On moduli spaces of Ricci-flat 4-manifolds

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Abstract of the Dissertation

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Einstein metrics have long been considered as the canonical metrics in Riemannian geometry. The moduli space of Einstein metrics constitutes a diffeomorphism invariant of the underlying closed smooth manifold. In dimension four, they exhibit a balance between the rigidity of the constant sectional curvature metrics in low dimensions and the flexibility coming from higher dimensions. We show that the moduli spaces of Einstein metrics for a certain family of closed 4-manifolds, the ones which admit a locally hyperKaehler metric, are all path-connected. This is achieved by defining a period map and proving a Torelli-type theorem. In addition, we investigate the existence of almost complex structures and semi-complex structures on these 4-manifolds. Using the representation of the holonomy group, the Teichmüller spaces of all closed oriented flat 4-manifolds are also computed.

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1 Introduction

1.1 Einstein metrics

A Riemannian metric g on a closed smooth manifold M is said to be Einstein if it satisfies the Einstein equation

$$\mathrm{Ric} = \lambda g, \tag{1.1}$$

for some constant λ. In the context of general relativity, this condition is equivalent to the vacuum Einstein field equation with cosmological constant.

According to [1], [8] and [27], it has always been geometers' dream to find a canonical metric on a given smooth manifold M so that all the topological data can be captured by the geometry. The dream came true in dimension 2, for the uniformization theorem states that every closed oriented surface admits a Riemannian metric of constant sectional curvature. In dimension 3, however, a general 3-manifold does not carry any Einstein metrics. Thurston's geometrization conjecture asserts that a 3-manifold can be decomposed into a family of components, each equipped with a geometric structure. The three geometric structures with Einstein metrics are of paramount importance among the eight Thurston geometries. If the dimension is at least 5, the quest for canonical metric might very well be intractable. As pointed out by Gromov in [27], the topology of the group of self-diffeomorphisms $\mathcal{D}(M)$ is usually be very wild and disorderly. There is also the computational issue of the fundamental group, which causes the gradient flow of any appropriate functional on closed curves to badly behave. The final blow is the existence of exotic smooth structures, as brought to light by Milnor in [65]. This somehow shows that the Einstein condition is too flexible and predicts a proliferation of Einstein metrics. This is partially confirmed by the discovery of many inequivalent families of Einstein structures on higher dimensional spheres and other familiar manifolds by Böhm [13] and Boyer-Galicki-Kollár [14].

There is a slim chance that such a geometrization program can be carried out for 4-manifolds and one expects the geometric structures with Einstein metrics to play a central role. Although the structure of $\mathcal{D}(M)$ is more rigid than the higher dimensional case, we still run into the aforementioned difficulties of the existence of exotic smooth structure and the computation of fundamental group. Another reason Einstein metrics in dimension 4 exhibit much more flexibility than in low dimensions is that they do not necessarily have constant sectional curvature. Thus the geometry and topology of 4-manifolds are vastly more complicated. On the flip side, in order for Einstein metric to qualify as a candidate for canonical metric, we hope that there are not too many of them on a given closed smooth 4-manifold. This paper is a very modest attempt to show that the moduli spaces of Einstien metrics are indeed rigid in some sense for a special class of Ricci-flat 4-manifolds. Sadly, despite the success in low dimensions, whether a geometrization program exists for 4-manifolds remains an enigma. The geometry of gauge fields, comprises Donaldson's theory for self-dual instantons and Seiberg-Witten theory for monopoles, is still the most effective tool to study the topology of 4-manifolds.

For a compact smooth manifold M, the *Einstein moduli space* is defined as the quotient

$$\mathcal{E}(M) = \{\text{Einstein metrics on } M\}/(\mathcal{D}(M) \times \mathbb{R}_+),$$

where the group of self-diffeomorphisms $\mathcal{D}(M)$ acts by pulling back and $\mathbb{R}_+$ acts by rescaling. The Einstein moduli space $\mathcal{E}(M)$ admits a ramified cover known as the *Teichmüller space* $\mathcal{T}(M)$, obtained from the above definition if we replace $\mathcal{D}(M)$ by the subgroup of diffeomorphisms isotopic to the identity $\mathcal{D}_0(M)$

$$\mathcal{T}(M) = \{\text{Einstein metrics on } M\}/(\mathcal{D}_0(M) \times \mathbb{R}_+).$$

There are three fundamental questions an inquisitive geometer could pose.

Question 1. Existence: Is $\mathcal{E}(M)$ non-empty? If so, can one produce an explicit construction?

Question 2. Uniqueness: What is the global structure of $\mathcal{E}(M)$?

Quesiton 3. Compactification: What are the different completions or degenerations of $\mathcal{E}(M)$? What are the relations between different compactifications?

Let us now do a quick survey of some known results that address the above three questions. Besson-Courtois-Gallot showed, via a volume entropy comparison theorem, that the standard constant sectional curvature metric on a compact quotient of the 4-dimensional real hyperbolic space $\mathbb{H}^4$ is the unique Einstein metric up to homothety and isometry [11]. In particular, their work gave a new proof of the Mostow rigidity theorem. LeBrun obtained an analogous uniqueness result in [53] for the compact quotient of complex hyperbolic space $\mathbb{CH}^2/\Gamma$ by showing a Riemannian generalization of the Bogomolov-Miyauka-Yau inequality. We remark here that this paper marked the beginning of a series of papers by LeBrun, who pioneered the approach of using Seiberg-

Witten theory to show many rigidity and non-existence results for Einstein metrics. The Einstein moduli spaces for the compact quotients of real and complex hyperbolic space in dimension 4 therefore consist of a single point.

Regarding Question 2, if the Einstein metric is not unique, then the next question we ask is the number of components of M. This modified question is unfortunately not very easy to answer. Consider the unit 4-sphere with the standard round metric g. It is known that any Einstein metric isotropic to g must coincide with g. Thus asking whether we can find a new Einstein metric is equivalent to asking if there are other new components in $\mathcal{E}(S^4)$.

We state here two cases in which the Einstein moduli space is explicitly known. An Einstein metric on a 4-torus T^4 must be flat. The Einstein moduli space is then the moduli space of flat metrics, which can be identified with the quotient $\mathrm{SO}(4)\backslash\mathrm{SL}(4,\mathbb{R})/\mathrm{SL}(4,\mathbb{Z})$ [8]. The other case is that of K3 surfaces. Kodaira showed that all K3 surfaces are diffeomorphic to a quartic in $\mathbb{P}^3$, so it makes sense to consider $\mathcal{E}(M)$ for a K3 surface M. From the work of Kobayashi-Todorov in [40], the Einstein moduli space of a K3 surface is an open subset in $\Gamma\backslash\,\mathrm{SO}(3,19)/\,\mathrm{SO}(3)\times\mathrm{SO}(19)$, where Γ is the automorphism group of the the integral lattice $H^2(M,\mathbb{Z})$. These two classes of manifolds constitute the Calabi-Yau manifolds in complex dimension 2.

Now we review some constructions of Einstein metrics on Fano surfaces, more precisely the 10 different 4-manifolds that arise as del Pezzo surfaces. Page constructed in [71] an explicit Einstein metric with cohomogeneity 1 on $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ and an isometric U(2)-action. An interesting new Einstein metric with an isometric T^2-symmetry was later discovered by Chen-LeBrun-Weber [19] on the two-point blow-up $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$. This metric is Hermitian-Einstein and conformal to a Kähler metric but is non-Kähler itself. Odaka-Spotti-Sun gave a beautiful solution to Question 3 for del Pezzo surfaces in [70]. They showed that the Gromov-Hausdorff compactification of degree d Kähler-Einstein del Pezzo surfaces is homeomorphic to certain algebro-geometric moduli space. Their theorem recovers Tian's result [78] on the existence of Kähler-Einstein metric for a certain family of del Pezzo surfaces. It was proved by LeBrun in [56] and [57] that all the known examples of del Pezzo surfaces sweep out a connected component of $\mathcal{E}(M)$.

In the realm of Kähler geometry, the existence of Kähler-Einstein metric for compact Kähler manifolds is the most important problem. Let (M,J) be a compact Kähler manifold with Kähler form ω. The Einstein equation is commonly written as

$$\mathrm{Ric}(\omega)=\lambda\omega, \tag{1.2}$$

where $\mathrm{Ric}(\omega)$ is the Ricci form of the metric. A Riemannian metric that is both Kähler and Einstein is called a Kähler-Einstein metric. The de Rham cohomology class of ω is proportional to the first Chern class of M, $2\pi c_1(M) = [\mathrm{Ric}(\omega)]$. Calabi conjectured around 1954 that a large class of complex manifolds with $c_1(M) < 0$ or $c_1(M) = 0$ admit Einstein metrics. This conjecture was later extended to the case $c_1(M) > 0$ by Yau, Tian and Donaldson.

Theorem 1.1. *Let M be a compact Kähler manifold with Kähler form ω so that $2\pi c_1(M) = \lambda[\omega]$.*

1. (Aubin, Yau). If $c_1(M) < 0$, then there is a unique Kähler metric $\omega' \in [\omega]$ such that $\mathrm{Ric}(\omega') = \lambda\omega'$.
2. (Yau). If $c_1(M) = 0$, then there is a unique Kähler metric in $\omega' \in [\omega]$ such that $\mathrm{Ric}(\omega') = 0$.
3. (Chen-Donaldson-Sun) If $c_1(M) > 0$, then there is a unique Kähler metric $\omega' \in [\omega]$ such that $\mathrm{Ric}(\omega') = \lambda\omega'$ if and only if M is K-stable.

The complex manifolds in the first case are of general type, while in the second case a manifold with a Ricci-flat Kähler metric is called a Calabi-Yau manifold. The complex manifolds in the last case are the Fano varieties. Theorem 1.1 has far-reaching consequences in algebraic geometry, especially in the moduli theory of varieties. Since the advent of string theory and supersymmetry, we have seen a revitalized interest in the study of Einstein metrics, in particular the Calabi-Yau and G2-metrics. They appear as the small fibers of 10 or 11-dimensinal supersymmetric space-time. Ongoing vibrant and intense investigations about conjectural relations between Calabi-Yau manifolds in mirror symmetry further consolidates the role of Calabi-Yau manifolds as the primary objects of study in modern geometry. Despite considerable attention and effort, we still have no explicit construction of non-trivial Calabi-Yau metrics on compact manifolds at present. All the available examples are done by gluing construction.

