Limits of Hodge structures via D-modules

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Abstract of the Dissertation

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This dissertation contains two parts. In the first part, we construct the limiting mixed Hodge structure of a degeneration of compact Kähler manifolds over the unit disk with a possibly non-reduced normal crossing central fiber via holonomic $\mathscr{D}$-modules, which generalizes Steenbrink's geometric construction of limits of Hodge structures. Our limiting mixed Hodge structure does not carry a $\mathbb{Q}$-structure; instead, we use sesquilinear pairings on $\mathscr{D}$-modules to construct a canonical polarization on the limiting mixed Hodge structure as a replacement. The associated graded quotient of the weight filtration of the limiting mixed Hodge structure can be computed by the cohomology of the cyclic coverings of certain intersections of components of the central fiber. We also generalize the local invariant cycle theorem to this setting.

In the second part, we study how the V-filtration along a subvariety of arbitrary codimension and the Hodge filtration on a mixed Hodge module interact with each other, generalizing the theory for hypersurfaces. In particular, we can describe Hodge module theoretic restriction functors in terms of this V-filtration. As applications, we give a Hodge theoretic proof of Skoda's theorem on multiplier ideals.

To Jin and Our families

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Chapter 1

Introduction

Based on the work of Hodge [Hod41], Hodge theory studies the linear algebra data, called Hodge structure, on cohomology groups of complex varieties, developed by Deligne [Del71a; Del71b; Del74], Griffiths [GS75] and others. It was Schmid who started the study of the asymptotic behavior of degeneration of variation of Hodge structure [Sch73]. For a 1-parameter family of compact Kähler manifolds, the cohomology of each smooth fiber carries a polarizable Hodge structure. This leads to the following two interesting questions:

1. How does the family of Hodge structures on the cohomology groups of smooth fibers degenerate?

2. How does the cohomology of the central fiber relate to that of nearby fibers?

These are two classical and central questions in Hodge theory. Before Saito's theory of mixed Hodge modules [Sai88; Sai90], Schmid showed the existence of a limiting mixed Hodge structure for an abstract polarized variation of Hodge structure over the unit disk [Sch73] using Lie theoretic methods. For the variation of Hodge structure coming from a semistable family of Kähler manifolds over a 1-dimensional base, the limiting mixed Hodge structure was first established by Steenbrink [Ste76] whose construction is equivalent to Schmid's in [Sch73] but purely geometric. A consequence of Steenbrink's construction is the local invariant cycle theorem, which is a piece in the Clemens-Schmid sequence [Cle77]. It says that for a semistable degeneration of compact Kähler manifolds, the monodromy invariant cohomology as a mixed Hodge structure of the smooth fiber is coming from the cohomology of the total space. The local invariant cycle theorem was first proved by Deligne in an algebraic setting when the base is a scheme [Del71b, Theorem 4.1.1] and later treated in [Ste76], [Cle77] and [GN90] for a semistable Kähler degeneration. The local invariant cycle theorem puts a strong constraint on the topology of the degeneration and it reads off the geometric information of the possible central fiber. For example, it was used to classify the semistable degeneration of K3 surfaces [Kul77].

The theme of this thesis is to study the degeneration of variation of Hodge structure via the theory of $\mathscr{D}$-modules. Invented in Japan and France, $\mathscr{D}$-modules, are modules over the ring $\mathscr{D}$ of differential operators. It has its origins in the field of algebraic analysis, which means the study of partial differential equations with algebraic tools. The famous Riemann-Hilbert correspondence proved by Kashiwara and Mebkhout [Kas84; Meb84] states there is an equivalence of categories between the category of regular holonomic $\mathscr{D}$-modules and the category of perverse sheaves. It builds a bridge from algebra and analysis to topology leading us to several applications in various fields in mathematics. Saito's theory [Sai88; Sai90] of mixed Hodge modules relates Hodge theory and $\mathscr{D}$-modules.

We give a conceptually simpler construction of the limiting mixed Hodge structure for the degenerations of Kähler manifolds over the unit disk, using the theory of holonomic $\mathscr{D}$-modules in Chapter 2. Although the $\mathbb{Q}$-structure is absent, our method enables us to bypass the semistable reduction. This means we can compute the limiting mixed Hodge structure for arbitrary degeneration of Kähler manifolds over the unit disk by embedded log resolution of the central fiber. We also prove the local invariant cycle theorem in this more general setting.

Chapter 3 is contained in joint work with Bradley Dirks [CD21], where the objects we focus on are two interesting filtrations of mixed Hodge modules: Hodge filtration and

V-filtration. A (mixed) Hodge module, roughly speaking, is a filtered regular holonomic $\mathscr{D}$-module which is a (graded-)polarizable variation of (mixed) Hodge structure over a locally closed subset. Hodge filtration is useful in algebraic geometry since it allows one to study canonical sheaves by the package of Hodge modules. V-filtration is topological filtration indexed by the eigenvalues of the Euler vector field along a submanifold. Deligne came up with a formal algebraic way of formalizing and generalizing the classical ideas of studying the degeneration of algebraic varieties to perverse sheaves [Del68], which led to the notions of nearby and vanishing cycles functors. Then V-filtration was introduced by Kashiwara [Kas83] and Malgrange [Mal83] to translate nearby and vanishing cycles to the regular holonomic $\mathscr{D}$-modules. Recently, the relation between Hodge filtration and V-filtration become more interesting because the projects started by Mustaţă and Popa on Hodge ideals [MP19]. One of the technical tools used by Mustaţă and Popa is the compatiblity of Hodge filtration and V-filtration in codimension 1. We generalize this compatibility to Hodge filtration and V-filtration in higher codimension in this thesis.

Our result is also interesting internal to the theory of mixed Hodge modules. The definition of mixed Hodge modules is given inductively by "restriction" to hypersurfaces using V-filtrations in codimension 1. This makes restriction of Hodge modules to subvarieties in higher codimension have to be done in terms of hypersurfaces. However, the V-filtration exists in any codimension and we wanted to know if we can characterize the restriction functors 1-step directly by V-filtration in higher codimension. We generalize what Saito did in codimension 1 to higher codimension and describe of the restriction functors in terms of V-filtration in higher codimension.

We proceed to introduce the two parts of this thesis in more detail.

1.1 Limits of Hodge structures

Before stating the main theorem, we briefly review the relative log de Rham complex for a proper holomorphic morphism $f : X \to \Delta$ from a complex manifold of dimension $n+1$ to the unit disk smooth away from the origin. Let Y be the central fiber and suppose that Y only has simple normal crossing support. Then we define $\Omega_X(\log Y)$ (resp. $\Omega_\Delta(\log 0)$) to be the sheaf of one-forms with log pole along Y (resp. 0). Let $\Omega_{X/\Delta}(\log Y) = \Omega_X(\log Y)/f^*\Omega_\Delta(\log 0)$ be the sheaf of relative log one-forms, which is locally free. Then the *relative log de Rham complex* is

$$\Omega^{\bullet+n}_{X/\Delta}(\log Y) = \{\mathscr{O}_X \to \Omega^1_{X/\Delta}(\log Y) \to \cdots \to \Omega^n_{X/\Delta}(\log Y)\}$$

placed in degrees $-n, -n+1, \ldots, 0$. Steenbrink proved that $\mathbf{R}^k f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is the Deligne's canonical extension of the flat connection $\mathbf{R}^k f_* \Omega^{\bullet+n}_{X/\Delta}|_{\Delta^*}$ over the punctured disk Δ^* with eigenvalues of the residue operator R in $[0,1)$. It follows the limiting mixed Hodge struture lives on the central fiber $\mathbf{R}^k f_* \Omega^{\bullet+n}_{X/\Delta}(\log Y) \otimes \mathbb{C}(0)$, where $\mathbb{C}(0)$ is the residue field of the origin. Our first theorem is as follows:

Theorem A. *Notation as above and assume that X is Kähler. Let R_n (resp. R_s) denote the nilpotent (resp. semisimple) part of the Jordan-Chevalley decomposition of the residue operator R on $\bigoplus_k H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$. Then each eigenspace of R_s on*

$$\bigoplus_{k,\ell} \mathrm{gr}^W_\ell H^k(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$$

underlies a limiting polarized bigraded Hodge-Lefschetz structure over $\mathbb{C}$ of central weight n, where $W_\bullet = W_\bullet(R_n)$ is the monodromy filtration associated to R_n.

A polarized bigraded Hodge-Lefschetz structure is essentially a direct sum of polarized Hodge structures of different weights preseerved by an $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-action. In the setting of Theorem A, the $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-action is induced by the operator R_n and $2\pi\sqrt{-1}L$ where $L = \omega\wedge$ is the Lefschetz operator for a Kähler form ω. In particular, each summand

$\mathrm{gr}_\ell^W H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ is a Hodge structure of weight $n+k+\ell$ and there are two Hard Lefschetz type isomorphisms of Hodge structures:

- for $k \geq 0, \ell \in \mathbb{Z}$

$$\left(2\pi\sqrt{-1}L\right)^k : \mathrm{gr}_\ell^W H^{-k}(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \to \mathrm{gr}_\ell^W H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(k) \text{ and}$$

- for $\ell \geq 0, k \in \mathbb{Z}$

$$R_n^\ell : \mathrm{gr}_\ell^W H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \to \mathrm{gr}_{-\ell}^W H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(-\ell).$$

Theorem A implies that each $H^k(X,\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)$ still underlies a limiting mixed Hodge structure of weight $n+k$ whose weight filtration is given by $W_\bullet = W_\bullet(R_n)$ when the central fiber is non-reduced. The associated graded quotient of the weight filtration of the limiting mixed Hodge structure can be computed by the cohomology of the cyclic coverings of certain intersections of components of the central fiber.

We will prove that there exists a filtered holonomic $\mathscr{D}$-module $(\mathcal{M},F)$ whose de Rham complex is filtered isomorphic to the $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$. Indeed, $\mathcal{M}$ is the cokernel of a canonical morphism

$$\Omega^{n-1}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X \to \Omega^{n}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X.$$

Locally, choosing a trivialization of $\Omega^n_{X/\Delta}(\log Y)$, $(\mathcal{M},F)$ is isomorphic to

$$\mathscr{D}_X/(t,D_1,D_2,\ldots,D_k,\partial_{k+1},\ldots,\partial_n)\mathscr{D}_X$$

with the filtration induced by the order filtration on $\mathscr{D}_X$ shifted by degree $-n$ where $t = z_0^{e_0}z_1^{e_1}\cdots z_k^{e_k}$ is the locally defining equation of Y and $D_i = e_i^{-1}z_i\partial_i - e_0^{-1}z_0\partial_0$. The monodromy logarithm is the left multiplication by $e_0^{-1}z_0\partial_0$ in the local presentation. The main difficulty of Steenrbink's approach is to construct the monodromy filtration on $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ over $\mathbb{Q}$. With the help of $\mathscr{D}$-modules, the monodromy filtration is easy to derive by local calculation on the single $\mathscr{D}$-module $\mathcal{M}$.

Instead of proving the monodromy filtration defined over $\mathbb{Q}$, we provide a sesquilinear pairing on $\mathcal{M}$, taking values in the sheaf of currents $\mathfrak{C}_X$ on X, by a device of Mellin transform, which only involves symbolic computation, to avoid a messy topological argument. The sesquilinear pairing can be viewed as a renormalization of the intersection pairing $\int_{X_t} : \omega_{X_t} \otimes_{\mathbb{C}} \overline{\omega_{X_t}} \to \mathscr{C}_{X_t}$ on the nearby fibers X_t for $t \in \Delta^*$; for example, if Y is reduced, the pairing on $\mathcal{M}$ is induced by

$$\operatorname{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \frac{d\bar{t}}{\bar{t}} \int_{X_t} : \Omega^n_{X/\Delta}(\log Y) \otimes_{\mathbb{C}} \overline{\Omega^n_{X/\Delta}(\log Y)} \to \mathfrak{C}_X,$$

where the constant scalar $\varepsilon(n+2)(2\pi\sqrt{-1})^{-(n+1)}$ depending on the dimension is used to make the pairing independent of the choice of orientation.

As an application of Theorem A, we establish the local invariant cycle theorem when Y is non-reduced.

Theorem B. *Suppose we are in the same setting as in Theorem A. Then the following sequence of mixed Hodge structures is exact:*

$$H^{\ell+n}(Y, \mathbb{C}) \longrightarrow H^\ell(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \xrightarrow{R} H^\ell(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y)(-1).$$

In other words, all cohomology classes invariant under the monodromy action comes from the cohomologies of Y.

Steenbrink later pointed out that the limiting mixed Hodge structure he constructed only depends on the log structure associated with the semistable family $f : X \to \Delta$ [Ste95]. Inspired by the idea in [Ste95], Fujisawa extended Steenbrink's results in [Ste76; Ste95] to semistable Kähler families over the polydisk and to the log geometry setting [Fuj99; Fuj08; Fuj14]. Recently, Nakkajima announced a simpler proof of Fujisawa's results [Nak21].

Assume that X is Kähler of dimension $n+1$ and $Y = \sum_{i \in I} e_i Y_i$ where the Y_i's are smooth components and I a finite index set. The strategy for proving Theorem A is as follows.

We shall first give a different proof of the local freeness of $R^k f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ which only uses the fact that the residue along the origin has eigenvalues in $[0,1)$ (Theorem 2.2.2). Then we translate the data of the relative log de Rham complex to the $\mathscr{D}$-module side (see §2.3):

Theorem C. *There exists a filtered holonomic $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ whose de Rham complex $\mathrm{DR}_X\mathcal{M}$ with the induced filtration $F_\bullet\mathrm{DR}_X\mathcal{M}$ is isomorphic to $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ with the stupid filtration in the derived category of filtered complex of $\mathbb{C}$-vector spaces. Moreover, there exists an operator $R:(\mathcal{M},F_\bullet\mathcal{M})\to(\mathcal{M},F_{\bullet+1}\mathcal{M})$ whose eigenvalues are in $[0,1)\cap\mathbb{Q}$ such that $\mathrm{DR}_X R$ can be identified with the residue operator on $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ via the above isomorphism.*

Next we will investigate the Jordan block of the operator R. Let $\mathcal{M}_{\geq\alpha}$ (resp. $\mathcal{M}_{>\alpha}$) be the submodule of $\mathcal{M}$ spaned by the generalized eigen-modules $\ker(R-\lambda)^\infty$ for $\lambda\geq\alpha$ (resp. $\lambda>\alpha$). Let $\mathcal{M}_\alpha=\mathcal{M}_{\geq\alpha}/\mathcal{M}_{>\alpha}$. Note that $\mathcal{M}_\alpha$ is canonically isomorphic to $\ker(R-\alpha)^\infty$ and therefore $R_\alpha=R-\alpha$ acts nilpotently on $\mathcal{M}_\alpha$. Using an idea of Saito [Sai90], we filter $\mathcal{M}_\alpha$ by

$$F_\ell\mathcal{M}_\alpha=\frac{F_\ell\mathcal{M}\cap\mathcal{M}_{\geq\alpha}+\mathcal{M}_{>\alpha}}{\mathcal{M}_{>\alpha}},\qquad\text{for }\ell\in\mathbb{Z}.$$

The filtration $F_\bullet\mathcal{M}_\alpha$ is different from the naive one $F_\bullet\mathcal{M}\cap\ker(R-\alpha)^\infty$. The reason why we do not use the naive filtration is that $F_\bullet\mathcal{M}_\alpha$ not only gives the correct weight but is also easy to work out. We prove that any power of the operator R_α is strict with respect to $F_\bullet\mathcal{M}_\alpha$. Namely, for every $\ell\geq0$, we have the relation $R^\ell_\alpha F_\bullet\mathcal{M}_\alpha=F_{\bullet+\ell}\mathcal{M}\cap R^\ell_\alpha\mathcal{M}_\alpha$ (Theorem 2.4.1 for the case Y is reduced and Theorem 2.6.5 for the general case). This implies that the monodromy filtration $W_\bullet\mathcal{M}_\alpha$ and $F_\bullet\mathcal{M}_\alpha$ interacts very well. Note that the monodromy filtration associated to R_α is the same as the one of R_n on $\mathcal{M}_\alpha$, the nilpotent part of R in Jordan-Chevalley decomposition. We have the induced good filtrations

$$F_\bullet W_r\mathcal{M}_\alpha=F_\bullet\mathcal{M}\cap W_r\mathcal{M}_\alpha\quad\text{ and }\quad F_\bullet\mathrm{gr}^W_r\mathcal{M}_\alpha=F_\bullet W_r\mathcal{M}_\alpha/F_\bullet W_{r-1}\mathcal{M}_\alpha.$$

Denote by $\mathcal{P}_{\alpha,\ell}=\ker R^{\ell+1}_\alpha\cap\mathrm{gr}^W_\ell\mathcal{M}_\alpha$ the ℓ-th primitive for $\ell\geq0$, which is isomorphic to

$$\frac{\ker R^{\ell+1}_\alpha}{\ker R^\ell_\alpha+\mathrm{im}\,R_\alpha\cap\ker R^{\ell+1}_\alpha}.$$

We endow it with the induced good filtration $F_\bullet \mathcal{P}_{\alpha,\ell} = \mathrm{im}\,(F_\bullet \mathcal{M} \cap \ker R_\alpha^{\ell+1} \to \mathcal{P}_{\alpha,\ell})$. As a corollary of the strictness of every power of R_α, the Lefschetz decomposition of $\mathrm{gr}^W \mathcal{M}_\alpha$ respects the good filtrations, i.e.

$$F_\bullet \mathrm{gr}_r^W \mathcal{M}_\alpha = \bigoplus_{\ell \geq 0, -\frac{r}{2}} R_\alpha^\ell F_{\bullet - \ell} \mathcal{P}_{\alpha, r+2\ell} \quad \text{for } r \geq 0.$$

See Theorem 2.4.6 for the case Y is reduced and Theorem 2.6.8 for the general case. This corollary suggests that it suffices to study the hypercohomology of each primitive part. The primitive parts will be the source for the pure polarized Hodge structures.

We will construct a sesquilinear pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha} \to \mathfrak{C}_X$ using the Mellin transformation [Sab02], where $\overline{\mathcal{M}_\alpha}$ is the naive conjugation of $\mathcal{M}_\alpha$ and $\mathfrak{C}_X$ is the sheaf of currents. Both $\mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha}$ and $\mathfrak{C}_X$ canonically carry $\mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}}$-module structures where $\mathscr{D}_{\overline{X}}$ denotes the sheaf of anti-holomorphic differential operators and the sesquilinear pairing is just a morphism of $\mathscr{D}_X \otimes_{\mathbb{C}} \overline{\mathscr{D}_X}$-modules. See the MHM project [SS] by Sabbah and Schnell for systematical treatment of complex variation of Hodge structure via sesquilinear pairings. The sesquilinear pairings on $\mathcal{M}_\alpha$ is an analogy of a polarization on a Hodge structure: a complex polarized Hodge structure of weight n can be described as a filtered vector space $(V, F^\bullet)$ with a Hermitian pairing S such that $(-1)^{n-p} S$ is a Hermitian inner product on $F^p \cap G^{n-p}$ where G^{n-p} is the S-orthogonal complement of F^{p+1}. The sesquilinear pairing S_α induces the second filtration on the hypercohomology of $\mathrm{DR}_X \mathcal{M}_\alpha$. We refer to the §2.1.1 for the definition of sesquilinear pairings on $\mathscr{D}$-module.

The operator R_α is self-adjoint with respect to the pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}}_\alpha \to \mathfrak{C}_X$, i,e, $S_\alpha(-, R_\alpha -) = S_\alpha(R_\alpha -, -)$. See §2.5 for the case that Y is reduced §2.7 for the general case. This implies we have an induced pairing on the associated graded modules:

$$S_{\alpha,r} : \mathrm{gr}_r^W \mathcal{M}_\alpha \otimes_{\mathbb{C}} \mathrm{gr}_{-r}^W \mathcal{M}_\alpha \to \mathfrak{C}_X.$$

Then $P_{R_\alpha} S_{\alpha,r} = S_{\alpha,r} \circ (\mathrm{id} \otimes_{\mathbb{C}} R_\alpha^r)$ defines a sesquilinear pairing on the primitive part $\mathcal{P}_{\alpha,r}$.

Theorem D. *The cohomologies of the de Rham complex of* $\mathcal{P}_{\alpha,r}$

$$\bigoplus_{\ell\in\mathbb{Z}} H^{\ell}(X, \mathrm{DR}_X\mathcal{P}_{\alpha,r})$$

together with the filtration induced by $F_\bullet\mathcal{P}_{\alpha,r}$ *and the sesquilinear pairing induced by* $P_{R_\alpha}S_{\alpha,r}$ *determine a polarized Hodge-Lefschetz structure of central weight* $n+r$ *with* $\mathfrak{sl}_2(\mathbb{C})$*-action induced by* $2\pi\sqrt{-1}L$.

A polarized Hodge-Lefschetz structure basically is a direct sum of Hodge structures of different weights preserved by an $\mathfrak{sl}_2(\mathbb{C})$-action. This notion is motivated by the direct sum of all the cohomologies of a compact Kähler manifold. We refer to §2.1.3 for the definition of polarized Hodge-Lefschetz structures. To illustrate the idea of Theorem D, assume for a moment that Y is reduced so the endomorphism R is nilpotent and this implies that $\mathcal{M} = \mathcal{M}_0$. Denote by $Y^J = \bigcap_{i\in J} Y_i$ for any non-empty subset J of I. Let $\tau^J : Y^J \to X$ be the closed embedding and $\tau^{(r+1)} : \tilde{Y}^{(r+1)} = \coprod_{\#J=r+1} Y^J \to X$ be the natural morphism for every $r \geq 0$. For simplicity, suppose $\mathcal{P}_r = \mathcal{P}_{0,r}$. We will show that there exists a filtered isomorphism (Theorem 2.4.7)

$$\phi_r : (\mathcal{P}_r, F_\bullet\mathcal{P}_r) \to \tau_+^{(r+1)}\omega_{\tilde{Y}^{(r+1)}}(-r).$$

Here, the Tate twist of a filtered $\mathscr{D}$-module is $(\mathcal{N}, F_\bullet\mathcal{N})(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$. Moreover, the isomorphism respects the pairing P_RS_r on $\mathcal{P}_r$ (Theorem 2.5.5):

$$P_RS_r(-,-) = \frac{(-1)^r}{(r+1)!}\tau_+^{(r+1)}S_{\tilde{Y}^{(r+1)}}(\phi_r-, \phi_r-),$$

where $S_{\tilde{Y}^{(r+1)}}$ is the standard pairing on $\omega_{\tilde{Y}^{(r+1)}}$. Therefore, the k-th hypercohomology of the de Rham complex $\mathrm{DR}_X\mathcal{P}_r$ is isomorphic to $H^{n-r+k}(\tilde{Y}^{(r+1)}, \mathbb{C})(-r)$ as polarized Hodge structures of weight $n+r+k$. Summing all the hypercohomology groups of $\mathrm{DR}_X\mathcal{P}_r$, we get a polarized Hodge-Lefschetz structure of central weight $n+r$ with $\mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$.

In contrast to the case when Y is reduced, if Y is non-reduced, we shall construct cyclic coverings of Y^J whose degree depends on the multiplicity of Y_j in Y for $j \in J$. Then the primitive part $\mathcal{P}_{\alpha,r}$ will be identified with the eigenspace of the intersection cohomology of the cyclic coverings under the Galois action (Theorem 2.6.13), and the identification also respects the sesquilinear pairing (Theorem 2.7.10). As a direct consequence, we obtain

Theorem E. *Let $V^{\alpha}_{\ell,k} = H^{\ell}(X, \mathrm{gr}^W_k \mathrm{DR}_X \mathcal{M}_{\alpha})$ be the relabelling of the first page of the weight spectral sequence. Then $V^{\alpha} = \bigoplus_{k,\ell\in\mathbb{Z}} V^{\alpha}_{\ell,k}$ is a polarized bigraded Hodge-Lefschetz structure of central weight n with the polarization induced by S_{α} and $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$ and R_{α}. Moreover, the differential d_1 of the first page of weight spectral is a differential of polarized bigraded Hodge-Lefschetz structure.*

By a formal argument of Guillén and Navarro Aznar [GN90], which follows some ideas of Deligne and Saito, we have

Corollary F. *We have the following statements:*

1. *the Hodge spectral sequence degenerates at ${}^{F}E_1$;*

2. *the weight spectral sequence degenerates at ${}^{W}E_2$;*

3. *the α-generalized eigenspace of the bigraded vector space*

$$ {}^{W}E_2 = \bigoplus_{\ell,k\in\mathbb{Z}} \mathrm{gr}^W_{\ell} H^k(Y, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) $$

with respect to R is a polarized bigradged Hodge-Lefschetz structure of central weight n with polarization induced by S_{α} and $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-action induced by $2\pi\sqrt{-1}L$ and R_{α}.

Note that the third statement in the above Corollary is equivalent to the Theorem A; therefore, we finish the proof of Theorem A. See Theorem 2.5.6 and Corollary 2.5.7, when Y is reduced. See Theorem 2.7.11 and Corollary 2.7.12, when Y is allowed to be non-reduced.

1.2 On Hodge filtration and V-filtration

The original Kashiwara and Malgrange's theory of V-filtration is in codimension one. Let $t : X \to \mathbb{A}^1$ be a regular function and Z be the central fiber. For any regular holonomic right $\mathscr{D}$-module $\mathcal{M}$, we can associate it with a functorial filtration $V_\bullet\mathcal{M}$ along Z such that $\mathscr{D}_Z$-module $\mathrm{gr}^V_\alpha\mathcal{M}$ is regular holonomic. Indeed, the nearby and vanishing cycle of $\mathcal{M}$ is given by $\mathrm{gr}^V_\alpha\mathcal{M}$ for $\alpha \in [-1,0]$ and the index α is determined by the eigenvalues of the monodromy. The nearby cycle and vanishing cycle of filtered $\mathscr{D}_X$-modules is an input in the definition of mixed Hodge modules [Sai88; Sai90]. If a filtered $\mathscr{D}_X$-module $(\mathcal{M}, F)$ underlies a mixed Hodge module, then

(V1) $t : F_pV_\alpha\mathcal{M} \to F_pV_{\alpha-1}\mathcal{M}$ is bijective for $\alpha < 0$,

(V2) $\partial_t : F_p\mathrm{gr}^V_{\alpha-1}\mathcal{M} \to F_{p+1}\mathrm{gr}^V_\alpha\mathcal{M}$ is isomorphism for $\alpha > 0$.

We also have two distinguished triangles in the derived category of mixed Hodge modules:

$$i^*\mathcal{M} \longrightarrow \mathrm{gr}^V_{-1}\mathcal{M} \xrightarrow{\partial_t} \mathrm{gr}^V_0\mathcal{M} \longrightarrow i^*\mathcal{M}[1] \qquad \text{and} \qquad i^!\mathcal{M} \longrightarrow \mathrm{gr}^V_0\mathcal{M} \xrightarrow{t} \mathrm{gr}^V_{-1}\mathcal{M} \longrightarrow i^!\mathcal{M}[1]$$

where $i : Z \to X$ is the closed embedding; see also the nice survey [Sch14].

The V-filtration along a higher codimension submanifold is induced by deformation to the normal cone. However, $\mathrm{gr}^V_\alpha\mathcal{M}$ is even not coherent in general for a holonomic $\mathscr{D}$-module $\mathcal{M}$. This is a major difference in the theory of higher codimension. The generalization of (V1), (V2) and the above distinguished triangles to higher codimension was not known and we formulate and prove the generalization in higher codimension in the second part of this thesis.

Now we give the general definition of V-filtration. Let $t = (t_1, t_2, \ldots, t_r) : X \to \mathbb{A}^r$ be a smooth morphism from a smooth variety to the affine r-space $\mathbb{A}^r$ and let Z be the fiber over the origin. Assume there exist global vector fields $\partial_1, \partial_2, \ldots, \partial_r$ on X dual to the 1-forms $dt_1, dt_2, \ldots, dt_r$. We define a $\mathbb{Z}$-indexed filtration on $\mathscr{D}_X$ by

$$V_k\mathscr{D}_X = \{P \in \mathscr{D}_X : P \cdot \mathscr{I}_Z^j \subseteq \mathscr{I}_Z^{j-k} \text{ for all } j\},$$

where $\mathscr{I}_Z$ is the ideal sheaf of Z. Then the V-filtration on a $\mathscr{D}$-module $\mathcal{M}$ along Z is the exhaustive, increasing $\mathbb{Q}$-indexed filtration uniquely characterized by the following:

1. $V_\alpha\mathcal{M}\cdot V_k\mathscr{D}_X \subseteq V_{\alpha+k}\mathcal{M}$ for all $k\in\mathbb{Z}, \alpha\in\mathbb{Q}$,
2. $V_\alpha\mathcal{M}\cdot V_k\mathscr{D}_X = V_{\alpha+k}\mathcal{M}$ for all $k\in\mathbb{Z}_{\le 0}$, $\alpha \ll 0$,
3. each $V_\alpha\mathcal{M}$ is coherent over $V_0\mathscr{D}_X$,
4. the operator $\theta-\alpha$ is nilpotent on $\mathrm{gr}^V_\alpha\mathcal{M}$, where $\theta := \sum_{i=1}^r t_i\partial_i$ is the Eular vector field.

We generalize the above properties of V-filtration in codimension 1 to the V-filtration along subvarieties of arbitrary codimension and the statement is formulated by certain Koszul-type complexes. For any filtered regular holonomic $\mathscr{D}_X$-module $\mathcal{M}$, define Koszul-type complexes

$$A_\alpha(\mathcal{M}) = \left\{(V_\alpha\mathcal{M},F)\xrightarrow{t}\bigoplus_{i=1}^r(V_{\alpha-1}\mathcal{M},F)\xrightarrow{t}\cdots\xrightarrow{t}(V_{\alpha-r}\mathcal{M},F)\right\}$$

placed in degrees $0,1,\ldots,r$,

$$B_\alpha(\mathcal{M}) = \left\{\left(\mathrm{gr}^V_\alpha\mathcal{M},F\right)\xrightarrow{t}\bigoplus_{i=1}^r\left(\mathrm{gr}^V_{\alpha-1}\mathcal{M},F\right)\xrightarrow{t}\cdots\xrightarrow{t}\left(\mathrm{gr}^V_{\alpha-r}\mathcal{M},F\right)\right\}$$

as the quotient $A_\alpha/A_{>\alpha}$ and

$$C_\alpha(\mathcal{M}) = \left\{(\mathrm{gr}^V_{\alpha-r}\mathcal{M},F[r])\xrightarrow{\partial_t}\bigoplus_{i=1}^r(\mathrm{gr}^V_{\alpha-r+1}\mathcal{M},F[r-1])\xrightarrow{\partial_t}\cdots\xrightarrow{\partial_t}(\mathrm{gr}^V_\alpha\mathcal{M},F)\right\}$$

in degrees $-r,-r+1,\ldots,0$, where $V_\bullet\mathcal{M}$ is the V-filtration along Z and $F[i]_k = F_{k-i}$. Our first theorem in this direction is a generalization of (V1) and (V2):

Theorem G. *If the filtered $\mathscr{D}_X$-module $(\mathcal{M},F)$ underlies a mixed Hodge module, then the Koszul-like complexes*

1. *the complex $A_\alpha(\mathcal{M})$ is filtered exact if $\alpha<0$;*
2. *the complex $C_\alpha(\mathcal{M})$ is filtered exact if $\alpha>0$.*

As a very special case of Theorem G, we give a Hodge-theoretic proof of a theorem of Skoda. See [Laz04] for the background on the multiplier ideal sheaves and a proof of Skoda's theorem. Let $f_1, f_2, \ldots, f_r$ be the generators of a coherent ideal $\mathfrak{a}$ on X and let $\iota : X \to X \times \mathbb{A}^r$ be the graph of $f_1, \ldots, f_r$. Then by [BMS06, Theorem 1], the $\mathscr{O}_X$-module $F_r V^{c+\varepsilon}\iota_+\mathscr{O}_X$ is the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^c)$ for $\varepsilon > 0$ sufficient small where $V^\bullet\iota_+\mathscr{O}_X$ is the V-filtration along $X \times \{0\}$. The the exactness of $A^{c-n+\varepsilon}(\iota_+\mathscr{O}_X)$ when $c \geq n$ by Theorem G gives:

Corollary H (Skoda)**.** *Let $\mathfrak{a}$ be a coherent ideal of $\mathscr{O}_X$ and $\mathcal{J}(X, \mathfrak{a}^c)$ be the multiplier ideal. Then we have*

$$\mathcal{J}(X, \mathfrak{a}^c) = \mathfrak{a}\mathcal{J}(X, \mathfrak{a}^{c-1})$$

for any $c \geq \dim X$.

To simplify the notation, denote $B(\mathcal{M}) := B_0(\mathcal{M})$ and $C(\mathcal{M}) := C_0(\mathcal{M})$. The second main theorem says that we can give a comparison between $i^!\mathcal{M}$ (resp. $i^*\mathcal{M}$) in the derived category of mixed Hoge modules and $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) where $i : Z \to X$ is the embedding of the central fiber of $f : X \to \mathbb{A}^r$.

Theorem I. *Let $(\mathcal{M}, F, L, \mathcal{K})$ be a mixed Hodge module where F is the Hodge filtration, L is the weight filtration and $\mathcal{K}$ is the $\mathbb{Q}$-structure of the $\mathscr{D}_X$-module $\mathcal{M}$ i.e. $\mathrm{DR}_X\mathcal{M} \simeq \mathcal{K} \otimes_{\mathbb{C}} \mathbb{Q}$. Then we have:*

1. *the complexes $B(\mathcal{M})$ and $C(\mathcal{M})$ together with the filtrations W induced by the relative monodromy filtration $W = W(\theta - \alpha, \mathrm{gr}^V_\alpha L_\bullet\mathcal{M})$ on $\mathrm{gr}^V_\alpha\mathcal{M}$ are mixed Hodge complexes, i.e. the $\mathscr{D}_Z$-modules $\mathcal{H}^\ell\mathrm{gr}^W_k B(\mathcal{M})$ and $\mathcal{H}^\ell\mathrm{gr}^W_k C(\mathcal{M})$ are polarizable Hodge modules of weight $k + \ell$ for any k, ℓ and*

$$\mathrm{gr}^W_k B(\mathcal{M}) \simeq \bigoplus_\ell \mathcal{H}^\ell\mathrm{gr}^W_k B(\mathcal{M})[-\ell] \quad \text{and} \quad \mathrm{gr}^W_k C(\mathcal{M}) \simeq \bigoplus_\ell \mathcal{H}^\ell\mathrm{gr}^W_k C(\mathcal{M})[-\ell]$$

   in the derived category of filtered $\mathscr{D}$-modules;

2. *the complex $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) is isomorphic to $(i^!\mathcal{M}, F)$ (resp. $(i^*\mathcal{M}, F)$) in the derived category of filtered $\mathscr{D}$-modules with $\mathbb{Q}$-structures;*

3. *moreover,*

$$\mathrm{gr}^W_k \mathcal{H}^\ell B(\mathcal{M}) \simeq \mathrm{gr}^W_{k+\ell}\mathcal{H}^\ell i^! \mathcal{M} \quad \textit{and} \quad \mathrm{gr}^W_k \mathcal{H}^{-\ell} C(\mathcal{M}) \simeq \mathrm{gr}^W_{k-\ell}\mathcal{H}^{-\ell} i^* \mathcal{M}$$

as polarizable Hodge modules.

The reason why we do not get the distinguished triangles in the derived category of mixed Hodge modules is that we directly use the monodromy filtrations relative to $L\mathrm{gr}^V_0\mathcal{M}$ without the shift in Saito's definition of vanishing cycles.

Theorem I simplifies, in a way, the calculation of the functors $i^!$ and i^* of mixed Hodge modules [Sai90]. For example, if i is the embedding of the origin in $\mathbb{A}^2$, Saito's definition of $i^!$ is

$$i_+ i^! \mathcal{M} = \{\mathcal{M} \to \mathcal{M}(*D_1) \oplus \mathcal{M}(*D_2) \to \mathcal{M}(*D_1 + D_2)\}$$

placed in degrees $0, 1, 2$ where D_1, D_2 are the two coordinate axes. The weight filtration of $\mathcal{M}(*D_i)$ is uniquely determined by some gluing conditions on the weight filtration on $\mathcal{M}$ and the relative monodromy filtration on the unipotent vanishing cycle of $\mathcal{M}$ along D_i. Theorem I says one can bypass the gluing construction of the weight filtration on $\mathcal{M}(*D_i)$ by looking at the V-filtration directly.

To prove Theorem G, we first do the case when $(\mathcal{M}, F_\bullet)$ underlies a polarizable pure Hodge module. Because pure Hodge modules have strict support decomposition, we are in two situations:

(a) the support of $\mathcal{M}$ is contained in Z;

(b) there is no sub-Hodge module of $\mathcal{M}$ whose support is contained in Z.

Case (a) will directly follow from the definition. For case (b), we will pass to the blow-up and reduce the problem to the codimension one case. Let $\pi : \hat{X} \to X$ be the blow-up of Z and E be the exceptional divisor. Let $(\hat{\mathcal{M}}, F_\bullet\hat{\mathcal{M}})$ be the minimal extension of $(\mathcal{M}, F_\bullet\mathcal{M})|_{X\setminus Z}$ along E, which also underlies a pure Hodge module by the structure theorem of Hodge modules [Sch14]. By the direct image theorem of Hodge modules, $(\mathcal{M}, F_\bullet\mathcal{M})$ is a direct summand of $\pi_+(\hat{\mathcal{M}}, F_\bullet\hat{\mathcal{M}})$. Therefore, it suffices to prove the statement for $\pi_+(\hat{\mathcal{M}}, F_\bullet\hat{\mathcal{M}})$. Then we factor $\pi : \hat{X} \to X$ into the graph embedding $i_\pi : \hat{X} \to \hat{X} \times X$ and the second projection $p : \hat{X} \times X \to X$ and study the direct images of $(\hat{\mathcal{M}}, F_\bullet\mathcal{M})$ under these two morphisms. The graph embedding case has no homological algebra involved and in the case of the projection, we use the bistrctness proved by Budur, Mustaţă and Saito [BMS06] and Hard Lefschetz [Sai88, p. 2.14] on the direct images.

The strategy of proof for the pure case does not work for mixed Hodge modules because there is no decomposition theorem for mixed Hodge modules. Instead, we use deformation to the normal cone to get the compatibility among the Hodge filtration, V-filtration, and weight filtration. From the compatibility, we reduce the proof to the pure case.

As for the proof of Theorem I, we first deal with the case when $(\mathcal{M}, F)$ underlies a polarizable Hodge module as we did in the proof of the pure case for Theorem G. In this case, we heavily use the semisimplicity of polarizable pure Hodge modules. To do the mixed case we need a theorem of Deligne [Del93] in his personal letter to Cattani and Kaplan, which roughly states that there exists a unique functorial splitting of the associated graded of the relative monodromy filtration. The proof reduces to the pure case by Deligne's Theorem.

Chapter 2

Limits of Hodge structures

2.1 Preliminaries

2.1.1 Filtered $\mathscr{D}$-modules with sesquilinear pairings

We will work with right $\mathscr{D}$-modules unless further specified. Let Z be a complex manifold of dimension n and denote by Ω_Z^p the sheaf of holomorphic p-forms and $\mathscr{T}_Z$ the sheaf of holomorphic tangent vectors fields. For a filtered $\mathscr{D}_Z$-module we mean a pair $(\mathcal{N}, F_\bullet\mathcal{N})$ where $\mathcal{N}$ is a coherent $\mathscr{D}_Z$-module and $F_\bullet\mathcal{N}$ is a good filtration. Occasionally we will abuse notations and say $\mathcal{N}$ also denotes the filtered $\mathscr{D}_Z$-module if the filtration is clear. Denote by $\mathrm{gr}^F\mathscr{D}_Z = \bigoplus_{\ell\in Z}\mathrm{gr}_\ell^F\mathscr{D}_X$ the associated graded algebra and $\mathrm{gr}^F\mathcal{N} = \bigoplus_{\ell\in\mathbb{Z}}\mathrm{gr}_\ell^F\mathcal{N}$ the associated graded module. Note that $\mathrm{gr}^F\mathcal{N}$ is a coherent $\mathrm{gr}^F\mathscr{D}_Z$-module. Let $T^*Z = \mathrm{Spec}_Z\mathrm{gr}^F\mathscr{D}_X$ be the algebraic cotangent bundle and T_V^*Z the geometric conormal bundle of a subvariety V in Z. The *characteristic variety* of $\mathcal{N}$ is the support of $\mathrm{gr}^F\mathcal{N}$ on T^*Z and is denoted by $char(\mathcal{N})$. The *characteristic cycle* of $\mathcal{N}$ is the cycle associated to the coherent sheaf $\mathrm{gr}^F\mathcal{N}$ on T^*Z and is denoted by $cc(\mathcal{N})$. Neither the characteristic variety nor the characteristic cycle depend on the choice of the filtration [HTT08]. For example, the canonical bundle ω_Z

is naturally a holonomic $\mathscr{D}_Z$-module with action

$$\alpha.\xi = -d(\xi \lrcorner \alpha)$$

for local sections $\xi \in \mathscr{T}_Z$ and $\alpha \in \omega_Z$. It also naturally has a good filtration

$$F_\ell \omega_Z = \begin{cases} \omega_Z, & \ell \geq -n; \\ 0, & \ell < -n. \end{cases} \tag{2.1.1}$$

Then one can compute $cc(\omega_Z) = [T^*_Z Z]$ which is the cycle of the zero section of the cotangent bundle. We call $\mathcal{N}$ a *holonomic* $\mathscr{D}_Z$-module if $\dim char(\mathcal{N}) = n$. See more details in [HTT08]. A *Tate twist* of filtered $\mathscr{D}_Z$-module is defined to be $\mathcal{N}(-r) = (\mathcal{N}, F_{\bullet+r}\mathcal{N})$ for any $r \in \mathbb{Z}$.

Denote by $\mathbf{D}^b(Z, \mathbb{C})$ the bounded derived category of complexes with values in finite dimensional $\mathbb{C}$-vector spaces and $\mathbf{D}^b(Z, \mathscr{D})$ the bounded derived category of $\mathscr{D}_Z$-modules. Denote by $\mathbf{D}^b_h(Z, \mathscr{D})$ the full subcategory of $\mathbf{D}^b(Z, \mathscr{D})$ whose objects are complexes with holonomic cohomologies. For a morphism $f : Z \to W$ between complex manifolds, denote by $Rf_*, Rf_! : \mathbf{D}^b(Z, \mathbb{C}) \to \mathbf{D}^b(W, \mathbb{C})$ the derived pushforward and proper pushforward functors respectively and $R^k f_*, R^k f_!$ the k-th cohomology functors respectively. For any $\mathcal{N}^\bullet \in \mathbf{D}^b(Z, \mathscr{D})$, the pushforward functor and the proper pushfoward functor $f_+, f_\dagger : \mathbf{D}^b(Z, \mathscr{D}) \to \mathbf{D}^b(W, \mathscr{D})$ are by definition, respectively

$$f_+\mathcal{N}^\bullet = Rf_*\left(\mathcal{N}^\bullet \overset{L}{\underset{\mathscr{D}_Z}{\otimes}} \mathscr{D}_{Z\to W}\right) \quad \text{and} \quad f_\dagger\mathcal{N}^\bullet = Rf_!\left(\mathcal{N}^\bullet \overset{L}{\underset{\mathscr{D}_Z}{\otimes}} \mathscr{D}_{Z\to W}\right),$$

where $\mathscr{D}_{Z\to W} = f^*\mathscr{D}_W$ is the transfer module. In fact, the functor $f_\dagger$ preserves the holonomicty, i.e., $f_\dagger : \mathbf{D}^b_h(Z, \mathscr{D}) \to \mathbf{D}^b_h(W, \mathscr{D})$ (see [HTT08]). Of course if f is proper or proper on the support of $\mathcal{N}$ then $f_+ = f_\dagger$. The *de Rham complex* of $\mathcal{N}$ is

$$\mathrm{DR}_Z\mathcal{N} =_{\mathrm{def}} \mathcal{N} \otimes \bigwedge^{-\bullet} \mathscr{T}_Z = \{\mathcal{N} \otimes \bigwedge^{n} \mathscr{T}_Z\mathcal{N} \to \mathcal{N} \otimes \bigwedge^{n-1} \mathscr{T}_Z \to \cdots \to \mathcal{N}\}$$

with $\mathcal{N}$ is in degree 0. If without further indication, tensor products are always taken over $\mathscr{O}$-modules. Some authors also call it Spencer complex. The de Rham complex of ω_Z

$$\omega_Z \otimes \bigwedge^{-\bullet} \mathscr{T}_Z = \{\omega_Z \otimes \bigwedge^{n} \mathscr{T}_Z\omega_Z \to \omega_Z \otimes \bigwedge^{n-1} \mathscr{T}_Z \to \cdots \to \omega_Z\}$$

is isomorphic to the usual de Rham complex $\mathrm{DR}_Z\mathscr{O}_Z = \Omega_Z^{n+\bullet}$ of Z under the isomorphisms

$$\omega_Z \otimes \bigwedge^p \mathscr{T}_Z \to \Omega_Z^{n-p},\ \ \omega \otimes \partial_J \mapsto (-1)^{n-j_1+\cdots+n-j_p} dz_{\bar{J}}, \tag{2.1.2}$$

where ∂_J is a local section of $\bigwedge^p \mathscr{T}_Z$, J is ordered index set and $\bar{J}$ is the complement with the natural ordering, and $\omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. If $F_\bullet\mathcal{N}$ is a good filtration, the de Rham complex is also filtered:

$$F_\ell \mathrm{DR}_Z\mathcal{N} = F_{\ell+\bullet}\mathcal{N} \otimes \bigwedge^{-\bullet} \mathscr{T}_Z = \left\{ F_{\ell-n}\mathcal{N} \otimes \bigwedge^{n} \mathscr{T}_Z\mathcal{N} \to F_{\ell-n+1}\mathcal{N} \otimes \bigwedge^{n-1} \mathscr{T}_Z \to \cdots \to F_\ell\mathcal{N} \right\}.$$

The direct image functor and the de Rham functor commute : $Rf_! \circ \mathrm{DR}_Z = \mathrm{DR}_W \circ f_\dagger$ [MS, Corollary 4.4.4].

A *sesquilinear pairing* S on $\mathscr{D}_Z$-module $\mathcal{N}$ is a $\mathscr{D}_{Z,\overline{Z}}$-module morphism $S : \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_Z$. Here, $\mathscr{D}_{Z,\overline{Z}} = \mathscr{D}_Z \otimes_{\mathbb{C}} \mathscr{D}_{\overline{Z}}$ for $\overline{\mathscr{D}_Z}$ is the sheaf antiholomorphic differential operators, $\overline{\mathcal{N}}$ is the stupid conjugate of $\mathcal{N}$ as a $\overline{\mathscr{D}}_Z$-module and $\mathfrak{C}_Z$ is the sheaf of currents on Z with natural $\mathscr{D}_{Z,\overline{Z}}$-module structure. We have the proper pushforward functor similarly as above on $\mathscr{D}_{Z,\overline{Z}}$-modules and also call it $f_\dagger$:

$$f_\dagger(-) =_{\mathrm{def}} Rf_!(- \overset{L}{\underset{\mathscr{D}_{Z,\overline{Z}}}{\otimes}} \mathscr{D}_{Z,\overline{Z}\to W,\overline{W}}),$$

where the transfer module $\mathscr{D}_{Z,\overline{Z}\to W,\overline{W}} =_{\mathrm{def}} f^*\mathscr{D}_{W,\overline{W}}$. Because of the natural morphism $f_\dagger\mathfrak{C}_Z \to \mathfrak{C}_W$, we can pushforward the sesquilinear pairing to get

$$\mathscr{H}^0 f_\dagger S_k : \mathscr{H}^k f_\dagger\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathscr{H}^{-k} f_\dagger\mathcal{N}} \to \mathscr{H}^0 f_\dagger\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_W.$$

If f is a closed embedding then $f_+S : f_+\mathcal{N} \otimes_{\mathbb{C}} f_+\overline{\mathcal{N}} \to \mathfrak{C}_W$. If W is a point, then we have an induced pairing on the complex

$$f_\dagger S : \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z \simeq \mathbb{C}[2n],$$

where $\mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \simeq \mathrm{DR}_Z\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathrm{DR}_Z\mathcal{N}}$. Taking cohomology at 0-th degree yields, for each $k \in \mathbb{Z}$,

$$H_c^k(Z, \mathrm{DR}_Z\mathcal{N}) \otimes H_c^{-k}(Z, \overline{\mathrm{DR}_Z\mathcal{N}}) \to H_c^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}) \to H_c^{2n}(Z, \mathbb{C}) \simeq \mathbb{C}. \tag{2.1.3}$$

Example 2.1.1. The $\mathscr{D}_Z$-module ω_Z carries a natural pairing $S_Z : \omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} \to \mathfrak{C}_Z$,

$$\langle S_Z(m', m''), \eta \rangle = \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z \eta m' \wedge \overline{m''}, \tag{2.1.4}$$

for m', m'' local sections of ω_Z, η a test function on Z and $\varepsilon(k) = (-1)^{\frac{k(k-1)}{2}}$. The coefficient $\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n}$ in the definition is chosen so that $\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} m \wedge \overline{m} = |m|^2$ is a positive current for any local section m of ω_Z and elimination the choice of orenation (see more details in §2.1.3). The pairing $S_Z : \omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} \to \mathfrak{C}_Z$ yields a collection of pairings

$$H^k_c(Z, \mathrm{DR}_Z \omega_Z) \otimes_{\mathbb{C}} \overline{H^{-k}_c(Z, \mathrm{DR}_Z \omega_Z)} \to \mathbb{C}.$$

2.1.2 Logarithmic connections

If $D = \sum a_i D_i$ is a simple normal crossing divisor on Z for $a_i \geq 0$, denote by $\Omega_Z(\log D)$ the sheaf of meromorphic differential 1-forms with logarithmic poles along $D_{\mathrm{red}} = \sum D_i$ and denote by $\Omega^p_Z(\log D) = \bigwedge^p \Omega_Z(\log D)$ the meromophic p-forms with logarithmic pole along D. Each $\Omega^p_Z(\log D)$ is a locally free $\mathscr{O}_Z$-module.

In our convention, the *de Rham complex* of Z is $\mathrm{DR}_Z \mathscr{O}_Z$

$$\Omega_Z^{\bullet+n} = \{\mathscr{O}_Z \to \Omega_Z \to \Omega_Z^2 \to \cdots \to \Omega_Z^n\}[n].$$

The *log de Rham complex* is

$$\Omega_Z^{\bullet+n}(\log D) = \{\mathscr{O}_Z \to \Omega_Z(\log D) \to \Omega_Z^2(\log D) \to \cdots \to \Omega_Z^n(\log D)\}[n].$$

We will follow the Koszul sign rule: for a chain complex $C^\bullet$ with differential d, the shifted complex $C^{\bullet+n} = C^\bullet[n]$ equipped with differential $(-1)^n d$. We define residue along D_i by (see [EV92, p. 2.5])

$$\mathrm{Res}_{D_i} : \Omega_Z^{\bullet+n}(\log D) \to \Omega_{D_i}^{\bullet+\dim D_i}(\log(D - D_i)|_{D_i}), \ \frac{dz_i}{z_i} \wedge \alpha \mapsto \alpha|_{D_i},$$

where z_i is the local defining equation of D_i and $\frac{dz_i}{z_i} \wedge \alpha$ is a local section of $\Omega_Z^{\bullet+n}(\log D)$. It factors through

$$\Omega_Z^{\bullet+n}(\log D)|_{D_i} \to \Omega_{D_i}^{\bullet+\dim D_i}(\log(D - D_i)|_{D_i}).$$

By abuse of notations, we still call the above morphism Res_{D_i}. Let $D^J = \cap_{j\in J} D^J$ and $D_J = \sum_{j\in J} D_j$. Then we have a collection of residue maps, by choosing an order on the indices and successively applying Res_{D_j} for $j \in J$,

$$\mathrm{Res}_{D^J} : \Omega_Z^{\bullet+n}(\log D) \to \Omega_{D^J}^{\bullet+\dim D^J}(\log(D - D_J)|_{D^J}).$$

A *log connection* ∇ with poles along D on a coherent $\mathscr{O}_Z$-module $\mathcal{F}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{F} \to \Omega_Z(\log D) \otimes \mathcal{F}$ satisfying the Leibniz rule $\nabla f s = df \otimes s + f\nabla s$ for f local section of $\mathscr{O}_Z$ and s local section of $\mathcal{F}$. One can extend standardly ∇ to a complex

$$\mathcal{F} \xrightarrow{\nabla} \Omega_Z(\log D) \otimes \mathcal{F} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega_Z^n(\log D) \otimes \mathcal{F}.$$

If the above is a chain complex, i.e., $\nabla^2 = 0$ we say $(\mathcal{F}, \nabla)$ is an *integrable* log connection. For any integrable log connection $\nabla : \mathcal{F} \to \Omega_Z(\log D) \otimes \mathcal{F}$, we call the morphism $\mathrm{Res}_{D_i}\nabla : \mathcal{F} \to \mathcal{F}|_{D_i}$ induced by $\mathrm{Res}_{D_i} : \Omega_Z(\log D) \to \mathscr{O}_{D_i}$ its *residue* along D_i. Note that Res_{D_i} is $\mathscr{O}_Z$-linear and factors through again $\mathcal{F}|_{D_i} \to \mathcal{F}|_{D_i}$.

An integrable log connection is same as a left $\mathscr{D}_Z(\log D)$-module, where $\mathscr{D}_Z(\log D)$ is the sub-algebra of $\mathscr{D}_Z$ generated locally by the differential operators P such that $P \cdot \mathscr{I}_D \subset \mathscr{I}_D$. Here, we denote by $\mathscr{I}_D$ the ideal sheaf of the normal crossing divisor D. Then we can extend the definition of residues of a log connection as follows. The sheaf $\mathscr{O}_{D_i} = \mathscr{O}_Z / \mathscr{I}_{D_i}$ naturally has a left $\mathscr{D}_Z(\log D)$-module structure because $\mathscr{I}_{D_i}$ is also stable under by the $\mathscr{D}_Z(\log D)$-action by the naive reason. Let $\mathcal{F}^\bullet$ be a complex of integrable log connections. Then the complex

$$\mathcal{F}^\bullet \overset{L}{\underset{\mathscr{O}_Z}{\otimes}} \mathscr{O}_{D_i}$$

is a complex of $\mathscr{D}_Z(\log D)$-modules because taking tensor products over $\mathscr{O}_Z$ is closed in the category of $\mathscr{D}_Z(\log D)$-modules and one can resolve either $\mathcal{F}^\bullet$ or $\mathscr{O}_{D_i}$ using locally $\mathscr{D}_Z(\log D)$-free resolutions. The ℓ-th cohomology $\mathscr{H}^\ell(\mathcal{F}^\bullet \otimes^L \mathscr{O}_{D_i})$ is indeed $\mathscr{O}_{D_i}$-module equipped with an integrable log connection. The residue of of this log connection is $\mathscr{O}_{D_i}$-linear and is called the the ℓ-th *residue* of the complex $\mathcal{F}^\bullet$.

