SYZ for Index 1 Fano Hypersurfaces in the Projective Space

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Mohamed El Alami

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Mohamed El Alami

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Mark McLean - Dissertation Advisor
Associate Professor, Department of Mathematics

Aleksey Zinger - Chairperson of Defense
Professor, Department of Mathematics

Kenji Fukaya - Member
Professor, Department of Mathematics

Mohammed Abouzaid - Outside Member
Professor, Department of Mathematics, Columbia University

This dissertation is accepted by the Graduate School.

Eric Wertheimer

Dean of the Graduate School

Abstract of the Dissertation

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We study aspects of homological mirror symmetry for the singular hypersurface $X_0 = V(t^{n+1} - x_0 \cdots x_n) \subseteq \mathbb{P}^{n+1}$. Following an SYZ type approach, we produce a Landau-Ginzburg model, whose Fukaya-Seidel category recovers line bundles on X_0. As a byproduct of our approach, we answer a conjecture of N. Sheridan about split-generating the small component of the Fukaya category of the *smooth* index 1 Fano hypersurface in $\mathbb{P}^{n+1}$, without bounding co-chains.

Table of Contents

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Chapter 1

Introduction

In his seminal work [She16], N. Sheridan studied homological mirror symmetry for all Fano hypersurfaces X_d of degree d in projective space $\mathbb{P}^{n+1}$, where $1 \leq d \leq n+1$. When d is fixed, all such hypersurfaces are symplectomorphic and that constitutes the A-side. The B-side in his work is a Landau-Ginzburg model (Y_d, W_d), and the main theorem of [She16] is an exact equivalence of triangulated categories:

$$\mathrm{D}^{\pi}\mathrm{Fuk}(X_d) \cong \mathrm{D}^{b}_{\mathrm{sing}}(Y_d, W_d). \tag{1.1}$$

The key component of (1.1) is a chain of Lagrangian spheres in X_d that N. Sheridan constructs building upon his earlier work in [She11; She15]. As these Lagrangians are geometrically rigid, he resorts to studying their algebraic deformations using weak bounding co-chains in order to compute the mirror LG-model (Y_d, W_d).

In the present work, we mostly investigate the other direction of mirror symmetry, i.e when X_d is the B-side. In doing so, we explore a more direct approach following the lines of Strominger-Yau-Zaslow. We limit our attention to the index 1 Fano case, i.e. $d = n+1$.

There is a simple construction of a *partial* SYZ-fibration on $X_{n+1} \subseteq \mathbb{P}^{n+1}$, which is obtained by projecting away from a point to a hyperplane $\mathbb{P}^n \subseteq \mathbb{P}^{n+1}$. When the branch locus is sufficiently close to the toric boundary of $\mathbb{P}^n$, one can lift some of the Clifford tori $L_{\mathrm{cl}} \subseteq \mathbb{P}^n$ to Lagrangian tori $L \subseteq X_{n+1}$. Though it is partial, this fibration can be made arbitrarily large by pushing the branch locus of the projection closer to the toric boundary of $\mathbb{P}^n$. At its limit, this process degenerates X_{n+1} to a singular toric hypersurface X_0, which has the following defining equation in homogeneous coordinates:

$$X_0 = V(t^{n+1} - x_0 \cdots x_n) \subseteq \mathbb{P}^{n+1}.$$

Our first main result is a computation of the super-potential $W : (\mathbb{C}^*)^n \to \mathbb{C}$ associated with this partial SYZ-fibration. The Laurent polynomial W packages all the counts of holomorphic discs in X_{n+1}, bounded by Lagrangian fibers, and whose of Maslov index is 2.

Theorem 1.1. *There is a partial SYZ-fibration on X_{n+1} whose associated super-potential has the formula:*

$$W = \frac{(1 + y_1 + \cdots + y_n)^{n+1}}{y_1 \cdots y_n} - (n+1)!.$$

We note that, up to the $-(n+1)!$ translation term, this result agrees with the expected Hori-Vafa mirror for the toric hypersurface X_0.

Our counts of Maslov index 2 discs shed some light on a question regarding the HMS equivalence in (1.1). To put it in context, recall that $\mathrm{D}^\pi\mathrm{Fuk}(X_{n+1})$ splits into components corresponding with the eigenvalues of quantum multiplication by $c_1(TX_{n+1})$. There are two such eigenvalues: A small one w_s which is a non-degenerate singularity in the mirror, and a big one w_b which is a more complicated singularity. The statement in (1.1) is therefore made of an equivalence over the small eigenvalue (also called the small component), and another one over the big eigenvalue (similarly called the big component). Sheridan's Lagrangian spheres naturally see the big component, and there they generate. However, in order to get them to see the small component, they require algebraic deformations using weak bounding co-chains. At the end of his paper [She16], conjecture B.2, the author contemplates the possibility of covering the small component using honest monotone Lagrangians without bounding co-chains. It turns out that the *partial* SYZ fibration we produce has a central monotone fiber, and we use it to show the following result.

Theorem 1.2. *The smooth index* 1 *Fano hypersurface* $X_{n+1} \subseteq \mathbb{P}^{n+1}$ *contains a monotone Lagrangian torus that split-generates the small component of its Fukaya category.*

Note that the case $n = 2$ of the theorem above has been established in the work of D. Tonkonog and J. Pascaleff on Lagrangian mutations, see [PT20].

Next, we view the super-potential W as the A-side, and we study a homological mirror symmetry correspondence between the singular toric limit X_0, and the Landau-Ginzburg model $((\mathbb{C}^*)^n, W)$. Our results in this direction can be summarized as follows.

Theorem 1.3. *There is a collection of Lefschetz thimbles* L_i *in* $((\mathbb{C}^*)^n, W)$ *such that:*

$$HW(L_i, L_j) \simeq \hom_{X_0}(\mathcal{O}_{X_0}(i), \mathcal{O}_{X_0}(j)).$$

Furthermore, the isomorphisms above are compatible with the relevant product structures.

Our approach relies on understanding how the branched covering map $\phi : X_0 \to \mathbb{P}^n$ corresponds (under mirror symmetry) to an *unbranched* quotient map:

$$\pi : ((\mathbb{C}^*)^n, W_{\text{cl}}) \to ((\mathbb{C}^*)^n, W),$$

where W_{cl} is the super-potential associated with a Clifford torus in projective space $L_{\text{cl}} \subseteq \mathbb{P}^n$.

Outline of the thesis. In chapter 2, we recall some facts about Maslov classes, their behavior with respect to anti-canonical divisors and branched coverings. We use these ideas to construct a monotone Lagrangian torus L in a *nearby* smoothing of the singular hypersurface X_0. In chapter 3, we compute the super-potential W associated with L. This computation has two parts. First, we make an educated *guess* of the correct count $m_{0,\beta}(L)$, by mapping the relevant Maslov index 2 discs down to projective space $\mathbb{P}^n$, using the cyclic covering map $\phi : X_0 \to \mathbb{P}^n$. Then, we explain a transversality argument that confirms that our guesses are indeed actual counts of Fredholm regular curves. In chapter 4, we view the smooth index 1 Fano hypersurface as the A-side, and we compute Fukaya's A_∞-algebra associated with the monotone Lagrangian torus L. We show in particular that L split-generates the small component. In chapter 5, the super-potential W is placed on the A-side. We compute the (partially) wrapped Floer cohomology of Lagrangian thimbles in the Fukaya-Seidel category associated with W, and we explain how they correspond with line bundles on X_0.

Chapter 2

Construction of Lagrangian tori

2.1 topological preliminaries

2.1.1 Intersection numbers

We begin by recalling and setting notation for intersection numbers, as this will be used extensively throughout this section. Let (X, ω_X) be a smooth compact Kähler manifold and let $D_X \subseteq X$ be a divisor. We always think of D_X as the zero locus of a section $s \in \Gamma(X, \mathscr{L})$ of a holomorphic line bundle $\mathscr{L} \to X$. Let $u : \Sigma \to X$ be a smooth map from a compact Riemann surface Σ such that

$$u(\partial\Sigma) \cap D_X = \emptyset. \tag{2.1}$$

The intersection number $u \cdot D_X$ is defined to be the signed counted of zeroes of the restriction u^*s of the section s to Σ. This may require a small perturbation of u to ensure that the pullback u^*s is transverse to the zero section of $u^*\mathscr{L} \to \Sigma$. This intersection number does not change under homotopies of u that preserve the boundary condition (2.1). When the Riemann surface Σ has no boundary, the intersection number has the following integral formula:

$$u \cdot D_X = \langle c_1(\mathscr{L}), [u] \rangle.$$

As an example of how these intersection numbers work, we present a quick proof of the Riemann-Hurwitz theorem. Let $\phi : X \to Y$ be a finite map between smooth projective varieties ramified along $R \subseteq X$. Let $u : \Sigma \to X$ be a holomorphic map from a closed Riemann surface Σ. Then

$$c_1^X(u) = c_1^Y(\phi \circ u) - u \cdot R,$$

where $c_1^X = c_1(TX)$ is the first Chern class of the tangent bundle and

$$c_1^X(u) = \int_\Sigma u^* c_1^X.$$

Indeed, the ramification locus is the zero set of the section $\wedge^n d\phi$ of the line bundle:

$$\mathscr{L} = (\wedge^n TX)^{-1} \otimes \wedge^n \phi^* TY.$$

Therefore

$$u \cdot R = \langle c_1(\wedge^n TX \otimes (\wedge^n \phi^* TY)^{-1}), u \rangle$$

and the classical Riemann-Hurwitz formula follows.

This formula has a relative analogue as well. Let $L \subseteq X$ and $K \subseteq Y$ be totally real sub-manifolds such that $R \cap L = \emptyset$ and $\phi(L) \subseteq K$. Let $u : \Sigma \to X$ be a map from a Riemann surface with boundary Σ such that $u(\partial\Sigma) \subseteq L$. Then

$$\mu_L^X(u) = \mu_K^Y(\phi \circ u) - 2u \cdot R,$$

where μ is the Maslov class, which we will recall soon. The proof is identical.

The next lemma will be used implicitly in our calculations. The proof is a direct application of (and in fact the reason we recalled) the definition of intersection numbers.

Lemma 2.1. *Let $\phi : X \to Y$ be a finite morphism of projective varieties with branch locus B, and let $D_X = \phi^{-1}(B)$ be its (possibly non-reduced) pre-image. Then, for any disc map $u : (D, \partial D) \to X$ with $u(\partial D) \cap D_X = \emptyset$, we have:*

$$u \cdot D_X = (\phi \circ u) \cdot B.$$

□

2.1.2 Maslov numbers

Let (X, ω_X) be a Kähler manifold. The *primary* Maslov class associated with an oriented totally real subspace $L \subseteq X$, is a $\mathbb{Z}$-module homomorphism

$$\mu_L^X : H_2(X, L) \to \mathbb{Z}.$$

Given a map $u : (\Sigma, \partial\Sigma) \to (X, L)$, it is defined as a relative Euler characteristic:

$$\mu_L^X(u) = \chi((\wedge_{\mathbb{C}}^n u^* TX)^{\otimes 2}, (\wedge_{\mathbb{R}}^n u^* TL)^{\otimes 2}).$$

This means counting zeros of a generic section of the complex line bundle $(\wedge_{\mathbb{C}}^n u^* TX)^{\otimes 2}$ over Σ, whose restriction to $\partial\Sigma$ belongs to the real sub-bundle $(\wedge_{\mathbb{R}}^n u^* TL)^{\otimes 2}$. In particular, when u is the class of a closed Riemann surface, the Maslov number is twice the Chern number:

$$\mu_L^X(u) = 2\langle c_1(X), [u]\rangle.$$

In our context, it will be equally important to consider a *secondary* Maslov class:

$$\eta_L^{X,H} : H_1(L, \mathbb{Z}) \to \mathbb{Q}. \tag{2.2}$$

This one is more relevant in the complement of a hypersurface $H \subseteq X$, that is a multiple of an anti-canonical divisor. In other words,

$$\mathcal{O}(H) = K_X^{-N}$$

for some positive integer N. To construct the class η, we need to choose a smooth trivialization s of $K_{X\backslash H}^{-N}$ and an orientation n-form α for L. After comparing s and α using the embedding $L \hookrightarrow X$, we obtain an argument function $\arg_L : L \to \mathbb{C}^*$ defined by

$$\arg_L(x) = \alpha_x^{\otimes N} / s_x. \tag{2.3}$$

Set $A = X \backslash H$, then the *secondary* Maslov class of the pair (A, L), viewed as a cohomology element $\eta_L^A \in H^1(L, \mathbb{Q})$ is

$$\eta_L^A = \frac{2}{N} \arg_L^*(d\theta).$$

Note that the compactification X of A plays no role in the construction so far; all we needed is an affine variety A whose first Chern class is torsion. Assuming L is connected, the class we have constructed depends only on the choice of the trivialization s, which is sometimes called a *grading* for A (see [Sei08], for instance).

Lemma 2.2. *Let $\phi : A \to B$ be an unbranched covering map of smooth affine varieties such that $Nc_1(B) = 0$, for some positive integer N. Let $L \subseteq A$ and $K \subseteq B$ be totally real sub-manifolds such that $\phi(L) \subseteq K$. Then,*

$$\phi^*(\eta_K^B) = \eta_L^A$$

for appropriately chosen trivializations.

Proof. Any trivialization of K_B^{-N} can be pulled back to a trivialization for K_A^{-N}, and the same goes for the orientations of K. With such choices, we ensure that $\arg_L = \phi^* \arg_K$, and the lemma follows. $\square$

In the presence of a compactification (X, H) of the variety $A = X \backslash H$, such that $H_1(X) = 0$, it is possible to arrange for η to be choice-independent. Instead of a trivialization of K_A^{-N}, one chooses a smooth section s of K_X^{-N} that is nowhere vanishing on A. The previous construction now results in a choice-independent Maslov class (2.2). This is made evident by the next result.

Lemma 2.3. *Let X be a smooth projective variety, $H \subseteq X$ a hypersurface, and $L \subseteq X$ an oriented totally real submanifold such that $L \cap H = \emptyset$. Furthermore, assume that there exists a natural number N such that*

$$\mathcal{O}(H) = K_X^{-N}.$$

Then for any disc $u : (D, \partial D) \to (X, L)$,

$$\mu_L^X(u) = \eta_L^{X,H}(\partial u) + \frac{2}{N} u \cdot H.$$

Proof. We start by fixing an orientation form α for L. Let s be a smooth section of K_X^{-N} vanishing along H. Then, the secondary Maslov number of a disc $u : (D, \partial D) \to (X, L)$ is:

$$\eta_L^{X,H}(\partial u) = \frac{2}{N} \deg(\arg_L \circ \partial u : \partial D \to \mathbb{C}^*).$$

Let $\arg_L^u : D \to \mathbb{C}$ be an extension of $\arg_L \circ \partial u$. Then $\arg_L^u \cdot s$ is a relative section of the bundle pair:

$$((\wedge_{\mathbb{C}}^n u^* TX)^{\otimes N}, (\wedge_{\mathbb{R}}^n u^* TL)^{\otimes N}).$$

It follows that:

$$\begin{aligned} \mu_L^X(u) &= \frac{2}{N} \chi((\wedge_{\mathbb{C}}^n u^* TX)^{\otimes N}, (\wedge_{\mathbb{R}}^n u^* TL)^{\otimes N}) \\ &= \frac{2}{N} \#(\arg_L^u \cdot s)^{-1}(0) \\ &= \frac{2}{N} \left(\deg(\arg_L \circ \partial u : \partial D \to \mathbb{C}^*) + u \cdot H\right). \end{aligned}$$

The Maslov number formula then follows. □

Remark 2.4. *Most of this section's content has appeared in the literature before. For example:*

- *When defining the secondary Maslov class, the choice of trivialization of (a multiple of) the canonical bundle is called a* grading*, and the construction we describe appears for example in [Sei08].*

- *The Maslov number formula we produced also has analogues in the literature pertaining to mirror symmetry in complements of anti-canonical divisors. It appears for instance in [Aur07].*

2.2 Monotone Lagrangian tori in branched covers

2.2.1 Maslov numbers and branched covers

Let X be an n-dimensional smooth projective variety. We assume it is given as a finite covering

$$\phi : X \to \mathbb{P}^n \text{ such that } \phi^* \mathcal{O}_{\mathbb{P}^n}(1) = m K_X^{-1}. \tag{2.4}$$

Let $B \subseteq \mathbb{P}^n$ be the branch locus of ϕ and $D_X = \phi^{-1}(B)$ its (possibly non-reduced) pre-image. Let $L \subseteq \mathbb{P}^n$ be a totally real torus that is disjoint from the branch locus and let $L_X \subseteq X$ be (a component of) its pre-image. Then L_X is itself a totally real torus.

Lemma 2.5. *For any disc map $u : (D, \partial D) \to (X, L_X)$,*

$$\mu^X_{L_X}(u) = \mu^{\mathbb{P}^n}_L(v) - \frac{2}{\deg(B)}\left(n+1-\frac{1}{m}\right) v \cdot B, \tag{2.5}$$

where $v = \phi \circ u : (D, \partial D) \to (\mathbb{P}^n, L)$ is the pushforward of u by the covering map ϕ.

Proof. Using the results of Lemmas 2.2-2.3 (applied both to X and to $\mathbb{P}^n$):

$$\begin{aligned}
\mu^X_{L_X}(u) &= \eta^{X,D_X}_{L_X}(\partial u) + \frac{2}{m\deg(B)} u \cdot D_X \\
&= \eta^{\mathbb{P}^n,B}_{L}(\partial v) + \frac{2}{m\deg(B)} v \cdot B \\
&= \mu^{\mathbb{P}^n}_L(v) - \frac{2(n+1)}{\deg(B)} v \cdot B + \frac{2}{m\deg(B)} v \cdot B.
\end{aligned}$$

Rearranging some of the terms results in the desired identity. □

2.2.2 Weakly monotone tori

So far, our discussion does not involve the Kähler structure. We keep it that way by introducing the notion of *weakly monotone* totally real sub-manifolds.