We now return to Question 3. The study of the compactification of the Einstein moduli space for 4-manifolds were initiated by Anderson, Bando, Kasue and Nakajima. They showed that the completion $\overline{\mathcal{E}}_{GH}(M)$ of $\mathcal{E}(M)$ with respect to the Gromov-Hausdorff topology is locally compact. $\overline{\mathcal{E}}_{GH}(M)$ comprises the unit-volume Einstein orbifold associated to M. Intuitively, this completion should be understood as filling in the missing lower dimensional components in the interior of $\mathcal{E}(M)$, as opposed to reaching a boundary where

$\mathcal{E}(M)$ comes to an end. Moreover, $\overline{\mathcal{E}}(M)$ need not be compact, as exemplified by $\overline{\mathcal{E}}_{GH}(T^4)$. Thus, to obtain genuine compactifiaction, one has to allow the Gromov-Hausdorff distance to go in infinity. Motivated by the Teichmüller theory for Riemann surfaces, Anderson developed a theory of completion with respect to the L^2 metric on $\mathcal{E}(M)$. The completion $\overline{\mathcal{E}}_{L^2}(M)$ now contains not only the Einstein orbifolds, but also new components known as cusps, where for the latter case convergence is done through the pointed Gromov-Hausdorff topology. For a K3 surface M, the L^2 completion is precisely the quotient $\Gamma\backslash \mathrm{SO}(3,19)/\mathrm{SO}(3)\times\mathrm{SO}(19)$, which is non-compact.

1.2 Local structure of the moduli space

In this section, we will endeavor to expound on the deformation theory for Einstein metrics. Thanks to the works of Berger, Ebin and Koiso, the Einstein moduli space is locally a finite-dimensional Hausdorff real analytic subset with singularities. The key property at play here that ensures the finiteness is the ellipticity of our operator involved.

We begin by giving a brief acccount of some major ingredients in the Kodaira-Spencer-Kuranishi theory for deformation of complex structures. Let M be a compact complex manifold. Denote the holomorphic tangent bundle of M by $\mathcal{T}_M$ and the sheaf of sections of $\mathcal{T}_M$ by Θ_M. Assume $\pi : \mathfrak{X} \to B$ is a *family of deformations* of M, in other words, π is a proper holomorphic submersion whose central fiber $\pi^{-1}(0)$ is isomorphic to M. An application of the Ehressmann theorem shows that all fibers are diffeomorphic, but with possibly varying almost complex structures. The data about complex structures of each fiber can be succinctly encoded by a section φ of $\Gamma^{0,1}(\mathcal{T}_M)$, the space of $\mathcal{T}_M$-valued $(0,1)$-form. Here the section φ depends holomorphically on $t \in B$ by our set-up. Locally, with respect to coordinates z^i on M, it has the form

$$\varphi(t) = \sum_{\alpha,\beta} \varphi^{\beta}_{\alpha}(z,t) d\bar{z}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}} \tag{1.3}$$

If we differentiate this section $\varphi(t)$ with respect to a tangent vector $v \in T_0B$, the resulting section is $\bar{\partial}$-closed, hence is an element in $H^1(M,\Theta_M)$. Thus, we obtain the Kodaira-Spencer map $f : T_0B \to H^1(M,\Theta_M)$, defined by

$$v = \sum_{\gamma} a^{\gamma} \frac{\partial}{\partial t^{\gamma}} \mapsto v\cdot\varphi = \sum_{\alpha,\beta,\gamma} a^{\gamma} \frac{\partial \varphi^{\alpha}_{\beta}}{\partial t}(z,0) d\bar{z}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}, \tag{1.4}$$

which is often interpreted as the derivative of complex structures for a deformation.

The first question one could ask is whether the almost complex structure arises from $\varphi(t)$ is an honest complex structure. By the Newlander-Nirenberg theorem, the integrability condition is equivalent to φ solving the Maurer-Cartan equation

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0. \tag{1.5}$$

One approach to tackle this problem is to first express φ as a formal power series and solve the Maurer-Cartan equation inductively. The second step is then to show that this formal solution converges in a small neighborhood of $0 \in B$. This is indeed the method adopted in the proof of the main existence theorem [44].

Theorem 1.2 (Kodaira, Nirenberg, Spencer)**.** *Let M be a compact complex manifold. If $H^2(M, \Theta_M) = 0$, then there exists a family of deformations of M, $\pi : \mathfrak{X} \to B$ such that*

1. *B is a small ball in $\mathbb{C}^n$ centered at 0 and $\pi^{-1}(0) = M$, where dim $H^1(M, \Theta_M) = n$.*
2. *The Kodaira-Spencer map is an isomorphism.*

The next natural question is if one could always find a universal family for a compact complex manifold M that contains arbitrary small deformations of M. To be more precise, a family of deformations $\pi : \mathfrak{X} \to B$ is said to be *complete* if for any other deformation $\nu : \mathfrak{X}' \to B'$ of M with $\nu^{-1}(a) = M$, there exists a neighborhood U of a and a map $h : U \to B$ such that $\nu^{-1}(U)$ is biholomorphic to the pullback $B \times_B \mathfrak{X}'$. Kodaira and Spencer [45] gave a sufficient and necessary condition as follows.

Theorem 1.3 (Kodaira, Spencer)**.** *A family of deformations of a compact complex manifold M, $\pi : \mathfrak{X} \to B$ with central fiber $\pi^{-1}(0) = M$ is complete at* 0 *if and only if the Kodaira-Spencer map is surjective at* 0*.*

For a complete family of deformations, if in addition the map h is unique, then we call such a family $\pi : \mathfrak{X} \to B$ a *universal* deformation. However, if only the derivative of h is unique, then the complete family is said to be a *versal* deformation. The main theorem is as below [43], [50].

Theorem 1.4 (Kodaira, Spencer, Kuranishi)**.** *A compact complex manifold M always has a versal deformation.*

One could define a map $K : H^1(M, \Theta_M) \to H^2(M, \Theta_M)$, sending a representative $\varphi \in \Gamma^{0,1}(\mathcal{T}_M)$ to $\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi]$, the left hand side of the Maurer-Cartan equation 1.5. Kuranishi's tactic is to apply the Banach space implicit function theorem to show that the preimage $K^{-1}(0)$ in a neighborhood of 0 in $H^1(M, \Theta_M)$, a possibly singular complex analytic subset, is the versal deformation of M. This complex analytic subset is commonly referred to as the *Kuranishi family* of M by algebraic geometers and the map K is called the Kuranishi map.

Due to the above structure theorems, we hope the reader would readily accept the nomenclature of calling $H^1(M, \Theta_M)$ the *deformation space* and $H^2(M, \Theta_M)$ the *obstruction space.*

Let us now return to the study of Einstein moduli space on a closed smooth manifold M. The following material is mostly taken from [8]. Let $\mathcal{M}$ be the set of Riemannian metrics on M while let $\mathcal{M}_1 \subset \mathcal{M}$ be the subspace consists of metrics with unit-volume . In order to describe the local structure of $\mathcal{E}(M)$, a preliminary step is to obtain a submanifold transversal to the orbit of $\mathcal{D}(M)$ in $\mathcal{M}_1$. The ensuing is the slice theorem of Ebin [25], proved in his thesis.

Theorem 1.5 (Ebin)**.** *Fix a Riemannian metric g on M, there exits a real analytic submanifold S_g containing g such that the following properties are satisfied.*

1. $\mathrm{Iso}(g) \cdot S_g = S_g$.
2. *For any $f \in \mathcal{D}(M)$, if $f^* S_g \cap S_g \neq \varnothing$, then $f \in \mathrm{Iso}(g)$.*
3. *There is a neighborhood U of the identity coset in $\mathcal{D}(M)/\mathrm{Iso}(g)$ and a local cross-section $\varphi : U \to \mathcal{D}$ such that $\varphi : U \times S_g \to \mathcal{M}_1$ is a diffeomorphism.*

We call S_g a slice to the action of $\mathcal{D}(M)$. Assume from now on g is an Einstein metric. The collection of Einstein metrics lying in S_g is said to be *the premoduli space of Einstein metrics* around g, denoted by P_g. It is clear that dividing this premoduli space by the action of the isometry group would recover the Einstein moduli space.

For any Riemannian metric $g \in \mathcal{M}$, the tangent space of $\mathcal{M}$ at g is the space of symmetric 2-forms $\Gamma(S^2T^*M)$. The action of the diffeomorphism group $\mathcal{D}(M)$

breaks this space into two orthogonal pieces. Indeed, for any vector field X on M, differentiating the one parameter of metrics $\exp(tX)^*g$ with respect to t yields $L_X g$. Notice that if ζ is a 1-form, then we have the identity $\delta_g^*\zeta = \frac{1}{2}L_{\zeta^\#}g$, where $\delta_g^* : \Gamma(T^*M) \to \Gamma(S^2T^*M)$ is the adjoint of the divergence operator δ_g. So the tangent space to the orbit of $\mathcal{D}(M)$ is simply $\mathrm{Im}\delta_g^*$. The operator δ_g^* is known to be over-determined elliptic. The image $\mathrm{Im}\delta_g^*$ is therefore closed and

$$T_g\mathcal{M} = \mathrm{Im}(\delta_g^*) \oplus \mathrm{Ker}(\delta). \tag{1.6}$$

The extra condition on the normalization of volume means that a tangent vector to $\mathcal{M}_1$ corresponds to a $h \in \Gamma(S^2T^*M)$ whose total trace $\int_M \mathrm{tr}_g h \mathrm{vol}_g$ vanishes. As a consequence, $\mathrm{Im}(\delta_g^*)$ is contained in $T_g\mathcal{M}_1$ and we have a decomposition on $T_g\mathcal{M}_1$ as

$$T_g\mathcal{M}_1 = \mathrm{Im}(\delta_g^*) \oplus (\mathrm{Ker}(\delta) \cap T_g\mathcal{M}_1). \tag{1.7}$$

The slice transversal to the action of $\mathcal{D}(M)$ has the second component as its tangent space.

Before proceeding further, we introduce two more definitions. We call $E(g) = \mathrm{Ric}_g - \frac{1}{n}S(g)g$ the Einstein operator, where $S(g)$ is the total scalar curvature of g. The premoduli space is then $S_g \cap E^{-1}(0)$. An *infinitesimal Einstein deformation* of an Einstein metric g is a tensor $h \in S^2T^*M$ contained in $\mathrm{Ker}(\delta_g) \cap T_g\mathcal{M}_1 \cap \mathrm{Ker}(E')$, or equivalently h satisfies the three conditions

$$E_g'(h) = 0, \quad \delta_g(h) = 0, \quad \int_M \mathrm{tr}_g h \mathrm{vol}_g = 0. \tag{1.8}$$

We denote the space of such tensors as $\epsilon(g) = 0$.

Berger and Ebin showed in [10] that the derivative of the Einstein operator $E_g'(h) = 0$ can be rewritten as $(\nabla^*\nabla - 2\mathring{R}_g)h = 0$, where ∇ is the Levi-Civita connection and R_g is the curvature acting on S^2T^*M. A quick corollary of this result is $\epsilon(g)$ must be finite-dimensional, for the operator $\nabla^*\nabla - 2\mathring{R}_g$ is elliptic.