As in the case of $\mathscr{D}$-module, the sheaf $\omega_Z(\log D) = \Omega^n_Z(\log D)$ carries a canonical right $\mathscr{D}_Z(\log D)$-module structure and we have the left to right transformation $\mathcal{F} \mapsto \omega_Z(\log D) \otimes \mathcal{F}$ for any left $\mathscr{D}_Z(\log D)$-module $\mathcal{F}$. Moreover, we have the following analog

Theorem 2.1.2. *The log de Rham complex of* $\mathscr{D}_Z(\log D)$

$$\{\mathscr{D}_Z(\log D) \to \Omega_Z(\log D) \otimes \mathscr{D}_Z(\log D) \to \cdots \to \Omega^n_Z(\log D) \otimes \mathscr{D}_Z(\log D)\}\,[n]$$

is a resolution of $\omega_Z(\log D)$ *as right* $\mathscr{D}_Z(\log D)$*-modules. The Spencer complex of* $\mathscr{D}_Z(\log D)$

$$\mathscr{D}_Z(\log D) \otimes \bigwedge^{n} \mathscr{T}_Z(\log D) \to \mathscr{D}_Z(\log D) \otimes \bigwedge^{n-1} \mathscr{T}_Z(\log D) \to \cdots \to \mathscr{D}_Z(\log D)$$

is a resolution of $\mathscr{O}_Z$ *as left* $\mathscr{D}_Z(\log D)$*-modules.*

For any integrable log connection $\mathcal{F}$, it induces a complex of right $\mathscr{D}_Z$-modules,

$$\{\mathcal{F} \otimes \mathscr{D}_Z \to \Omega_Z(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z \to \cdots \to \Omega^n_Z(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z\}[n]. \tag{2.1.5}$$

In fact, it is nothing but the log de Rham complex of $\mathcal{F} \otimes \mathscr{D}_Z$ as a left $\mathscr{D}_Z(\log D)$-module.

Lemma 2.1.3. *The log de Rham complex of* $\mathcal{F} \otimes \mathscr{D}_Z$ *is a* $\mathscr{D}_Z$*-module resolution of*

$$\omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \mathscr{D}_Z.$$

Proof. By the above theorem, we have

$$\begin{aligned}\omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \mathscr{D}_Z &\simeq \omega_Z(\log D) \otimes \mathcal{F} \underset{\mathscr{D}_Z(\log D)}{\otimes} \left(\mathscr{D}_Z(\log D) \otimes \bigwedge^{-\bullet} \mathscr{T}_Z(\log D)\right) \otimes \mathscr{D}_Z \\ &= \omega_Z(\log D) \otimes \mathcal{F} \otimes \bigwedge^{-\bullet} \mathscr{T}_Z(\log D) \otimes \mathscr{D}_Z \\ &\simeq \Omega^{\bullet+n}_Z(\log D) \otimes \mathcal{F} \otimes \mathscr{D}_Z.\end{aligned}$$

The last isomorphism follows from that the contraction $\omega_Z(\log D) \otimes \bigwedge^{-\bullet} \mathscr{T}_Z(\log D) \simeq \Omega^{\bullet+n}_Z(\log D)$.

□

Example 2.1.4. We will use the following fact: the complex of right $\mathscr{D}_Z$-modules

$$\{\mathscr{D}_Z \to \Omega_Z(\log D) \otimes \mathscr{D}_Z \to \cdots \to \Omega^n_Z(\log D) \otimes \mathscr{D}_Z\}[n]$$

is a filtered resolution of $\omega_Z(*D) = \cup_{k\in\mathbb{Z}}\omega_Z(kD)$, equipped the induced filtration by $\Omega_Z^{n+\bullet}(\log D)\otimes F_{\ell+n+\bullet}\mathscr{D}_Z$. In fact, it is well-known that the inclusion $\Omega_Z^{n+\bullet}(\log D) \to \Omega_Z^{n+\bullet}(*D)$ is a filtered quasi-isomorphism [Del71b]. The inclusion extends to a filtered quasi-isomorphism $\Omega_Z^{n+\bullet}(\log D)\otimes\mathscr{D}_Z \to \Omega_Z^{n+\bullet}(*D)\otimes\mathscr{D}_Z$. Since $\Omega_Z^{n+\bullet}(*D)\otimes\mathscr{D}_Z$ is a filtered resolution of $\omega_Z(*D)$, we conclude the proof. It follows that, for $f: Z\to W$,

$$f_\dagger\omega_Z(*D) = Rf_!(\omega_Z(*D)\otimes^L_{\mathscr{D}_Z}\mathscr{D}_{Z\to W}) = Rf_!\Omega_Z^{n+\bullet}(\log D)\otimes\mathscr{D}_W.$$

In particular, if f is a closed embedding then $f_! = f_+$ is right exact and $f_\dagger = \mathscr{H}^0 f_\dagger$, which means

$$\{\mathscr{D}_W \to f_+\Omega_Z(\log D)\otimes\mathscr{D}_W \to \cdots \to f_+\Omega_Z^n(\log D)\otimes\mathscr{D}_W\}[n]$$

is a resolution of $f_\dagger\omega_Z(*D)$. We put the induced filtration to make it a filtered resolution and denote by

$$f_\dagger(\omega_Z(*D), F_\bullet\omega_Z(*D)) = (f_\dagger\omega_Z(*D), F_\bullet f_\dagger\omega_Z(*D)),$$

or for simplicity just $f_\dagger\omega_Z(*D)$.

The $\mathscr{D}_Z$-module looks like $\mathcal{L}\otimes\mathscr{D}_Z$ for $\mathcal{L}$ is a $\mathscr{O}_Z$-module is called *induced* $\mathscr{D}_Z$-module. For example, we have seen $\Omega_Z^{\dim Z+\bullet}\otimes\mathscr{D}_Z$ and $\Omega(\log D)_Z^{\dim Z+\bullet}\otimes\mathscr{D}_Z$ are complexes of induced $\mathscr{D}_Z$-modules.

2.1.3 Polarized Hodge-Lefschetz structures

The goal of this subsection is to introduce polarized bigraded Hodge-Lefschetz structures. The prototype of polarized Hodge-Lefschetz structures one should keep in mind is the graded vector space consisting of cohomologies of a compact Kähler manifold. Polarized bigraded Hodge-Lefschetz structures are the degenerations of polarized Hodge-Lefschetz structures. We begin with the convention on Hodge structures and we only consider complex Hodge structures.

A *Hodge structure* of *weight* n is a finite dimensional vector space V with two decreasing filtrations $F^\bullet$ and $G^\bullet$ satisfying

$$V = F^p \oplus G^{n+1-p},$$

for each $p \in \mathbb{Z}$. Let $V^{p,q} = F^p \cap G^q$ for $p + q = n$. Then the above definition is equivalent to

$$V = \bigoplus_{p+q=n} V^{p,q}.$$

A morphism of Hodge structures is just a morphism of vector spaces such that it preserves the two filtrations. A *polarization* on the Hodge structure $(V, F^\bullet, G^\bullet)$ is a non-degenerated hermitian pairing $S : V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that

1. F^p is orthogonal to G^{n+1-p} with respect to S for every $p \in \mathbb{Z}$;
2. $(-1)^q S(-,-)$ is hermitian inner product on $V^{p,q}$.

Remark 2.1.5. A polarized Hodge structure of weight n is completely determined by the triple $(V, F_\bullet V, S)$ because

$$G^{n+1-p}V = \{a \in V : S(a,b) = 0 \text{ for all b in } F^pV\} = \overline{F^pV^{\perp_S}}.$$

We will also call the triple $(V, F_\bullet V, S)$ a polarized Hodge structure.

Remark 2.1.6. A Tate twist $(V, F^\bullet, S)(r)$ on a polarized Hodge structure $(V, F^\bullet, S)$ is the triple $(V, F^{\bullet+r}, (-1)^r S)$, for any integer r.

Now let us move on to the geometric case. It is well-known that the k-th cohomology group of a compact Kähler manifold Z has Hodge decomposition

$$H^k(Z, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(Z)$$

and thus it is a Hodge structure of weight k. Fix a choice of $\sqrt{-1}$. Let Z be a compact Kähler manifold of dimension n, and let h be any Kähler metric on Z. We denote the Kähler form by $\omega = -\mathrm{Im}\, h \in A^2(Z, \mathbb{R})$ and denote its cohomology class by $[\omega] \in H^2(Z, \mathbb{R})$; note that this depends on the choice of $\sqrt{-1}$ through the function $\mathrm{Im}:\mathbb{C}\to\mathbb{R}$. The choice of $\sqrt{-1}$ endows the two-dimensional real vector space $\mathbb{C}$ with an orientation on Z. The induced orientation on Z has the property that

$$\int_Z \frac{\omega^n}{n!} = \mathrm{vol}(Z) > 0.$$

The integral also depends on the orientation, hence on the choice of $\sqrt{-1}$. To remove the dependence, instead of the usual integral, we should use

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_Z : A^{2n}(Z,\mathbb{C}) \to \mathbb{C}.$$

Of course we still have

$$\frac{1}{(2\pi\sqrt{-1})^n}\int_Z \frac{(2\pi\sqrt{-1}\omega)^n}{n!} = \mathrm{vol}(Z).$$

Let $L = [w]\wedge$ be the Lefschetz operator for a Kähler class $[w]$. Then for $k \le \dim Z$ the primitive part

$$P_L H^k(Z,\mathbb{C}) =_{\mathrm{def}} \ker L^{\dim Z - k} \cap H^k(X,\mathbb{C})$$

is a polarized Hodge structure of weight k with the polarization

$$S(a,b) = \frac{\varepsilon(n-k+1)}{(2\pi\sqrt{-1})^n}\int_Z (2\pi\sqrt{-1}L)^{n-k} a \wedge \bar{b},$$

for $a, b \in P_L H^k(Z,\mathbb{C})$ because of the Hodge-Riemman bilinear relation.

If we consider the cohomology groups all together, we will get the Hodge-Lefschetz strcuture of central weight n. Denote by $(\mathsf{X},\mathsf{Y},\mathsf{H})$ the $\mathfrak{sl}_2(\mathbb{C})$-triple, i.e.,

$$[\mathsf{X},\mathsf{Y}] = \mathsf{H},\ [\mathsf{H},\mathsf{X}] = 2\mathsf{X},\ [\mathsf{H},\mathsf{Y}] = -2\mathsf{Y}.$$

In the Lie group $\mathrm{SL}_2(\mathbb{C})$, we have the Weil element $\mathsf{w} = e^{\mathsf{X}}e^{-\mathsf{Y}}e^{\mathsf{X}}$ with the property that $\mathsf{w}^{-1} = -\mathsf{w}$, and under the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on its Lie algebra, one has the identities

$$\mathsf{w}\mathsf{H}\mathsf{w}^{-1} = -\mathsf{H}, \quad \mathsf{w}\mathsf{X}\mathsf{w}^{-1} = -\mathsf{Y}, \quad \mathsf{w}\mathsf{Y}\mathsf{w}^{-1} = -\mathsf{X}$$

From this, one deduces that $e^{\mathsf{X}} = \mathsf{w}e^{-\mathsf{X}}e^{\mathsf{Y}} = e^{\mathsf{Y}}\mathsf{w}e^{\mathsf{Y}}$. Now $A^\bullet(Z)$ becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ if we set

$$\mathsf{X} = 2\pi\sqrt{-1}L \quad \text{and} \quad \mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$$

and let H act as multiplication by $k-n$ on the subspace $A^k(Z)$. The reason for this (non-standard) definition is that it makes the representation not depend on the choice of $\sqrt{-1}$. It is easy to see how w acts on primitive forms. Suppose that $\alpha \in A^{n-k}(Z)$ satisfies $\mathsf{Y}\alpha = 0$. Then $\mathsf{w}\alpha \in A^{n+k}(Z)$. If we now expand both sides of the identity

$$e^{\mathsf{X}}\alpha = e^{\mathsf{Y}}\mathsf{w}e^{\mathsf{Y}}\alpha = e^{\mathsf{Y}}\mathsf{w}\alpha$$

into power series, and then compare terms in degree $n+k$, we get

$$\mathsf{w}\alpha = \frac{\mathsf{X}^k}{k!}\alpha.$$

This formula is the reason for using w (instead of the otherwise w^{-1}): there is no sign on the right-hand side.

A Hodge-Lefschetz structure is linear algebra data encoding both representation theoretic and Hodge theoretic information. Recall that a finite dimensional $\mathfrak{sl}_2(\mathbb{C})$-representation is a graded vector space $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$ satisfying the following three equivalent conditions.

1. each graded piece V_ℓ is the ℓ-eigenspace of H;
2. the morphism $\mathsf{X}^\ell : V_{-\ell} \to V_\ell$ is an isomorphism for each $\ell \geq 0$;
3. the morphism $\mathsf{Y}^\ell : V_\ell \to V_{-\ell}$ is an isomorphism for each $\ell \geq 0$.

Example 2.1.7. For any finite dimensional vector space V together with a nilpotent operator N, there exists a so-called monodromy filtration $W_\bullet$ uniquely determined by the following two conditions

- for each $\ell \in \mathbb{Z}$, $N : W_\ell \to W_{\ell-2}$;
- the induced operator $N^\ell : \mathrm{gr}^W_\ell \to \mathrm{gr}^W_{-\ell}$ is an isomorphism for each $\ell \geq 0$.

Let $\mathrm{gr}^W = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}^W_\ell$. The ℓ-th primitive part $P_N\mathrm{gr}^W_\ell = \ker N^{\ell+1} \cap \mathrm{gr}^W_\ell$ consists of the classes of generators of cyclic subspaces of V of dimension ℓ as $\mathbb{C}[N]$-modules for $\ell \geq 0$. For each generator v, we have $N^{\ell+1}v = 0$ but $N^{\ell}v \neq 0$ and also v is not a image of N. Therefore, we have the identification

$$P_N \mathrm{gr}_\ell^W = \frac{\ker N^{\ell+1}}{\ker N^{\ell} + \operatorname{im} N \cap \ker N^{\ell+1}}.$$

Furthermore, we have the Lefschetz decomposition $\mathrm{gr}_\ell^W = \bigoplus_{k\geq 0} N^k P_N V_{\ell+2k}$. Taking $N = \mathsf{Y}$, the Lefschetz structure and the grading uniquely determines the operator X such that $(\mathsf{X}, \mathsf{Y}, \mathsf{H})$ is a $\mathfrak{sl}_2(\mathbb{C})$-triple by the relation $\mathsf{X}\mathsf{Y}^k = k(\ell - k + 1)\mathsf{Y}^{k-1}$ on $P_N \mathrm{gr}_\ell^W$. Thus gr^W naturally is a representation of $\mathfrak{sl}_2(\mathbb{C})$.

By Hard Lefschetz theorem, for any compact Kähler manifold the vector space

$$\bigoplus_{\ell\in\mathbb{Z}} H^{\dim Z+\ell}(Z,\mathbb{C})$$

is a representation of $\mathfrak{sl}_2(\mathbb{C})$ by setting $\mathsf{X} = 2\pi\sqrt{-1}L$ the Lefschetz operator, $\mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$ the adjoint operator. But because of the Lefschetz operator of is of type $(1,1)$, we actually have $\mathsf{X} : H^k(Z,\mathbb{C}) \to H^{k+1}(Z,\mathbb{C})(1)$ is a morphism of Hodge structures and $\mathsf{X}^\ell : H^{\dim Z-\ell}(Z,\mathbb{C}) \to H^{\dim Z+\ell}(Z,\mathbb{C})(\ell)$ is an isomorphism of Hodge structures. This leads to the following definition: a *Hodge-Lefschetz* structure of *central weight* n is a $\mathfrak{sl}_2(\mathbb{C})$-representation $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$ with two filtrations $F^\bullet V$ and $G^\bullet V$ such that

1. each graded piece $(V_\ell, F^\bullet V_\ell, G^\bullet V_\ell)$ is a Hodge structure of weight $n + \ell$;

2. the operator $\mathsf{X} : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{\ell+2}, F^{\bullet+1}V_{\ell+2},, G^{\bullet+1}V_{\ell+2})$ is a morphism of Hodge structures such that

$$\mathsf{X}^\ell : (V_{-\ell}, F^\bullet V_{-\ell}, G^\bullet V_{-\ell}) \to (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell)(\ell)$$

is an isomorphism of Hodge structures;

3. the operator $\mathsf{Y} : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{\ell-2}, F^{\bullet-1}V_{\ell-2}, G^{\bullet-1}V_{\ell-2})$ is a morphism of Hodge structures such that

$$\mathsf{Y}^\ell : (V_\ell, F^\bullet V_\ell, G^\bullet V_\ell) \to (V_{-\ell}, F^\bullet V_{-\ell}, G^\bullet V_{-\ell})(-\ell)$$

is an isomorphism of Hodge structures.

It follows from the definition the primitive part $P_X V_\ell$ is a sub-Hodge structure for each $\ell < 0$. Let $V_\ell = H^{\dim Z+\ell}(Z,\mathbb{C})$ and $V = \bigoplus_{\ell\in\mathbb{Z}} V_\ell$. It follows that V is a Hodge-Lefschetz structure of central weight $\dim Z$. Hodge-Lefschetz structure interplays well with the Hodge-Riemann bilinear relation. A *polarization* on a Hodge-Lefschetz structure V of central weight n is a hermitian symmetric paring $S : V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that

1. the restriction $S|_{V_\ell \otimes_{\mathbb{C}} \overline{V_{-k}}}$ is zero for $\ell + k \neq 0$;
2. $S(\mathsf{X}-,-) = S(-,\mathsf{X}-)$ and $S(-,\mathsf{Y}-) = S(\mathsf{Y}-,-)$;
3. $S_{-\ell}(\mathsf{X}^\ell-,-)$ is a polarization on $P_X V_{-\ell}$, or equivalently, $S_\ell \circ (\mathrm{id} \otimes \mathsf{w})$ is a polarization on V_ℓ where $S_\ell : V_\ell \otimes \overline{V_{-\ell}} \to \mathbb{C}$ is the restriction of S.

Note that $\mathsf{w} : V_k \to V_{-k}(-k)$ is automatically an isomorphism of Hodge structures (of weight $n + k$). We first prove an auxiliary formula. Suppose that $a \in V_{-\ell}$ is primitive, in the sense that $\mathsf{X}^{\ell+1} a = 0($ and $\ell \geq 0)$. Then $\mathsf{Y} a = 0$, and from $\mathsf{w} e^{-\mathsf{X}} = e^{\mathsf{X}} e^{-\mathsf{Y}}$, we get $\mathsf{w} e^{-\mathsf{X}} a = e^{\mathsf{X}} a$, and after expanding and comparing terms in degree $\ell - 2j$, also

$$\mathsf{w}\frac{\mathsf{X}^j}{j!} a = (-1)^j \frac{\mathsf{X}^{\ell-j}}{(\ell-j)!} a \tag{2.1.6}$$

since w^2 acts on $V_{-\ell+2j}$ as $(-1)^{-\ell+2j} = (-1)^\ell$, this formula is actually symmetric in j and $\ell - j$,.

Lemma 2.1.8. *If V is a Hodge-Lefschetz structure, then $\mathsf{w} : V_k \to V_{-k}(-k)$ is an isomorphism of Hodge structures.*

Proof. Any $a \in V_k$ has a unique Lefschetz decomposition

$$a = \sum_{j \geq \max(k,0)} \frac{\mathsf{X}^j}{j!} a_j$$

where $a_j \in V_{k-2j}$ satisfies $Y a_j = 0$. (We only need to consider $j \geq k$ in the sum because $\mathsf{X}^{2j-k+1} a_j = 0$, which implies that $\mathsf{X}^j a_j = 0$ for $j < k$.) Suppose further that $a \in V_k^{p,q}$, where $p + q = n + k$. Then $\mathsf{X}^i a_j \in V_{k+2i}^{p+i,q+i}$, and by descending induction on $j \geq \max(k,0)$, we deduce that $a_j \in V_{k-2j}^{p-j,q-j}$. In other words, the Lefschetz decomposition holds in the category of Hodge structures.

We can now check what happens when we apply w. Using (2.1.6), we find that

$$\mathsf{w}a = \sum_{j \geq \max(k,0)} \mathsf{w}\frac{\mathsf{X}^j}{j!}a_j = \sum_{j \geq \max(k,0)} (-1)^j \frac{\mathsf{X}^{j-k}}{(j-k)!}a_j \in V_{-k}^{p-k,q-k}$$

and so w is a morphism of Hodge structures. The same calculation shows that w^{-1} is also a morphism of Hodge structures. It follows that w is an isomorphism of Hodge structures. □

The definition of polarized Hodge-Lefschetz structure of central weight n is redundant. In fact the definition is equivalent to a tuple $(V, \mathsf{X}, F^\bullet, S)$ for $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$, $F^\bullet$ is a decreasing filtration, $\mathsf{X} : (V_\ell, F^\bullet) \to (V_{\ell+2}, F^{\bullet+1})$, and S is a Hermitian pairing such that

(pHL1) for each $\ell \geq 0$, $\mathsf{X}^\ell : F^\bullet V_{-\ell} \to F^{\bullet+\ell} V_\ell$ is an isomorphism;

(pHL2) $S(\mathsf{X}-,-) = S(-,\mathsf{X}-)$ and $S|_{V_\ell \otimes_{\mathbb{C}} \overline{V_{-k}}}$ vanishes except for $k = -\ell$;

(pHL3) the triple $(P_{\mathsf{X}} V_j, F_\bullet, S \circ (\mathsf{X}^j \circ \mathrm{id}))$ is a porlarized Hodge structure of weight $n - j$.

The condition (pHL1) in the above definition indicates the Lefschetz decomposition respects the filtration $F^\bullet$. Therefore Y is determined uniquely and also filtered. The second condition implies that $S(\mathsf{Y}-,-) = S(-,\mathsf{Y}-)$. The third condition says that $S \circ (\mathrm{id} \otimes \mathsf{w})$ is non-degenerate on $F^p V_\ell \otimes \overline{F^p V_{-\ell}}$. Therefore, we also get the following concrete description of the Hodge structure on V_ℓ: for $p + q = n + \ell$

$$V_\ell^{p,q} = \{a \in F^p V_\ell : S_\ell(a,b) = 0 \text{ for all } b \in F^{p-\ell+1} V_{-\ell}\},$$

$$G^q V_\ell = \{a \in V_\ell, S_\ell(a,b) = 0 \text{ for all } b \in F^{n-q+1} V_{-\ell}\}.$$

Example 2.1.9. For a compact Kähler manifold Z of dimension n, let $V_\ell = H^{n+\ell}(Z, \mathbb{C})$ and $V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$. Then V together with $\mathsf{X} = 2\pi\sqrt{-1}L$ and $\mathsf{Y} = (2\pi\sqrt{-1})^{-1}\Lambda$ and with the natural filtration is a Hodge-Lefschetz structure of central weight n. By Hodge-Riemann bilinear relation, taking

$$S_\ell(a,b) = \frac{\varepsilon(n+\ell+1)}{(2\pi\sqrt{-1})^n} \int_Z a \wedge \bar{b} = \varepsilon(\ell)(-1)^{\ell n} \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_Z a \wedge \bar{b} \tag{2.1.7}$$

for $a \in V_\ell$ and $b \in V_{-\ell}$ gives a polarization on V. The polarized Hodge-Lefschetz structure V is determined by the filtered $\mathscr{D}_Z$-module ω_Z together with the sesquilinear pairing S_Z. The graded piece V_ℓ is just ℓ-th hypercohomology of $\mathrm{DR}_Z\omega_Z$ with induced filtration $F^\bullet V_\ell$ given by the image of $H^\ell(Z, F_{-\bullet}\mathrm{DR}_Z\omega_Z)$. And the polarization S_k is given by $\varepsilon(k)$ times the pairing

$$H^k(Z, \mathrm{DR}_Z\omega_Z) \otimes \overline{H^{-k}(Z, \mathrm{DR}_Z\omega_Z)} \longrightarrow H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z}) \xrightarrow{S_Z} H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z) \simeq \mathbb{C}$$

We can work out the pairing explicitly. Note that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{DR}_{Z,\overline{Z}}\omega_Z \otimes_{\mathbb{C}} \overline{\omega_Z} & \longrightarrow & \mathrm{DR}_{Z,\overline{Z}}\mathscr{O}_Z \otimes_{\mathbb{C}} \overline{\mathscr{O}_Z} \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle D} \\ \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z & \longrightarrow & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z \end{array}$$

where the upper horizontal arrow is the isomorphism induced by (2.1.2) and similarly the lower horizontal arrow is defined on the terms in degree $-k$,

$$\mathfrak{C}_Z \otimes_{\mathscr{O}_{Z,\bar{Z}}} \bigwedge^k \mathscr{T}_{Z,\bar{Z}} \to \Omega^{2n-k}_{Z,\bar{Z}} \otimes_{\mathscr{O}_{Z,\bar{Z}}} \mathfrak{Db}_Z$$

by the following rule: write a current locally as $D\omega \wedge \bar{\omega}$, with a distribution D and denote by $\partial_J = \bigwedge_J \partial_j$ and $dx_{\bar{J}} = \bigwedge_{i \notin J} dx_i$ for an ordered index subset J of I; then

$$(D\omega \wedge \bar{\omega}) \otimes \partial_J \wedge \bar{\partial}_K \mapsto (-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)}(-1)^{nq} dx_{\bar{J}} \wedge \overline{dx}_{\bar{K}} \otimes D \tag{2.1.8}$$

where $\#J = p$ and $\#K = q$, and $p+q=k$. The sign factor is explained by the number of swaps that are needed to move everything into the right place, which is $(2n-j_1)+\cdots+(2n-j_p)+(n-k_1)+\cdots+(n-k_q)$. We can now derive a formula for the induced pairing

$$\mathrm{DR}_Z\mathscr{O}_Z \otimes_{\mathbb{C}} \overline{\mathrm{DR}_Z\mathscr{O}_Z} \to \mathrm{DR}_{Z,\bar{Z}}\mathfrak{Db}_Z. \tag{2.1.9}$$

For the two local sections $\alpha = dx_{\bar{J}}$ and $\beta = dx_{\bar{K}}$, under the isomorphism $\mathrm{DR}_Z\mathscr{O}_Z \cong \mathrm{DR}_Z\omega_Z$ in (2.1.2), the $(n-p)$-form α goes to

$$(-1)^{np}(-1)^{j_1+\cdots+j_p} \cdot \omega \otimes \partial_J.$$

and the $(n-q)$-form β goes to

$$(-1)^{nq}(-1)^{k_1+\cdots+k_q}\cdot\omega\otimes\partial_K.$$

The pairing S_Z on $\mathrm{DR}_Z\omega_Z$ takes those two sections to

$$(-1)^{n(p+q)}(-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)}S(\omega,\omega)\otimes\partial_J\wedge\bar{\partial}_K \tag{2.1.10}$$

where S_Z is defined in (2.1.4). Now $S_Z(\omega,\omega)=D_Z\omega\wedge\bar{\omega}$, where D is the distribution

$$D_Z=\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n}\int_Z$$

Under the isomorphism in (2.1.8) the section (2.1.10) therefore goes to

$$(-1)^{np}dx_{\bar{J}}\wedge\overline{dx}_{\bar{K}}\otimes D_Z=(-1)^{n(\deg\alpha-n)}\alpha\wedge\bar{\beta}\otimes D_Z$$

The formula we have just derived also works for smooth forms, of course. In other words, the same formula can be used to extend (2.1.9) to a pairing on the de Rham complex of smooth forms. The resulting pairings on cohomology are, assuming Z is compact

$$H^{n+k}(Z,\mathbb{C})\otimes\overline{H^{n-k}(Z,\mathbb{C})}\to\mathbb{C},\quad(\alpha,\beta)\mapsto(-1)^{n(\deg\alpha-n)}\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n}\int_Z\alpha\wedge\bar{\beta}, \tag{2.1.11}$$

which coincides with the pairing (2.1.7) precisely.

2.1.4 Polarized bigraded Hodge-Lefschetz structures

In the paper, what we really consider is the degeneration of "variation of Hodge-Lefschetz structures" of a family of compact Kähler manifolds. As it turns out the limit of the degeneration is a bigraded Hodge-Lefschetz structure. We begin to define polarized bigraded Hodge-Lefschetz structures. Similarly to the case of $\mathfrak{sl}_2(\mathbb{C})$-representation, a $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{sl}_2(\mathbb{C})$-representation is a bigraded vector space $V=\bigoplus_{\ell,k\in\mathbb{Z}}V_{\ell,k}$ satisfying the following three equivalent conditions:

1. each bigraded piece $V_{\ell,k}$ is the ℓ-th eigenspace of H_1 and k-th eigenspace of H_2;

2. for each $\ell, k \in \mathbb{Z}$ we have $\mathsf{X}_1 : V_{\ell,k} \to V_{\ell+2,k}$ and $\mathsf{X}_2 : V_{\ell,k} \to V_{\ell,k+2}$ plus isomorphisms

$$\mathsf{X}_1^\ell : V_{-\ell,k} \to V_{\ell,k} \text{ and } \mathsf{X}_2^k : V_{\ell,-k} \to V_{\ell,k};$$

3. for each $\ell, k \in \mathbb{Z}$ we have $\mathsf{Y}_1 : V_{\ell,k} \to V_{\ell-2,k}$ and $\mathsf{Y}_2 : V_{\ell,k} \to V_{\ell,k-2}$ plus the isomorphism

$$\mathsf{Y}_1^\ell : V_{\ell,k} \to V_{-\ell,k} \text{ and } \mathsf{Y}_2^k : V_{\ell,k} \to V_{\ell,-k}.$$

A *bigraded Hodge-Lefschetz structure* of *central weight* n is a $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-representation $V = \bigoplus_{\ell,k\in\mathbb{Z}} V_{\ell,k}$ with two filtrations $F^\bullet V$ and $G^\bullet V$ such that

1. the bifiltered vector space $(V_{\ell,k}, F^\bullet V_{\ell,k}, G^\bullet V_{\ell,k})$ is a Hodge structure of weight $n + \ell + k$;

2. the two operators $\mathsf{X}_1 : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell+2,k}, F^{\bullet+1}, G^{\bullet+1})$ and $\mathsf{X}_2 : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell,k+2}, F^{\bullet+1}, G^{\bullet+1})$ are morphisms of Hodge structures such that

$$\mathsf{X}_1^\ell : (V_{-\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell,k}, F^\bullet, G^\bullet)(\ell) \quad \text{and} \quad \mathsf{X}_2^k : (V_{\ell,-k}, F^\bullet, G^\bullet) \to (V_{\ell,k}, F^\bullet, G^\bullet)(k)$$

are isomorphisms of Hodge structures.

3. the two operators $\mathsf{Y}_1 : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell-2,k}, F^{\bullet-1}, G^{\bullet-1})$ and $\mathsf{Y}_2 : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell,k-2}, F^{\bullet-1}, G^{\bullet-1})$ are morphisms of Hodge structures such that

$$\mathsf{Y}_1^\ell : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{-\ell,k}, F^\bullet, G^\bullet)(-\ell) \quad \text{and} \quad \mathsf{Y}_2^k : (V_{\ell,k}, F^\bullet, G^\bullet) \to (V_{\ell,-k}, F^\bullet, G^\bullet)(-k)$$

are isomorphisms of Hodge structures.

A *polarization* on a bigraded Hodge-Lefschetz structure $V = \bigoplus_{\ell,k\in\mathbb{Z}} V_{\ell,k}$ of central weight n is a hermitian symmetric pairing $S : V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ such that

1. the restriction $S|_{V_{\ell,k} \otimes_{\mathbb{C}} \overline{V_{i,j}}} : V_{\ell,k} \otimes_{\mathbb{C}} \overline{V_{i,j}} \to \mathbb{C}$ vanishies except for $\ell = -i$ and $k = -j$;

2. $S(\mathsf{X}_1 -, -) = S(-, \mathsf{X}_1 -)$ and $S(-, \mathsf{Y}_2 -) = S(\mathsf{Y}_2 -, -)$;

3. $S_{\ell,k}(\mathsf{X}_1^{\ell}-,(-\mathsf{Y}_2)^k-)$ is a polarization on the bi-primitive part $P_{-\ell,k} = \ker \mathsf{X}_1^{\ell+1} \cap \ker \mathsf{Y}_2^{k+1} \cap V_{-\ell,k}$, or equivalently, $S_{\ell,k}(-,\mathsf{w}_1\mathsf{w}_2-)$ is a polarization on $V_{\ell,k}$, where $S_{\ell,k}$ is the restriction of S on $V_{\ell,k} \otimes \overline{V_{-\ell,k}}$ and $\mathsf{w}_i = e^{\mathsf{X}_i}e^{-\mathsf{Y}_i}e^{\mathsf{X}_i}$ for $i = 1,2$.

This is the practical definition because in the later application X_1 will be the $2\pi\sqrt{-1}L$ and Y_2 will be, up to a scalar, the logarithmic of the monodromy for the degeneration. Similiarly to the case of Hodge-Lefschetz structure, we have a simpler definition.

Theorem 2.1.10. *A polarized bigraded Hodge-Lefschetz structure of central weight n on a filtered bigraded vector space $(V = \bigoplus_{\ell,k} V_{\ell,k}, F^\bullet V)$ is uniquely determined by the following:*

(pbHL1) *for every $\ell, k \in \mathbb{Z}$ we have two operators $\mathsf{X}_1 : (V_{\ell,k}, F^\bullet) \to (V_{\ell+2,k}, F^{\bullet+1})$ and $\mathsf{Y}_2 : (V_{\ell,k}F^\bullet) \to (V_{\ell,k-2}, F^{\bullet-1})$ such that*

$$\mathsf{X}_1^{\ell} : F^\bullet V_{-\ell,k} \to F^{\bullet+\ell}V_{\ell,k} \quad \text{and} \quad \mathsf{Y}_2^k : F^\bullet V_{\ell,k} \to F^{\bullet-k}V_{\ell,-k} \text{ are isomorphisms;}$$

(pbHL2) *a collection of Hermitian pairings $S_{\ell,k} : V_{\ell,k} \otimes_{\mathbb{C}} \overline{V_{-\ell,-k}} \to \mathbb{C}$ such that*

$$S_{\ell,k}(\mathsf{X}_1-,-) = S_{\ell+2,k}(-,\mathsf{X}_1-) \quad \text{and} \quad S_{\ell,k}(-,\mathsf{Y}_2-) = S_{\ell,k-2}(\mathsf{Y}_2-,-);$$

(pbHL3) *the triple $\left(P_{-\ell,k}, F^\bullet P_{-\ell,k}, S \circ \left(\mathsf{X}_1^{\ell} \otimes (-\mathsf{Y}_2)^k\right)\right)$ is a polarized Hodge structure of weight $n-\ell+k$ where $F^\bullet P_{-\ell,k} = \ker \mathsf{X}_1^{\ell} \cap \ker \mathsf{Y}_2^k \cap F^\bullet V_{-\ell,k}$ is the bi-primtive part.*

Then the Hodge structure on $V_{j,k}$ can be described as: for $p+q = n+j+k$

$$V_{j,k}^{p,q} = \{a \in F^pV_{j,k} : S_{j,k}(a,b) = 0 \text{ for all } b \in F^{p-j-k+1}V_{-j-k}\},$$
$$G^qV_{j,k} = \{a \in V_{j,k} : S_{j,k}(a,b) = 0 \text{ for all } b \in F^{n-q+1}V_{-j,-k}\}.$$

The proof is simple and is left to the reader. Later when we construct the limiting mixed Hodge structure, the polarized bigraded Hodge-Lefschetz structure naturally comes up from the first page of weight spectral sequence associated to a mixed Hodge complex. Modeled on the properties of the differential of spectral sequence we give the following definition:

A *differential* of a polarized bigraded Hodge Lefschetz structure $(V, F^\bullet, \mathsf{X}_1, \mathsf{Y}_2, S)$ is a linear map $d: V \to V$ such that

1. $d: (V_{j,k}, F^\bullet) \to (V_{j+1,k-1}, F^\bullet)$ and $d^2 = 0$;
2. d is skew-symmetrc with respect to S, i.e., $S(d-,-) + S(-,d-) = 0$;
3. $[\mathsf{X}_1, d] = 0$ and $[\mathsf{Y}_2, d] = 0$.

Remark 2.1.11. In fact, the above three conditions imply that d is a morphism of Hodge structures $d: V_{j,k}^{p,q} \to V_{j+1,k-1}^{p,q}$. A vector $a \in G^q V_{j,k}$ means that $S(a,b) = 0$ for all $b \in F^{n-q+1}V_{-j,-k}$. Then $S(da,b) = S(a,db) = 0$ for all $b \in F^{n-q+1}V_{-j-1,-k+1}$, indicating da belongs to $G^q V_{j+1,k-1}$.

The main result of this subsection is the following version of Deligne's lemma, showed by Guillén and Navarro Aznar.

Theorem 2.1.12 ([GN90, (4.5)])**.** *The cohomology* $\ker d/\operatorname{im} d$ *of a polarized differential bigraded Hodge-Lefschetz struture is again a polarized bigraded Hodge-Lefschetz structure.*

Proof. Let $C: V \to V$ be the operator that acts as $(-1)^q$ on the subspace $V_{j,k}^{p,q}$ in the Hodge decomposition of each $V_{j,k}$. Since d is a morphism of Hodge structures, we have $[d, C] = 0$. The fact that S is a polarization means that the Hermitian pairing

$$h^+ : V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}, \quad h^+(a,b) = S\left(Ca, \mathsf{w}_1\mathsf{w}_2 b\right)$$

is positive-definite on V. Let d^* be the adjoint of d with respect to h^+. Fix $a \in V_{j,k}$ and $b \in V_{j,k}$,:

$$\begin{aligned} h^+(da,b) &= S\left(Cda, \mathsf{w}_1\mathsf{w}_2 b\right) = S\left(dCa, \mathsf{w}_1\mathsf{w}_2 b\right) \\ &= -S\left(Ca, d\mathsf{w}_1\mathsf{w}_2 b\right) = -S\left(Ca, \mathsf{w}_1\mathsf{w}_2 \cdot \mathsf{w}_2^{-1}\mathsf{w}_1^{-1} d\mathsf{w}_1\mathsf{w}_2 \cdot b\right) = h^+\left(a, d^* b\right), \end{aligned}$$

i.e. the adjoint $d^* = -\mathsf{w}_2^{-1}\mathsf{w}_1^{-1} d\mathsf{w}_1\mathsf{w}_2$.

In addition to the two relations in the definition of differential

$$[\mathsf{X}_1, d] = 0 \quad \text{and} \quad [\mathsf{Y}_2, d] = 0$$

we obtain from the grading another two relations

$$[\mathsf{H}_1, d] = d \quad \text{and} \quad [\mathsf{H}_2, d] = -d.$$

With respect to the $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$-action on $\operatorname{End}_{\mathbb{C}}(V)$, the element d therefore has weight $(+1,-1)$, and is primitive with respect to the action by Y_1 and X_2. Define

$$d_1 = [\mathsf{Y}_1, d] \quad \text{and} \quad d_2 = -[\mathsf{X}_2, d].$$

The reason for the minus sign is that we have $[\mathsf{Y}_2, d] = 0$. Then d_1 has weight $(-1,-1)$, and is primitive with respect to the action by X_1 and X_2; this gives

$$\begin{aligned} &[\mathsf{H}_1, d_1] = -d_1, \quad [\mathsf{X}_1, d_1] = d, \quad [\mathsf{Y}_1, d_1] = 0, \quad \mathsf{w}_1 d_1 \mathsf{w}_1^{-1} = d \\ &[\mathsf{H}_2, d_1] = -d_1, \quad\quad\quad\quad\quad\quad\quad\; [\mathsf{Y}_2, d_1] = 0. \end{aligned}$$

Similarly, d_2 has weight $(+1,+1)$, and therefore

$$\begin{aligned} &[\mathsf{H}_2, d_2] = d_2, \quad [\mathsf{X}_2, d_2] = 0, \quad [\mathsf{Y}_2, d_2] = -d, \quad \mathsf{w}_2 d_2 \mathsf{w}_2^{-1} = d \\ &[\mathsf{H}_1, d_2] = d_2, \quad [\mathsf{X}_1, d_2] = 0. \end{aligned}$$

Therefore, $d^* = -[\mathsf{Y}_1, d_2] = [\mathsf{X}_2, d_1] \in \operatorname{End}_{\mathbb{C}} V$. It has weight $(-1,+1)$, and is primitive with respect to X_1 and Y_2. From this, and the identities we already have, we deduce the following set of relations:

$$\begin{aligned} &[\mathsf{H}_1, d^*] = -d^*, \quad [\mathsf{X}_1, d^*] = d_2, \quad [\mathsf{Y}_1, d^*] = 0, \quad \mathsf{w}_1 d^* \mathsf{w}_1^{-1} = -d_2 \\ &[\mathsf{H}_2, d^*] = d^*, \quad [\mathsf{X}_2, d^*] = 0, \quad [\mathsf{Y}_2, d^*] = -d_1, \quad \mathsf{w}_2 d^* \mathsf{w}_2^{-1} = -d_1. \end{aligned}$$

We can check that the (formal) Laplace operator

$$\Delta = dd^* + d^*d \in \operatorname{End}_{\mathbb{C}}(V)$$

is invariant under the action of $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$. For example,

$$\begin{aligned} &[\mathsf{X}_1, dd^*] = \mathsf{X}_1 dd^* - dd^* \mathsf{X}_1 = d\mathsf{X}_1 d^* - d(\mathsf{X}_1 d^* + d_2) = -dd_2 \\ &[\mathsf{X}_1, d^*d] = \mathsf{X}_1 d^* d - d^* d \mathsf{X}_1 = (d^* \mathsf{X}_1 - d_2)\, d - d^* \mathsf{X}_1 d = -d_2 d \end{aligned}$$

from which we conclude, using $d^2 = 0$, that

$$[\mathsf{X}_1, \Delta] = -(dd_2 + d_2 d) = -(d(d\mathsf{X}_2 - \mathsf{X}_2 d) + (d\mathsf{X}_2 - \mathsf{X}_2 d)d) = 0$$

The other three commutators can be checked similarly. On the other hand, Δ is also a morphism of Hodge structures: the reason is that

$$d : V_{j,k} \to V_{j+1,k-1}, \quad \mathsf{Y}_1 : V_{j,k} \to V_{j-2,k}(-1), \quad \mathsf{X}_2 : V_{j,k} \to V_{j,k+2}(1)$$

are all morphisms of Hodge structures, and Δ is obtained by composing them in some order. It follows that $\ker \Delta \subseteq V$ is a bigraded Hodge-Lefschetz structure, polarized by the restriction of S. Because of the canonical isomorphism $\ker \Delta \simeq \ker d/\operatorname{im} d$ as bigraded Hodge-Lefschetz structures, the induced pairing by S on $\ker d/\operatorname{im} d$ is also a polarization. This concludes the proof. □

2.2 Log relative de Rham complex

Let $f : X \to \Delta$ be a proper holomorphic morphism smooth away from the origin whose central fiber Y is simple normal crossing but not necessarily reduced. Assume X is Kähler of dimension $n+1$ and $Y = \sum_{i \in I} e_i Y_i$ where Y_i's are smooth components and I a finite index set. Let t be a parameter on Δ and $z_0, z_1, z_2, ..., z_n$ a local coordinate system on X such that $t = z_0^{e_0} z_1^{e_1} \cdots z_k^{e_k}$ such that $e_0, e_1, ..., e_k \geq 1$. Then we have $\Omega_\Delta(\log 0) = \mathscr{O}_\Delta \cdot \frac{dt}{t}$ and $\Omega_X(\log Y)$ is locally generated by

$$e_0 \frac{dz_0}{z_0}, e_1 \frac{dz_1}{z_1}, ..., e_k \frac{dz_k}{z_k}, dz_{k+1}, dz_{k+2}, ..., dz_n$$

over $\mathscr{O}_X$. Denote by $\xi_0, \xi_1, ..., \xi_n$ the image of the above generators in $\Omega_{X/\Delta}(\log Y)$, respectively. As a quotient of $\Omega_X(\log Y)$, the sheaf $\Omega_{X/\Delta}(\log Y)$ is generated by $\xi_0, \xi_1, ..., \xi_n$, but under the relation

$$\xi_0 + \xi_1 + \cdots + \xi_n = 0 \quad \text{because} \quad f^* \frac{dt}{t} = e_0 \frac{dz_0}{z_0} + e_1 \frac{dz_1}{dz_1} + \cdots + e_k \frac{dz_k}{z_k}.$$

Let $\mathscr{T}_{X/\Delta}(\log Y)$ be the dual bundle of $\Omega_{X/\Delta}(\log Y)$. It is a subsheaf of $\mathscr{T}_X$, generated by

$$D_i = \begin{cases} \dfrac{1}{e_i} z_i \partial_i - \dfrac{1}{e_0} z_0 \partial_0, & 1 \le i \le k \\ \\ \partial_i, & i > k, \end{cases} \tag{2.2.1}$$

where ∂_i is the local section of $\mathscr{T}_X$ dual to dz_i in Ω_X. It follows that $D_1, D_2, ..., D_n$ is the dual frame of $\xi_1, \xi_2, ..., \xi_n$.

2.2.1 A "log connection"

We shall construct an operator in $\mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}\left(Rf_* \Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$ which should be regarded a "log connection". Note that we have the following short exact sequence of $\mathscr{O}_X$-modules

$$0 \to f^*\Omega_\Delta(\log 0) \otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n+1}_X(\log Y) \to \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \to 0.$$

Under the identification $\frac{dt}{t}\wedge : \mathscr{O}_X \to f^*\Omega_\Delta(\log 0)$, the above short exact sequence becomes

$$0 \longrightarrow \Omega^{\bullet+n}_{X/\Delta}(\log Y) \xrightarrow{\frac{dt}{t}\wedge} \Omega^{\bullet+n+1}_X(\log Y) \longrightarrow \Omega^{\bullet+n+1}_{X/\Delta}(\log Y) \longrightarrow 0.$$

Here, the morphism $\frac{dt}{t}\wedge : \Omega^k_{X/\Delta}(\log Y) \to \Omega^{k+1}_X(\log Y)$ works as $[\alpha] \mapsto \frac{dt}{t} \wedge \alpha$ which does not depend on the representative of $[\alpha]$. Let $\mathrm{Cone}^\bullet = \Omega^{\bullet+n}_X(\log Y) \oplus \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ be the mapping cone of $\frac{dt}{t}\wedge : \Omega^{\bullet+n-1}_{X/\Delta}(\log Y) \to \Omega^{\bullet+n}_X(\log Y)$. In our convention, the differential δ of the mapping cone works as $\delta(\alpha, [\beta]) = \left((-1)^n d\alpha + \frac{dt}{t} \wedge \beta, (-1)^n d[\beta]\right)$, where d is the usual exterior derivative on $\Omega^\bullet_X(\log)$ and by abuse of notation, also d denotes the induced differential on $\Omega^\bullet_{X/\Delta}(\log Y)$. Then we have the following diagram:

$$\begin{array}{ccc} \mathrm{Cone}^\bullet & \xrightarrow{\quad q \quad} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) \\ \Big\downarrow p & \swarrow_{p \circ q^{-1}} & \\ \Omega^{\bullet+n}_{X/\Delta}(\log Y) & & \end{array} \tag{2.2.2}$$

where $q : \mathrm{Cone}^\bullet \to \Omega^{\bullet+n}_{X/\Delta}(\log Y), (\alpha, [\beta]) \mapsto [\alpha]$ is a quasi-isomorphism and p is the second projection. Therefore we have the morphism $p \circ q^{-1}$ in $\mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\right)$. For any local section $g \in \mathscr{O}_\Delta$, the multiplication by g is an endomorphism of $\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ because it is $f^{-1}\mathscr{O}_\Delta$-linear.

Lemma 2.2.1. *The operator* $\nabla = (-1)^{n-1} p \circ q^{-1}$ *satisfies* $[\nabla, g] = tg'$ *in* $\operatorname{End}_{\mathbf{D}^b(X,\mathbb{C})}\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$, *where* g' *denotes the derivative of* $g \in \mathscr{O}_\Delta$.

Proof. It is equivalent to show that $[p \circ q^{-1}, g] = (-1)^n tg'$. Define $g(\alpha, [\beta]) = (g\alpha, g[\beta] + (-1)^{n-1} tg'[\alpha])$ for any $(\alpha, [\beta]) \in \mathrm{Cone}^\bullet$ and $g \in f^{-1}\mathscr{O}_\Delta$. We shall show that g is an endomorphism of $\mathrm{Cone}^\bullet$, i.e., $g\delta(\alpha, [\beta]) = \delta g(\alpha, [\beta])$. This follows from that

$$\begin{aligned} g\delta(\alpha, [\beta]) &= g\left((-1)^n d\alpha + \frac{dt}{t} \wedge \beta, (-1)^n d[\beta]\right) \\ &= \left((-1)^n g d\alpha + g\frac{dt}{t} \wedge \beta, (-1)^n g d[\beta] - tg' d[\alpha]\right) \end{aligned}$$

and

$$\begin{aligned} \delta g(\alpha, [\beta]) &= \delta\left(g\alpha, g[\beta] + (-1)^{n-1} tg'[\alpha]\right) \\ &= \left((-1)^n dg\alpha + \frac{dt}{t} \wedge (g\beta + (-1)^{n-1} tg'\alpha), (-1)^n d(g[\beta] + (-1)^{n-1} tg'[\alpha])\right) \\ &= \left((-1)^n g d\alpha + g\frac{dt}{t} \wedge \beta, (-1)^n g d[\beta] - tg' d[\alpha]\right). \end{aligned}$$

It is easy to see that $g \circ q = q \circ g$ so that $q^{-1} \circ g = g \circ q^{-1}$. Therefore,

$$[p \circ q^{-1}, g] = p \circ q^{-1} \circ g - g \circ p \circ q^{-1} = [p, g] \circ q^{-1}$$

But $[p, g](\alpha, [\beta]) = p(g\alpha, g[\beta] + (-1)^{n-1} tg'[\alpha]) - g[\beta] = (-1)^{n-1} tg'[\alpha]$. It follows that

$$[p \circ q^{-1}, g] \circ q(\alpha, [\beta]) = [p, g](\alpha, [\beta]) = (-1)^{n-1} tg' \circ q(\alpha, [\beta]).$$

By inverse q we prove the statement. □

Because of the identification $\frac{dt}{t}\wedge : \mathscr{O}_\Delta \to \Omega_\Delta(\log 0)$, what we really get is a morphism in $\mathbf{D}^b(X, \mathbb{C})$

$$\nabla : \Omega^{\bullet+n}_{X/\Delta}(\log Y) \to f^*\Omega_\Delta(\log 0) \otimes \Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

such that $\nabla g = g\nabla + \frac{dt}{t} \otimes tg' \in \operatorname{End}_{\mathbf{D}^b(X,\mathbb{C})}(\Omega^{\bullet+n}_{X/\Delta}(\log Y))$ for any local section $g \in \mathscr{O}_\Delta$. Running the similar construction, we obtain an induced $\mathbb{C}$-linear (in fact $f^{-1}\mathscr{O}_\Delta$-linear) endomorphism $[\nabla]$ on $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ in $\mathbf{D}^b(X, \mathbb{C})$ satisfying the following diagram.

$$\begin{array}{ccccccc}
\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle [\nabla]} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & \Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1]
\end{array}$$

Since $\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is $f^{-1}\mathscr{O}_\Delta$-linear, each cohomolgy $R^k f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is a coherent $\mathscr{O}_\Delta$-module. Taking direct image, we get $\mathbb{C}$-linear morphisms between distinguished triangles in $\mathbf{D}^b_{\mathrm{coh}}(\Delta, \mathscr{O}_\Delta)$:

$$\begin{array}{ccccccc}
Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)[1] \\
\downarrow{\scriptstyle Rf_*\nabla+1} & & \downarrow{\scriptstyle Rf_*\nabla} & & \downarrow{\scriptstyle Rf_*[\nabla]} & & \downarrow{\scriptstyle Rf_*(\nabla+1)[1]} \\
Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \xrightarrow{t} & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) & \longrightarrow & Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y & \longrightarrow & \Omega^{\bullet+n}_{X/\Delta}(\log Y)[1]
\end{array} \tag{2.2.3}$$

where the morphism

$$Rf_*\nabla : Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \to Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$$

satisfies $[Rf_*\nabla, g] = tg' \in \mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}\left(Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)\right)$ for any local sections $g \in \mathscr{O}_\Delta$.

2.2.2 Residue

In the above situation, one should regard $Rf_*[\nabla]$ as the residue of $Rf_*\nabla$. More generally, let $\mathcal{F}^\bullet$ be a complex of $\mathscr{O}_\Delta$-modules with a morphism $\nabla \in \mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{F}^\bullet)$ such that $[\nabla, g] = tg'$ for any $g \in \mathscr{O}_\Delta$. Let $\mathcal{G}^\bullet$ be the mapping cone of $t : \mathcal{F}^\bullet \to \mathcal{F}^\bullet$, which computes to $\mathcal{F}^\bullet \otimes^L \mathbb{C}(0)$. Then by the axioms of triangulated categories [HTT08], there exists an operator $R \in \mathrm{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{G}^\bullet)$ making the following diagram commute in $\mathbf{D}^b(\Delta, \mathbb{C})$.

$$\begin{array}{ccccccc}
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^L \mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{G}^\bullet = \mathcal{F}^\bullet \otimes^L \mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1]
\end{array}$$

We call the operator R a *residue* of ∇. Note that the axioms of triangulated categories cannot guarantee that the filling is unique. However, the eigenvalues of R_ℓ only depends on

∇, where R_ℓ denotes the induced operator on the cohomology $\mathscr{H}^\ell(\mathcal{F}^\bullet \otimes^L \mathbb{C}(0))$. First, every object in $\mathbf{D}^b_{\mathrm{coh}}(\Delta, \mathscr{O})$ splits, meaning that $\mathcal{F}^\bullet \simeq \bigoplus_{\ell\in\mathbb{Z}} \mathscr{H}^\ell \mathcal{F}^\bullet[-\ell]$, since there are no Ext^i for $i \geq 2$ between two coherent sheaves over a curve. It follows that the morphism ∇ breaks up into sum of morphism consisting of diagonal morphism $\nabla_\ell : \mathscr{H}^\ell\mathcal{F}^\bullet[-\ell] \to \mathscr{H}^\ell\mathcal{F}^\bullet[-\ell]$ which is an actual log connection and off-diagonal morphism $\mathscr{H}^\ell\mathcal{F}^\bullet[-\ell] \to \mathscr{H}^m\mathcal{F}^\bullet[-m]$ but only for $\ell > m$. Thus the eigenvalues of R_ℓ are determined by ∇_ℓ and $\nabla_{\ell+1}$. When $\mathcal{F}^\bullet$ is a locally free sheaf centered at degree zero and ∇ is the usual log connection. Then above definition coincides with the usual definition of the residue of ∇.

Returning to our case, the natural choice of a residue of $Rf_*\nabla$ is $R = Rf_*[\nabla]$ because of the diagram (2.2.3): by the projection formula, we have

$$Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \overset{L}{\underset{\mathscr{O}_\Delta}{\otimes}} \mathbb{C}(0) = Rf_*\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y) \overset{L}{\underset{f^{-1}\mathscr{O}_\Delta}{\otimes}} f^{-1}\mathbb{C}(0)\right) = Rf_*\left(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y\right).$$

Our main result concerning the relative log de Rham complex is the following.

Theorem 2.2.2. *The higher direct image $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is locally free for each $\ell \in \mathbb{Z}$. Moreover, there exists a canonical isomorphism for every $p \in \Delta$*

$$R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y) \otimes \mathbb{C}(p) \simeq H^\ell(X, \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_{X_p}), \quad \text{where } \mathbb{C}(p) \text{ is the residue filed at } p.$$

We first present two preliminary theorems.

Theorem 2.2.3. *The operator R_ℓ has eigenvalues in $[0,1) \cap \mathbb{Q}$ for each $\ell \in \mathbb{Z}$.*

Proof. Later in §2.3 (Theorem 2.3.3) we will show that in fact $[\nabla]$ satisfies $p([\nabla]) = 0$ for

$$p(\lambda) = \prod_{i\in I}\prod_{j=0}^{e_i-1}(\lambda - \frac{j}{e_i}).$$

Hence so is $R^\ell f_*[\nabla]$ and this implies the eigenvalues are in $[0,1) \cap \mathbb{Q}$.

