[p_2, p_3] \times \langle e_1 \rangle.$$

Let $\pi_1 : X \times \mathbb{C} \to \mathbb{R}^2$ be the map defined by $\pi_1(x, z) = \pi(x)$, where $\pi : X \to \mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Let $\pi_2 : X \times \mathbb{C} \to \mathbb{R}^2 = \mathbb{C}$ be the map defined by $\pi_2(x, z) = z$. We would get the following degenerate figure.

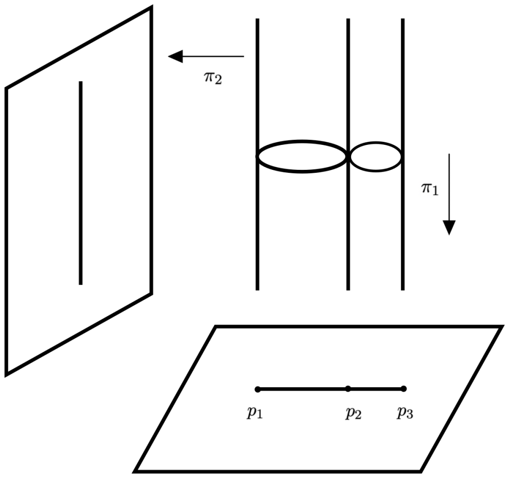

Figure 4.9: Singular Special Lagrangian

In fact, we expect there would be no smooth Donaldson-Scaduto special Lagrangian in this degenerate case. Let $L^0_{\text{sing},p_1,p_2,p_3}$ be the singular special Lagrangian, illustrated in 4.4.3, which is the union of $L^0_{1,2}$ and $L^0_{2,3}$. This special Lagrangian is singular along a line, which is $\{p_2\} \times \langle e_1 \rangle$. Note that the intersection

$$L^0_{p_1,p_2} \cap L^0_{p_2,p_3} = \{p_2\} \times \langle e_1 \rangle,$$

is not a transversal intersection.

For each $t \in (0,1)$, let p_1^t, p_2^t, p_3^t be three points which are not collinear, but they converge to the collinear ones p_1, p_2, p_3 as $t \to 0$. In fact, by a affine transformation of $\mathbb{R}^2$, we can assume p_1^t and p_3^t are fixed and p_2^t is getting closer to the segment connecting p_1^t and p_3^t. As one moves p_2^t closer and closer to this line segment, we would expect to get a family of smooth Donaldson-Scaduto special Lagrangians $L^0_{p_1^t,p_2^t,p_3^t}$ converging to the singular special Lagrangian $L^0_{\text{sing},p_1,p_2,p_3}$.

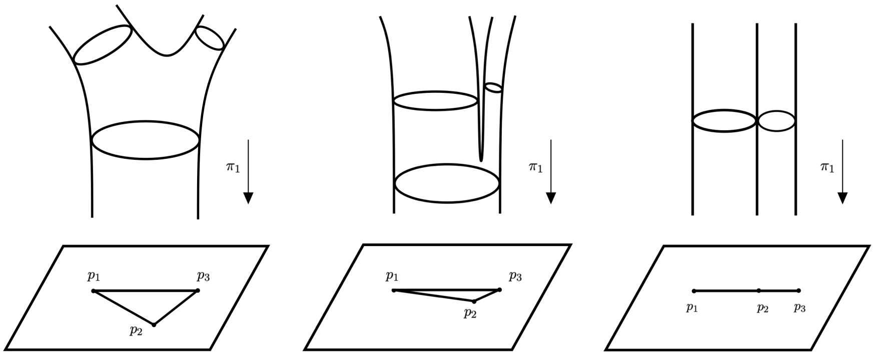

Figure 4.10: Formation of Singular Special Lagrangian

The main deformation problem here is to start with the singular special Lagrangian $L^0_{\text{sing},p_1,p_2,p_3}$, and show that this special Lagrangian can be disingularized as we move the point p_2 slightly off the segment connecting p_1 and p_3. This is also motivated by the method of continuity for proving the Donaldson-Scaduto conjecture, since we can consider this singular case as the special case of the method of continuity. In fact, proving this would show the space of smooth Donaldson-Scaduto special Lagrangians is non-empty.