Theorem 1.6 (Berger-Ebin)**.** *The space of infinitesimal Einstein deformations is finite-dimensional.*

Having examined the deformation theory of complex structures, one reasonably expects to show analogous local structure theorems for Einstein metrics. In particular, we anticipate an existence theorem for Einstein metrics by using formal power series or the implicit function theorem to find solutions to the equation $E(g) = 0$. But there is one more constraint at play here, the image of the Einstein operator satisfies

$$\beta_g(E(g))) = 0, \tag{1.9}$$

where β_g is the Bianchi operator $\delta_g + \frac{1}{2}d\mathrm{tr}g$. Thus, the above equation is also considered as part of the integrability condition, in addition to $E(g) = 0$.

Let $g(t) = \sum_{m=0}^{\infty} t^m/m! \cdot g_m$ be a one-parameter formal series of metrics and let

$$E(g(t)) = \sum_{m=0}^{\infty} \frac{t^m}{m!} E^m(g_0, \cdots, g_m) = E(g_0) + \sum_{m=1}^{\infty} E^m_{g_0}(g_1, \cdots, g_m),$$

where E^m is a polynomial in g_i for each m, with degree one in g_m. Differentiating E^m with respect to t at 0 gives

$$\begin{aligned} E^{m+1}(g_0, \cdots, g_{m+1}) &= \sum_{i=0}^{m-1} \frac{\partial}{\partial g_i} E^m(g_0, \cdots, g_m) \cdot g_{i+1} + E^m(g_0, \cdots, g_{m-1}, g_{m+1}) \\ &= A^{m+1}(g_0, \cdots, g_m) + E^1(g_0, g_{m+1}), \end{aligned}$$

where we have used induction and A^{m+1} is a polynomial in variables $g_0, \cdots, g_m$. If E^1 is surjective, then we could solve for $E^m = 0$, which of course gives a formal solution.

But $E^1_{g_0}$ need not be surjective. In this case, suppose there exists a linear operator Φ_{g_0} depending smoothly on g such that $\mathrm{Im}(E^1_{g_0}) \subset \mathrm{Ker}(\Phi_{g_0})$. Thus every formal power series $g(t)$ starting with g_0 satisfies

$$\Phi_{g(t)}(\frac{d}{dt} E(g(t)) = 0. \tag{1.10}$$

We want to construct a formal solution to the Einstein equation under the restriction 1.10. Assume we have found $g_0, \cdots, g_m$ so that $E^i_{g_0}(g_1, \cdots, g_i) = 0$ for $i = 0, \cdots, m$. In order to find g_{m+1} that solves $E^{m+1}(g_0, \cdots, g_m + 1) = 0$, we plug $g(t) = \sum_0^m t^i/i! \cdot g_i$ into

$$\frac{d^m}{dt^m} \Phi_{g(t)}(\frac{d}{dt} E(g(t))) = 0. \tag{1.11}$$

A straightforward computation applying the Leibniz's rule and the inductive hypothesis yields

$$\Phi_{g_0}(A^{m+1}_{g_0}(g_1, \cdots, g_m)) = 0. \tag{1.12}$$

So $E^{m+1}(g_0, g_1, \cdots, g_{m+1}) = 0$ is equivalent to

$$E^1_{g_0}(g_{m+1}) = -A^{m+1}_{g_0}(g_1, \cdots, g_m). \tag{1.13}$$

For each m, it has a solution whenever the image $\mathrm{Im}(E^1_{g_0})$ coincides with $\mathrm{Ker}(\Phi_{g_0})$. The space $\mathrm{Ker}(\Phi_{g_0})/\mathrm{Im}(E^1_{g_0})$ is called *the obstruction space* for the Einstein condition, where Φ_{g_0} is usually chosen to be the Bianchi operator. Yet the situation here is a little unsatisfactory as we do not have a similar result as Theorem 1.2 in the story for complex structure. This is demonstrated by the next result of Koiso.

Theorem 1.7 (Koiso)**.** *Within $T_g\mathcal{M}_1$ for a unit-volume Einstein metric g, there is an orthogonal decomposition $\mathrm{Ker}(\beta_g) = \mathrm{Im}(E'_g) \oplus \epsilon(g)$.*

What immediately follows is that the deformation space is isomorphic to the obstruction space. In most cases, the obstruction space is non-vanishing. By considering the second order derivative of $E(g)$, Koiso showed in [47] that the symmetric metric on $\mathbb{CP}^{2n} \times S^2$ is isolated in the moduli space $\mathcal{E}(M)$. But the deformation space is non-zero and all the infinitesimal deformations are not integrable. This result shows that in general one should not expect the deformation space $\epsilon(g)$ to be the tangent space of the premoduli space P_g. The tangent space to P_g is a subspace of $\epsilon(g)$, as confirmed by the next result, proved using the implicit function theorem.

Theorem 1.8 (Koiso)**.** *Let g be a unit-volume Einstein metric on M, then the space of infinitesimal Einstein deformations $\epsilon(g)$ can be exponentiated into, within S_g, a finite-dimensional real analytic submanifold W such that W contains the premoduli space at g, P_g as a real analytic subset.*

However, under the assumption that our metric is Kähler-Einstein with vanishing first Chern class, each infinitesimal Einstein deformation is integrable and the deformation space agrees with the tangent space to P_g, see [46].

Theorem 1.9 (Koiso)**.** *Suppose g is a Kähler-Einstein metric on (M, J) such that $c_1(M) = 0$ and (M, J) has a smooth deformation in $H^1(M, \Theta_M)$. In this case, the Einstein deformation space $\epsilon(g)$ is the tangent space of the premoduli space P_g. In a small neighborhood of g in P_g, any Einstein metric is Kähler with respect to some nearby complex structure of J.*

If the scalar curvature s is strictly negative or identically zero, the identity component of the isometry group $\mathrm{Iso}^0(g)$ acts trivially on P_g. The Einstein moduli space can then be obtained as the quotient of the premoduli space by the finite group $\mathrm{Iso}(g)/\mathrm{Iso}^0(g)$, so it has the structure of a finite-dimensional Hausdorff orbifold.

1.3 Summary of results

Our goal in this paper is to investigate the topological properties of the Einstein moduli space for each manifold appearing in the rigidity case above. More precisely, the main result is the following.

Theorem 1.10. *Let M be a closed oriented smooth 4-manifold which admits a locally hyperKähler metric, then the Einstein moduli space $\mathcal{E}(M)$ is connected.*

Let us give a short summary of each section. In section 2 we introduce the locally hyperKähler manifolds and explain how they appear naturally in the rigidity case of the Hitchin-Thorpe inequality. In section 3, we compute the Teichmüller spaces of flat metrics on all closed oriented Riemannian manifolds in dimension 2,3 and 4, starting from the ones on flat tori. In section 4, we investigate the existence of almost complex structure and semi-complex structure on flat manifolds. In sections 5, we review the moduli theory of complex structures and the theory of hyperKähler metrics on K3 surfaces. In section 6, we show a Torelli theorem and surjectivity for the polarized moduli space on an Enriques surface. Using this result, we prove that the Einstein moduli space on an Enriques surface must be path-connected. In section 7, we show a similar connectedness result for Einstein moduli space on a Hitchin manifold. The strategy is to perform a hyperKähler rotation to reduce the case to the one on Enriques surface. Moreover, a Hitchin manifold does not admit any complex structure, but we show there is a Torelli theorem for semi-complex structure.

2 Locally hyperKähler manifolds

2.1 The Ricci decomposition of curvature tensor

For a general Riemannian n-manifold (M, g), the curvature operator is a self-adjoint operator $\mathfrak{R} : \Lambda^2 \to \Lambda^2$ defined in terms of the Riemann curvature tensor

$$g(\mathfrak{R}(u \wedge v), x \wedge y) = R(u, v, x, y), \tag{2.1}$$

for $u, v, x, y \in TM$. The curvature operator $\mathfrak{R}$ is a section of $S^2\Lambda^2T^*M$ since it is self-adjoint. There is a natural way of breaking up $\mathfrak{R}$ into three irreducible pieces for the action of $\mathrm{O}(n)$ as follows.

Suppose (V, g) is an n-dimensional vector space with a metric. This vector space V should be identified with the cotangent space T_p^*M at any point $p \in M$. Due to the Bianchi identity, the curvature tensor lives in the kernel of the map $b : S^2\Lambda^2V \to S^2\Lambda^2V$ defined as

$$b(R)(u, v, x, y) = \frac{1}{3}(R(u, v, x, y) + R(v, x, u, y) + R(x, u, v, y)). \tag{2.2}$$

We call $\mathfrak{R}V := \mathrm{Ker}(b)$ the space of algebraic curvature tensors. $\mathfrak{R}(V)$ is an $\mathrm{O}(n)$-module as b is $\mathrm{GL}(n)$-equivariant.

Another operator we need for the decomposition is the Ricci contraction $c : S^2\Lambda^2V \to S^2V$, given by

$$c(R)(u, v) = \mathrm{tr}R(u, \cdot, v, \cdot). \tag{2.3}$$

The Kulkarni-Nomizu product, on the other hand, yields an element of $S^2\Lambda^2V$ by pairing two elements of S^2V, defined as

$$(a \owedge b)(u, v, x, y) = a(u, x)b(v, y)+a(v, y)b(u, x)-a(u, y)b(v, x)-a(v, x)b(u, y).$$

For $n \geq 4$, we have a decomposition of the $\mathrm{O}(n)$-module $\mathfrak{R}V$ into three irreducible components as

$$\mathfrak{R}V = \mathfrak{S}V \oplus \mathfrak{E}V \oplus \mathfrak{W}V, \tag{2.4}$$

where

$$\mathfrak{S}V = \mathbb{R}g \owedge g,$$
$$\mathfrak{E}V = g \owedge S_0^2 V,$$
$$\mathfrak{W}V = \mathrm{Ker}c \cap \mathrm{Ker}b.$$

Here $S_0^2 V$ consists of traceless symmetric 2-tensors. $\mathfrak{W}V$ is called the space of Weyl tensors.

According to 2.4, the curvature operator can be decomposed into

$$\mathfrak{R} = \frac{s}{2n(n-1)} g \owedge g + \frac{1}{n-2} \mathring{r} \owedge g + W, \tag{2.5}$$

where $\mathring{r} = r - \frac{s}{n} g$ is the traceless Ricci tensor, s is the scalar curvature and W is the Weyl part of $\mathfrak{R}$, known as the Weyl curvature tensor.