Alternatively, by Grothendieck spectral sequence

$$E_2^{p,q} = R^p f_*\mathscr{H}^q(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y) \Rightarrow R^{p+q} f_*(\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y),$$

it suffices to show that the induced operator $R^p f_* \mathscr{H}^q[\nabla]$ on $R^p f_* \mathscr{H}^q \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ has eigenvalues in $[0,1) \cap \mathbb{Q}$ for each $q \in \mathbb{Z}$ since $E_\infty^{p,q}$ is a sub-quotient of $E_2^{p,q}$. The following is proved by Steenbrink [Ste76, Proposition 1.13]:

Lemma 2.2.4. *The stalk of $\mathscr{H}^q \Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ at a point u is generated by the germs $(t^{\frac{a}{e}} \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \xi_{i_{q+n}})_u$ for all $0 \le a < e$ and all $0 \le i_1, i_2, ..., i_{q+n} \le n$ over the ring $\mathbb{C}\{t^{\frac{1}{e}}\}/t\mathbb{C}\{t^{\frac{1}{e}}\}$ where e is the* gcd *of $e_0, e_1, ..., e_k$ and $\mathbb{C}\{t^{\frac{1}{e}}\}$ is the ring of convergent power series with the variable $t^{\frac{1}{e}}$.*

We will elaborate the proof of the lemma later. Temporarily admitting the lemma, then

$$\mathscr{H}^q[\nabla]_u (t^{\frac{a}{e}} \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \xi_{i_{q+n}})_u = (\frac{a}{e} t^{\frac{a}{e}} \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_{q+n}})_u,$$

meaning that the eigenvalues of $\mathscr{H}^q[\nabla]$ are $0, \frac{1}{e}, \frac{2}{e}, ..., \frac{e-1}{e} \in [0,1) \cap \mathbb{Q}$ in a neighborhood of u. This implies that there exists an open neighborhood U containing u and a polynomial $p_U(\lambda)$ whose roots are in $[0,1) \cap \mathbb{Q}$ such that $p_U(\mathscr{H}^q[\nabla]) = 0$ over U. By the properness of Y, we can take a finite open covering $\mathcal{U} = \{U_i\}$ of Y such that $p(\mathscr{H}^q[\nabla]) = \prod_i p_{U_i}(\mathscr{H}^q[\nabla]) = 0$. It follows that $p(R^p f_* \mathscr{H}^q[\nabla]) = 0$, meaning eigenvalues of $R^p f_* \mathscr{H}^q[\nabla]$ in $[0,1) \cap \mathbb{Q}$. □

Proof of Lemma 2.2.4. We will actually prove the original statement of [Ste76, Proposition 1.13] that, in the same notations as in the lemma, the stalk at a point u of $\mathscr{H}^q \Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is generated by germs

$$\left(t^{\frac{a}{e}} \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \xi_{i_{q+n}}\right)_u$$

for all $a \in \mathbb{Z}_{\ge 0}$ and all tuples $0 \le i_1, i_2, ..., i_{q+n} \le n$ over $\mathbb{C}\{t^{\frac{1}{e}}\}$ The lemma is a direct corollary.

The complex of stalks $\Omega^{\bullet+n}_{X/\Delta}(\log Y)_u$ can be identified with the Kozul complex of operators $D_1, D_2, ..., D_n$ on $\mathscr{O}_{X,u}$ putting in degree $-n, -n+1, ..., 0$. Define $G^j \Omega^\ell_{X/\Delta}(\log Y)_u$ to be the submodules of $\Omega^\ell_{X/\Delta}(\log Y)_u$ spanned by the germs

$$\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_\ell} \quad \text{for} \quad \#\{m : i_m \le k\} \ge j.$$

Then $\{G^\ell\Omega^\bullet_{X/\Delta}(\log Y)_u\}_{\ell\in\mathbb{Z}}$ is a decreasing filtration of $\Omega^\bullet_{X/\Delta}(\log Y)_u$. The associated spectral sequence has $E_0^{r,\bullet} = \mathrm{gr}^r_G\Omega^{r+\bullet}_{X/\Delta}(\log Y)_u$. Notice that $\mathrm{gr}^r_G\Omega^{r+\bullet}_{X/\Delta}(\log Y)_u$ can be identified with direct sums of Koszul complex of operators $D_{k+1}, D_{k+2}, ..., D_n$ on $\mathscr{O}_{X,u}$, so $E_1^{r,\ell} = H^{r+\ell}(\mathrm{gr}^r_G\Omega^\bullet_{X/\Delta}(\log Y)) = 0$ for $\ell \neq 0$ and $E_1^{r,0}$ is spanned by germs

$$\xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_\ell} \text{ such that } \#\{i_m \leq k\} = j$$

over $\mathbb{C}\{z_0, z_1, ..., z_k\}$, thanks to the usual Poincaré lemma. Consequently, the spectral sequence degenerates at E_2 with $E_2^{r,0} = \mathscr{H}^r(\Omega^\bullet_{X/\Delta}(\log Y))_u$. Now $E_1^{\bullet,0}$ is the Koszul complex of operators $D_1, D_2, ..., D_k$ on $\mathbb{C}\{z_0, z_1, ..., z_k\}$. Because each D_i for $0 \leq i \leq k$ is a homogenous differential operator, E_2 can be computed monomial by monomial.

For simplicity let $\xi_{i_1,i_2,...,i_r} = \xi_{i_1} \wedge \xi_{i_2} \wedge \cdots \wedge \xi_{i_r}$. Now I claim that a cocycle

$$v = \sum_{i_1<i_2,...<i_r} c_{i_1,i_2,...,i_r} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,..,i_r} \in E_1^{r,0}$$

is cohomologous to zero if $A_j := a_j/e_j - a_0/e_0 \neq 0$ for some $1 \leq j \leq k$. Note that $D_j(z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}) = A_j z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}$ for every $1 \leq j \leq k$. Since v is a cocycle, the coefficients satisfy

$$\sum_{\ell=1}^{r} (-1)^\ell c_{i_1,i_2,...,\hat{i_\ell},...,i_{r+1}} A_{i_\ell} = 0. \tag{2.2.4}$$

Assume that not all A_j's are zero for $1 \leq j \leq k$ then $A = \sum A_i^2$ is non-zereo. Then the number

$$d_{i_1,i_2,...,i_{r-1}} = \sum_{\alpha=1}^{k} \frac{A_\alpha}{A} c_{\alpha,i_1,i_2,...,i_{r-1}}.$$

is well-defined. Here we extend standardly that $c_{\sigma(i_1),\sigma(i_2),..,\sigma(i_r)} = \mathrm{sign}(\sigma) c_{i_1,i_2,...,i_r}$ for any permutation σ. Then the element

$$\sum_{i_1<i_2<...<i_{r-1}} d_{i_1,i_2,...,i_{r-1}} z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k} \xi_{i_1,i_2,...,i_{r-1}}$$

in $E_1^{r-1,0}$ has coboundary

$$\begin{aligned}
&\sum_{\alpha=1}^{k}\sum_{i_1<\dots<i_{r-1}} A_\alpha d_{i_1,i_2,\dots,i_{r-1}} z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}\xi_{\alpha,i_1,i_2,\dots,i_{r-1}} \\
&= \sum_{i_1<\dots<i_r}\sum_{\ell=1}^{r}(-1)^\ell A_{i_\ell} d_{i_1,i_2,\dots,\hat{i_\ell},\dots,i_r} z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}\xi_{i_1,i_2,\dots,i_r} \\
&= \sum_{i_1<\dots<i_r}\sum_{\alpha=1}^{k}\sum_{\ell=1}^{r}(-1)^\ell \frac{A_{i_\ell}A_\alpha}{A} c_{\alpha,i_1,i_2,\dots,\hat{i_\ell},\dots,i_r} z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}\xi_{i_1,i_2,\dots,i_r} \\
\text{applying (2.2.4)} \;&= \sum_{i_1<\dots<i_r}\sum_{\alpha=1}^{k} \frac{A_\alpha^2}{A} c_{,i_1,i_2,\dots,i_r} z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}\xi_{i_1,i_2,\dots,i_r} = v.
\end{aligned}$$

We conclude the claim. Therefore, $E_2^{r,0}$ is generated over $\mathbb{C}$ by $z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}\xi_{i_1,i_2,..,i_r}$ with

$$D_i(z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k}) = 0.$$

That is, $z_0^{a_0} z_1^{a_1}\cdots z_k^{a_k} = t^{a/e}$ for some a. Hence, we conclude the lemma. □

Theorem 2.2.5. *Let $\mathcal{F}^\bullet$ be a complex of $\mathscr{O}_\Delta$-modules with coherent cohomologies, equipped with a log connection, i.e an operator*

$$\nabla \in \operatorname{End}_{\mathbf{D}^b(\Delta,\mathbb{C})}(\mathcal{F}^\bullet) \quad \text{such that } [\nabla, g] = tg'$$

for ant local holomorphic function g where g' is the derivative of g. Assume that the residue R_ℓ of ∇ defined in the beginning of this subsection acting on each cohomology $\mathscr{H}^\ell(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0))$ has eigenvalues in $[0,1)$. Then every $\mathscr{H}^\ell(\mathcal{F}^\bullet)$ is locally free.

Proof. By the definition of residue, we have the morphism of distinguished triangles

$$\begin{array}{ccccccc}
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^\bullet\otimes^L\mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1] \\
\downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle (\nabla+1)[1]} \\
\mathcal{F}^\bullet & \xrightarrow{t} & \mathcal{F}^\bullet & \longrightarrow & \mathcal{F}^\bullet\otimes^L\mathbb{C}(0) & \longrightarrow & \mathcal{F}^\bullet[1]
\end{array}$$

in $\mathbf{D}^b(\Delta,\mathbb{C})$. Taking cohomologies gives

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & \mathscr{H}^{\ell-1}(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \xrightarrow{t} & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)) & \longrightarrow\cdots \\
& \downarrow{\scriptstyle R_\ell} & & \downarrow{\scriptstyle \nabla+1} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle R_{\ell+1}} & \\
\cdots\longrightarrow & \mathscr{H}^{\ell-1}(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \xrightarrow{t} & \mathscr{H}^\ell(\mathcal{F}^\bullet) & \longrightarrow & \mathscr{H}^\ell(\mathcal{F}^\bullet\otimes^L\mathbb{C}(0)) & \longrightarrow\cdots
\end{array} \tag{2.2.5}$$

For simplicity, fix ℓ and let $\mathscr{H} = \mathscr{H}^\ell(\mathcal{F}^\bullet)$ and denote by $\ker t$ the kernel of the morphism $t : \mathscr{H} \to \mathscr{H}$. It suffices to prove that $\ker t$ is trivial on $\mathscr{H}$. We are going to show that $\ker t$ is a subset of $t^k\mathscr{H}$ for all $k \geq 0$ and thus, by Krull's theorem $\ker t$ is zero.

It follows from the diagram (2.2.5) that $\nabla + 1$ on $\ker t$ and ∇ on $\mathscr{H}/t\mathscr{H}$ have eigenvalues in $[0,1)$. Therefore, there exists a polynomial $b_1(s) \in \mathbb{C}[s]$ with roots in $[0,1)$ such that

$$b_1(\nabla)\mathscr{H} \subset t\mathscr{H},$$

and another a polynomial $b_2(s) \in \mathbb{C}[s]$ with eigenvalues in $[0,1)$ such that

$$b_2(\nabla + 1)\ker t = 0.$$

Suppoe v is an element in $\ker t \cap t^k\mathscr{H}$ for some $k \geq 0$. It follows that $v = t^k v_1$ for some $v_1 \in \mathscr{H}$. Because the roots of $b_1(s-k)$ are bigger then the roots of $b_2(s+1)$, the two polynomials $b_1(s-k)$ and $b_2(s+1)$ are relative prime. We deduce that there exist $p(s), q(s) \in \mathbb{C}[s]$ such that

$$1 = p(s)b_1(s-k) + q(s)b_2(s+1).$$

Therefore, combining the fact that $b_2(\nabla+1)v$ vanishes,

$$v = p(\nabla)b_1(\nabla - k)v + q(\nabla)b_2(\nabla+1)v = p(\nabla)b_1(\nabla-k)t^k v_1.$$

Because of the identity $(\nabla - k)t^k = t^k\nabla$, the above is equivalent to

$$v = t^k p(\nabla + k)b_1(\nabla)v_1.$$

Because $b_1(\nabla)v_1 = tv_2$ for some $v_2 \in \mathscr{H}$, substituting in the last equality yields

$$v = t^k p(\nabla+k)b_1(\nabla)v_1 = t^k p(\nabla+k)b_1(\nabla)tv_2 = t^{k+1}p(\nabla+k+1)b_1(\nabla+1)v_2 \in t^{k+1}\mathscr{H}.$$

We proved that v is also an element in $t^{k+1}\mathscr{H}$. By induction and Krull's theorem we conclude the proof. □

Now we can immediately finish

Proof of Theorem 2.2.2. The complex $Rf_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ with $Rf_*\nabla$ satisfies the condition of Theorem 2.2.5. Therefore, each cohomology $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ is locally free. The second statement in the theorem follows from the the locally freeness of $R^\ell f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)$ plus the Grauert's base change theorem. □

2.3 Transfer to $\mathscr{D}$–modules

Lemma 2.2.4 implies the restriction of the relative log de Rham complex on Y is semi-perverse. Indeed, it is even perverse, showed in [Ste76, §2]. Therefore, there should be a regular holonomic $\mathscr{D}$-module whose de Rham complex is the restriction of the relative log de Rham complex on Y, in the view of Riemann-Hilbert correspondence established by Kashiwara [Kas84] and Mebkhout [Meb84]. The stupid filtration should also translates to a coherent filtration from Hodge theoretic point of view. Then the endomorphism $[\nabla]$ in the derived category can be captured by an endomorphism of a $\mathscr{D}$-module. This enable us to study the relation between the filtration and $[\nabla]$ much easier and cleaner. In this section, we will construct the filtered $\mathscr{D}$-module and the endomorphism.

2.3.1 Construction of filtered holonomic $\mathscr{D}_X$-modules

Since $\mathscr{T}_{X/\Delta}(\log)$ is a subsheaf of $\mathscr{T}_X$, the multiplication by sections in $\mathscr{T}_{X/\Delta}(\log Y)$ induces a morphism $\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, with $P \mapsto \sum_{i=1}^k \xi_i \otimes D_i P$ locally. The morphism extends to a filtered complex of $\mathscr{D}_X$-modules

$$\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X = \{\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y) \otimes \mathscr{D}_X \to \cdots \to \Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\}[n] \qquad (2.3.1)$$

with filtration $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$ given by

$$\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes F_{\ell+n+\bullet}\mathscr{D}_X = \{F_\ell\mathscr{D}_X \to \Omega_{X/\Delta}(\log Y)\otimes F_{\ell+1}\mathscr{D}_X \to \cdots \to \Omega^n_{X/\Delta}(\log Y)\otimes F_{\ell+n}\mathscr{D}_X\}[n].$$

Let $\tilde{\mathcal{M}}$ be the 0-th cohomology of $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ and $F_\ell\mathcal{M}$ be the $\mathscr{O}_X$-submodule induced by the the filtration $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$.

Theorem 2.3.1. *The complex* $\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$ *is a filtered resolution of a filtered* $\mathscr{D}_X$*-module* $(\tilde{\mathcal{M}},F_\bullet\tilde{\mathcal{M}})$.

Proof. Notice that $\mathrm{gr}^F\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)=\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathrm{gr}^F\mathscr{D}_X$, can be identified locally with the Koszul complex associated to the regular sequence $D_1,D_2,...,D_n$ over the ring $\mathrm{gr}^F\mathscr{D}_X$. It follows that $\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathrm{gr}^F\mathscr{D}_X$ is acyclic. Therefore, each graded peace $\mathrm{gr}^F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)$ is acyclic. We deduce inductively that $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)$ is also acyclic; this can be seen from the long exact sequence associated to the short exact sequence

$$0\to F_{\ell-1}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\to F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\to\mathrm{gr}^F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\to 0.$$

Taking direct limit, we conclude that $\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$ is a resolution of $\tilde{\mathcal{M}}$. The long exact sequence also implies the 0-th cohomology of $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)$ is isomorphic to $F_\ell\tilde{\mathcal{M}}$. This completes the proof. □

Remark 2.3.2. Note that $\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$ is a complex of $(f^{-1}\mathscr{O}_\Delta,\mathscr{D}_X)$-bimodules because $\Omega^{n+\bullet}_{X/\Delta}(\log Y)$ is $f^{-1}\mathscr{O}_\Delta$-linear. It follows that $\tilde{\mathcal{M}}$ is also a $(f^{-1}\mathscr{O}_\Delta,\mathscr{D}_X)$-bimodule. Note we have two different actions of t on $\tilde{\mathcal{M}}$ due to the bimodule structure. We usually use the left multiplication by t. One can think of $\tilde{\mathcal{M}}$ as a flat family assembling the $\mathscr{D}$-module $i_{X_p+}\omega_{X_p}$ of the smooth fibers X_p for $p\in\Delta$ and a specialization $\mathcal{M}=\tilde{\mathcal{M}}/t\tilde{\mathcal{M}}$ because using the left $f^{-1}\mathscr{O}_\Delta$ structure, we have filtered isomorphisms

$$\mathbb{C}(p)\otimes\tilde{\mathcal{M}}\simeq\mathbb{C}(p)\otimes\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\simeq\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_{X_p}\otimes\mathscr{D}_X\simeq i_{X_p*}\Omega^{n+\bullet}_{X_p}\otimes\mathscr{D}_X\simeq i_{X_p+}\omega_{X_p},$$

where $i_{X_p}:X_p\to X$ is the closed embedding of the smooth fiber X_p.

Remark 2.3.3. The theorem also says by choosing the local trivialization $\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n$ of $\Omega^n_{X/\Delta}(\log Y)$, $\tilde{\mathcal{M}}$ can be identified locally with $\mathscr{D}_X/(D_1,D_2,...,D_n)\mathscr{D}_X$ and $\mathrm{gr}^F\tilde{\mathcal{M}}$ can be identified locally with $\mathrm{gr}^F\mathscr{D}_X/(D_1,D_2,...,D_n)\mathrm{gr}^F\mathscr{D}_X$.

Remark 2.3.4. Let $\mathscr{D}_{X/\Delta}(\log Y)$ be the subalgebra of $\mathscr{D}_X$ generated by $\mathscr{T}_{X/\Delta}(\log Y)$. One can show that $\tilde{\mathcal{M}}$ is nothing but

$$\omega_{X/\Delta}(\log Y) \underset{\mathscr{D}_{X/\Delta}(\log Y)}{\otimes} \mathscr{D}_X.$$

And the filtration $F_\bullet \tilde{\mathcal{M}}$ is induced from $F_\bullet \omega_{X/\Delta}(\log Y)$, where $F_\ell \omega_{X/\Delta}(\log Y)$ is $\omega_{X/\Delta}(\log Y)$ for $\ell \geq -n$ and is zero otherwise. To keep the proof elementary, we avoid talking about $\mathscr{D}_{X/\Delta}(\log Y)$-modules.

Theorem 2.3.5. *The complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X$ is a filtered resolution of a filtered holonimic $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet \mathcal{M})$.*

Proof. Because of the bimodule structure, we have $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y \otimes \mathscr{D}_X$ is the cokernel of the left multiplication by t on $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$. Therefore, the first part of the statement is equivalent to $t : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is injective. It suffices to prove that $t : \mathrm{gr}^F \tilde{\mathcal{M}} \to \mathrm{gr}^F \tilde{\mathcal{M}}$ is injective because the multiplication by t is a filtered morphism. But this follows from $t, D_1, D_2, ..., D_n$ is a regular sequence over the ring $\mathrm{gr}^F \mathscr{D}_X$. It also follows that $\mathrm{gr}^F \mathcal{M}$ is isomorphic locally to $\mathrm{gr}^F \mathscr{D}_X / (t, D_1, D_2, ..., D_n) \mathrm{gr}^F \mathscr{D}_X$. This means the characteristic variety of $\mathcal{M}$ is cut out by $t, D_1, D_2, ..., D_n \in \mathscr{O}_{T^*X}$ and thus, the characteristic variety is of dimension $n+1$. This proves the holonomicity of $\mathcal{M}$. $\square$

Remark 2.3.6. Similarly to the case of $\tilde{\mathcal{M}}$, the $\mathscr{D}_X$-module $\mathcal{M}$ is just

$$\omega_{X/\Delta}(\log Y)|_Y \underset{\mathscr{D}_{X/\Delta}(\log Y)}{\otimes} \mathscr{D}_X$$

with the filtration $F_\bullet \mathcal{M}$ induced by $\big(F_\bullet \omega_{X/\Delta}(\log Y)\big)|_Y$.

2.3.2 Properties of $\mathcal{M}$

We first calculate the characteristic cycle of $\mathcal{M}$ which is important for later when we identifying the primitive part of $\mathrm{gr}^W \mathcal{M}$. Then we prove that the de Rham complex of $\mathcal{M}$ with the induced filtration recover $\Omega^{\bullet+n}_{X/\Delta}(\log Y)|_Y$ with the stupid filtration. Lastly, we translate the operator $[\nabla] \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}(\Omega^{\bullet+n}_{X/\Delta}(\log Y))|_Y$ to an operator R on $\mathcal{M}$

Theorem 2.3.7. *The characteristic cycle of $\mathcal{M}$ is*

$$cc(\mathcal{M}) = \sum_{J\subset I}\sum_{j\in J} e_j \left[T^*_{Y^J}X\right],$$

where $[T^*_{Y^J}X]$ is the cycle of the conormal bundle of Y^J in T^*X and e_i is the multiplicity of Y along each component Y_i for $i \in I$.

Proof. The statement is local and we identify $\mathcal{M}$ with $\mathscr{D}_X/(t, D_1, D_2, ..., D_n)$. We first describe the characteristic variety of $\mathcal{M}$. The support of $\mathrm{gr}^F\mathcal{M}$ as a sheaf on T^*X is defined by the radical of the ideal $(t, D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X$. In fact, $z_i\partial_i$ for $0 \le i \le k$ is in the radical because

$$(z_i\partial_i)^{e_0+e_1+\cdots+e_k} \equiv (z_0\partial_0)^{e_0}(z_1\partial_1)^{e_1}\cdots(z_k\partial_k)^{e_k} \equiv t\partial_0^{e_0}\partial_1^{e_1}\cdots\partial_k^{e_k} \equiv 0 \,\mathrm{mod}\, (t, D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X.$$

Therefore, $char(\mathcal{M})$ is cut out by $t_{\mathrm{red}}, z_0\partial_0, z_1\partial_1, ..., z_k\partial_k, \partial_{k+1}, ..., \partial_n$, where $t_{\mathrm{red}} = z_0z_1\cdots z_k$. It follows that $char(\mathcal{M}) = \bigcup_{J\subset I} T^*_{Y^J}X$.

Denote by $\mathfrak{p}(Z)$ the prime ideal defining a integral subvariety Z. Let m_J be the length of $\mathrm{gr}^F\mathcal{M}_{\mathfrak{p}(T^*_{Y^J}X)}$ as an Artinian $\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}$-module . Then $cc(\mathcal{M}) = \sum_{J\in I} m_J\left[T^*_{Y^J}X\right]$. For simplicity let us assume $J = \{0, 1, 2, .., \mu\}$ and by abuse of notation we also the prime ideal $\mathfrak{p} = \mathfrak{p}(T^*_{Y^J}X)$ of the variety $T^*_{Y^J}X$ is locally generated by $z_0, z_1, ..., z_\mu, \partial_{\mu+1}, \partial_{\mu+2}, ..., \partial_n$ over $\mathrm{gr}^F\mathscr{D}_X$ in some local coordinate system. Notice that

$$\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}/(t, D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}} = \mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}/(D'_0, D'_1, ..., D'_n)\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}$$

where

$$D'_i = \begin{cases} z_0^{e_0+e_1+\cdots+e_\mu}, & \text{for } i = 0 \\ \dfrac{1}{e_i}z_i - \dfrac{1}{e_0}z_0\dfrac{\partial_0}{\partial_i}, & \text{for } 1 \le i \le \mu \\ \dfrac{1}{e_i}\partial_i - \dfrac{1}{e_0}z_0\dfrac{\partial_0}{z_i}, & \text{for } \mu+1 \le i \le k \\ \partial_i, & \text{for } i > k, \end{cases} \tag{2.3.2}$$

because $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}, z_{\mu+2}, ..., z_k$ are invertible in $\mathrm{gr}^F\mathscr{D}_{X,\mathfrak{p}}$. Therefore, $\mathrm{gr}^F\mathcal{M}_{\mathfrak{p}}$ can be identifies with

$$\mathbb{C}\{z_0\}/(z_0^{e_0+e_1+\cdots+e_\mu}).$$

Then $m_J = \dim_{\mathbb{C}} \mathbb{C}\{z_0\}/(z_0^{e_0+e_1+\cdots+e_\mu}) = \sum_{j\in J} e_j$. This completes the computation. □

Remark 2.3.8. The above theorem verifies that $cc(\mathcal{M}) = \lim_{p\to 0} cc(i_{p_+}\omega_{X_p}) = \lim_{p\to 0}\left[T^*_{X_p}X\right]$ as cycles in algebraic cotangent space T^*X for $p \in \Delta^*$ where $i_p : X_p \to X$ the closed embedding of the smooth fiber. In fact, one can show that $\mathbb{C}(p) \otimes \mathrm{gr}^F \tilde{\mathcal{M}}$, using the left $f^{-1}\mathscr{O}_\Delta$-module structure, is isomorphic to $\mathrm{gr}^F i_{p_+}\omega_{X_p}$ as in Remark 2.3.2. Refer to [Gin86a] for general results about the characteristic cycles of specializations of holonomic $\mathscr{D}$-modules.

Corollary 2.3.9. *The de Rham complex* $\mathrm{DR}_X\mathcal{M}$ *together with filtration* $F_\bullet\mathrm{DR}_X\mathcal{M}$ *is isomorphic to* $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y$ *with the stupid filtration in the derived category of filtered complexes of sheaves of* $\mathbb{C}$*-vector spaces.*

Proof. We have showed that $F_\ell\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)$ is a resolution of $F_\ell\mathcal{M}$. Therefore, the total complex of $F_{\ell+*}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\otimes\bigwedge^{-*}\mathscr{T}_X$ is quasi-isomorphic to $F_{\ell+*}\mathcal{M}\otimes\bigwedge^{-*}\mathscr{T}_X$, which is exactly $F_\ell\mathrm{DR}_X\mathcal{M}$. It remains to show the total complex also quasi-isomorphic to $F_\ell\Omega^{n+\bullet}_{X/\Delta}(\log Y)$. This follows from that

$$
\begin{aligned}
F_{\ell+*}\left(\Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X\right)\otimes\bigwedge^{-*}\mathscr{T}_X &= \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes F_{\ell+n+\bullet}\left(\mathscr{D}_X\otimes\bigwedge^{-*}\mathscr{T}_X\right)\\
&\simeq \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes F_{\ell+n+\bullet}\mathscr{O}_X\\
&= F_\ell\Omega^{n+\bullet}_{X/\Delta}(\log Y).
\end{aligned}
$$

Here, $F_\ell\mathscr{O}_X = \mathscr{O}_X$ for $\ell \geq 0$ and otherwise it is zero. □

Theorem 2.3.10. *The endomorphism* $\nabla \in \mathrm{End}_{\mathbf{D}^b(X,\mathbb{C})}\Omega^{n+\bullet}_{X/\Delta}(\log Y)$ *in Lemma 2.2.1 transfers to a filtered morphism*

$$\nabla : (\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}}) \to (\tilde{\mathcal{M}}, F_{\bullet+1}\tilde{\mathcal{M}}), \quad [[\alpha]\otimes P] \mapsto (\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}$$

where $\alpha \in \Omega^n_X(\log Y)$ *and* $P \in \mathscr{D}_X$ *so that* $[\alpha]\otimes P \in \Omega^n_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$*. Moreover, restriction on* Y *yields a filtered morphism*

$$R : (\mathcal{M}, F_\bullet\mathcal{M}) \to (\mathcal{M}, F_{\bullet+1}\mathcal{M})$$

such that

$$\prod_{i\in I}\prod_{j=0}^{e_i-1}(R-\frac{j}{e_i})=0. \tag{2.3.3}$$

Proof. The morphism $\frac{dt}{t}\wedge : \Omega^{n+\bullet}_{X/\Delta}(\log Y) \to \Omega^{n+1+\bullet}_X(\log Y)$ extends to the corresponding complexes of induced $\mathscr{D}_X$-modules

$$\frac{dt}{t}\wedge : \Omega^{n+\bullet}_{X/\Delta}(\log Y)\otimes\mathscr{D}_X \to \Omega^{n+1+\bullet}_X(\log Y)\otimes\mathscr{D}_X.$$

Let $\mathrm{Cone}^\bullet\otimes\mathscr{D}_X$ be the mapping cone of the above morphism. We get a diagram of complexes of $\mathscr{D}_X$-modules similarly to (2.2.2) and taking 0-th cohomology we get the following.

$$\begin{array}{ccc} \mathscr{H}^0(\mathrm{Cone}^\bullet\otimes\mathscr{D}_X) & \xrightarrow{q} & \tilde{\mathcal{M}} \\ \downarrow p & \swarrow_{p\circ q^{-1}} & \\ \tilde{\mathcal{M}} & & \end{array} \tag{2.3.4}$$

where abuse of notation, still denote by p and q the induced morphisms from diagram (2.2.2). Now q is an isomorphism of $\mathscr{D}_X$-modules. Let $[\alpha\otimes P,[\beta]\otimes Q]$ be a class in $\mathscr{H}^0(\mathrm{Cone}^\bullet\otimes\mathscr{D}_X)$ for any $\alpha\otimes P\in\Omega^n_X(\log Y)\otimes\mathscr{D}_X$ and $[\beta]\otimes Q\in\Omega^n_{X/\Delta}(\log Y)\otimes\mathscr{D}_X$. Then

$$\delta(\alpha\otimes P,[\beta]\otimes Q)=\Big((-1)^n d(\alpha\otimes P)+\frac{dt}{t}\wedge\beta\otimes Q,(-1)^n d([\beta]\otimes Q)\Big)=0.$$

Here, the sign factor $(-1)^n$ shows up due to we follow the Koszul sign rule. Because $\frac{dt}{t}\wedge:\Omega^n_{X/\Delta}(\log Y)\to\Omega^{n+1}_X(\log Y)$ is an isomorphism, we have

$$[\beta]\otimes Q=(-1)^{n-1}(\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}.$$

Therefore, $q^{-1}:\tilde{\mathcal{M}}\to\mathscr{H}^0(\mathrm{Cone}^\bullet\otimes\mathscr{D}_X)$ is given by $[[\alpha]\otimes P]\mapsto[\alpha\otimes P,(-1)^n(\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}]$. Then we have

$$\nabla=(-1)^{n-1}p\circ q^{-1}:[[\alpha]\otimes P]\mapsto(\frac{dt}{t}\wedge)^{-1}\{d(\alpha\otimes P)\}.$$

Restricting to Y we have the induced operator R on $\mathcal{M}$. If $\alpha=\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n$ then

$$\begin{aligned} R[\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n\otimes P] &= (\frac{dt}{t}\wedge)^{-1}\left(d\left(e_1\frac{dz_1}{z_1}\wedge e_2\frac{dz_2}{z_2}\wedge\cdots\wedge dz_n\otimes P\right)\right)\\ &= (\frac{dt}{t}\wedge)^{-1}\left(e_0\frac{dz_0}{z_0}\wedge e_1\frac{dz_1}{z_1}\wedge e_2\frac{dz_2}{z_2}\wedge\cdots\wedge dz_n\otimes\frac{1}{e_0}z_0\partial_0 P\right)\\ &= \left[\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n\otimes\frac{1}{e_0}z_0\partial_0 P\right]. \end{aligned}$$

We see that $RF_\bullet\mathcal{M} \subset F_{\bullet+1}\mathcal{M}$. The reason for $\nabla F_\bullet\tilde{\mathcal{M}} \subset F_{\bullet+1}\tilde{\mathcal{M}}$ is similar. To prove the last statement, we work locally and identify $\mathcal{M}$ with $\mathscr{D}_X/(t, D_1, ..., D_n)$ via the local trivialization $\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n$ of $\Omega^n_{X/\Delta}(\log Y)$. Then for $P \in \mathscr{D}_X$, $R[P] = [\frac{1}{e_0}z_0\partial_0 P]$. In fact, because of the relation $D_1, D_2, ..., D_n$, the left multiplication by $\frac{1}{e_0}z_0\partial_0$ on $\mathcal{M}$ is the same as the multiplication by $\frac{1}{e_i}z_i\partial_i$ for $1 \le i \le k$. It follows from the identity

$$(z\partial)(z\partial - 1)\cdots(z\partial - \ell) = z^{\ell+1}\partial^{\ell+1}$$

for any $\ell \ge 0$ that

$$\prod_{i\in I}\prod_{j=0}^{e_i-1}(R - \frac{j}{e_i})[P] = \prod_{i\in I}\prod_{j=0}^{e_i-1}(\frac{1}{e_i}z_i\partial_i - \frac{j}{e_i})[P] = \prod_{i\in I}\frac{1}{e_i^{e_i}}z_i^{e_i}\partial_i^{e_i}[P] = t\prod_{i\in I}\frac{1}{e_i^{e_i}}\partial_i^{e_i}[P]$$
$$= 0 \in \mathscr{D}_X/(D_1, D_2, ..., D_n, t)\mathscr{D}_X.$$

This completes the proof. □

Remark 2.3.11. Note that $\nabla : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ is also can be identified with the left multiplication by $\frac{1}{e_i}z_i\partial_i$ for $i \le k$, by choosing the trivialization of $\Omega^n_{X/\Delta}(\log Y)$, because of the relations $D_i = \frac{1}{e_i}z_i\partial_i - \frac{1}{e_0}z_0\partial_0$ for $1 \le i \le k$. This means for any function $g \in f^{-1}\mathscr{O}_\Delta$, we have $[\nabla, g] = tg'$ where t and g are local sections of $f^{-1}\mathscr{O}_\Delta$ acting on the left of $\tilde{\mathcal{M}}$. This makes $\tilde{\mathcal{M}}$ a $(f^{-1}\mathscr{D}_\Delta(\log 0), \mathscr{D}_X)$-bimodule. Using Godement resolution, the direct image $Rf_*\mathrm{DR}_X\tilde{\mathcal{M}}$ is a complex of left $\mathscr{D}_\Delta(\log 0)$-modules. Similarly, as we already saw in the proof, locally the morphism $R : \mathcal{M} \to \mathcal{M}$ can be identified with left multiplication by $\frac{1}{e_i}z_i\partial_i$ for $0 \le i \le k$, meaning $[R, g] = tg' = 0$ for g local sections of $f^{-1}\mathscr{O}_\Delta$ acting left on $\mathcal{M}$.

Remark 2.3.12. The $\mathscr{D}_X$-module $\mathcal{M}$ is even regular holonomic. Even though it is irrelevant for our purpose, we can also check $\mathcal{M}$ is regular using the definition. Recall that a holonomic right $\mathscr{D}_Z$-module $\mathcal{N}$ is called *regular* if there exists a good filtration $F_\bullet\mathcal{N}$ such that for any $\sigma \in \mathrm{gr}^F\mathscr{D}_Z$ vanishing on the charateristic variety of $\mathcal{N}$ one has $\mathrm{gr}^F\mathcal{N}\sigma = 0$. In the case of $\mathcal{M}$, define locally

$$G_\ell\mathcal{M} = \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M}t^r_{\mathrm{red}}$$

where $t_{\text{red}} = z_0 z_1 \cdots z_k$. This is a finite sum because $\mathcal{M}$ is supported on $t = 0$ and R has a characteristic polynomial. It follows that $G_\bullet$ is a good filtration for $\mathcal{M}$. I claim that $G_\bullet\mathcal{M}$ gives the filtration in the definition of the regularity. Since the characteristic variety of $\mathcal{M}$ is locally cut out by $t_{\text{red}}, z_0\partial_0, ..., z_k\partial_k, \partial_{k+1}, ..., \partial_n$ (see Theorem 2.3.7) it suffices to check that $G_\ell\mathcal{M}t_{\text{red}} \subset G_{\ell-1}\mathcal{M}$, $G_\ell\mathcal{M}z_i\partial_i \subset G_\ell\mathcal{M}$ for $0 \le i \le k$ and $G_\ell\mathcal{M}\partial_i \subset G_\ell\mathcal{M}$ for $k+1 \le i \le n$. It is clear that $G_\ell\mathcal{M}t_{\text{red}} \subset G_{\ell-1}\mathcal{M}$. Due to locally $\text{gr}^F\mathcal{M} = \text{gr}^F\mathscr{D}_X/(t, D_1, D_2, ..., D_n)\text{gr}^F\mathcal{M}$, it follows that $\text{gr}^F\mathcal{M}D_i = 0$ for $1 \le i \le n$. In particular, $\text{gr}^F\mathcal{M}\partial_i = 0$ for $k+1 \le i \le n$, i.e. $F_\ell\mathcal{M}\partial_i \subset F_\ell\mathcal{M}$ for $k+1 \le i \le n$. Therefore, for $k+1 \le i \le n$, because $[t_{\text{red}}, \partial_i] = 0$,

$$G_\ell\mathcal{M}\partial_i = \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M}t^r_{\text{red}}\partial_i \subset \sum_{r,k\ge 0} R^k F_{\ell+r}\mathcal{M}t^r_{\text{red}} = G_\ell\mathcal{M}.$$

Since $[t^r_{\text{red}}, z_i\partial_i] = (z_i\partial_i - r)\, t^r_{\text{red}}$, and $[z_i\partial_i, F_\ell\mathscr{D}_X] \subset F_\ell\mathscr{D}_X$, we have

$$R^k F_{\ell+r}\mathcal{M}t^r_{\text{red}}z_i\partial_i = R^k F_{\ell+r}\mathcal{M}\,(z_i\partial_i - r)\, t^r_{\text{red}} \subset R^k\,(z_i\partial_i F_{\ell+r}\mathcal{M} + F_{\ell+r}\mathcal{M})\, t^r_{\text{red}}.$$

But locally R has the same effect as the left multiplication by one of $\frac{1}{e_i}z_i\partial_i$ for $0 \le i \le k$. Hence,

$$R^k\,(z_i\partial_i F_{\ell+r}\mathcal{M} + F_{\ell+r}\mathcal{M})\, t^r_{\text{red}} = R^{k+1} F_{\ell+r}\mathcal{M}t^r_{\text{red}} + R^k F_{\ell+r}\mathcal{M}t^r_{\text{red}}.$$

It follows that $G_\ell\mathcal{M}z_i\partial_i \subset G_\ell\mathcal{M}$ for $0 \le i \le k$.

In fact, later we will see that $\mathcal{M}$ is an extensions of regular holonomic $\mathscr{D}_X$-modules which will again prove that $\mathcal{M}$ is regular (see Theorem 2.4.7 for the reduced case and Theorem 2.6.13 for the general case).

2.4 Reduced case: Strictness and the weight filtration

We begin to study the weight filtration $W_\bullet\mathcal{M}$ induced R on $\mathcal{M}$. For simplicity to state the results and illustrate the ideas, we assume Y is reduced in §2.4 and §2.5. The general case will be treated in §2.6 and §2.7. Since Y is reduced, the multiplicity e_i of irreducible component Y_i is 1 and R is nilpotent. Recall that the weight filtration of the nilpotent operator R is uniquely characterized by the following two properties:

- for each $\ell \in \mathbb{Z}$, $R : W_\ell \mathcal{M} \to W_{\ell-2}\mathcal{M}$;
- the induced operator $R^\ell : \mathrm{gr}^W_\ell \mathcal{M} \to \mathrm{gr}^W_{-\ell}\mathcal{M}$ is an isomorphism for each $\ell \geq 0$.

2.4.1 Strictness of R

Let $F_\bullet W_r \mathcal{M} = F_\bullet \mathcal{M} \cap W_r \mathcal{M}$ be the induced filtration for every integer r. In fact, the good filtration and the weight filtration interact nicely because of the following theorem.

Theorem 2.4.1. *The power of R is strict on $(\mathcal{M}, F_\bullet \mathcal{M})$, i.e., $R^a F_b \mathcal{M} = F_{a+b} R^a \mathcal{M}$.*

Proof. The strictness is a local property; therefore, we can assume $\mathcal{M} = \mathscr{D}_X/(t, D_1, D_2, ..., D_n)\mathscr{D}_X$ and R is left multiplication by $z_0\partial_0$ on it, recalling that $D_i = z_i\partial_i - z_0\partial_0$ for $1 \leq i \leq k$ and $D_i = \partial_i$ for $k+1 \leq i \leq n$. It is clear that $R^a F_b \mathcal{M}$ is contained in $F_{a+b}R^a\mathcal{M}$. It suffices to show that for every $R^a P \in F_{a+b}\mathcal{M}$, we can find an element $Q \in F_b\mathcal{M}$ such that $R^a P = R^a Q$. Assume $P \in F_\ell \mathcal{M}$. If $\ell \leq b$ then there is nothing to prove. Thus, we consider the situation that $\ell > b$. Then the class of $R^a P$ vanishes in $\mathrm{gr}^F_{a+\ell}\mathcal{M}$. In fact, we have the following lemma:

Lemma 2.4.2. *Denote by $[R]$ the induced operator on $\mathrm{gr}^F\mathcal{M}$. Then $\ker[R]^{r+1}$ is locally generated by the classes of all degree $k-r$ monomials dividing $t = z_0 z_1 \cdots z_k$.*

We can easily check that monomials of degree $k-r$ dividing t is in $\ker[R]^{r+1}$. Indeed, it is already true that monomials of degree $k-r$ dividing t is in $\ker R^{r+1}$. Without loss of generality, we only need to check this for the monomial $z_{r+1}z_{r+2}\cdots z_k$:

$$R^{r+1} z_{r+1}z_{r+2}\cdots z_k = z_0\partial_0 z_1\partial_1 \cdots z_r\partial_r z_{r+1}z_{r+2}\cdots z_k = t\partial_0\cdots\partial_k = 0 \in \mathcal{M}.$$

We will prove the opposite direction after finishing the proof of the theorem. Going back to the proof of the theorem, by the above lemma,

$$P = \sum_{\substack{J \subset I, \\ \#J = k-a+1}} z_J Q_J + Q_{\ell-1}$$

where $z_J = \prod_{j\in J} z_j$, $Q_J \in F_\ell \mathcal{M}$ and $Q_{\ell-1} \in F_{\ell-1}\mathcal{M}$. But R^a kills the monomials z_J of degree $k-a+1$ dividing t. It follows that $R^a P = R^a Q_{\ell-1}$. Iterating the procedure, we eventually find an element $Q \in F_b\mathcal{M}$ such that $R^a P = R^a Q$ with $Q \in F_b \mathcal{M}$. □

Proof of Lemma 2.4.2. Note that we are over the commutative ring $\mathrm{gr}^F \mathscr{D}_X$. We proceed by induction on r. Let $P \in \mathrm{gr}^F\mathscr{D}_X$ be a representative of an element in $\ker[R]^{r+1}$. When $r=0$, we have

$$z_0\partial_0 P = tQ_0 + \sum_{i=1}^n D_i Q_i.$$

Then $tQ_0 \in (\partial_0, \partial_1, ..., \partial_n)\mathrm{gr}^F\mathscr{D}_X$. Notice that $t, \partial_0, \partial_1, ..., \partial_n$ is a regular sequence over $\mathrm{gr}^F\mathscr{D}_X$. We have $Q_0 = \sum_{i=0}^n \partial_i Q_i'$. This implies

$$\begin{aligned} z_0\partial_0 P &= \sum_{i=0}^k \frac{t}{z_i} z_i \partial_i Q_i' + \sum_{j=k+1}^n t\partial_j Q_j' + \sum_{i=1}^n D_i Q_i \\ &= \sum_{i=0}^k \frac{t}{z_i} z_0\partial_0 Q_i' + \sum_{i=1}^k D_i\Big(Q_i + \frac{t}{z_i}Q_i'\Big) + \sum_{j=k+1}^n D_j(Q_j + tQ_j'), \end{aligned}$$

from which we conclude that $z_0\partial_0(P - \sum_{i=0}^k \frac{t}{z_i}Q_i') \in (D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X$. Because $z_0\partial_0, D_1, D_2, ..., D_n$ is again a regular sequence, we see that $P - \sum_{i=0}^k \frac{t}{z_i}Q_i' \in (D_1, D_2, ..., D_n)\mathrm{gr}^F\mathscr{D}_X$. This concludes the base case for the induction.

Assume the statement is true for the cases when the exponent is less then $r+1$. Let $z_J = \prod_{j\in J} z_j$. Now for $[P] \in \ker[R]^{r+1}$, we have $[R][P]$ is in $\ker[R]^r$. By induction,

$$z_0\partial_0 P = \sum_{\substack{\#J=k-r+1,\\ J\subset I}} z_J Q_J + \sum_{i=1}^n D_i Q_i. \tag{2.4.1}$$

Fix an index subset J of I such that $\#J = k-r+1$. Then $z_J Q_J$ is in the submodule generated by z_i for $i \in I \smallsetminus J$ and ∂_j for $j\in J$ and $k<j\le n$ over $\mathrm{gr}^F\mathscr{D}_X$. Since z_i for $i\in I\smallsetminus J$, ∂_j for $j\in J$ and $k<j\le n$ and z_J form a regular sequence, we have

$$Q_J = \sum_{i\in I\smallsetminus J} z_i Q_i' + \sum_{j\in J}\partial_j Q_j' + \sum_{k<\ell\le n}\partial_\ell Q_\ell'.$$

Therefore, it follows that

$$z_J Q_J = \sum_{i\in I\smallsetminus J} z_J z_i Q_i' + \sum_{j\in J}\left(\frac{z_J}{z_j} z_0\partial_0 Q_j' + D_j \frac{z_J}{z_j} Q_j'\right) + \sum_{k<\ell\le n} D_\ell z_J Q_\ell'.$$

Then substuiting in (2.4.1), we deduce that

$$z_0\partial_0\left(P - \sum_{j\in J}\frac{z_J}{z_j}Q'_j\right) - \sum_{i\in I\setminus J} z_J z_i Q'_i$$

is in the submodule generated by degree $k-r+1$ monomials dividing t except z_J, and $D_1, D_2, ..., D_n$ over $\mathrm{gr}^F\mathscr{D}_X$. It follows that we can reduce the monomials of degree $k-r+1$ dividing t in the right-hand side equation (2.4.1) one by one and at the last step, we get $z_0\partial_0(P-P')-Q'$, where P' is a linear combination of degree $k-r$ monomials dividing t and Q' is a linear combination of $k-r+2$ monomials dividing t, is in the submodule generated by $D_1, ..., D_n$ over $\mathrm{gr}^F\mathscr{D}_X$. But $\ker[R]^{r-1}$ is generated by classes represented by degree $k-r+2$ monomials dividing t by induction hypothesis. It says that the class of $P-P'$ is in $\ker[R]^r$ and by induction it is generated by degree $k-r+1$ monomials dividing t. Therefore, P is a linear combination of degree $k-r$ monomials dividing t. This completes the proof. □

Corollary 2.4.3. *The* $\ker R^{r+1}$ *is also generated by degree $k-r$ monomials dividing t if one identifies $\mathcal{M}$ locally with $\mathscr{D}_X/(t, D_1, D_2, ..., D_n)\mathscr{D}_X$.*

Proof. It suffices to show that $\mathrm{gr}^F \ker R^{r+1}$ is generated by degree $k-r$ monomials dividing t. Notice that $\mathrm{gr}^F \ker R^{r+1}$ is contained in $\ker[R]^{r+1}$, since $[R]^{r+1}$ vanishes on $\mathrm{gr}^F \ker R^{r+1}$. In fact, we have $\mathrm{gr}^F \ker R^{r+1} = \ker[R]^{r+1}$ because degree $k-r$ monomials dividing t are also in $\mathrm{gr}^F \ker R^{r+1}$. □

2.4.2 The weight filtration

The results concerning the weight filtration and Lefschetz decomposition are formal and we will work on the abstract setting.

Theorem 2.4.4. *Let $N : (\mathcal{G}, F_\bullet) \to (\mathcal{G}, F_{\bullet+1})$ be a nilpotent operator on a filtered $\mathscr{D}$-module $(\mathcal{G}, F_\bullet)$. Asume that every power of N satisfies strictness, i.e., $N^a F_b\mathcal{G} = F_{a+b}N^a\mathcal{G}$ for $a \geq 0$ and $b \in \mathbb{Z}$. Then the induced operator $N^r : F_\ell \mathrm{gr}^W_r \mathcal{G} \to F_{\ell+r}\mathrm{gr}^W_{-r}\mathcal{G}$ is an isomorphism for $r \geq 0$, where $W_\bullet$ is the weight filtration induced by N.*

Proof. It suffices to prove that for any $b \in F_{\ell+r}W_{-r}\mathcal{G}$, we could find $a' \in F_\ell W_r\mathcal{G}$ such that $a = N^r a'$. Because $W_{-r}\mathcal{G} \subset N^r\mathcal{G}$, let $N^r a = b$ for some a. Then by strictness, there exists $a' \in F_\ell\mathcal{G}$ such that $N^r a' = N^r a \in W_{-r}\mathcal{G}$. It follows that $a' \in W_r\mathcal{G}$. Indeed, if $a' \in W_{r+k}\mathcal{G}$ for some $k > 0$ such that $a' \neq 0 \in \mathrm{gr}^W_{r+k}\mathcal{G}$. Then $N^{r+k}a' = 0 \in \mathrm{gr}^W_{-r-k}\mathcal{G}$ because $N^r a' = 0 \in \mathrm{gr}^W_{-r+k}\mathcal{G}$, from which we conclude that $a' \in F_\ell W_{r+k-1}\mathcal{G}$. Thus, iterating the procedure, a' is actually in $F_\ell W_r\mathcal{G}$. We conclude the proof. □

Let $\mathcal{P}_r =_{\mathrm{def}} \ker\left(N^{r+1} : \mathrm{gr}^W_r\mathcal{G} \to \mathrm{gr}^W_{-r-2}\mathcal{G}\right)$ be the primitive part of $\mathrm{gr}^W\mathcal{G}$, which can be identified with

$$\frac{\ker N^{r+1}}{\ker N^r + N\ker N^{r+2}}.$$

See Example 2.1.7. Recall the Lefschetz decomposition:

$$\mathrm{gr}^W_r\mathcal{G} = \bigoplus_{\ell\geq 0, -\frac{r}{2}} N^\ell\mathcal{P}_{r+2\ell} \text{ for any } r \in \mathbb{Z}.$$

There are two possible ways to define the filtration on $\mathcal{P}_r$: first we have the natural filtration $F_\ell\mathcal{P}_r$ induced from the inclusion $\mathcal{P}_r \to \mathrm{gr}^W_r\mathcal{G}$ and second we can also define the filtration using

$$\frac{F_\ell \ker N^{r+1} + \ker N^r + N\ker N^{r+2}}{\ker N^r + N\ker N^{r+2}}.$$

But indeed, the two different methods result in the same filtration because of the strictness. Let $m \in F_\ell W_r + W_{r-1}$ such that $N^{r+1}m \in W_{-r-3}$ so that represents a class in $F_\ell\mathcal{P}_r$. It suffices to find an element in $F_\ell\ker N^{r+1}$ representing the same class as m in $F_\ell\mathcal{P}_r$. Let $m = m_1 + m_2$ for $m_1 \in F_\ell W_r$ and $m_2 \in W_{r-1}$. It follows that $N^{r+1}m_1 \in F_{\ell+r+1}W_{-r-3}$ because both $N^{r+1}m, N^{r+1}m_2 \in W_{-r-3}$ and $m_1 \in F_\ell W_r$. Since $N^{r+3} : F_{\ell-2}W_{r+3} \to F_{\ell+r+1}W_{-r-3}$ is surjective, there exists $x \in F_{\ell-2}W_{r+3}$ such that $N^{r+3}x = N^{r+1}m_1 \in F_{\ell+r+1}W_{-r-3}$. See the proof of the above theorem. It follows that $m_1 - N^2x \in F_\ell\ker N^{r+1}$ represents the same element as m in $F_\ell\mathcal{P}_r \subset F_\ell\mathrm{gr}^W_r$.

Corollary 2.4.5. *The Lefschetz decomposition of* $\mathrm{gr}^W\mathcal{G}$ *respects filtrations, i.e.*

$$F_\bullet\mathrm{gr}^W_r\mathcal{G} = \bigoplus_{\ell\geq 0, -\frac{r}{2}} N^\ell F_{\bullet-\ell}\mathcal{P}_{r+2\ell} \textit{ for any } r \in \mathbb{Z}.$$

Returning to our situation, it follows that:

Theorem 2.4.6. *The induced operator* $R^r : F_\ell \mathrm{gr}_r^W \mathcal{M} \to F_{\ell+r} \mathrm{gr}_{-r}^W \mathcal{M}$ *is an isomorphism. Therefore, the Lefschetz decomposition of* $\mathrm{gr}^W \mathcal{M}$ *respects filtrations, i.e.*

$$F_\bullet \mathrm{gr}_r^W \mathcal{M} = \bigoplus_{\ell \geq 0, -\frac{r}{2}} R^\ell F_{\bullet - \ell} \mathcal{P}_{r+2\ell} \textit{ for any } r \in \mathbb{Z}.$$

2.4.3 Identifying the primitive part $\mathcal{P}_r$

Recall that $Y^J = \cap_{j \in J} Y_j$ for a subset J of the index set I and $\tilde{Y}^{(r+1)}$ is the disjoint union of Y^J such that the cardinality of J is $r+1$. The morphism $\tau^{(r+1)} : \tilde{Y}^{(r+1)} \to X$ is the natural morphism induced by the closed embeddings $\tau^J : Y^J \to X$.

Theorem 2.4.7. *There exists a canonical filtered isomorphism* $\phi_r : (\mathcal{P}_r, F_\bullet \mathcal{P}_r) \to \tau_+^{(r+1)} \omega_{\tilde{Y}^{(r+1)}}(-r)$.