Definition 2.6. *Given a pair (X, D_X) of a smooth projective variety together with a hypersurface, we say that a totally real submanifold $L_X \subseteq X \backslash D_X$ is weakly monotone, if there is a rational number $\lambda \in \mathbb{Q}$ such that for any disc $u : (D, \partial D) \to (X, L_X)$,*

$$\mu^X_{L_X}(u) = 2\lambda u \cdot D_X.$$

Remark 2.7. *This is analogous to L_X being monotone with respect to a Kähler form that is a Dirac-Delta along D_X.*

The previous definition only makes sense when D_X is (numerically) a multiple of the anti-canonical class. If so, the constant λ must be the inverse of the said multiple.

A known way of obtaining weakly-monotone Lagrangians comes from toric geometry. We take the example of $(\mathbb{P}^n, \omega_{\mathrm{FS}})$ with homogeneous coordinates $[x_0 : x_1 : \cdots : x_n]$, equipped with the Fubini-Study Kähler form. It admits a Hamiltonian action by the n-torus which is free in the complement of a union of $n+1$ hyperplanes

$$H = \bigcup_{i=0}^{n} \{x_i = 0\}.$$

The moment map of this Hamiltonian action is

$$\begin{aligned} M : \mathbb{P}^n &\to \Delta \\ [x_0 : \cdots : x_n] &\mapsto \left(\frac{|x_0|^2}{|x|^2}, \ldots, \frac{|x_n|^2}{|x|^2} \right), \end{aligned} \tag{2.6}$$

where $|x|^2 = |x_0|^2 + \cdots + |x_n|^2$ and $\Delta \subseteq \mathbb{R}^{n+1}$ is the n-dimensional simplex:

$$\Delta = \left\{ (r_0, \ldots, r_n) \in (\mathbb{R}_{\geq 0})^{n+1} \mid \sum_{i=0}^{n} r_i = 1 \right\}.$$

The fibers of the moment map over the interior of the simplex are Lagrangian tori parametrized by $\boldsymbol{r} \in \mathrm{int}(\Delta)$:

$$L_{\boldsymbol{r}} = \{[x_0 : \cdots : x_n] \mid r_0^{-1} |x_0| = \cdots = r_n^{-1} |x_n|\}.$$

All of these tori are totally real and weakly monotone in $(\mathbb{P}^n, H)$. To see that, we can use a generating set of group $H_2(\mathbb{P}^n, L_{\boldsymbol{r}})$, such as the collection of holomorphic discs given by

$$\begin{aligned} u_k(\boldsymbol{r}) : (D, \partial D) &\to (\mathbb{P}^n, L_{\boldsymbol{r}}) \\ z &\mapsto [r_0 : \cdots : r_k z : \cdots : r_n]. \end{aligned} \tag{2.7}$$

Note that these classes add up to the spherical class that generates $H_2(\mathbb{P}^n)$. They each have Maslov number 2 and intersect H exactly once. It follows that for all discs $[u] \in H_2(\mathbb{P}^n, L_{\boldsymbol{r}})$,

$$\mu_{L_{\boldsymbol{r}}}^{\mathbb{P}^n}(u) = 2u \cdot H.$$

The torus L_{cl} corresponding to $\boldsymbol{r} = (1, 1, \ldots, 1)$ is usually called the Clifford torus. It is the only one among these tori that is monotone with respect to the Fubini-Study Kähler form. When studying the Clifford torus, the holomorphic discs $u_k(\boldsymbol{r})$ will be denoted by u_k for ease of notation.

The previous construction of weakly monotone tori extends to other hypersurfaces $B \subseteq \mathbb{P}^n$ that are close to H.

Definition 2.8. *We call a hypersurface $B \subseteq \mathbb{P}^n$ nearly degenerate if it is a 'small' perturbation of H.*

We can actually quantify how small the perturbation needs to be. Let

$$f_0 = x_0 \cdots x_n \tag{2.8}$$

be the defining equation of H. A small perturbation would be a hypersurface $B_f = V(f)$ that is defined by an equation

$$f = f_0 + h,$$

where h is a homogeneous polynomial of degree $d = n + 1$, satisfying the following inequality:

$$|h(x_0 : \cdots : x_n)| < \frac{|x_0|^{n+1} + \cdots + |x_n|^{n+1}}{n+1}. \tag{2.9}$$

Lemma 2.9. *Suppose that $B_f \subseteq \mathbb{P}^n$ is a nearly degenerate hypersurface. Then, the Clifford torus L_{cl} is disjoint from B_f and is weakly monotone in $(\mathbb{P}^n, B_f)$.*

Proof. Indeed, if $[x_0 : \cdots : x_n]$ is an intersection point of B_f and L_{cl}, then:

$$\begin{aligned} |x_0|^{n+1} &= |x_0 \cdots x_n| \\ &= |h(x_0 : \cdots : x_n)| < |x_0|^{n+1} \end{aligned}$$

which is a contradiction. Therefore, $L_{\text{cl}} \cap B_f = \emptyset$. Next, we compute the intersection numbers $u_k \cdot B_f$ by counting the zeros of

$$f \circ u_k : D \to \mathbb{C}.$$

For any $z \in D$,

$$\begin{aligned} |f \circ u_k(z) - f_0 \circ u_k(z)| &= |h \circ u_k(z)| \\ &< \frac{|z|^{n+1} + n}{n+1}. \end{aligned}$$

When $z \in \partial D$, this inequality implies that

$$|f \circ u_k(z) - f_0 \circ u_k(z)| < |f_0 \circ u_k(z)| \, .$$

It follows (by Rouché's theorem) that $f \circ u_k(z)$ and $f_0 \circ u_k(z)$ have the same number of zeros and therefore,

$$\begin{aligned} \mu(u_k) &= 2u_k \cdot B_0 \\ &= 2u_k \cdot B_f. \end{aligned}$$

Because the collection of discs (u_k) generates the group $H_2(\mathbb{P}^n, L_{\mathrm{cl}})$, the statement of the lemma follows. $\square$

Suppose now that we have a finite map $\phi : X \to \mathbb{P}^n$ as in the setup of Lemma 2.5, whose branch locus B is nearly degenerate. Then, B is disjoint from L_{cl}. Let L_X be (a connected component of) its pre-image $\phi^{-1}(L_{\mathrm{cl}})$.

Lemma 2.10. *The totally real torus $L_X \subseteq (X, R)$ is weakly monotone, where $R = \phi^{-1}(B)$ is the (extended) ramification locus.*

Proof. Lemma 2.9 asserts that $L_{\mathrm{cl}} \subseteq (\mathbb{P}^n, B)$ is weakly monotone. Therefore, using the Maslov number formula from Lemma 2.5, we deduce that for any disc $u : (D, \partial D) \to (X, R)$,

$$\begin{aligned} \mu_L(u) &= \frac{2(n+1)}{\deg(B)} v \cdot B - \frac{2}{\deg(B)} \left(n + 1 - \frac{1}{m} \right) v \cdot B \\ &= \frac{2}{m \deg(B)} u \cdot R, \end{aligned}$$

where $v = \phi \circ u$. It follows that $L_X \subseteq (X, R)$ is weakly monotone. $\square$

2.2.3 Partial Lagrangian fibration

We now explain how to construct suitable Kähler structres on branched covers, so as to make the weakly monotone Lagrangians in our previous discussion into genuine monotone Lagrangians.

Lemma 2.11. *Let (Y, ω) be a Kähler variety with $[\omega] \in H^2(Y, \mathbb{Z})$, and let $\phi : X \to Y$ be a finite branched cover. Then, for any neighborhood U of the ramification locus, there exists a Kähler form ω_X on X, and a real valued function $\rho : X \to \mathbb{R}$ with support in U, such that*

$$\omega_X = \phi^*\omega + dd^c\rho.$$

Proof. Indeed $[\omega] = c_1(\mathcal{L})$ is the curvature of some ample line bundle $\mathcal{L} \to Y$ with respect to some Hermitian metric. Since ϕ is a finite morphism, the pullback $\phi^*\mathcal{L} \to X$ is necessarily ample. Therefore, it admits a positively curved Hermitian metric of its own. Its curvature 2-form ω_X is

$$\omega_X = \phi^*\omega + dd^c\psi,$$

where ψ is the rescaling from the metric we pullback from $\mathcal{L}$ to the new positively curved metric. Next, let U_1 be an open neighborhood of the ramification locus $R \subseteq X$ such that

$$\overline{U}_1 \subset U.$$

Choose a smooth function $f : X \to \mathbb{R}$ such that $f = 1$ on U_1 and $f = 0$ outside of U. We claim that there is a constant C such that:

$$\omega_{X,C} = \phi^*\omega + \frac{1}{C} dd^c(f\psi) > 0.$$

Indeed, as long as $C > 1$, the Hermitian 2-form $\omega_{X,C}$ is positive except possibly on $U \backslash U_1$: this is clear outside of U, while inside of U_1 it can be seen from the equation:

$$\omega_{X,C} = \left(1 - \frac{1}{C}\right) \phi^*\omega + \frac{1}{C}\left(\phi^*\omega + dd^c\psi\right).$$

Moreover, when C is sufficiently large, the closed form $\omega_{X,C}$ is also positive on the compact region $\overline{U \backslash U_1}$. □

Remark 2.12. *Versions of Lemma 2.11 appear in the literature, e.g. [She15], or [Aur00]. The advantage we have in our case is that we don't need to worry about the types of singularities of the branch locus.*

The symplectic form constructed in the Lemma 2.11 has the following key property. Let $L \subseteq X$ be a Lagrangian that is disjoint from the ramification locus. Then for any disc $u : (D, \partial D) \to (X, L)$, we have an area formula:

$$\mathrm{Area}_{\omega_X}(u) = \mathrm{Area}_{\omega}(\phi \circ u).$$

We now specialize all of our previous discussion to the case of an index 1 Fano hypersurface in projective space X.

Let f be a homogeneous polynomial of degree $n+1$ that is a *generic, small* perturbation of f_0 as in the setup of Lemma 2.9. The genericity assumption is to ensure that the zero locus $V(f) \subseteq \mathbb{P}^n$ is a smooth Calabi-Yau hypersurface. With such f, we associate the smooth projective variety

$$X_f = V(t^{n+1} - f) \subseteq \mathbb{P}^{n+1},$$

which is an index 1 Fano hypersurface of dimension n.

By projecting away from the point $[1 : 0 : \cdots : 0] \in \mathbb{P}^{n+1}$ onto the hyperplane

$$P = \{t = 0\} \cong \mathbb{P}^n,$$

we produce a finite map $\phi : X_f \to \mathbb{P}^n$ given by

$$\phi([t : x_0 : \cdots : x_n]) = [x_0 : \cdots : x_n]. \tag{2.10}$$

It is cyclic covering map of degree $n+1$ that is branched along the Calabi-Yau hypersurface $V(f) \subseteq \mathbb{P}^n$.

Lemma 2.13. *The pre-image $\phi^{-1}(L_{cl}) \subseteq X_f$ is an n-dimensional torus.*

Proof. By Lemma 2.9, the torus L_{cl} is disjoint from the branch locus. Therefore, it is enough to show its pre-image is connected. Indeed, away from the branch locus, the map ϕ restricts to an unbranched cyclic covering $\hat{\phi} : X_f \backslash R \to \mathbb{P}^n \backslash B$ of degree $n+1$. Since $B \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree $n+1$,

$$\pi_1(\mathbb{P}^n \backslash B) \cong \mathbb{Z}_{n+1}.$$

This is a classical result. As we have stated it, it is an application of Seifert-Van-Kampen's theorem and Poincaré duality. The isomorphism above is given by linking numbers

$$\pi_1(\mathbb{P}^n \backslash B) \to \mathbb{Z}_{n+1} : \quad \mathrm{lk}(\gamma) = u_\gamma \cdot B,$$

where $u_\gamma : D \to \mathbb{P}^n$ is a disc whose boundary is γ. It follows (using the discs from (2.7)) that the map

$$\pi_1(L_{\mathrm{cl}}) \to \pi_1(\mathbb{P}^n \backslash B)$$

is surjective. In particular, $L_{\mathrm{cl}} \subseteq \mathbb{P}^n \backslash B$ has a connected pre-image in the universal cover $X_f \backslash R$. □

Proposition 2.14. *Let $(X, \omega_X) \subseteq \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $n+1$, viewed as a monotone symplectic manifold. Then there are sequences of anti-canonical symplectic divisors $D_i \subseteq X$, and of open neighborhoods U_i of D_i with the following properties:*

- *$U_{i+1} \subseteq U_i$ and $Vol_{\omega_X}(U_i) \to 0$.*

- *$X \backslash U_i$ has a Lagrangian torus fibration with a monotone central fiber.*

Proof. We use the same setup preceding Lemma 2.13. We construct a nested sequence of open sets $V_{i+1} \subseteq V_i$, as the pre-images via the moment map (2.6), of a shrinking sequence of open neighborhoods of the toric boundary

$$P \cap V(f_0) \subseteq P \cong \mathbb{P}^n.$$

Then, the open sets $V_i \subseteq P$ shrink to $V(f_0) \cap P$, and the complements $P \backslash V_i$ are all fibered by Lagrangian tori for the Fubini-Study metric ω_P. Next, we construct a sequence f_i of regular homogeneous polynomials in $x_0, \ldots, x_n$ of degree $n+1$ converging to f_0, fast enough to ensure that $V(f_i) \cap P \subseteq V_i$. Define

$$X_i = V(t^{n+1} - f_i) \quad \text{and} \quad D_i = X_i \cap P.$$

By forgetting the t variable, we get a branched covering map $\phi_i : X_i \to P$ ramified along D_i, which is contained in the open set $U_i = \phi_i^{-1}(V_i)$. We then apply the construction of Kähler metrics in Lemma 2.11 to produce a Kähler form

$$\omega_i = \phi_i^* \omega_H + d\alpha_i,$$

where α is compactly supported in U_i. In particular,

$$\mathrm{Vol}_{\omega_i}(U_i) = (n+1)\mathrm{Vol}_{\omega_P}(V_i)$$

decreases to 0. Moreover, $X_i \backslash U_i$ is an unbranched covering of $H \backslash V_i$, and as such, it inherits a Lagrangian torus fibration (see Lemma 2.13 above). As a consequence of the Maslov number formula (2.5), and of our choice of the symplectic form, the lift of the monotone Clifford torus $L_{\mathrm{cl}} \subseteq (P, \omega_P)$ to (X_i, ω_i) will then be monotone. To complete the proof, one can use a Moser argument to trivialize the family (X_i, ω_i). $\square$

Because the Lagrangian fibration from the previous proposition covers most of the symplectic manifold X, we expect it to carry non-trivial Floer theoretic data to probe the mirror of X. The next section is dedicated to computing the super-potential associated with this partial Lagrangian fibration.

Chapter 3

Computation of the super-potential

3.1 Computation of the super-potential

Let $\mathcal{H}$ be the vector space of homogeneous polynomials of degree $n+1$ in the variables $x_0, \ldots, x_n$. Let $f \in \mathcal{H}$ be a generic and small perturbation of

$$f_0 = x_0 \cdots x_n.$$

We denote by X_f the index 1 Fano hypersurface associated with f, which is given as

$$X_f = V(t^{n+1} - f) \subseteq \mathbb{P}^{n+1}.$$

In Proposition 2.14, we used the cyclic quotient map (see (2.10))

$$\phi : X_f \to \mathbb{P}^n$$

to produce a Lagrangian torus fibration away from the ramification locus of ϕ. We now count (pseudo-)holomorphic discs of Maslov index 2, with boundary on a Lagrangian torus fiber L, and passing through a fixed point in L. This count does not actually depend on the choice of the torus fiber. We therefore choose L to be the monotone torus fiber. In the context of Proposition 2.14, it arises as the pre-image of the Clifford torus $L_{\mathrm{cl}} \subseteq \mathbb{P}^n$:

$$L = \phi^{-1}(L_{\mathrm{cl}}).$$

The strategy is to count pseudo-holomorphic discs in $\mathbb{P}^n$ with a prescribed tangency to the hypersurface $V(f)$.

3.2 Disc Endomorphisms

Let $\mathscr{E}_d(D)$ be the space of degree d maps $(D, \partial D) \to (D, \partial D)$. Note that we are referring here to the topological degree of v. It can be computed from the pullback

$$v^* : H^1(\partial D, \mathbb{Z}) \to H^1(\partial D, \mathbb{Z}),$$

or equivalently using the integral formula

$$\deg(v) = \int_{\partial D} v^*(d\theta).$$

Lemma 3.1. *Any element $v \in \mathscr{E}_d(D)$ is a product of d Möbius transformations*

$$v(z) = \xi \prod_{k=1}^{d} \left(\frac{z - a_k}{1 - \overline{a}_k z} \right),$$

where ξ is a unitary complex number and $a_k \in int(D)$, for $k = 1, \ldots, d$. The complex numbers (a_k) will often called the Möbius centers.

Proof. This result can be proved by induction on d. Because v is holomorphic, the topological degree formula above simplifies to

$$\deg(v) = \frac{1}{2\pi i} \int_{\partial D} \frac{v'(z)}{v(z)} dz.$$

When $\deg(v) = 0$, the argument principle then implies that $v(z) = 0$ has no solutions. We claim now that v must be constant. If it were not, the open mapping theorem would imply that $v(D)$ is an open subset of D. But D is compact, so $v(D)$ would also be closed and so $v(D) = D$; this is a contradiction.

The induction step goes as follows. Given v of degree $d \geq 1$, the argument principle implies that there exists $a \in D$ such that $v(a) = 0$. We now pre-compose v with the inverse

ϕ^{-1} of the Möbius transformation

$$\phi(z) = \frac{z-a}{1-\overline{a}z}.$$

The result is a disc endomorphism $g = v \circ \phi^{-1}$ with the property that $g(0) = 0$. Therefore, there exists a holomorphic function $h : D \to \mathbb{C}$ such that $g(z) = zh(z)$. Note h still restricts to a map $h : \partial D \to \partial D$. Since D is compact, $h(D)$ is compact, and by the maximum principle we have $\partial h(D) \subseteq h(\partial D)$. It follows that h is again a disc endomorphism whose degree is $d-1$. $\square$

One can extract a set of global coordinates on the space $\mathscr{E}_d(D)$ from the previous Lemma: The set of elementary symmetric polynomials on the *Möbius centers* (a_k), together with the angular coordinate ξ.