2.2 Special feature of four-dimensional geometry

Let $(V, \langle\, , \rangle)$ be an oriented vector space of dimension n with a positive-definite inner product. This in turn induces inner products on the p-forms $\Lambda^p V$. Denote the volume form of unit norm by ω. The Hodge star operator $* : \Lambda^p V \to \Lambda^{n-p} V$ is defined by the identity

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \omega, \tag{2.6}$$

where α, $\beta \in \Lambda^p V$.

For an oriented Riemannian manifold (M, g) of dimension n, the above pointwise construction can be applied to the bundle of p-forms Λ^p on M. If the dimension is even, say $n = 2m$, then the Hodge star operator acting on Λ^m satisfies $*^2 = (-1)^m$. In the particular case of dimension four, $*$ is an involution on the bundle of two-forms Λ^2. Λ^2 then decomposes into

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-, \tag{2.7}$$

where $\Lambda^{\pm}$ are the ± 1-eigenspaces of $*$. Sections of Λ^+ are called self-dual 2-forms, whereas sections of Λ^- are called anti-self-dual 2-forms.

This decomposition is intrinsically connected with the fact that $\mathrm{SO}(4)$ is not a simple Lie group. Under the adjoint action of SO(4), the Lie algebra

$$\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3) \tag{2.8}$$

splits into two irreducible components. Now $\Lambda^2\mathbb{R}^4$ is isomorphic to $\mathfrak{so}(4)$ as SO(4)-modules, so 2.8 induces a decomposition of Λ^2. Since the Hodge star operator is SO(4)-equivariant, this splitting coincides with 2.7 by the Schur's lemma.

As is beautifully explained in [4], [8] and [59], this decomposition has a profound impact on the geometry of dimension four. From the previous section, we have seen that $\mathfrak{R}$ can be broken into three irreducible pieces. Thanks to 2.7 and 2.8, the curvature operator can be further partitioned into four blocks

$$\mathfrak{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W^- + \frac{s}{12} \end{array} \right), \tag{2.9}$$

where $W^\pm$ are the trace-free part of the corresponding blocks, acting on $\Lambda^\pm$. To sum up, it allows the Weyl curvature tensor defined in 2.5 to split into self-dual and anti-self-dual parts as $W = W^+ \oplus W^-$.

2.3 Locally hyperKähler metrics

A hyperKähler manifold is a Riemannian manifold (M, g) of dimension $4m$ whose holonomy group $\mathrm{Hol}(g)$ is contained in the compact symplectic group $\mathrm{Sp}(m)$. The Lie group $\mathrm{Sp}(m)$ can be described as the subgroup of $\mathrm{GL}(m, \mathbb{H})$ which perserves the standard Hermitian form on $\mathbb{H}^m$, where $\mathbb{H}$ here denotes the quaternions. The metric g is called a hyperKähler metric. A second characterization of a hyperKähler manifold is a Riemannian manifold (M, g) which is Kähler with respect to three complex structures I, J, K, satisfying the identity for imaginary quaternions

$$I^2 = J^2 = K^2 = IJK = -1. \tag{2.10}$$

Such a quadruplet (I, J, K, g) is said to be a hyperKähler structure on M. For any $a, b, c \in \mathbb{R}$ such that $a^2+b^2+c^2 = 1$, the linear combination $aI+bJ+cK$ is also a covariantly constant complex structure on M compatible with g. There is therefore an S^2-worth of complex structures on M, with respect to which g is Kähler. We call the operation of changing complex structures within this 2-sphere while keeping the underlying metric g fixed a hyperKähler rotation on M. The three associated Kähler forms are

$$\omega_1(X,Y) = g(IX,Y), \quad \omega_2(X,Y) = g(JX,Y), \quad \omega_3(X,Y) = g(KX,Y). \tag{2.11}$$

The complex 2-form $\omega_+ = \omega_2 + i\omega_3$ is a holomorphic symplectic form of type $(2,0)$ with respect to the complex structure I. The equivalence of the above two definition can be seen as follows. Let $i, j, k \in \mathbb{H}$ be the basic quaternions satisfying 2.10, then $h = g + i\omega_1 + j\omega_2 + k\omega_3$ is a $\mathbb{H}$-valued Hermitian form. The subgroup of $\mathrm{GL}(m, \mathbb{H})$ that preserves h is precisely $\mathrm{Sp}(m)$, so $\mathrm{Hol}(g)$ is contained in $\mathrm{Sp}(m)$. Conversely, if a tensor on $\mathbb{R}^{4m}$ is invariant under $\mathrm{Sp}(m)$, it gives rise to a covariantly constant tensor on M via parallel transport. One can then construct three complex structures I, J, K on M as in 2.10, see [38] for more details.

In the particular case of dimension 4, the symplectic group $\mathrm{Sp}(1)$ conincides with the special unitary group $\mathrm{SU}(2)$. In other words, the Calabi-Yau 4-manifolds are precisely the hyperKähler manifolds. The following theorem is well-known.

Theorem 2.1. *Let (M, g) be a compact hyperKähler 4-manifold, then M endowed with any complex structure $aI + bJ + cK$ compatible with g is biholomorphic to a complex torus or a K3 surface.*

Proof. First, the canonical bundle K_M of M with respect to $aI + bJ + cK$ is trivial, so all the plurigenera $P_n(M) = h^0(M, \mathcal{O}(K^n))$ equal 1. It implies the Kodaira dimension of M, $\kappa(M)$ is zero. By the Enriques-Kodaira classification of compact complex surfaces, M is either a K3 surface, an Enriques surface, a complex torus, a hyperelliptic surface, or a Kodaira surface. However, a Kodaira surface is never Kähler and the canonical bundle of an Enriques surface or a hyperelliptic surface is non-trivial. Thus we conclude that M is a complex torus or a K3 surface. □

An oriented Riemannian manifold (M, g) is said to be locally hyperKähler if the universal cover $(\tilde{M}, \tilde{g})$ equipped with the covering metric is hyperKähler. From Theorem 2.1, each such $(\tilde{M}, \tilde{g})$ must be a complex torus or a K3 surface with a hyperKähler metric. Our main interest in this paper is the class of locally hyperKähler 4-manifolds. If a Riemannian 4-manifold is simply connected, then the metric is hyperKähler if and only if it is Ricci-flat Kähler. This specifically reveals that a locally hyperKähler metric must be Ricci-flat, hence Einstein. Since the Kähler property is not preserved by covering map in general, we will see that some locally hyperKähler 4-manifolds are indeed non-Kähler. In fact, there exist locally hyperKähler 4-manifolds that do not admit any integrable complex structures. Nonetheless, most of them do admit almost complex structures, except for the class of Hitchin manifolds, which will be defined in the next section.

2.4 The Hitchin-Thorpe Inequality

A striking result of Hitchin in this direction gives a restriction on the topology of Einstein manifolds.

Theorem 2.2 (Hitchin-Thorpe)**.** *Let M be a compact oriented four-dimensional smooth manifold. If M carries an Einstein metric, then*

$$2\chi + 3\tau \geq 0, \tag{2.12}$$

where χ denotes the Euler characteristic of M and τ denotes the signature of M. The equality is satisfied if and only if M covered by a flat torus or M is covered by a K3 surface equipped with a hyperKähler metric.

The above inequality was independently discovered by Hitchin and Thorpe [77], whereas the equality case was proved by Hitchin [32].

The proof of Theorem 2.2 relies on the use of Gauss-Bonnet theorem in dimension four

$$\chi(M^4) = \frac{1}{8\pi^2}\int_M (\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - |\mathring{r}|^2) d\mu, \tag{2.13}$$

and the signature formula

$$\tau(M^4) = \frac{1}{12\pi^2}\int_M (|W^+|^2 - |W^-|^2) d\mu. \tag{2.14}$$

Here W^+ is the self-dual Weyl curvature and $\mathring{r}$ is the trace-free Ricci tensor as defined in section 2.2. By the above identities, one sees that

$$2\chi + 3\tau = \frac{1}{4\pi^2}\int_M (\frac{s^2}{24} + 2|W^+|^2 - \mathring{r}^2) \geq 0, \tag{2.15}$$

since $\mathring{r} = 0$ by the Einstein condition.

It is worth pointing out that prior to the discovery of the inequality 2.15, Berger showed that $\chi(M) \geq 0$ for every Einstein 4-manifold M with equality if and only if M is flat. This observation is a quick corollary of the Gauss-Bonnet formula above and the decomposition 2.9.

Combining Berger's theorem with Theorem 2.1 from the last section, we can rephrase the rigidity result of Hitchin as below.

Proposition 2.3 (Hitchin)**.** *A closed oriented Einstein 4-manifold M satisfies $2\chi + 3\tau = 0$ if and only if M is locally hyperKähler.*

Lemma 2.4. *If an oriented Einstein 4-manifold saturates the inequality 2.15, then the bundle of self-dual 2-forms Λ^+ is flat.*

Proof. First, from the expression in 2.15, W^+ and s must vanish. The curvature operator $\mathfrak{R}$ of the Levi-Civita connection on TM then maps $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ to Λ^-. Let $\widetilde{\mathfrak{R}} \in \Gamma(\Lambda^2 \otimes \mathrm{End}(\Lambda^2))$ be the curvature of the induced connection on Λ^2. For any 2-form σ on M and any vector fields $u, v \in \Gamma(TM)$,

$$[\widetilde{\mathfrak{R}}(u \wedge v)\sigma](\cdot,\cdot) = -[\sigma(\mathfrak{R}(u \wedge v)\cdot,\cdot) + \sigma(\cdot, \mathfrak{R}(u \wedge v)\cdot)]. \tag{2.16}$$

Using the orthogonality of Λ^+ and Λ^-, we have the decomposition $\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{R}}^+ + \widetilde{\mathfrak{R}}^-$, where $\widetilde{\mathfrak{R}}^+$ and $\widetilde{\mathfrak{R}}^-$ are the curvatures of Λ^+ and Λ^- respectively. If σ is a self-dual 2-form, one checks easily that $\widetilde{\mathfrak{R}}^+(u \wedge v)\sigma = 0$, so Λ^+ is flat. □

Consider the pull-back metric on the universal cover $(\tilde{M}, \tilde{g})$. Since $\tilde{M}$ is simply-connected and Λ^+ is flat, Λ^+ is a trivial bundle spanned by three linearly independent parallel 2-forms. In section 2.2, we have seen that there is a splitting of $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ as an SO(4)-module, or equivalently $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with respect to the adjoint action. We now show that the holonomy group of $\tilde{M}$ is a subgroup of SU(2). To this end, $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on $\mathfrak{so}(4)$ via adjoint representation, in the same way as SO(4). To be more precise, we have the factorization

$$\begin{tikzcd} \mathrm{Spin}(4) \arrow[r, "\pi"] \arrow[dr, "\mathrm{Ad}"'] & \mathrm{SO}(4) \arrow[d, "\mathrm{Ad}'"] \\ & \mathrm{Aut}(\mathfrak{so}(4)), \end{tikzcd} \tag{2.17}$$

where π is the natural quotient map with kernel $\{(1,1), (-1,-1)\}$. Thus, the pre-image $G = \pi^{-1}\,\mathrm{Hol}(\tilde{g})$ acts trivially on the first factor of $\mathfrak{su}(2)$ and we deduce that $G \subset \{\pm 1\} \times \mathrm{SU}(2)$. The holonomy group $\mathrm{Hol}(\tilde{g}) = \pi(G)$ is then indeed a subgroup of SU(2). $\tilde{M}$ is therefore a hyperKähler manifold.