Proof. Denote by D^J the normal crossing divisor $Y^J \cap Y_{I \smallsetminus J}$ on Y^J. The residue morphism

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \Omega_X^{\bullet + n+1}(\log Y)|_Y \to \bigoplus_{\# J = r+1} \Omega_{Y^J}^{\bullet + n - r}(\log D^J)$$

extends to a morphism of complexes of filtered induced $\mathscr{D}_X$-modules

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \Omega_X^{\bullet + n+1}(\log Y)|_Y \otimes \mathscr{D}_X \to \bigoplus_{\# J = r+1} \Omega_{Y^J}^{\bullet + n - r}(\log D^J) \otimes \mathscr{D}_X.$$

Denote by $\mathcal{H}^k$ the k-th cohomology $\mathscr{H}^k\left(\Omega_X^{\bullet + n+1}(\log Y)|_Y \otimes \mathscr{D}_X\right)$. Taking 0-th cohomology of the above yields, by Example 2.1.4

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} : \mathcal{H}^0 \to \bigoplus_{\# J = r+1} \tau_+^J \omega_{Y^J}(*D^J)(-r).$$

Since the morphism $\frac{dt}{t} \wedge : \Omega_{X/\Delta}^{\bullet + n}(\log Y) \to \Omega_X^{\bullet + n+1}(\log Y)$ also extends to the complexes of induced $\mathscr{D}_X$-modules, we have a short exact sequence of $\mathscr{D}_X$-modules

$$0 \longrightarrow \Omega_{X/\Delta}^{\bullet + n}(\log Y)|_Y \otimes \mathscr{D}_X \xrightarrow{\frac{dt}{t}\wedge} \Omega_X^{\bullet + n+1}(\log Y)|_Y \otimes \mathscr{D}_X \longrightarrow \Omega_{X/\Delta}^{\bullet + n+1}(\log Y)|_Y \otimes \mathscr{D}_X \longrightarrow 0.$$

Considering the associated long exact sequence

$$0 \longrightarrow \mathcal{H}^{-1} \longrightarrow \mathcal{M} \xrightarrow{R} \mathcal{M} \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}^0 \longrightarrow 0 \tag{2.4.2}$$

we have the morphism $\frac{dt}{t}\wedge : \mathcal{M} \to \mathcal{H}^0$ and it vanishes on the image of R. To motivate the proof, let me do some local calculation. Let $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ represent a local frame of $\Omega^n_{X/\Delta}(\log Y)|_Y$. Then a local section of $\mathcal{M}$ is represented by $\zeta \otimes P$ for P a local section $\mathscr{D}_X$. Then $\mathrm{Res}_{\tilde{Y}^{(r+1)}} \frac{dt}{t} \wedge \zeta \otimes P$ is a section of $\bigoplus_{\#J=r+1} \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X$. Post-composing with the projection

$$\bigoplus_{\#J=r+1} \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X \to \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r),$$

we make the morphism explicit:

$$\mathrm{Res}_{\tilde{Y}^{(r+1)}} \circ \frac{dt}{t}\wedge : \mathcal{M} \to \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r), \quad [\zeta \otimes P] \mapsto [\mathrm{Res}_{\tilde{Y}^{(r+1)}} \frac{dt}{t} \wedge \zeta \otimes P].$$

Let $\zeta \otimes z_{\overline{J}} P$ represent a class in $\ker R^{r+1}$ for some fixed ordered index subset J with $\#J = r+1$, where $z_{\overline{J}} = \prod_{j\in I\smallsetminus J} z_j$ (Corollary 2.4.3). Its image under the above morphism only contained in the component $\tau^J_+ \omega_{Y^J}(*D^J)(-r)$ because $z_{\overline{J}}$ vanishes on other components. Thus, the image is the class represented by

$$\mathrm{Res}_{\tilde{Y}^{r+1}} \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n \otimes z_{\overline{J}} P = \pm \frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes z_{\overline{J}} P \in \Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{D}_X, \tag{2.4.3}$$

where $\frac{dz_{\overline{J}}}{z_{\overline{J}}} = \bigwedge_{j\in I\smallsetminus J} \frac{dz_j}{z_j}$ and the sign depends on the order of J. In fact, from the calculation we see that the image does not have any pole along D^J, so it is contained in the subsheaf consisting of classes represented by $\Omega^{n-r}_{Y^J} \otimes \mathscr{D}_X$. This means that the class of (2.4.3) in $\tau^J_+ \omega_{Y^J}(*D^J)(-r)$ is also contained in the image of the inclusion

$$\tau^J_+ \omega_{Y^J}(-r) \to \tau^J_+ \omega_{Y^J}(*D^J)(-r), \quad [dz_{\overline{J}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes P] \mapsto [\frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge dz_{k+1} \wedge \cdots dz_n \otimes z_{\overline{J}} P].$$

See Example 2.1.4. It follows that we obtain a factorization $\rho_r : \ker R^{r+1} \to \tau^{(r+1)}_+ \omega_{Y^{(r+1)}}(-r)$. In conclusion, we have the following commutative diagram.

$$\begin{array}{ccc} \ker R^{r+1} & \overset{\rho_r}{\dashrightarrow} & \tau^{(r+1)}_+ \omega_{Y^{(r+1)}}(-r) \\ \downarrow & & \downarrow \\ \mathcal{M} \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}^0 & \xrightarrow{\mathrm{Res}} & \bigoplus_{\#J=r+1} \tau^J_+ \omega_{Y^J}(*D^J)(-r) \end{array}$$

For a local section $\zeta \otimes z_K P$ where $z_K = \prod_{i\in K} z_i$ a monomial of degree $k - r + 1$, representing a class in $\ker R^r$, its image under ρ_r is indeed zero because z_K annihilates all $\Omega^{n-r}_{Y^J}(\log D^J)$ for index subset J such that $\#J = r + 1$. This implies the morphism ρ_r kills $\ker R^r$. The morphism ρ_r also kills $R \ker R^{r+2}$, because by (2.4.2) $\frac{dt}{t}\wedge$ vanishes on the image of R. Thus it factors through

$$\phi_r : \mathcal{P}_r = \frac{\ker R^{r+1}}{\ker R^r + R\ker R^{r+2}} \to \tau_+^{(r+1)} \omega_{Y^{(r+1)}}(-r).$$

The morphism ϕ_r is filtered surjective because for $dz_{\bar{J}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes P \in \Omega^{n-r}_{Y^J} \otimes F_\ell \mathscr{D}_X$ representing a class in $F_\ell \tau_+^J \omega_{Y^J}(-r)$ with $\#J = r + 1$, we can find a lifting class represented by $\zeta \otimes z_{\overline{J}} P$ in $F_\ell \ker R^{r+1}$. It follows that

$$cc(\mathcal{P}_r) \geq cc(\tau_+^{(r+1)} \omega_{Y^{(r+1)}}) = \sum_{\#J=r+1} \left[T^*_{Y^J} X\right].$$

Summing up the inequalities gives

$$\sum_{r\geq 0}(r+1)cc(\mathcal{P}_r) \geq \sum_{r\geq 0}(r+1) \sum_{\#J=r+1} \left[T^*_{Y^J} X\right] = \sum_{J\subset I}(\#J)\left[T^*_{Y^J} X\right].$$

On the other hand, by the Lefschetz decomposition and Theorem 2.3.7, we have

$$\sum_{J\subset I}(\#J)\left[T^*_{Y^J} X\right] = cc(\mathcal{M}) = cc(\mathrm{gr}^W \mathcal{M}) = \sum_{r\geq 0}(r+1)cc(\mathcal{P}_r).$$

Therefore, all inequalities must be equalities, i.e. $cc(\mathcal{P}_r) = cc(\tau_+^{(r+1)} \omega_{\tilde{Y}^{(r+1)}})$. It follows that ϕ_r is a filtered isomorphism [HTT08, Proposition 3.1.2]. $\square$

2.5 Reduced case: Sesquilinear pairing on $\mathcal{M}$ and limiting mixed Hodge structure

2.5.1 Sesquilinear pairing

We begin to construct the last data we need for the limiting mixed Hodge structure – Sesquilinear pairing. In the sense that $\mathcal{M}$ is the specialization of $i_{X_{t+}} \omega_{X_t}$ for $t \neq 0$, the sesquilinear $S : \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ should also be the specialization of $i_{X_t+}S_{X_t}$, where S_{X_t} is defined in §2.1. Presumably one would expect that the pairing

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle &= \lim_{t\to 0}\langle i_{X_t+}S_{X_t}(\zeta_1 \otimes P_1, \zeta_2 \otimes P_2), \eta\rangle \\ &= \lim_{t\to 0} \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1\overline{P_2}\eta \wedge \zeta_1 \wedge \overline{\zeta_2}\end{aligned}$$

should work on $\mathcal{M}$ for $\zeta_i \otimes P_i$, $i = 1, 2$ sections of $\Omega^n_{X/\Delta} \otimes \mathscr{D}_X$ over local chart U representing classes of $\mathcal{M}$, and η is a test function over U. But one could check that the integral $\int_{X_t} P_1\overline{P_2}\,(\eta)\,\zeta_1\overline{\wedge\zeta_2}$ could have order $(-\log|t|^2)^k$ near the origin where $k+1$ is the number of components that intersect in U, so the limit may not exist. To avoid the issue, we use a Mellin transform device (see [Sab02, 4.E]): locally

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle &=_{\text{def}} \operatorname{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1\overline{P_2}\eta\frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2} \\ &= \operatorname{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s}\frac{dt}{t} \wedge \overline{\frac{dt}{t}} \left(\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1\overline{P_2}\eta \wedge \zeta_1 \wedge \overline{\zeta_2}\right) \\ &= \operatorname{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s}\frac{dt}{t} \wedge \overline{\frac{dt}{t}} \langle i_{X_t+}S_{X_t}(\zeta_1 \otimes P_1, \zeta_2 \otimes P_2), \eta\rangle.\end{aligned}$$

The last expression in the definition in some extent explains that S is the specialization of $i_{X_t+}S_{X_t}$ and the 0-current $\operatorname{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s}\frac{dt}{t} \wedge \overline{\frac{dt}{t}}$ is doing the job of renormalization of $i_{X_t+}S_{X_t}$ for $t \neq 0$. In fact, for any test function g on Δ, we have

$$\operatorname{Res}_{s=0} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s}\frac{dt}{t} \wedge \overline{\frac{dt}{t}} g = g(0).$$

We have not check that S is well-defined, but let us do an example to see how the Mellin transform works.

Example 2.5.1. Suppose Y is smooth, then R is identical zero and $\mathcal{M} \simeq i_{Y+}\omega_Y$, by Theorem 2.4.7. Thus, the pairing S should recover the natural pairing S_Y. In local coordinates $t = z_0$ and for any local sections $\zeta_i \otimes P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes P_i$ of $\Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, $i = 1, 2$

over local chart U,

$$\begin{aligned}\langle S([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle &= \mathrm{Res}_{s=0} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1 \overline{P_2}(\eta) \frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2} \\ &= \mathrm{Res}_{s=0} \int_X |t|^{2s-2} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} \\ \text{integration by parts on } t \text{ and } \bar{t} &= \mathrm{Res}_{s=0} \int_X \frac{|t|^{2s}}{s^2} \partial_0 \overline{\partial_0} \left(P_1 \overline{P_2}(\eta)\right) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}.\end{aligned}$$

Because the Laurent expansion of $s^{-2}|t|^{2s}$ is $\sum_{\ell=0}^{\infty} (\log |t|^2)^{\ell} s^{\ell-2}$, the above continuously equals to, by Poincaré-Lelong equation [GH14, Page 388]

$$\begin{aligned}\int_X \log |t|^2 \partial_0 \overline{\partial_0} \left(P_1 \overline{P_2}(\eta)\right) \bigwedge_{i=0}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} &= \int_Y P_1 \overline{P_2}(\eta) \bigwedge_{i=1}^{n} \frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i} \\ &= \frac{\varepsilon(n+1)}{(2\pi\sqrt{1})^n} \int_Y P_1 \overline{P_2}(\eta) \zeta_1 \wedge \overline{\zeta_2} \\ &= \langle i_{Y+} S_Y([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle.\end{aligned}$$

We can take a cleaner point of view. In the case Y is smooth, the form $P_1 \overline{P_2}(\eta) \zeta_1 \overline{\wedge \zeta_2}$ is smooth in the neighborhood of Y. It follows that $i_{X_t+} S_{X_t}$ extends smoothly to $t = 0$ and the limit of $i_{X_t+} S_{X_t}$ is exactly $i_{Y+} S_Y$.

When Y has several smooth irreducible components, the idea of computation is similar to the above. Now we begin to establish the statements needed to ensure S is well-defined. For any test function η over an arbitrary open subset U of X and two sections m_1, m_2 in $H^0\left(U, \Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X\right)$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta)$ is smooth away from Y but with poles along Y supported in U. Locally, say $m_i = \zeta \otimes P_i$ for $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$ and $i = 1, 2$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta)$ is just $P_1 \overline{P_2}(\eta) \frac{dt}{t} \wedge \zeta \wedge \overline{\frac{dt}{t} \wedge \zeta}$. Let $F(s) = F(s, m_1, m_2, \eta)$ be the meromorphic continuation via integration by parts of the following function

$$\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\eta).$$

The function $F(s)$ is holomorphic when $\mathrm{Re}\, s > 0$ and has potential poles at non-positive integers. Note that $F(s)$ is independent of local coordinates. We are only interested in the polar part of the function $F(s)$ at $s = 0$.

Theorem 2.5.2. *The polar part of $F(s)$ at $s = 0$ only depends on the classes of m_1 and m_2 in $\mathcal{M}$.*

Proof. Let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts. Then

$$F(s) = \sum_\lambda \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_{U_\lambda} |t|^{2s} \frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}(\rho_\lambda \eta).$$

Since $\rho_\lambda \eta$ is a test function over U_λ, without loss of generality, we can assume U itself is a local chart. It follows that we can assume that $m_i = \zeta \otimes P_i$ for $i = 1,2$ and $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n$. We begin with some properties of $F(s)$.

Lemma 2.5.3. *Under the assumption that $m_i = \zeta \otimes P_i$ for $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ and for $i = 1,2$, the followings are valid.*

1. *the order of the pole of $F(s)$ at $s = 0$ is at most $k+1$;*

2. *if $P_i = tP_i'$ for one of $i = 1,2$, then $F(s)$ is holomorphic at $s = 0$;*

3. *for $0 \le j \le k$ we have,*

$$F(s, \zeta_1 \otimes P_1, \zeta_2 \otimes z_j \partial_j P_2, \eta) = F(s, \zeta_1 \otimes z_j \partial_j P_1, \zeta_2 \otimes P_2, \eta) = -sF(s, \zeta_1 \otimes P_1, \zeta_2 \otimes P_2, \eta).$$

Proof of the lemma. The Laurent expansion of $F(s)$ at $s = 0$ is

$$\begin{aligned} F(s) &= \int_X |z_I|^{2s-2} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right), \quad \text{where } z_I = \prod_{i \in I} z_i \\ &= \int_X \frac{|z_I|^{2s}}{s^{2k+2}} \partial_I \overline{\partial_I} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right), \quad \text{where } \partial_I = \prod_{i=0}^k \partial_i \\ &= \sum_{\ell=0}^\infty \frac{s^{\ell-(2k+2)}}{\ell!} \int_X \left(\log |z_I|^2\right)^\ell \partial_I \overline{\partial_I} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right). \end{aligned}$$

The order of the pole at $s = 0$ is at most $k+1$: if $\ell < k+1$, the form

$$\left(\log |z_I|^2\right)^\ell \partial_I \overline{\partial_I} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^n \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right)$$

is actually exact because one of a_i's must be 0 in the expansion of $(\log |z_I|^2)^\ell$ into a linear combination of $\prod_{i=0}^k (\log |z_i|^2)^{a_i}$ with $\sum_{i=0}^k a_i = \ell < k+1$. This proves (1).

Suppose that $P_1 = tP_1'$. Then the function

$$F(s) = \int_X |z_I|^{2s-2} t P_1' \overline{P_2}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}).$$

is well-defined at $s = 0$ because the form

$$\frac{1}{\overline{z_I}} P_1' \overline{P_2}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i})$$

is integrable. The same argument works for the case when $P_2 = tP_2'$. This proves (2).

Now we turn to the last statement

$$\begin{aligned}
& F(s, \zeta \otimes P_1, \zeta \otimes \overline{z_j \partial_j} P_2, \eta) \\
= & \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} \overline{z_j \partial_j} (P_1 \overline{P_2} \eta) \frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge dz_n \wedge \overline{\frac{dz_0}{z_0} \wedge \frac{dz_1}{z_1} \wedge \cdots \wedge dz_n} \\
= & \int_X |z_{I \setminus \{j\}}|^{2s-2} z_j^{s-1} \overline{z_j^s \partial_j} P_1 \overline{P_2} \eta \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}) \\
\text{integration by part on } dz_j = & -s \int_X |z_I|^{2s-2} P_1 \overline{P_2} \eta \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
= & -sF(s, \zeta \otimes P_1, \zeta \otimes P_2, \eta).
\end{aligned}$$

The same argument works for $F(s, \zeta \otimes z_j \partial_j P_1, \zeta \otimes P_2, \eta) = -sF(s, \zeta \otimes P_1, \zeta \otimes P_2, \eta)$. This proves (3). □

Returning to the proof of the theorem, if one of $\zeta \otimes P_i$ is $\frac{dz_1}{z_1} \wedge \frac{dz_2}{dz_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes tP_i'$, the above lemma (2) says $F(s)$ is holomorphic. If one of $\zeta \otimes P_i$ is $\frac{dz_1}{z_1} \wedge \frac{dz_2}{dz_2} \wedge \cdots \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes D_i P$, then the (3) above lemma says $F(s)$ is in fact 0. □

For any sections $\alpha, \beta \in \mathcal{M}$, let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts such that α, β lifts to $\tilde{\alpha}_\lambda, \tilde{\beta}_\lambda$ over U_λ in $\Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$. The above theorem just says that the pairing $S : \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ given by

$$\langle S(\alpha, \beta), \eta \rangle =_{\text{def}} \operatorname{Res}_{s=0} \sum_\lambda F(s, \tilde{\alpha}_\lambda, \tilde{\beta}_\lambda, \rho_\lambda \eta)$$

is well-defined and does not depend on the choice of partition of unity. By the above lemma we also have the following.

Corollary 2.5.4. *The operator R is self-adjoint with respect to S, i.e.* $S\circ(R\otimes_{\mathbb{C}}\mathrm{id})=S\circ(\mathrm{id}\otimes_{\mathbb{C}}R)$.

Because the self-adjointness, we have induced pairings on the graded quotient $S_r : \mathrm{gr}_r^W\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathrm{gr}_{-r}^W\mathcal{M}}\to\mathfrak{C}_X$ for every integer r. Denote by P_RS_r the pairing

$$S_r\circ(\mathrm{id}\otimes_{\mathbb{C}}R^r):\mathcal{P}_r\otimes_{\mathbb{C}}\overline{\mathcal{P}_r}\to\mathfrak{C}_X.$$

Theorem 2.5.5. *The isomorphism $\phi_r:(\mathcal{P}_r,F_\bullet\mathcal{P}_r)\to\tau_+^{(r+1)}\omega_{\tilde{Y}^{(r+1)}}(-r)$ in Theorem 2.4.7 respects the sesquilinear pairings up to a constant $(-1)^r(r+1)!^{-1}$, i.e.*

$$P_RS_r(\alpha,\beta)=\frac{(-1)^r}{(r+1)!}\tau_+^{(r+1)}S_{\tilde{Y}^{(r+1)}}(\phi_r\alpha,\phi_r\beta)$$

for any local sections $\alpha,\beta\in\mathcal{P}_r$.

Proof. Because the problem is local, it suffices to prove the theorem for α and β are represented by

$$\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes z_{K_i}$$

and $\#K_i=k-r$ for $i=1,2$ over a local chart U respectively. Recall that $z_K=\prod_{j\in K}z_j$. Let η be a test function over U. We have

$$\langle P_RS_r(\alpha,\beta),\eta\rangle=\langle S(\alpha,R^r\beta),\eta\rangle=\mathrm{Res}_{s=0}(-s)^r\int_X|z_I|^{2s-2}z_{K_1}\overline{z_{K_2}}(\eta)\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).$$

If $\alpha\neq\beta$, the above is in fact zero. Indeed, for $v\in K_2\smallsetminus K_1$, by choosing $R^r=\prod_{i\in I\smallsetminus K_1\smallsetminus\{v\}}z_i\partial_i$,

$$\langle P_RS_r(\alpha,\beta),\eta\rangle=\langle S(R^r\alpha,\beta),\eta\rangle=\mathrm{Res}_{s=0}\int_X|z_I|^{2s-2}\frac{t}{z_v}\overline{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}),$$

where $\tilde{\eta}=\partial_{I\smallsetminus(K_1\smallsetminus\{v\})}\overline{z_{K_2}}(\overline{z_v})^{-1}\eta$ is a smooth test function. The function

$$\int_X|z_I|^{2s-2}\frac{t}{z_v}\overline{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is holomorphic at $s=0$ because

$$\frac{1}{\overline{z_I}}\frac{\overline{z_v}}{z_v}\tilde{\eta}\bigwedge_{i=0}^n(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is integrable.

Therefore, we reduce the proof to the case when $\alpha = \beta$ represented by

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n \otimes z_K.$$

We shall prove that

$$P_R S_r(\alpha,\alpha) = \frac{(-1)^r}{(r+1)!}\tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\phi_r\alpha, \phi_r\alpha),$$

where $\overline{K}$ is the complement of K in I. Without loss of generality, we can assume $K = \{r+1, r+2, ..., k\}$. Then

$$\begin{aligned}
P_R S_r(\alpha,\alpha) &= \mathrm{Res}_{s=0}(-s)^r \int_X |z_{\overline{K}}|^{2s-2} \prod_{j=r+1}^{k} |z_j|^{2s} \eta \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
&= (-1)^r \mathrm{Res}_{s=0} s^{-(r+2)} \int_X \prod_{i=0}^{k} |z_i|^{2s} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \text{ where } \partial_{\bar{K}} = \prod_{i=0}^{r} \partial_i \\
&= \frac{(-1)^r}{(r+1)!} \int_X \left(\log \prod_{i=0}^{k} |z_i|^2\right)^{r+1} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
(\star) &= \frac{(-1)^r}{(r+1)!} \int_X \prod_{i=0}^{r} \log |z_i|^2 \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \\
&= \frac{(-1)^r}{(r+1)!} \int_{Y^{\overline{K}}} \eta \bigwedge_{i=r+1}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) \quad \text{(Poincaré-Lelong equation [GH14, Page 388])} \\
&= \frac{(-1)^r}{(r+1)!} \tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha, \mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha).
\end{aligned}$$

The equality $(\star)$ holds because if we expand $\left(\log \prod_{i=0}^{k} |z_i|^2\right)^{r+1}$ as a linear combination of $\prod_{i=0}^{k} \left(\log |z_i|^2\right)^{a_i}$ with $\sum_{i=0}^{k} a_i = r+1$, the only possible non-exact form among

$$\prod_{i=0}^{k} \left(\log |z_i|^2\right)^{a_i} \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}),$$

is $\left(\prod_{i=0}^{r} \log |z_i|^2\right) \partial_{\bar{K}} \overline{\partial_{\bar{K}}}(\eta) \bigwedge_{i=0}^{n} (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$. Note that while $\mathrm{Res}_{Y^{\overline{K}}}$ depends on the order of the index sets K and I, the pairing

$$\frac{(-1)^r}{(r+1)!} \tau_+^{(r+1)} S_{\tilde{Y}^{(r+1)}}(\phi_r\alpha, \phi_r\beta) = \frac{(-1)^r}{(r+1)!} \tau_+^{\overline{K}} S_{Y^{\overline{K}}}(\mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha, \mathrm{Res}_{Y^{\overline{K}}} \frac{dt}{t} \wedge \alpha)$$

does not because the sign will cancel out. We complete the proof. □

2.5.2 Constructure of the limiting mixed Hodge structure

We are going to show that the triple $(\mathrm{DR}_X\mathcal{M}, F, W)$ gives a mixed Hodge complex. Unlike the $\mathbb{Q}$-mixed Hodge complex considered by Deligne [Del71b], where the rational structure is a required input, we do not have this piece of information in our situation. We will redo the Deligne's argument on mixed Hodge complex by sesquilinear pairings. It also worths to point out that the sesqiuilinear pairing makes one check the first page weight spectral sequence of $\mathrm{DR}_X\mathcal{M}$ is a polarzed bigraded Hodge-Lefschetz structure easier than the case in [GN90], where they need to decompose the differential d_1 on the first page into a combinatorial differential and a sum of Gysin morphisms.

We first set up the pairing on each page of the weight spectral sequence abstractly. Let $\mathcal{N}$ be a holonomic $\mathscr{D}_Z$-module equipped with a sesquilinear pairing $S : \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_Z$ on a complex manifold Z. Assume that N has compact support. Let N be a nilpotent operator on $\mathcal{N}$ such that $S \circ (\mathrm{id} \otimes_{\mathbb{C}} N) = S \circ (N \otimes_{\mathbb{C}} \mathrm{id})$. Let $W_\bullet\mathcal{N}$ be the monodromy filtration associated to N on $\mathcal{N}$. Denote by $E_r^{i,j}$ be the weight spectral sequence convergent to $\mathrm{gr}^W_{-i} H^{i+j}(Z, \mathrm{DR}_Z\mathcal{N})$ with $E_1^{i,j} = H^{i+j}(Z, \mathrm{gr}^W_{-i}\mathrm{DR}_Z\mathcal{N})$. By abuse of notation, denote by S_k the induced pairing

$$H^k(Z, \mathrm{DR}_Z\mathcal{N}) \otimes_{\mathbb{C}} \overline{H^k(Z, \mathrm{DR}_Z\mathcal{N})} \to H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}) \to H^0_c(Z, \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z) \simeq \mathbb{C}$$

multiplying a sign factor $\varepsilon(k)$. Let a be a local section of $(\mathrm{DR}_Z\mathcal{N})^{-j-1}$ and b be a local section of $(\mathrm{DR}_Z\mathcal{N})^i$. Then

$$D(a \otimes_{\mathbb{C}} b) = da \otimes_{\mathbb{C}} b + (-1)^{-j-1} a \otimes_{\mathbb{C}} db$$

for D a differential on $\mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}$. Applying S, we find that

$$DS(a,b) = S(da, b) + (-1)^{-j-1} S(a, db). \tag{2.5.1}$$

Since the differential d is compatible with the weight filtration, we have an induced pairing $E_1(S)_k$ on the first page $E_1^{i,j}$ of the weight spectral sequence by the pairing

$$H^k(Z, \mathrm{gr}^W_{-i}\mathrm{DR}_Z\mathcal{N}) \otimes_{\mathbb{C}} \overline{H^k(Z, \mathrm{gr}^W_{i}\mathrm{DR}_Z\mathcal{N})} \to H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathrm{gr}^W_{-i}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathrm{gr}^W_{i}\mathcal{N}}) \to H^0(Z, \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z)$$

multiplying a sign factor $\varepsilon(k)$. Then by equation (2.5.1) we obtain

$$0 = \varepsilon(-j)E_1(S)_{-j}(d_1 a, b) + \varepsilon(-j-1)(-1)^{-j-1}E_1(S)_{-j-1}(a, d_1 b),$$

since $DSa \otimes_C \overline{b}$ is cohmologous to zero. Working out the sign, the above is equivalent to

$$E_1(S)_{-j}(d_1 a, b) + E_1(S)_{-j-1}(a, d_1 b) = 0,$$

i.e. the differential d_1 is skew-symmetrc with respect to $E_1(S)$. It follows that we have an induced pairing on the second page: $E_2(S)_k : E_2^{i,k-i} \otimes \overline{E_2^{-i,-k+i}} \to \mathbb{C}$ since $E_2 = \ker d_1/\operatorname{Im} d_1$. Again, it follows from the equation (2.5.1), the differential d_2 is skew-symmetric with respect to $E_2(S)$. By an inductive argument, we get the induced pairing $E_r(S) : E_r \otimes \overline{E_r} \to \mathbb{C}$ on the r-th page of the weight spectral sequence $E_r \otimes \overline{E_r} \to \mathbb{C}$ such that d_r is skew-symmetric with respect to $E_r(S)$ for every $r \geq 1$.

Next, let $L = [\omega]\wedge$ be a Lefschetz operator for a Kähler class $[\omega] \in H^1(Z, \Omega_Z) \cap H^2(Z, \mathbb{R})$ on Z which can be thought as a morphism $L : \mathbb{C} \to \mathbb{C}[2]$ in $\mathbf{D}^b(Z, \mathbb{C})$ and so is $\mathsf{X} = 2\pi\sqrt{-1}L$. Therefore, we obtain a morphism $\mathsf{X} : \mathrm{DR}_Z\mathcal{N} \to \mathrm{DR}_Z\mathcal{N}[2]$. Let us work out the relation between the sesquilinear pairing S_k and the operator X. By funtorailty, we have the following commutative diagram in $\mathbf{D}^b(Z, \mathbb{C})$.

$$\begin{array}{ccccccc}
\mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} & \xrightarrow{S} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z & \xrightarrow{\simeq} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z & \xrightarrow{\simeq} & \mathcal{A}_Z^\bullet \otimes \mathfrak{Db}_Z[2\dim Z] \\
\downarrow{\scriptstyle \mathsf{X}\otimes_{\mathbb{C}}\mathrm{id}} & & \downarrow{\scriptstyle \mathsf{X}} & & \downarrow{\scriptstyle \mathsf{X}} & & \downarrow{\scriptstyle \mathsf{X}} \\
\mathrm{DR}_{Z,\overline{Z}}\mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}}[2] & \xrightarrow{S[2]} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{C}_Z[2] & \xrightarrow{\simeq} & \mathrm{DR}_{Z,\overline{Z}}\mathfrak{Db}_Z[2] & \xrightarrow{\simeq} & \mathcal{A}_Z^\bullet \otimes \mathfrak{Db}_Z[2\dim Z + 2]
\end{array}$$

Similarly, we have $S[2] \circ (\mathrm{id} \otimes_{\mathbb{C}} \mathsf{X}) = \overline{\mathsf{X}}S$. It follows from $\mathsf{X} + \overline{\mathsf{X}} = 0$ on $\mathcal{A}_Z^\bullet \otimes \mathfrak{Db}[2\dim Z]$ that

$$\varepsilon(k)S_k(\mathsf{X}-,-) + \varepsilon(k-2)S_{k-2}(-,-) = 0, \quad \text{i.e. } S_k(\mathsf{X}-,-) = S_{k-2}(-,\mathsf{X}-). \tag{2.5.2}$$

Returning to our situation, we begin to construct a polarized bigraded Hodge-Lefschetz structure on

$$\mathrm{gr}^W H^\bullet(X, \mathrm{DR}_X\mathcal{M}).$$

Fix a Kähler class $[\omega]$ on X and let $L = [\omega]\wedge : \mathrm{DR}_X\mathcal{M} \to \mathrm{DR}_X\mathcal{M}[2]$ be the Lefschetz operator and $\mathsf{X}_1 = 2\pi\sqrt{-1}L$ as the discussion above. Relabel the first page of the weight spectral sequence by

$$V_{\ell,k} = H^\ell(X, \mathrm{gr}_k^W \mathrm{DR}_X\mathcal{M}) = {}^W E_1^{-k,\ell+k}.$$

Let $V = \bigoplus_{\ell,k\in\mathbb{Z}} V_{\ell,k}$ with filtration $F_\bullet V$ induced by $F_\bullet\mathcal{M}$. Denote by $E_i(R)$ the induced operator by R on ${}^W E_i$ and let $\mathsf{Y}_2 = E_1(R)$. Denote by $S_{\ell,k}$ for $\ell, k \in \mathbb{Z}$, the induced pairing on $V_{\ell,k} \otimes \overline{V_{-\ell,-k}}$

$$H^\ell(X, \mathrm{gr}_k^W \mathrm{DR}_X\mathcal{M}) \otimes \overline{H^{-\ell}(X, \mathrm{gr}_{-k}^W \mathrm{DR}_X\mathcal{M})} \to H^0(X, \mathrm{DR}_{X,\overline{X}} \mathrm{gr}_k^W \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathrm{gr}_{-k}^W \mathcal{M}}) \to H_c^0(X, \mathrm{DR}_{X,\overline{X}} \mathfrak{C}_X) \simeq \mathbb{C}.$$

multiplying a sign factor $\varepsilon(\ell)$. Let d_1 be the differential of E_1. In terms of relabeling, we have

$$d_1 : (V_{\ell,k}, F_\bullet V_{\ell,k}) \to (V_{\ell+1,k-1}, F_\bullet V_{\ell+1,k-1}).$$

Theorem 2.5.6. *The tuple* $(V, \mathsf{X}_1, \mathsf{Y}_2, F_\bullet V, \bigoplus S_{j,k}, d_1)$ *gives a differential polarized bigraded Hodge-Lefschetz structure of central weight* n.

Proof. Let us first check the conditions in Theorem 2.1.10 one by one. It is clear that two operators $\mathsf{X}_1, \mathsf{Y}_2$ are commute. Moreover, we have $\mathsf{Y}_2 : (V_{\ell,k}, F_\bullet V_{\ell,k}) \to (V_{\ell,k-2}, F_{\bullet+1} V_{\ell,k-2})$ such that

$$\mathsf{Y}_2^k : F_\bullet V_{\ell,k} \to F_{\bullet+k} V_{\ell,-k},$$

is an isomorphism by Theorem 2.4.6. Denote by $P_{\mathsf{Y}_2} V_{-j,r}$ the Y_2-primitive part $\ker \mathsf{Y}_2^{r+1} \cap V_{-j,r} = H^{-j}(X, \mathrm{DR}_X \mathcal{P}_r)$. It follows from Theorem 2.4.7 that $(P_{\mathsf{Y}_2} V_{-j,r}, F_\bullet P_{\mathsf{Y}_2} V_{-j,r})$ is filtered isomorphic to $H^{-j}(\tilde{Y}^{(r+1)}, \mathrm{DR}_{\tilde{Y}^{(r+1)}} \omega_{\tilde{Y}^{(r+1)}})(-r)$ via ϕ_r. Therefore, $\mathsf{X}_1 F_\bullet P_{\mathsf{Y}_2} V_{-j,r} \subset F_{\bullet-1} P_{\mathsf{Y}_2} V_{-j+2,r}$ and by Hard Lefschetz,

$$\mathsf{X}_1^j : F_\bullet P_{\mathsf{Y}_2} V_{-j,r} \to F_{\bullet-j} P_{\mathsf{Y}_2} V_{j,r}$$

is an isomorphism. It follows from the Lefschetz decomposition of Y_2 that $\mathsf{X}_1^j : F_\bullet V_{-j,r} \to F_{\bullet-j} V_{j,r}$ is an isomorphism. This proves (pbHL1) in Theorem 2.1.10. (pbHL2) follows from the equation (2.5.2).

Because the operator R self-adjoin with respect to S by Corollary 2.5.4, we have $S_{j,r}(-,\mathsf{Y}_2-) = S_{j,r+2}(\mathsf{Y}_2-,-)$. By Theorem 2.5.5, the morphism ϕ_r identifies $P_{\mathsf{Y}_2}S_{-j,r} =_{\text{def}} S_{-j,r}(-,\mathsf{Y}_2^r-)$ with $\frac{(-1)^r}{(r+1)!}S_{\tilde{Y}^{(r+1)},-j}$. Recall that

$$S_{\tilde{Y}^{(r+1)},j}(a,b) = \frac{\varepsilon(n-r+j+1)}{(2\pi\sqrt{-1})^{n-r}}\int_{\tilde{Y}^{(r+1)}} a\wedge\bar{b},\ \text{for } a\in H^{n-r+j}(\tilde{Y}^{(r+1)}) \text{ and } b\in H^{n-r-j}(\tilde{Y}^{(r+1)}),$$

and that $S_{\tilde{Y}^{(r+1)},j}(\mathsf{X}_1^j-,-)$ is a polarization on $H^{n-r-j}_{\text{prim}}(\tilde{Y}^{(r+1)},\mathbb{C})$. The bi-primitive part $P_{-j,r} = \ker\mathsf{X}_1^j\cap\ker\mathsf{Y}_2^r\cap V_{-j,r}$ together with the induced filtration $F_\bullet P_{-j,r}$ and the sesquilinear pairing $S_{j,r}(\mathsf{X}_1^j-,(-\mathsf{Y})_2^r-)$ is identified with the polarized Hodge structure $H^{n-r-j}_{\text{prim}}(\tilde{Y}^{(r+1)},\mathbb{C})(-r)$ via ϕ_r. This proves (pbHL3).

It remains to prove that d_1 is a differential of the bigraded Hodge-Lefschetz structure V. Clearly, we have

$$[d_1,\mathsf{X}_1] = [d_1,\mathsf{Y}_2] = 0$$

because d_1 is induced by the differential of $\mathrm{DR}_X\mathcal{M}$ and d_1 preserves $F_\bullet$. The differential d_1 is skew-symmetric with respect to $\bigoplus_{j,r} S_{j,r}$ is formally follows the discussion at the beginning of this subsection. Thus, we finished checking that d_1 is a differential. ☐

Corollary 2.5.7. *We have the following*

1. *the Hodge spectral sequence degenerates at* ${}_FE_1$*,*
2. *the weight spectral sequence degenerates at* ${}^WE_2$*,*
3. *The tuple* $(\bigoplus_{\ell\in\mathbb{Z}}\mathrm{gr}^W H^\ell(X,\mathrm{DR}_X\mathcal{M}),F,\mathsf{X}_1,\mathsf{Y}_2)$ *together with the pairing induced by* $\bigoplus S_{j,k}$ *is a polarized bigradged Hodge-Lefschetz structure of central weight* n*.*

Proof. We slightly modify the idea of cohomological mixed Hodge complex in [Del71b] for statement (1) and (2). I claim that the k-th weight spectral sequence $V^k_{\ell,r} =_{\text{def}} {}^WE_k^{-r,\ell+r}$ together with the induced filtration $F_\bullet$ and the induced pairing $S^k_{\ell,r}\circ(\mathrm{id}\otimes\mathsf{w}) : V^k_{\ell,r}\otimes\overline{V^k_{\ell,r}}\to\mathbb{C}$ is a polarized Hodge structure of weight $n+\ell+r$ and the differential $d_k : V^k_{\ell,r}\to V^k_{\ell+1,r-k}$ is a morphism of Hodge structures. Indeed, the differential d_k is skew-symmetric with respect to the sesquilinear pairing, i.e. $S^k_{\ell,r}(d_k-,-)+S^k_{\ell+1,r-k}(-,d_k-)=0$. Therefore, if $(-1)^q S^k_{\ell,r}\circ(\mathrm{id}\otimes \mathsf{w})$ for $q=n+\ell+r-p$ is a Hermitian inner product on

$$\left(V^k_{\ell,r}\right)^{p,q}=\{a\in F^pV^k_{\ell,r}:S^k_{\ell,r}(a,b)=0 \text{ for all } b\in F^{p-\ell-r+1}V^k_{-\ell,-r}\}$$

then $(-1)^q S^{k+1}_{\ell,r}\circ(\mathrm{id}\otimes \mathsf{w})$ is also a Hermitian inner product on

$$\left(V^{k+1}_{\ell,r}\right)^{p,q}=\{a\in F^pV^{k+1}_{\ell,r}:S^{k+1}_{\ell,r}(a,b)=0 \text{ for all } b\in F^{p-\ell-r+1}V^{k+1}_{-\ell,-r}\}.$$

In particular, we have the decomposition

$$V^{k+1}_{\ell,r}=\bigoplus_{p+q=n+\ell+r}\left(V^{k+1}_{\ell,r}\right)^{p,q}$$

and the morphism $d_k:\left(V^k_{\ell,r}\right)^{p,q}\to\left(V^{k+1}_{\ell,r}\right)^{p,q}$ is compatible with the decomposition. See Remark 2.1.11. By induction the claim is proved. It follows that d_k vanishes for $k\geq 2$ by it is a morphism of Hodge structures of different weights, which proves (2).

Since each bigraded piece $V_{\ell,r}=H^\ell\left(X,\mathrm{gr}^W_r\mathrm{DR}_X\mathcal{M}\right)$ is pure Hodge structure of weight $n+r+\ell$, the two vector spaces $H^\ell\left(X,\mathrm{gr}^F\mathrm{gr}^W_r\mathrm{DR}_X\mathcal{M}\right)$ and $V_{\ell,r}$ is isomorphic. Moreover, the isomorphism is compatible with d_1, because d_1 respects $F_\bullet$ and

$$\mathrm{gr}^W_r\mathrm{gr}^F\mathrm{DR}_X\mathcal{M}=\mathrm{gr}^F\mathrm{gr}^W_r\mathrm{DR}_X\mathcal{M}.$$

Taking cohomology of d_1, we obtain that $\mathrm{gr}^W_r H^\ell\left(X,\mathrm{gr}^F\mathrm{DR}_X\mathcal{M}\right)$ is isomorphism to $\mathrm{gr}^W_r H^\ell\left(X,\mathrm{DR}_X\mathcal{M}\right)$. It follows from the dimension reason that $H^\ell\left(X,\mathrm{gr}^F\mathrm{DR}_X\mathcal{M}\right)$ is isomorphic to $H^\ell\left(X,\mathrm{DR}_X\mathcal{M}\right)$, which is exactly the degeneration of Hodge spectral sequence at ${}_FE_1$.

The statement (3) follows from Theorem 2.1.12. □

The third statement in the above corollary ensures that the weight filtration on the hypercohomology of $\mathrm{DR}_X\mathcal{M}$ is the monodromy weight filtration of the nilpotent operator R, i.e. $RW_\bullet H^\ell(X,\mathrm{DR}_X\mathcal{M})\subset W_{\bullet-2}H^\ell(X,\mathrm{DR}_X\mathcal{M})(-1)$ and $R^r:\mathrm{gr}^W_r H^\ell(X,\mathrm{DR}_X\mathcal{M})\to\mathrm{gr}^W_{-r}H^\ell(X,\mathrm{DR}_X\mathcal{M})(-r)$ is a filtered isomorphism. We proved Theorem A for the case when Y is reduced.

2.6 Non-reduced case: Generalized eigenspace $\mathcal{M}_\alpha$ and the weight filtration

Now we move to the general situation. Recall that we have introduced the notations: the index set I consisting of indices of irreducible components of Y and e_i is the multiplicity of Y along the component Y_i.

2.6.1 The generalized eigen-modules $\mathcal{M}_\alpha$

We begin with studying the generalized eigen-modules $\ker(R-\alpha)^\infty$ of the morphism R in the category of filtered $\mathscr{D}_X$-modules. The generalized eigen-modules are naturally sub-modules of $\mathcal{M}$ and one can put the induced filtration on it. However, this filtration does not match with the expected weight of the mixed Hodge structure and is difficult to study. Instead, we use the idea of Saito in [Sai90]: one regards the generalized eigen-module as a sub-quotient of $\mathcal{M}$ and puts the induced filtration on it. It turns out the filtration behaves nice. Now let us begin to settle some definitions.

Define $\mathcal{M}_{\geq\alpha} = \ker\prod_{\lambda\geq\alpha}(R-\lambda)^\infty$, $\mathcal{M}_{>\alpha} = \ker\prod_{\lambda>\alpha}(R-\lambda)^\infty$ and $\mathcal{M}_\alpha = \mathcal{M}_{\geq\alpha}/\mathcal{M}_{>\alpha}$. Then $\mathcal{M}_\alpha$ is canonically isomorphic to the generalized eigen-module $\ker(R-\alpha)^\infty$. Endow $\mathcal{M}_\alpha$ the filtration $F_\bullet\mathcal{M}_\alpha$ induced from $(\mathcal{M}, F_\bullet\mathcal{M})$,

$$F_\bullet\mathcal{M}_\alpha = \frac{\mathcal{M}_{\geq\alpha}\cap F_\bullet\mathcal{M}}{\mathcal{M}_{>\alpha}\cap F_\bullet\mathcal{M}}.$$

There are parallel definitions on the relative log de Rham complex. Denote by $C^\bullet = \Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathscr{O}_Y$ for simplicity. Define sub-complexes of $C^\bullet$ by

$$C^\bullet_{\geq\alpha} = C^\bullet\otimes\mathscr{O}_X(-\lceil\alpha Y\rceil), \quad C^\bullet_{>\alpha} = C^\bullet\otimes\mathscr{O}_X(-\lfloor\alpha Y\rfloor - Y_{\mathrm{Red}}) \quad \text{and } C^\bullet_\alpha = C^\bullet_{\geq\alpha}/C^\bullet_{>\alpha},$$

where Y_{Red} is the associated reduced divisor of Y. Notice that if we let I_α be the subset of I consisting of all i such that αe_i is an integer, then

$$C^\bullet_\alpha = C^\bullet_{\geq\alpha}\otimes\mathscr{O}_{Y_{I_\alpha}}, \quad \text{where } Y_{I_\alpha} = \sum_{i\in I_\alpha} Y_i.$$

One can check $C^\bullet_\alpha$ is a generalized eigen-perverse sheaves of the residue $[\nabla]$. Since $\mathscr{O}_X(-\lceil \alpha Y \rceil)$ is preserved by relative log differential $\mathscr{T}_{X/\Delta}(-\log Y)$, the multiplication by relative log differentials gives a morphism, recalling that $D_1, D_2, ..., D_n$ are local generators of $\mathscr{T}_{X/\Delta}(-\log Y)$ dual to the local generators $\xi_1, \xi_2, ..., \xi_n$ of $\Omega_{X/\Delta}(\log Y)$,

$$\begin{aligned} &\mathscr{O}_X(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X \to \Omega_{X/\Delta}(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X, \\ &\qquad z_I^{\lceil \alpha \mathbf{e} \rceil} \otimes P \mapsto \sum_j \xi_j \otimes D_j z_I^{\lceil \alpha \mathbf{e} \rceil} \otimes P = \sum_j \xi_j \otimes z_I^{\lceil \alpha \mathbf{e} \rceil}(D_j + \alpha_j) \otimes P, \end{aligned} \tag{2.6.1}$$

where, using the multi-index notation, $z_I^{\lceil \alpha \mathbf{e} \rceil} = \prod_{i \in I} z_i^{\lceil \alpha e_i \rceil}$ denotes the local generator of $\mathscr{O}_X(-\lceil \alpha Y \rceil)$ and define $\alpha_i = [D_i, z_I^{\lceil \alpha \mathbf{e} \rceil}]/z_I^{\lceil \alpha \mathbf{e} \rceil} = \lceil \alpha e_i \rceil / e_i - \lceil \alpha e_0 \rceil / e_0$. The morphism extends to a complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y)(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X$, which is a subcomplex of $\Omega^{n+\bullet}_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$ (see (2.3.1)). Tensoring $\mathscr{O}_Y$ on the left gives $C^\bullet_{\geq \alpha} \otimes \mathscr{D}_X$ by the above definition. Further tensoring $\mathscr{O}_{Y_{I_\alpha}}$ on the left, we obtain the complex of induced $\mathscr{D}_X$-modules $C^\bullet_\alpha \otimes \mathscr{D}_X$ with the filtration defined by

$$F_\ell \left(C^\bullet_\alpha \otimes \mathscr{D}_X \right) = C^\bullet_\alpha \otimes F_{\ell+n+\bullet} \mathscr{D}_X.$$

The following two theorems give the description of the generalized eigen-modules in terms of complexes of the induced $\mathscr{D}_X$-modules.

Theorem 2.6.1. *The complex $C^\bullet_\alpha \otimes \mathscr{D}_X$ is filtered acyclic and the characteristic cycle of the 0-th cohomology is*

$$cc \left(\mathscr{H}^0 \left(C^\bullet_\alpha \otimes \mathscr{D}_X \right) \right) = \sum_{J \subset I} \left(\# I_\alpha \cap J \right) \left[T^*_{Y^J} X \right].$$

Proof. Similarly to the proof of Theorem 2.3.1 and Theorem 2.3.5, the associated graded $\mathrm{gr}^F \left(C^\bullet_\alpha \otimes \mathscr{D}_X \right)$ locally is the Koszul complex of the regular sequence $(t_\alpha, D_1, D_2, ..., D_n)$ over $\mathrm{gr}^F \mathscr{D}_X$, where $t_\alpha = \prod_{i \in I_\alpha} z_i$ is the defining equation of Y_{I_α}. It follows that $\mathrm{gr}^F (C^\bullet_\alpha \otimes \mathscr{D}_X)$ is acyclic and therefore, $C^\bullet_\alpha \otimes \mathscr{D}_X$ is filtered acyclic. We also get that $\mathrm{gr}^F \mathscr{H}^0 (C^\bullet_\alpha \otimes \mathscr{D}_X)$ is locally represented by

$$\zeta_\alpha \otimes \mathrm{gr}^F \mathscr{D} / (t_\alpha, D_1, D_2, ..., D_n) \mathrm{gr}^F \mathscr{D}_X, \quad \text{where } \zeta_\alpha = z_I^{\lceil \alpha \mathbf{e} \rceil} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_n. \tag{2.6.2}$$

As the calculation in Theorem 2.3.7, we get the characteristic cycle is $\sum_{J\subset I}(\#I_\alpha\cap J)\left[T^*_{Y^J}X\right]$.

$\square$

Theorem 2.6.2. *There exists a canonical filtered isomorphism*

$$\psi_\alpha:(\mathscr{H}^0(C^\bullet_\alpha\otimes\mathscr{D}_X),F_\bullet\mathscr{H}^0(C^\bullet_\alpha\otimes\mathscr{D}_X))\xrightarrow{\sim}(\mathcal{M}_\alpha,F_\bullet\mathcal{M}_\alpha). \tag{2.6.3}$$

In particular, the characteristic cycle $cc(\mathcal{M}_\alpha)=\sum_{J\subset I}(\#I_\alpha\cap J)\left[T^*_{Y^J}X\right]$.

We first study $\mathcal{M}_{\geq\alpha}$ and $\mathcal{M}_{>\alpha}$ locally by pointing out their cyclic generator. In principal, this always can be done because every holonomic $\mathscr{D}_X$-module locally is cyclic.

Lemma 2.6.3. *Locally,* $\mathcal{M}_{\geq\alpha}$ *is generated by* $z_I^{\lceil\alpha\mathbf{e}\rceil}$, *and* $\mathcal{M}_{>\alpha}$ *is generated by* $z_I^{\lfloor\alpha\mathbf{e}\rfloor+\mathbf{1}}$ *where* $\mathbf{1}=(1,1,...,1)\in\mathbb{Z}^I$.

Proof. Let us first check that $z_I^{\lceil\alpha\mathbf{e}\rceil}\in\mathcal{M}_{\geq\alpha}$. It suffices to check that it is in

$$\ker\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i}).$$

This is follows from direct calculation:

$$\begin{aligned}\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i})z_I^{\lceil\alpha\mathbf{e}\rceil}&=\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(R-\frac{j}{e_i})z_i^{\lceil\alpha e_i\rceil}=\prod_{i\in I}\prod_{j=\lceil\alpha e_i\rceil}^{e_i-1}(\frac{1}{e_i}z_i\partial_i-\frac{j}{e_i})z_i^{\lceil\alpha e_i\rceil}\\&=\prod_{i\in I}\frac{1}{e_i^{e_i-\lceil\alpha e_i\rceil}}z_i^{e_i}\partial_i^{e_i-\lceil\alpha e_i\rceil}=t\prod_{i\in I}\frac{1}{e_i^{e_i-\lceil\alpha e_i\rceil}}\partial_i^{e_i-\lceil\alpha e_i\rceil}=0\in\mathcal{M}.\end{aligned}$$

Because R satisfies the identity (2.3.3), $\mathcal{M}_{\geq\alpha}$ is also equal to the image of $\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(R-\frac{j}{e_i})$. It follows from

$$\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(R-\frac{j}{e_i})(1)=\prod_{i\in I}\prod_{j=0}^{\lceil\alpha e_i\rceil-1}(\frac{1}{e_i}z_i\partial_i-\frac{j}{e_i})=z_I^{\lceil\alpha\mathbf{e}\rceil}\prod_{i\in I}\frac{1}{e_i^{\lceil\alpha e_i\rceil}}\partial_i^{\lceil\alpha e_i\rceil}$$

that $z_I^{\lceil\alpha\mathbf{e}\rceil}\prod_{i\in I}\partial_i^{\lceil\alpha e_i\rceil}$ generates $\mathcal{M}_{\geq\alpha}$. We deduce that $z_I^{\lceil\alpha\mathbf{e}\rceil}$ generates $\mathcal{M}_{\geq\alpha}$. The similar argument works for $\mathcal{M}_{>\alpha}$. $\square$

Proof of Theorem 2.6.2. It follows from the above lemma that $\mathcal{M}_\alpha$ is locally isomorphic to

$$\zeta\otimes\left(z_I^{\lceil\alpha\mathbf{e}\rceil},D_1,D_2,...,D_n\right)\mathscr{D}_X/\left(z_I^{\lfloor\alpha\mathbf{e}\rfloor+\mathbf{1}},D_1,D_2,...,D_n\right)\mathscr{D}_X$$

where $\zeta = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ so that $\zeta_\alpha = z_I^{\lceil \alpha \mathbf{e} \rceil}\zeta$. Since $\mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X)$ by (2.6.1) is locally isomorphic to

$$\zeta_\alpha \otimes \mathscr{D}_X / (t_\alpha, D_1 + \alpha_1, D_2 + \alpha_2, ..., D_n + \alpha_n)\mathscr{D}_X,$$

the multiplication $\mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X) \to \mathcal{M}_\alpha$, $\zeta_\alpha \otimes P \mapsto \zeta \otimes z_I^{\lceil \alpha \mathbf{e} \rceil} P$ is well-defined, does not depend on the coordinate and therefore, gives a filtered morphism

$$\psi_\alpha : (\mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X), F_\bullet \mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X)) \longrightarrow (\mathcal{M}_\alpha, F_\bullet \mathcal{M}_\alpha).$$

The surjectivity is clear from the local description. It follows that $cc\left(\mathscr{H}^0\left(C_\alpha^\bullet \otimes \mathscr{D}_X\right)\right) \geq cc(\mathcal{M}_\alpha)$. Summing over all the rational numbers α in $[0,1)$ gives

$$\sum_\alpha cc\left(\mathscr{H}^0\left(C_\alpha^\bullet \otimes \mathscr{D}_X\right)\right) \geq \sum_\alpha cc(\mathcal{M}_\alpha) = cc(\mathcal{M}).$$

On the other hand, by Theorem 2.3.5 and Theorem 2.6.1, the $\mathscr{D}_X$-module $\mathcal{M}$ is also successive extensions of $\mathscr{H}^0\left(C_\alpha^\bullet \otimes \mathscr{D}_X\right)$ for $\alpha \in \mathbb{Q} \cap [0,1)$. Thus,

$$\sum_\alpha cc\left(\mathscr{H}^0\left(C_\alpha^\bullet \otimes \mathscr{D}_X\right)\right) = cc(\mathcal{M}).$$

This forces that ψ_α must be isomorphism and therefore, filtered injective.

It remains to show that

$$F_\ell \psi_\alpha : F_\ell \mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X) \longrightarrow F_\ell \mathcal{M}_\alpha, \tag{2.6.4}$$

is sujective. Suppose that $z_I^{\lceil \alpha \mathbf{e} \rceil} P \in \mathscr{D}_X$ is a representative of a class in $F_\ell \mathcal{M}_\alpha$. Then we can write

$$z_I^{\lceil \alpha \mathbf{e} \rceil} P = P' + \sum_{i=1}^n D_i Q_i + z_I^{\lfloor \alpha \mathbf{e} \rfloor + \mathbf{1}} T$$

for $P' \in F_{\ell+n}\mathscr{D}_X$ and $T, Q_i \in \mathscr{D}_X$. It follows that

$$z_I^{\lceil \alpha \mathbf{e} \rceil}(P - t_\alpha T) = P' + \sum_{i=1}^n D_i Q_i$$

By the regular sequence argument of Theorem 2.3.5, we can assume that $P - t_\alpha T$ is in $F_{\ell+n}\mathscr{D}_X$. Then the class represented by $P - t_\alpha T$ in $\mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X)$ is actually in $F_\ell \mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X)$ by the local formula. Therefore, we find a lifting represented by P in $F_\ell \mathscr{H}^0(C_\alpha^\bullet \otimes \mathscr{D}_X)$ of the class of $z_I^{\lceil \alpha \mathbf{e} \rceil} P$ in $F_\ell \mathcal{M}_\alpha$. We conclude the proof. □

Without loss of generality, we can assume by abuse of notation that locally $I_\alpha = \{0, 1, ..., \mu\}$ so that $t_\alpha = z_0 z_1 \cdots z_\mu$. Let R_α be the induced operator $(R - \alpha)$ on $(\mathcal{M}_\alpha, F_\bullet \mathcal{M}_\alpha)$. One easily gets a nice local formula of R_α:

Corollary 2.6.4. *The endormorphism R_α of $\mathcal{M}_\alpha$ acts locally as $\psi_\alpha \circ (\mathrm{id} \otimes \frac{1}{e_j} z_j \partial_j) \circ (\psi_\alpha)^{-1}$ for any $j \in I_\alpha$.*

Proof. Because $R - \alpha$ acts on the left hand side of the identification (2.6.2) by the left multiplication by $\frac{1}{e_0} z_0 \partial_0 - \alpha$, the statement follows from

$$\begin{aligned} R_\alpha \left[\zeta \otimes z_I^{\lceil \alpha \mathbf{e} \rceil} \right] &= \left[\zeta \otimes \left(\frac{1}{e_j} z_j \partial_j - \alpha \right) \left(z_I^{\lceil \alpha \mathbf{e} \rceil} \right) \right] \\ &= \left[\zeta \otimes \left(\left(\frac{1}{e_j} \lceil \alpha e_j \rceil - \alpha \right) z_I^{\lceil \alpha \mathbf{e} \rceil} + z_I^{\lceil \alpha \mathbf{e} \rceil} \left(\frac{1}{e_j} z_j \partial_j \right) \right) \right] \\ &= \psi_\alpha \left[\zeta z_I^{\lceil \alpha \mathbf{e} \rceil} \otimes \left(\frac{1}{e_j} z_j \partial_j \right) \right] = \psi_\alpha \circ \left(\mathrm{id} \otimes \frac{1}{e_j} z_j \partial_j \right) \circ \psi_\alpha^{-1} \left[\zeta_\alpha \otimes 1 \right]. \end{aligned}$$

This completes the proof. □

By the local formula of R_α, it is obvious that $R_\alpha : (\mathcal{M}_\alpha, F_\bullet \mathcal{M}_\alpha) \to (\mathcal{M}_\alpha, F_{\bullet+1} \mathcal{M}_\alpha)$ is a filtered morphism.