For each integer $d \geq 1$ and interior point $z_0 \in \mathrm{int}(D)$, there is well defined jet map $j_{0,d-1} : \mathscr{E}_d(D) \to \mathbb{C}^d$. It is given by the formula

$$j_{z_0,d-1}(h) = (h(z_0), h'(z_0), \ldots, h^{\langle d-1\rangle}(z_0)). \tag{3.1}$$

In some of our calculations, we will fix the angular coordinate ξ by restricting to the subspace

$$\mathscr{E}_{d,1}(D) = \{v \in \mathscr{E}_d(D) | \ \ v(1) = 1\}.$$

Lemma 3.2. *Let d be a positive integer. Then, $0 \in \mathbb{C}^{d-1}$ is a regular value of the jet map (see (3.1))*

$$j_{0,d-1} : \mathscr{E}_{d,1}(D) \to \mathbb{C}^d.$$

Proof. This is a direct computation. An element $v \in \mathscr{E}_{1,d}(D)$, according to Lemma 3.1, must have the form

$$v(z) = \prod_{k=1}^{d}\left(\frac{1-\overline{a}_k}{1-a_k}\right)\prod_{k=1}^{d}\left(\frac{z-a_k}{1-\overline{a}_k z}\right).$$

As we have alluded to before, the elementary symmetric polynomials on $(a_1, \ldots, a_d)$ provide a complete set of coordinates on the space $\mathscr{E}_{d,1}(D)$. Let λ_i be the elementary symmetric polynomial of degree i. Then,

$$\prod_{d=1}^{d}(z-a_k)=\sum_{i=0}^{d}(-1)^i\lambda_i z^{d-i}.$$

If we think of each $v\in\mathscr{E}_{1,d}(D)$ in terms of its coordinates $\lambda=(\lambda_1,\ldots,\lambda_d)$ that define it, we see that the jet map takes the form

$$\lambda\mapsto j_{0,d-1}\left(\frac{R(\overline{\lambda},1)}{R(\overline{\lambda},z)}\cdot\frac{P(\lambda,z)}{P(\lambda,1)}\right),$$

where

$$P(\lambda,z)=\sum_{i=0}^{d}(-1)^i\lambda_i z^{d-i}\quad\text{and}\quad R(\overline{\lambda},z)=\sum_{i=0}^{d}(-1)^i\overline{\lambda}_i z^i.$$

We show that $\lambda=0$ is a regular point. Since we described the domain with complex coordinates, we find it easier to compute complex derivatives, even though $j_{0,d-1}$ is not holomorphic. The derivatives at $\lambda=0$ are

$$(dj_{0,d-1})_{\lambda=0}\,(\partial_{\bar{i}})=0\quad\text{and}\quad(dj_{0,d-1})_{\lambda=0}(\partial_i)=(0,\ldots,(-1)^i,\ldots,0),$$

where the non-zero entry corresponds to the $(d-i)^{\text{th}}$ derivative. It follows that $0\in\mathbb{C}^d$ is indeed a regular value of $j_{0,d-1}$. $\square$

3.3 Discs with tangency condition

Let $\alpha\in H_2(\mathbb{P}^n,L_{\text{cl}})$ be relative homology class. It is determined by the intersection numbers

$$\alpha_k=\alpha\cdot(x_k=0).$$

These numbers determine the Maslov index of α through the equation

$$\frac{1}{2}\mu(\alpha)=\alpha_0+\cdots+\alpha_n.$$

We now recall a description of the space $\mathcal{M}(L_{\text{cl}},\alpha)$ of parametrized holomorphic discs $v:(D,\partial D)\to(\mathbb{P}^n,L_{\text{cl}})$ in the class α.

Lemma 3.3. *For each $v \in \mathcal{M}(L_{cl}, \alpha)$, there exist holomorphic maps $v_k : (D, \partial D) \to (D, \partial D)$ of degree α_k such that*

$$v(z) = [v_0(z) : v_1(z) : \cdots : v_n(z)].$$

Proof. The claim on degrees is automatic once we have the required description of v in homogeneous coordinates, refer to Lemma 3.1. Because v intersects the hyperplane $\{x_0 = 0\}$ in a finite subset $A_0 \subset D$,

$$v(z) = [1 : g_1(z) : \cdots : g_n(z)] \tag{3.2}$$

where the g_k are disc endomorphisms, with singularities only at the points of A_0. It suffices to show that these singularities, if they arise, are at worst poles. Let $z_c \in A_0$ be a singularity for g_1. By definition this means that $v(z_c)$ belongs to the hyperplane $\{x_0 = 0\}$. But since the hyperplanes $(x_i = 0)$ are linearly independent, one of them shouldn't contain $v(z_c)$. Without loss of generality, assume $v(z_c) \notin \{x_1 = 0\}$. Similarly to (3.2), we can then use an expression of v in the complement of the hyperplane $\{x_1 = 0\}$:

$$v(z) = [h_0(z) : 1 : \cdots : h_n(z)]$$

where the functions h_k are holomorphic outside of a subset $A_1 \subset D$, corresponding to the intersection of v with $\{x_1 = 0\}$. In particular, h_0 is holomorphic near z_c and $g_1 = 1/h_0$. It follows that g_1 is a meromorphic function as claimed. The same arguments applies to the remaining $g_2, g_3, \ldots, g_n$. □

By the work of Cho-Oh in [CO06], the moduli space $\mathcal{M}(L_{\mathrm{cl}}, \alpha)$ for the integrable complex structure of $\mathbb{P}^n$ is Fredholm regular. Its dimension is computed using the Riemann-Roch formula

$$\dim \mathcal{M}(L_{\mathrm{cl}}, \alpha) = n + \mu(\alpha).$$

It can also be verified in this case using the Lemmas 3.1 and 3.3. We will always assume $\alpha_k \geq 0$ for $k = 0, 1, \ldots, n$. Otherwise, the corresponding moduli space is empty for the integrable complex structure.

Let us now introduce the relevant tangency moduli space. For each homogeneous polynomial of $f \in \mathcal{H}$ of degree $n+1$, define

$$\tau_\alpha^f = \{(v, z_0) \in \mathcal{M}(L_{\mathrm{cl}}, \alpha) \times D \mid \ j_{z_0,n}(f \circ v) = 0\}. \tag{3.3}$$

The tangency condition (3.3) imposes a minimal Maslov number constraint when f is near f_0. Indeed:

$$\begin{aligned} \mu(\alpha) &= 2v \cdot V(f_0) \\ &= 2v \cdot V(f) \\ &\geq 2(n+1). \end{aligned}$$

More importantly, it means that v comes from a holomorphic disc in the branched covering X_f, see Lemma 3.12 below.

In our counting problems, we will always assume

$$\mu(\alpha) = 2(n+1). \tag{3.4}$$

We want to count elements of (3.3) with 1 boundary constraint. To that end, we define

$$\hat{\tau}_{\alpha,1}^f = \tau_\alpha^f \times \partial D / Aut(D).$$

It comes with a boundary evaluation map

$$ev : \hat{\tau}_{\alpha,1}^f \to L_{\mathrm{cl}}. \tag{3.5}$$

Ideally, the space $\hat{\tau}_{\alpha,1}^f$ will be an oriented closed manifold so that one can compute the degree n_α of the evaluation map (3.5). This is not always true. The goal of the remainder of this section is to highlight and resolve the difficulties that arise.

One of the relative homology classes with Maslov number $2(n+1)$ is actually spherical:

$$\alpha_s = (1, \ldots, 1).$$

The class α_s behaves somewhat differently from all the others, so we treat it separately.

Proposition 3.4. *If the homogeneous polynomial $f \in \mathcal{H}$ is sufficiently close to f_0, the moduli space $\tau^f_{\alpha_s}$ is empty.*

Proof. Suppose we have a sequence $f_i \to f_0$, and elements $(v_i, z_i) \in \tau^{f_i}_{\alpha_s}$. Up to composition with Möbius transformations, We may assume $z_i = 0$. By Gromov compactness, the sequence v_i sub-converges to a genus 0 nodal curve with boundary on L_{cl}. This limit is must be tangent to the toric boundary f_0 to order n. Let v_∞ be the component of this nodal curve that is tangent to f_0. Since $\mu(\alpha_s) = 2(n+1)$ is the minimal Maslov number for this order of tangency, all other components must in fact be constant. Now, the irreducible component v_∞ is either a genuine disc with boundary on L_{cl}, or a projective line arising as a spherical bubble in the Gromov limit.

The former case can be ruled out using the description of discs in terms of Möbius transformations, see Lemma 3.3. Indeed:

$$v_\infty = [\phi_0 : \cdots : \phi_n].$$

Then, $f_0 \circ v_\infty = \phi_0 \dots \phi_n$ would be a degree $n+1$ disc endomorphism that vanishes at 0 to order n, as implied by the tangency condition. This can happen only if all ϕ_k are multiples of z. In such a case, v_∞ would be constant, which is a contradiction. See Lemma 3.1 as well.

In the latter case, v_∞ would be a projective line tangent to the toric divisor to order n. This means that it is a line that intersects all components $\{x_i = 0\}$ of the toric divisor simultaneously. However, the intersection of these hyperplanes is empty. □

In the remainder of this section, we present a systematic method for computing n_α for all non-spherical classes. From now on, $\alpha \neq \alpha_s$ is a relative homology class for the pair $(\mathbb{P}^n, L_{\mathrm{cl}})$ whose Maslov index equals $2(n+1)$.

3.4 Compactness and counting

According to our computations, it seems that τ_α^f (for some particular classes α) is not necessarily regular. Not even if we allow perturbations of $f \in \mathcal{H}$ near f_0. In other words, 0 is not a regular value of the universal jet map

$$\mathcal{M}(L_{\text{cl}}, \alpha) \times D \times \mathcal{H} \to \mathbb{C}^{n+1} : \quad (v, z_0, f) \mapsto j_{z_0,n}(f \circ v).$$

Nonetheless, it is still possible to calculate n_α if we interpret it as the degree of the map

$$\begin{aligned} \Phi : \mathcal{M}(L_{\text{cl}}, \alpha) \times \mathcal{H} &\to \mathbb{C}^{n+1} \times \mathcal{H} \times L \\ (v, f) &\mapsto (j_{0,n}(f \circ v), f, v(1)) \, . \end{aligned} \tag{3.6}$$

Indeed, if one fixes a point $p \in L$, then the pre-image $\Phi^{-1}(0, f, p)$ counts holomorphic discs $v : (D, \partial D) \to (\mathbb{P}^n, L_{\text{cl}})$ in the homology class α that are tangent to $V(f) \subseteq \mathbb{P}^n$ to order n at $z = 0$ and such that $v(1) = p$. This fiber is essentially the same as $ev^{-1}(p)$ in (3.5); the only difference is that we are taking a slice of the action of the automorphism group $\text{Aut}(D)$ by choosing $z = 0$ to be the tangency point with $V(f)$ and $z = 1$ to be the boundary marked point.

Remark 3.5. *There is an ambiguity in the definition of the jet map $j_{n,0}$ in (3.6): it depends on the choice of a representation of the holomorphic disc v in homogeneous coordinates on $\mathbb{P}^n$. However, when a class $\alpha \neq \alpha_s$ satisfying (3.4) is fixed, there is a systematic way to produce such representations across the moduli space $\mathcal{M}(L_{cl}, \alpha)$. The reason is that for some $0 \leq i \leq n$, we have vanishing of the intersection number*

$$\alpha_i = (x_i = 0) \cdot v = 0.$$

Therefore, all holomorphic discs in the moduli space $\mathcal{M}(L_{cl}, \alpha)$ actually land in the open set $\mathbb{P}^n \backslash \{x_i = 0\} = \{x_i = 1\}$.

In order to ensure that the map in (3.6) has a well defined degree, we need the following compactness result.

Lemma 3.6. *There is an open neighborhood U of $\{0\} \times \{f_0\} \times L$ such that the restriction of Φ to U is a proper map.*

Proof. Because L is compact, the only potential cause of non-properness of the map Φ is the non-compact space $\mathcal{M}(L_{\mathrm{cl}}, \alpha)$. We can remedy this by using the previously established relationship between this space and Möbius transformations. First of all, using Lemma 3.1 and the remark thereafter, we can endow $\mathcal{M}(L_{\mathrm{cl}}, \alpha)$ with smooth coordinates using the following parametrization:

$$\mathscr{E}_{\alpha_0,1}(D) \times \mathscr{E}_{\alpha_1}(D) \cdots \times \mathscr{E}_{\alpha_n}(D) \to \mathcal{M}(L_{\mathrm{cl}}, \alpha) \tag{3.7}$$
$$(v_0, v_1, \ldots, v_n) \mapsto [v_0 : v_1 : \cdots : v_n],$$

where

$$\mathscr{E}_{\alpha_0,1}(D) = \{v \in \mathscr{E}_{\alpha_0}(D) | \;\; v(1) = 1\}.$$

This allows us to compactify $\mathcal{M}(L_{\mathrm{cl}}, \alpha)$ by allowing the Möbius centers a_k from Lemma 3.1 to reach the boundary ∂D. We will denote the resulting compactification by $\overline{\mathcal{M}}(L_{\mathrm{cl}}, \alpha)$. Note that discs in the boundary have strictly smaller Maslov numbers.

With this set-up in mind, we can prove the Lemma by way of contradiction. If it weren't true, there would exist an unbounded sequence $v_i \in \mathcal{M}(L_{\mathrm{cl}}, \alpha)$ and $f_i \in \mathcal{H}$ such that

$$f_i \to f_0 \;\text{ and }\; j_{0,n}(f_i \circ v_i) = 0.$$

After possibly passing to a sub-sequence, the maps v_i will converge to an element v_∞ of the boundary of $\overline{\mathcal{M}}(L_{\mathrm{cl}}, \alpha)$, and we would still have the tangency equation

$$j_{n,0}(f_0 \circ v_\infty) = 0.$$

But since $\mu(v_\infty) < 2(n+1)$, the disc map $f_0 \circ v_\infty : (D, \partial D) \to (D, \partial D)$ is non-constant and has degree at most n, and as such, it cannot vanish at 0 to order n. $\square$

Remark 3.7. *This proof can also be rephrased using Gromov compactness, and then tracking the tangency point in the Gromov limit of the sequence v_i, in a similar spirit to the proof of Proposition 3.4.*

For the purpose of studying the degree of Φ, we recall some useful computational tools from differential topology.

Lemma 3.8. *Let $f : X \to Y$ be a proper smooth map between smooth oriented manifolds of the same dimension such that Y is connected. Let $Z \subseteq Y$ be a smooth connected oriented submanifold that is transverse to f. Then $f^{-1}(Z)$ is a smooth oriented manifold and the degree of f agrees with that of its restriction $f_{|Z} : f^{-1}(Z) \to Z$.*

Proof. The transversality assumption ensures that when $z \in Z$ is a regular value of $f_{|Z}$, it is also a regular value of f. $\square$

Lemma 3.9. *Let X_0 and X_1 be smooth oriented manifolds and let $\mathfrak{X}$ be a smooth oriented manifold with boundary such that*

$$\partial \mathfrak{X} = -X_0 \sqcup X_1.$$

Let $F : \mathfrak{X} \to Y$ be a proper smooth map to an oriented smooth connected manifold Y. Then the degrees of the restrictions of $F_{|X_0}$ and $F_{|X_1}$ agree.

Proof. See Lemma 1 in Chapter 5 of Milnor's book [MW97]. $\square$

We can now compute the desired degree.

Lemma 3.10. *If a relative homology class $\alpha = (\alpha_0, \dots, \alpha_n)$ has Maslov number $2(n+1)$ and is different from α_s, then*

$$\deg(\Phi) = \frac{(n+1)!}{\alpha_0! \dots \alpha_n!}.$$

Proof. We start by applying Lemma 3.8 to restrict Φ to the submanifold $\mathbb{C}^{n+1} \times \{f_0\} \times \{p\}$, where $p = [1 : \dots : 1] \in L_{\mathrm{cl}}$. The required transversality condition follows from the fact that the evaluation map

$$\mathcal{M}(L_{\mathrm{cl}}, \alpha) \to L_{\mathrm{cl}} : \quad v \mapsto v(1)$$

is a submersion, see Lemmas 3.1 and 3.3. Therefore, we may compute $\deg(\Phi)$ from the map

$$\hat{\Phi} : \mathcal{M}_p(L_{\mathrm{cl}}, \alpha) \to \mathbb{C}^{n+1}, \quad v \mapsto j_{0,n}(f_0 \circ v),$$
$$\text{where} \quad \mathcal{M}_p(L_{\mathrm{cl}}, \alpha) = \{v \in \mathcal{M}(L_{\mathrm{cl}}, \alpha) \ \mid \ v(1) = p\}.$$

By Lemmas 3.1 and 3.3, we have a product decomposition

$$\mathcal{M}_p(L_{\mathrm{cl}}, \alpha) = \mathscr{E}_{\alpha_0,1}(D) \times \cdots \times \mathscr{E}_{\alpha_n,1}(D) \tag{3.8}$$

using the same parametrization as (3.7). Moreover, the map $\hat{\Phi}$ is the composition of a product map and a jet map:

$$\mathscr{E}_{\alpha_0,1}(D) \times \cdots \times \mathscr{E}_{\alpha_n,1}(D) \xrightarrow{\pi_\alpha} \mathscr{E}_{n+1,1}(D) \xrightarrow{j_{0,n}} \mathbb{C}^{n+1}$$
$$v = (v_0, \ldots, v_n) \mapsto v_0 \cdot v_1 \cdots v_n \mapsto j_{0,n}(\pi_\alpha(v)).$$

By Lemma 3.1, the integer $\deg(\pi_\alpha)$ is the number of ways to partition a set S of $n+1$ Möbius transformations into $n+1$ sets S_i of size α_i:

$$\deg(\pi_\alpha) = \frac{(n+1)!}{\alpha_0! \ldots \alpha_n!}.$$

It remains to show that the jet map

$$j_{0,n} : \mathscr{E}_{n+1,1}(D) \to \mathbb{C}^{n+1}$$

has degree 1. This follows from Lemma 3.2 and (the proof of) Lemma 3.1. The former ensures that $0 \in \mathbb{C}^{n+1}$ is a regular value, while the latter implies that $j_{0,n}^{-1}(0) = \{z^{n+1}\}$. $\square$

Remark 3.11. *One might ask what goes wrong in the proof of Lemma 3.10 when $\alpha = \alpha_s$ is the spherical class. The main difference is that we no longer have the parametrization in (3.8) due to the cancellations that occur in homogeneous coordinates when all the Möbius transformations are equal to each other.*

3.5 Transversality

Now that we have our numbers n_α, we need to justify that they in fact correspond to generic counts of Maslov index 2 discs in the branched covering (X_f, L).