Assume that $\tilde{M}$ is non-compact. Then $\tilde{M}$ must contain a geodesic line because $\tilde{M}$ covers a compact manifold. Using the Cheeger-Gromoll splitting theorem, $\tilde{M} = N \times \mathbb{R}$ decomposes into a direct Riemannian product, where N is Ricci-flat and simply connected. But a Ricci-flat metric is necessarily flat in dimension 3, so $\tilde{M}$ is flat. This result, together with the Bieberbach theorem, shows that M is finitely covered by a flat 4-torus.

Now we are left with the case that $(\tilde{M}, \tilde{g})$ is a compact Ricci-flat manifold. Fix a complex structure J compatible with $\tilde{g}$, and let $\rho(\cdot,\cdot) = \mathrm{Ric}(J\cdot,\cdot)$ be the Ricci form. Since $(\tilde{M}, J, \tilde{g})$ is Kähler, the Levi-Civita connection coincides with the Chern connection on $T\tilde{M}$. The curvature of the corresponding induced connection on the canonical bundle $K_{\tilde{M}}$ satisfies

$$\mathfrak{R}(u,v) = i \cdot \rho(u,v), \tag{2.18}$$

so we immediately get $K_{\tilde{M}}$ is flat. Our assumption on the simply-connectedness of $\tilde{M}$ then implies $K_{\tilde{M}}$ can be trivialised by a parallel section. Hence $\tilde{M}$ is a K3 surface.

As we will see in section 3, finding the complete list of compact flat 4-manifolds requires one to classify the so-called torsion-free 4-dimensional crystallographic groups. While this is a seemingly daunting group theoretic task, the class of locally hyperKähler manifolds finitely covered by a K3 surface comprises only three diffeomorphism types. To resolve the latter problem, suppose $p : \tilde{M} \to M$ is a covering map of degree d such that $\tilde{M}$ is a K3 surface. Here d is also the order of the group Γ of deck transformations. The constraints imposed by the following the identities

$$24 = \chi(\tilde{M}) = d \cdot \chi(M), \quad -16 = \tau(\tilde{M}) = d \cdot \tau(M) \tag{2.19}$$

limit the values of d to be either 1,2,4 or 8. If $d = 8$, then $\chi(M) = 3$ and $b_2 = 1$. But this contradicts $\tau(M) = 2$.

For any complex structure J compatible with $\tilde{g}$ on $\tilde{M}$, by a point-wise computation, the Kähler form $\omega = \tilde{g}(J\cdot,\cdot)$ can be verified to be a self-dual 2-form. In this way, the sphere bundle $S\Lambda^+$ parametrizes the set of complex structures compatible with $\tilde{g}$ on $\tilde{M}$. By the Bochner's formula 3.3, $b_1(\tilde{M})$ is the dimension of parallel harmonic 1-forms on $\tilde{M}$. Taking into account the action of Γ, $b_1(M)$ is the dimension of parallel harmonic 1-forms on $\tilde{M}$ preserved by Γ. Thus both $b_1(\tilde{M})$ and $b_1(M)$ vanish. Likewise, $b_+(\tilde{M})$ is the dimension of parallel harmonic self-dual 2-forms on $\tilde{M}$ and $b_+(M)$ is the dimension of the subspace of the former preserved by Γ. As $b_1(M) = 0$, the dimension is

$$b_+(M) = \frac{\chi + \tau}{2} - 1. \tag{2.20}$$

If $d = 2$, then $b_+(M) = 1$. This means that, by the correlation pointed out above, there is exactly a pair of complex structures $\{\pm J\}$ preserved by Γ. The group Γ is then generated by a holomorphic involution with respect to the complex structures $\pm J$. M is the quotient of a K3 surface $(\tilde{M}, J)$ by a free holomorphic $\mathbb{Z}_2$-action, so M is an Enriques surface.

For the case $d = 4$, we will show that Γ must be the Klein four-group. Assume the contrary that Γ is the cyclic group $\mathbb{Z}_4$, say generated by γ. Because $\tilde{M}$ is spin, the oriented orthonormal frame bundle $P_{\mathrm{SO}}(\tilde{M}) = P_{\mathrm{SO}}(T\tilde{M})$ admits an equivariant 2-fold cover $\xi : P_{\mathrm{Spin}}(\tilde{M}) \to P_{\mathrm{SO}}(\tilde{M})$, which is itself a principal Spin(4)-bundle. γ acts on $\tilde{M}$ by isometry, hence it also acts on $P_{\mathrm{SO}}(\tilde{M})$. This in turn induces a lifting of γ, denoted by γ' on $P_{\mathrm{Spin}}(\tilde{M})$. Yet this induced action does not satisfy $\gamma'^4 = 1$. Indeed, if this were true, $P_{\mathrm{Spin}}(\tilde{M})/\Gamma$ would be a principal Spin(4)-bundle for M and M would be a spin manifold. However, $\tau(M) = -4$ should then be divisble by 16 due to the Rokhlin's theorem. Thus M is spin^c but not spin. Under such circumstance, the induced action is subject to $\gamma'^4 = -1$. We can define an action on the principal Spin^c(4)-bundle $P_{\mathrm{Spin}(\tilde{M})} \times_{\mathbb{Z}_2} S^1$ by sending $(x, y) \mapsto (\gamma' x, e^{i\pi 4} y)$. What follows is that we have an associated Dirac operator on M whose index equals the $\hat{A}$-genus. But $\hat{A}(M)$ can be easily computed to be $-1/2$, so there is no free isometric $\mathbb{Z}_4$-action on a K3 surface with an Einstein metric.

Remark 2.5. There have been several improvements of the Hitchin-Thorpe inequality, see [28], [41], [55] and [59].

3 Flat Riemannian manifolds

3.1 The Bieberbach theorems

We start by reviewing some basic facts in Riemannian geometry.

Theorem 3.1. *Let (M^n, g) be a compact flat n-dimensional Riemannian manifold and let $\mathbb{R}^n$ be equipped with the standard flat metric g_{st}. Then (M^n, g) is isometric to $\mathbb{R}^n/\Gamma$, for some subgroup Γ of the isometry group of $(\mathbb{R}^n, g_{st})$, which acts freely and properly discontinuously.*

Choose an orthonormal basis in T_pM, then we can identify T_pM with $(\mathbb{R}^n, g_{st})$. Since M is flat, the exponential map $\exp_p : T_pM \to M$ becomes a local isometry. Hence $(T_pM, \langle\cdot,\cdot\rangle)$ can be viewed as the universal cover of M with the fundamental group of M, Γ acting freely and properly discontinuously. This yields two Γ-equivariant isometries, which fit into the commutative diagram below

$$\begin{tikzcd}
(T_pM, \langle\cdot,\cdot\rangle) \arrow[r, "\tilde{\phi}"] \arrow[d, "\exp_p"'] & (\mathbb{R}^n, g_{st}) \arrow[d] \\
(M, g) \arrow[r, "\phi"] & (\mathbb{R}^n/\Gamma, g_{st})).
\end{tikzcd} \tag{3.1}$$

Note that each element in Γ must act by isometry on $\mathbb{R}^n$, so Γ is a subgroup of the isometry group of $(\mathbb{R}^n, g_{st})$, i.e. the Euclidean group $\mathrm{E}(n) := \mathrm{O}(n) \ltimes \mathbb{R}^n$. By the above theorem, finding all compact flat Riemannian manifolds is equivalent to determining all such $\Gamma \subset \mathrm{E}(n)$ that acts properly discontinuously and freely.

This problem is a special case of the first part of the three well-known questions under the Hilbert's eighteenth problem. It was answered affirmatively by Bieberbach in 1910. It asks whether there are finitely many essentially different discrete subgroups $\Gamma \subset \mathrm{E}(n)$ such that the coset space $\mathrm{E}(n)/\Gamma$ is compact. We call such $\Gamma \subset \mathrm{E}(n)$ an n-dimensional *crystallographic group* or an *n-space group*.

The action of a generic crystallographic group contains fixed points, and its quotient $\mathbb{R}^n/\Gamma$ is an *orbifold*. Space groups whose quotients are Riemannian manifolds correspond to those which are torsion-free. Suppose a discrete group acts isometrically and properly, then the action free exactly when it is torsion-free.

Consider now the Euclidean group $\mathrm{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$. There is a natural projection map $\alpha : \mathrm{E}(n) \to \mathrm{O}(n)$. The image of a space group Γ is a finite group, called the *point group* of Γ, or the *holonomy group* if Γ is torsion-free. The kernel of α is the translation part $\mathbb{Z}^n \subset \mathbb{R}^n$ of Γ. Thus Γ admits a short exact sequence

$$1 \longrightarrow \mathbb{Z}^n \xrightarrow{\ i\ } \Gamma \xrightarrow{\ \alpha\ } \Phi \longrightarrow 1.$$

What we just mentioned is part of the three Bieberbach theorems.

Theorem 3.2 (Bierberbach). *Let Γ be an n-space group, then*

1. *Γ can be described as an extension of a finite subgroup $\Phi \subset \mathrm{O}(n)$ by n linearly independent translations $\mathbb{Z}^n \subset \mathbb{R}^n$.*
2. *Let Γ' be a space group that is isomorphic to Γ, then Γ' is related to Γ via conjugation by an affine transformation.*
3. *For each n, there are only finitely many isomorphism classes of space groups.*

Conversely, a theorem of Zassenhaus showed that each Γ obtained from such extension can be embedded as a discrete subgroup of $\mathrm{E}(n)$. So Hilbert's problem reduces to figuring out the number of extension of Φ by $\mathbb{Z}^n$, which is the size of $H^2(\Phi, \mathbb{Z}^n)$.

Example 3.3. In dimenison 1, there are two crystallographic groups, the translations $\mathbb{Z}$ and the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$. The orbit spaces of which are the 1-torus and the 1-orbifold with two singularities that resemble $\mathbb{R}/\mathbb{Z}_2$ respectively.