2.6.2 Striness of R_α

Similar to the reduced case, every power of R_α is strict.

Theorem 2.6.5. *The power of the endomorphism R_α on $(\mathcal{M}_\alpha, F_\bullet \mathcal{M}_\alpha)$ is strict:*

$$R_\alpha^a F_b \mathcal{M}_\alpha = F_{a+b} R_\alpha^a \mathcal{M}_\alpha, \quad \text{for any } a \in \mathbb{Z}_{\geq 0} \text{ and } b \in \mathbb{Z}. \tag{2.6.5}$$

Let $[R_\alpha]$ be the endomorphism on $\mathrm{gr}^F \mathcal{M}_\alpha$ induced by R_α. To prove the above theorem, we need the following statement on $\ker [R_\alpha] \subset \mathrm{gr}^F \mathcal{M}_\alpha$.

Lemma 2.6.6. $\ker [R_\alpha]^{r+1}$ *is locally generated by monomials of degree $\mu - r$ that divid t_α.*

Proof of Theorem 2.6.5. Temporarily admitting this lemma, let $R_\alpha^{r+1}m$ be an element in $F_{\ell+r+1}\mathcal{M}_\alpha$. Assume that $m \in F_k\mathcal{M}_\alpha$. If $k > \ell$ then the projection of $R_\alpha^{r+1}m$ vanishes in $\mathrm{gr}^F_{k+r+1}\mathcal{M}_\alpha$. It follows from the lemma that m can be written as

$$m = \sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} z_J m_J + \sum_{i=1}^n D_i Q_i + m', \qquad \text{for } z_J = \prod_{j\in J} z_j$$

where $Q_i, m' \in F_{k-1}\mathcal{M}_\alpha$. Because for every $J \subset I_\alpha$ of cardinality $r+1$ we can arrange

$$R_\alpha^{r+1} z_J = \prod_{j\in I_\alpha\smallsetminus J} \frac{1}{e_j} z_j \partial_j z_J = t_\alpha \prod_{j\in I_\alpha} \frac{1}{e_j}\partial_j = 0 \in \mathcal{M}_\alpha$$

it follows that $R_\alpha^{r+1}m$ is equal to,

$$\begin{aligned}\sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} R_\alpha^{r+1} z_J m_J + R_\alpha^{r+1}(\sum_{i=1}^n D_i Q_i + m') = &\sum_{\substack{\#J=\mu-r,\\ J\subset I_\alpha}} t_\alpha m'_J + \sum_{i=1}^n (D_i+\alpha) R_\alpha^{r+1} Q_i + R_\alpha^{r+1}(m' - \sum_{i=1}^n \alpha Q_i)\\ =&R_\alpha^{r+1}(m' - \sum_{i=1}^n \alpha Q_i) \in \mathcal{M}_\alpha.\end{aligned}$$

But now $m' - \sum_{i=1}^n \alpha Q_i \in F_{k-1}\mathcal{M}_\alpha$. Iterating the above argument one can find $\tilde{m} \in F_\ell \mathcal{M}_\alpha$ such that

$$R_\alpha^{r+1} m = R_\alpha^{r+1}\tilde{m}.$$

This completes the proof of the theorem. □

Proof of the lemma. The proof is essentially the same as the reduced case. Note that we are now working over the commutative ring $\mathrm{gr}^F\mathscr{D}_X$. We prove by induction on r. Let $P \in \mathrm{gr}^F\mathscr{D}_X$ represent an element of $\ker[R_\alpha]^{r+1}$. When $r = 0$, we have

$$\frac{1}{e_0} z_0 \partial_0 P = t_\alpha Q_0 + \sum_{i=1}^n D_i Q_i \ \text{ recalling that } \ t_\alpha = z_0 z_1 \cdots z_\mu. \tag{2.6.6}$$

Then $t_\alpha Q_0$ is in the ideal generated by $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}\partial_{\mu+1}, z_{\mu+2}\partial_{\mu+2}, ..., z_k\partial_k, \partial_{k+1}, ...\partial_n$ over $\mathrm{gr}^F\mathscr{D}_X$. Because t_α together with $\partial_0, \partial_1, ..., \partial_\mu, z_{\mu+1}\partial_{\mu+1}, z_{\mu+2}\partial_{\mu+2}, ..., z_k\partial_k, \partial_{k+1}, ...\partial_n$ form a regular sequence in $\mathrm{gr}^F\mathscr{D}_X$, Q_0 can be written as,

$$Q_0 = \sum_{a=0}^{\mu} \partial_a Q_a + \sum_{b=\mu+1}^{k} z_b \partial_b Q_b + \sum_{c=k+1}^{n} \partial_c Q_c.$$

Substuiting in (2.6.6)

$$\frac{1}{e_0}z_0\partial_0\left(P-\sum_{a=0}^{\mu}e_a\frac{t_\alpha}{z_a}Q_a-\sum_{b=\mu+1}^{k}e_bt_\alpha Q_b\right)\in(D_1,D_2,...,D_n)\mathrm{gr}^F\mathscr{D}_X.$$

Now because $(z_0\partial_0,D_1,D_2,...,D_n)$ is a regular sequence in $\mathrm{gr}^F\mathscr{D}_X$, P is a linear combination of t_α/z_a for $a\in\{0,1,...,\mu\}$ and $D_1,D_2,...,D_n$ over $\mathrm{gr}^F\mathscr{D}_X$. This concludes the case when $r=0$.

Assume the statement is true for the case when the exponent is less than r. Because $[R_\alpha]$ sends the class of P to $\ker[R_\alpha]^r$, by induction hypothesis we have

$$\frac{1}{e_0}z_0\partial_0P=\sum_{\substack{\#J=\mu-r+1,\\ J\subset I_\alpha}}z_JQ_J+\sum_{i=1}^{n}D_iQ_i\quad\text{recalling that }z_J=\prod_{j\in J}z_j.\tag{2.6.7}$$

Fixing a subset J, then z_JQ_J is in the submodule generated by z_a for $a\in I_\alpha\smallsetminus J$, ∂_b for $b\in J$, $z_c\partial_c$ for $c\in I\smallsetminus I_\alpha$ and ∂_d for $d\notin I$ over $\mathrm{gr}^F\mathscr{D}_X$. Because the elements $z_a,\partial_b,z_c\partial_c,\partial_d$ for $a\in I_\alpha\smallsetminus J,b\in J,c\in I\smallsetminus I_\alpha,d\notin I$ together with z_J form a regular sequence in $\mathrm{gr}^F\mathscr{D}_X$, we deduce that

$$Q_J=\sum_{a\in I_\alpha\smallsetminus J}z_aQ_a+\sum_{b\in J}\partial_bQ_b+\sum_{c\in I\smallsetminus I_\alpha}z_c\partial_cQ_c+\sum_{d\notin I}\partial_dQ_d.$$

Substituting in (2.6.7), we deduce that

$$\frac{1}{e_0}z_0\partial_0\left(P-\left(\sum_{b\in J}e_b\frac{z_J}{z_b}Q_b+\sum_{c\in I\smallsetminus I_\alpha}e_cz_JQ_c\right)\right)-\sum_{a\in I_\alpha\smallsetminus J}z_Jz_aQ_a$$

is in the submodule generated by degree $\mu-r+1$ monomials dividing t_α except z_J and by $D_1,D_2,...,D_n$ over $\mathrm{gr}^F\mathscr{D}_X$. This means we can reduce z_JQ_J one by one for each J on the right-hand side of the equation (2.6.7) and at the last step we find that $\frac{1}{e_0}z_0\partial_0(P-P')$ is a linear combination of degree $\mu-r+2$ monomials dividing t_α and $D_1,D_2,...,D_n$, where P' is a linear combination of degree $\mu-r$ monomials dividing t_α.

Note that the left multiplication by $\frac{1}{e_0}z_0\partial_0$ has the same effect as applying $[R_\alpha]$ on $\mathrm{gr}^F\mathcal{M}_\alpha$. Therefore, the class represented by $P-P'$ is in $\ker[R_\alpha]^r$ since degree $\mu-r+2$ monomials dividing t_α is in $\ker[R_\alpha]^{r-1}$. By induction hypothesis the class represented $P-P'$ is a linear combination of degree $\mu-r+1$ monomials dividing t_α. Therefore, the class represented by P

in $\mathrm{gr}^F\mathcal{M}_\alpha$ is a linear combination of degree $\mu - r$ monomials dividing t_α. This completes the proof. □

Corollary 2.6.7. *The* $\ker R_\alpha^{r+1}$ *is also generated by degree* $\mu - r$ *monomials dividing* t_α *if one identifies* $\mathcal{M}_\alpha$ *locally with* $\mathscr{D}_X/(t_\alpha, D_1, D_2, ..., D_n)\mathscr{D}_X$.

The proof is the same as the one of Corollary 2.4.3

2.6.3 The weight filtration

Now the weight filtration of each generalized eigen-modules interacts well with the good filtration because of the strictness. Recall that since R_α is nilpotent on $\mathcal{M}_\alpha$, it induces a $\mathbb{Z}$-indexed filtration $W_\bullet\mathcal{M}_\alpha$. We filtered the sub-module $W_r\mathcal{M}_\alpha$ by the induced filtration $F_\bullet W_r\mathcal{M}_\alpha = F_\bullet\mathcal{M}_\alpha \cap W_r\mathcal{M}_\alpha$. Let

$$\mathcal{P}_{\alpha,r} = \frac{\ker R_\alpha^{r+1}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}$$

be the r-th primitive part of $\mathrm{gr}^W\mathcal{M}_\alpha$ with the filtration defined by

$$F_\ell\mathcal{P}_{\alpha,r} = \frac{F_\ell \ker R_\alpha^{r+1} + \ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}}.$$

As the formal proof in Theorem 2.4.6, we have

Corollary 2.6.8. *The induced operator* $R_\alpha^r : F_\ell \mathrm{gr}_r^W\mathcal{M}_\alpha \to F_{\ell+r}\mathrm{gr}_{-r}^W\mathcal{M}_\alpha$ *is an isomorphism. Therefore, the Lefschetz decomposition of* $\mathrm{gr}^W\mathcal{M}_\alpha$ *respects filtrations, i.e.*

$$F_\bullet \mathrm{gr}_r^W\mathcal{M}_\alpha = \bigoplus_{\ell\geq 0, -\frac{r}{2}} R_\alpha^\ell F_{\bullet-\ell}\mathcal{P}_{\alpha,r+2\ell} \ \textit{for any} \ r \in \mathbb{Z}.$$

2.6.4 Summands of the primitive part $\mathcal{P}_{\alpha,r}$

Recall that $Y^J = \bigcap_{j\in J} Y_j$ and $Y_J = \bigcup_{j\in J} Y_j$ for any subset J of I and e_j is the multiplicity of Y_j in Y. Like the reduced case that $\mathcal{P}_r$ decomposes into the direct images of $\omega_{Y^J}(-r)$ for all index subset sJ of cardinality $r+1$ (Theorem 2.4.7), the primitive part $\mathcal{P}_{\alpha,r}$ of the generalized

α-eigemodule also decomposes into direct images of certain filtered $\mathscr{D}_{Y^J}$-modules $\mathcal{V}_{\alpha,J}(-r)$ for all J of cardinality $r+1$ such that $e_j\alpha$ for every $j \in J$ is an integer. The filtered $\mathscr{D}_{Y^J}$-modules $\mathcal{V}_{\alpha,J}$ comes from cyclic coverings so that $\mathcal{P}_{\alpha,r}$ carries the Hodge theory of the cyclic coverings. In fact, by a well-know construction in [EV92, §3] the direct image of the de Rham complex of a cyclic covering decomposes into log de Rham complexes of line bundles. A line bundle with an integrable log connection also can be viewed as a log $\mathscr{D}$-module. This suggests that the $\mathscr{D}$-modules $\mathcal{V}_{\alpha,J}$ is generated by a certain log $\mathscr{D}$-module $\mathscr{V}_{\alpha,J}$. If Y is reduced and $\alpha = 0$, $\mathcal{V}_{\alpha,J}$ is just ω_{Y^J}. We shall construct auxiliary log $\mathscr{D}$-modules $\mathscr{V}_{\alpha,J}$ whose log de Rham complex will be used to construct the $\mathscr{D}$-module $\mathcal{V}_{\alpha,J}$, without using cyclic cover. The cyclic coverings are involved only when we study the Hodge theory of those $\mathscr{D}$-modules. We fix a rational number $\alpha \in [0,1)$ to simplify the notations and let I_α be a subset of indices consisting of i such that αe_i is an integer.

Denote by $\mathcal{L}$ the line bundle $\mathscr{O}_X\left(-\sum_{i\in I_\alpha}\frac{e_i}{N}Y_i\right)$, where N is the greatest common divisor of e_i for $i \in I_\alpha$. In this notation, $\mathscr{O}_X(-\lceil \alpha Y\rceil) = \mathcal{L}^{\alpha N}\left(-\sum_{i\in I\smallsetminus I_\alpha}\lceil \alpha e_i Y_i\rceil\right)$. Because the line bundle $\mathscr{O}_X(Y)$ can be trivialized by a global section, we get an isomorphism of $\mathscr{O}_X$-modules:

$$\mathcal{L}^N = \mathscr{O}_X\left(-\sum_{i\in I_\alpha} e_iY_i\right) \to \mathscr{O}_X\left(\sum_{i\in I\smallsetminus I_\alpha} e_iY_i\right). \tag{2.6.8}$$

Choose a local section l of $\mathcal{L}$ such that $l^N \mapsto \prod_{i\in I\smallsetminus I_\alpha} z_i^{-e_i}$ under (2.6.8). Now we shall put a log connection ∇ on

$$\mathscr{O}_X(-\lceil \alpha Y\rceil) = \mathcal{L}^{\alpha N}\left(-\sum_{i\in I\smallsetminus I_\alpha}\lceil \alpha e_iY_i\rceil\right).$$

First we define, using the product rule

$$\frac{\nabla l^N}{l^N} = N\frac{\nabla l}{l} = \sum_{i\in I\smallsetminus I_\alpha} -e_i\frac{dz_i}{z_i} \tag{2.6.9}$$

due to (2.6.8). Then, let $s = l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil \alpha e_i\rceil}$ be the local frame of $\mathscr{O}_X(-\lceil\alpha Y\rceil)$. Noting that αN is a non-negative integer, the induced log connection works as

$$\begin{aligned}\frac{\nabla s}{s} = \frac{\nabla\left(l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil \alpha e_i\rceil}\right)}{l^{\alpha N}\prod_{i\in I\smallsetminus I_\alpha} z_i^{\lceil \alpha e_i\rceil}} &= \alpha N\frac{\nabla l}{l} + \sum_{i\in I\smallsetminus I_\alpha}\lceil \alpha e_i\rceil\frac{dz_i}{z_i} \\ &= \sum_{i\in I\smallsetminus I_\alpha}\left(\lceil \alpha e_i\rceil - \alpha e_i\right)\frac{dz_i}{z_i} = \sum_{i\in I\smallsetminus I_\alpha}\{-\alpha e_i\}\frac{dz_i}{z_i},\end{aligned} \tag{2.6.10}$$

where $\{-\}$ denotes the function of taking fractional part. Putting in more standard form,

$$\nabla s = \sum_{i \in I \setminus I_\alpha} \{-\alpha e_i\} \frac{dz_i}{z_i} \otimes s.$$

This log connection is integrable and has poles along Y_i for $i \in I \setminus I_\alpha$ with eigenvalues $\{-\alpha e_i\}$. We endow the line bundle $\mathscr{O}_X(-\lceil \alpha Y \rceil)$ with this integrable log connection ∇.

Fix a subset J of I_α with $\#J = r+1$ so that $\dim Y^J = n-r$. The pullback of $(\mathscr{O}_X(\lceil -\alpha Y \rceil), \nabla)$ by the inclusion $\tau^J : Y^J \to X$ gives an integrable log connection $(\mathscr{V}, \nabla) = (\mathscr{V}_{\alpha,J}, \nabla)$ on Y^J with poles along $E = E^{\alpha,J}$ the pullback of $Y_{I \setminus I_\alpha}$. Moreover, the log de Rham complex of $(\mathscr{V}, \nabla)$

$$\{\mathscr{V} \to \Omega_{Y^J}(\log E) \otimes \mathscr{V} \to \cdots \to \Omega^{n-r}_{Y^J}(\log E) \otimes \mathscr{V}\}[n-r],$$

induces a complex of $\mathscr{D}_{Y^J}$-modules

$$\{\mathscr{V} \otimes \mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log E) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log E) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J}\}[n-r], \qquad (2.6.11)$$

which is nothing but the log de Rham complex of $\mathscr{V} \otimes \mathscr{D}_{Y^J}$. It follows from Lemma 2.1.3 that the complex is a resolution of

$$\mathcal{V} = \mathcal{V}_{\alpha,J} =_{\mathrm{def}} \omega_{Y^J}(\log E) \otimes \mathscr{V} \underset{\mathscr{D}_{(Y^J,E)}}{\otimes} \mathscr{D}_{Y^J}.$$

We endow $\mathcal{V}$ with the filtration $F_\ell \mathcal{V} = F_\ell \mathcal{V}_{\alpha,J}$ induced the subcomplex

$$\{\mathscr{V} \otimes F_\ell \mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+1}\mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{D}_{Y^J}\}[n-r].$$

It is clear that $F_\bullet \mathcal{V}$ is a good filtration. For example, if $\alpha = 0$, then E is empty and $\mathscr{V}$ is just $\mathscr{O}_{Y^J}$ so that $\mathcal{V} = \omega_{Y^J}$ as $\mathscr{D}_{Y^J}$-modules. Since the eigenvalues of the log connection are in $(0,1)$ if poles exist, the log de Rham complex of $(\mathscr{V}, \nabla)$ is the minimal extension $R_{!*}\mathbb{V}$ of the local system $\mathbb{V}$ consisting of the flat sections of ∇ on $\mathscr{V}$ over $Y^J \setminus Y_{I \setminus J}$ (see [EV92, p. 1.6]). Later we will put a sesquilinear pairing on $\mathcal{V}$ and all the data will yield a pure Hodge structure of the log de Rham complex of $\mathscr{V}$.

Lemma 2.6.9. *The de Rham complex* $\mathrm{DR}_{Y^J}\mathcal{V}$ *together with the filtration* $F_\bullet \mathrm{DR}_{Y^J}\mathcal{V}$ *is isomorphic to the log de Rham complex* $\Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V}$ *with the stupid filtration in the* *derived category of filtered complexes of $\mathbb{C}$-vector spaces. In addition, $\mathcal{V}$ is holonomic and the characteristic cycle of $\mathcal{V}$ is*

$$cc(\mathcal{V}) = \sum_{K \subset I \smallsetminus I_\alpha} \left[T^*_{Y^{K \cup J}} Y^J\right].$$

Proof. We can choose the local frame s of $\mathscr{V}$ such that

$$\nabla s = \sum_{i \in I \smallsetminus I_\alpha} \frac{dz_i}{z_i} \otimes \{-\alpha e_i\} s$$

where z_i is the defining equation of E_i for each i. Therefore, the complex (2.6.11) locally is the Koszul complex over $\mathscr{D}_{Y^J}$ associated to the sequence

$$x_1\partial_1 + \{-\alpha e_1\}, x_2\partial_2 + \{-\alpha e_2\}, ..., x_p\partial_p + \{-\alpha e_p\}, \partial_{p+1}, \partial_{p+2}, ..., \partial_{n-r},$$

for some rearrangement of coordinates and under the trivialization of $\mathscr{V}$ given by s. It follows that the associated graded of (2.6.11) is the Koszul complex associated to the regular sequence

$$x_1\partial_1, x_2\partial_2, ..., x_p\partial_\nu, \partial_{p+1}, \partial_{p+2}, ..., \partial_{n-r}$$

over $\mathrm{gr}^F \mathscr{D}_{Y^J}$. Thus, the complex (2.6.11) is filtered acyclic. By the similar argument in Theorem 2.3.5, the $\mathscr{D}_{Y^J}$-module $\mathcal{V}$ is holonomic and the charateristic cycle $cc(\mathcal{V}) = \sum_{K \subset I \smallsetminus I_\alpha} \left[T^*_{Y^{K \cup J}} Y^J\right]$.

Moreover, we have isomorphisms in the derived category of complexes of $\mathbb{C}$-vector spaces:

$$\begin{aligned} F_\ell \mathrm{DR}\mathcal{V} = F_{\ell+*}\mathcal{V} \otimes \bigwedge^{-\star} \mathscr{T}_{Y^J} &\simeq \Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r+\bullet+*}\mathscr{D}_{Y^J} \otimes \bigwedge^{-\star} \mathscr{T}_{Y^J} \\ &\simeq \Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r+\bullet}\mathscr{O}_{Y^J}. \end{aligned}$$

Since $F_\ell \mathscr{O}_{Y^J}$ is $\mathscr{O}_{Y^J}$ or vanishes if $\ell < 0$, the complex $\Omega^{n-r+\bullet}_{Y^J}(\log E) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{O}_{Y^J}$ is the stupid filtration on the log de Rham complex on $\mathscr{V}$. We conclude the proof. □

We also need an auxiliary $\mathscr{D}_{Y^J}$-module $\mathcal{V}^*_{\alpha,J}$ to identify the primitive part $\mathcal{P}_{\alpha,r}$ which plays the role as $\omega_{Y^J}(*D^J)$ in the counterpart for the reduced case (Theorem 2.4.7). The log de Rham complex of $(\mathscr{V}, \nabla)$ can be enlarged into

$$\{\mathscr{V} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V}\}[n-r], \quad \text{for } D = D^J \text{ the pullback of the divisor } Y_{I \smallsetminus J}.$$

It is quasi-isomorphic to $Rj_*\mathbb{V}$ for $j : Y^J \smallsetminus Y_{I_\alpha} \to Y^J$ is the open immersion. By the similar process of the above, it induces a filtered acyclic complex of $\mathscr{D}_{Y^J}$-modules

$$\{\mathscr{V} \otimes \mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V} \otimes \mathscr{D}_{Y^J}\}[n-r]. \qquad (2.6.12)$$

Let $\mathcal{V}^* = \mathcal{V}^*_{\alpha,J}$ be the 0-th cohomology of the above complex and endow it with the filtration such that $F_\ell \mathcal{V}^* = F_\ell \mathcal{V}^*_{\alpha,J}$ is induced by the subcomplex

$$\{\mathscr{V} \otimes F_\ell\mathscr{D}_{Y^J} \to \Omega_{Y^J}(\log D) \otimes \mathscr{V} \otimes F_{\ell+1}\mathscr{D}_{Y^J} \to \cdots \to \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V} \otimes F_{\ell+n-r}\mathscr{D}_{Y^J}\}[n-r].$$

We naturally get an induced morphism $(\mathcal{V}, F_\bullet\mathcal{V}) \to (\mathcal{V}^*, F_\bullet\mathcal{V}^*)$ from the inclusion of the log de Rham complexes.

Lemma 2.6.10. *The canonical morphism $(\mathcal{V}, F_\bullet\mathcal{V}) \to (\mathcal{V}^*, F_\bullet\mathcal{V}^*)$ is injective, whose image is generated by the monomials defining $D - E$.*

Proof. Suppose $x_1x_2\cdots x_p$ is the local defining equation of E and $x_1x_2\cdots x_q$ is the local defining equation of D for $q \geq p+1$. Since $\mathcal{V}$ is locally generated by the class of

$$\bigwedge_{i=1}^{p} \frac{dx_i}{x_i} \wedge dx_{p+1} \wedge \cdots \wedge dx_{n-r} \otimes s \otimes 1$$

and $\mathcal{V}^*$ is locally generated by the class of

$$\bigwedge_{i=1}^{q} \frac{dx_i}{x_i} \wedge dx_{q+1} \wedge \cdots \wedge dx_{n-r} \otimes s \otimes 1,$$

the image is generated by the class of $\bigwedge_{i=1}^{q} \frac{dx_i}{x_i} \wedge dx_{q+1} \wedge \cdots \wedge dx_{n-r} \otimes s \otimes x_{p+1}x_{p+2}\cdots x_q$. The morphism locally is

$$\mathscr{D}_{Y^J}/(x_1\partial_1 + r_1, \ldots, x_p\partial_p + r_p, \partial_{p+1}, \ldots, \partial_{n-r})\mathscr{D}_{Y^J} \to \mathscr{D}_{Y^J}/(x_1\partial_1 + r_1, \ldots, x_q\partial_q + r_q, \partial_{q+1} \ldots, \partial_{n-r})\mathscr{D}_{Y^J},$$

with $[P] \mapsto [x_{p+1}x_{p+2}\cdots x_q P]$ where $r_1, r_2, \ldots, r_p$ are the eigenvalues of ∇ on $\mathscr{V}$ and $r_{p+1} = r_{p+2} = \cdots = r_q = 0$. Since

$$\Omega^{n-r}_{Y^J}(\log E) \otimes \mathscr{V} = F_{-(n-r)}\mathcal{V} \to F_{-(n-r)}\mathcal{V}^* = \Omega^{n-r}_{Y^J}(\log D) \otimes \mathscr{V}$$

is injective, by induction, it suffices to show that $\mathrm{gr}^F\mathcal{V} \to \mathrm{gr}^F\mathcal{V}^*$ is injective. Due to the complexes (2.6.11) and (2.6.12) is filtered acyclic, the morphism on the associated graded modules works as, in the local representation,

$$\mathrm{gr}^F\mathscr{D}_{Y^J}/(x_1\partial_1,...,x_p\partial_p,\partial_{p+1},...,\partial_{n-r})\mathrm{gr}^F\mathscr{D}_{Y^J} \to \mathrm{gr}^F\mathscr{D}_{Y^J}/(x_1\partial_1,...,x_q\partial_q,\partial_{q+1}...,\partial_{n-r})\mathrm{gr}^F\mathscr{D}_{Y^J},$$

with $[P] \mapsto [x_{p+1}x_{p+2}\cdots x_q P]$. By induction on the number of components of $D-E$, we can assume $q = p+1$. Let $P \in \mathrm{gr}^F\mathscr{D}_{Y^J}$ represent a class in the kernel. Then

$$x_qP = \sum_{i=1}^{q} x_i\partial_i P_i + \sum_{j=q+1}^{n-r} \partial_j P_j \in \mathrm{gr}^F\mathscr{D}_{Y^J}.$$

Subtracting $x_q\partial_q P_q$ on the both sides yeilds

$$x_q(P-\partial_q P_q) = \sum_{i=1}^{q-1} x_i\partial_i P_i + \sum_{j=q+1}^{n-r} \partial_j P_j \in \mathrm{gr}^F\mathscr{D}_{Y^J}.$$

Since $x_q, x_1\partial_1, ..., x_{q-1}\partial_{q-1}, \partial_{q+1}, ..., \partial_{n-r}$ is a regular sequence over $\mathrm{gr}^F\mathscr{D}_{Y^J}$,

$$(P-\partial_q P_q) = \sum_{i=1}^{q-1} x_i\partial_i P_i' + \sum_{j=q+1}^{n-r} \partial_j P_j' \in \mathrm{gr}^F\mathscr{D}_{Y^J}.$$

We find that P is a linear combination of $x_1\partial_1, x_2\partial_2, ..., x_p\partial_p, \partial_{p+1}, ..., \partial_{n-r}$ over $\mathrm{gr}^F\mathscr{D}_{Y^J}$. We conclude the proof. □

Remark 2.6.11. One can use Riemann-Hilbert correspondence to conclude that $\mathcal{V}$ is the minimal extension of $\mathscr{V}|_{Y^J\setminus D}$ and $\mathcal{V}^*$ is the $*$-extension of $\mathscr{V}|_{Y^J\setminus D}$, which is overkill in our situation. The above argument also showed the strictness, i.e., $F_\ell\mathcal{V} = F_\ell\mathcal{V}^* \cap \mathcal{V}$.

Putting in more general notations and summarizing what we have proved in the above two lemmas:

Theorem 2.6.12. *The filtered $\mathscr{D}_{Y^J}$-module $(\mathcal{V}_{\alpha,J}, F_\bullet)$ is holonomic whose de Rham complex $\mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}$ together with the induced filtration is isomorphic to the log de Rham complex $\Omega_{Y^J}^{n-r+\bullet}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J}$ with the stupid filtration in the derived category of filtered complexes of $\mathbb{C}$-vector spaces and whose characteristic cycle is*

$$cc(\mathcal{V}_{\alpha,J}) = \sum_{K\subset I\setminus I_\alpha} \left[T^*_{Y^{K\cup J}}Y^J\right].$$

The canonical filtered morphism $(\mathcal{V}_{\alpha,J}, F_\bullet\mathcal{V}_{\alpha,J}) \to (\mathcal{V}^*_{\alpha,J}, F_\bullet\mathcal{V}^*_{\alpha,J})$ is injective and the image is generated by the monomial defining the divisor $D^J - E^{\alpha,J}$.

2.6.5 Identifying the primitive part $\mathcal{P}_{\alpha,r}$

Now we are going to identify the r-th primitive part $(\mathcal{P}_{\alpha,r}, F_\bullet\mathcal{P}_{\alpha,r})$ with a direct sum of $\mathcal{V}_{\alpha,J}(-r)$ for J ranging over subsets I_α of cardinality $r+1$. The argument is parallel to the one of the reduced case (Theorem 2.4.7), replacing $\mathcal{M}$ by $\mathcal{M}_\alpha$, R by R_α, ω_{Y^J} by $\mathcal{V}_{\alpha,J}$, $\omega_{Y^J}(*D^J)$ by $\mathcal{V}^*_{\alpha,J}$, the complex $\Omega^{n+\bullet}_{X/\Delta}(\log Y)|_Y$ by $C^\bullet_\alpha = \Omega^{n+\bullet}_{X/\Delta}(\log Y)(-\lceil\alpha Y\rceil)|_{Y_{I_\alpha}}$ and the log de Rham complex $\Omega^{n-r+\bullet}_{Y^J}(\log D^J)$ by $\Omega^{n-r+\bullet}_{Y^J}(\log D^J)\otimes\mathscr{V}_{\alpha,J}$.

Theorem 2.6.13. *Let $\mathcal{V}_{\alpha,r} = \bigoplus_J \tau^J_+\mathcal{V}_{\alpha,J}$ for J running over the subsets of I_α of cardinality $r+1$, where $\tau^J : Y^J \hookrightarrow X$ is the closed embedding. Then there exists an isomorphism $\phi_{\alpha,r} : (\mathcal{P}_{\alpha,r}, F_\bullet\mathcal{P}_{\alpha,r}) \to \mathcal{V}_{\alpha,r}(-r)$ in the category of filtered $\mathscr{D}_X$-modules.*

Proof. Because the log connection (2.6.8) we constructed on $\mathscr{O}_X(-\lceil\alpha Y\rceil)$ has zero residue on Y_i for $i\in I_\alpha$, we have the residue morphism between log de Rham complexes.

$$\mathrm{Res}_{Y^J} : \Omega^{\bullet+n+1}_X(\log Y)\otimes\mathscr{O}_X(-\lceil\alpha Y\rceil)|_{Y_{I_\alpha}} \to \Omega^{\bullet+n-r}_{Y^J}(\log D^J)\otimes\mathscr{V}_{\alpha,J},\ \text{where } D^J \text{ is the pull back of } Y_{I\smallsetminus J}$$

for $J\subset I_\alpha$ of cardinality $r+1$, up to a sign depending on the order of the indices. Denote by $B^\bullet_\alpha$ the log de Rham complex $\Omega^{\bullet+n+1}_X(\log Y)\otimes\mathscr{O}_X(-\lceil\alpha Y\rceil)$ of $\mathscr{O}_X(-\lceil\alpha Y\rceil)$. The residue morphism Res_{Y^J} extends to a morphism of the complexes of induced $\mathscr{D}_X$-modules

$$\mathrm{Res}_{Y^J} : B^\bullet_\alpha|_{Y_{I_\alpha}}\otimes\mathscr{D}_X \to \Omega^{\bullet+n-r}_{Y^J}(\log D^J)\otimes\mathscr{V}_{\alpha,J}\otimes\mathscr{D}_X.$$

Let $\mathcal{H}^\ell_\alpha$ be the ℓ-th cohomology of $B^\bullet_\alpha|_{Y_{I_\alpha}}\otimes\mathscr{D}_X$. Then we have an induced morphism $\mathrm{Res}_{Y^J}:\mathcal{H}^0_\alpha \to \mathcal{V}^*_{\alpha,J}$ by taking cohomology. Let $\mathrm{Res}_{\alpha,r} = \bigoplus \mathrm{Res}_{Y^J} : \mathcal{H}^0_\alpha \to \mathcal{V}^*_{\alpha,r}(-r)$ where $\mathcal{V}^*_{\alpha,r} = \bigoplus_J \mathcal{V}^*_{\alpha,J}$ for J running over cardinality $r+1$ subsets of I_α. Because $\frac{dt}{t}\wedge : \Omega^{\bullet+n}_{X/\Delta}(\log Y)(-\lceil\alpha Y\rceil) \to \Omega^{\bullet+n+1}_X(\log Y)(-\lceil\alpha Y\rceil)$ also extends to the complexes of the induced $\mathscr{D}_X$-modules, we obtain a short exact sequence

$$0 \longrightarrow C^{\bullet-1}_\alpha\otimes\mathscr{D}_X \xrightarrow{\frac{dt}{t}\wedge} B^{\bullet-1}|_{Y_{I_\alpha}}\otimes\mathscr{D}_X \longrightarrow C^\bullet_\alpha\otimes\mathscr{D}_X \longrightarrow 0.$$

The associated long exact sequence gives

$$0 \longrightarrow \mathcal{H}_\alpha^{-1} \longrightarrow \mathcal{M}_\alpha \longrightarrow \mathcal{M}_\alpha \xrightarrow{\frac{dt}{t}\wedge} \mathcal{H}_\alpha^0 \xrightarrow{R_\alpha} 0. \tag{2.6.13}$$

By pre-composing $\frac{dt}{t}\wedge$, we get a morphism

$$\mathrm{Res}_{\alpha,J} \circ \frac{dt}{t}\wedge : \mathcal{M}_\alpha \to \mathcal{V}^*_{\alpha,r}(-r), \quad [\zeta_\alpha \otimes P] \to [\mathrm{Res}_{\alpha,J}\frac{dt}{t} \wedge \zeta_\alpha \otimes P].$$

Recall that every element in $\mathcal{M}_\alpha$ is locally represented by $\zeta_\alpha \otimes P$ for $\zeta_\alpha = z_I^{[\alpha \mathbf{e}]}\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n$ given that locally $I = \{0, 1, \ldots, k\}$, and $P \in \mathscr{D}_X$. By Corollary 2.6.7, every class in $\ker R_\alpha^{r+1}$ is represented by $\zeta_\alpha \otimes z_{\overline{J}}P$ for some ordered index subset J of I_α of cardinality $r+1$ and $\overline{J}$ is the complement of J in I_α and $z_{\overline{J}} = \prod_{j \in \overline{J}} z_j$. Thus, its image under the above morphism only contained in the component $\mathcal{V}^*_{\alpha,J}(-r)$ because $z_{\overline{J}}$ vanishes on other components. The image is the class represented by

$$\mathrm{Res}_{\alpha,J}\frac{dz_0}{z_0}\wedge\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge\cdots\wedge dz_n z_I^{[\alpha\mathbf{e}]}\otimes z_{\overline{J}}P = \pm\frac{dz_{I\smallsetminus J}}{z_{I\smallsetminus J}}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes s_{\alpha,J}\otimes z_{\overline{J}}P \in \Omega^{n-r}_{Y^J}\otimes\mathscr{V}_{\alpha,J}\otimes\mathscr{D}_X, \tag{2.6.14}$$

where $s_{\alpha,J}$ is the local frame of $\mathscr{V}_{\alpha,J}$ by restricting $z_I^{[\alpha\mathbf{e}]}$ and the sign is depending on the order of J. It also follows from the calculation that the image does not have pole along the pull-back of $Y_{\overline{J}}$. So it is contained in the subsheaf consisting of classes represented by $\Omega^{n-r}_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} \otimes \mathscr{D}_X$, where $E^{\alpha,J}$ is the pull-back of $Y_{I\smallsetminus I_\alpha}$ so that $D^J - E^{\alpha,J}$ is the pull-back of $Y_{\overline{J}}$. This means that the image of the class represented by (2.6.14) is also in the image of the canonical inclusion:

$$\tau_+^J\mathcal{V}_{\alpha,J}(-r) \hookrightarrow \tau_+^J\mathcal{V}^*_{\alpha,J}(-r),$$

$$[dz_{\overline{J}} \wedge \frac{dz_{I\smallsetminus I_\alpha}}{z_{I\smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes P] \mapsto [\frac{dz_{\overline{J}}}{z_{\overline{J}}} \wedge \frac{dz_{I\smallsetminus I_\alpha}}{z_{I\smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes z_{\overline{J}}P].$$

See Theorem 2.6.12. Therefore, the morphism $\ker R_\alpha^{r+1} \to \mathcal{V}^*_{\alpha,r}(-r)$ constructed above factors through $\mathcal{V}_{\alpha,r}(-r)$. Summarizing, we have the following diagram.

$$\begin{array}{ccccc}
\ker R_\alpha^{r+1} & & \overset{\rho_{\alpha,r}}{\dashrightarrow} & & \mathcal{V}_{\alpha,r}(-r) \\
\big\downarrow & & & & \big\downarrow \\
\mathcal{M}_\alpha & \xrightarrow{\frac{dt}{t}\wedge} & \mathcal{H}_\alpha^0 & \xrightarrow{\mathrm{Res}_{\alpha,r}} & \mathcal{V}^*_{\alpha,r}(-r)
\end{array}$$

In fact, the kernel of ρ_r contains $\ker R_\alpha^r$: for an element in $\ker R_\alpha^r$ locally represented by $\zeta_\alpha \otimes z_K P$ for K a subset of I_α such that the cardinality of $I_\alpha \smallsetminus K$ is r, its image under $\rho_{\alpha,r}$ is zero because z_K annihilates all $\Omega^{n-r}_{Y^J}(\log D^J) \otimes \mathscr{V}_{\alpha,J}$ for any $J \subset I_\alpha$ of cardinality $r+1$. The morphism $\rho_{\alpha,r}$ also kills $R_\alpha \ker R_\alpha^{r+2}$ because $\frac{dt}{t}\wedge$ vanishes on the image of R_α by (2.6.13). It follows that $\rho_{\alpha,r}$ factors through a filtered morphism

$$\phi_{\alpha,r} : \mathcal{P}_{\alpha,r} = \frac{\ker R_\alpha^{r+1}}{\ker R_\alpha^r + R_\alpha \ker R_\alpha^{r+2}} \to \mathcal{V}_{\alpha,r}(-r).$$

For $dz_{\overline{J}} \wedge \frac{dz_{I\smallsetminus I_\alpha}}{z_{I\smallsetminus I_\alpha}} \wedge dz_{k+1} \wedge \cdots \wedge dz_n \otimes s_{\alpha,J} \otimes P \in \Omega^{n-r}_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} \otimes F_\ell \mathscr{D}_X$ representing a class in $F_\ell \tau_+^J \mathcal{V}_{\alpha,J}(-r)$ where $J \subset I_\alpha$ of cardinality $r+1$, we can find a lifting represented by $\zeta_\alpha \otimes z_{\overline{J}} P$ in $F_\ell \ker R_\alpha^{r+1}$, which means

$$F_\ell \ker R_\alpha^{r+1} \to F_{\ell+r} \mathcal{V}_{\alpha,r}$$

is surjective, i.e. the morphism $\phi_{\alpha,r}$ is filtered surjective. It remains to prove that $\phi_{\alpha,r}$ is injective. We prove that $\phi_{\alpha,r}$ is an isomorphism by counting the characteristic cycles as in Theorem 2.4.7. Because $\phi_{\alpha,r}$ is surjective, one gets

$$cc(\mathcal{P}_{\alpha,r}) \geq cc(\mathcal{V}_{\alpha,r}).$$

It follows from Corollary 2.6.12 that

$$cc(\mathcal{V}_{\alpha,r}) = \sum_{\substack{J\subset I_\alpha,\\ \#J=r+1}} cc(\tau_+^J \mathcal{V}_{\alpha,J}) = \sum_{\substack{J\subset I_\alpha,\\ \#J=r+1}} \sum_{K\subset I\smallsetminus I_\alpha} \left[T^*_{Y^{J\cup K}} X\right] = \sum_{\substack{J\subset I,\\ \#J\cap I_\alpha=r+1}} \left[T^*_{Y^J} X\right].$$

One the other hand, by the Lefschetz decomposition and Theorem 2.6.2,

$$\begin{aligned}\sum_{J\subset I} \#(J\cap I_\alpha)\left[T^*_{Y^J}X\right] = cc(\mathcal{M}_\alpha) = cc(\mathrm{gr}^W \mathcal{M}_\alpha) = \sum_{r\geq 0}(r+1)cc(\mathcal{P}_{\alpha,r}) \geq \sum_{r\geq 0}(r+1)cc(\mathcal{V}_{\alpha,r})\\ = \sum_{r\geq 0} \sum_{\substack{J\subset I,\\ \#J\cap I_\alpha=r+1}} (r+1)\left[T^*_{Y^J}X\right] = \sum_{J\subset I} \#(J\cap I_\alpha)\left[T^*_{Y^J}X\right].\end{aligned}$$

It follows that all inequalities above are equalities and in particular,

$$cc(\mathcal{P}_{\alpha,r}) = cc(\mathcal{V}_{\alpha,r})$$

from which we conclude that $\phi_{\alpha,r}$ is an isomorphism between the underlying $\mathscr{D}_X$-modules. Plus it is filtered surjective, we conclude that $\phi_{\alpha,r}$ is filtered isomorphism. □

2.7 Non-reduced case: Sesquilinear pairing and limiting mixed Hodge structure

2.7.1 Kähler package of cyclic covering

To accomplish our goal, we need to show that the sum of all hypercohomologies of the complex

$$\Omega^{\bullet}_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J}[n-r]$$

has a polarized Hodge-Lefschetz structure and hard Lefschetz so that the hypercohomology of the de Rham complex of the primitive part $\mathcal{P}_{\alpha,r}$ will inherit the properties by Theorem 2.6.12 and Theorem 2.6.13. For this, we need to use the geometry of cyclic coverings.

We first give another description of the integrable log connection (2.6.8) using cyclic coverings. Fix a rational number α in $[0,1)$, Because the isomorphism,

$$\mathcal{L}^N = \mathscr{O}_X\left(-\sum_{i\in I_\alpha} e_i Y_i\right) \to \mathscr{O}_X\left(\sum_{i\in I\smallsetminus I_\alpha} e_i Y_i\right),$$

we obtain a cyclic covering $\pi_\alpha : X_\alpha \to X$ by taking the N-th roots out of $\sum_{i\in I\smallsetminus I_\alpha} e_i Y_i$ and normalizing it. The direct image $\pi_{\alpha*}\mathscr{O}_{X_\alpha}$ decomposes into eigenspaces with respect the Galois action as well as the direct image of exterior differential $\pi_{\alpha*}\mathscr{O}_{X_\alpha} \to \pi_{\alpha*}\Omega_{X_\alpha}$ [EV92, Theorem 3.2]. The line bundle

$$\mathcal{L}^{\alpha N}\left(-\sum_{i\in I\smallsetminus I_\alpha} \lceil \alpha e_i Y_i \rceil\right),$$

is the α-eigenspaces of $\pi_{\alpha*}\mathscr{O}_{X_\alpha}$ for some suitable choice of a generator of the Galois group. Because the decomposition respects the exterior differential, we obtained an integrable log connection with eigenvalues $\{\alpha e_i\}$ along Y_i for each $i \in I_\alpha$. Note that X_α might not be smooth.

Let $J \subset I_\alpha$ of cardinality $r+1$. Since Y^J is not contained in $Y_{I\smallsetminus I_\alpha}$, the fiber product $Y^J_\alpha = X_\alpha \times_X Y^J$ is again a cyclic covering of Y^J by taking the N-th roots out of $\sum_{i\in I\smallsetminus I_\alpha} e_i Y_i \cap Y^J$.

Let $\pi_\alpha^J : Y_\alpha^J \to Y^J$ be the second projection.

$$\begin{array}{ccc} Y_\alpha^J & \longrightarrow & X_\alpha \\ \downarrow{\scriptstyle \pi_\alpha^J} & & \downarrow{\scriptstyle \pi_\alpha} \\ Y^J & \xrightarrow{\tau^J} & X \end{array} \tag{2.7.1}$$

We conclude that $(\mathscr{V}_{\alpha,J},\nabla)$ is the α-eigenspace of $\pi_{\alpha*}^J(\mathscr{O}_{Y_\alpha^J},d)$. The log de Rham complex of $(\mathscr{V}_{\alpha,J},\nabla)$ is a summand of the direct image of the de Rham compolex $\pi_{\alpha*}^J\Omega_{Y_\alpha^J}^{\bullet+n-r}$ of Y_α^J.

We shall work in the general setting and adopt the convention in [EV86] and [EV92]. Let $\mathcal{L}$ be a line bundle on a Kähler manifold Z with a Kähler form ω and $D = \sum_i \nu_i D_i$ be a simple normal crossings divisor such that for some $N > 1$ one has $\mathcal{L}^N = \mathscr{O}_Z(D)$. Define $\mathcal{L}^{(j)} = \mathcal{L}^j(-\lfloor \frac{jD}{N} \rfloor)$ for $1 \le j \le N-1$. One puts an integrable logarithmic connection on $\mathcal{L}^{(j)}$ with poles along $D^{(j)}$, where

$$D^{(j)} = \sum_{\frac{j\nu_i}{N} \notin \mathbb{Z}} D_i.$$

Let $\iota : U \hookrightarrow Z$ be the complement of D and $\mathbb{V}$ is the underlying local system of $\mathcal{L}|_U$. Let $\tau : Z' \to Z$ be the cyclic covering obtained by first taking N-th root out of D then taking the normalization and $\pi : \tilde{Z} \to Z'$ be a log resolution of singularity equivariant with respect to the Galois group $\mathrm{Gal}(Z'/Z) = \langle\sigma\rangle$ and let E be the simple normal crossing exceptional divisor.

$$\tilde{Z} \xrightarrow{\pi} Z' \xrightarrow{\tau} Z, \qquad \eta = \tau \circ \pi$$

Note that $\tilde{Z}$ is Kähler because it is a resolution of subvariety of the geometric line bundle of $\mathcal{L}$, which is Kähler, although the induced Kähler class does not relate well with ω on X. The pullback $\eta^*\omega$ is only positive over $\tilde{U} = \eta^{-1}(U)$, but one can still cook up a Kähler class by adding a small multiple of the first Chern class $\Theta \in H^2(\tilde{Z},\mathbb{Z}(1))$ of the relative ample line bundle of the projective morphism $\pi : \tilde{Z} \to Z'$. We can assume Θ is invariant under σ by averaging it.

Lemma 2.7.1. *Notations as above, the cohomology class* $[\eta^*\omega] + \lambda(2\pi\sqrt{-1})^{-1}\Theta \in H^{1,1}(Z) \cap H^2(Z,\mathbb{R})$ *is an invariant Kähler class for* λ *is a sufficient small positive number.*

Proof. Let $\tilde{D}_i$ be the strict transformation of $\tau^{-1}(D_i)$ and $s_i \in H^0(\tilde{Z}, \mathscr{O}_{\tilde{Z}}(\tilde{D}_i))$ whose zero locus is $\tilde{D}_i$. Let h_i be a Hermitian metric on each line bundle $\mathscr{O}_{\tilde{Z}}(\tilde{D}_i)$ and ρ_i be sufficient small positive bump function supported in a small neighborhood of $\tilde{D}_i$ for each i. Then the $(1,1)$-form

$$\eta^*\omega + \sum_i \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_i h_i(s_i, s_i)$$

is positive on $\tilde{Z} - E$ but only semi-positive over E. However, the class $(2\pi\sqrt{-1})^{-1}\Theta$ is positive over E. Therefore, for λ sufficient small positive, the class of

$$\eta^*\omega + \sum_i \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\rho_i h_i(s_i, s_i) + \lambda(2\pi\sqrt{-1})^{-1}\Theta$$

is a Kähler class. But $\partial\bar{\partial}\rho_i h_i(s_i, s_i)$ is exact. The cohomology class of above just equals $[\eta^*\omega] + \lambda(2\pi\sqrt{-1})^{-1}\Theta$ in $H^{1,1}(\tilde{Z}) \cap H^2(Z, \mathbb{R})$. It is invariant because both $[\eta^*\omega]$ and Θ are invariant. □

Lemma 2.7.2. *The hypercohomology* $H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$ *is a summand of* ξ^{-j}*-eigenspace of* $H^k(\tilde{Z})$*, and thus it is a sub-Hodge structure of weight* k*.*

Proof. It follows from (1.6) in [EV86] that $R\iota_!\mathbb{V}^{-j}$, $R\iota_*\mathbb{V}^{-j}$ and $\Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}$ are all quasi-isomorphic. Taking hypercohomology gives canonical isomorphisms

$$H^k(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}) \simeq H_c^k(U, \mathbb{V}^{-j}) \simeq H^k(U, \mathbb{V}^{-j}).$$

Because η is étale over U, $H^k(U, \mathbb{V}^j)$ (resp. $H_c^k(U, \mathbb{V}^j)$) is a ξ^j-eigenspace of $H^k(\tilde{U}, \mathbb{C})$ (resp. $H_c^k(\tilde{U}, \mathbb{C})$) for the cyclic action σ, where ξ is a N-th root of unity. Then the canonical morphisms of mixed Hodge structures

$$H_c^k(\tilde{U}) \to H^k(\tilde{Z}) \to H^k(\tilde{U}) \tag{2.7.2}$$

respect the eigenspaces decomposition because we make $\tilde{Z}$ equivariant. We complete the proof. □

Lemma 2.7.3. *Let* $\mathsf{X} = 2\pi\sqrt{-1}L$ *where* $L = [\omega]\wedge$ *is the Lefschetz operator on* Z*. The following two statements hold:*

1. *Hard Lefschetz is valid on the hypercohomolgy, i.e.*

$$\mathsf{X}^k : H^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z + k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k)$$

is an isomorphism of Hodge structures.

2. *The pairing*

$$(m', m'') \mapsto \frac{\varepsilon(\dim Z + k + 1)}{(2\pi\sqrt{-1})^{\dim Z}} \int_{\tilde{Z}} \eta^*\left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right) \tag{2.7.3}$$

is a polarization on the primitive part of $H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$*, where* $\eta^*\left(\mathsf{X}^{\dim Z - k}\alpha \wedge \overline{\beta}\right)$ *is the top form determined by the inclusion* $\eta^*\Omega_Z^{\dim Z}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \subset \omega_{\tilde{Z}}$.

Proof. Let $\tilde{L} = [\eta^*\omega + \lambda\Theta]\wedge$ be the Lefschetz operator on $\tilde{Z}$. Then the Hard Lefschetz on $\tilde{Z}$ says

$$\tilde{\mathsf{X}}^k : H^{\dim Z - k}(\tilde{Z}) \to H^{\dim Z + k}(\tilde{Z})(k)$$

is an isomorphism, where $\tilde{\mathsf{X}} =_{\text{def}} 2\pi\sqrt{-1}\tilde{L}$. Because $\tilde{L}$ is invariant and respects the morphisms in (2.7.2), the above isomorphism is compatible with eigenspaces decomposition, it follows that

$$\tilde{\mathsf{X}}^k : H^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z + k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k) \tag{2.7.4}$$

is injective by Lemma 2.7.2. In fact, the ξ^i-eigenspace of $H_c^k(\tilde{U})$ is orthogonal to the ξ^j-eigenspace of $H^{2\dim Z - k}(\tilde{U})$ with respect to Poincaré pairing unless $i + j \equiv 0 \pmod N$: for a in the ξ^i-eigenspace of $H_c^k(\tilde{U})$ and b in the ξ^j-eigenspace of $H^{2\dim Z - k}(\tilde{U})$ then

$$\xi^i \int_{\tilde{U}} a \wedge b = \int_{\tilde{U}} \sigma^* a \wedge b = \int_{\tilde{U}} a \wedge (\sigma^{-1})^* b = \xi^{-j} \int_{\tilde{U}} a \wedge b,$$

which means $\int_{\tilde{U}} a \wedge b$ is zero unless $i + j \equiv 0 \pmod N$. It follows from Poincaré duality on $H_c^k(\tilde{U}) \times H^{2\dim Z - k}(\tilde{U})$ that the ξ^i-eigenspace of $H_c^k(\tilde{U})$ is Poincaré dual to the ξ^{-i}-eigenspace of $H^{2\dim Z - k}(\tilde{U})$. On the other hand, since the ξ^i-eigenspace is complex conjugate to the ξ^{-i}-eigenspace, the ξ^i-eigenspace of $H_c^k(\tilde{U})$ and the ξ^i-eigenspace of $H^{2\dim Z - k}(\tilde{U})$ have the same dimension. It follows that the morphism (2.7.4) is an isomorphism.

The operator $\tilde{L}$ has the same effect as $\eta^* L$ over $H_c^\bullet(\tilde{U})$, because Θ is supported on E. Therefore,

$$\mathsf{X}^k : H^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z + k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)(k)$$

is an isomorphism. We conclude (1). It also follows that η^* identifies the primitive part of X

$$H_{\mathrm{prim}}^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$$

with the primitive part of $\tilde{\mathsf{X}}$

$$\ker\left(\tilde{\mathsf{X}}^{k+1} : H^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right) \to H^{\dim Z + k + 2}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)\right).$$

Thus, $H_{\mathrm{prim}}^{\dim Z - k}\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$ is a sub-Hodge structure of $H_{\mathrm{prim}}^{\dim Z - k}(\tilde{Z})$. And the restriction of the polarization is again a polarization. This proves (2). □

The above two lemmas indicate that the sum of hypercohomologies

$$\bigoplus_{k \in \mathbb{Z}} H^k\left(Z, \Omega_Z^\bullet(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}\right)$$

is a polarized sub-Hodge-Lefschetz structure of $\bigoplus_{k \in \mathbb{Z}} H^k(\tilde{Z}, \mathbb{C})$. In practice, it is more convenient to make the polarization independent of the resolution of singularities and intrinsic on Z. Heuristically, the local system $\mathbb{V}^{-j}$ over U inherits a pairing from $\mathbb{C}_{\tilde{U}}$ and it has a Hodge theoretic extension on its canonical extension. First, we can resolve $\Omega_Z^\bullet(\log D^{(j)})$ using $\mathcal{A}_Z^\bullet(\log D^{(j)})$, the complex of $\mathscr{C}^\infty$-forms with log poles along $D^{(j)}$. Note that we have the inclusion of sheaves

$$\mathcal{A}_Z^{\dim Z + k}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}} \wedge \overline{\mathcal{A}_Z^{\dim Z - k}(\log D^{(j)}) \otimes \mathcal{L}^{(j)^{-1}}} \subset \mathcal{A}_Z^{2\dim Z} \otimes \mathcal{L}^{(j)^{-1}}(D^{(j)}) \underset{\mathbb{C}}{\otimes} \overline{\mathcal{L}^{(j)^{-1}}(D^{(j)})}.$$

Since $\mathcal{L}^N \simeq \mathscr{O}_Z(D)$, picking local section of l such that $l^N = \prod_i x_i^{-\nu_i}$ we can put a canonical singular Hermitian metric on $\mathcal{L}$ by setting the weight function as

$$|l|_h = \prod_i |x_i|^{-\nu_i/N}, \quad \text{where } x_i \text{ is the local defining equation of } D_i.$$

Then the induced singular Hermitian metric on ${\mathcal{L}^{(j)}}^{-1}(D^{(j)}) = \mathcal{L}^{-j}(\lfloor \frac{jD}{N} \rfloor + D^{(j)})$ locally is

$$\left| l^{-j} \prod_i x_i^{-\lfloor j\mu_i/N \rfloor} \prod_{j\nu_i/N \notin \mathbb{Z}} x_i^{-1} \right|_h = \prod_i |x_i|^{j\nu_i/N - \lfloor j\nu_i/N \rfloor} \prod_{j\nu_i/N \notin \mathbb{Z}} |x_i|^{-1} = \prod_i |x_i|^{-\{-j\nu_i/N\}}.$$

For any smooth top form Υ with values in ${\mathcal{L}^{(j)}}^{-1}(D^{(j)}) \otimes_{\mathbb{C}} \otimes \overline{{\mathcal{L}^{(j)}}^{-1}(D^{(j)})}$ we can associate an integrable top form $(\Upsilon)_h = f\overline{g}|s|_h^2 \operatorname{vol}(Z)$ by fixing a volume form $\operatorname{vol}(Z)$ on Z and writing locally $\Upsilon = fs \otimes \overline{gs} \operatorname{vol}(Z)$ for s a local fram of ${\mathcal{L}^{(j)}}^{-1}(D^{(j)})$. Therefore, we obtain a well-defined pairing,

$$\begin{aligned} \mathcal{A}_Z^k(\log D^{(j)}) \otimes {\mathcal{L}^{(j)}}^{-1} \wedge \overline{\mathcal{A}_Z^k(\log D^{(j)}) \otimes {\mathcal{L}^{(j)}}^{-1}} &\to \mathbb{C}, \\ (m', m'') &\mapsto \frac{\varepsilon(\dim Z + k + 1)}{(2\pi\sqrt{-1})^{\dim Z}} \int_Z \left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right)_h. \end{aligned} \tag{2.7.5}$$

Since $\eta : \tilde{Z} \to Z$ is generic finite, it follows from

$$\int_{\tilde{Z}} \eta^* \left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right) = N \int_Z \left(\mathsf{X}^{\dim Z - k} m' \wedge \overline{m''}\right)_h$$

that (2.7.5) induces the same polarization in the statement (2) of the above lemma except for the constant N.