Fix a Maslov index 2 homology class $\beta \in H_2(X_f, L)$ and consider its corresponding moduli space

$$\mathcal{M}(L,\beta) = \left\{u : (D, \partial D) \to (X_f, L) \ \mid \ [u] = \beta \text{ and } \overline{\partial}_J u = 0\right\}.$$

where J is the integrable complex structure of X_f. We then form the moduli space of unparametrized discs with 1 boundary marked point,

$$\hat{\mathcal{M}}_1(L,\beta) = \frac{\mathcal{M}(L,\beta) \times \partial D}{\mathrm{Aut}(D)}.$$

Our goal is to compute the degree $m_{0,\beta}(L)$ of the evaluation map

$$ev : \hat{\mathcal{M}}_1(L,\beta) \to L, \quad (u, e^{i\theta}) \mapsto u(e^{i\theta}).$$

Because $\mu(\beta) = 2$, each holomorphic disc u intersects the ramification divisor $\{t = 0\} \subseteq X_f$ exactly once. We can use this fact to take a slice of the action of the group $\mathrm{Aut}(D)$ above: fix the boundary point to be 1 and the intersection point with R to be $u(0)$. In other words, $m_{0,\beta}(L)$ is also the degree of the map

$$\begin{aligned} &ev_1 : \hat{\mathcal{M}}_0(L,\beta) \to L, \quad u \mapsto u(1), \\ \text{where} \quad &\hat{\mathcal{M}}_0(L,\beta) = \{u \in \mathcal{M}(L,\beta) \mid t(u(0)) = 0\}. \end{aligned} \tag{3.9}$$

The plan is to compute the above degree by pushing down to $\mathbb{P}^n$ using the branch covering map (see (2.10))

$$\phi : X_f \to \mathbb{P}^n.$$

The branch locus of this map is $B_f = V(f) \subseteq \mathbb{P}^n$ and the ramification locus is $R_f = \{t = 0\} = \phi^{-1}(B_f) \subseteq X_f$. For technical transversality reasons, we need to work with an *enlarged* ramification locus

$$R_+ = \phi^{-1}(B_f \cup V(f_0)).$$

To ensure the moduli space $\hat{\mathcal{M}}_0(L,\beta)$ is Fredholm regular, we keep J fixed and perturb the J-holomorphic equation with a domain dependent Hamiltonian term,

$$(du - Y)^{0,1} = 0.$$

The perturbation datum $Y \in \Omega^1(D, \Gamma_0(TX_f))$ is a 1-form on D with values in the space of Hamiltonian vector fields on X_f that have compact support in the complement of R_+ in X_f. This class of perturbations is large enough to achieve transversality; see for example [Sei08], Section (9k). We therefore fix a sufficiently small Y for which the perturbed moduli space

$$\begin{aligned}\hat{\mathcal{M}}_0^Y(L,\beta) = \{u : (D, \partial D) \to (X_f, L) \mid (du - Y)^{0,1} = 0,\\ \text{and } t(u(0)) = 0,\ [u] = \beta\}\end{aligned}$$

is a regular. Next, we push this setup down to $\mathbb{P}^n$. Let $\alpha = \phi_*(\beta)$ and $Z = \phi_* Y \in \Omega^1(D, \Gamma_0(T\mathbb{P}^n))$, where now $\Gamma_0(T\mathbb{P}^n)$ stands for vector fields with compact support in the complement of the branch locus $V(f) \subseteq \mathbb{P}^n$.

Our choice of perturbation data ensures that an element $u \in \hat{\mathcal{M}}_0^Y(L,\beta)$ is genuinely holomorphic near the ramification locus. Therefore, its pushforward $v = u \circ \phi$ is also holomorphic near the branch locus V_f. Moreover,

$$j_{n,0}(f \circ v) = 0. \tag{3.10}$$

This pushforward $u \mapsto v$ is an *unbranched* covering map of degree $n+1$. To see that, we consider the perturbed tangency moduli space

$$\begin{aligned}\hat{\tau}_0^Z(L_{\text{cl}},\alpha) = \{v : (D, \partial D) \to (\mathbb{P}^n, L_{\text{cl}}) \mid (dv - Z)^{0,1} = 0,\\ \text{and } j_{0,n}(f \circ v) = 0,\ [v] = \alpha\}.\end{aligned}$$

Lemma 3.12. *Every disc map $v \in \hat{\tau}_0^Z(L_{cl},\alpha)$ has $n+1$ distinct lifts $(u^i)_{0\le i\le n} \in \hat{\mathcal{M}}_0^Y(L,\phi^{-1}(\alpha))$, where*

$$\hat{\mathcal{M}}_0^Y(L,\phi^{-1}(\alpha)) = \bigcup_{\beta\in\phi^{-1}(\alpha)} \hat{\mathcal{M}}_0^Y(L,\beta).$$

Furthermore,

$$\sum_{\beta \in \phi^{-1}(\alpha)} m_{0,\beta}(L) = \deg\left(\hat{\tau}_0^Z(L_{cl}, \alpha) \xrightarrow{ev_1} L_{cl}\right). \tag{3.11}$$

Proof. By abuse of notation, we will identify v with its component-wise description in homogenous coordinates in order to study the composition $f \circ v : D \to \mathbb{C}$, the ambiguity in this choice is the subject of Remark 3.5. In a small disc $\{|z| < r\}$, this function is holomorphic and vanishes to degree $n+1$ at 0, and hence there exists a function $t : \{|z| < r\} \to \mathbb{C}$ such that

$$t(z)^{n+1} = f \circ v(z).$$

This produces a lift $u = [t : v]$ of v, but only on the smaller domain $t : \{|z| < r\} \to \mathbb{C}$. Because $f \circ v$ only vanishes at 0 (otherwise the Maslov number of α would be bigger than $2(n+1)$), the lift u above can be extended to the whole of D using the path lifting property of the unbranched covering map $X_f \backslash R \to \mathbb{P}^n \backslash V_f$. The number of lifts is $n+1$ because on $\{|z| < r\}$, the equation $t(z)^{n+1} = f \circ v(z)$ has exactly $n+1$ solutions in t. Moreover, the unique continuation principle for solutions of the perturbed J-holomorphic equation

$$(du - Y)^{0,1} = 0$$

ensures that two solutions u_1 and u_2 that agree on a non-empty open set, must in fact be identical. Finally, the degree formula follows from the diagram

$$\begin{array}{ccc} \hat{\mathcal{M}}_0^Y(L, \phi^{-1}(\alpha)) & \xrightarrow{ev_1} & L \\ \downarrow & & \downarrow \\ \hat{\tau}_0^Z(L_{\mathrm{cl}}, \alpha) & \xrightarrow{ev_1} & L_{\mathrm{cl}} \end{array}$$

of covering spaces, because the vertical maps both have degree $n+1$. □

Remark 3.13. *When $n > 2$, the pushforward map $\phi_* : H_2(X_f, L) \to H_2(\mathbb{P}^n, L)$ is injective. This is because $H_2(X_f)$ is generated by a hyperplane section, which is fixed by Deck transformations of the branched covering map $\phi : (X_f, L) \to (\mathbb{P}^n, L)$. In particular, Deck transformations act trivially on $H_2(X_f, L)$. This property fails in dimension 2. This is not an* *issue however, because when the Maslov index 2 classes $\beta_1, \beta_2 \in H_2(X_f, L)$ are in the same orbit of the $\mathbb{Z}_3$-action, we in fact have $m_{0,\beta_1}(L) = m_{0,\beta_2}(L)$.*

The previous argument omits at least one important technical detail, and that is the regularity of the perturbed tangency space. In order to address this issue, we introduce the deformed moduli space:

$$\mathcal{M}^Z(L_{\text{cl}}, \alpha) = \left\{ v : (D, \partial D) \to (\mathbb{P}^n, L_{\text{cl}}) \mid [v] = \alpha,\ (dv - Z)^{0,1} = 0 \right\}.$$

Note that $Z = 0$ corresponds to the unperturbed moduli space of holomorphic discs with boundary on the Clifford torus in the class α, and with the standard complex structure of projective space. Recall that this moduli space is Fredholm regular, see theorem 6.1 [CO06]. Since we are perturbing using a small Y (and hence a small $Z = \phi_*(Y)$), this Fredholm regularity is not lost. On this moduli space, we can define a jet map:

$$\begin{aligned} j^f_{0,n} : \mathcal{M}^Z(L_{cl}, \alpha) &\to \mathbb{C}^{n+1} \\ v &\mapsto j_{0,n}(f \circ v). \end{aligned} \tag{3.12}$$

Just like in Remark 3.5, there is an ambiguity in defining this map, but it is resolved by the same argument: Indeed, the perturbation datum Z vanishes near the toric divisor $V(f_0)$ in projective space, and that implies that for any $v \in \mathcal{M}^Z(L_{cl}, \alpha)$, the disc v intersects the hyperplane $(x_i = 0)$ finitely many times, and that all the intersection points have positive contributions to the intersection number $v \cdot (x_i = 0)$. But again, the Maslov number constraint (3.4) (together with $\alpha \neq \alpha_s$) forces one of these intersection numbers to be 0. In particular, we can fix an i such that $\alpha_i = 0$, and then all elements $v \in \mathcal{M}^Z(L_{cl}, \alpha)$ will have a unique representation in homogeneous coordinates where the i^{th} coordinate is constantly equal to 1, and this coordinate representation makes $j^f_{0,n}$ well defined.

Remark 3.14. *From now on, we assume that we have fixed i such that $\alpha_i = 0$, so that all discs in the moduli space $\mathcal{M}^Z(L_{cl}, \alpha)$ have image in the open set $\{x_i = 1\}$, and we think of f as a function on this open set.*

We are now in position to state the main regularity theorem of this section.

Proposition 3.15. *In the jet map (3.12),* $0 \in \mathbb{C}^{n+1}$ *is a regular value.*

Proof. Let $v \in (j^f_{0,n})^{-1}(0)$. By the work of Lemma 3.12, there exists $u \in \mathcal{M}^Y_0(L,\beta)$ such that $\phi \circ u = v$. By differentiating the maps:

$$\mathcal{M}^Y_0(L,\beta) \xrightarrow{\phi} \mathcal{M}^Z(L_{cl},\alpha) \xrightarrow{j^f_{0,n}} \mathbb{C}^{n+1},$$

we obtain a sequence of vector spaces:

$$T_u\mathcal{M}^Y_0(L,\beta) \xrightarrow{\phi_*} T_v\mathcal{M}^Z(L_{cl},\alpha) \xrightarrow{(dj^f_{0,n})_v} \mathbb{C}^{n+1}. \tag{3.13}$$

We prove regularity by showing that this is actually a short exact sequence, which we call the regularity sequence.

Recall that the tangent space $T_v\mathcal{M}^Z(L_{cl},\alpha)$ is the kernel of the linearization of the perturbed $\overline{\partial}$-equation. This looks like:

$$D_v : \Gamma\left(D, v^*T\mathbb{P}^n, v^*_{\partial D}TL_{\mathrm{cl}}\right) \to \Omega^{0,1}(D, v^*T\mathbb{P}^n).$$

The same applies to $T_u\mathcal{M}^Y_0(L,\beta)$, except that the constraint $t(u(0)) = 0$ restricts the domain of the linearized operator a bit:

$$D_u : \Gamma_0\left(D, u^*TX_f, u^*_{\partial D}TL\right) \to \Omega^{0,1}(D, u^*TX_f),$$

where:

$$\Gamma_0\left(D, u^*TX_f, u^*_{\partial D}TL\right) = \left\{\xi \in \Gamma\left(D, u^*TX_f, u^*_{\partial D}TL\right) \;\mid\; \xi_0 \in T_{u(0)}R_f\right\}.$$

The best way to understand the regularity sequence (3.13) is by examining the sheafy versions of $\ker(D_u)$ and $\ker(D_v)$. For an open set $U \subseteq D$, let:

$$\mathcal{E}_u(U) = \{\xi \in \Gamma\left(U, u^*TX_f, u^*_{\partial D}TL\right) \;\mid\; \xi_0 \in T_{u(0)}R_f \;\text{ and }\; D_u(\xi) = 0\}$$
$$\mathcal{E}_v(U) = \{\xi \in \Gamma\left(U, v^*T\mathbb{P}^n, v^*_{\partial D}TL_{\mathrm{cl}}\right) \;\mid\; D_v(\xi) = 0\}.$$

To obtain the regularity sequence, it suffices to show that we have a short exact sequence of sheaves:

$$0 \to \mathcal{E}_u \xrightarrow{\phi_*} \mathcal{E}_v \xrightarrow{dj^f_{0,n}} \underline{\mathbb{C}}^{n+1} \to 0, \tag{3.14}$$

where $\underline{\mathbb{C}}^{n+1}$ is a skyscrapper sheaf at $0 \in D$. Indeed, Fredholm regularity of D_u means that $H^1(\mathcal{E}_u) = 0$ and so, by appealing to the long exact sequence in sheaf cohomology, we get the short exact sequence in (3.13).

Looking at the sequence (3.14), observe that the map ϕ_* restricts to a sheaf isomorphism on the punctured disc $D\backslash 0$. It means that both kernel and cokernel are supported at 0. Because of the identity principle for solutions of the Cauchy-Riemann equations:

$$D_u(\xi) = 0,$$

we can already deduce that the sheaf map $\phi_* : \mathcal{E}_u \to \mathcal{E}_v$ is injective.

Next, as the cokernel is supported at 0, we can compute it by trivializing near $u(0) \in R_f$. In (resp. below) a small neighborhood of $u(0)$, the perturbation data vanishes and D_u (resp. D_v) is the Dolbeaux operator. Furthermore, following the conventions of Remark 3.14, f defines a regular function in a neighborhood of $v(0)$, and the branched covering has the local model:

$$(t^{n+1} - x_1) \subseteq \mathbb{C}^{n+1} \to \mathbb{C}^n$$
$$(t, x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n).$$

in which $u(0) = 0 \in \mathbb{C}^{n+1}$, $v(0) \in \mathbb{C}^n$ and $f = x_1$. In this local model, we have a trivializing frame for both TX_f and $T\mathbb{P}^n$. The first is given by the vector fields $\partial_t, \partial_{x_2},\ldots,\partial_{x_n}$ and the later is given by $\partial_{x_1}, \partial_{x_2},\ldots,\partial_{x_n}$. Moreover, the action of ϕ_* on this frame is:

$$\phi_*(\partial_t) = (n+1)t^n\partial_{x_1} \quad \text{and} \quad \phi_*(\partial_{x_k}) = \partial_{x_k} \text{ for } 2 \leq k \leq n.$$

The holomorphic disc u has a coordinate description in this chart:

$$u(z) = (t(z), x_1(z), \ldots, x_n(z)).$$

This is defined over a small open set $0 \in U \subseteq D$. Moreover:

$$\mathcal{E}_u(U) = \{f_0(z)\partial_t + f_2(z)\partial_{x_2} + \cdots + f_n(z)\partial_{x_n} \, |\overline{\partial} f_i = 0 \ \text{ and } \ f_0(0) = 0\},$$

and at the same time:

$$\mathcal{E}_v(U) = \{f_1(z)\partial_{x_1} + f_2(z)\partial_{x_2} + \cdots + f_n(z)\partial_{x_n} \, |\overline{\partial} f_i = 0\}.$$

Furthermore, the jet map (recall the conventions of Remark 3.14) has the formula:

$$\begin{aligned} dj^f_{0,n} &: \mathcal{E}_v(U) \to \mathbb{C}^{n+1} \\ \sum_{k=1}^{n} & f_k(z)\partial_{x_k} \mapsto j_{0,n}(f_1(z)). \end{aligned}$$

Therefore, by taking stalks at $0 \in D$ in the sequence (3.14), we reduce our transversality problem to the following short exact sequence:

$$0 \to \{f \in \mathscr{H}_0 \mid f(0) = 0\} \xrightarrow{\times t(z)^n} \mathscr{H}_0 \xrightarrow{j_{0,n}} \mathbb{C}^{n+1} \to 0, \tag{3.15}$$

where $\mathscr{H}_0$ is the stalk at 0 of the sheaf of holomorphic functions on D. The sequence in (3.15) above is exact because the vanishing order of t at 0 is exactly 1, i.e $t(z) = z \cdot \epsilon$, where ϵ is an invertible element of $\mathscr{H}_0$. □

As a consequence of Proposition 3.15, of Lemma 3.12, and Lemma 3.8, we can actually see that $m_{0,\beta}(L)$ is the local degree of the map:

$$\begin{aligned} \Psi_1 : \mathcal{M}^Z(L_{\text{cl}}, \alpha) \times \mathcal{H} &\to \mathbb{C}^{n+1} \times \mathcal{H} \times L \\ (v, f) &\mapsto (j_{0,n}(f \circ v), f, v(1)) \end{aligned} \tag{3.16}$$

near $\{0\} \times \{f_0\} \times L$ (recall that we only have properness in this region, see Lemma 3.6). We now have all the necessary ingredients to prove the main result of this section.