Example 3.4. In dimension 2, the 17 crystallographic groups are also known as the wallpaper groups. These wallpaper groups arise as the symmetries of 2-dimensional repetitive tiling. There are two torsion-free crystallographic groups: $\mathbb{Z}^2$ which yields the torus T^2, and $\Gamma := \langle e_1, e_2\ , \frac{1}{2}e_1 + \mathrm{diag}(1, -1)\rangle$ which gives the Klein bottle.

Example 3.5. In dimension 3, the 219 space groups are of great interest in the study of crystals. They were first classified by Barlow, Fedorov and Schoenfliess in the 1890s. The 10 torsion-free spaces groups were found by Hantzsche and Wendt.

Example 3.6. In dimension 4, Brown, Bülow, Neubüser, Wondratschek and Zassenhaus showed that there are 4783 examples. The 74 torsion-free space groups were computed by Calabi, Lambert and Wolf.

Remark 3.7. Each statement in Theorem 3.2 has a geometric manifestation in terms of orbifolds, see [12].

3.2 Teichmüller space of metrics

Now given any flat metric g on T^n, we know by Theorem 3.1 that g is isometric to $g_{st} := dy_1 \otimes dy_1 + \cdots + dy_n \otimes dy_n$ via a diffeomorphism $T^n \cong \mathbb{R}^n / \langle v_1, \cdots, v_n \rangle$. Here $\langle v_1, \cdots, v_n \rangle$ denotes the group of translations generated by $v_i \in \mathbb{R}^n$ and y_i denotes the coefficient of v_i. Let $\{e_1, \cdots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$ and let x_i be the coefficient of e_i, then we can construct a map $\phi : \mathbb{R}^n / \langle e_1, \cdots, e_n \rangle \to \mathbb{R}^n / \langle v_1, \cdots, v_n \rangle$ by sending

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto (v_1, \cdots, v_n) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Note $v_j = \alpha_{ij}$ is meant to be a column vector. The pullback of g now has the form

$$\begin{aligned}
\tilde{g} := \phi^* g &= \sum_{i,j} \alpha_{1i} dx_i \otimes \alpha_{1j} dx_j + \cdots + \sum_{i,j} \alpha_{ni} dx_i \otimes \alpha_{nj} dx_j \\
&= \sum_j \sum_i \alpha_{ij}^2 dx_j \otimes dx_j + \sum_{i \neq j} \sum_k \alpha_{ki} \alpha_{kj} dx_i \otimes dx_j \\
&= \sum_i (v_i \cdot v_i) dx_i \otimes dx_i + \sum_{i \neq j} (v_i \cdot v_j) dx_i \otimes dx_j.
\end{aligned}$$

The above computation shows that flat metrics on T^n can be described by $n(n+1)/2$ parameters in $(\mathbb{R}^+)^n \times \mathbb{R}^{n(n-1)/2}$. In matrix form, the parameters are as follows:

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ & \zeta_{22} & \cdots & \zeta_{2n} \\ & & \ddots & \vdots \\ & & & \zeta_{nn} \end{pmatrix},$$

with $v_i \cdot v_i = \zeta_{ii} \in \mathbb{R}^+$ and $v_i \cdot v_j = \zeta_{ij} \in \mathbb{R}$ for $i \neq j$. Denote this space by $\mathcal{M}(T^n)$.

To obtain the Teichmüller space, we need to divide the space $\mathcal{M}(T^n)$ by the action of $\mathcal{D}_0(T^n) \times \mathbb{R}_+$. Yet it is not hard to see that $D_0(T^n)$ acts trivially on $\mathcal{M}(T^n)$, so the Teichmüller space coincides with $\mathcal{M}(T^n)$, up to homothety,

$$\mathcal{T}(T^n) = \mathcal{M}(T^n)/\mathbb{R}_+. \tag{3.2}$$

Let M^n be an n-dimensional compact flat Riemannian manifold. By the first Bierberbach theorem 3.2, M^n is finitely covered by a flat n-torus T^n. So, flat metrics on M^n are the flat metrics descends from T^n that are invariant under the action of the holonomy group Φ of M^n.

3.3 Affine structures

Our discussion can be understood in the context of affine structures. An affine structure on a smooth manifold M is a maximal atlas $\{(U_\alpha, \psi_\alpha)\}$ of charts $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ whose transition functions $\psi_\alpha \circ \psi_\beta^{-1}$ are affine transformations. Parallel translation on M yields a flat torsion-free connection on the tangent bundle TM. Conversely, taking the exponential map of a torsion-free flat connection recovers the above affine charts. The third characterization of an affine structure is given by using the notion of a (G, X)-structure on M. Let X be a real affine space and let $G = \mathrm{Aff}(X)$. An affine structure is equivalent to the datum of a group homomorphism $H : \pi_1(M, p) \to G$ and a developing map $D : \tilde{M} \to X$, where $\tilde{M}$ is the universal cover of M, H is the monodromy of the (G, X)-structure. The developing map is obtained via analytic continuation of charts along paths in M and is equivariant $D(\gamma \cdot p) = H(\gamma) \cdot D(p)$. It is unique up to composition with an element of G.

We say that a (G, X)-structure is complete if $D : \tilde{M} \to X$ is a covering map. It is convenient to work with a complete (G, X)-structure since we can reconstruct the underlying manifold M as the quotient X/H if X is connected, where H is the monodromy group. For an affine manifold M, completeness of the (G, X)-structure corresponds precisely to the geodesic completeness of the torsion-free flat connection on TM. If, in addition, X is a Riemannian manifold and $G \subset \mathrm{Iso}(X)$, then a (G, X)-structure is complete if and only if X is geodesically complete with respect to the Riemannian metric. Thus, every Bieberbach manifold is a complete affine manifold.

Remark 3.8. The map $H : \pi_1(M) \to G$ is called the holonomy in [79], but we avoid this terminology here since the image of H differs from the holonomy group of the corresponding affine connection.

Remark 3.9. There exist closed affine manifolds which are not complete.

Example 3.10. Let $X := \mathbb{C}^2 \setminus \{0\}$ and let $\Phi = \{\lambda^n : |\lambda| > 1\}$ be the group generated by multiplication by λ. Since Φ lies in $\mathrm{Aff}(\mathbb{C}^2)$, the quotient space $M := X/\Phi$ inherits an affine structure. M is known in complex geometry as the Hopf surface. However, this affine manifold M is not complete as the geodesics directed towards the origin can not be extended. As a consequence, there is no Riemannian metric on M that is compatible with the affine connection ∇. Notice that the holonomy group $\mathrm{Hol}(M, \nabla) = \Phi \cong \mathbb{Z}$ is infinite.

Example 3.11. Let $P = \square ABCD$ be a quadrilateral in $\mathbb{R}^2$. By performing orientation-preserving similar transformations, we can identify the opposite edges of P, gluing $\overrightarrow{AB}$ with $\overrightarrow{DC}$ and $\overrightarrow{AD}$ with $\overrightarrow{BC}$. This produces an affine structure on the two-torus as similar transformations are affine. For a generic choice of P, the affine structure is not complete with $\mathrm{Hol}(\nabla) \cong \mathbb{Z}^2$. We refer the reader to [79] for a picture of the developing map.

We now give a more conceptual justification of our computation in section 3.2. Let V be an n-dimensional vector space, then V inherits a natural affine structure. Given a closed flat manifold M^n, the associated crystallographic group Γ can be regarded as a subgroup of $\mathrm{Aff}(V)$. Once we fix the affine structure, which is equivalent to a torsion-free affine connection, a compatible metric on V can be identified with a positive definite inner product on V. Each compatible metric is flat with ∇ as the Levi-Civita connection. The Teichmüller space of flat metrics compatible with this affine structure is therefore a convex open subset of $\mathrm{Sym}^2 V^*$ invariant under the action of the holonomy group $\mathrm{Hol}(\Gamma)$. Fix a basis of V, then a positive-definite inner product on V is then a positive-definite symmetric matrix. $\mathrm{GL}(n)$ acts on an inner-product by similarity with stabilizer a conjugate of the orthogonal group $\mathrm{O}(n)$. Each orbit can then be represented by a unique upper triangular matrix.

3.4 Low dimension

3.4.1 Dimension two

From the previous section, the only compact flat 2-manifolds are the tori and the Klein bottles.

1. Torus T^2: The Teichmüller space $\mathcal{T}_{T^2} \cong (\mathbb{R}_+^2 \times \mathbb{R})/\mathbb{R}_+ \cong \mathbb{R}^2$. From now on, denote the $g = \alpha dx_1 \otimes dx_1 + \beta dx_1 \otimes dx_2 + \beta dx_2 \otimes dx_1 + \gamma dx_2 \otimes dx_2$ by the following matrix:

$$\begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix}.$$

2. Klein bottle K: A space group whose orbit space equals K is $\Gamma := \langle e_1, e_2\ , \frac{1}{2}e_1 + \mathrm{diag}(1, -1)\rangle$. Let $\phi := \frac{1}{2}e_1 + \mathrm{diag}(1, -1)$, then a metric on K satisfies $\phi^* g = g$, i.e.

$$\begin{pmatrix} \alpha & -\beta \\ & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix}.$$

So $\beta = 0$ and $\mathcal{T}_K \cong \mathbb{R}_+^2/\mathbb{R}_+ \cong \mathbb{R}$.

Remark 3.12. The metrics on flat 2-orbifolds can be computed in the same way.

3.4.2 Dimension three

There are 10 homeomorphism classes of closed flat 3-manifolds. Denote the six orientable ones by $\mathcal{M}_1, \cdots, \mathcal{M}_6$ and the four non-orientable ones by $\mathcal{N}_1, \cdots, \mathcal{N}_4$. We order these flat 3-manifolds according to the numbering given by Hantzsche-Wendt and Wolf, see [81].

1. $\mathcal{M}_1 = T^3$: $\mathcal{T}_{T^3} \cong (\mathbb{R}_+^3 \times \mathbb{R}^3)/\mathbb{R}_+ \cong \mathbb{R}^5$. As in the two-dimensional case, denote a metric g by

$$\begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix}.$$

2. $\mathcal{M}_2$: a crystallographic group can be chosen to be $\Gamma := \langle e_1, e_2, e_3, \gamma = \mathrm{diag}(1,1,-1) + (\frac{1}{2}, 0, 0)\rangle$. $\gamma^* g = g$ gives

$$\begin{pmatrix} \alpha & -\beta & -\gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix}.$$

It follows that $\beta = 0$ and $\gamma = 0$ and $\mathcal{T}_{\mathcal{M}_2} \cong (\mathbb{R}_+^3 \times \mathbb{R})/\mathbb{R}_+ \cong \mathbb{R}^3$.