Applying to our situation yields that $\mathscr{V}_{\alpha,J}(E^{\alpha,J})$ carries a canonical singular Hermitian metric $|-|_h$ with local weight functions $\prod_{j \in I \setminus I_\alpha} |z_j|^{-\{\alpha e_j\}}$ restricted on Y^J, where z_i is the defining equation of Y_i. Provided the above two lemmas, the sum of hypercohomologies

$$\bigoplus_{k \in \mathbb{Z}} H^k \left(Y^J, \Omega_{Y^J}^{\bullet + \dim Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J}\right)$$

is a polarized Hodge-Lefschetz structure of central weight $\dim Y^J$ for any non-empty subset J of I_α. Similarly to Example 2.1.9 this is also determined by the filtered $\mathscr{D}_{Y^J}$-module $(\mathcal{V}_{\alpha,J}, F_\bullet \mathcal{V}_{\alpha,J})$ with the sesquilinear pairing $S_{\alpha,J} : \mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}} \to \mathfrak{C}_{Y^J}$ is given by

$$([s_1 \otimes P_1], [s_2 \otimes P_2]) \mapsto \frac{\varepsilon(\dim Y^J + 1)}{(2\pi\sqrt{-1})^{\dim Y^J}} \int_{Y^J} (P_1 \overline{P_2} -) (s_1 \wedge \overline{s_2})_h \tag{2.7.6}$$

for local sections of $\mathcal{V}_{\alpha,J}$ (see (2.7.1)) represented by $s_i \otimes P_i$ such that s_i local sections of

$$\omega_{Y^J}(\log E^{\alpha,J}) \otimes \mathscr{V}_{\alpha,J} = \omega_{Y^J} \otimes \mathscr{V}_{\alpha,J}(E^{\alpha,J})$$

and P_i is a differential operator $i = 1, 2$. Here, $(s_1 \wedge \overline{s_2})_h$ is the top form induced by the singular Hermitian metric on $\mathscr{V}_{\alpha,J}(E^{\alpha,J})$. Summarizing the results we proved in this subsection:

Corollary 2.7.4. *With notations as above, the direct sum of all hypercohomologies of the de Rham complex of $(\mathcal{V}_{\alpha,J}, F_\bullet \mathcal{V}_{\alpha,J})$ underlies a polarized Hodge-Lefschetz structure of central weight $\dim Y^J$ with the Hodge filtration induced by $F_\bullet \mathcal{V}_{\alpha,J}$ and with the polarization, on degree k, given by the following induced pairing scaled by $\varepsilon(k)$,*

$$H^k(Y^J, \mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}) \otimes H^{-k}(Y^J, \mathrm{DR}_{Y^J}\mathcal{V}_{\alpha,J}) \longrightarrow H^0(Y^J, \mathrm{DR}_{Y^J,\overline{Y^J}}\mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}}) \hookrightarrow H^0(Y^J, \mathrm{DR}_{Y^J,\overline{Y^J}}\mathfrak{C}_{Y^J}) \xrightarrow[\simeq]{S_{\alpha,J}} \mathbb{C}.$$

Remark 2.7.5. We cannot make the Hodge structure in the above corollary over $\mathbb{Q}$ because there is no eigenvalue decomposition of $\mathbb{Q}$-structure.

2.7.2 Sesquilinear pairing

As in the reduced case, we need a sesquilinear pairing to construct the limiting mixed Hodge structure. In fact, the construction for the reduced case still works with a little modification. Note that for any test function η over a local chart U and two local sections $\zeta_1 \otimes P_1, \zeta_2 \otimes P_2$ of $H^0\left(U, \Omega^n_{X/\Delta}(\log Y)(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X\right)$, the function

$$t \mapsto \frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1 \overline{P_2}(\eta) \zeta_1 \wedge \overline{\zeta_2}.$$

may have order approximately at most $|t|^{2\alpha} \left(-\log |t^2|\right)^k$ near $t = 0$ where $k + 1$ is the number of components of Y_{I_α} that intersect in U. This suggests that we can define the pairing S_α on $\mathcal{M}_\alpha$ by

$$\begin{aligned} \langle S_\alpha([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta \rangle &=_{\mathrm{def}} \mathrm{Res}_{s=-\alpha} \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}} \int_X |t|^{2s} P_1 \overline{P_2}(\eta) \frac{dt}{t} \wedge \zeta_1 \wedge \overline{\frac{dt}{t} \wedge \zeta_2} \\ &= \mathrm{Res}_{s=-\alpha} \frac{\varepsilon(2)}{2\pi\sqrt{-1}} \int_\Delta |t|^{2s} \frac{dt}{t} \wedge \overline{\frac{dt}{t}} \left(\frac{\varepsilon(n+1)}{(2\pi\sqrt{-1})^n} \int_{X_t} P_1 \overline{P_2}(\eta) \zeta_1 \wedge \overline{\zeta_2} \right). \end{aligned}$$

Again, we have not check that S_α is well-defined but let us do some local calculations to see what is going on.

Example 2.7.6. Suppose $Y = 2Y_0$ for Y_0 is a smooth manifold and t is equal to z_0^2 on X. Then R satisfies the equation $R(R-\frac{1}{2}) = 0$. We deduce that $\mathcal{M}$ has two eigenspaces $\mathcal{M}_0$ and $\mathcal{M}_{\frac{1}{2}}$ by (2.3.3). Then for any local sections $\zeta_i \otimes P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes P_i$ of $\Omega^n_{X/\Delta}(\log Y) \otimes \mathscr{D}_X$, $i = 1, 2$ representing classes of $\mathcal{M}_0$, the calculation of the pairing $S_0([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2])$ is exactly as in the reduced case and as it turned out

$$S_0([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]) = i_{Y_0+} S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]).$$

By Theorem 2.6.3 $\mathcal{M}_{\frac{1}{2}}$ is locally generated by the class represented by $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes z_0$. Now for any local sections $\zeta \otimes z_0 P_i = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \otimes z_0 P_i$ representing classes of $\mathcal{M}_{\frac{1}{2}}$, we have

$$\begin{aligned}
\langle S_{\frac{1}{2}}([\zeta \otimes z_0 P_1], [\zeta \otimes z_0 P_2]), \eta\rangle &= \operatorname{Res}_{s=-\frac{1}{2}} \int_X |z_0|^{4s} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right) \\
&= \int_X \frac{1}{2} \log|z_0|^2 \partial_0 \overline{\partial_0} P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right) \\
\text{by Poincaré-Lelong equation [GH14, Page 388]} &= \int_{Y_0} \frac{1}{2} P_1 \overline{P_2}(\eta) \bigwedge_{i=1}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right) \\
&= \frac{1}{2} \langle i_{Y_0+} S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle \\
&= \frac{1}{2} \langle i_{Y_0+} S_{\frac{1}{2},\{0\}}([\zeta_1 \otimes z_0 P_1], [\zeta_2 \otimes z_0 P_2]), \eta\rangle,
\end{aligned}$$

Recall $S_{\frac{1}{2},\{0\}}$ defined in (2.7.6): since we have the isomorphism $\mathscr{O}_{Y_0}(2Y_0) = \mathscr{O}_{Y_0}(Y) \simeq \mathscr{O}_{Y_0}$ there exists a canonical singular Hermitian metric (this case is smooth) $|-|_h$ on $\mathscr{O}_{Y_0}(-Y_0)$ by setting the local frame z_0 has norm 1 so that

$$\begin{aligned}
&i_{Y_0+} S_{\frac{1}{2},\{0\}}([\zeta_1 \otimes z_0 P_1], [\zeta_2 \otimes z_0 P_2]), \eta\rangle \\
&\quad = \int_X |z_0|_h^2 P_1 \overline{P_2}(\eta) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}\right) = i_{Y_0+} S_{Y_0}([\zeta_1 \otimes P_1], [\zeta_2 \otimes P_2]), \eta\rangle.
\end{aligned}$$

The above equality can also be explained as follows: the cyclic covering constructed by taking out of the second root of the constant section of $\mathscr{O}_{Y_0}(2Y_0) \simeq \mathscr{O}_{Y_0}$ has two connected components and each component is isomorphic to Y_0.

Let η be a test function over an open subset U. For any two sections $m_1, m_2 \in H^0(U, \Omega^n_{X/\Delta}(\log Y)(-\lceil \alpha Y \rceil) \otimes \mathscr{D}_X)$, the $(2n+2)$-form $\frac{dt}{t} \wedge m_1 \wedge \overline{\frac{dt}{t} \wedge m_2}$ is smooth of out-side Y and has pole along Y. Locally, the $(2n+2)$-form just is $|z_I|^{2\lceil\alpha\mathbf{e}\rceil}P_1\overline{P_2}(\eta)\frac{dt}{t}\wedge\zeta\wedge\overline{\frac{dt}{t}\wedge\zeta}$, where $m_j=\zeta\otimes z_I^{\lceil\alpha\mathbf{e}\rceil}P_j$ for $\zeta=\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge\cdots\wedge dz_n$ and $j=1,2$. Let $F(s)=F(s,m_1,m_2,\eta)$ be the meromorphic extension of

$$\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}}\int_X|t|^{2s}\frac{dt}{t}\wedge m_1\wedge\overline{\frac{dt}{t}\wedge m_2}(\eta)$$

via integration by parts. The function $F(s)$ is well defined when $\mathrm{Re}\,s>-\alpha$ and has a pole at $s=-\alpha$. We only care about the polar part of $F(s)$ at $s=-\alpha$.

Theorem 2.7.7. *The polar part of $F(s)$ at $s=-\alpha$ is only depends on the classes of m_1 and m_2 in $\mathcal{M}_\alpha$.*

Proof. Let $\{\rho_\lambda\}$ be a partition of unity of the open covering $\{U_\lambda\}$ by local charts. Then

$$F(s)=\sum_\lambda\frac{\varepsilon(n+2)}{\left(2\pi\sqrt{-1}\right)^{n+1}}\int_{U_\lambda}|t|^{2s}\frac{dt}{t}\wedge m_1\wedge\overline{\frac{dt}{t}\wedge m_2}(\rho_\lambda\eta).$$

Since $\rho_\lambda\eta$ is a test function over local chart U_λ, we can assume U itself is a local chart. We assume $k+1$ components of Y intersect in U.

Lemma 2.7.8. *Under the assumption that $m_i=\zeta_\alpha\otimes P_i$ for $\zeta_\alpha=z_I^{\lceil\alpha\mathbf{e}\rceil}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n$ and for $i=1,2$, the followings are valid.*

1. *the order of the pole of $F(s)$ at $s=-\alpha$ is at most $k+1$;*

2. *if $P_i=t_\alpha P_i'$ for one of $i=1,2$, then $F(s)$ is holomorphic at $s=-\alpha$;*

3. *for $0\le j\le k$ we have,*

$$F\left(s,\zeta_\alpha\otimes P_1,\zeta_\alpha\otimes\frac{1}{e_j}z_j\partial_jP_2,\eta\right)=F\left(s,\zeta_\alpha\otimes\frac{1}{e_j}z_j\partial_jP_1,\zeta_\alpha\otimes P_2,\eta\right)=-\left(s+\frac{\lceil\alpha e_j\rceil}{e_j}\right)F(s,\zeta_1\otimes P_1,\zeta_2\otimes P_2$$

Proof of the lemma. We work out Laurent series of $F(s)$ at $s=-\alpha$:

$$\begin{aligned}F(s) &= \int_X |z_I|^{2s\mathbf{e}+2\lceil\alpha\mathbf{e}\rceil-2\cdot\mathbf{1}} P_1\overline{P_2}(\eta)\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\ &= \int_X |z_I|^{2(s+\alpha)\mathbf{e}-2\cdot\mathbf{1}}|z_I|^{2\{-\alpha\mathbf{e}\}} P_1\overline{P_2}(\eta)\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\ &= \int_X (s+\alpha)^{-2(k+1)}|z_I|^{2(s+\alpha)\mathbf{e}}\eta'\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}) \quad \text{where } \eta'=\partial_I\overline{\partial_I}\left(|z_I|^{2\{-\alpha\mathbf{e}\}}P_1\overline{P_2}\eta\right)\\ &= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}(s+\alpha)^{\ell-2(k+1)}\int_X(\log|z_I|^{2\mathbf{e}})^{\ell}\eta'\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).\end{aligned}$$

When $\ell<k+1$, then the form

$$(\log|z_I|^{2\mathbf{e}})^{\ell}\eta'\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}).$$

is actually exact because one of the a_i must be zero in the expansion of $(\log|z_I|^{2\mathbf{e}})^{\ell}$ into the linear combination of $\prod_{i=0}^{k}(\log|z_i|^{2e_i})^{a_i}$ such that $\sum_{i=0}^{k}a_i=\ell$. Therefore, the order of the pole at $s=-\alpha$ is at most $k+1$.

When $P_1=t_\alpha P_1'$, the form

$$|z_I|^{2(s+\alpha)\mathbf{e}-2\cdot\mathbf{1}}|z_I|^{2\{-\alpha\mathbf{e}\}}t_\alpha P_1'\overline{P_2}(\eta)\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

is integrable when $s=-\alpha$ where $\{-\alpha\mathbf{e}\}$ is the multi-index such that $\{-\alpha\mathbf{e}\}_i=\{-\alpha e_i\}$. Therefore, $F(s)$ is holomorphic at $s=-\alpha$. It is the same when $P_2=t_\alpha P_2'$.

Lastly, by linearity we can assume that $P_1=P_2=1$.

$$\begin{aligned}F(s,\zeta_\alpha\otimes 1,\zeta_\alpha\otimes\frac{1}{e_j}z_j\partial_j,\eta) &= \frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}}\int_X|t|^{2s}\left(\frac{1}{e_j}\overline{z_j\partial_j\eta}\right)\frac{dt}{t}\wedge\zeta_\alpha\wedge\overline{\frac{dt}{t}\wedge\zeta_\alpha}\\ &= \int_X\prod_{i\in I\setminus\{j\}}|z_i|^{2se_i+2\lceil\alpha e_i\rceil-2}z_j^{se_j+\lceil\alpha e_j\rceil-1}\frac{1}{e_j}\overline{z_j^{se_j+\lceil\alpha e_j\rceil}\partial_0\eta}\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\ &= \int_X-\left(s+\frac{\lceil\alpha e_j\rceil}{e_j}\right)\prod_{i\in I}|z_i|^{2se_i+2\lceil\alpha e_i\rceil-2}\eta\bigwedge_{i=0}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})\\ &= -\left(s+\frac{\lceil\alpha e_j\rceil}{e_j}\right)\frac{\varepsilon(n+2)}{(2\pi\sqrt{-1})^{n+1}}\int_X|t|^{2s}\eta\frac{dt}{t}\wedge\zeta_\alpha\wedge\overline{\frac{dt}{t}\wedge\zeta_\alpha}.\\ &= -\left(s+\frac{\lceil\alpha e_j\rceil}{e_j}\right)F(s,\zeta_\alpha\otimes 1,\zeta_\alpha\otimes 1,\eta).\end{aligned} \tag{2.7.7}$$

The other equality in (3) holds similarly. We complete the proof of the lemma. □

Returning to the proof of theorem. Since $\mathcal{M}_\alpha$ is locally represented by

$$\zeta_\alpha \otimes \mathscr{D}_X/(t_\alpha, D_1+\alpha_1, D_2+\alpha_2, \ldots, D_n+\alpha_n)\mathscr{D}_X$$

(see the proof of Theorem 2.6.2), and (2) and (3) in the lemma say that when one of m_1 and m_2 is in

$$\zeta_\alpha \otimes (t_\alpha, D_1+\alpha_1, D_2+\alpha_2, \ldots, D_n+\alpha_n)\mathscr{D}_X$$

then $F(s)$ is holomorphic since α_i equals $\lceil \alpha e_i \rceil / e_i - \lceil \alpha e_0 \rceil / e_0$ for $1 \le i \le k$ and equals zero otherwise. □

For two sections $\gamma_1, \gamma_2 \in H^0(U, \mathcal{M})$ and any test function η over U, we define the pairing $S_\alpha : \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathcal{M}_\alpha} \to \mathfrak{C}_X$ by

$$\langle S_\alpha(\gamma_1, \gamma_2), \eta \rangle = \operatorname{Res}_{s=-\alpha} \sum_\lambda F(s, \tilde{\gamma}_1, \tilde{\gamma}_2, \rho_\lambda \eta),$$

where $\{\rho_\lambda\}$ is a partition of unity with respect to an open covering by local charts $\{U_\lambda\}$ such that γ_i has a local lifting of $\tilde{\gamma}_i$ over U_λ for $i = 1, 2$. It is obvious that S_α is $\mathscr{D}_{X,\overline{X}}$-linear. Thus, it is a sesquilinear pairing. As a corollary of Lemma 2.7.8, we have

Corollary 2.7.9. *We have* $S_\alpha \circ (\mathrm{id} \otimes_{\mathbb{C}} R_\alpha) = S_\alpha \circ (R_\alpha \otimes_{\mathbb{C}} \mathrm{id})$.

Because of the corollary, the sesquilinear pairing S_α induces pairings on the associated graded quotient of the weight filtration

$$S_\alpha : \mathrm{gr}_k^W \mathcal{M}_\alpha \otimes_{\mathbb{C}} \overline{\mathrm{gr}_{-k}^W \mathcal{M}_\alpha} \to \mathfrak{C}_X,$$

as well as on the primitive part

$$P_{R_\alpha} S_r = S_\alpha \circ (\mathrm{id} \otimes_{\mathbb{C}} R_\alpha^r) : \mathcal{P}_{\alpha,r} \otimes_{\mathbb{C}} \overline{\mathcal{P}_{\alpha,r}} \to \mathfrak{C}_X.$$

Theorem 2.7.10. *The isomorphism* $\phi_{\alpha,r} : (\mathcal{P}_{\alpha,r}, F_\bullet \mathcal{P}_{\alpha,r}) \to \mathcal{V}_{\alpha,r}(-r)$ *in Theorem 2.6.13 respects the sesquilinear pairings up to a constant scalar. More concretely,*

$$P_{R_\alpha} S_r(m_1, m_2) = \bigoplus_{\substack{J \subset I_\alpha, \\ \#J = r+1}} \frac{(-1)^r}{(r+1)! C_J} \tau_+^J S_{\alpha,J}(\phi_{\alpha,r} m_1, \phi_{\alpha,r} m_2)$$

for any local sections $m_1, m_2 \in \mathcal{P}_{\alpha,r}$ *and* $C_J = \prod_{j\in J} e_j$*, where the pairing* $S_{\alpha,J} : \mathcal{V}_{\alpha,J} \otimes_{\mathbb{C}} \overline{\mathcal{V}_{\alpha,J}} \to \mathfrak{C}_{Y^J}$ *is defined in* (2.7.6).

Proof. Because of the linearity and the generators of $\mathcal{P}_{\alpha,r}$ are all monomials dividing t_α of degree $\mu - r$ Corollary 2.6.7, it suffices to prove the theorem in the case when m_i is represented by

$$\zeta_\alpha \otimes z_{K_i} = z_I^{\lceil \alpha \mathbf{e} \rceil} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k} \wedge \cdots \wedge dz_n \otimes z_{K_i}$$

where $K_i \subset I_\alpha$ with $\# K_i = \mu - r$ and $i = 1, 2$. Let η be a test function over U. Then we have

$$\langle S_\alpha(m_1, R_\alpha^r m_2), \eta \rangle = \mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X |z_I|^{2s\mathbf{e} + 2\lceil \alpha \mathbf{e} \rceil - 2\cdot\mathbf{1}} z_{K_1} \overline{z_{K_2}} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge \overline{dz_i}).$$

If $m_1 \neq m_2$, then the above is zero. Indeed, for $v \in K_2 \setminus K_1$ by choosing $R_\alpha^r = 1 \otimes \prod_{i\in I\setminus K_1 \setminus \{v\}} \frac{1}{e_i} z_i \partial_i$,

$$\langle S(R_\alpha^r m_1, m_2), \eta \rangle = \mathrm{Res}_{s=-\alpha} \int_X |z_I|^{2s\mathbf{e} - 2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

where $\tilde{\eta} = C_{I\setminus K_1 \setminus \{v\}}^{-1} \partial_{I\setminus K_1\setminus\{v\}} \overline{z_{K_2}} (\overline{z_v})^{-1} \eta$ is a smooth function with compact support. The function

$$\int_X |z_I|^{2s\mathbf{e} - 2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

is holomorphic at $s = -\alpha$ because by setting $s = -\alpha$ the form

$$|z_I|^{-2\alpha\mathbf{e} - 2\cdot\mathbf{1}} |z_I|^{2\lceil \alpha \mathbf{e} \rceil} \frac{t_\alpha}{z_v} \overline{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}) = |z_{I\setminus I_\alpha}|^{-2\{\alpha \mathbf{e}\}} \frac{1}{\overline{t_\alpha}} \frac{\overline{z}_v}{z_v} \tilde{\eta} \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i})$$

is integrable.

Therefore, we reduce the proof to the case when $m_1 = m_2 = m$ represented by $\zeta_\alpha \otimes z_K$. We shall prove that

$$S_\alpha(m, R_\alpha^r m) = \frac{(-1)^r}{(r+1)! C_{\overline{K}}} \tau_+^{\overline{K}} S_{\alpha,\overline{K}}(\phi_{\alpha,r} m, \phi_{\alpha,r} m),$$

where $\overline{K}$ is the complement of K in I_α. Without loss of generality, we can assume that $K = \{r+1, r+2, ..., \mu\}$ and $\overline{K} = \{0, 1, ..., r\}$ so that $z_K = z_{r+1} z_{r+2} \cdots z_\mu$. We have

$$\langle S(m, R_\alpha^r m), \eta \rangle = \mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X |z_K|^{2(s+\alpha)\mathbf{e}_K} |z_{I\setminus K}|^{2s\mathbf{e}_{I\setminus K} + 2\lceil \alpha \mathbf{e}_{I\setminus K} \rceil - 2} \eta \bigwedge_{i=0}^n (\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i}), \tag{2.7.8}$$

where, for any index subset $J \subset I$, the j-th component the multi-index $\mathbf{e}_J$ is e_j if $j \in J$ or zero otherwise, and the j-th component of $\lceil \alpha \mathbf{e}_J \rceil$ is $\lceil \alpha e_j \rceil$ if $j \in J$ or zero otherwise. Integration by parts for $\{dz_i, d\bar{z}_i\}_{i \in \overline{K}}$, the identity (2.7.8) equals to

$$\mathrm{Res}_{s=-\alpha}(-(s+\alpha))^r \int_X \frac{|z_{I_\alpha}|^{2(s+\alpha)\mathbf{e}_{I_\alpha}}}{C_{\overline{K}}^2 (s+\alpha)^{2r+2}} |z_{I \smallsetminus I_\alpha}|^{2s\mathbf{e}_{I \smallsetminus I_\alpha} + 2\lceil \alpha \mathbf{e}_{I \smallsetminus I_\alpha} \rceil - 2} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta \right) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right) \tag{2.7.9}$$

$$= \mathrm{Res}_{s=-\alpha} \frac{(-1)^r}{C_{\overline{K}}^2 (s+\alpha)^{r+2}} \int_X |t|^{2(s+\alpha)} \prod_{j \in I \smallsetminus I_\alpha} |z_j|^{-2\{\alpha e_j\}} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta \right) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right), \tag{2.7.10}$$

where $\partial_{\overline{K}} \overline{\partial_{\overline{K}}} = \prod_{j \in \overline{K}} \partial_j \overline{\partial_j}$. Because of the expansion

$$|t|^{2(s+\alpha)} = \exp\left(\log |t|^2 \, (s+\alpha) \right) = \sum_{\ell=0}^{\infty} \frac{\left(\log |t|^2 \right)^{\ell} (s+\alpha)^{\ell}}{\ell!},$$

we find that (2.7.10) is equal to

$$\frac{(-1)^r}{C_{\overline{K}}^2 (r+1)!} \int_X \left(\log |t|^2 \right)^{r+1} \prod_{j \in I \smallsetminus I_\alpha} |z_j|^{-2\{\alpha e_j\}} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta \right) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right) \tag{2.7.11}$$

The expansion of $(\log |t|^2)^{r+1}$ is a linear combination of

$$\prod_{i \in I} \left(\log |z_i|^2 \right)^{a_i}$$

for all partitions $\sum_{i \in I} a_i = r+1$, but the differential form

$$\prod_{i \in I} \left(\log |z_i|^2 \right)^{a_i} \prod_{j \in I \smallsetminus I_\alpha} |z_j|^{-2\{\alpha e_j\}} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta \right) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right)$$

is exact unless $a_i \neq 0$ for any $i \in \overline{K}$, which is equivalent to $a_i = 1$ for $i \in \overline{K}$ and $a_i = 0$ for $i \notin \overline{K}$. It follows that (2.7.11) is equal to

$$\frac{(-1)^r}{C_{\overline{K}} (r+1)!} \int_X \prod_{j \in \overline{K}} \log |z_j|^2 \prod_{j \in I \smallsetminus I_\alpha} |z_j|^{-2\{\alpha e_j\}} \left(\partial_{\overline{K}} \overline{\partial_{\overline{K}}} \eta \right) \bigwedge_{i=0}^{n} \left(\frac{\sqrt{-1}}{2\pi} dz_i \wedge d\overline{z_i} \right).$$

We deduce from Poincáre-Lelong equation [GH14, Page 388] that the above continues to equal to

$$\frac{(-1)^r}{(r+1)!C_{\overline{K}}}\int_{Y^{\overline{K}}}\prod_{j\in I\setminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\eta\bigwedge_{i=r+1}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i}) \tag{2.7.12}$$

Since $\phi_{\alpha,\overline{K}}m=\pm\frac{dz_{I\setminus K}}{z_{I\setminus K}}\wedge dz_{k+1}\wedge\cdots\wedge dz_n\otimes s_{\alpha,\overline{K}}\in\omega_{Y^{\overline{K}}}(E^{\alpha,\overline{K}})\otimes\mathscr{V}_{\alpha,\overline{K}}$, it follows that

$$(\phi_{\alpha,\overline{K}}m\wedge\overline{\phi_{\alpha,\overline{K}}m})_h=\prod_{j\in I\setminus I_\alpha}|z_j|^{-2\{\alpha e_j\}}\bigwedge_{i=r+1}^{n}(\frac{\sqrt{-1}}{2\pi}dz_i\wedge d\overline{z_i})$$

from which we conclude that (2.7.12) is equal to

$$\frac{(-1)^r}{(r+1)!C_{\overline{K}}}\int_{Y^{\overline{K}}}\eta(\phi_{\alpha,\overline{K}}m\wedge\overline{\phi_{\alpha,\overline{K}}m})_h=\frac{(-1)^r}{(r+1)!C_{\overline{K}}}\langle S_{\alpha,\overline{K}}(\phi_{\alpha,\overline{K}}m,\phi_{\alpha,\overline{K}}m),\eta\rangle.$$

See (2.7.6). The theorem is proved. □

2.7.3 Construction of the limiting mixed Hodge structure

We begin to construct a polarized bigraded Hodge-Lefschetz structure on $\mathrm{gr}^W H^\bullet(X,\mathrm{DR}_X\mathcal{M}_\alpha)$. Fix a Kähler class ω on X and let $L=\omega\wedge:\mathrm{DR}_X\mathcal{M}_\alpha\to\mathrm{DR}_X\mathcal{M}_\alpha[2]$ be the Lefschetz operator and $\mathsf{X}_1=2\pi\sqrt{-1}L$. Relabel the graded pieces of the first page of the weight spectral sequence by

$$V^\alpha_{\ell,k}=H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M}_\alpha)={}^WE_1^{-k,\ell+k}.$$

Let $V^\alpha=\bigoplus_{\ell,k\in\mathbb{Z}}V^\alpha_{\ell,k}$ with the filtration $F_\bullet V^\alpha$ induced by $F_\bullet\mathcal{M}_\alpha$. Denote by $E_i(R_\alpha)$ the induced operator by R_α on ${}^WE_i$ and let $\mathsf{Y}_2=E_1(R_\alpha)$. Denote by $S_{\ell,k}$ the induced pairing on $V^\alpha_{\ell,k}\otimes\overline{V^\alpha_{-\ell,-k}}$

$$H^\ell(X,\mathrm{gr}^W_k\mathrm{DR}_X\mathcal{M}_\alpha)\otimes\overline{H^{-\ell}(X,\mathrm{gr}^W_{-k}\mathrm{DR}_X\mathcal{M}_\alpha)}\to H^0(X,\mathrm{DR}_{X,\overline{X}}\mathrm{gr}^W_k\mathcal{M}_\alpha\otimes_{\mathbb{C}}\overline{\mathrm{gr}^W_{-k}\mathcal{M}_\alpha})\to H^0_c(X,\mathrm{DR}_{X,\overline{X}}\mathfrak{C}_X)\simeq\mathbb{C},$$

modifying by a sign factor $\varepsilon(\ell)$. Let d_1 be the differential of the first page of the spectral sequence. In terms of relabeling we have

$$d_1:(V^\alpha_{\ell,k},F_\bullet V^\alpha_{\ell,k})\to(V^\alpha_{\ell+1,k-1},F_\bullet V^\alpha_{\ell+1,k-1}).$$

Exactly same to Theorem 2.5.6 and Corollary 2.5.7 in the reduced case, we conclude that

Theorem 2.7.11. *The tuple* $(V^\alpha, \mathsf{X}_1, \mathsf{Y}_2, F_\bullet V, \bigoplus S_{j,k}, d_1)$ *gives a differential polarized bigraded Hodge-Lefschetz structure of central weight* n.

Corollary 2.7.12. *We have the following*

1. *Hodge spectral sequence degenerates at* ${}_F E_1$;
2. *the weight spectral sequence degenerates at* ${}^W E_2$;
3. *the tuple* $(\bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}^W H^\ell(X, \mathrm{DR}_X \mathcal{M}_\alpha), \mathsf{X}_1, \mathsf{Y}_2, F_\bullet)$ *together with the pairing induced by* S_α *is a polarized bigradged Hodge-Lefschetz structure of central weight* n.

The last statement in the above corollary implies that the induced weight filtration on $H^\ell(X, \mathrm{DR}_X\mathcal{M}_\alpha)$ is the monodromy filtration associated to R_α on $H^\ell(X, \mathrm{DR}_X\mathcal{M}_\alpha)$. We established Theorem A.

2.8 Application

2.8.1 Hard Lefschetz

The following is a consequence of the bigraded Hodge-Lefschetz structure

Theorem 2.8.1. *The Lefschetz operator induces an isomorphism between* $\mathscr{O}_\Delta$*-modules*

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell R^{-k}\Omega^{\bullet+n}_{X/\Delta}(\log Y) \simeq F_{\ell-k}R^k\Omega^{\bullet+n}_{X/\Delta}(\log Y) \quad \textit{for any integer } \ell.$$

As a result, we have the following decomposition in the derived category of coherent $\mathscr{O}_\Delta$*-modules:*

$$Rf_*F_\ell\Omega^{\bullet+n}_{X/\Delta}(\log Y) \simeq \bigoplus_{k\in\mathbb{Z}} F_\ell R^k f_*\Omega^{\bullet+n}_{X/\Delta}(\log Y)[-k] \quad \textit{for any integer } \ell.$$

Proof. The first statement follows from the Hard Lefschetz on each fiber

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell R^{-k}\Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathbb{C}(p) \simeq F_{\ell-k}R^k\Omega^{\bullet+n}_{X/\Delta}(\log Y)\otimes\mathbb{C}(p),$$

for every $p \in \Delta$. The second statement follows from the first one plus the main theorem in [Del68]. □

2.8.2 Invariant cycle theorem

Now we shall give the proof of Theorem B, which is equivalently to the following statement:

Theorem 2.8.2. *We have the following exact sequence of mixed Hodge structures*

$$H^{\ell} + n(Y, \mathbb{C}) \longrightarrow H^{\ell}(X, \mathrm{DR}_X \mathcal{M}) \xrightarrow{R} H^{\ell}(X, \mathrm{DR}_X \mathcal{M})(-1).$$

Of course one can try to show that $\ker R$ is the filtered $\mathscr{D}_X$-module such that the hypercohomologies of its de Rham complex computes the cohomologies of Y. But we would like to keep the proof elementary so we will just show that the first page of the weight spectral sequence computing the hypercohomology of $\mathrm{DR}_X \ker R$ is the same to the one computing the cohomology of Y up to a constant scalar; this will prove the theorem because both weight spectral sequences degenerate at the second page. See [GS75, (4.2)] or [Ste76, (3.5)] for the weight filtration of $H^{\ell}(Y, \mathbb{C})$

Proof. Note that $\ker R$ is contained in $\mathcal{M}_0$. Therefore, $W_{-j}\ker R = R^j \ker R^{j+1}$ for $j \geq 0$ and vanishes for $j < 0$ where $W = W(R)$ on $\mathcal{M}_0$. It follows that $\mathrm{gr}^W_{-j}\ker R$ is isomorphic to $\omega_{\tilde{Y}^{(j+1)}}$ for $j \geq 0$ by Theorem 2.6.13. Because $\mathrm{gr}^W_{-j}\ker R$ is a summand of $\mathrm{gr}^W_{-j}\mathcal{M}_0$ for $j \geq 0$ by the Lefschetz decomposition on $\mathrm{gr}^W \mathcal{M}_0$, we have the following short exact sequence of Hodge structures on the first page of the weight spectral sequences:

$$0 \longrightarrow H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-\bullet}\mathrm{DR}_X \ker R) \longrightarrow H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-\bullet}\mathrm{DR}_X \mathcal{M}_0) \xrightarrow{R} H^{\ell+\bullet}(X, \mathrm{gr}^W_{-j-2-\bullet}\mathrm{DR}_X \mathcal{M}_0)(-1) \longrightarrow 0.$$

The associated long exact sequence gives the relation between the second page of the spectral sequences:

$$\cdots \longrightarrow \mathrm{gr}^W_{-j} H^{\ell}(X, \mathrm{DR}_X \ker R) \longrightarrow \mathrm{gr}^W_{-j} H^{\ell}(X, \mathrm{DR}_X \mathcal{M}_0) \longrightarrow \mathrm{gr}^W_{-j-2} H^{\ell}(X, \mathrm{DR}_X \mathcal{M}_0)(-1) \longrightarrow \cdots.$$

Now it remains to prove that $H^\ell(X,\mathrm{DR}_X\ker R)$ and $H^\ell + n(Y,\mathbb{C})$ are isomorphic as mixed Hodge structures. It suffices to check that they coincide at the first page of weight spectral sequence since they degenerate at the second page. We have the following commutative diagram where the leftmost column is the E_1-page spectral sequence of $\ker R$ and all the horizontal arrows are isomorphisms of mixed Hodge structures.

$$\begin{array}{ccccc}
H^\ell(X,\mathrm{gr}^W_{-j}\mathrm{DR}_X\ker R) & \xrightarrow{\phi_{0,r}\circ(R^j)^{-1}} & H^\ell(X,\mathrm{DR}_X\tau_+^{j+1}\omega_{\tilde{Y}^{(j+1)}}) & \xleftarrow[\simeq]{} & H^\ell(\tilde{Y}^{(j+1)},\Omega^{n-j+\bullet}_{\tilde{Y}^{(j+1)}}) \\
\Big\downarrow d_1 & & \Big\downarrow & & \Big\downarrow \\
H^{\ell+1}(X,\mathrm{gr}^W_{-(j+1)}\mathrm{DR}_X\ker R) & \xrightarrow{\simeq} & H^{\ell+1}(X,\mathrm{DR}_X\tau_+^{j+2}\omega_{\tilde{Y}^{(j+2)}}) & \xleftarrow[\simeq]{} & H^{\ell+1}(\tilde{Y}^{(j+2)},\Omega^{n-j+1+\bullet}_{\tilde{Y}^{(j+2)}})
\end{array} \tag{2.8.1}$$

We shall identify the the rightmost vertical arrow with the differential of the first page of the weight spectral sequence of $H^{\ell+n}(Y,\mathbb{C})$ via diagram chasing.

$$\begin{array}{ccccc}
\mathrm{gr}^W_{-j}\ker R\otimes\bigwedge^p\mathscr{T}_X & \xrightarrow{\simeq} & \tau_+^K\omega_{Y^K}\otimes\bigwedge^p\mathscr{T}_X & \longleftarrow & \Omega^{n-j-p}_{Y^K} \\
\Big\downarrow d & & \Big\downarrow & & \Big\downarrow \\
\mathrm{gr}^W_{-(j+1)}\ker R\otimes\bigwedge^{p-1}\mathscr{T}_X & \xrightarrow{\simeq} & \bigoplus_{j_i\in J}\tau_+^{K\cap\{j_i\}}\omega_{Y^{K\cap\{j_i\}}}\otimes\bigwedge^{p-1}\mathscr{T}_X & \longleftarrow & \bigoplus_{j_i\in J}\Omega^{n-j-p}_{Y^{K\cap\{j_i\}}}
\end{array}$$

$$\begin{array}{ccccc}
[\pm R^j\zeta_0\otimes z_Iz_K^{-1}\otimes\partial_J] & \longrightarrow & \pm dz_{\bar{K}}\otimes\partial_J & \longleftarrow & dz_{\bar{K}\smallsetminus J} \\
\Big\downarrow d & & \Big\downarrow & & \Big\downarrow \\
\left[\pm R^{j+1}\zeta_0\otimes\sum_{j_i\in J}e_{j_i}z_Iz_Kz_{j_i}{}^{-1}\otimes\partial_{J\smallsetminus\{j_i\}}\right] & \longrightarrow & \bigoplus_{j_i\in J}\pm dz_{\bar{K}}\otimes\partial_{J\smallsetminus\{j_i\}} & \longleftarrow & \pm\sum_{j_i\in J}e_{j_i}dz_{\bar{K}\smallsetminus J}
\end{array}$$

Starting from the upper-right corner, let $dz_{\bar{K}\smallsetminus J}=\bigwedge_{i\in\bar{K}\smallsetminus J}dz_i$ be a local section of $\Omega^{n-j-p}_{Y^K}$ where K is an ordered index set of cardinality $j+1$, $\bar{K}$ is the complement of K in I and $J\subset\bar{K}$ of cardinality p. Then $\pm dz_{\bar{K}}\otimes\partial_J$ is the image in $\tau_+^K\omega_{Y^K}\otimes\bigwedge^p\mathscr{T}_X$ via the inclusion

$$\Omega^{n-j-p}_{Y^K}=\omega_{Y^K}\otimes\bigwedge^p\mathscr{T}_{Y^K}\to\tau_+^K\omega_{Y^K}\otimes\bigwedge^p\mathscr{T}_X,$$

where $\partial_J=\bigwedge_{j\in J}\partial_j$. Its preimage under the isomorphism

$$\phi_{0,K}\circ\left(R^j\right)^{-1}:\mathrm{gr}^W_j\ker R\otimes\bigwedge^p\mathscr{T}_X=R^j\ker R^{j+1}\otimes\bigwedge^p\mathscr{T}_X\to\mathcal{P}_{0,-j}\otimes\bigwedge^p\mathscr{T}_X\to\tau_+^K\omega_{Y^K}\otimes\bigwedge^p\mathscr{T}_X$$

is the class represented by $\pm R^j\zeta_0\otimes z_Iz_K^{-1}\otimes\partial_J$, where $\zeta_0=\frac{dz_0}{z_0}\wedge\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_k}{z_k}\wedge dz_{k+1}\wedge\cdots\wedge dz_n$ and $\mathcal{P}_{0,-j}$ is the $(-j)$th-primitive part of $\mathrm{gr}^W\mathcal{M}_0$. It maps to the class of $\pm R^{j+1}\zeta_0\otimes$

$\sum_{j_i \in J} e_{j_i} z_I (z_K z_{j_i})^{-1} \otimes \partial_{J \setminus \{j_i\}}$ by the differential of $\mathrm{DR}_X \mathrm{ker} R$. By reverse the above procedure, $\pm R^{j+1} \zeta_0 \sum_{j_i \in J} e_{j_i} z_I (z_K z_{j_i})^{-1} \otimes \partial_{J \setminus \{j_i\}}$ corresponds to $\pm \sum_{j_i \in J} e_{j_i} dz_{\bar{K} \setminus J}$ restricting on $\bigoplus_{j_i \in J} \Omega^{n-j-i-p}_{Y^{K \cap \{j_i\}}}$. Therefore, the morphism d_1 in the diagram (2.8.1), up to a scalar factor, can be identified with the pullback

$$H^{\ell}\left(\tilde{Y}^{(j+1)}, \Omega^{n-j+\bullet}_{\tilde{Y}^{(j+1)}}\right) \to H^{\ell+1}\left(\tilde{Y}^{(j+2)}, \Omega^{n-j-1+\bullet}_{\tilde{Y}^{(j+2)}}\right),$$

which is the differential of the ${}^{W}E_1$-page of $H^{\ell+n}(Y, \mathbb{C})$. This completes the proof. □

Chapter 3

Hodge filtration and V-filtration

3.1 Preliminaries

3.1.1 Kashiwara-Malgrange V-filtrations

We begin with a review of the theory of V-filtrations introduced by Kashiwara and Malgrange. For more details, see [Sai88, Section 3.1] and [Sch14, Section 9] for the case of a hypersurface and [BMS06, Section 1.1] for the case of higher codimension.

Let $(t_1, \dots, t_r) : X \to \mathbb{A}^r$ be a smooth regular function, with fiber Z over the origin. We define a $\mathbb{Z}$-indexed filtration on $\mathscr{D}_X$ by

$$V_k \mathscr{D}_X = \{P \in \mathscr{D}_X : P \cdot \mathscr{I}_Z^j \subseteq \mathscr{I}_Z^{j-k} \text{ for all } j\}.$$

A $\mathbb{Q}$-indexed filtration $V^\bullet \mathcal{M}$ is *discrete and left-continuous* if $\bigcap_{\alpha<\beta} V^\alpha = V^\beta$ for all $\beta \in \mathbb{Q}$, and if there exists some $\ell \in \mathbb{Z}_{>0}$ such that the subspace V^α is constant for all $\alpha \in (\frac{m}{\ell}, \frac{m+1}{\ell}]$, for any $m \in \mathbb{Z}$.

Given a coherent left $\mathscr{D}_X$-module $\mathcal{M}$, a Kashiwara-Malgrange V-filtration on $\mathcal{M}$ along Z (see [Kas83], [Mal83]) is an exhaustive, decreasing $\mathbb{Q}$-indexed filtration which is discrete and left-continuous such that, if $\theta := \sum_{i=1}^r t_i \partial_{t_i}$ is any locally defined Euler vector field along Z, the filtration must satisfy:

1. $V_\alpha \mathcal{M} \cdot V_k \mathscr{D}_X \subseteq V_{\alpha+k}\mathcal{M}$ for all $k \in \mathbb{Z}, \alpha \in \mathbb{Q}$,

2. $V_\alpha \mathcal{M} \cdot V_k \mathscr{D}_X = V_{\alpha+k}\mathcal{M}$ for all $k \in \mathbb{Z}_{\leq 0}$, $\alpha \ll 0$,

3. each $V_\alpha \mathcal{M}$ is coherent over $V_0 \mathscr{D}_X$,

4. the operator $\theta - \alpha$ is nilpotent on $\mathrm{gr}^V_\alpha \mathcal{M}$, where $\theta := \sum_{i=1}^r t_i \partial_i$ is the Eular vector field.

It is an easy exercise to see that there can be at most one V-filtration on any coherent $\mathscr{D}_X$-module $\mathcal{M}$. We say that a module $\mathcal{M}$ which has a $\mathbb{Q}$-indexed V-filtration is $\mathbb{Q}$-specializable. Any morphism between $\mathbb{Q}$-specializable modules is strict with respect to the V-filtration. Moreover, if

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is a short exact sequence of $\mathscr{D}_X$-modules, and $\mathcal{M}$ has a V-filtration, then the induced filtrations on $\mathcal{M}'$ and $\mathcal{M}''$ satisfy the properties of the V-filtration.

Example 3.1.1. (a) Let $\mathcal{E}$ be an $\mathscr{O}_X$-coherent $\mathscr{D}_X$-module. Then $V^k\mathcal{E} := \mathcal{I}_Z^{k-r} \cdot \mathcal{E}$ satisfies the properties of the V-filtration. For example,

$$\theta t^\alpha m = (|\alpha| + \theta) t^\alpha m,$$

(b) (Kashiwara's equivalence) Assume $\mathcal{M}$ is supported on Z, so by Kashiwara's equivalence (see [HTT08, Section 1.6]), there exists a coherent $\mathscr{D}_Z$-module $\mathcal{N}$ such that $\mathcal{M} = \sum_{\alpha \in \mathbb{N}^r} \mathcal{N}\partial_t^\alpha$. Then

$$V_k \mathcal{M} = \sum_{|\alpha| \leq k} \mathcal{N}\partial_t^\alpha.$$

For us, it will also be important to understand the case when $(\mathcal{M}, F) \cong i_+(\mathcal{N}, F)$ as a filtered $\mathscr{D}$-module. For left $\mathscr{D}$-modules, the pushforward of a filtered module has filtration defined as

$$F_p i_+(\mathcal{N}, F) = \sum_{\alpha \in \mathbb{N}^r} F_{p-|\alpha|-r}\mathcal{N}\partial_t^\alpha.$$

From this, we see easily that

$$F_p V_k i_+(\mathcal{N}, F) = \sum_{|\alpha| \leq k} F_{p-|\alpha|-r}\mathcal{N}\partial_t^\alpha.$$

This last example leads to an important property of the V-filtration.

Lemma 3.1.2. *Assume $\varphi : \mathcal{N} \to \mathcal{M}$ is a morphism between two specializable modules, such that $\varphi|_U : \mathcal{N}|_U \to \mathcal{M}|_U$ is an isomorphism, where $U = X - Z$. Then $\varphi : V^{>0}\mathcal{N} \to V^{>0}\mathcal{M}$ is an isomorphism.*

Proof. Let $K = \ker(\varphi), C = \operatorname{coker}(\varphi)$. The assumption implies these are supported on Z, so by the previous example, $V^{>0}K = 0$ and $V^{>0}C = 0$. Hence, taking $V^{>0}$ of the long exact sequence

$$0 \to K \to \mathcal{N} \to \mathcal{M} \to C \to 0,$$

we get

$$0 = V_{<0}K \to V_{<0}\mathcal{N} \to V_{<0}\mathcal{M} \to V_{<0}C = 0,$$

proving the claim. □

3.1.2 Saito's Main Theorems about Hodge Modules

In this section, we state two essential theorems in Saito's theory of mixed Hodge modules.

The first main result is the behavior of mixed Hodge modules with respect to the pushforward functor for a projective morphism $f : Y \to X$. For more details and proofs, see [Sch14, Section 16] or [Sai88, Section 5.3].

We say a morphism $\varphi : (\mathcal{M}, F) \to (\mathcal{N}, F)$ is *strict* if $F_p\mathcal{N} \cap im(\varphi) = \varphi(F_p\mathcal{M})$. We say that a filtered complex $(K^\bullet, F)$ is *strict* if all differentials are strict.

For example, a monomorphism $i : A \hookrightarrow B$ is strict iff the filtration on A is the induced filtration from B. The main utility of strictness is that, if $(K^\bullet, F)$ is a filtered complex with strict differentials, then $\mathcal{H}^k(F_pK^\bullet) \to \mathcal{H}^k(K^\bullet)$ is injective for all $k \in \mathbb{Z}$. Hence, we can define a filtration F on $\mathcal{H}^k(K^\bullet)$, and strictness allows us to commute $\mathcal{H}^k$ with F_p.

We begin now with the statement of the direct image theorem in the pure case:

Theorem 3.1.3 ([Sai88, Thm 5.3.1]). *Let $f : Y \to X$ be a projective morphism of smooth complex varieties, let M be a pure Hodge module on Y of weight w. Let $\ell \in H^2(X, \mathbb{Z})$ be the class of a relatively ample divisor over Y. Then*

1. *$f_+(\mathcal{M}, F)$ is strict and $\mathcal{H}^i f_+(\mathcal{M}, F)$ underlies a Hodge module on X of weight $w + i$.*

2. *$\ell^i : \mathcal{H}^{-i} f_+(\mathcal{M}, F) \to \mathcal{H}^i f_+(\mathcal{M}, F)(i)$ is an isomorphism for all $i \geq 0$.*

As an application, if X is a smooth projective variety, $f : X \to *$ is the constant map, then the strictness of $f_+(\mathcal{M}, F)$ recovers the fact that the Leray spectral sequence degenerates at E_1.

Also, as a formal consequence of the second part of the theorem (see [Del68, Prop. 2.1]), one recovers the decomposition theorem, i.e., an isomorphism in the derived category

$$f_+(\mathcal{M}, F) \cong \bigoplus_{k \in \mathbb{Z}} \mathcal{H}^k f_+(\mathcal{M}, F)[-k].$$

Remark 3.1.4. The strictness of $f_+(\mathcal{M}, F)$ in part (a) of Theorem 3.1.3 still holds if we assume $\mathcal{M}$ is a mixed Hodge module. One particular application of Theorem 3.1.3 will be when the map $f : Y = Z \times X \to X$ is a smooth, projective projection from a product and $(\mathcal{M}, F)$ underlies a mixed Hodge module. In this case, the $\mathscr{D}$-module pushforward $f_+(\mathcal{M})$ is given by the relative de Rham complex (see [HTT08, Prop. 1.5.28])

$$K^\bullet = \left\{ \mathcal{M} \otimes \bigwedge^{\dim Z} \mathscr{T}_Z \xrightarrow{d} \mathcal{M} \otimes \bigwedge^{\dim Z - 1} \mathscr{T}_Z \xrightarrow{d} \dots \xrightarrow{d} \mathcal{M} \right\}$$

and this complex is filtered, given by

$$F_p K^\bullet = \left\{ F_{p - \dim Z} \mathcal{M} \otimes \bigwedge^{\dim Z} \mathscr{T}_Z \xrightarrow{d} F_{p - \dim Z + 1} \mathcal{M} \otimes \bigwedge^{\dim Z - 1} \mathscr{T}_Z \xrightarrow{d} \dots \xrightarrow{d} F_p \mathcal{M} \right\}.$$

Then strictness tells us that the induced map

$$R^k f_*(F_p K^\bullet) \to R^k f_*(K^\bullet) = \mathcal{H}^k f_*(\mathcal{M})$$

is injective, and defines the Hodge filtration on this cohomology module.

The second main theorem is called the "structure theorem for polarizable Hodge modules".

Let $Z \subseteq X$ be an irreducible closed subset. A Hodge module M on X has strict support Z if the underlying $\mathscr{D}$-module has no sub or quotient $\mathscr{D}$ modules supported on a proper subset of Z. See [Sch14, Exercise 10.2] for a characterization of this property in terms of the V-filtration along a hypersurface. See also our generalization of this property to higher codimension in Corollary 3.3.3 and Corollary 3.3.4.

Built into the definition of the category of pure Hodge modules is the property that *every pure Hodge module has a decomposition by strict support*, meaning, for any M pure on X, we have

$$M = \bigoplus_{Z \subseteq X} M_Z,$$

where the direct sum ranges over irreducible closed subsets of Z, $M_Z \neq 0$ for only finitely many Z, and each M_Z is a pure Hodge module with strict support Z. See [Sch14, Theorem 11.7] for a characterization of this property in terms of the V-filtration. See our generalization of this property to higher codimension in Corollary 3.3.5.

The structure theorem gives a description of those pure Hodge modules with strict support Z: they are generically given by (polarizable) variations of Hodge structure on Z. See [Sch14, Section 15].

Theorem 3.1.5. *Let X be a smooth complex algebraic variety, $Z \subseteq X$ an irreducible subset. Then*

1. *Every polarizable variation of Hodge structure of weight $w - \dim Z$ on a Zariski open subset of Z extends uniquely to a polarizable Hodge module on X of weight w with strict support Z.*

2. *Every Hodge module with strict support Z arises in this way.*

The difficult claim is to extend a polarizable VHS to a Hodge module with strict support on Z. This result will be used to identify certain Hodge modules as strict support direct summands of other Hodge modules.

3.1.3 Conventions for Shifting the Hodge Filtration

We refer to [Sch14] for all conventions regarding the Hodge filtration and weight filtration when applying functors to mixed Hodge modules when considering *right* $\mathscr{D}$-modules. As noted at the beginning of Section 2.1, these conventions may differ if we want to use left $\mathscr{D}$-modules instead. For convenience, we will list here those conventions for left $\mathscr{D}$-modules.

Tate Twist: Let $(\mathcal{M}, F)$ be a filtered $\mathscr{D}_X$-module. Then we define $(\mathcal{M}, F)(k)$ for any $k \in \mathbb{Z}$, the *Tate twist* of $(\mathcal{M}, F)$ by k, to be $(\mathcal{M}, F[k])$, where $F[k]_p(\mathcal{M}) = F_{p-k}(\mathcal{M})$.

Smooth pullbacks: See Remark (4.4.2) and Formula (2.17.3) in [Sai90]. Let $p : X \times Y \to Y$ be a smooth surjective morphism of relative dimension $r = \dim X$ between smooth varieties. Let $\widetilde{\mathcal{M}} = p^*(\mathcal{M})$ as an $\mathscr{O}$-module (which is also the $\mathscr{D}$-module pullback, see [HTT08, Sect. 1.3]). If $(\mathcal{M}, F)$ is a filtered left $\mathscr{D}_Y$-module, let $F_p\widetilde{\mathcal{M}} = p^*(F_p\mathcal{M})$.