Theorem 3.16. *Let $X_f = V(t^{n+1} - f)$ be a smooth hypersurface of degree $n+1$, $\phi : X_f \to \mathbb{P}^n$ the linear projection onto the hyperplane $\{t = 0\}$. Let $L_{cl} \subset H$ be the Clifford torus. If f is generic and nearly degenerate, then L_{cl} lifts to a totally real torus L in X_f. Moreover,* *counts of Maslov index* 2 *discs with respect to an anti-canonical Kähler form are given by the formula:*

$$\sum_{\beta \in \phi^{-1}(\alpha)} m_{0,\beta}(L) = \frac{(n+1)!}{\alpha_0! \ldots \alpha_n!},$$

for any class $\alpha = (\alpha_0, \ldots, \alpha_n) \in H_2(\mathbb{P}^n, L_{cl})$ *of Maslov index* $2(n+1)$*, except when* $\alpha = \alpha_s$ *is the spherical class. In that case:*

$$m_{0,\beta}(L) = 0,$$

for all $\beta \in \phi^{-1}(\alpha_s)$*.*

Proof. The only part of the theorem above that we haven't proved yet is the degree formula when β is not spherical. The key is that we can scale down the perturbation datum Z by a real number $s \in [0, 1]$, without losing regularity of the moduli space $\mathcal{M}^{sZ}(L_{\mathrm{cl}}, \alpha)$, because Z is small and $\mathcal{M}^0(L_{\mathrm{cl}}, \alpha)$ is Fredholm regular. We can therefore deform the map (3.17) through a cobordism:

$$\begin{aligned} \Psi : \mathcal{M}^Z_{[0,1]}(L_{\mathrm{cl}}, \alpha) \times \mathcal{H} &\to \mathbb{C}^{n+1} \times \mathcal{H} \times L \\ (v, f) &\mapsto (j_{0,n}(f \circ v), f, v(1)), \end{aligned} \tag{3.17}$$

where

$$\mathcal{M}^Z_{[0,1]}(L_{\mathrm{cl}}, \alpha) = \{(v, s) \mid s \in [0, 1] \text{ and } v \in \mathcal{M}^{sZ}(L_{\mathrm{cl}}, \alpha)\}.$$

By applying Lemma 3.9, we deduce that $m_{0,\beta}(L)$ agrees with the degree n_α of the jet map defined in (3.6). Finally, the formula for n_α was obtained in Lemma 3.10, and the corresponding formula for $m_{0,\beta}(L)$ follows. □

3.6 Super-potential

Recall that the super-potential associated with a Lagrangian torus $L \subseteq X_f$ is a function on its mirror space:

$$M_L = \mathrm{Spec}(\mathbb{C}[H_1(L, \mathbb{Z})]),$$

where q is a formal parameter. For an Abelian group A, the algebra $\mathbb{C}[A]$ has a generator z_a for each element $a \in A$, subject to the relation:

$$z_{a+a'} = z_a z_a'$$

The potential function is then given by the formula

$$W = \sum_{\mu(\beta)=2} m_{0,\beta}(L) z_{\partial\beta}.$$

It is somewhat easier to write the potential function on the mirror of L_{cl} first. By construction, loops in $H_1(L_{\text{cl}})$ lift to L if and only if they link trivially around the toric boundary. In other words, if they lie in the kernel of the map:

$$H_1(L_{\text{cl}}) \to \mathbb{Z}_{n+1}$$
$$\gamma \mapsto u_\gamma \cdot D_0,$$

where u_γ is any a disc whose boundary is γ, and $D_0 = V(x_0 \cdots x_n) \subseteq \mathbb{P}^n$ is the toric boundary. As a consequence, we have a short-exact sequence:

$$0 \to H_1(L) \to H_1(L_{\text{cl}}) \to \mathbb{Z}_{n+1} \to 0.$$

Passing to group algebras, we obtain:

$$0 \to \mathbb{C}[H_1(L)] \to \mathbb{C}[H_1(L_{\text{cl}})] \to \mathbb{C}[z]/[z^{n+1} - 1] \to 0. \tag{3.18}$$

This short exact sequence describes a cyclic $n+1$ covering map $\pi : M_{L_{\text{cl}}} \to M_L$. The space $M_{L_{\text{cl}}}$ has natural coordinates coming from the paths $\gamma_k = \partial u_k$, where the discs u_k are the generators that we defined in (2.7). We therefore set $z_k = z_{\gamma_k}$, and we note that these elements satisfy the equation:

$$z_0 \dots z_n = 1.$$

With this set of coordinates, we can compute the pullback of the potential-function using Theorem 3.16:

$$\pi^* W = (z_0 + \cdots + z_n)^{n+1} - (n+1)!. \tag{3.19}$$

To compute W itself, we need a set of coordinates in M_L. This is not difficult once we understand that the quotient map in the short exact sequence (3.18) is:

$$\mathbb{C}[H_1(L_{\mathrm{cl}})] \to \mathbb{C}[z]/[z^{n+1}-1]$$
$$z_\alpha \mapsto z^{\sigma(\alpha)},$$

where $\sigma(\alpha) = \alpha_0 + \cdots + \alpha_n$. Therefore, by setting $y_k = z_k/z_0$, we produce a homomorphism of algebras:

$$\mathbb{C}[y_1, \ldots, y_n] \to \mathbb{C}[H_1(L)].$$

Using the equation $y_1 \ldots y_n = z_0^{-(n+1)}$, we see that the morphism above is exactly localization at the product $y_1 \ldots y_n$. As a conclusion of this analysis, we obtain the following result:

Proposition 3.17. *There is a natural embedding of M_L in $\mathbb{C}^n$ as the complement of the standard set of axes. In this coordinate system, the potential function is given by*

$$W = \frac{(1 + y_1 + \cdots + y_n)^{n+1}}{y_1 \cdots y_n} - (n+1)!. \tag{3.20}$$

Remark 3.18. *The super-potential above agrees with Givental's Landau-Ginzburg model associated with X_f, which is typically computed from its Gromov-Witten invariants. See for instance [KP14] for an overview, and section 3 of [Prz21] for some explicit formulae.*

The potential function W has the expected critical values:

$$w_b = -(n+1)! \ \text{ and } \ w_s = (n+1)^{n+1} - (n+1)!.$$

These are the eigenvalues of multiplication by c_1 on the quantum cohomology ring of X_f. We call w_s the small critical value; the fiber there has an isolated non-degenerate singularity, and we often call w_s the non-degenerate critical value. We call w_b the big critical value, and the fiber there is not reduced, but it's reduction is the smooth $(n-1)$-dimensional pair of pants.

Finally, we note that the relationship between M_L and $M_{L_{\mathrm{cl}}}$ runs even deeper. Indeed, the potential function W_{cl} for L_{cl} is known in the literature. In the same set of coordinates used in equation (3.19), this super-potential has the formula:

$$W_{\mathrm{cl}} = z_0 + \cdots + z_n.$$

With that in mind, We obtain a commutative diagram:

$$\begin{tikzcd} M_{L_{\mathrm{cl}}} \arrow[r, "W_{\mathrm{cl}}"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "z^{n+1}"] \\ M_L \arrow[r, "\hat{W}"] & \mathbb{C} \end{tikzcd}$$

where:

$$\hat{W} = W + (n+1)!. \tag{3.21}$$

We will explore this relationship in detail and use it to study homological mirror symmetry for the super-potential W.

Chapter 4

Generation of the small component

The goal of this chapter is to prove that the monotone Lagrangian torus at the center of our (partial) SYZ fibration generates the small component of the Fukaya category. This chapter contains no new results, but is instead a compilation of all the ingredients needed to establish homological mirror symmetry over the small component.

4.1 Monotone Floer theory, review

Let (X, ω) be a monotone symplectic manifold, such that ω an anti-canonical form, and let $L \subseteq X$ be a monotone Lagrangian brane. As far as we know, there are two main approaches to associating an A_∞-algebra $A = CF(L)$ with L. One approach is to count holomorphic polygons with boundaries on small push-offs of L, following the same lines of [Sei08]; this method makes use of the monotonicity assumption to achieve transversality and compactness. Another more general approach, carried out by Fukaya-Oh-Ohta-Ono in [Fuk+10b], relies on chain-level intersection theory in the moduli spaces $\mathcal{M}_{d+1}(L)$ of discs with boundary on L . The former approach is the one adopted by N. Sheridan in [She16] and has its own advantages: It yields an A_∞-algebra over $\mathbb{Z}$, and its underlying $\mathbb{Z}$-module is small, generated only by the intersection points of L with one of its nearby perturbations. In our work however, we find it more convenient to work with the later approach, as it comes with a *divisor axiom*, which makes our Floer cohomology computations easier. We therefore recall the main characteristics of this construction.

In [Fuk10], K. Fukaya constructs an A_∞-algebra structure $(m_k)_{k\geq 1}$ on the $\mathbb{Z}$-graded vector space:

$$A = H^*(L, \mathbb{C}[[q]]),$$

where q is a formal parameter of degree 2. Ignoring all analytical, topological and algebraic complications, the A_∞-structure maps:

$$m_k : A^{\otimes k} \to A[2-k],$$

have a sum decomposition:

$$m_k = \sum_{\beta \in H_2(X,L)} q^{\langle \omega, \beta \rangle} m_{k,\beta}, \tag{4.1}$$

with respect to topological types $\beta \in H_2(X, L)$, and each term $m_{k,\beta}$ is a cohomological Fourier-Mukai transform based on the correspondence:

$$\begin{tikzcd} & \overline{\mathcal{M}_{k+1}(L,\beta)} \arrow[dl, "ev_k \times \cdots \times ev_1"'] \arrow[dr, "ev_0"] & \\ L^k & & L \end{tikzcd}$$

where $\overline{\mathcal{M}_{k+1}(L,\beta)}$ is the Gromov compactification of the space of holomorphic discs in the class β, with boundary on L and carrying $k+1$ boundary marked points. The non-constant discs are responsible for terms of m_k that involve non-constant powers of q, these are sometimes called instanton corrections. For the reader's convenience, we recall the index formula that shows which Maslov numbers are relevant in each term of m_k:

$$\dim \mathcal{M}_{k+1}(L, \beta) = k - 2 + n + \mu_L(\beta). \tag{4.2}$$

In Lemma 13.2 of [Fuk10], it is proved that A is strictly unital and that the structure maps satisfy a divisor axiom: Given $b \in A^1$, an integer $k \geq 0$, elements $x_1, \ldots, x_k \in A$, and $s \geq 0$ another integer then:

$$\sum_{s_0 + \cdots + s_k = s} m_{k+s,\beta}(b^{\otimes s_0}, x_1, b^{\otimes s_1}, \ldots, x_k, b^{\otimes s_k}) = \frac{1}{s!} (\partial\beta \cap b)^s \, m_{k,\beta}(x_1, \ldots, x_k).$$

Recall that when $k = 0$, the element $m_{0,\beta} \in \mathbb{C}$ is simply the regular count of isolated holomorphic discs with boundary on L.

The A_∞-algebra can be deformed using *bounding co-chains*, which are elements $b \in H^1(L, \mathbb{C})$ for which we have an equation:

$$m_1(b) + m_2(b, b) + \cdots = p(b)\mathbf{1}_A, \tag{4.3}$$

where $p(b)$ is an element of $\mathbb{C}[[q]]$. The b-deformed A_∞-structure is given by the equation:

$$m_k^b(x_1, \ldots, x_k) = \sum_{s_0+\cdots+s_k=s} m_{k+s}(b^{\otimes s_0}, x_1, b^{\otimes s_1}, \ldots, x_k, b^{\otimes s_k}).$$

In our setting, equation (4.3) holds automatically, and the assignment:

$$p : H^1(L, \mathbb{C}) \to \mathbb{C}[[q]],$$

is called the *potential function* of L.

4.2 Monotone Floer theory, calculation

We now apply the general framework above to compute Fukaya's A_∞-algebra associated with the monotone Lagrangian torus $L \subset X \backslash D$ constructed in Proposition 2.14. Recall that:

$$X = V(t^{n+1} - f) \subseteq \mathbb{P}^{n+1},$$

where f is a generic homogeneous polynomial of degree $n+1$ in the variables $x_0, \ldots, x_n$, sufficiently close to the product:

$$f_0 = x_0 \cdot x_1 \cdots x_n,$$

which in turn is the defining equation of the toric boundary of $\mathbb{P}^n$. The index 1 Fano hypersurface X above comes with a cyclic covering map (drop t):

$$\phi : X \to \mathbb{P}^n,$$

branched over the zero locus of f. The appropriate Kähler form on X is constructed in Lemma 2.11, the monotone Lagrangian torus of interest is the pre-image $L = \phi^{-1}(L_{\mathrm{cl}})$, and it lives in the complement of the ramification (anti-canonical) divisor D. In particular we have an area formula for discs $\beta \in H_2(X, L)$:

$$\langle \omega, \beta \rangle = \beta \cdot D.$$

This formula explains in particular why we only need the power series ring $\mathbb{C}[[q]]$, as opposed to the Novikov ring $\Lambda_{\mathbb{C}}$.

Next, if we use the dimension formula (4.2), one sees that only discs of Maslov number 2 contribute to the potential function p. As one expects, this potential function is tightly related to the Landau-Ginzburg potential W that we computed in (3.20). The only difference is that when we defined W, we did not take areas into account, and as such we don't have the extra parameter q. Indeed, each bounding co-chain $b \in H^1(L, \mathbb{C})$ gives a local system ξ_b on L:

$$\begin{aligned} \xi_b : \pi_1(L) &\to \mathbb{C}^* \\ \gamma &\mapsto \exp(\gamma \cdot b). \end{aligned}$$

Using the divisor axiom, it can be seen that:

$$p(b) = qW(L, \xi_b). \tag{4.4}$$

In fact, when we compute the b-deformed A_∞-algebra structure, it is the same as computing Fukaya's A_∞-algebra structure for (L, ξ_b).

Let $x, y \in H^1(L, \mathbb{C}) \subseteq A^1$ be degree 1 elements in our A_∞-algebra A. Observe that we have:

$$m_1^b(x) = (dp)_b(x) \quad \text{and} \quad m_2^b(x, y) = (d^2 p)_b(x, y). \tag{4.5}$$

In fact, similar formulae hold for higher A_∞-products as well.

Lemma 4.1. *When b is a non-degenerate critical point of p, $m_1^b = 0$, and we have an isomorphism of associative algebras:*

$$(A, m_2^b) \cong \mathrm{Cl}(H^1(L, \mathbb{C})) \otimes \mathbb{C}[[q]].$$

Proof. The two equations in (4.5) already give the desired result on A^1, and it suffices to show that A is generated in degree 1.

Notice that if we drop the instanton corrections, the resulting A_∞-algebra $A_0 = A \otimes \mathbb{C}[[q]]/(q)$ computes Fukaya's A_∞-algebra of the exact Lagrangian manifold L in the exact symplectic manifold $X \backslash D$, which is a formal exterior algebra on its degree 1 part.

Now let $A_+ = \oplus_{i \geq 1} A^i$ be the ideal of all elements of positive degree, and consider the product map:

$$m_2^b : A_+^{\oplus 2} \to A_+.$$

This is a map of finitely generated $\mathbb{C}[[q]]$-modules and, by our previous observation, it is surjective when restricted to the fiber at 0; the unique maximal ideal of $\mathbb{C}[[q]]$. By Nakayama's lemma, we deduce that m_2^b is surjective. Next, using the Leibniz rule, one sees that the differential m_1^b vanishes identically, and that A is generated in degree 1. Therefore, the product structure is that of the usual Clifford algebra associated with $(d^2p)_b$. But recall that b was assumed to be a non-degenerate critical point, so the lemma follows. $\square$

Next, we need to compute the A_∞-category associated with L over $\mathbb{C}$. The underlying (now $\mathbb{Z}_2$-graded) vector space is:

$$\mathcal{A} = H^*(L, \mathbb{C}),$$

and the A_∞-structure maps $(\mu_k)_{k\geq 1}$ are the evaluations of $(m_k)_{k \geq 1}$ (from (4.1)) at $q = 1$. There are no convergence issues to worry about because L is monotone.

Proposition 4.2. *The $\mathbb{Z}_2$-graded A_∞-algebra $\mathcal{A}$ is the formal Clifford algebra* $\mathrm{Cl}_n(\mathbb{C})$ *.*

Proof. By combining the identity (4.4), and the formula of W from (3.20), we see that $b = 0$ is a critical point of the the potential function p. Going back to the proof of Lemma 4.1, we have seen that $m_1^0 = 0$, and that m_2^0 is given by the Hessian of p at 0. By setting $q = 1$, we get that $\mu^1 = 0$, and that μ^2 follows the Hessian of a non degenerate function on $H^1(L, \mathbb{C})$. It follows that $H(\mathcal{A})$ is the Clifford algebra $Cl_n(\mathbb{C})$, which is known to be intrinsically formal: see for example [She16], Corollary 6.4. $\square$

Remark 4.3. *This method of computing Floer cohomology appears in the work of Sheridan (see [She16], Theorem 4.3) and also in the work of Fukaya-Ohta-Ono-Oh (see [Fuk+10a], Theorem 5.5), and before them in the work of Cho (see [Cho05], theorem 5.6, also corollary 6.4).*

4.3 The B-side and HMS

The homological algebra of isolated hypersurface singularities is greatly studied in the work of Dyckerhoff [Dyc11]. It is shown there that $\mathrm{D}^{\pi}_{\mathrm{sg}}(W^{-1}(w_s))$ is generated by the skyscraper sheaf $\mathcal{O}_p$ of the singular point. It is also shown in [Dyc11] (see also [Orl11]) that this category only depends on the formal completion of a neighborhood of the singular point. In particular, we have an equivalence of triangulated categories:

$$\mathrm{D}^{\pi}_{\mathrm{sg}}(W^{-1}(w_s)) = D^{\pi}\mathrm{MF}(\mathbb{C}[[z_1, \dots, z_n]], z_1^2 + \cdots + z_n^2).$$

In the equivalence above, passing to matrix factorizations requires a stabilization procedure explained in section 2 of [Dyc11]. The category MF of matrix factorizations is a $\mathbb{Z}_2$-graded category and in this case it is generated by the (*stablization* of the) residue field $\mathbb{C}$. In section 5.5 of that same paper, the self-hom space is computed to be $\mathrm{Cl}_n(\mathbb{C})$ with an identically vanishing differential.