3. $\mathcal{M}_3$: $\Gamma = \langle e_1, e_2, e_3, \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ 0 \end{pmatrix}\rangle$. $\gamma^* g = g$ is equivalent to

$$\begin{pmatrix} \alpha & \gamma & -\beta - \gamma \\ & \theta & -\nu - \theta \\ & & \mu + 2\nu + \theta \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix}.$$

We get $\gamma = \beta = 0$ and $\theta = \mu = -2\nu$, so $\mathcal{T}_{\mathcal{M}_3} \cong \mathbb{R}_+^2/\mathbb{R}_+ \cong \mathbb{R}$.

4. $\mathcal{M}_4 : \Gamma = \langle e_1, e_2, e_3, \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \\ 0. \end{pmatrix}\rangle$. $\gamma^* g = g$ gives

$$\begin{pmatrix} \alpha & \gamma & -\beta \\ & \theta & -\nu \\ & & \mu \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix}.$$

Solving the system, one obtains $\beta = \gamma = \nu = 0$ and $\mu = \theta$, so $\mathcal{T}_{\mathcal{M}_4} \cong \mathbb{R}_+^2/\mathbb{R}_+ \cong \mathbb{R}$.

5. $\mathcal{M}_5 : \Gamma = \langle e_1, e_2, e_3, \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{6} \\ 0 \\ 0 \end{pmatrix}\rangle$. $\gamma^* g = g$ gives

$$\begin{pmatrix} \alpha & \gamma & -\beta + \gamma \\ & \theta & -\nu + \theta \\ & & \mu - 2\nu + \theta \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix}.$$

So $\beta = \gamma$ and $\theta = \mu = 2\nu$ and $\mathcal{T}_{\mathcal{M}_5} \cong \mathbb{R}_+^2/\mathbb{R}_+ \cong \mathbb{R}$.

6. $\mathcal{M}_6$: $\Gamma = \langle e_1, e_2, e_3, \gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \rangle$.

   $\gamma_i^* g = g$ gives

$$\begin{pmatrix} \alpha & -\beta & -\gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ & \mu & \nu \\ & & \theta \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & \gamma \\ & \mu & -\nu \\ & & \theta \end{pmatrix}.$$

   This is also known as the Hantzsche-Wendt manifold. From the above equalities, we deduce $\beta = \gamma = \nu = 0$ and so $\mathcal{T}_{\mathcal{M}_6} \cong \mathbb{R}^3_+/\mathbb{R}_+ \cong \mathbb{R}^2$.

Remark 3.13. The Teichmüller space of flat metrics in dimension three was first computed by Kang [39].

3.5 Dimension four

Now we consider the family of compact oriented flat four-manifolds. Given such a closed flat $M = \mathbb{R}^4/\Gamma$, to compute the Teichmüller space $\mathcal{T}_M$, we need to know the action of its holonomy group Φ on $\mathbb{R}^4$. By the Maschke's theorem, the group algebra $\mathbb{R}[\Phi]$ is semisimple, in other words, every finite-dimensional representation of Φ over $\mathbb{R}$ is completely reducible. So it suffices to figure out the irreducible representations of Φ.

The representation theory of finite groups over $\mathbb{R}$ is similar to the representation theory over its splitting field $\mathbb{C}$, but is slightly more involved. A representation over $\mathbb{R}$ or $\mathbb{C}$ is determined by its character theory. As in the complex case, there is a decomposition $\mathbb{R}[\Phi] \cong \oplus_i \mathrm{End}(V_i)$ known as the Artin-Wedderburn theorem. This gives the sum of squares formula

$$|\Phi| = \sum_i \frac{\dim^2 V_i}{\|\chi_{V_i}\|^2},$$

where V_i's are the irreducible representations. Recall that a complex representation is irreducible if and only if $\|\chi_V\|^2 = 1$. But for a real irreducible representation, $\|\chi_V\|^2 = 1$ is not always true. The corresponding irreducibility criterion for a real representation is

$$\|\chi\|^2 + \nu(\chi) = 2,$$

where $\nu(\chi) := \frac{1}{|G|}\sum_{g\in G}\chi(g^2)$ is the Frobenius-Schur indicator. Using the above formulas and the orthogonality relation of characters, we can check if a list of irreducible representations are complete. There is a more elaborate theory of Brauer that tells us the number of irreducible representations over $\mathbb{R}$ equals the sum of the number of irreducibles over $\mathbb{C}$ with real character values and the number of pairs of irreducibles over $\mathbb{C}$ with conjugate non-real character values.

Let b_1 denote the first Betti number of M. Recall the Bochner's formula for 1-form μ_a

$$(dd^* + d^*d)\mu_a = \nabla^*\nabla\mu_a + R_{ab}g^{bc}\mu_c, \tag{3.3}$$

where R_{ab} is the Ricci curvature of g. If μ is harmonic, then $(dd^*+d^*d)\mu_a = 0$. Since we are considering flat manifolds, the Ricci curvature R_{ab} is identically zero. Hence we obtain $\nabla^*\nabla\mu_a = 0$. A straightforward integration by parts then shows that μ_a must be parallel. This implies b_1 is precisely the dimension of parallel harmonic 1-forms on M. So the holonomy group Φ acts trivially on this subspace of $\mathbb{R}^4$ of dimension b_1. It turns out in dimension 4, all the 27 orientable closed flat manifolds have $b_1 \geq 1$. So Φ acts non-trivially on a subspace of dimension $n \leq 3$. We show that under such circumstance, the irreducible representations are uniquely determined by the holonomy group Φ of Γ. An immediate corollary is that any two compact oriented flat four-manifolds with isomorphic holonomy groups have diffeomorphic Teichmüller spaces. Note that this does not hold in higher dimension in general since a crystallographic group can have different irreducible decompositions.

For the 27 closed oriented flat four-manifolds, we use the holonomy groups and $H_1(\mathbb{Z})$ computed by Lambert, Ratcliffe and Tschantz in [51].

Theorem 3.14. *Let (M, g) be a compact oriented flat four-manifold. Then the Teichmüller space of flat metrics on M is determined by its holonomy group Φ. More precisely, the following holds*

1. *If $\Phi = \{1\}$, then $\mathcal{T}_M \cong \mathbb{R}^9$.*
2. *If $\Phi = \mathbb{Z}_2$, then $\mathcal{T}_M \cong \mathbb{R}^5$.*
3. *If Φ is one of $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$, then $\mathcal{T}_M \cong \mathbb{R}^3$.*
4. *If $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{T}_M \cong \mathbb{R}^3$.*

5. *If Φ is one of the dihedral groups D_3, D_4 or D_6, then $\mathcal{T}_M \cong \mathbb{R}^2$.*

6. *If $\Phi = A_4$ the alternating group of order 12, then $\mathcal{T}_M \cong \mathbb{R}$.*

The proof goes in the same way as in dimension 2 and 3, except we use only the holonomy group and the first betti number of M to deduce the action of Γ up to equivalence.

1. $\mathcal{O}_1 = T^4$: $\mathcal{T}_{T^4} \cong (\mathbb{R}^6 \times \mathbb{R}^4_+)/\mathbb{R}_+ \cong \mathbb{R}^9$. As before denote a generic metric by the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix}$$

2. $\mathcal{O}_2, \mathcal{O}_3$: The holonomy group $\Phi \cong \mathbb{Z}_2 = \langle r \rangle$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}^2$. There are two one-dimensional irreducible representations of $\mathbb{Z}_2$, given by $r \mapsto 1$ and $r \mapsto -1$. We can assume a generator is $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. $r^*g = g$ gives

$$\begin{pmatrix} \alpha & \beta & -\gamma & -\eta \\ & \mu & -\nu & -\zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix},$$

so $\gamma = \eta = \nu = \zeta = 0$ and $\mathcal{T}_{\mathcal{O}_2} \cong (\mathbb{R}^4_+ \times \mathbb{R}^2)/\mathbb{R}_+ \cong \mathbb{R}^5$.

3. $\mathcal{O}_4, \mathcal{O}_5$: The holonomy group $\Phi \cong \mathbb{Z}_3 = \langle r \rangle$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}^2$. The irreducibles include the trivial representation and the two-dimensional anti-clockwise rotation with angle $2\pi/3$. After changing basis, a generator can be chosen to be $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$. $r^*g = g$ gives

$$\begin{pmatrix} \alpha & \beta & \eta & -\gamma-\eta \\ & \mu & \zeta & -\nu-\zeta \\ & & \tau & -\omega-\varphi \\ & & & \tau+2\varphi+\omega \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix}.$$

The parameters then satisfy $\gamma = \eta = 0$, $\zeta = \nu = 0$, $\tau = \omega = -2\varphi$, so $\mathcal{T}_{\mathcal{O}_4} \cong (\mathbb{R}^3_+ \times \mathbb{R})/\mathbb{R}_+ \cong \mathbb{R}^3$.

4. $\mathcal{O}_6, \mathcal{O}_7$: The holonomy group $\Phi \cong \mathbb{Z}_4 = \langle r \rangle$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}^2$. There are two one-dimensional irreducibles and one two-dimensional irreducible: $r \mapsto 1$, $r \mapsto -1$ and $r \mapsto$ anti-clockwise rotation by angle $\pi/4$. A generator can be chosen to be $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. $r^*g = g$ requires

$$\begin{pmatrix} \alpha & \beta & \eta & -\gamma \\ & \mu & \zeta & -\nu \\ & & \omega & -\varphi \\ & & & \tau \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix},$$

so $\gamma = \eta = \nu = \zeta = \varphi = 0$ and $\tau = \omega$. We obtain $\mathcal{T}_{O_6} \cong (\mathbb{R}^3_+ \times \mathbb{R})/\mathbb{R}_+ \cong \mathbb{R}^3$.

5. $\mathcal{O}_8$: The holonomy group is $\mathbb{Z}_6 = \langle r \rangle$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}^2$. There are two one-dimensional irreducibles: $r \mapsto \pm 1$ and two two-dimensional irreducibles: the anti-clockwise rotation by $\pi/6$ and $\pi/3$. Since the action of Φ is faithful and orientation-preserving, we can pick $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ as a generator. Then equating the pullback metric $r^*g = g$,

$$\begin{pmatrix} \alpha & \beta & \eta & -\gamma \\ & \mu & \zeta & -\nu+\zeta \\ & & \omega & -\varphi+\omega \\ & & & \tau-2\varphi+\omega \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix}.$$

Solving the system yields $\gamma = \eta = \nu = \zeta = 0$ and $\tau = \omega = 2\varphi$, so $\mathcal{T}_{\mathcal{O}_8} \cong (\mathbb{R}^3_+ \times \mathbb{R})/\mathbb{R}_+ \cong \mathbb{R}^3$.