If M is a mixed Hodge module with underlying filtered $\mathscr{D}_Y$-module $\mathcal{M}$, then the pullback $p^*(M) \in D^b\mathrm{MHM}(X \times Y)$ has underlying filtered $\mathscr{D}_{X\times Y}$-module

$$(\widetilde{\mathcal{M}}, F_\bullet) \tag{3.1.1}$$

lying in cohomological degree r, and $p^!(M) \in D^b\mathrm{MHM}(Y)$ has underlying filtered $\mathscr{D}_{X\times Y}$-module given by

$$(\widetilde{\mathcal{M}}, F_\bullet[r]) \tag{3.1.2}$$

lying in cohomological degree $-r$. The weight filtration is given by

$$W_\bullet p^*(\mathcal{M})[r] = p^*(W_{\bullet-r}\mathcal{M})$$

$$W_\bullet p^!(\mathcal{M})[-r] = p^*(W_{\bullet+r}\mathcal{M}).$$

Nearby and Vanishing Cycles: Let $X = \{t = 0\} \subseteq Y$ be a smooth hypersurface defined by the global function t. Let $\mathcal{M}$ be a holonomic $\mathscr{D}_Y$-module. We define

$\psi_{t,\lambda}(\mathcal{M}) = \mathrm{gr}^V_\lambda(\mathcal{M})$ for $\lambda \in [-1, 0)$, $\quad \phi_{t,\lambda}(\mathcal{M}) = \psi_{t,\lambda}(\mathcal{M})$ for $\lambda \in (-1, 0)$ $\quad$ and $\quad \phi_{t,1}(\mathcal{M}) = \mathrm{gr}^V_0(\mathcal{M})$, where $V^\bullet\mathcal{M}$ is the V-filtration of $\mathcal{M}$ along X.

If $(\mathcal{M}, F)$ is a filtered holonomic $\mathscr{D}_X$-module, then the filtration on nearby and vanishing cycles is defined to be

$$F_p\psi_{t,\lambda}(\mathcal{M}) = \frac{F_pV_\lambda\mathcal{M}}{F_pV_{<\lambda}\mathcal{M}} \text{ for } \lambda \in [-1,0], \quad \text{and} \quad F_p\phi_{t,1}(\mathcal{M}) = \frac{F_{p+1}V^0\mathcal{M}}{F_{p+1}V_{<0}\mathcal{M}}. \tag{3.1.3}$$

Just as the Hodge filtration includes a shift based on if $\lambda = 1$ or $\lambda \in (0,1)$, so does the weight filtration (see [Sch14, Sect. 20]. We make note of it here for later use: the weight filtration $W_\bullet\phi_{t,\lambda}(\mathcal{M})$ for $(\mathcal{M}, W_\bullet)$ a $\mathscr{D}$-module underlying a mixed Hodge module is defined to be the relative monodromy filtration (as defined in Subsection 3.5 above) of $L_\bullet\phi_{t,\lambda}(\mathcal{M})$ along the nilpotent operator $N = \partial_t t - \lambda$. here, $L_\bullet\phi_{t,\lambda}(\mathcal{M})$ is defined as

$$L_k\phi_{t,1}(\mathcal{M}) = \mathrm{gr}_V^0(W_k\mathcal{M}), \quad \text{and} \quad L_k\phi_{t,\lambda}(\mathcal{M}) = \mathrm{gr}_\lambda^V(W_{k+1}\mathcal{M}) \text{ for } \lambda \in (-1,0). \tag{3.1.4}$$

3.2 Normal crossing type

For the codimension one case, it is essentially immediate from the definition that the maps $t : V^\alpha\mathcal{M} \to V^{\alpha+1}\mathcal{M}$ (resp. $\partial_t : \mathrm{gr}_V^{\alpha+1}\mathcal{M} \to \mathrm{gr}_V^\alpha\mathcal{M}$) are isomorphisms for all $\alpha \neq 0$. The following example shows that, for codimension larger than one, the correct generalization of this property should concern Koszul-like complexes in the $t_1, \dots, t_r$ (resp. $\partial_{t_1}, \dots, \partial_{t_r}$).

Let $\mathcal{M}$ be an algebraic regular holonomic *left* D_2-module of normal crossing type along the two axes on $\mathbb{A}^2$, where D_2 is the Weyl algebra over $\mathbb{A}^2$. For details on normal crossing type modules, see [Sai90, Section 3]. Let (x, y) be the coordinate system on $\mathbb{A}^2$. Define $\mathcal{M}^{\alpha,\beta} = \ker(\partial_x x - \alpha)^\infty \cap \ker(\partial_y y - \beta)^\infty$ for $(\alpha, \beta) \in \mathbb{Q}^2$. Because of the assumption that $\mathcal{M}$ is of normal crossing type, we have the identity

$$\bigoplus_{\alpha,\beta\in\mathbb{Q}^2} \mathcal{M}^{\alpha,\beta} = \mathcal{M}$$

and each $\mathcal{M}^{\alpha,\beta}$ is a finite dimensional vector space over $\mathbb{C}$. Then one can easily check the

V-filtration along the origin is given by

$$V^k\mathcal{M} = \bigoplus_{\alpha+\beta\geq k} \mathcal{M}^{\alpha,\beta},$$

and $\mathrm{gr}^{\alpha}_{V_x}\mathrm{gr}^{\beta}_{V_y}\mathcal{M} = \mathcal{M}^{\alpha,\beta}$ where $V_x\mathcal{M}$ is the V-filtration along $\{x = 0\}$ and $V_y\mathcal{M}$ is the V-filtration along $\{y = 0\}$. Then the double complex

$$\begin{array}{ccc} \mathrm{gr}^k_V\mathcal{M} & \xrightarrow{x} & \mathrm{gr}^{k+1}_V\mathcal{M} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y} \\ \mathrm{gr}^{k+1}_V\mathcal{M} & \xrightarrow{x} & \mathrm{gr}^{k+2}_V\mathcal{M} \end{array} = \bigoplus_{\alpha+\beta=k} \left(\begin{array}{ccc} \mathcal{M}^{\alpha,\beta} & \xrightarrow{x} & \mathcal{M}^{\alpha+1,\beta} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y} \\ \mathcal{M}^{\alpha,\beta+1} & \xrightarrow{x} & \mathcal{M}^{\alpha+1,\beta+1} \end{array} \right) \tag{3.2.1}$$

is exact if $k \neq 0$ because one of x and y must be bijective in a summand by the properties of V-filtration in codimension one. If $k = 0$, the above double complex is quasi-isomorphic to

$$\begin{array}{ccc} \mathcal{M}^{0,0} & \xrightarrow{x} & \mathcal{M}^{1,0} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y} \\ \mathcal{M}^{0,1} & \xrightarrow{x} & \mathcal{M}^{1,1} \end{array}$$

which is isomorphic to $i^!_Z\mathcal{M}$. Since the total complex of the double complex is just the Koszul complex

$$\mathrm{gr}^k_V\mathcal{M} \xrightarrow{(x,y)} (\mathrm{gr}^{k+1}_V\mathcal{M})^2 \xrightarrow{\binom{y}{-x}} \mathrm{gr}^{k+2}_V\mathcal{M},$$

we proved a version of generalization of the properties of V-filtration in codimension one that the above Koszul complex is isomorphic to $i^!_Z\mathcal{M}$ when $k = 0$ and is exact when $k \neq 0$. The similar statement regarding the complex

$$\mathrm{gr}^{k+2}_V\mathcal{M} \xrightarrow{(\partial_x,\partial_y)} (\mathrm{gr}^{k+1}_V\mathcal{M})^2 \xrightarrow{\binom{\partial_y}{-\partial_x}} \mathrm{gr}^V_k\mathcal{M}$$

is left to the readers.

If $(\mathcal{M}, L)$ underlies a mixed Hodge module of normal crossing type where L is the weight filtration then $\mathcal{M}^{\alpha,\beta}$ carries a relative mondromy filtration $W = W(\partial_x x + \partial_y y - \alpha - \beta, L\mathcal{M}^{\alpha,\beta})$. In fact, we have the relation $W = W(\partial_x x - \alpha, W(\partial_y y - \beta, L))$ by [Sai90, p. 3] since we assume $\mathcal{M}$ is of normal crossing type. It follows that, if $k = 0$, the result of applying gr^W to the complex (3.2.1) is quasi-isomorphic to

$$\begin{array}{ccc} \mathrm{gr}^W\mathcal{M}^{0,0} & \xrightarrow{x} & \mathrm{gr}^W\mathcal{M}^{1,0} \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle y} \\ \mathrm{gr}^W\mathcal{M}^{0,1} & \xrightarrow{x} & \mathrm{gr}^W\mathcal{M}^{1,1} \end{array}$$

but the upper-horizontal and left-vertical morphisms are zero by [Sai90, p. 1]. This is the motivation for using mixed Hodge complexes in Theorem I.

3.3 Topological properties of V-filtration

In this section we first prove some basic properties of V-filtrations along a smooth subvariety. The analogous statements for a codimension 1 subvariety appear in [Sai88, Section 3]. Now let us fix the notation. Let X be a smooth variety and Z be a smooth subvariety of codimension r globally defined by regular functions $t_1, t_2, \dots, t_r$. Assume there exist global vector fields $\partial_1, \partial_2, \dots, \partial_r$ dual to the 1-forms $dt_1, dt_2, \dots, dt_r$. Let $\mathcal{M}$ be a *right* holonomic $\mathscr{D}_X$-module along Z and $V_\bullet\mathcal{M}$ be the V-filtration along Z. Recall that we have introduced the following notation: for a *right* holonomic $\mathscr{D}_X$-module $\mathcal{M}$, we define

$$A_\alpha(\mathcal{M}) = \{V_\alpha\mathcal{M} \to (V_{\alpha-1}\mathcal{M})^r \to \cdots \to V_{\alpha-r}\mathcal{M}\}, \qquad \text{in degrees } 0, 1, \dots, r;$$

$$B_\alpha(\mathcal{M}) = \left\{\mathrm{gr}^V_\alpha\mathcal{M} \to (\mathrm{gr}^V_{\alpha-1}\mathcal{M})^r \to \cdots \to \mathrm{gr}^V_{\alpha-r}\mathcal{M}\right\}, \qquad \text{in degrees } 0, 1, \dots, r;$$

$$C_\alpha(\mathcal{M}) = \left\{\mathrm{gr}^V_{\alpha-r}\mathcal{M} \to (\mathrm{gr}^V_{\alpha-r+1}\mathcal{M})^r \to \cdots \to \mathrm{gr}^V_{\alpha}\mathcal{M}\right\}, \quad \text{in degrees } -r, -r+1, \dots, 0.$$

Theorem 3.3.1. *The complexes $B_\alpha(\mathcal{M})$ and $C_\alpha(\mathcal{M})$ are exact for $\alpha \neq 0$.*

Proof. We shall construct a retraction on the complex $B_\alpha(\mathcal{M})$, i.e. a series of morphisms

$$s_\ell : \left(\mathrm{gr}^V_{\alpha-\ell}\mathcal{M}\right)^{\binom{r}{\ell}} \to \left(\mathrm{gr}^V_{\alpha-\ell+1}\mathcal{M}\right)^{\binom{r}{\ell-1}}$$

such that $s_{\ell+1} \circ d_\ell + d_{\ell-1} \circ s_\ell = \theta + \ell$ where d is the differential of the complex $B_\alpha(\mathcal{M})$. Note that the collection $\{\theta + \ell\}$ gives an endomorphism of the complex $B_\alpha(\mathcal{M})$. Let

$$(\mathrm{gr}^V_{\alpha-1}\mathcal{M})^r = \bigoplus_{i=1}^{r} \mathrm{gr}^V_{\alpha-1}\mathcal{M}e_i$$

where $e_1, e_2, ..., e_r$ is a standard basis such that the Koszul differential works as

$$d_\ell\,(\eta\, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}) = \sum_{i=1}^{r} \eta t_i\, e_i \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell},$$

where η is a local section of $\mathrm{gr}^V_{\alpha-\ell}\mathcal{M}$. Now we can define the morphism

$$s_\ell\,(\eta\, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}) = \sum_{j=1}^{r} \eta \partial_j\, e_j^*(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}),$$

where $\{e_1^*, e_2^*, \ldots, e_r^*\}$ is the dual basis. It follows that

$$\begin{aligned}
&(s_{\ell+1} \circ d_\ell + d_{\ell-1} \circ s_\ell)\eta\, e_1 \wedge e_2 \wedge \cdots \wedge e_\ell \\
=&s_{\ell+1} \sum_{i=1}^{r} \eta t_i\, e_i \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_\ell + d_{\ell-1} \sum_{j=1}^{r} \eta \partial_j p(\theta + \ell - 1)\, e_j^*(e_1 \wedge e_2 \wedge \cdots \wedge e_\ell) \\
=&\sum_{k=1}^{r}\sum_{i=1}^{r} \eta t_i \partial_k\, e_k^*(e_i \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_\ell) + \sum_{a=1}^{r}\sum_{j=1}^{r} \eta \partial_j t_a\, e_a \wedge e_j^*(e_1 \wedge e_2 \wedge \cdots \wedge e_\ell) \\
=&\eta \left(\sum_{i=1}^{r} t_i \partial_i + \ell \right) e_1 \wedge e_2 \wedge \cdots \wedge e_\ell \\
=&\eta(\theta + \ell)\, e_1 \wedge e_2 \wedge \cdots \wedge e_\ell.
\end{aligned}$$

Because $\theta + \ell = (\theta - (\alpha - \ell)) + \alpha$, the scalar multiplication by α is equal to the nilpotent operator $\theta - (\alpha - \ell)$ on the ℓ-th cohomology of $B_\alpha(\mathcal{M})$. This can happen for $\alpha \neq 0$ if and only if the ℓ-th cohomology vanishes. We conclude that the complex $B_\alpha(\mathcal{M})$ is exact for $\alpha \neq 0$.

The proof of the exactness of the complex $C_\alpha(\mathcal{M})$ is similar and we leave the rest of the proof to the readers. □

Theorem 3.3.2. *The complex $A_\alpha(\mathcal{M})$ is exact for $\alpha < 0$.*

Proof. Let H be the hypersurface defined $t_1 = 0$, let $i : H \to X$ be the closed immersion and $j : X \smallsetminus H \to X$ be the open immersion. Considering the distinguished triangle

$$i_+ i^! \mathcal{M} \to \mathcal{M} \to j_+ j^* \mathcal{M} \to i_+ i^! \mathcal{M}[1]$$

in the derived category of holonomic $\mathscr{D}_X$-modules, the problem is reduced to two cases: (a) $\mathcal{M} = \mathcal{M}(*H)$ and (b) $\mathcal{M} = \mathcal{H}^0_H \mathcal{M}$.

(a) Suppose that $\mathcal{M} = \mathcal{M}(*H)$, then the right multiplication by t_1 is a bijection on $\mathcal{M}$. Consider another filtration $U_\alpha\mathcal{M} = V_{\alpha-1}\mathcal{M}t_1^{-1}$. We find that $U_\alpha\mathcal{M}$ also satisfies the definition of the V-filtration, which forces, by the uniqueness of V-filtration,

$$U_\alpha\mathcal{M} = V_\alpha\mathcal{M} = V_{\alpha-1}\mathcal{M}t_1^{-1}.$$

In other words, we have a bijection $t_1 : V_\alpha\mathcal{M} \to V_{\alpha-1}\mathcal{M}$. By the property of Koszul complex, it follows that $A_\alpha(\mathcal{M})$ is exact for any α.

(b) Suppose that $\mathcal{M} = \mathcal{H}^0_H\mathcal{M}$, then by Kashiwara's equivalence, we have $\mathcal{M} = \mathcal{N}[\partial_1]$ for some holonomic $\mathscr{D}_H$-module $\mathcal{N}$. It is obvious to verify (see Example 3.1.1) the V-filtration of $\mathcal{M}$ is given by

$$V_\alpha\mathcal{M} = \sum_{k\geq 0} V_{\alpha-k}\mathcal{N}\partial_1^k$$

for any α, where $V_\bullet\mathcal{N}$ is the V-filtration of $\mathcal{N}$ along Z. The complex in $A_\alpha(\mathcal{M})$ is the same as the total complex of of the double complex

$$\sum_{k\geq 0}\left(\begin{array}{ccccccc} V_{\alpha-k}\mathcal{N}\partial_1^k & \longrightarrow & (V_{\alpha-k-1}\mathcal{N}\partial_1^k)^{r-1} & \longrightarrow & \cdots & \longrightarrow & V_{\alpha-k-r+1}\mathcal{N}\partial_1^k \\ \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_1} & & & & \downarrow{\scriptstyle t_1} \\ V_{\alpha-k-1}\mathcal{N}\partial_1^k & \longrightarrow & (V_{\alpha-k-2}\mathcal{N}\partial_1^k)^{r-1} & \longrightarrow & \cdots & \longrightarrow & V_{\alpha-k-r}\mathcal{N}\partial_1^k. \end{array}\right)$$

Notice that the horizontal complexes are the Koszul complexes induced by $t_2, t_3, \dots, t_r$

$$A_{\alpha-i}(\mathcal{N}) = \{V_{\alpha-i}\mathcal{N} \longrightarrow (V_{\alpha-i-1}\mathcal{N})^{r-1} \longrightarrow \cdots \longrightarrow V_{\alpha-i-r+1}\mathcal{N}\}.$$

for $i = k, k+1$. By an induction argument on the dimension, we conclude the proof. □

We give some elementary applications of Theorem 3.3.1 and Theorem 3.3.2. As a consequence we give a criterion for when $\mathcal{M}$ has strict support decomposition along Z.

Corollary 3.3.3. *A $\mathscr{D}_X$-module $\mathcal{M}$ with a V-filtration along Z has no submodules supported on Z if and only if* $\mathrm{gr}^V_0\mathcal{M} \xrightarrow{t} \oplus_{i=1}^r \mathrm{gr}^V_{-1}\mathcal{M}$ *is injective.*

Proof. If $m \in \mathcal{M}$ is such that $mt_i = 0$ for all i, then $m \in V_0\mathcal{M}$. Indeed, $m \in V_\lambda\mathcal{M}$ for some $\lambda \in \mathbb{Q}$. If $\lambda \leq 0$, we are done. Otherwise, considering the short exact sequence

$$0 \to A_{<\lambda}(\mathcal{M}) \to A_\lambda(\mathcal{M}) \to B_\lambda(\mathcal{M}) \to 0,$$

by acyclicity of $B_{-\lambda}(\mathcal{M})$ for $\lambda \neq 0$, the left-most map being injective implies $m \in V_{<\lambda}\mathcal{M}$. Since the V-filtratoin is desrete, by induction we know that $m \in V_0\mathcal{M}$. This means that $\mathcal{M}$ has no submodules supported on Z if and only if $\bigcap_{t=1}^r \ker(t_i : V_0\mathcal{M} \to V_{-1}\mathcal{M})$ vanishes.

Since $A_{>0}(\mathcal{M})$ is acyclic, it follows from the short exact sequence and the snake lemma

$$0 \to A_{<0}(\mathcal{M}) \to A_0(\mathcal{M}) \to B_0(\mathcal{M}) \to 0.$$

that $\bigcap_{t=1}^r \ker(t_i : \mathrm{gr}_0^V \mathcal{M} \to \mathrm{gr}_{-1}^V \mathcal{M}) = \bigcap_{t=1}^r \ker(t_i : V_0\mathcal{M} \to V_{-1}\mathcal{M})$, which concludes the proof. □

Corollary 3.3.4. *Let $\mathcal{M}'$ be the smallest submodule of $\mathcal{M}$ such that $\mathcal{M}'|_U \cong \mathcal{M}|_U$. Then*

$$\mathcal{M}/\mathcal{M}' \cong i_+ \mathrm{coker}\left(\bigoplus_{i=1}^r \mathrm{gr}_{-1}^V \mathcal{M} \xrightarrow{\partial_t} \mathrm{gr}_0^V \mathcal{M} \right).$$

In particular, the morphism $\bigoplus_{i=1}^r \mathrm{gr}_{-1}^V \mathcal{M} \to \mathrm{gr}_0^V \mathcal{M}$ is surjective if and only if $\mathcal{M}$ has no quotients supported on Z.

Proof. Note that $\mathcal{M}' = V_\lambda\mathcal{M} \cdot \mathscr{D}_X$ for any $\lambda < 0$. Indeed, we know that $V_\lambda\mathcal{M}' = V_\lambda\mathcal{M}$ if $\lambda < 0$, as they restrict to the same module on $X - Z$. Thus, $V_\lambda\mathcal{M} \cdot \mathscr{D}_X = V_\lambda\mathcal{M}' \cdot \mathscr{D}_X \subseteq \mathcal{M}'$. For the other inclusion, note that $(V_\lambda\mathcal{M} \cdot \mathscr{D}_X)|_U = \mathcal{M}|_U$, because the V-filtration is all of $\mathcal{M}$ away from Z. Hence, by minimality of $\mathcal{M}'$, we get the desired equality.

Note that $\mathcal{M}/\mathcal{M}'$ is supported on Z, so by Kashiwara's equivalence $\mathcal{M}/\mathcal{M}' = i_+\mathrm{gr}_0^V(\mathcal{M}/\mathcal{M}')$, where $i : Z \to X$ is the inclusion. We know $\mathrm{gr}_0^V(\mathcal{M}/\mathcal{M}') = \mathrm{gr}_0^V(\mathcal{M})/\mathrm{gr}_0^V(\mathcal{M}')$ and

$$\mathrm{gr}_0^V(\mathcal{M}') = \frac{V_0\mathcal{M} \cap \mathcal{M}'}{V_{<0}\mathcal{M}},$$

because $V_{<0}\mathcal{M} = V_{<0}\mathcal{M}'$ and $V_\bullet\mathcal{M} \cap \mathcal{M}' = V_\bullet\mathcal{M}'$ by the uniqueness of the V-filtration. Thus, the claim reduces to proving

$$V_0\mathcal{M} \cap \mathcal{M}' = \sum_{i=1}^r V_{-1}\mathcal{M}\partial_{t_i} + V_{<0}\mathcal{M}.$$

In fact, we can define inductively a filtration $U_\bullet\mathcal{M}'$ by $U_\lambda\mathcal{M}' = \sum_{i=1}^r U_{\lambda-1}\mathcal{M}\partial_{t_i} + U_{<\lambda}\mathcal{M}$ for $\lambda \geq 0$ and $U_\lambda\mathcal{M}' = V_\lambda\mathcal{M}'$ for $\lambda < 0$. Note that $V_\lambda\mathcal{M}' = V_\lambda\mathcal{M}$ for $\lambda < 0$ is discrete so $U_\bullet\mathcal{M}'$

is well-defined. Since $\mathcal{M}' = V_{<0}\mathcal{M} \cdot \mathscr{D}_X$, the filtration $U_\bullet \mathcal{M}$ is exhausted. Then it is easy to check that $U_\bullet \mathcal{M}'$ satisfies all the characterization of V-filtration, i.e. $U_\bullet \mathcal{M}' = V_\bullet \mathcal{M}'$ which concludes the proof. □

We prove here an analogue of the fact from the codimension one case that you can test if a module has a strict support decomposition by looking at $\phi_{f,1}$ as $f \in \mathscr{O}_X$ varies.

Corollary 3.3.5. *Let $\mathcal{M}$ be a $\mathscr{D}_X$-module admitting a V-filtration along Z. Then there exists a decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ with supp$(\mathcal{M}') \subseteq Z$ and $\mathcal{M}''$ having no submodules or quotient modules supported on Z if and only if*

$$\mathrm{gr}_0^V(\mathcal{M}) = \left(\bigcap_{i=1}^r \ker(t_i : \mathrm{gr}_0^V \mathcal{M} \to \mathrm{gr}_{-1}^V \mathcal{M}\right) \bigoplus \left(\sum_{i=1}^r \mathrm{gr}_{-1}^V \mathcal{M} \partial_{t_i}\right).$$

Proof. For the "only if" part, by the previous lemma we know $\mathrm{gr}_0^V \mathcal{M}'' = \mathrm{im}(\partial_{z_i})$ and $\bigcap_{i=1}^r \ker(t_i : \mathrm{gr}_0^V \mathcal{M}'' \to \mathrm{gr}_{-1}^V \mathcal{M}'') = 0$. Also, by Kashiwara's equivalence, we know $\mathcal{M}'$ satisfies $gr_{-1}^V \mathcal{M}' = 0$. By taking gr_V^0 of the equality $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$, we conclude.

For the other implication, note that we must certainly set $\mathcal{M}' = \mathcal{H}_Z^0(\mathcal{M})$, as this is the maximal submodule of $\mathcal{M}$ supported on Z. Let $\mathcal{M}'' = V_{<0}\mathcal{M} \cdot \mathscr{D}_X$, which we know is the smallest submodule such that $\mathcal{M}''|_U = \mathcal{M}|_U$, and satisfies

$$\mathcal{M}/\mathcal{M}'' = i_+(\mathrm{coker}\left(\bigoplus_{i=1}^r \mathrm{gr}_{-1}^V \mathcal{M} \xrightarrow{\partial_{t_i}} \mathrm{gr}_0^V \mathcal{M}\right).$$

By the assumption, this cokernel is isomorphic to $\bigcap_{i=1}^r \ker(t_i : \mathrm{gr}_0^V \mathcal{M} \to \mathrm{gr}_{-1}^V \mathcal{M})$, and so $\mathcal{M}/\mathcal{M}'' \cong \mathcal{M}'$. But the inclusion $\mathcal{M}' \to \mathcal{M}$ splits this quotient map, yielding the direct sum

$$\mathcal{M} \cong \mathcal{M}' \oplus \mathcal{M}'',$$

which proves the claim. □

For convenience, denote by $B(\mathcal{M}) = B_0(\mathcal{M})$ and $C(\mathcal{M}) = C_0(\mathcal{M})$. To close this section, we give a comparison of the restriction $i^*\mathcal{M}$ and $i^!\mathcal{M}$ with $B(\mathcal{M})$ and $C(\mathcal{M})$ for $i : Z \to X$.

Theorem 3.3.6. *With notation as above, the complex $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) is isomorphic to $i_Z^!\mathcal{M}$ (resp. $i^*\mathcal{M}$) in $D^b_{rh}(\mathscr{D}_Z)$, where $i_Z : Z \to X$ is the closed embedding.*

Proof. Let Z_i be the hypersurface defined by $t_i = 0$. Then the complex $i_{Z+}i_Z^!\mathcal{M}$ can be expressed by the Koszul complex

$$K(\mathcal{M}, Z_1, Z_2, \dots, Z_r) = \left\{\mathcal{M} \to \bigoplus_{i=1}^{r} \mathcal{M}(*Z_i) \to \cdots \to \mathcal{M}\left(*\sum_{i=1}^{r} Z_i\right)\right\} \tag{3.3.1}$$

placed in degrees $0, 1, \dots, r$ where the morphism is induced by natural morphisms $\mathcal{N} \to \mathcal{N}(*Z_i)$ for any regular holonomic $\mathscr{D}_X$-module $\mathcal{N}$. Similarly, the complex $i_{Z+}i_Z^*\mathcal{M}$ can be expressed by the Koszul complex

$$K_!(\mathcal{M}, Z_1, Z_2, \dots, Z_r) = \left\{\mathcal{M}\left(!\sum_{i=1}^{r} Z_i\right) \to \cdots \to \bigoplus_{i=1}^{r} \mathcal{M}(!Z_i) \to \mathcal{M}\right\} \tag{3.3.2}$$

placed in degree $-r, -r+1, \dots, 0$, where the morphism is induced by the natural morphisms $\mathcal{N}(!Z_i) \to \mathcal{N}$ for any regular holonomic $\mathscr{D}_X$-module $\mathcal{N}$.

Lemma 3.3.7. *Let $\gamma : X \to X \times \mathbb{A}^r$ be the graph embedding of f and $i_H : H = X \times \{0\} \to X \times \mathbb{A}^r$ be the closed embedding of the central fiber. Then we have natural isomorphisms*

1. $\gamma_+ K(\mathcal{M}, Z_1, Z_2, \dots, Z_r) \simeq i_{H+} K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)$ *and*

2. $\gamma_+ K_!(\mathcal{M}, Z_1, Z_2, \dots, Z_r) \simeq i_{H+} K_!(\mathcal{M}, Z_1, Z_2, \dots, Z_r)$

in the derived category of regular holonomic $\mathscr{D}_{X\times\mathbb{A}^r}$-modules.

Proof of the lemma. Let $\tilde{\mathcal{M}} = \mathcal{M} \boxtimes \omega_{\mathbb{A}^r}$ be the pullback of $\mathcal{M}$ to $X \times \mathbb{A}^r$. Denote by D_j be the divisor on $X \times \mathbb{A}^r$ defined by $f_j - t_j = 0$ for $j = 1, 2, \dots, r$ and denote by H_j be the divisor on $X \times \mathbb{A}^r$ defined by $t_j = 0$. Then we have

$$K\left(\tilde{\mathcal{M}}, D_1, D_2, \dots, D_r\right) \simeq \gamma_+\mathcal{M} \quad \text{and} \quad K\left(\tilde{\mathcal{M}}, H_1, H_2, .., H_r\right) \simeq i_{H+}\mathcal{M}.$$

It follows that

$$\begin{aligned} K(\tilde{\mathcal{M}}, D_1, D_2, \dots, D_r, H_1, H_2, \dots, H_r) &= K\left(K\left(\tilde{\mathcal{M}}, D_1, D_2, \dots, D_r\right), H_1, H_2, \dots, H_r\right) \\ &\simeq K(\gamma_+\mathcal{M}, H_1, H_2, \dots, H_r) \\ &\simeq \gamma_+ K(\mathcal{M}, Z_1, Z_2, \dots, Z_r). \end{aligned}$$

On the other hand,

$$\begin{aligned} K(\tilde{\mathcal{M}}, D_1, D_2, \dots, D_r, H_1, H_2, \dots, H_r) &= K\left(K\left(\tilde{\mathcal{M}}, H_1, H_2, \dots, H_r\right), D_1, D_2, \dots, D_r\right) \\ &\simeq K(i_{H+}\mathcal{M}, D_1, D_2, \dots, D_r) \\ &\simeq i_{H+} K(\mathcal{M}, Z_1, Z_2, \dots, Z_r). \end{aligned}$$

We conclude the first statement of the lemma. The second statement is similar, we leave it to the reader. □

Returning to the proof of the theorem, denote by $B_S(\mathcal{N}) = B(\mathcal{N})$ if we want to emphasize the V-filtration is along a subvariety S. Since taking gr^V_α is exact for the V-filtration along H, by the above lemma,

$$\mathrm{gr}^V_\alpha \gamma_+ K(\mathcal{M}, Z_1, Z_2, \dots, Z_r) \simeq \mathrm{gr}^V_\alpha i_{H+} K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)$$

It follows from the fact that $i_{H+}K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)$ is supported on H that

$$\mathrm{gr}^V_\alpha i_{H+} K(\mathcal{M}, Z_1, Z_2, \dots, Z_r) = \begin{cases} 0, & \alpha < 0; \\ K(\mathcal{M}, Z_1, Z_2, \dots, Z_r), & \alpha = 0. \end{cases}$$

Therefore, the complex $B_H(\gamma_+ K(\mathcal{M}, Z_1, Z_2, \dots, Z_r))$ is isomorphic to $K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)$. Due to the relation $B_H \gamma_+ = i_{Z+} B_Z$, we have the isomorphism

$$i_{Z+} B_Z(K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)) \simeq K(\mathcal{M}, Z_1, Z_2, \dots, Z_r).$$

Then the theorem follows from $B_Z(K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)) \simeq B_Z(\mathcal{M})$. This is because

$$B_Z\left(K(\mathcal{M}, Z_1, Z_2, \dots, Z_r)\right) = \left\{ B_Z(\mathcal{M}) \to \bigoplus_{i=1}^{r} B_Z(\mathcal{M}(*Z_i)) \to \cdots \to B_Z\left(\mathcal{M}\left(*\sum_{i=1}^{r} Z_i\right)\right)\right\}$$

and $B_Z(\mathcal{N}(*Z_i))$ is exact for any regular holonomic $\mathscr{D}_X$-module $\mathcal{N}$ and any $i = 1, 2, \ldots, r$ by part (a) in the proof of Theorem 3.3.2.

The statement about $C(\mathcal{M})$ just follows from applying Proposition 3.3.8 to $T_Z X \to Z$ and Theorem 3.3.1. Indeed, $Sp(\mathcal{M})$ is monodromic on $T_Z X$, and it is not hard to show that $\sigma^*(Sp(\mathcal{M})) = i^*(\mathcal{M})$, where $\sigma : Z \to T_Z X$ is the zero section of the normal bundle. □

Proposition 3.3.8 ([Gin86b, Proposition 10.4])**.** *For a monodromic $\mathscr{D}_E$-module $\mathcal{M}$, there are quasi-isomorphisms*

$$p_+\mathcal{M} \simeq i^*\mathcal{M}, \quad p_\dagger\mathcal{M} \simeq i^!\mathcal{M}$$

where $p : E \to Z$ is a vector bundle and $i : Z \to E$ is the zero section.

Remark 3.3.9. Lemma 3.3.7 also holds in the derived category of mixed Hodge modules. If $\mathcal{M}$ underlies a mixed Hodge module, then $\tilde{\mathcal{M}}$ in the proof of Lemma 3.3.7 underlies a mixed Hodge module as well. It follows that (3.3.1) and (3.3.2) are complexes of mixed Hodge modules by Saito's theory [Sai90] so every isomorphism in the proof of Lemma 3.3.7 extends to the derived category of mixed Hodge modules.

Remark 3.3.10. Using the previous theorem, we can rephrase the results of Lemma 3.3.5 and Lemma 3.3.4 respectively as $\mathcal{H}^0 i^! \mathcal{M} = 0$ iff $\mathrm{Hom}(i_+\mathcal{N}, \mathcal{M}) = 0$ for all $\mathcal{N}$ supported on Z, and $\mathcal{H}^0 i^* \mathcal{M} = 0$ iff $\mathrm{Hom}(\mathcal{M}, i_+\mathcal{N}) = 0$ for all $\mathcal{N}$ supported on Z.

We can describe the vanishing of other cohomologies in terms of Ext groups, similar to the characterization of vanishing of local cohomology for $\mathscr{O}$-modules. Specifically, the result is

$$\mathcal{H}^{-j} i^* \mathcal{M} = 0 \text{ for all } 0 \le j \le k \iff \mathrm{Ext}^j(\mathcal{M}, i_+\mathcal{N}) = 0 \text{ for all } \mathcal{N} \text{ supported on } Z, 0 \le j \le k$$

$$\mathcal{H}^{j} i^! \mathcal{M} = 0 \text{ for all } 0 \le j \le k \iff \mathrm{Ext}^j(i_+\mathcal{N}, \mathcal{M}) = 0 \text{ for all } \mathcal{N} \text{ supported on } Z, 0 \le j \le k.$$

The proofs of these are not hard, and we leave them to the reader.

3.4 Deformation to the Normal Cone

This section is devoted to studying the specialization construction, which goes through the deformation to the normal cone. See for example, Section 2.30 of [Sai90] and Section 1.3 of [BMS06].

Let $Z \subseteq X$ be defined by the ideal sheaf $\mathcal{I}_Z \subseteq \mathscr{O}_X$, and consider the variety

$$\widetilde{X} := \mathcal{S}pec_X \left(\bigoplus_{i \in \mathbb{Z}} \mathcal{I}_Z^{-i} \otimes u^i \right),$$

along with the smooth morphism $u : \widetilde{X} \to \mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[u])$. The fiber $u^{-1}(0)$ is isomorphic to $T_Z X$, the normal cone of Z in X, and so we call this a *deformation to the normal cone*. Over the open subset $\mathbf{G}_m := \mathbb{A}^1 - \{0\}$, the map is isomorphic to the smooth projection $X \times \mathbf{G}_m \to \mathbf{G}_m$. Wee will also consider the smooth morphism $p : X \times \mathbf{G}_m \to X$ of relative dimension 1. Let $j : X \times \mathbf{G}_m \hookrightarrow \widetilde{X}$ be the open immersion. It is the complement of the smooth divisor $T_Z X = u^{-1}(0)$.

$$\begin{array}{ccccc} X \times \mathbf{G}_m = \tilde{X}^* & \xrightarrow{j} & \tilde{X} & \longleftarrow & T_Z X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{G}_m & \longrightarrow & \mathbb{A} & \longleftarrow & \{0\} \end{array} \qquad \begin{array}{ccc} \tilde{X}^* & \xrightarrow{j} & \tilde{X} \\ & \searrow^{p} & \downarrow{\scriptstyle \rho} \\ & & X \end{array}$$

For any $M \in \operatorname{MHM}(X)$, define $Sp(M) := \psi_u j_+(p^*(M)[-1]) \in \operatorname{MHM}(T_Z X)$. Here the shift by $[-1]$ comes from the relative dimension of p. As explained in [BMS06], the underlying $\mathscr{D}$-module is

$$Sp(\mathcal{M}) = \bigoplus_{\chi \in \mathbb{Q} \cap [0,1)} \operatorname{gr}_V^{\chi} \mathcal{M},$$

where we take the associated graded of $V^\bullet \mathcal{M}$, the V-filtration along Z of the $\mathscr{D}_X$-module $\mathcal{M}$ underlying M.

3.5 Admissiblity

For convenience, we recall the definition of the relative monodromy filtration, see Section 1 of [Sai90] for details.

Let L be a finite increasing filtration on an object $M \in \mathcal{C}$, an exact category which we take to be embedded in some abelian category $\mathcal{A}$. Let $S : \mathcal{C} \to \mathcal{C}$ be an additive automorphism of the category, which extends to $\mathcal{A}$.

Let $N : (M, L) \to S^{-1}(M, L)$ be a filtered morphism such that $N^i = 0$ for $i \gg 0$. Here the filtration L on $S^j M$ is defined as $L_k(S^j M) = S^j(L_k M)$ for any $j \in \mathbb{Z}, k \in \mathbb{Z}$. Then there is at most one finite, increasing filtration $W = W(N, L)$ of (M, L), called the *relative monodromy filtration* which satisfies:

1. $N : (M; L, W) \to S^{-1}(M; L, W[2])$ is a filtered morphism,

2. $N^i : \mathrm{gr}^W_{k+i} \mathrm{gr}^L_k M \to \mathrm{gr}^W_{k-i} \mathrm{gr}^L_k M$ is an isomorphism for all $i > 0$.

Here, recall that an increasing filtration is shifted as $W[j]_\bullet = W_{\bullet - j}$. We shall take $\mathcal{C}$ the category of filtered $\mathscr{D}$-modules and S the shifting of the filtration.

In the theory of mixed Hodge modules, the objects are defined to satisfy the *admissble condition*: if $(\mathcal{M}, W)$ is a mixed Hodge module with its weight filtration and $g \in \mathscr{O}_X$ is any locally defined regular function, then

1. the relative monodromy filtration for $\psi_g(M, W)$ exists for the nilpotent monodromy operator on this nearby cycle, with $L_i = \psi_g(W_{i+1} M)$. Similarly, one assumes the existence of the relative monodromy filtration on $\phi_{g,1}(M, W)$, with $L_i = \phi_{g,1}(W_i M)$ defined without a shift.

2. the three filtrations are compatible

$$0 \to F_\ell V_\alpha W_{i-1} \mathcal{M} \to F_\ell V_\alpha W_i \mathcal{M} \to F_\ell V_\alpha \mathrm{gr}^W_i \mathcal{M} \to 0,$$

where V is the V-filtration along g.

In the setting of higher codimension, say Z is a smooth subvariety defined by $t_1, \ldots, t_r$, it is an easy exercise using the specialization construction to see that the V-filtration along

Z satisfies a similar property. The associated graded modules $\mathrm{gr}^V_\chi(\mathcal{M})$ also have nilpotent operators, given by $\theta - \chi = \sum_{i=1}^r t_i\partial_{t_i} - \chi$.

Lemma 3.5.1. *Suppose that the triple $(\mathcal{M}, F, W)$ underlies a graded polarizable mixed Hodge module, then the three filtrations F, V, W are compatible, i.e., the following sequence is exact*

$$0 \to F_\ell V_\alpha W_{i-1}\mathcal{M} \to F_\ell V_\alpha W_i\mathcal{M} \to F_\ell V_\alpha \mathrm{gr}^W_i \mathcal{M} \to 0.$$

Proof. We first recall the setting in Section 3.4: let $\tilde{X} = \mathcal{S}pec_X\left(\sum_{i\in\mathbb{Z}} \mathcal{I}_Z^i \cdot u^{-i}\right)$ be the deformation to the normal cone along Z, where $\mathcal{I}_Z$ is the ideal sheaf of Z and $\mathcal{I}_Z^i = 0$ where $i < 0$. Let $\rho : \tilde{X} \to X$, $p : \tilde{X}^* \to X$ be the two structure morphisms and $j : \tilde{X}^* \to \tilde{X}$ is the open immersion. Let $\tilde{\mathcal{M}} = j_+p^*\mathcal{M}$. Then by Saito's theory [Sai90], there exist filtrations $F_\bullet\tilde{\mathcal{M}}$ and $W_\bullet\tilde{\mathcal{M}}$ on $\tilde{\mathcal{M}}$ such that the triple $(\tilde{\mathcal{M}}, F_\bullet\tilde{\mathcal{M}}, W_\bullet\tilde{\mathcal{M}})$ underlies a graded polarizable mixed Hodge module and that $j^*F_\bullet\tilde{\mathcal{M}} = p^*F_{\bullet+1}\mathcal{M}$ and $j^*W_\bullet\tilde{\mathcal{M}} = p^*W_\bullet\mathcal{M}$. It follows from the compatibility for mixed Hodge modules of the codimension-one case that

$$0 \to F_\ell V_\alpha W_{i-1}\tilde{\mathcal{M}} \to F_\ell V_\alpha W_i\tilde{\mathcal{M}} \to F_\ell V_\alpha \mathrm{gr}^W_i \tilde{\mathcal{M}} \to 0, \tag{3.5.1}$$

where $V_\bullet$ is the V-filtration along T_ZX. Since $V_{<0}$ only depends on the restriction of a $\mathscr{D}$-module to $\tilde{X}^*$, it follows that $V_\alpha W_i\tilde{\mathcal{M}} = V_\alpha j_+p^*W_i\mathcal{M}$ for $\alpha < 0$. On the other hand, the Hodge filtration on V_α for $\alpha < 0$ can be calculated by

$$F_\ell V_\alpha W_k\tilde{\mathcal{M}} = F_\ell V_\alpha W_k j_+p^*\tilde{\mathcal{M}} = j_*p^*F_{\ell+1}W_k\mathcal{M} \cap V_\alpha j_+p^*W_k\mathcal{M}.$$

We obtain, for $\alpha < 0$,

$$\rho_*F_\ell V_\alpha W_i\tilde{\mathcal{M}} = \sum_{k\in\mathbb{Z}} F_{\ell+1}V_{\alpha+k+1}W_i\mathcal{M}\cdot u^k.$$

Similarly, we have, for $\alpha < 0$,

$$\rho_*F_\ell V_\alpha \mathrm{gr}^W_i\tilde{\mathcal{M}} = \sum_{k\in\mathbb{Z}} F_\ell V_{\alpha+k+1}\mathrm{gr}^W_i\mathcal{M}\cdot u^k.$$

Applying ρ_* to the sequence (3.5.1) for $\alpha > 0$ yields an exact sequence on X:

$$0 \to \sum_{k\in\mathbb{Z}} F_\ell V_{\alpha+k+1}W_{i-1}\mathcal{M}\cdot u^k \to \sum_{k\in\mathbb{Z}} F_\ell V_{\alpha+k+1}W_i\mathcal{M}\cdot u^k \to \sum_{k\in\mathbb{Z}} F_\ell V_{\alpha+k+1}\mathrm{gr}^W_i\mathcal{M}\cdot u^k \to 0.$$

Since the morphisms in the above sequence respect the grading, we have

$$0 \to F_\ell V_\alpha W_{i-1}\mathcal{M} \to F_\ell V_\alpha W_i \mathcal{M} \to F_\ell V_\alpha \mathrm{gr}_i^W \mathcal{M} \to 0$$

for every $\alpha \in \mathbb{Q}$. We conclude the proof. □

Lemma 3.5.2. *If $(\mathcal{M}, F, W)$ is a bifiltered $\mathscr{D}_X$-module underlying a mixed Hodge module with the weight filtration W, then the relative monodromy filtration $W(\theta - \chi, L)$ on $\mathrm{gr}_\chi^V \mathcal{M}$ exists where $L_\bullet \mathrm{gr}_\chi^V \mathcal{M} = \mathrm{gr}_\chi^V (W_\bullet \mathcal{M})$ is induced by the weight filtration.*

Proof. The relative monodromy filtration $W = W(u\partial_u - \alpha, L)$ exists on $\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}$ for $\alpha \in [-1, 0]$ because $\tilde{\mathcal{M}}$ is a mixed Hodge module. Then applying ρ_*, since $W_k \mathrm{gr}_\alpha^V \tilde{\mathcal{M}}$ is invariant under the $\mathbb{C}^*$-action $u\partial_u$,

$$\rho_* W_k \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} = \sum_{i \in \mathbb{Z}} W_k \mathrm{gr}_{\alpha+i+1}^V \mathcal{M} \cdot u^i.$$

induces a filtration W on each $\mathrm{gr}_{\alpha+i+1}^V \mathcal{M}$. We easily check that $W \mathrm{gr}_{\alpha+i+1}^V \mathcal{M}$ is the relative monodromy filtration $W(\theta - \alpha - i - 1, L)$ if $\alpha < 0$. Indeed, we have seen that, for $\alpha < 0$

$$\rho_* \mathrm{gr}_{k+i}^W \mathrm{gr}_i^L \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} = \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}_{k+i}^W \mathrm{gr}_i^L \mathrm{gr}_{\alpha+i+1}^V \mathcal{M} \cdot u^i.$$

The isomorphism $(u\partial_u - \alpha)^k : \mathrm{gr}_{k+i}^W \mathrm{gr}_i^L \mathrm{gr}_\alpha^V \tilde{\mathcal{M}} \to \mathrm{gr}_{-k+i}^W \mathrm{gr}_i^L \mathrm{gr}_\alpha^V \tilde{\mathcal{M}}$ commutes with the $\mathbb{C}^*$-action so it induces an isomorphism on each graded piece after we apply ρ_*. □

Lemma 3.5.3. *Let $(\mathcal{M}, F)$ be a filtered $\mathscr{D}$-module underlying a mixed Hodge modules over projective smooth variety $Y \times X$. Let $p : Y \times X \to X$ be the second projection, Then*

1. *The spectral sequence associated to the relative monodromy filtration on $p_+ \mathrm{gr}_\chi^V \mathcal{M}$ degenerates at the second page E_2 in the category of filtered $\mathscr{D}$-modules.*

2. *If $(\mathcal{M}, F)$ underlies a polarizable Hodge module, then $E_2^{p,q}$ is a filtered summand of $E_1^{p,q}$.*

3. *If $(\mathcal{M}, F)$ underlies a polarizable Hodge module and $W \mathrm{gr}_\chi^V \mathcal{M}$ is the monodromy filtration, then the image of $\mathcal{H}^\ell p_+ W_k \mathrm{gr}_\chi^V \mathcal{M}$ in $\mathcal{H}^\ell p_+ \mathrm{gr}_\chi^V \mathcal{M}$ is the monodromy filtration of*

$$\mathrm{gr}_\chi^V \mathcal{H}^\ell p_+ \mathcal{M} = \mathcal{H}^\ell p_+ \mathrm{gr}_\chi^V \mathcal{M}.$$

4. We have the decomposition in the filtered derived category of $\mathscr{D}$-modules

$$p_+(\mathrm{gr}_k^W \mathrm{gr}_\chi^V \mathcal{M}, F) \simeq \bigoplus_\ell (\mathcal{H}^\ell p_+ \mathrm{gr}_k^W \mathrm{gr}_\chi^V \mathcal{M}, F)[-\ell]$$

where $W \mathrm{gr}_\chi^V \mathcal{M}$ is the relative monodromy filtration.

Proof. Because the spectral sequence associated to the relative monodromy filtration $W\mathrm{gr}_\chi^V \tilde{\mathcal{M}}$ degenerates at the second page, the same is true for $\mathrm{gr}_\chi^V \mathcal{M}$ thanks to the fact that ρ_* is an exact functor. Since polarizable Hodge modules are semisimple [Sai88, p. 5.2.13], $E_2^{p,q}$ is a summand of $E_1^{p,q}$ for the spectral sequence associated to the relative monodromy filtration on $\mathrm{gr}_\chi^V \tilde{\mathcal{M}}$. Again because ρ_* is exact, the same is true for the spectral sequence associatd to $W\mathrm{gr}_\chi^V \mathcal{M}$. Lastly, the image of $\mathcal{H}^\ell W_\bullet \mathrm{gr}_\chi^V \tilde{\mathcal{M}}$ in $\mathcal{H}^\ell \mathrm{gr}_\chi^V \mathcal{M}$ is the monodromy filtration [Sai88, p. 5.3.4]. Applying ρ_* for $\chi > 0$ we conclude (c). For (d), since $\mathrm{gr}^W \mathrm{gr}_\chi^V \tilde{\mathcal{M}}$ is a polarizable Hodge module, we have the decomposition theorem

$$\tilde{p}_+(\mathrm{gr}_k^W \mathrm{gr}_\chi^V \tilde{\mathcal{M}}, F) \simeq \bigoplus_\ell (\mathcal{H}^\ell p_+ \mathrm{gr}_k^W \mathrm{gr}_\chi^V \tilde{\mathcal{M}}, F)[-\ell].$$

Applying ρ_* for $\chi > 0$ we conclude the proof. □

Lemma 3.5.4. *For any short exact sequence of mixed Hodge modules*

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0,$$

the induced sequence

$$0 \to (\mathrm{gr}_\alpha^V \mathcal{M}', F, W) \to (\mathrm{gr}_\alpha^V \mathcal{M}, F, W) \to (\mathrm{gr}_\alpha^V \mathcal{M}'', F, W) \to 0$$

is bifiltered exact, where W is the relative monodromy filtration.

Proof. By the assumption and [Sai90, p. 2.5], we have

$$0 \to (\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}', F, W) \to (\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}, F, W) \to (\mathrm{gr}_\alpha^V \tilde{\mathcal{M}}'', F, W) \to 0$$

is exact for $\alpha \in [-1, 0)$. Then the remaining goes like the proof of the above two Lemmas. □

3.6 Proof of the Theorem G

Recall our setting: let $X \to \mathbb{A}^r$ be a smooth regular map of smooth varietes where $\mathbb{A}^r$ is the affine space of dimension r and let Z be the fiber over the origin. Suppose $(t_1, t_2, \ldots, t_r)$ is a coordinate system on the $\mathbb{A}^r$ term and assume there exist global vector fields $\partial_1, \partial_2, \ldots, \partial_r$ on X dual to the one-forms $dt_1, dt_2, \ldots, dt_r$.

We restate Theorem G in terms of *right* $\mathscr{D}$-modules: for any *right* filtered regular holonomic and $\mathscr{D}_X$-module $\mathcal{M}$ and rational number α, define Koszul-type filtered complexes

$$A_\alpha(\mathcal{M}) = \left\{(V_\alpha \mathcal{M}, F) \xrightarrow{t} \bigoplus_{i=1}^{r} (V_{\alpha-1}\mathcal{M}, F) \xrightarrow{t} \cdots \xrightarrow{t} (V_{\alpha+r}\mathcal{M}, F)\right\}$$

placed in degrees $0, 1, \ldots, r$,

$$B_\alpha(\mathcal{M}) = \left\{(\mathrm{gr}^V_\alpha \mathcal{M}, F) \xrightarrow{t} \bigoplus_{i=1}^{r} (\mathrm{gr}^V_{\alpha-1}\mathcal{M}, F) \xrightarrow{t} \cdots \xrightarrow{t} (\mathrm{gr}^V_{\alpha-r}\mathcal{M}, F)\right\}$$

as the quotient $A_\alpha / A_{>\alpha}$ and

$$C_\alpha(\mathcal{M}) = \left\{(\mathrm{gr}^V_{\alpha-r} \mathcal{M}, F[r]) \xrightarrow{\partial_t} \bigoplus_{i=1}^{r} (\mathrm{gr}^V_{\alpha-r+1}\mathcal{M}, F[r-1]) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} (\mathrm{gr}^V_{\alpha}\mathcal{M}, F)\right\}$$

in degrees $-r, -r+1, \ldots, 0$, where $V_\bullet \mathcal{M}$ is the V-filtration along Z and $F[i]_k = F_{k-i}$.

Theorem 3.6.1. *With the above notation, assume that $(\mathcal{M}, F_\bullet \mathcal{M})$ is a filtered holonomic $\mathscr{D}_X$-module underlying a mixed Hodge module. Then*

1. *the complex $F_\ell A_\alpha(\mathcal{M})$ is exact for $\alpha < 0$;*

2. *the complex $F_\ell C_\alpha(\mathcal{M})$ is exact for $\alpha > 0$.*

Proof. By Lemma 3.5.1, we only need to prove the case when $(\mathcal{M}, F)$ underlies a polarizable Hodge module. If the support of $\mathcal{M}$ is contained in Z, then by Kashiwara's equivalence, there exists a Hodge module $(\mathcal{N}, F_\bullet \mathcal{N})$ on Z such that $(\mathcal{M}, F_\bullet \mathcal{M}) = i_+(\mathcal{N}, F_\bullet \mathcal{N})$. One can easily check that (see Example 3.1.1)

$$F_\ell V_\alpha \mathcal{M} = \begin{cases} \displaystyle\sum_{i_1+i_2+\cdots+i_r \le \alpha} F_{\ell-i_1-i_2-\cdots-i_r} \mathcal{N} \partial_1^{i_1} \partial_2^{i_2} \cdots \partial_r^{i_r}, & \alpha \ge 0; \\ 0, & \alpha < 0. \end{cases}$$

Thus, $(\mathrm{gr}_0^V\mathcal{M}, F_\bullet\mathrm{gr}_0^V\mathcal{M})$ recovers the filtered $\mathscr{D}_Z$-module $(\mathcal{N}, F_\bullet\mathcal{N})$ and $\mathrm{gr}_\alpha^V\mathcal{M}$ vanishes for $\alpha < 0$. The statement (a) is clear now. The statement (b) follows from the fact that $\partial_1, \partial_2, \ldots, \partial_r$ form a regular sequence on the polynomial ring $\mathbb{C}[\partial_1, \partial_2, \ldots, \partial_r]$.

Now we are in the case that no submodule of $\mathcal{M}$ is supported in Z. Let $\widehat{X}$ denote the blowup of X along Z, with exceptional divisor E. Let $(\widehat{\mathcal{M}}, F_\bullet\widehat{\mathcal{M}})$ be the minimal extension of $(\mathcal{M}, F_\bullet\mathcal{M})|_{X\smallsetminus Z}$ over E on $\widehat{X}$. By the structure theorem of Hodge modules (see Theorem 3.1.5), $(\widehat{\mathcal{M}}, F_\bullet\widehat{\mathcal{M}})$ underlies a polarizable Hodge module. Then by the decomposition theorem of polarizable Hodge modules, the filtered holonomic $\mathscr{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ is a direct summand of $\mathcal{H}^0\pi_+(\widehat{\mathcal{M}}, F_\bullet\widehat{\mathcal{M}})$. Thus, it suffices to prove the theorem for $\mathcal{H}^0\pi_+(\widehat{\mathcal{M}}, F_\bullet\widehat{\mathcal{M}})$. Let $\pi : \widehat{X} \to X$ be the blow up of X along Z and $E = \pi^{-1}Z$ be the exceptional divisor. Consider the factorization $\pi = i_\pi \circ p$ and the Cartesian diagram

$$\begin{array}{ccccc} E & \longrightarrow & \widehat{X} \times Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X} & \xrightarrow{i_\pi} & \widehat{X} \times X & \xrightarrow{p} & X, \end{array}$$
$$\widehat{X} \xrightarrow{\ \pi\ } X$$

where $i_\pi : \widehat{X} \to \widehat{X} \times X$ is the graph embedding and $p : \widehat{X} \times X \to X$ is the second projection. Denote by Γ_π the graph of π. Since the problem is local on X, we can assume that X is affine and that $(t_1, t_2, \ldots, t_r)$ extends to a coordinate system $(t, s) = (t_1, t_2, \ldots, t_r, s_1, s_2, \ldots, s_{n-r})$ on X. Note that the blow-up is given by

$$\widehat{X} = \mathrm{Proj}_X \bigoplus_{i\geq 0} \mathcal{I}_Z^i, \quad \text{where } \mathcal{I}_Z \text{ is generated by } t_1, t_2, \ldots, t_r.$$

Let $u = [u_1 : u_2 : \cdots : u_r]$ be the homogeneous coordinates on $\mathbf{P}^{r-1}$. Then $\widehat{X}$ is a subvariety of $\mathbf{P}_X^{r-1}$ defined by $u_i t_j - u_j t_i = 0$ for any $1 \leq i, j \leq r$. Denote also by $(x, y) = (x_1, x_2, \ldots, x_r, y_1, \ldots, y_{n-r})$ the parameter (t, s) on X so that

$$\pi(u, t, s) = (t, s) = (x, y).$$

Define a subvariety

$$H = \{(u, t, s, x, y) \in \widehat{X} \times X : u_i x_j - u_j x_i = 0 \text{ for any } 1 \leq i, j \leq r\}$$

with codimension $r-1$ in $\widehat{X}\times X$. Since the graph Γ_π is defined by equations $t=x$ and $s=y$, it is contained in H. Therefore, we can further factor the graph embedding $i_\pi = f\circ g$ to get a Cartesian diagram

$$\begin{array}{ccccc} E & \longrightarrow & \widehat{X}\times Z & = & \widehat{X}\times Z \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{X} & \xrightarrow{g} & H & \xrightarrow{f} & \widehat{X}\times X \end{array} \qquad (i_\pi : \widehat{X} \to \widehat{X}\times X)$$

where $g:\widehat{X}\to H$ and $f: H\to \widehat{X}\times X$ are the natural embeddings. Note that $\widehat{X}\times Z$ is a hypersurface in H.