Combining all of this together, we get:

Lemma 4.4. *There is an equivalence of triangulated categories:*

$$\mathrm{D}^{\pi}_{sg}(W^{-1}(w_s)) = \mathrm{D}^{\pi}\, Cl_n(\mathbb{C}).$$

Remark 4.5. *We refer the reader to the work of J. Smith in [Smi19], for a recent treatment of the homological algebra of isolated hypersurface singularities that is more adapted to homological mirror symmetry.*

We have now collected all the necessary ingredients to prove the main result of this chapter.

proof of Theorem 1.2 Recall that the eigenspace corresponding to w_s in:

$$c_1 \star (-) : \mathrm{QH}(X) \to \mathrm{QH}(X),$$

has dimension 1, and as a consequence, any object in $\mathrm{Fuk}(X)_{w_s}$ with non-zero Floer cohomology will split-generate. Refer to Corollary 2.19 and Proposition 7.11 of [She16] for more details. In particular, the monotone Lagrangian torus L split-generates. We have already computed its associated Fukaya A_∞-algebra in Proposition 4.2. Combining that with the result of Lemma 4.4, we deduce the desired equivalence:

$$\mathrm{D}^\pi \mathrm{Fuk}(X)_{w_s} \cong \mathrm{D}^\pi_{\mathrm{sg}}(W^{-1}(w_s)).$$

$\square$

Chapter 5

HMS in the toric limit

In Proposition 3.17, we counted Maslov index 2 discs with boundary on the monotone Lagrangian torus $L \subseteq X_f$ constructed in Proposition 2.14, where X_f is the smooth index 1 Fano hypersurface in projective space $\mathbb{P}^{n+1}$, cut-out by an equation of the form:

$$X_f = V(t^{n+1} - f(x_0, \ldots, x_n)),$$

where f is a homogeneous polynomial of degree $n+1$, that is sufficiently close to the toric boundary $f_0 = x_0 \cdots x_n$. The limit of these index 1 Fano hypersurfaces is the singular toric Fano variety:

$$X_0 = V(t^{n+1} - x_0 \ldots x_n).$$

The super-potential function associated with L has the following formula:

$$W_L = \frac{(1 + y_1 + \cdots + y_n)^{n+1}}{y_1 \cdots y_n} - (n+1)!. \tag{5.1}$$

In the mirror symmetry literature, the pair (X_0, W) is a called a toric Landau-Ginzburg model for the index 1 Fano hypersurface, we refer the reader to [KP14] for more context.

Our goal for this chapter is to study homological mirror symmetry for X_0, which we view as the B-side, and its mirror super-potential W_L, which we view as the A-side. While the translation term $(n+1)!$ is crucial in the full HMS story of index 1 Fano hypersurfaces, it actually has no bearing on the particular version of HMS we consider in the present chapter. Because of that, we simply drop the translation term and work with the following instead:

$$W = \frac{(1 + y_1 + \cdots + y_n)^{n+1}}{y_1 \cdots y_n}. \tag{5.2}$$

We associate with W a Fukaya-Seidel A_∞-category $\mathrm{FS}((\mathbb{C}^*)^n, W)$ using the Lagrangian thimbles of W. We explain how this Fukaya-Seidel category recovers the homogeneous coordinate ring of X_0. More precisely, we prove the following result:

Theorem 5.1. *There is a collection of Lefschetz thimbles L_i in $((\mathbb{C}^*)^n, W)$ such that:*

$$HW(L_i, L_j) \simeq \hom(\mathcal{O}_{X_0}(i), \mathcal{O}_{X_0}(j)).$$

Furthermore, the isomorphisms above are compatible with the relevant product structures.

The main insight we use is a base-cover relationship between $((\mathbb{C}^*)^n, W)$ and $((\mathbb{C}^*)^n, W_{\mathrm{cl}})$, together with a folklore result on homological mirror symmetry for projective space $\mathbb{P}^n$. Indeed, recall that the Landau-Ginzburg model associated with projective space is $((\mathbb{C}^*)^n, W_{\mathrm{cl}})$, where:

$$W_{\mathrm{cl}} = y_1 + \cdots + y_n + \frac{1}{y_1 \cdots y_n}. \tag{5.3}$$

There is a free action of $\mathbb{Z}_{n+1}$ on $(\mathbb{C}^*)^n$ that rotates the coordinates by $(n+1)^{\mathrm{th}}$-roots of unity:

$$\zeta \cdot (y_1, \ldots, y_n) \mapsto (\zeta \cdot y_1 \ldots, \zeta \cdot y_n).$$

The potential function W_{cl} is not $\mathbb{Z}_{n+1}$-invariant, but its power W_{cl}^{n+1} is, and in fact:

$$((\mathbb{C}^*)^n, W) = ((\mathbb{C}^*)^n/\mathbb{Z}_{n+1}, W_{\mathrm{cl}}^{n+1}). \tag{5.4}$$

The quotient map is:

$$\begin{aligned} \pi : (\mathbb{C}^*)^n &\to (\mathbb{C}^*)^n \\ (y_1, \ldots, y_n) &\mapsto (y_1 Y, \ldots, y_n Y), \end{aligned} \tag{5.5}$$

where $Y = y_1 \cdots y_n$. The unbranched covering map π seems to mirror the branched covering map $\phi : X_0 \to \mathbb{P}^n$. This mirror correspondence looks like:

$$\pi^{-1}(-) \longleftrightarrow \phi_*(-)$$
$$\pi(-) \longleftrightarrow \phi^*(-).$$

Our approach to proving Theorem 5.1 is guided by this correspondence: the methods we use suggest that there exists a commutative diagram of triangulated categories:

$$\begin{tikzcd}
D^\pi \mathrm{FS}((\mathbb{C}^*)^n, W) \arrow[r, "\mathrm{HMS}_{X_0}"] \arrow[d, "\psi"] & \mathrm{Perf}(X_0) \arrow[d, "\phi"] \\
D^\pi \mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}) \arrow[r, "\mathrm{HMS}_{\mathbb{P}^n}"] & D^b\mathrm{Coh}(\mathbb{P}^n),
\end{tikzcd}$$

such that both horizontal arrows are equivalences.

5.1 Partially wrapped Floer theory

We fix a base field $k = \mathbb{C}$. The Fukaya-Seidel A_∞-category associated with the Landau-Ginzburg model $((\mathbb{C}^*)^n, W)$ is constructed by counting holomorphic polygons with boundary on (wrappings of) a collection of Lagrangians. The role of W is to *stop* (in the sense of Z.Sylvan [Syl19]) the wrapping at a regular fiber of W. When W has only non-degenerate singularities, this is exactly the Fukaya-Seidel category defined for example in [Sei08]. Because in our case, one of the two singularities of W is non-degenerate, we instead resort to the more recent work of Ganatra-Pardon-Shende in [GPS17] and [GPS18], although our set-up is actually closer to [Abo06]. The Liouville structure on $(\mathbb{C}^*)^n$ comes from the 1-form:

$$\theta = \sum r_i d\theta_i,$$

where (r_i, θ_i) are the radial and angular components of i^{th}-coordinate $y_i \in \mathbb{C}^*$. It can also be seen as the Stein structure coming from the pluri-subharmonic function:

$$h = |y_1| + \cdots + |y_n| . \tag{5.6}$$

Let R be a fixed, large enough positive number. The objects of $FS((\mathbb{C}^*)^n, W)$ are Lefschetz thimbles L_γ corresponding to embedded paths $\gamma : [0,1] \to \mathbb{C}$ such that:

- $|\gamma(1)| = R$ but $\gamma(1) \neq -R$.

- $\gamma(0)$ is a non-degenerate critical value of W.

The first condition means that we will stop our wrapped Floer theory at the Weinstein hypersurface $W^{-1}(-R)$. Such γ is sometimes called a *vanishing path*.

Because we are only restricting to Lefschetz thimbles, we note that this category (even after taking triangulated split-closures) is a-priori *smaller* than the stopped category $\mathrm{WF}((\mathbb{C}^*)^n, W^{-1}(R))$ in the language of [GPS17]. For example, when W is the Laurent polynomial in (5.3), thimbles are enough to recover the full stopped category. However, when W is the Laurent polynomial from (5.1), they are not.

Let L_1 and L_2 be two objects in $\mathrm{FS}((\mathbb{C}^*)^n, W)$. The holomorphic convexity of $(\mathbb{C}^*)^n$, together with exactness of the Lagrangians L_i, ensure that we have the necessary compactness to define a Floer cohomology vector space $HF(L_1, L_2)$ over k. However, these vector spaces fails to be independent of Hamiltonian isotopies. Indeed, as L_1 is wrapped positively to L_1^+ (or L_2 wrapped negatively to L_2^-), the pair (L_1^+, L_2) will likely acquire more intersection points and the vector space $HF(L_1^+, L_2)$ "grows" bigger as a consequence. More accurately, there is a continuation map:

$$c : HF(L_1, L_2) \to HF(L_1^+, L_2).$$

One therefore defines (see [GPS17]) a wrapped Floer cohomology group by the following recipe:

$$HW(L_1, L_2) = \varinjlim_{w} HF(L_1^w, L_2), \tag{5.7}$$

where the limit is taken over all positive wrappings L_1^w that do not cross the stop $W^{-1}(-R)$. This is now invariant under Hamiltonian isotopies, up to canonical isomorphism.

In the case of a pair $(L_{\gamma_1}, L_{\gamma_2})$ of Lefschetz thimbles, this recipe simplifies: we can get positive wrappings of L_{γ_1} by instead wrapping the underlying vanishing path γ_1 around the boundary of the disc of radius R. Notice however that once we wrap γ_1 to a path γ_1^+ whose end-point $\gamma_1^+(1)$ is closer to the stop $-R$ (in the anti-clockwise direction) than $\gamma_2(1)$, we no longer gain any new intersection points by positively wrapping γ even further. As a consequence:

$$HW(L_{\gamma_1}, L_{\gamma_2}) = HF(L_{\gamma_1^+}, L_{\gamma_2}). \tag{5.8}$$

This is basically how stopped Floer cohomology was defined for Fukaya-Seidel categories before Z. Sylvan introduced stops in [Syl19]. See for example [Abo06] section 2, or [Sei08] section 3. These vector spaces can be upraded into an A_∞-category by counting holomorphic polygons:

$$\mu^d : CF(L_{\gamma_{d-1}}, L_{\gamma_d}) \otimes \cdots \otimes CF(L_{\gamma_0}, L_{\gamma_1}) \to CF(L_{\gamma_0}, L_{\gamma_d})[2-d],$$

whenever the sequence of boundary points $\gamma_0(1), \gamma_1(1), \ldots, \gamma_d(1)$ is ordered clock-wise in the arc $\{|z| = R\} \backslash \{-R\}$. Finally, because the Lagrangians L_γ are contractible, they carry canonical spin structures to orient the moduli spaces of holomorphic polygons, and grading data to make $\mathrm{FS}((\mathbb{C}^*)^n, W)$ a k-linear, $\mathbb{Z}$-graded A_∞-category.

In the previous construction, we may stop the wrapping in Floer cohomology even further by adding more stops of the form $W^{-1}(z)$, where z spans a finite subset I of the circle $\{|z| = R\}$. This means that in equations (5.7) and (5.8), the positive wrappings stop before running into either one of the fibers in $W^{-1}(I)$. We denote the resulting A_∞-category by $\mathrm{FS}((\mathbb{C}^*)^n, W, I)$. For example:

$$\mathrm{FS}((\mathbb{C}^*)^n, W, -R) = \mathrm{FS}((\mathbb{C}^*)^n, W).$$

Given two finite collections of stops $I \subseteq J \subseteq \{|z| = R\}$, the extra wrapping the may occur in $\mathrm{FS}((\mathbb{C}^*)^n, W, I)$, produces continuation elements:

$$c_{I \subseteq J} : HW_J(L_1, L_2) \to HW_I(L_1, L_2).$$

These continuation elements can in fact be upgraded to an A_∞-functor:

$$c : \mathrm{FS}((\mathbb{C}^*)^n, W, J) \to \mathrm{FS}((\mathbb{C}^*)^n, W, I),$$

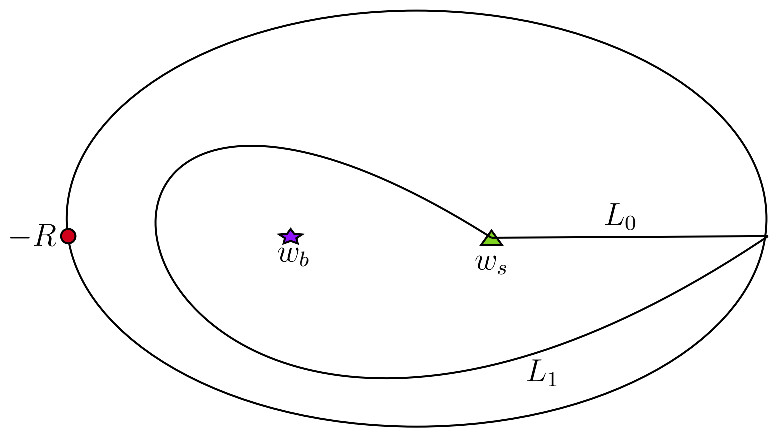
which is sometimes called *stop-removal*. This functor is carefully constructed in [GPS17] and thoroughly studied in [GPS18].

Remark 5.2. *In our presentation here, we work as though k is a field of characteristic 2, so as to avoid cluttering the main ideas with notation. In reality, intersection points of Lagrangians should be interpreted as trivializations of orientation operators coming from the Fredholm theory of the $\overline{\partial}$-equation. We refer the reader to [Sei08], section 11 for the exact details on how this works.*

5.2 The A-side, unbranched coverings

We now restrict our discussion of Fukaya categories to the context of the base-cover relationship in (5.4). The potential function W from (5.2) has one non-degenerate critical value at $w_s = (n+1)^{n+1}$, and then a big critical value $w_b = 0$. Therefore, the Lefschetz thimbles L_γ in $\mathrm{FS}((\mathbb{C}^*)^n, W)$ are classified by their monodromy around 0, which also can be thought of as the intersection number of γ with the segment $(-R, 0)$.

Definition 5.3. *For an integer $i \in \mathbb{Z}$, the Lagrangian $L_i \in ((\mathbb{C}^*)^n, W)$ is the Lefschetz thimble associated with an embedded path $\gamma : [0,1] \to \mathbb{C}\backslash\{0\}$, such that $\gamma(1) = R$, $\gamma(0)$ is the non-degenerate critical value $w_s = (n+1)^{n+1}$, and the path's clockwise winding number around 0, relative to the endpoints w_s and R, is i.*



Some Lefschetz thimbles for W.

The unbranched covering map π from (5.5) induces an A_∞-functor:

$$\pi : \mathrm{FS}((\mathbb{C}^*)^n, W, -R)) \to \mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}, J)),$$

where the collection J of stops is:

$$J = \{z \in \mathbb{C} \mid z^{n+1} = -R\}.$$

At the level of objects, this functor maps a Lagrangian thimble to its pre-image. At the level of hom spaces, the chain map:

$$\pi^1 : CW(L_i, L_j) \to CW_J(\pi^{-1}L_i, \pi^{-1}L_j), \tag{5.9}$$

takes an intersection point $p \in L_i \cap L_j$ to the sum of its pre-images. As an A_∞-functor, the higher components all vanish, i.e $\pi^d = 0$ for all $d \geq 2$. The reason that π^1 above is a chain map (and in fact respects the A_∞-structures) is because the pre-images $\pi^{-1}(L_i)$ have $n+1$ connected components lying in different sheets of the covering map, one for each critical value of W_{cl}. By the homotopy lifting property, a holomorphic strip with boundary on (L_0, L_1) has exactly $n+1$-lifts via π, which again lie each in a different sheet of the covering map.

Remark 5.4. *A few observations regarding the previous definition are in order:*

- *For the picture above to work perfectly, we need to choose the pluri-subharmonic function on the bottom $(\mathbb{C}^*)^n$ to be the descent of h (as in (5.6)) through the covering map.*

- *In the map (5.9), the point p should be replaced by its orientation line $o(p)$. The pre-images $\pi^{-1}(L_i)$ and $\pi^{-1}(L_j)$ inherit their brane structures from those of L_i and L_j. Because π is unbranched, for each intersection point $q \in \pi^{-1}(L_i) \cap \pi^{-1}(L_j)$, there is a canonical isomorphism of orientation lines $o(q) \simeq o(\pi(q))$, this is what should be used to define π^1.*

Next, we push our Lagrangian thimbles to $((\mathbb{C}^*)^n, W_{\mathrm{cl}})$ using the acceleration functor:

$$c : \mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}, J) \to \mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}, s_1),$$

where the stop s_1 is the one located immediately after $\sqrt[n+1]{R}$ in the counter-clockwise direction:

$$s_1 = R^{\frac{1}{n+1}} e^{\frac{\pi i}{n+1}}.$$

Finally we define the A_∞-functor ψ as the composition of π and c:

$$\psi : \mathrm{FS}((\mathbb{C}^*)^n, W)) \to \mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}, s_1). \tag{5.10}$$

We refer the reader to the figure below for some intuition. It turns out that the functor ψ mirrors the pushforward map ϕ_* on perfect complexes.

The Landau-Ginzburg model $((\mathbb{C}^*)^n, W_{\mathrm{cl}})$ has been extensively studied in the literature as the mirror to projective space. P.Seidel studied the case $n = 2$ in [Sei01b], section 3. M. Abouzaid then proved HMS for all smooth toric Fano varieties in [Abo09], and a quick summary of that story in the case of $\mathbb{P}^n$ can be found in D.Auroux's speculations [Aur17], section 7. We will rely on the more recent treatment of Futaki-Ueda in [FU14]. We now briefly recall the elements of that story that are most pertinent to our work.