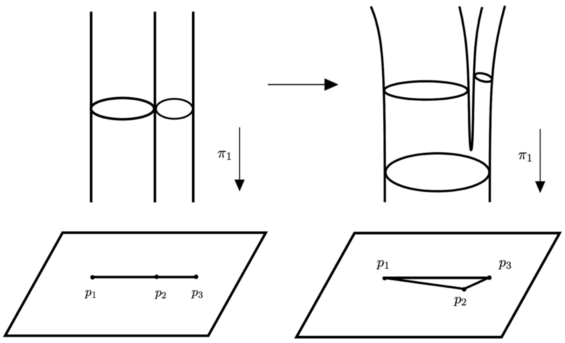

Figure 4.11: Disingularization

The picture in the symplectic reduction space is clearer. Let $\pi_1 : \mathbb{R}^4_{(u_1,u_2,x_3,y_3)} \to \mathbb{R}^2_{(u_1,u_2)}$ and $\pi_2 : \mathbb{R}^4_{(u_1,u_2,x_3,y_3)} \to \mathbb{R}^2_{(x_3,y_3)}$ be the projection maps on the first two components and the last two components, respectively. In the degenerate case, we would have two $\underline{J}$-holomorphic bands $\underline{L}^0_{p_1,p_2}$ and $\underline{L}^0_{p_2,p_3}$. The $\underline{J}$-holomorphic curve conjectured by the dimensionally reduced Donaldson-Scaduto conjecture is simply their union, which unlike the special Lagrangian case, is a smooth object $[p_1, p_3] \times \langle e_1 \rangle$; however with boundary.

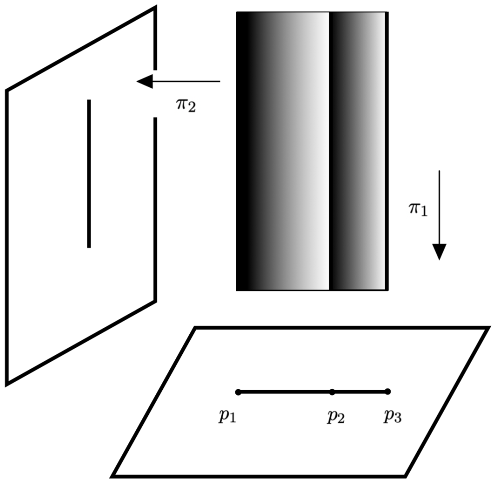

Figure 4.12: The Limiting Band

The deformation problem in this case is to start with the smooth band $[p_1, p_3] \times \langle e_1 \rangle$, and prove as we move the point p_2 slightly off the segment connecting p_1 and p_3, we would get a smooth Donaldson-Scaduto $\underline{J}$-holomorphic curve with three ends.

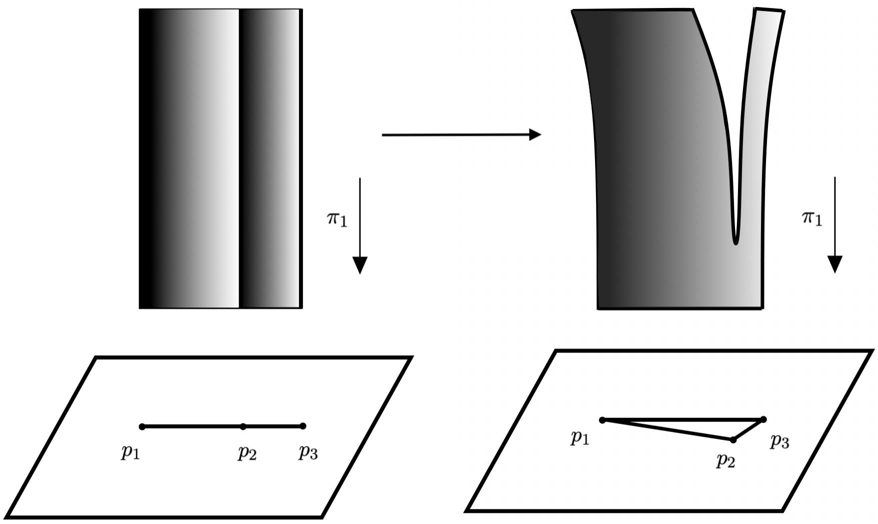

Figure 4.13: Deformation of the Limiting Case

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