The claim is that the Koszul complex

$$F_\ell A_\alpha(i_{\pi+}\widehat{\mathcal{M}}) = \left\{F_\ell V_\alpha i_{\pi+}\widehat{\mathcal{M}} \to \left(F_\ell V_{\alpha-1} i_{\pi+}\widehat{\mathcal{M}}\right)^r \to \cdots \to F_\ell V_{\alpha-r} i_{\pi+}\widehat{\mathcal{M}}\right\} \tag{3.6.1}$$

is exact if $\alpha<0$ where $V_\bullet i_{\pi+}\widehat{\mathcal{M}}$ is the V-filtration of $\widehat{\mathcal{M}}$ along $\widehat{X}\times Z$. The exactness of the complex 3.6.1 is local so without loss of generality, we restrict everything to the open subset $U\times X$ where U is the open subset of $\widehat{X}$ defined $u_1\neq 0$. The blow-up over U is given in coordinates by

$$\pi : (t_1, u_2, u_3, \ldots, u_r, s_1, s_2, \ldots, s_{n-r}) \mapsto (t_1, t_1u_2, t_1u_2, \ldots, t_1u_r, s_1, s_2, \ldots, s_{n-r}).$$

To give a concrete description of $i_{\pi+}\widehat{\mathcal{M}}$, we make the following local coordinate charge:

$$w_i = \begin{cases} t_1 & \text{for } i=1 \\ u_i & \text{for } 2\le i\le r \end{cases}, \qquad p_i = s_i \quad \text{for } 1\le i\le n-r,$$

$$z_i = \begin{cases} x_1 & \text{for } i=1 \\ x_i - u_i x_1 & \text{for } 2\le i\le r \end{cases}, \qquad q_i = y_i \quad \text{for } 1\le i\le n-r$$

so that $z_2, z_3, \ldots, z_r$ are the local defining equations of H. It follows from $i_{\pi+}\widehat{\mathcal{M}} = f_+g_+\widehat{\mathcal{M}}$ that

$$i_{\pi+}\widehat{\mathcal{M}} = g_+\widehat{\mathcal{M}}[\partial_{z_2}, \partial_{z_3}, \ldots, \partial_{z_r}].$$

In fact, a simple calculation using the the chain rule indicates that

$$\partial_{z_2} = \partial_{x_2} = \partial_2, \quad \partial_{z_3} = \partial_{x_3} = \partial_3, \quad \ldots, \quad \partial_{z_r} = \partial_{x_r} = \partial_r.$$

Then $F_\ell V_\alpha i_{\pi+}\widehat{\mathcal{M}}$ can be written as

$$\sum_{k\geq 0}\sum_{a_2+a_3+\cdots a_r=k} F_{\ell-k}V_{\alpha-k}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r}, \tag{3.6.2}$$

for every α where $V_\bullet g_+\widehat{\mathcal{M}}$ is the V-filtration along $\widehat{X}\times Z$. Notice that the morphism

$$F_\ell V_\alpha g_+\widehat{\mathcal{M}} \xrightarrow{x_1} F_\ell V_{\alpha-1}g_+\widehat{\mathcal{M}}$$

is bijective when $\alpha<0$ because $V_\bullet g_+\widehat{\mathcal{M}}$ is the V-filtration along $\widehat{X}\times Z$ defined by $\{x_1=0\}$ in H. We deduce that the morphism

$$x_1 : F_{\ell-k}V_{\alpha-k}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r} \to F_{\ell-k}V_{\alpha-k-1}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r}$$

is also bijective for $\alpha<0$ and $k\geq 0$. It follows that the Koszul complex (3.6.1) is exact when $\alpha<0$.

Similarly, the complex

$$F_\ell C_\alpha(i_{\pi+}\widehat{\mathcal{M}}) = \left\{F_{\ell-r}\mathrm{gr}^V_{\alpha-r}i_{\pi+}\widehat{\mathcal{M}} \to \left(F_{\ell-r+1}\mathrm{gr}^V_{\alpha-r+1}i_{\pi+}\widehat{\mathcal{M}}\right)^r \to\cdots\to F_\ell\mathrm{gr}^V_\alpha i_{\pi+}\widehat{\mathcal{M}}\right\} \tag{3.6.3}$$

is exact for $\alpha>0$. By the expression (3.6.2),

$$F_\ell\mathrm{gr}^V_\alpha i_{\pi+}\widehat{\mathcal{M}} = \sum_{k\geq 0}\sum_{a_2+a_3+\cdots a_r=k} F_{\ell-k}\mathrm{gr}^V_{\alpha-k}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r}.$$

Since for each $2\leq i\leq r$ the morphism

$$\partial_i : F_{\ell-k}\mathrm{gr}^V_{\alpha-k}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r} \to F_{\ell-k}\mathrm{gr}^V_{\alpha-k}g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_i^{a_i+1}\cdots\partial_r^{a_r}$$

is bijective, the complex (3.6.3) is quasi-isomorphic to,

$$\{F_{\ell-1}\mathrm{gr}^V_{\alpha-1}g_+\widehat{\mathcal{M}} \xrightarrow{\partial_1} F_\ell\mathrm{gr}^V_\alpha g_+\widehat{\mathcal{M}}\}, \quad \text{placed in degrees } r-1, r.$$

which is exact for $\alpha>0$ also because again $V_\bullet g_+\widehat{\mathcal{M}}$ is the V-filtration along the hypersurface $\widehat{X}\times Z\subset H$.

It remains to prove the exactness of (3.6.1) and (3.6.3) are invariant under higher direct image of p. This is Theorem 3.6.2 below. Applying Theorem 3.6.2 to (3.6.1) gives us that the Koszul complex

$$\begin{aligned}&F_\ell A_\alpha(\mathcal{H}^k p_+ i_{\pi+}\widehat{\mathcal{M}})\\ &= \left\{F_\ell V_\alpha \mathcal{H}^k p_+ i_{\pi+}\widehat{\mathcal{M}} \to \left(F_\ell V_{\alpha-1}\mathcal{H}^k p_+ i_{\pi+}\widehat{\mathcal{M}}\right)^r \to \cdots \to F_\ell V_{\alpha-r}\mathcal{H}^k p_+ i_{\pi+}\widehat{\mathcal{M}}\right\}\end{aligned}$$

is exact for $\alpha < 0$ and every k where $V_\bullet \mathcal{H}^k p_+ i_{\pi+}\widehat{\mathcal{M}}$ is the V-filtration along Z. Due to

$$\mathcal{H}^k p_+ i_{\pi+} = \mathcal{H}^k \pi_+,$$

we have finished the proof of the first statement in Theorem G. The second statement follows similarly and we leave it to the readers. □

Theorem 3.6.2. *Let X be a nonsingular quasi-projective variety and Y be an affine space with Z an affine subspace defined by $x_1, x_2, \ldots, x_r$. Let $(\mathcal{M}, F)$ be a filtered holonomic $\mathscr{D}_{X\times Y}$-module underlying a polarizable Hodge module. Suppose that the second projection $p : X \times Y \to Y$ is projective on the support of $\mathcal{M}$. Let $V_\bullet\mathcal{M}$ be the V-filtration along $p^{-1}(Z)$. Let $V_\bullet \mathcal{H}^k p_+\mathcal{M}$ be the V-filtration along Z for every k.*

1. *If the complex*

$$F_\ell A_\alpha(\mathcal{M}) = \{F_\ell V_\alpha\mathcal{M} \to (F_\ell V_{\alpha-1}\mathcal{M})^r \to \cdots \to F_\ell V_{\alpha-r}\mathcal{M}\} \tag{3.6.4}$$

is exact for some α, then the complex $F_\ell A_\alpha(\mathcal{H}^k p_+\mathcal{M})$ is also exact for every k.

2. *Similarly, if the Koszul complex*

$$F_\ell C_\alpha(\mathcal{M}) = \left\{F_{\ell-r}\mathrm{gr}^V_{\alpha-r}\mathcal{M} \to \left(F_{\ell-r+1}\mathrm{gr}^V_{\alpha-r+1}\mathcal{M}\right)^r \to \cdots \to F_\ell \mathrm{gr}^V_\alpha\mathcal{M}\right\} \tag{3.6.5}$$

is exact for some α, then the complex $F_\ell C_\alpha(\mathcal{H}^k p_+\mathcal{M})$ is exact for every k.

Proof. Because of the bistrictness proved in [BMS06] on the complex $p_+(\mathcal{M}, V_\bullet, F_\bullet) =$

$$\left(\mathbf{R}p_*\left(\mathcal{M}\otimes \overset{-\star}{\bigwedge}\mathscr{T}_{X\times Y/Y}\right), \mathbf{R}p_*\left(V_\bullet\mathcal{M}\otimes \overset{-\star}{\bigwedge}\mathscr{T}_{X\times Y/Y}\right), \mathbf{R}p_*\left(F_{\bullet+\star}\mathcal{M}\otimes \overset{-\star}{\bigwedge}\mathscr{T}_{X\times Y/Y}\right)\right),$$

we know that the k-th cohomology of $\mathcal{H}^k F_\ell V_\alpha p_+\mathcal{M} = \mathbf{R}^k p_*\left(F_{\ell+\star} V_\alpha \mathcal{M} \otimes \wedge^{-\star} \mathscr{T}_{X\times Y/Y}\right)$ is canonically isomorphic to $F_\ell V_\alpha \mathcal{H}^k p_+\mathcal{M}$. It follows from the Hard Lefschetz theorem on the direct image of polarizable Hodge modules (see part (b) of Theorem 3.1.3) that the morphism

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell V_\alpha \mathcal{H}^{-k} p_+\mathcal{M} \to F_{\ell-k} V_\alpha \mathcal{H}^k p_+\mathcal{M}.$$

is an isomorphism induced by the Lefschetz operator $L = \omega\wedge$ of a hyperplane class ω on X. Therefore, we have the decomposition

$$F_\ell V_\alpha p_+\mathcal{M} \simeq \bigoplus_{k\in\mathbb{Z}} F_\ell V_\alpha \mathcal{H}^k p_+\mathcal{M}[-k]$$

in the bounded derived category $\mathbf{D}^b_{\mathrm{coh}}(Y, \mathscr{O}_Y)$ of Y. If we apply p_+ on (3.6.4), by the above decomposition, we obtain

$$F_\ell p_+ A_\alpha(\mathcal{M}) \simeq \bigoplus_{k\in\mathbb{Z}} F_\ell A_\alpha(\mathcal{H}^k p_+\mathcal{M})[-k]$$

in $\mathbf{D}^b_{\mathrm{coh}}(Y, \mathscr{O}_Y)$. But by the assumption of the lemma, the complex $F_\ell p_+ A_\alpha(\mathcal{M})$ is exact. It follows that each summand

$$F_\ell A_\alpha(\mathcal{H}^k p_+\mathcal{M}) = \left\{F_\ell V_\alpha \mathcal{H}^k p_+\mathcal{M} \to (F_\ell V_{\alpha-1}\mathcal{H}^k p_+\mathcal{M})^r \to \cdots \to F_\ell V_{\alpha-r}\mathcal{H}^k p_+\mathcal{M}\right\}$$

in the decomposition is exact. We have thus proved (a).

The proof of (2) is similar. Since we still have the isomorphism from the Hard Lefschetz theorem

$$\left(2\pi\sqrt{-1}L\right)^k : F_\ell \mathrm{gr}^V_\alpha \mathcal{H}^{-k} p_+\mathcal{M} \to F_{\ell-k}\mathrm{gr}^V_\alpha \mathcal{H}^k p_+\mathcal{M},$$

we get a decomposition

$$p_+ F_\ell C_\alpha(\mathcal{M}) \simeq \bigoplus_{k\in\mathbb{Z}} F_\ell C_\alpha(\mathcal{H}^k p_+\mathcal{M})[-k]$$

in $\mathbf{D}^b_{\mathrm{coh}}(Y, \mathscr{O}_Y)$. The remaining goes like in (a) and is left to the readers. □

Remark 3.6.3. One can bypass the decomposition theorem in the above proof by the argument in Theorem 3.7.5 and the double complexes (3.7.4) and (3.7.6)

3.7 Proof of the Theorem I

In this section we prove Theorem I and it is more convenient to work with *right* $\mathscr{D}$-modules. Recall that the convention for right $\mathscr{D}$-modules is that the V-filtration be indexed increasingly. The proof is split into three parts: Theorem 3.7.1, Theorem 3.7.5 and Theorem 3.7.7. For simplicity, we denote by $B_Z(\mathcal{M}) = B_0(\mathcal{M})$ and $C_Z(\mathcal{M}) = C_0(\mathcal{M})$ to emphasize the V-filtration is along the smooth subvariety Z. If the V-filtration is clear from the context, we will simply use the notation $B(\mathcal{M})$ or $C(\mathcal{M})$.

3.7.1 Mixed Hodge complex

We first prove that for $\mathcal{M}$ underlying a mixed Hodge module the complex $B(\mathcal{M})$ together with W induced by the relative monodromy filtration is a mixed Hodge complex. A *mixed Hodge complex*, roughly speaking, is a bifiltered complex of $\mathscr{D}$-modules (C, F, W), where F is a decreasing "Hodge" filtration by $\mathscr{O}$-submodules and W is an increasing "weight" filtration by $\mathscr{D}$-submodules with $\mathbb{Q}$-structure $(C_{\mathbb{Q}}, W_{\mathbb{Q}})$. These data should satisfy $\mathrm{DR}(C, W) \simeq (C_{\mathbb{Q}}, W_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ and that

$$\mathrm{gr}_k^W C \simeq \bigoplus_{\ell \in \mathbb{Z}} \mathcal{H}^\ell \mathrm{gr}_k^W C[-\ell]$$

in the derived category of filtered $\mathscr{D}$-modules. Moreover, $(\mathcal{H}^\ell \mathrm{gr}_k^W C, F)$ together with the induced $\mathbb{Q}$-structure underlies a polarizable Hodge module of weight $k + \ell$ for any k and ℓ. Theorem I(a) is restated as follows:

Theorem 3.7.1. *Let $M = (\mathcal{M}, F, L, \mathcal{K})$ be a mixed Hodge module on a smooth variety X as in Theorem I and let Z be a smooth subvariety of X. Then $B_Z(\mathcal{M})$ together with the relative monodromy filtration is a mixed Hodge complex.*

Proof. We first remark that $B(\mathcal{M})$ carries a $\mathbb{Q}$-structure. Indeed, by Theorem 3.3.1

$$\mathrm{DR}_Z B(\mathcal{M}) \simeq \mathrm{DR}_Z(i^! \mathcal{M}) \simeq i^! \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{C}.$$

In fact, if W is the filtration on $B(\mathcal{M})$ induced by the monodromy filtration on each $\mathrm{gr}_\alpha^V\mathcal{M}$ relative to $\mathrm{gr}_\alpha^V L_\bullet\mathcal{M}$ then $W_kB(\mathcal{M})$ also carries a $\mathbb{Q}$-structure. This is because

$$\mathrm{DR}_Z i_Z^! W_k\mathrm{Sp}(\mathcal{M}) \simeq i_Z^! W_k\mathrm{Sp}(\mathcal{K})\otimes_{\mathbb{Q}}\mathbb{C}, \quad i_Z : Z \to T_Z X$$

and $i_Z^! W_k\mathrm{Sp}(\mathcal{M}) \simeq W_kB(\mathcal{M})$ by the fact that the retraction constructed in the proof of Theorem 3.3.1 also preserves the filtration $WB(\mathcal{M})$. Recall that $\mathrm{Sp}(\mathcal{M})$ is the specialization of $\mathcal{M}$ introduced in 3.4.

Pure case. We first prove the case when $(\mathcal{M}, F, \mathcal{K})$ is a polarizable Hodge module of weight w. If $\mathcal{M}$ is supported on Z then $B(\mathcal{M}) \simeq i_+\mathrm{gr}_0^V\mathcal{M}$ in the (F, W)-bifiltered category and therefore, the theorem follows easily. Now assume that the support of $\mathcal{M}$ is not contained in Z. Let $\pi : \widehat{X} \to X$ be the blow up along Z and $\widehat{\mathcal{M}}$ be the minimal extension of $\mathcal{M}$ to $\widehat{X}$ from $\widehat{X} - E \cong X - Z$. Then we can factor the blow-up into the graph embedding followed by the smooth projection

$$\widehat{X} \xrightarrow{i_\pi} \widehat{X}\times X \xrightarrow{p} X$$

The proof consists of two steps:

Step 1. We show that $B_{p^{-1}Z}\left(i_{\pi+}\widehat{\mathcal{M}}\right)$ is a mixed Hodge complex.

In fact, the complex $B_{p^{-1}Z}\left(i_{\pi+}\widehat{\mathcal{M}}\right)$ together with the monodromy filtration is quasi-isomorphic to $B_E\left(\widehat{\mathcal{M}}\right)$ locally, where E is the exceptional divisor of π. Note that, although E is not defined by a global function, we can make the complex $B_E\left(\widehat{\mathcal{M}}\right)$ well-defined by

$$\mathrm{gr}_0^V\widehat{\mathcal{M}}\otimes\mathcal{O}(-E)|_E \to \mathrm{gr}_{-1}^V\widehat{\mathcal{M}}.$$

As we can see in the proof of Theorem 3.6.1: the formula (3.6.2) is compatible with the monodromy filtration, i.e.

$$F_\ell\mathrm{gr}^W\mathrm{gr}_\alpha^V i_{\pi+}\widehat{\mathcal{M}} = \sum_{k\geq 0}\sum_{a_2+a_3+\cdots a_r=k} F_{\ell-k}\mathrm{gr}^W\mathrm{gr}_{\alpha-k}^V g_+\widehat{\mathcal{M}}\partial_2^{a_2}\partial_3^{a_3}\cdots\partial_r^{a_r}$$

But since $B_E\left(\widehat{\mathcal{M}}\right)$ is a mixed Hodge complex, and this property (like the property of being a Hodge module) is local, it follows that $B_{p^{-1}Z}\left(i_{\pi+}\widehat{\mathcal{M}}\right)$ is also a mixed Hodge complex. Due to the decomposition theorem of polarizable Hodge modules, the module $\mathcal{M}$ is a summand of $\mathcal{H}^0 p_+ i_{\pi+}\mathcal{M}$. Therefore, we reduce the proof to the following.

Step 2. We prove that if $B_{p^{-1}Z}(\mathcal{M})$ is a mixed Hodge complex for a polarizable Hodge module $\mathcal{M}$ of weight w on $Y \times X$, where $p : Y \times X \to X$ is the second projection proper over the support of $\mathcal{M}$, then $B_Z(\mathcal{H}^\ell p_+\mathcal{M})$ is a mixed Hodge complex of weight $w+\ell$ for any $\ell \in \mathbb{Z}$.

In fact, we have

$$\begin{aligned} p_+\left(\mathrm{gr}_k^W B_{p^{-1}Z}(\mathcal{M})\right) &\simeq \bigoplus_{i\in\mathbb{Z}} p_+\left(\mathcal{H}^i \mathrm{gr}_k^W B_{p^{-1}Z}(\mathcal{M})\right)[-i] \\ &\simeq \bigoplus_{i,j\in\mathbb{Z}} \mathcal{H}^j p_+\left(\mathcal{H}^i \mathrm{gr}_k^W B_{p^{-1}Z}(\mathcal{M})\right)[-i-j] \end{aligned}$$

in the derived category of filtered $\mathscr{D}$-modules. On the other hand, we also have the decomposition in the derived category of filtered $\mathscr{D}$-modules by Lemma 3.5.3(d):

$$p_+\left(\mathrm{gr}_k^W B_{p^{-1}Z}(\mathcal{M})\right) \simeq \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_{k,\ell}[-\ell],$$

where $\mathcal{F}^i_{k,\ell} = \mathcal{H}^\ell p_+ \mathrm{gr}_k^W B^i_{p^{-1}Z}(\mathcal{M})$. This implies

$$\mathcal{F}_{k,\ell} \simeq \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i \mathcal{F}_{k,\ell}[-i] \tag{3.7.1}$$

and $\mathcal{H}^i\mathcal{F}_{k,\ell}$ is a polarizable Hodge module of weight $w+k+i+\ell$. For each k we have a weight spectral sequence

$$E_1^{i,j}(k) = \mathcal{H}^{i+j} p_+ \mathrm{gr}_{-i}^W B^k_{p^{-1}Z}(\mathcal{M}) \Rightarrow E_\infty^{i,j}(k) = \mathrm{gr}_{-i}^W \mathcal{H}^{i+j} p_+ B^k_{p^{-1}Z}(\mathcal{M})$$

so that $E_1^{i,j}(k) = \mathcal{F}^k_{-i,j+j}$. Note that by the bistrictness proved in [BMS06], we have

$$E_\infty^{i,j}(k) = \mathrm{gr}_{-i}^W B_Z^k(\mathcal{H}^{i+j} p_+\mathcal{M}).$$

We gather some facts deduced from the deformation to the normal bundle argument (Lemma 3.5.3):

1. the spectral sequence degenerates at the second page;

2. the induced filtration $W\mathcal{H}^{i+j}p_+B^k_{p^{-1}Z}(\mathcal{M})$ is the monodromy filtration on

$$\mathcal{H}^{i+j}p_+B^k_{p^{-1}Z}(\mathcal{M}) = \left(\mathrm{gr}^V_{-k}\mathcal{H}^{i+j}p_+\mathcal{M}\right)^{\binom{r}{k}};$$

3. lastly, $E_2^{i,j}(k)$ is a summand of $E_1^{i,j}(k)$ in the category of filtered $\mathscr{D}$-modules.

Therefore, the differential d_1 on the first page induces a double complex

$$\cdots \xrightarrow{d_1} \mathcal{F}_{k+1,\ell-1} \xrightarrow{d_1} \mathcal{F}_{k,\ell} \xrightarrow{d_1} \mathcal{F}_{k-1,\ell+1} \xrightarrow{d_1} \cdots.$$

Let T be the total complex of this double complex. Then by (3.7.1) and semisimplicity, T decomposes into

$$\begin{aligned}&\bigoplus_i \left\{\cdots \xrightarrow{d_1} \mathcal{H}^i\mathcal{F}_{k+1,\ell-1} \xrightarrow{d_1} \mathcal{H}^i\mathcal{F}_{k,\ell} \xrightarrow{d_1} \mathcal{H}^i\mathcal{F}_{k-1,\ell+1} \xrightarrow{d_1} \cdots\right\}[-i]\\ \simeq &\bigoplus_{i,j} \mathcal{H}^j_{d_1}\mathcal{H}^i\mathcal{F}_{k-\bullet,\ell+\bullet}[-i-j]\end{aligned} \tag{3.7.2}$$

in the derived category of filtered $\mathscr{D}$-modules. On the other hand, by the claim (c) above, we also have another decomposition in the derived category:

$$T \simeq \bigoplus_j \mathcal{H}^j_{d_1}\mathcal{F}_{k-\bullet,\ell+\bullet}[-j].$$

Since $\mathcal{H}^j_{d_1}\mathcal{F}_{k-\bullet,\ell+\bullet} = \mathrm{gr}^W_{k-j}B_Z(\mathcal{H}^{\ell+j}p_+\mathcal{M})$, the decomposition (3.7.2) implies $\mathrm{gr}^W_{k-j}B_Z(\mathcal{H}^{\ell+j}p_+\mathcal{M})$ decomposes into the direct sum of its cohomology in the derived category of filtered $\mathscr{D}$-modules and the cohomology $\mathcal{H}^i\mathrm{gr}^W_kB_Z(\mathcal{H}^\ell p_+\mathcal{M})$ is of weight $w+\ell+k+i$. It is easy to see that the decomposition is compatible with $\mathbb{Q}$-structures and therefore, we conclude the proof.

Mixed case. By Lemma 3.7.2 below, there exists a functorial splitting

$$\mathrm{gr}^W\mathrm{gr}^V_\alpha\mathcal{M} \simeq \mathrm{gr}^W\mathrm{gr}^L\mathrm{gr}^V_\alpha\mathcal{M},$$

with respect to $t_1, t_2, \ldots, t_r$ which implies $\mathrm{gr}^WB(\mathcal{M}) \simeq \mathrm{gr}^WB(\mathrm{gr}^L\mathcal{M})$. Therefore, we reduce the proof to the case where $\mathcal{M}$ underlies a pure Hodge module. □

We collect some corollaries of Deligne's Theorem which we have already applied in the previous theorem and will apply these results in the proof of Theorem 3.7.7. The proof is based on [Sai90, p. 1.5] and a Theorem of Deligne 3.7.10. For the purpose of the exposition, we postpone the proof to the end of this section.

Lemma 3.7.2. *Let $\mathcal{M}, \mathcal{M}'$ be mixed Hodge modules on a smooth variety X and V be the V-filtration along a smooth subvariety Z. Let L be the filtration on gr^V_α induced by the weight filtration and $W = W(\theta - \alpha, L)$ be the relative monodromy filtration on gr^V_α. Then we have:*

1. *For any local defining equation f of Z, the induced filtered morphism*

$$f : (\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, F) \to (\mathrm{gr}^W \mathrm{gr}^V_{\alpha-1} \mathcal{M}, F)$$

   splits into $f : \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_\alpha \mathcal{M} \to \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{\alpha-1} \mathcal{M}$.

2. *For any local vector fields ξ normal to Z, the induced filtered morphism*

$$\xi : (\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, F) \to (\mathrm{gr}^W \mathrm{gr}^V_{\alpha+1} \mathcal{M}, F[-1])$$

   splits into $\xi : (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_\alpha \mathcal{M}, F) \to (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{\alpha+1} \mathcal{M}, F[-1])$.

3. *If $T : \mathcal{M} \to \mathcal{M}'$ is a morphism of mixed Hodge modules, then the filtered morphism*

$$\mathrm{gr}^W T : (\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, F) \to (\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}', F)$$

   splits into $\mathrm{gr}^W T : (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_\alpha \mathcal{M}, F) \to (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_\alpha \mathcal{M}', F)$.

Now we turn to the complex $C(\mathcal{M})$. The filtration $W_k C(\mathcal{M})$ also carries a $\mathbb{Q}$-structure. In fact, it follows from Proposition 3.3.8 and the fact that the retraction constructed in Theorem 3.3.1 respects the filtration W that

$$\mathrm{DR}_Z W_k C(\mathcal{M}) \simeq \mathrm{DR}_Z p_+ W_k \mathrm{Sp}(\mathcal{M}) \simeq p_* W_k \mathrm{Sp}\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{C}$$

where $p : T_Z X \to Z$ is the projection. Therefore, we can simply modify the proof of Theorem 3.7.1 to prove the following.

Theorem 3.7.3. *Let $(\mathcal{M}, F, L, \mathcal{K})$ be a mixed Hodge module on a smooth variety X and Z is a smooth subvariety. Then $C_Z(\mathcal{M})$ together with the relative monodromy filtration is also a mixed Hodge complex.*

By a formal argument in [Del71b], we conclude:

Corollary 3.7.4. *The Hodge spectral sequences of $B(\mathcal{M})$ and $C(\mathcal{M})$ degenerate at the first page while the weight spectral sequences degenerate at the second page.*

3.7.2 Comparison to the restriction functors

The goal of this part is to prove Theorem I(b):

Theorem 3.7.5. *If $(\mathcal{M}, F)$ is a graded polarizable mixed Hodge module then the complex $B(\mathcal{M})$ (resp. $C(\mathcal{M})$) is isomorphic to $(i^!\mathcal{M}, F)$ (resp. $(i^*\mathcal{M}, F)$) in the derived category of filtered $\mathscr{D}$-modules with $\mathbb{Q}$-structures.*

Proof. Note that the $\mathbb{Q}$-structure has already been handled in Theorem 3.7.1.

1. We first deal with the complex $B(\mathcal{M})$. Recall that, as we introduced the proof of Theorem 3.3.6, the functor $i_+i^!\mathcal{M}$ can be defined by the the Koszul complex in the derived category of mixed Hodge modules (see the proof of [Sai90, Prop. 2.19]):

$$K(\mathcal{M}) = \left\{ \mathcal{M} \to \bigoplus \mathcal{M}(*Z_i) \to \cdots \to \mathcal{M}(*\sum_{i=1}^{r} Z_i) \right\} \tag{3.7.3}$$

placed in degrees $0, 1, \ldots, r$. Moreover, the complex $K(\mathcal{M})$ is isomorphic to $i_+\mathrm{gr}_0^V K(\mathcal{M})$ in the derived category of (F, W)-bifiltered $\mathscr{D}$-modules because Lemma 3.3.7 also holds in the derived category of mixed Hodge modules. Consider the double complex $BK(\mathcal{M})$:

$$\begin{array}{ccccccc}
\mathrm{gr}_0^V\mathcal{M} & \xrightarrow{\delta_0} & (\mathrm{gr}_{-1}^V\mathcal{M})^r & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_{r-1}} & \mathrm{gr}_{-r}^V\mathcal{M} \\
\downarrow & & \downarrow & & & & \downarrow \\
\bigoplus_{i=1}^r \mathrm{gr}_0^V\mathcal{M}(*Z_i) & \xrightarrow{\delta_0} & \bigoplus_{i=1}^r(\mathrm{gr}_{-1}^V\mathcal{M}(*Z_i))^r & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_{r-1}} & \bigoplus_{i=1}^r \mathrm{gr}_{-r}^V\mathcal{M}(*Z_i) \\
\downarrow & & \downarrow & & & & \downarrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{gr}_0^V\mathcal{M}(*\sum_{i=1}^r Z_i) & \xrightarrow{\delta_0} & (\mathrm{gr}_{-1}^V\mathcal{M}(*\sum_{i=1}^r Z_i))^r & \xrightarrow{\delta_1} & \cdots & \xrightarrow{\delta_{r-1}} & \mathrm{gr}_{-r}^V\mathcal{M}(*\sum_{i=1}^r Z_i)
\end{array} \tag{3.7.4}$$

whose uppermost row is $BK^0(\mathcal{M}) = B(\mathcal{M})$ and leftmost column is $B^0K(\mathcal{M}) = \mathrm{gr}_0^V K(\mathcal{M})$. The total complex of $BK(\mathcal{M})$ is (F,W)-bifiltered quasi-isomorphic to $\mathrm{gr}_0^V K(\mathcal{M})$ because $\mathrm{gr}_\alpha^V K(\mathcal{M})$ is (F,W)-bifiltered acyclic for $\alpha < 0$ and Lemma 3.5.4. On the other hand, the total complex of $BK(\mathcal{M})$ is also F-filtered quasi-isomorphic to $B(\mathcal{M})$ because each row $BK^i(\mathcal{M})$ is F-filtered acyclic when $i \neq 0$ by Corollary 3.7.4 and Theorem 3.3.6. We conclude that $\mathrm{gr}_0^V K(\mathcal{M})$ and $B(\mathcal{M})$ are isomorphic in the derived category of F-filtered $\mathscr{D}_Z$-modules. But $\mathrm{gr}_0^V K(\mathcal{M})$ is (F,W)-bifiltered quasi-isomorphic to $i^!(\mathcal{M},F,W)$. We conclude the proof of this part.

2. Next, we deal with the complex $C(\mathcal{M})$. The functor $i_+i^*\mathcal{M}$ can be computed by the the Koszul complex

$$K_!(\mathcal{M}) = \left\{\mathcal{M}\left(!\sum_{i=1}^r Z_i\right) \to \cdots \to \bigoplus_{i=1}^r \mathcal{M}(!Z_i) \to \mathcal{M}\right\} \tag{3.7.5}$$

placed in degrees $-r, -r+1, \ldots, 0$. Moreover, the complex $K_!(\mathcal{M})$ is isomorphic to $i_+\mathrm{gr}_0^V K_!(\mathcal{M})$ in the derived category of (F,W)-bifiltered $\mathscr{D}$-modules because Lemma 3.3.7 also holds in the derived category of mixed Hodge modules. Consider the double complex $CK_!(\mathcal{M})$

$$\begin{array}{ccccccc}
\mathrm{gr}_{-r}^V\mathcal{M} & \xrightarrow{\delta_{-r}} & \cdots & \xrightarrow{\delta_{-r+1}} & (\mathrm{gr}_{-r+1}^V\mathcal{M})^r & \xrightarrow{\delta_{-1}} & \mathrm{gr}_0^V\mathcal{M} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\bigoplus_{i=1}^r \mathrm{gr}_{-r}^V\mathcal{M}(!Z_i) & \xrightarrow{\delta_{-r}} & \cdots & \xrightarrow{\delta_{-r+1}} & \bigoplus_{i=1}^r(\mathrm{gr}_{-r+1}^V\mathcal{M}(!Z_i))^r & \xrightarrow{\delta_{-1}} & \bigoplus_{i=1}^r \mathrm{gr}_0^V\mathcal{M}(!Z_i) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{gr}_{-r}^V\mathcal{M}(!\sum_{i=1}^r Z_i) & \xrightarrow{\delta_{-r}} & \cdots & \xrightarrow{\delta_{-r+1}} & (\mathrm{gr}_{-r+1}^V\mathcal{M}(!\sum_{i=1}^r Z_i))^r & \xrightarrow{\delta_{-1}} & \mathrm{gr}_0^V\mathcal{M}(!\sum_{i=1}^r Z_i)
\end{array} \tag{3.7.6}$$

whose uppermost row is $CK_!^0(\mathcal{M}) = C(\mathcal{M})$ and leftmost column is $C^0K(\mathcal{M}) = \mathrm{gr}_0^V K_!(\mathcal{M})$. The total complex of $CK_!(\mathcal{M})$ is (F,W)-bifiltered quasi-isomorphic to $\mathrm{gr}_0^V K_!(\mathcal{M})$ because $\mathrm{gr}_\alpha^V K_!(\mathcal{M})$ is (F,W)-bifiltered acyclic for $\alpha < 0$. On the other hand, the total complex of $CK_!(\mathcal{M})$ is also F-filtered quasi-isomorphic to $C(K)$ because each row $CK_!^i(\mathcal{M})$ is F-filtered acyclic when $i \neq 0$ because of Corollary 3.7.4 and Theorem 3.3.6. We conclude that $\mathrm{gr}_0^V K_!(\mathcal{M})$ and $C(\mathcal{M})$ are isomorphic in the derived category of F-filtered $\mathscr{D}_Z$-modules. Finally, $\mathrm{gr}_0^V K_!(\mathcal{M})$ is bifiltered quasi-isomorphic to $i^*(\mathcal{M},F,W)$. We conclude the proof of this part. □

Remark 3.7.6. If one is just interested in the isomorphisms

$$(B(\mathcal{M}),F) \simeq (i^!\mathcal{M},F) \quad \text{and} \quad (C(\mathcal{M}),F) \simeq (i^*\mathcal{M},F)$$

in the derived category of filtered $\mathscr{D}$-modules, there is a way to bypass mixed Hodge complexes as are used in Theorem 3.7.1 and Theorem 3.7.3. To prove $(B(\mathcal{M}),F) \simeq (i^!\mathcal{M},F)$, we just need to show that $(B(\mathcal{M}(*Z_i)),F)$ is filtered acyclic for any Z_i as in the proof Theorem 3.7.5. For this we consider $\widehat{\mathcal{M}}(*\widehat{Z}_i+E)$ on the blow-up $\pi:\widehat{X}\to X$ along Z where $\widehat{\mathcal{M}}$ is the minimal extension of $\mathcal{M}|_{X-Z}$, $\widehat{Z}_i$ is the strict transform of Z_i and E is the exceptional divisor. Note that $\pi_+\widehat{\mathcal{M}}(*\widehat{Z}_i+E) = \mathcal{M}(*Z_i)$. It follows from the computation in the proof of Theorem 3.6.1 that $B(i_{\pi_+}\mathcal{M}(*\widehat{Z}_i+E))$ is filtered acyclic where $i_\pi:\widehat{X}\to\widehat{X}\times X$ is the graph embedding because of the fact that one of the Koszul differentials is filtered bijective. We can conclude by applying p_+ to $B(i_{\pi_+}\mathcal{M}(*\widehat{Z}_i+E))$ and the bistrictness result for smooth, projective morphisms. The same idea works for the filtered acyclicity of $(C(\mathcal{M}(*Z_i)),F)$.

3.7.3 Finishing the proof

We now prove the last part of Theorem I:

Theorem 3.7.7. *If $\mathcal{M}$ is a graded polarizable mixed Hodge module and W is the filtration on $B(\mathcal{M})$ and $C(\mathcal{M})$ induced by the relative monodromy filtration on $\mathrm{gr}_\alpha^V\mathcal{M}$, then*

$$\mathrm{gr}_k^W \mathcal{H}^\ell B(\mathcal{M}) \simeq \mathrm{gr}_{k+\ell}^W \mathcal{H}^\ell i_Z^! \mathcal{M} \quad \textit{and} \quad \mathrm{gr}_k^W \mathcal{H}^{-\ell} C(\mathcal{M}) \simeq \mathrm{gr}_{k-\ell}^W \mathcal{H}^{-\ell} i_Z^* \mathcal{M}$$

as polarizable Hodge modules for $\ell \geq 0$.

Proof. 1. We first focus on the complex $B(\mathcal{M})$. We shall prove the following as a preparation:

Lemma 3.7.8. *The complex $\mathcal{H}^\ell_\delta \mathrm{gr}^W_k BK(\mathcal{M})$ is exact for $\ell \neq 0$ and any $k \in \mathbb{Z}$ and the natural inclusion*

$$\mathcal{H}^0_\delta \mathrm{gr}^W_k BK(\mathcal{M}) = \ker \mathrm{gr}^W_k \delta_0 \to \mathrm{gr}^W_k \mathrm{gr}^V_0 K(\mathcal{M})$$

is a filtered quasi-isomorphism, where $BK(\mathcal{M})$ is defined in (3.7.4).

Proof of the lemma. We first prove that the inclusion

$$\ker \mathrm{gr}^W \delta_0 \to \mathrm{gr}^W \mathrm{gr}^V_0 K(\mathcal{M})$$

is a bifiltered quasi-isomorphism. By Lemma 3.7.2, the double complex $\mathrm{gr}^W BK(\mathcal{M})$ decomposes into

$$\begin{array}{ccccccc}
\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_0 \mathcal{M} & \longrightarrow & (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-1} \mathcal{M})^r & \longrightarrow & \cdots & \longrightarrow & \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-r} \mathcal{M} \\
\downarrow & & \downarrow & & & & \downarrow \\
\bigoplus_{i=1}^r \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_0 \mathcal{M}(*Z_i) & \to & \bigoplus_{i=1}^r (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-1} \mathcal{M}(*Z_i))^r & \to & \cdots & \to & \bigoplus_{i=1}^r \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-r} \mathcal{M}(*Z_i) \\
\downarrow & & \downarrow & & & & \downarrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_0 \mathcal{M}(* \sum_{i=1}^r Z_i) & \to & (\mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-1} \mathcal{M}(* \sum_{i=1}^r Z_i))^r & \to & \cdots & \to & \mathrm{gr}^W \mathrm{gr}^L \mathrm{gr}^V_{-r} \mathcal{M}(* \sum_{i=1}^r Z_i)
\end{array}$$

where L is the filtration induced by the weight filtration on $K(\mathcal{M})$. Since the category of polarizable Hodge modules on an algebraic variety is semisimple, the cohomology $\mathcal{H}^\ell \mathrm{gr}^L K(\mathcal{M})$ is a summand of $\mathrm{gr}^L K^\ell(\mathcal{M})$. It follows that $\mathrm{gr}^W \mathrm{gr}^V_0 \mathcal{H}^\ell \mathrm{gr}^L K(\mathcal{M})$ is contained in $\mathcal{H}^\ell \ker \mathrm{gr}^W \delta_0$ because the support of $\mathrm{gr}^W \mathrm{gr}^V_0 \mathcal{H}^\ell \mathrm{gr}^L K(\mathcal{M})$ is contained in Z. Then due to the fact that

$$\mathrm{gr}^W \mathrm{gr}^V_0 \mathcal{H}^\ell \mathrm{gr}^L K(\mathcal{M}) \to \mathcal{H}^\ell \ker \mathrm{gr}^W \delta_0$$

is injective, we conclude that $\ker \mathrm{gr}^W \delta_0 \to \mathrm{gr}^W \mathrm{gr}^V_0 K(\mathcal{M})$ is an isomorphism.

Next, we prove that the complex $\mathcal{H}^\ell_\delta \mathrm{gr}^W_k BK(\mathcal{M})$ is exact for $\ell > 0$. By Theorem 3.7.1, the total complex of $\mathrm{gr}^W BK(\mathcal{M})$ decomposes into

$$\bigoplus_{\ell\in\mathbb{Z}} \mathcal{H}^\ell_\delta \mathrm{gr}^W BK(\mathcal{M})[-\ell].$$

On the other hand, since $\mathrm{gr}^W B^i K(\mathcal{M})$ is filtered exact for all $i > 0$, the total complex of $\mathrm{gr}^W BK(\mathcal{M})$ is filtered quasi-isomorphic to $\mathrm{gr}^W \mathrm{gr}^V_0 K(\mathcal{M})$ which is also filtered quasi-isomorphic to $\mathcal{H}^0_\delta \mathrm{gr}^W BK(\mathcal{M})$ as we just proved. This completes the proof of the lemma. □

Returning to the proof of the theorem, we have a weight spectral sequence on $BK^j(\mathcal{M})$

$$E_1^{p,q} = \mathcal{H}^{p+q}_\delta \mathrm{gr}^W_{-p} BK^j(\mathcal{M}) \Rightarrow E_\infty^{p,q} = \mathrm{gr}^W_{-p} \mathcal{H}^{p+q}_\delta BK^j(\mathcal{M}).$$

which degenerates at $E_2^{p,q}$ by Theorem 3.7.1. The differential of the first page of the spectral sequence induces morphisms of complexes

$$S_{k,\ell} = \{\mathcal{H}^0_\delta \mathrm{gr}^W_{k+\ell} BK(\mathcal{M}) \to \mathcal{H}^1_\delta \mathrm{gr}^W_{k+\ell-1} BK(\mathcal{M}) \to \cdots \to \mathcal{H}^r_\delta \mathrm{gr}^W_{k+\ell-r} BK(\mathcal{M})\}$$

for any $\ell \in \mathbb{Z}$. By the above lemma, the total complex of $S_{k,\ell}$ is filtered isomorphic to $\mathcal{H}^0_\delta \mathrm{gr}^W_{k+\ell} BK(\mathcal{M})$ and thus, $\mathrm{gr}^W_{k+\ell} \mathrm{gr}^V_0 K(\mathcal{M})$. On the other hand, because of Theorem 3.7.1, the second page of the weight spectral sequence on $B(\mathcal{N})$ is zero if one of the x_i acts bijectively on a graded polarizable mixed Hodge module $\mathcal{N}$. This means $S_{k,\ell}$ is also filtered isomorphic to the first page of the weight spectral sequence of $B(\mathcal{M})$:

$$\mathcal{H}^0_\delta \mathrm{gr}^W_{k+\ell} B(\mathcal{M}) \to \mathcal{H}^1_\delta \mathrm{gr}^W_{k+\ell-1} B(\mathcal{M}) \to \cdots \to \mathcal{H}^r_\delta \mathrm{gr}^W_{k+\ell-r} B(\mathcal{M}),$$

which is filtered isomorphic to $\mathrm{gr}^W_{k+\ell} \mathrm{gr}^V_0 K(\mathcal{M})$. If we take cohomology at degree ℓ, we conclude that

$$\mathrm{gr}^W_k \mathcal{H}^\ell B(\mathcal{M}) \simeq \mathrm{gr}^W_{k+\ell} \mathcal{H}^\ell K(\mathcal{M})$$

as polarizable Hodge modules.

2. We deal with the complex $C(\mathcal{M})$. The proof of the following lemma is parallel to the one of Lemma 3.7.8 and therefore, we leave it to the readers.

Lemma 3.7.9. *The complex $\mathcal{H}^\ell_\delta \mathrm{gr}^W_k CK_!(\mathcal{M})$ is exact for $\ell \neq 0$ and any $k \in \mathbb{Z}$ and the natural quotient*

$$\mathrm{gr}^W_k \mathrm{gr}^V_0 K_!(\mathcal{M}) \to \mathcal{H}^0_\delta \mathrm{gr}^W_k CK_!(\mathcal{M}) = \mathrm{coker}\mathrm{gr}^W_k \delta_{-1}$$

is a filtered quasi-isomorphism.

We also have a weight spectral sequence

$$E_1^{p,q} = \mathcal{H}^{p+q}_\delta \mathrm{gr}^W_{-p} CK^j_!(\mathcal{M}) \Rightarrow E_\infty^{p,q} = \mathrm{gr}^W_{-p} \mathcal{H}^{p+q}_\delta CK^j_!(\mathcal{M}).$$

which degenerates at the second page by Theorem 3.7.3. The differential of the first page of the spectral sequence induces morphisms of complexes

$$T_{k,\ell} = \{\mathcal{H}^{-r}_\delta \mathrm{gr}^W_{k-\ell+r} CK_!(\mathcal{M}) \to \mathcal{H}^{-r+1}_\delta \mathrm{gr}^W_{k-\ell+r-1} CK_!(\mathcal{M}) \to \cdots \to \mathcal{H}^0_\delta \mathrm{gr}^W_{k-\ell} CK_!(\mathcal{M})\}$$

for any $\ell \in \mathbb{Z}$. By the above lemma, the total complex of $T_{k,\ell}$ is filtered isomorphic to $\mathcal{H}^0_\delta \mathrm{gr}^W_{k-\ell} CK_!(\mathcal{M})$ and thus, $\mathrm{gr}^W_{k-\ell} \mathrm{gr}^V_0 K_!(\mathcal{M})$. On the other hand, because of Theorem 3.7.3, the second page of weight spectral sequence on $B(\mathcal{N})$ is zero if $\mathcal{N} = \mathcal{N}(!Z)$. This means $T_{k,\ell}$ is also filtered isomorphic to the first page of the weight spectral sequence of $C(\mathcal{M})$:

$$\mathcal{H}^{-r}_\delta \mathrm{gr}^W_{k-\ell+r} C(\mathcal{M}) \to \mathcal{H}^{-r+1}_\delta \mathrm{gr}^W_{k-\ell+r-1} C(\mathcal{M}) \to \cdots \to \mathcal{H}^0_\delta \mathrm{gr}^W_{k-\ell} C(\mathcal{M})$$

which is filtered isomorphic to $\mathrm{gr}^W_{k-\ell} \mathrm{gr}^V_0 K_!(\mathcal{M})$. If we take cohomology at degree $-\ell$, we conclude that

$$\mathrm{gr}^W_k \mathcal{H}^{-\ell} C(\mathcal{M}) \simeq \mathrm{gr}^W_{k-\ell} \mathcal{H}^{-\ell} K_!(\mathcal{M})$$

as polarizable Hodge modules. □

3.7.4 Deligne's theorem

The aim of this part is to prove Lemma 3.7.2. For this purpose, we generalize, with little effort, the theorem on relative monodromy filtrations to the abstract setting, proved by Deligne in his personal letter to Cattani and Kaplan. Then Lemma 3.7.2 will be an immediate corollary.

Let $\mathcal{A}$ be an abelian category and V be an object in $\mathcal{A}$. Let L be a finite increasing filtration of V and N be a nilpotent endomorphism preserving the filtration L. We will now assume that the relative weight filtration $W = W(N, L)$ exists and that there is a splitting operator Y for W, i.e. Y is a semisimple operator on V with eigenvalues in $\mathbb{Z}$ such that $W_k = \bigoplus_{i \le k} E_i(Y)$ where $E_i(Y)$ is the i-eigenspace of Y. We say the splitting operator Y satisfies the *admissibility conditions* if

$$[Y, N] = -2N, \quad \text{and} \quad YL_i \subset L_i, \quad \text{for all } i. \tag{3.7.7}$$

Suppose that Y' is a splitting operator for L that commutes with Y. Then the pair $(N_0, Y - Y')$ determines an $\mathfrak{sl}_2$-representation on V. We will denote the standard $\mathfrak{sl}_2$-triple by (e^+, e^-, H):

$$[e^+, e^-] = H, \quad [H, e^-] = -2e^-, \quad [H, e^+] = 2e^+.$$

Then $e^- = N_0$ and $H = Y - Y'$. We call the collection (V, L, N, Y, Y') a *Deligne-system*, a notion introduced in [Sch01], if in addition

$$[e^+, N_j] = 0, \quad \text{for all } j \ne 0$$

where N_j is the j-th $\operatorname{ad} Y'$-homogenous component of N. In other words, N_j is $\operatorname{ad} e^-$-primitive in the adjoint representation for $j \ne 0$.

Theorem 3.7.10. *Let (V, N, L, Y) be as above and assume Y satisfies the admissibility condition* (3.7.7)*. If the set of splitting operators of L commuting with Y is not empty then there exists a unique splitting operator Y' of L such that (V, L, N, Y, Y') is a Deligne-system.*

Proof. Fix a splitting operator of L commuting with Y. We can modify the splitting of L by conjugating by an automorphism g such that g respects W and $(g - 1)L_i \subset L_{i-1}$, and consequently, g induces an automorphism on gr^L. We want to achieve that

$$\left[N - ge^-g^{-1}, ge^+g^{-1}\right] = 0,$$

or equivalently,

$$\left[g^{-1}Ng - e^{-}, e^{+}\right] = 0. \tag{3.7.8}$$

We find g by successive approximations: if $[N_i, e^+] = 0$ for $0 > i > -k$, we take $g = 1 + \gamma_{-k}$ for γ_{-k} of degree $-k$ with respect to the L-grading for $k \geq 1$. Then to make the k-th L-degree in (3.7.8) valid, we need

$$[-[\gamma_{-k}, e^-] + N_{-k}, e^+] = 0,$$

which is equivalent to

$$(\operatorname{ad} e^+)\left((\operatorname{ad} e^-)(\gamma_{-k}) + N_{-k}\right) = 0. \tag{3.7.9}$$

As $k - 2 \geq -1$, we can write uniquely $N_{-k} = N' + (\operatorname{ad} e^-)N''$, by the Lefschetz decomposition, such that N' is in the kernel of $\operatorname{ad} e^+$ and the $\operatorname{ad} H$-degree of N'' is k because N_{-k} is of $\operatorname{ad} H$-degree $k-2$. Then (3.7.9) becomes

$$(\operatorname{ad} e^+)(\operatorname{ad} e^-)(\gamma_{-k} + N'') = 0.$$

It follows from the fact that the H-degree of $\gamma_{-k} + N''$ is k that γ_{-k} has to equal $-N''$. It remains to show that $[\gamma_{-k}, Y] = 0$, i.e $[N'', Y] = 0$. By the admissible condition,

$$(\operatorname{ad} Y)N_{-k} = -2N_{-k}.$$

Substituting N_{-k} by $N' + (\operatorname{ad} e^-)N''$,

$$(\operatorname{ad} Y)N' + (\operatorname{ad} Y)(\operatorname{ad} e^-)N'' = (\operatorname{ad} Y)N' + (\operatorname{ad} e^-)(\operatorname{ad} Y)N'' - 2(\operatorname{ad} e^-)N'' = -2N' - 2(\operatorname{ad} e^-)N'.$$

Then we get

$$(\operatorname{ad} Y + 2)N' + (\operatorname{ad} e^-)(\operatorname{ad} Y)N'' = 0.$$

Applying $(\operatorname{ad} e^-)^{k-1}$ yields

$$(\operatorname{ad} e^-)^k(\operatorname{ad} Y)N'' = 0,$$

which forces $(\operatorname{ad} Y)N'' = 0$. This completes proof. □

The *morphisms* of a pair of Deligne-systems (V, L, N, Y, Y') and $(\widehat{V}, \widehat{L}, \widehat{N}, \widehat{Y}, \widehat{Y}')$ are the operators $T \in \mathrm{Hom}(V, \widehat{V})$ such that $\widehat{Y}T = TY$, $\widehat{N}T = TN$ and $TL \subset \widehat{L}$ for all i. In fact, the morphisms of Deligne-systems are functoral:

Corollary 3.7.11. *If T is a morphism of a pair of Deligne-systems*

$$(V, L, N, Y, Y') \quad \textit{and} \quad (\widehat{V}, \widehat{L}, \widehat{N}, \widehat{Y}, \widehat{Y}'),$$

then $\widehat{Y}'T = T\widehat{Y}'$.

Proof. Let $T = \sum_{i \le 0} T_i$ be the $\mathrm{ad}\, Y'$-homogenous decomposition of T. Then the H degree of T_i is $-i$ because $\widehat{Y}T = TY$. Suppose that T_i vanishes for $i = -1, 2, \ldots, -k+1$. Then $(\mathrm{ad}\, N)T = 0$ gives

$$[N_0, T_{-k}] + [N_{-k}, T_0] = 0.$$

It follows that $(\mathrm{ad}\, e^+)(\mathrm{ad}\, e^-)T_{-k}$ vanishes since

$$(\mathrm{ad}\, e^+)(\mathrm{ad}\, e^-)T_{-k} = [e^+, [e^-, T_{-k}]] = [e^+, [T_0, N_{-k}]] = [[e^+, T_0], N_{-k}] + [T_0, [e^+, N_{-k}]]$$

and $[e^+, T_0] = [e^+, N_{-k}] = 0$. Then T_{-k} must vanish because the H-degree of T_{-k} is $k > 0$. □

Finally we can give

Proof of Lemma 3.7.2. By [Sai90, p. 1.5], we have a canonical splitting

$$\mathrm{gr}^W_k \mathrm{gr}^V_\alpha \mathcal{M} \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{gr}^W_k \mathrm{gr}^L_i \mathrm{gr}^V_\alpha \mathcal{M}.$$

If we set $(V, L, N) = (\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, L\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, \theta - \alpha)$ and $Y = i$ on $\mathrm{gr}^W_i \mathcal{M}$, then we can apply Theorem 3.7.10 to this situation: there exists a unique splitting operator Y' for L such that (V, L, N, Y, Y') is a Deligne-system. As a consequence, for any local defining equation f of Z, it follows from Corollary 3.7.11 the induced morphism

$$f : \mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M} \to \mathrm{gr}^W \mathrm{gr}^V_{\alpha-1} \mathcal{M}$$

commute the splitting operator Y' which concludes (a).

For part (b), it is easy to see that the morphism $\mathrm{gr}^W T$ is a morphism of Deligne's systems $(\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, L\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}, \theta-\alpha)$ and $(\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}', L\mathrm{gr}^W \mathrm{gr}^V_\alpha \mathcal{M}', \theta-\alpha)$. Then by Corollary 3.7.11, $\mathrm{gr}^W T$ commutes with the splitting operator Y' which concludes (b). □

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