Following the set-up of the previous discussion, we consider Lagrangian thimbles $\hat{L}_\gamma$ whose underlying vanishing path is an embedding:

$$\gamma : [0, 1] \to \{n + 1 \leq |z| \leq R^{\frac{1}{n+1}}\},$$

satisfying the following properties:

- $|\gamma(1)| = R$ and $\gamma(1) \neq s_1$.

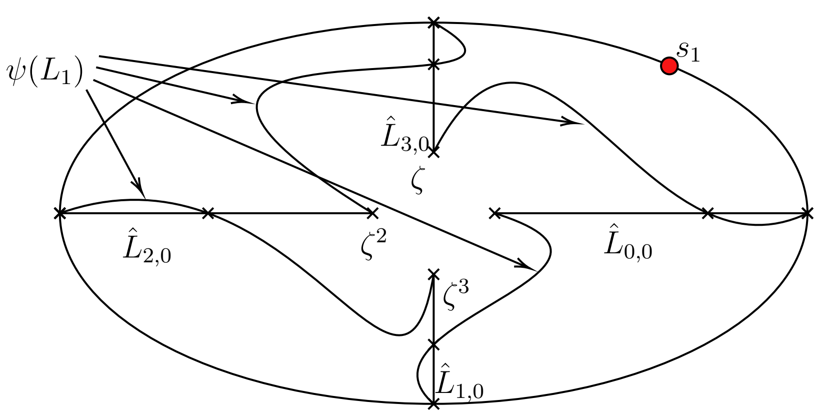
- $\gamma(0)$ is one of the $n+1$ critical values of W_{cl}.

These vanishing paths depend on 2 pieces of data. The first is the choice of a critical value:

$$\gamma(0) = w \in \{n+1, (n+1)\zeta, \ldots, (n+1)\zeta^n\}.$$

After $\gamma(0) = w$ has been fixed, γ only depends on the amount of winding it does with respect to the stop. To quantify this amount, we fix $\gamma_{w,0}$ to be the radial path from w to the circle $\{|z| = R^{\frac{1}{n+1}}\}$. Then $\gamma_{w,i}$ will be obtained from $\gamma_{w,0}$ by further winding the endpoint $\gamma_{w,0}(1)$ in the clockwise direction until it crosses the stop s_1, i times.

Definition 5.5. *Given a critical value $w = (n+1)\zeta^{-k}$ of W_{cl} and an integer $i \in \mathbb{Z}$, the Lagrangian $\hat{L}_{k,i}$ is the Lefschetz thimble associated with the path $\gamma_{w,i}$ as described above. See figure below for examples.*



The action of ψ on L_0 and L_1. This figure is for W_{cl}.

We now state a folklore result in homological mirror symmetry. It will facilitate the comparison between the A-side calculations we do next, with their B-side counterparts. We provide a more detailed discussion of this equivalence in section 5.3.

Theorem 5.6. *(see [FU14], [Abo09]) There is an A_∞-functor:*

$$\theta : FS((\mathbb{C}^*)^n, W_{cl}) \to Coh_{dg}(\mathbb{P}^n),$$

that induces a quasi-equivalence of split-closed triangulated categories:

$$\theta : D^\pi FS((\mathbb{C}^*)^n, W_{cl}) \to D^b(Coh(\mathbb{P}^n)). \tag{5.11}$$

At the level of objects, this functor maps $\hat{L}_{k,i}$ to $\mathcal{O}_{\mathbb{P}^n}(-k+i(n+1))$.

We now go back to the A_∞-functor ψ defined in (5.10). We start by computing its action on objects.

Lemma 5.7. *Let $j \in \mathbb{Z}$ be an integer given in the form $j = q(n+1) + r$ with $0 \leq r \leq n$, and let L_j be the exact Lagrangian from Definition 5.3. Then:*

$$\psi(L_j) = \bigoplus_{k=0}^{n} L_{k,j_k}, \tag{5.12}$$

where:

$$j_k = \begin{cases} q & \text{if } 0 \leq k \leq n-r, \\ q+1 & \text{if } k > n-r. \end{cases}$$

Proof. We assume $j \geq 0$ in order to simplify the phrasing of the argument. The Lagrangians $\hat{L}_{k,j_k}$ are the connected components of $\psi(L_j)$, so the direct sum decomposition is automatic. The only work that needs be done is in identifying the winding numbers j_k. In the base $((\mathbb{C}^*)^n, W)$, the wrapping $L_0 \rightsquigarrow L_j$ follows the angles $\exp(-2\pi i t)$, with $0 \leq t \leq j$. When this wrapping is lifted to $\hat{L}_{k,0} \rightsquigarrow \hat{L}_{k,j_k}$, it follows the angles:

$$\theta_t = \exp\left(\frac{2\pi i}{n+1}(n+1-k-t)\right).$$

The integer j_k is now simply the number of times this path of angles crosses the stop $s_1 = \exp\left(\frac{\pi i}{n+1}\right)$. This is the same as counting the number of elements in the set:

$$\left\{ t \in [0,j] \mid t + k + \frac{1}{2} \equiv 0 \mod (n+1)\mathbb{Z} \right\}.$$

Using the Euclidean division $j = q(n+1) + r$, we see that this number is q, plus however many multiples of $n+1$ are in the interval:

$$\left[k + \frac{1}{2}, k + r + \frac{1}{2}\right].$$

Because $k, r < n$, this interval either contains 1 such multiple (if $k + r > n$) or none at all (if $k + r \leq n$). The formula for j_k then follows. □

Remark 5.8. *In light of the homological mirror symmetry statement in Theorem 5.6, it is worth noting that the numbers j_k in the previous Lemma work out perfectly so that:*

$$\theta(\psi(L_j)) = \bigoplus_{k=0}^{n} \mathcal{O}_{\mathbb{P}^n}(j-k).$$

In the direct sum decomposition (5.12) above, the direct summand with index $k_+ = n+1-r$ is "more positive" than all the others. The next lemma makes this idea more precise.

Lemma 5.9. *In the context of the previous lemma, let $p \in \mathbb{Z}$ be another integer. Then the composition:*

$$HW(L_j, L_p) \to HW(\psi(L_j), \psi(L_p)) \to HW(\hat{L}_{k_+, j_{k_+}}, \psi(L_p)).$$

is an isomorphism.

Proof. Observe that the composition:

$$HW(L_j, L_p) \to HW(\pi^{-1}(L_j), \pi^{-1}(L_p)) \to HW(\hat{L}_{k,j_k}, \pi^{-1}(L_p)), \tag{5.13}$$

is an isomorphism for all $k = 0, \ldots, n$, because the intersection points in $CF(L_j, L_p)$ are in 1-to-1 correspondence with those of $CF(\hat{L}_{k,j_k}, \pi^{-1}(L_p))$, and the pair $(\hat{L}_{k,j_k}, \pi^{-1}(L_p))$ acquires no further wrapping in the category $\mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}, J))$. When we remove all the stops but s_1, many of the pairs $(\hat{L}_{k,j_k}, \pi^{-1}(L_p))$ will acquire more wrapping. This phenomenon can be studied by examining the angle where $\hat{L}_{k,j_k}$ hits the boundary. This angle is:

$$\frac{-2\pi}{n+1}(k+j).$$

Recall that the stop s_1 sits at an angle of $\pi/(n+1)$. In particular, when $k = k_+$, the boundary of $\hat{L}_{k,j_k}$ is as close to the stop as any $\hat{L}_{k,p_k}$ can be. In particular, the pair $(\hat{L}_{k_+,j_{k_+}}, \psi(L_p))$ is sufficiently wrapped, and the Lemma now follows from (5.13). $\square$

Remark 5.10. *In light of the homological mirror symmetry statement in Theorem 5.6, the previous Lemma mirrors the adjunction isomorphism:*

$$\hom_{X_0}(\mathcal{O}_{X_0}(i), \mathcal{O}_{X_0}(j)) \to \hom(\mathcal{O}_{\mathbb{P}^n}(i), \phi_*\mathcal{O}_{X_0}(j)).$$

The previous Lemma computes $HW(L_j, L_p)$ as a quotient (as opposed to a subspace) of $HW(\psi(L_j), \psi(L_p))$. While that is enough the compute these wrapped Floer cohomologies as vector spaces, it unfortunately loses most of the information in the product structure. In order to compute the embedding:

$$HW(L_j, L_p) \to HW(\psi(L_j), \psi(L_p)),$$

we will need to appeal to an extra grading datum that comes from topological aspects of Fukaya-Seidel categories.

5.3 HMS for projective space, review

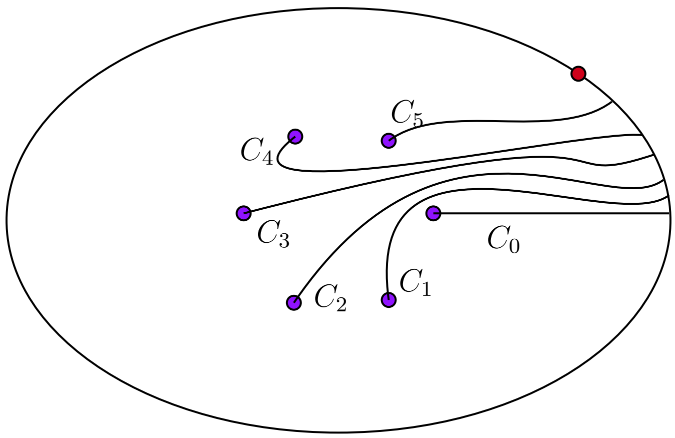
In this section, all vector spaces are defined over a fixed base field k. We review some of the literature pertaining to homological mirror symmetry for projective space $\mathbb{P}^n$. It was studied by Paul Seidel (when $n = 2$ in [Sei01a]), Abouzaid in [Abo09], and more recently by Futaki-Ueda in [FU14]. The folklore result discussed in all these references is an equivalence of triangulated categories:

$$\theta : D^\pi\mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}}) \to D^b\mathrm{Coh}(\mathbb{P}^n). \tag{5.14}$$

Because $\mathbb{P}^n$ is Fano, the equivalence above can be fixed (for example) by setting $\theta(\hat{L}_{0,0}) = \mathcal{O}_{\mathbb{P}^n}$, and then choosing homogeneous coordinates on $\mathbb{P}^n$. Note that $\hat{L}_{0,0}$ is a cotangent fiber of $(\mathbb{C}^*)^n$. This uniqueness of choice in θ sets some expectations on how the functor θ should behave, and this section is devoted to establishing some of them. In particular, we provide a more or less topological description of $D^\pi FS((\mathbb{C}^*)^n, W_{\mathrm{cl}})$.

5.3.1 Algebraic computations

In [FU14], Futaki and Ueda consider a collection of graded Lagrangian thimbles $C_0, C_1, \ldots, C_n$ in $FS((\mathbb{C}^*)^n, W_{\mathrm{cl}})$ that we can best describe with the following figure:



Futaki-Ueda thimbles for $n = 5$.

Their main theorem is the following computation:

Theorem 5.11. *(see [FU14]) Let V be a vector space in degree 0 of dimension $n+1$. Then for each pair of Lefschetz thimbles C_i and C_j, we have an isomorphism of graded vector spaces:*

$$HW(C_i, C_j) \simeq \bigwedge^{j-i} (V[-1]) \, . \tag{5.15}$$

Furthermore, these isomorphisms match the triangle product in the Fukaya-Seidel category with the wedge product. The higher A_∞-operations all vanish.

On the B-side of things, this collections mirrors (a twist of) Beilinson's dual collection, which classically is the full exceptional collection:

$$\mathcal{C}(-1) = \langle \Omega^n_{\mathbb{P}^n}(n)[n], \Omega^{n-1}_{\mathbb{P}^n}(n-1)[n-1], \ldots, \Omega^1_{\mathbb{P}^n}(1)[1], \mathcal{O}_{\mathbb{P}^n} \rangle.$$

Because of choices we made on the A-side, we twist this collection by $\mathcal{O}_{\mathbb{P}^n}(1)$, the resulting collection will then be denoted $\mathcal{C}$:

$$\mathcal{C} = \langle \Omega^n_{\mathbb{P}^n}(n+1)[n], \Omega^{n-1}_{\mathbb{P}^n}(n)[n-1], \dots, \Omega^1_{\mathbb{P}^n}(2)[1], \mathcal{O}_{\mathbb{P}^n}(1) \rangle.$$

The A_∞-equivalence between the full exceptional collections $\mathcal{C}$ and the Lefschetz thimbles $\langle C_0, \dots, C_n \rangle$, induces an equivalence of triangulated categories as in (5.14).

The relationship between the collection $\mathcal{C}$ and the collection of thimbles $\hat{L}_{k,0}$ we introduced earlier, is Koszul duality.

Lemma 5.12. *(see [Sei08], sections 18k,18l) In the A_∞-category $FS((\mathbb{C}^*)^n, W_{cl})$, the collection $\langle \hat{L}_{n,0}, \dots, \hat{L}_{0,0} \rangle$ is the Koszul dual collection to $\langle C_0, \dots, C_n \rangle$.*

Koszul duality is customarily denoted with an upper shriek, for example:

$$\hat{L}_{k,0} = C_k^!.$$

As a consequence, the equivalence θ from (5.14) above maps $\hat{L}_{k,0}$ to $\mathcal{O}_{\mathbb{P}^n}(-k)$, for each $k = 0, \dots, n$. This allows us in particular to compute the hom spaces between them:

$$HW(\hat{L}_{i,0}, \hat{L}_{j,0}) \simeq \mathrm{Sym}^{j-i}(V^\vee), \tag{5.16}$$

whenever $i \leq j$. In order the reach other Lefschetz thimbles of the form $\hat{L}_{k,d}$, the tool we need is Serre duality. On the B-side, the triangulated category $D^b\mathrm{Coh}(\mathbb{P}^n)$ has a Serre functor given by:

$$S(\mathcal{L}) = \mathcal{L}(-(n+1))[n].$$

On the A-side, the Serre functor takes a thimble L to its image under (counter-clockwise) monodromy near infinity, and then shifts the underlying grading by n. Another way to think of this monodromy near infinity is wrapping past the stop. A classical result (see for instance Lemma 1.30 in [Huy+06]) ensures that any triangulated equivalence has to commute with Serre functors. Therefore, it follows that the functor θ from (5.14) satisfies:

$$\theta(\hat{L}_{k,i}) = \mathcal{O}_{\mathbb{P}^n}(-k + i(n+1)).$$

To simplify notation a bit, we now will denote by $\hat{L}_d$ (for $d \in \mathbb{Z}$) any Lagrangian thimble whose image under θ is $\mathcal{O}_{\mathbb{P}^n}(d)$. By means of Serre duality, we can now compute the hom space between all thimbles $\hat{L}_d$. For example:

$$\hom(\hat{L}_0, \hat{L}_{-d}) \simeq \mathrm{Sym}^{d-(n+1)}(V)[n],$$

whenever $d \geq n+1$. We also note that these isomorphisms respect the product structures too.

5.3.2 Topological computations

We begin with the observation that the A_∞-category $\mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}})$ carries a topological grading by the relative homology group:

$$\hat{G} = H_1((\mathbb{C}^*)^n, \mathrm{Crit}(W_{\mathrm{cl}}), \mathbb{Z}), \tag{5.17}$$

where $\mathrm{Crit}(W_{\mathrm{cl}})$ is the (finite) collection of critical points of W_{cl}. This grading associates with each Hamiltonian $y : [0,1] \to (\mathbb{C}^*)^n$ from a Lefschetz thimble L to another Lefschetz thimble L', an element $\deg_{\hat{G}}(y) \in \hat{G}$ by connecting $y(0)$ to the vanishing point of L (without leaving L), and $y(1)$ to the vanishing point of L' (without leaving L') and then taking the homology class of the resulting path in $\hat{G}$. Because of its topological nature, this $\hat{G}$-grading is preserved by all Floer theoretic constructions. This includes continuation maps, TQFT structures, A_∞-operations, twists and mutations.

This topological grading however, is a bit too fine for our purposes: For example, in the computation of Futaki-Ueda 5.11, the vector space V inherits different $\hat{G}$-gradings from the different isomorphisms:

$$HW(C_k, C_{k+1}) \simeq V[-1].$$

We can remedy this issue by identifying all $n+1$ critical points of W_{cl} in the homology group defining $\hat{G}$ (see (5.17)). We do so by means of the projection map:

$$\pi : \hat{G} \to H_1((\mathbb{C}^*)^n, x_0),$$

where x_0 is the unique non-degenerate critical point of W. Observe that the group:

$$G = H_1((\mathbb{C}^*)^n, x_0),$$

naturally grades the Fukaya-Seidel category $\mathrm{FS}((\mathbb{C}^*)^n, W)$, and the collapsing map $\pi : \hat{G} \to G$ makes $\mathrm{FS}((\mathbb{C}^*)^n, W_{\mathrm{cl}})$ a G-graded A_∞-category as well.

Remark 5.13. *The group G is isomorphic to $\mathbb{Z}^n$ but we are not fixing an isomorphism yet. The G-grading on the A-side should be compared with the toric grading on $D^b Coh(\mathbb{P}^n)$ in the B-side (see [BKR01] for instance).*

Lemma 5.14. *There is a G-grading on the vector space V so that the isomorphisms in (5.15) are all G-graded.*

Proof. This is best seen from the isomorphisms in (5.16), because the G-grading in $HW(\hat{L}_{k+1,0}, \hat{L}_{k,0})$ is inherited from the one in $HW(L_0, L_1)$ via the map π, independently of $k = 0, 1, \ldots, n-1$. It follows that V has a G-grading such that the isomorphisms:

$$HW(\hat{L}_{k+1,0}, \hat{L}_{k,0}) \simeq V,$$

are G-graded for $k = 0, \ldots, n$. Using the Serre functor, we can take any integer $d \in \mathbb{Z}$, and isotope the pair (L_d, L_{d+1}) past the stop sufficiently many times in order to get get an isomorphism:

$$HW(\hat{L}_d, \hat{L}_{d+1}) \simeq HW(\hat{L}_{k+1,0}, \hat{L}_{k,0}),$$

for some $k = 0, \ldots, n$. As a consequence, the isomorphism:

$$HW(\hat{L}_d, \hat{L}_{d+1}) \simeq V,$$

is G-graded for all $d \in \mathbb{Z}$. Next, whenever $i < j$, we have a G-graded surjective map:

$$HW(\hat{L}_{j-1}, \hat{L}_j) \otimes \cdots \otimes HW(\hat{L}_i, \hat{L}_{i+1}) \to HW(\hat{L}_i, \hat{L}_j),$$

given by iterated composition (not to be confused with the A_∞-structure maps). Because this map is surjective, one deduces that the isomorphisms in (5.16) all respect the G-grading. Now the lemma follows from an application of Koszul duality to the collection $(\hat{L}_{n,0}, \ldots, \hat{L}_{0,0})$. $\square$

We now consider the *weight* decomposition of V with respect to G:

$$V = \ell_{g_0} \oplus \ell_{g_1} \oplus \cdots \oplus \ell_{g_n}, \tag{5.18}$$

where $g_0, \ldots, g_n$ are elements of G, and ℓ_g denotes a one dimensional vector space where all non-zero elements have degree g. We will see later that in this decomposition, all g_k are distinct, but we do not assume that for now.

Lemma 5.15. *In the group G, we have the following relation:*

$$g_0 + g_1 + \cdots + g_n = 0.$$

Proof. This Lemma is purely topological, but we exploit known Floer theoretic calculations to prove it. From the Koszul duality isomorphism in Lemma 5.15, (ii) of [Sei08], we have a G-graded isomorphism:

$$\hom(C_0, \hat{L}_n^!) \simeq \hom(\hat{L}_0, \hat{L}_1[n])^\vee,$$

because $\hat{L}_n^! = C_n$. We therefore get a G-graded isomorphism:

$$\wedge^n V \simeq V^\vee.$$

Now the lemma follows by comparing the sum of the weights (as in (5.18)) appearing on both sides of the isomorphism above. □

Lemma 5.16. *The group G has the following presentation:*

$$G = \mathbb{Z}g_0 \oplus \cdots \oplus \mathbb{Z}g_n / \langle g_0 + \cdots + g_n \rangle.$$

Proof. Because of the previous lemma, together with the fact that G is a free abelian group of rank n, it suffices to show that the elements g_i generate the group G. Let $G' \subseteq G$ be the subgroup generated by $g_0, \ldots, g_n$. Because of Lemma 5.14, all of the partially wrapped Floer cohomology vector spaces $HW(\hat{L}_i, \hat{L}_j)$ are G'-graded. Next, using the isomorphisms:

$$HW(L_i, L_j) \to HW(\hat{L}_{0,i}, \psi(\hat{L}_j)),$$

we deduce that the cohomology vector spaces $HW(L_i, L_j)$ are also G'-graded. At the same time, the wrapping sequence:

$$L_0 \to L_1 \to \cdots \to L_i \to \cdots$$

computes the fully (unstopped) wrapped Floer cohomology algebra $\mathcal{W}(L_0)$ of L_0, as the limit:

$$\varinjlim_i HW(L_0, L_i) = \mathcal{W}(L_0).$$

It follows that the (unstopped) wrapped Floer cohomology is also G'-graded. However, the later is canonically given by:

$$\mathcal{W}(L_0) \simeq k[G],$$

where the right hand side is the group algebra of G. As a consequence, $G' = G$ and the Lemma follows. $\square$

we reorganize all of the previous discussion in the following theorem.

Theorem 5.17. *There is a group isomorphism $\alpha : G \to \mathbb{Z}^n$ and an equivalence of triangulated categories:*

$$\theta : D^\pi FS((\mathbb{C}^*)^n, W_{cl}) \to D^b Coh(\mathbb{P}^n),$$

with the following properties:

- *At the level of objects, we have $\theta(\hat{L}_{k,i}) = \mathcal{O}_{\mathbb{P}^n}(-k + i(n+1))$. We also use the notation $\hat{L}_d = \hat{L}_{k,i}$ whenever $d = -k + i(n+1)$.*

- *At the level of hom-spaces, the linear isomorphisms:*

$$\theta : \hom(\hat{L}_i, \hat{L}_j) \to \hom(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(j))$$

map a Hamiltonian chord of topological degree $g \in G$, to the monomial $x^{\alpha(g)}$.

Remark 5.18. *In item 2 of the previous theorem, in the case where $j < i$, we still think of* $\hom(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(j))$ *as a vector space of monomials by means of Serre duality:*

$$\hom(\mathcal{O}_{\mathbb{P}^n}(i), \mathcal{O}_{\mathbb{P}^n}(j)) \simeq \hom(\mathcal{O}_{\mathbb{P}^n}(j), \mathcal{O}_{\mathbb{P}^n}(i-n-1))^\vee[n].$$

5.4 B-side calculations

We now carry out some calculations on the algebraic geometry side of homological mirror symmetry to understand the category $\mathrm{Perf}(X_0)$. We will heavily rely on the structure of the cyclic covering map $\phi : X_0 \to \mathbb{P}^n$ and the action of $\mathbb{Z}_{n+1}$ on X_0 as deck transformations. To begin with, observe that for any coherent sheaf $\mathcal{G}$ on X_0, we have a natural isomorphism of sheaf cohomology:

$$\hom^i(\mathcal{O}_{X_0}, \mathcal{G}) \to \hom^i(\mathcal{O}_{\mathbb{P}^n}, \phi_*\mathcal{G}). \tag{5.19}$$

It comes from a composition of the pushforward map:

$$\hom^i(\mathcal{O}_{X_0}, \mathcal{G}) \to \hom^i(\phi_*\mathcal{O}_{X_0}, \phi_*\mathcal{G}),$$

with the structure map $\iota : \mathcal{O}_{\mathbb{P}^n} \to \phi_*\mathcal{O}_{X_0}$. Because ϕ is a cyclic covering, we actually have an isomorphism of $\mathcal{O}_{\mathbb{P}^n}$-modules:

$$\phi_*\mathcal{O}_{X_0} \simeq \mathscr{E}, \tag{5.20}$$

where $\mathscr{E}$ is the locally free sheaf:

$$\mathscr{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-n).$$

This isomorphism endows $\mathscr{E}$ with the structure of a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-algebras, which in turn completely determines X_0. We also remind the reader that the vector bundle $\mathscr{E}$ split-generates the triangulated category $D^b\mathrm{Coh}(\mathbb{P}^n)$. We fix an injective resolution I of the structure sheaf $\mathcal{O}_{X_0}$, and we use it to build a dg-model $\mathscr{C}_{\mathrm{dg}}$ for $\mathrm{Perf}(X_0)$ as follows:

$$\mathscr{C}_{\mathrm{dg}}(i,j) = \hom^\bullet_{X_0}(I(i), I(j)). \tag{5.21}$$

Because ϕ is a finite map, the sheaf ϕ_*I is an injective resolution for $\mathscr{E}$. We can therefore use it to produce a dg-model for $\mathbb{P}^n$ as well:

$$\mathscr{A}_{\mathrm{dg}}(i,j) = \hom^\bullet_{\mathbb{P}^n}(\phi_*I(i), \phi_*I(j)).$$

Note in particular that we have a dg-pushforward map:

$$\phi_* : \mathscr{C}_{\mathrm{dg}} \to \mathscr{A}_{\mathrm{dg}}.$$

At the level of cohomology, this functor becomes a faithful (but not full) embedding $H(\phi) : H(\mathscr{C}_{\mathrm{dg}}) \to H(\mathscr{A}_{\mathrm{dg}})$. The next lemma shows an instance of how the image of $H(\phi)$ remembers the cyclic covering it came from.

Lemma 5.19. *Let $X_f = V(t^{n+1} - f(x_0, \ldots, x_n)) \subseteq \mathbb{P}^{n+1}$ be a degree $n+1$ hypersurface, and let $\phi : X_f \to \mathbb{P}^n$ be the branched covering map that "forgets t". The pushforward of the homomorphism $(-) \times t : \mathcal{O}_{X_f} \to \mathcal{O}_{X_f}(1)$ using the covering map ϕ has the formula:*

$$\phi_*((-) \times t) = \mathrm{id}_{\mathcal{O}} \oplus \mathrm{id}_{\mathcal{O}(-1)} \oplus \cdots \oplus \mathrm{id}_{\mathcal{O}(-n+1)} \oplus (\mathcal{O}(-n) \xrightarrow{(-)\times f} \mathcal{O}(1)).$$

Proof. Let $R = \mathbb{C}[x_0, \ldots, x_n]$ and $S = R[t]/(t^{n+1} - f)$ be the homogeneous coordinate rings defining the varieties $\mathbb{P}^n$ and X_f, respectively. Then the line bundle decomposition in (5.20) is the sheafy version of the direct sum decomposition of graded R-modules:

$$S = R \oplus R(-1) \oplus \cdots \oplus R(-n),$$

where the inclusion $R(-k) \to S$ is multiplication by t^k. The Lemma then follows from interpreting the map $(-) \times t \in \hom_S(S, S(1))$ in terms of this decomposition. $\square$

The previous lemma (at least in principle) is enough to determine the entire image of the functor $H(\phi)$. However, there is another approach that we favor in doing this computation, and it involves extra grading data that our categories come with.

We now explain how the categories $H(\mathscr{A}_{\mathrm{dg}})$ and $H(\mathscr{C}_{\mathrm{dg}})$ carry a grading by $\mathbb{Z}^n$ that we call the *toric grading*. We begin by fixing an action of $T = (\mathbb{C}^*)^n$ on $\mathbb{P}^n$ and X_0 as follows:

$$(\zeta_1, \ldots, \zeta_n) \cdot [x_0 : \cdots : x_n] = [\zeta^{-1} x_0 : \zeta_1 x_1 : \cdots : \zeta_n x_n] \quad \text{on } \mathbb{P}^n, \tag{5.22}$$
$$(\zeta_1, \ldots, \zeta_n) \cdot [t : x_0 : \cdots : x_n] = [t : \zeta^{-1} x_0 : \zeta_1 x_1 : \cdots : \zeta_n x_n] \quad \text{on } X_0,$$

where:

$$\zeta = \zeta_1 \zeta_2 \cdots \zeta_n.$$

Note in particular that $\phi : X_0 \to \mathbb{P}^n$ is T-equivariant.

Let Y be a projective variety with an action of T on it. This action produces a *consistent* choice of isomorphisms for all $\zeta \in T$:

$$\mathcal{O}_Y \to \zeta^*\mathcal{O}_Y$$
$$g \mapsto \zeta^* g,$$

that pulls-back regular functions on open subsets of Y using the torus action. This *consistent* choice of isomorphisms is called a *linearization*; we refer the reader to [BKR01] for a more detailed treatment of this idea. If $D \subseteq Y$ is T-invariant divisor, then we can similarly pull-back meromorphic functions to produce a linearization of $\mathcal{O}_Y(D)$. When two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ are linearized, the vector space $\hom_Y(\mathcal{F}, \mathcal{G})$ carries a T-action via the diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\sigma"] \arrow[d] & \mathcal{G} \arrow[d] \\ t^*\mathcal{F} \arrow[r, "t^*\sigma"] & t^*\mathcal{G}. \end{tikzcd} \tag{5.23}$$

As a consequence, the finite dimensional T-representation $\hom_Y(\mathcal{F}, \mathcal{G})$ carries a weight-decomposition, which is the toric grading by $\mathbb{Z}^n$ that we have alluded to before. By specializing the previous discussion to $Y = \mathbb{P}^n$, and then to $Y = X_0$, we deduce the following:

Lemma 5.20. *The categories $H(\mathscr{A}_{\mathrm{dg}})$ and $H(\mathscr{C}_{\mathrm{dg}})$ carry toric gradings by $\mathbb{Z}^n$. Furthermore, because ϕ is T-equivariant, the functor $H(\phi)$ respects this grading.* □

Going back to the discussion following Lemma 5.19, we get a practical description of the pushforward map as follows:

Lemma 5.21. *For each integer d, and $v \in \mathbb{Z}^n$, there is at most one monomial in $\hom_{X_0}(\mathcal{O}_{X_0}, \mathcal{O}_{X_0}(d))$ whose toric degree is v. Moreover, when such a monomial exists, its pushforward using ϕ is the sum of all $n+1$ monomials of degree v in the direct sum decomposition of $\hom_{\mathbb{P}^n}(\mathscr{E}, \mathscr{E}(d))$.*

Proof. Consider two degree $d \geq 0$ monomials on X_0:

$$t^\alpha x_0^{\alpha_0} \dots x_n^{\alpha_n} \quad \text{and} \quad t^\beta x_0^{\beta_0} \dots x_n^{\beta_n}.$$

Their toric degrees (respectively) are $(\alpha_1-\alpha_0,\dots,\alpha_n-\alpha_0)$ and $(\beta_1-\beta_0,\dots,\beta_n-\beta_0)$. For the two toric degrees to agree, we need the difference $\alpha_k-\beta_k$ to be independent of $k=0,1,\dots,n$. At the same, the two monomials have the same polynomial degree d. It follows that:

$$\beta-\alpha=(n+1)(\alpha_k-\beta_k),$$

for all $k=0,1,\dots,n$. We can however show using these identities that:

$$\frac{t^\beta x_0^{\beta_0}\dots x_n^{\beta_n}}{t^\alpha x_0^{\alpha_0}\dots x_n^{\alpha_n}}=\left(\frac{t^{n+1}}{x_0\dots x_n}\right)^{\alpha_0-\beta_0}.$$

It follows that the two monomials are equal in $\hom_{X_0}(\mathcal{O}_{X_0},\mathcal{O}_{X_0}(d))$.

The second part of the Lemma can be proved in exactly the same way as Lemma 5.19. Finally, the case $d<0$ follows from Serre duality which also respects the toric grading:

$$\hom_{X_0}(\mathcal{O}_{X_0}(d),\mathcal{O}_{X_0}(-1))\otimes\hom_{X_0}(\mathcal{O}_{X_0},\mathcal{O}_{X_0}(d))\to k[n].$$

$\square$

By identifying the toric grading on perfect complexes with the topological grading on Fukaya-Seidel categories, we prove the following upgrade of the isomorphism in Lemma 5.9.

Lemma 5.22. *For each pair of integers i and j, the two embeddings:*

$$HW(L_i,L_j)\xrightarrow{\theta\circ\psi}\hom_{\mathbb{P}^n}(\mathscr{E}(i),\mathscr{E}(j))\xleftarrow{\phi_*}\hom_{X_0}(\mathcal{O}_{X_0}(i),\mathcal{O}_{X_0}(j)),$$

have the same image.

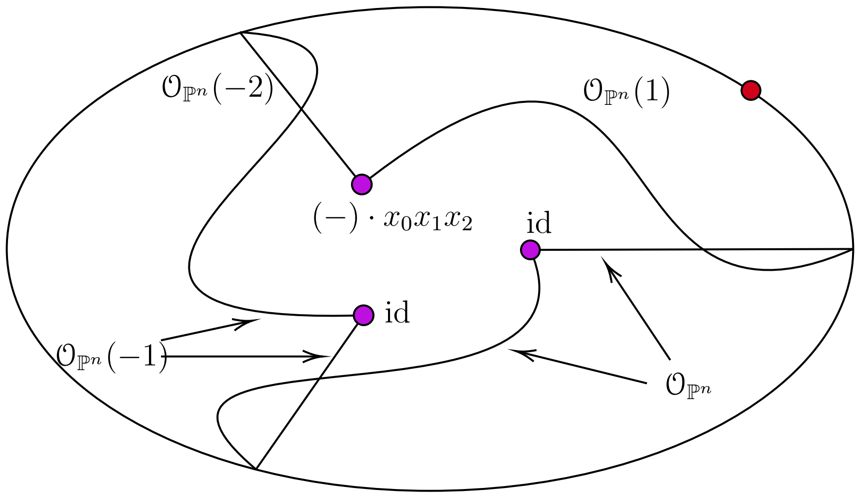
Proof. Indeed, let $p\in HW(L_i,L_j)$ be an intersection point of topological degree $g\in G$. By definition:

$$\psi(p)=p_0+p_1+\dots+p_n,$$

is the sum of all intersection points in $HW(\hat{L}_{k,i_k},\hat{L}_{l,j_l})$ of topological degree g. It follows that in the decomposition:

$$\hom_{\mathbb{P}^n}(\mathscr{E}(i),\mathscr{E}(j))=\bigoplus_{0\le k,l\le n}\hom_{\mathbb{P}^n}(\mathcal{O}(i-k),\mathcal{O}(j-k)),$$

the element $\theta \circ \psi(p)$ is the sum of all monomials of degree $\alpha(g) \in \mathbb{Z}^n$. But, as in Lemma 5.21, this is exactly the image under ϕ_* of the unique monomial in $\hom_{X_0}(\mathcal{O}_{X_0}(i), \mathcal{O}_{X_0}(j))$ whose degree toric degree is $\alpha(g) \in \mathbb{Z}^n$. $\square$



Computation of $\phi_*(t)$ on the A-side

when $n = 2$; compare with Lemma 5.19.

Proof of Theorem 5.1. In our setup, we have the following diagram of A_∞-functors:

$$FS((\mathbb{C}^*)^n, W) \xrightarrow{\theta \circ \psi} \mathscr{A}_{\mathrm{dg}} \xleftarrow{\phi_*} \mathscr{C}_{\mathrm{dg}}.$$

Recall that the differential graded categories $\mathscr{A}_{\mathrm{dg}}$ and $\mathscr{C}_{\mathrm{dg}}$ compute $D^b\mathrm{Coh}(\mathbb{P}^n)$ and $\mathrm{Perf}(X_0)$, respectively. Using the result of the previous lemma, the functors $H(\theta \circ \psi)$ and $H(\phi_*)$ have identical images inside $H(\mathscr{A}_{\mathrm{dg}})$, and the desired theorem follows as a consequence. $\square$

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