6. $\mathcal{O}_9, \cdots, \mathcal{O}_{17}$: The holonomy group is the Klein four-group $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle r, s\rangle$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}$. There is a total of four one-dimensional irreducibles: the four combinations of $r \mapsto \pm 1$ and $s \mapsto \pm 1$. We can assume $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1. \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1. \end{pmatrix}$ A generic metric on $\mathcal{O}_i$ is subject to

$$\begin{pmatrix} \alpha & \beta & -\gamma & -\eta \\ & \mu & -\nu & -\zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & \gamma & -\eta \\ & \mu & -\nu & \zeta \\ & & \tau & -\varphi \\ & & & \omega \end{pmatrix}.$$

So $\beta = \gamma = \eta = \nu = \zeta = \varphi = 0$ and $\mathcal{T}_{\mathcal{O}_i} \cong \mathbb{R}^4_+/\mathbb{R}_+ \cong \mathbb{R}^3$.

7. $\mathcal{O}_{18}, \cdots, \mathcal{O}_{20}$: The holonomy group is the dihedral group of order 6 $D_3 = \langle r, s\rangle$ with rotation r and reflection s. The first homology group is $H_1(\mathcal{O}_i) \cong \mathbb{R}$. There are two one-dimensinal irreducibles and one two-dimensional representation. The one-dimensional irreducibles are the trivial one and $r \mapsto 1, s \mapsto -1$. The two dimensional irreducible is the standard one. Up to equivalence, the generators must be $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $r^*g = g = s^*g$ yields two equations

$$\begin{pmatrix} \alpha & \beta & \frac{1}{2}(-\gamma + \sqrt{3}\eta) & -\frac{1}{2}(\sqrt{3}\gamma + \eta) \\ & \mu & \frac{1}{2}(-\nu + \sqrt{3}\zeta) & -\frac{1}{2}(\sqrt{3}\nu + \zeta) \\ & & \frac{1}{4}(\tau - 2\sqrt{3}\varphi + 3\omega) & \frac{1}{4}(\sqrt{3}\tau - 2\varphi - \sqrt{3}\omega) \\ & & & \frac{1}{4}(3\tau + 2\sqrt{3}\varphi + \omega) \end{pmatrix} =$$
$$\begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & -\gamma & \eta \\ & \mu & \nu & -\zeta \\ & & \tau & -\varphi \\ & & & \omega \end{pmatrix}.$$

Then $\gamma = \eta = \zeta = \nu = \varphi = \beta = 0$, so $\mathcal{T}_{\mathcal{O}_i} \cong \mathbb{R}^3_+/\mathbb{R}_+ \cong \mathbb{R}^2$.

8. $\mathcal{O}_{21} \cdots, \mathcal{O}_{24}$: The holonomy group is $D_4 = \langle r, s\rangle$ with r denotes the rotation and s denotes the reflection. The first homology group $H_1(\mathcal{O}_i) \cong \mathbb{R}$.

There are four one-dimensional irreducibles given by $r \mapsto \pm 1$, $s \mapsto \pm 1$ and a standard two-dimensional irreducible representation. The generators can be chosen to be
$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $r^*g = g = s^*g$ imposes

$$\begin{pmatrix} \alpha & \beta & \eta & -\gamma \\ & \mu & \zeta & -\nu \\ & & \omega & -\varphi \\ & & & \tau \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & \gamma & -\eta \\ & \mu & -\nu & \zeta \\ & & \tau & -\varphi \\ & & & \omega \end{pmatrix}.$$

So $\mathcal{T}_{\mathcal{O}_i} \cong \mathbb{R}^3_+/\mathbb{R}_+ \cong \mathbb{R}^2$.

9. $\mathcal{O}_{25}$: The holonomy group is $D_6 = \langle r, s \rangle$ and $H_1(\mathcal{O}_{25}) \cong \mathbb{R}$. D_6 has four one-dimensional irreducible representations, given by the combinations of $r \mapsto \pm 1$ and $s \mapsto \pm 1$, and two two-dimensional irreducibles. One is the standard two-dimensional representation of D_6 and the other is obtained from the composing $D_6 \mapsto D_3$ $(r \mapsto r^2, s \mapsto s)$ with the standard representation of D_3. The generators can be chosen to be
$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ and $s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $r^*g = g = s^*g$ gives

$$\begin{pmatrix} \alpha & \beta & \frac{1}{2}(\gamma + \sqrt{3}\eta) & \frac{1}{2}(-\sqrt{3}\gamma + \eta) \\ & \mu & \frac{1}{2}(\nu + \sqrt{3}\zeta) & \frac{1}{2}(-\sqrt{3}\nu + \zeta) \\ & & \frac{1}{4}(\tau + 2\sqrt{3}\varphi + 3\omega) & \frac{1}{4}(-\sqrt{3}\tau - 2\varphi + \sqrt{3}\omega) \\ & & & \frac{1}{4}(3\tau - 2\sqrt{3}\varphi + \omega) \end{pmatrix} =$$
$$\begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & -\gamma & \eta \\ & \mu & \nu & -\zeta \\ & & \tau & -\varphi \\ & & & \omega \end{pmatrix}.$$

We get $\beta = \gamma = \eta = \nu = \zeta = \varphi$ and $\tau = \omega$, so $\mathcal{T}_{\mathcal{O}_{25}} \cong \mathbb{R}^3_+/\mathbb{R}_+ \cong \mathbb{R}^2$.

10. $\mathcal{O}_{26}$, $\mathcal{O}_{27}$: The holonomy group is the alternating group $A_4 \cong V_4 \rtimes \mathbb{Z}_3$ and $H_1(\mathcal{O}_i) \cong \mathbb{R}$, where V_4 is the Klein four-group. If we view A_4 as subgroup of S_4 acting on $\{1, 2, 3, 4\}$, then $V_4 \trianglelefteq A_4$ can be identified with

$\{(12)(34),(13)(24),(14)(23),(1)\}$ and $\mathbb{Z}_3 = \langle(123)\rangle$. A_4 is generated by $r = (12)(34)$ and $s = (123)$. Over the reals, A_4 has three irreducibles, with degree one, two and three respectively. The action of Φ must be equivalent to the three-dimensional irreducible given by the permutation on $\{e_1, e_2, e_3, e_4\}$. This has one-dimensional fixed subspace $\mathrm{Span}\{e_1 + e_2 + e_3 + e_4\}$. So the action is equivalent to $r = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. $r^*g = g = s^*g$ gives

$$\begin{pmatrix} \mu & \beta & \zeta & \nu \\ & \alpha & \eta & \gamma \\ & & \omega & \varphi \\ & & & \tau \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \eta \\ & \mu & \nu & \zeta \\ & & \tau & \varphi \\ & & & \omega \end{pmatrix} = \begin{pmatrix} \tau & \gamma & \nu & \varphi \\ & \alpha & \beta & \eta \\ & & \mu & \zeta \\ & & & \omega \end{pmatrix}.$$

So we obtain $\alpha = \tau = \mu = \omega$ and $\beta = \gamma = \nu = \eta = \varphi = \zeta$. The Teichmüller space $\mathcal{T}_{\mathcal{O}} \cong (\mathbb{R} \times \mathbb{R}_+)/\mathbb{R}_+ \cong \mathbb{R}$.

An algebraic description of the Teichmüller space of flat metrics was given in the elegant work of Bettiol, Derdzinski and Piccione [12].

Theorem 3.15 (Bettiol, Derdzinski, Piccione)**.** *Let $\mathcal{O}$ be a closed flat orbifold and let W_i, $1 \leq i \leq l$, be the isotypical components of the orthogonal representation of its holonomy group. Then the Teichmüller space together with homothety is*

$$\mathbb{R}_+ \times \mathcal{T}_{\mathcal{O}} \cong \prod_{j=1}^{l} \frac{\mathrm{GL}(n_j, \mathbb{K}_j)}{\mathrm{O}(n_j, \mathbb{K}_j)},$$

where $\mathbb{R}^+$ acts by rescaling. Each $\mathbb{K}_j$ stands for $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, according to the irreducibles in W_i being real, complex or quaternionic type.

This theorem allows one to read off the Teichmüller space once the representation of the holonomy group Φ is known. We will use their method to verify our theorem about the Teichmüller spaces of closed oriented flat four-manifolds.

1. $\mathcal{O}_1 = T^4$: $\Phi = \{1\}$ acts trivially on $\mathbb{R}^4$. So Φ has one isotypical component consists of four trivial irreducibles and $\mathcal{T}_{T^4} \cong \frac{\mathrm{GL}(4,\mathbb{R})}{\mathrm{O}(4)}/\mathbb{R}_+ \cong \mathbb{R}^9$.

2. $\mathcal{O}_2, \mathcal{O}_3$: $\Phi \cong \mathbb{Z}_2$. Φ has two isotypical components. One consists of two one-dimensional trivial representations and the other consists of two one-dimensional sign representations. So $\mathcal{T}_{\mathcal{O}_i} \cong [\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{O}(2)} \times \frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{O}(2)}]/\mathbb{R}_+ \cong \mathbb{R}^5$.

3. $\mathcal{O}_4, \cdots, \mathcal{O}_8$: Φ is one of $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$. In this case, each of the action of the holonomy group has two one-dimensional trivial representations and a two-dimensional rotation, which is of complex type since $\|\chi_V\|^2 = 2$. So $\mathcal{T}_{\mathcal{O}_i} \cong [\frac{\mathrm{GL}(2,\mathbb{R})}{\mathrm{O}(2)} \times \frac{\mathrm{GL}(1,\mathbb{C})}{\mathrm{U}(1)}]/\mathbb{R}_+ \cong \mathbb{R}^3$.

4. $\mathcal{O}_9, \cdots, \mathcal{O}_{17}$: $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The action of Φ has four distinct one-dimensional irreducible representation. The Teichmüller space $\mathcal{T}_{\mathcal{O}_i} \cong [\frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)}]/\mathbb{R}_+ \cong \mathbb{R}^3$.

5. $\mathcal{O}_{18}, \cdots, \mathcal{O}_{25}$: Φ is one of the three dihedral groups D_3, D_4 and D_6. There are three isotypical components. Each action of Φ decomposes into the trivial one-dimensional representation, the sign representation and the standard two-dimensional representation. The standard two-dimensional representation is of real type since $\|\chi_V\|^2 = 1$. So $\mathcal{T}_{\mathcal{O}_i} \cong [\frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)}] \cong \mathbb{R}^2$.

6. $\mathcal{O}_{26}, \mathcal{O}_{27}$: The action of $\Phi \cong A_4$ decomposes into the trivial one-dimensional representation and the irreducible three dimension representation, which is of real type since $\|\chi_V\|^2 = 1$. So there are two isotypical components and $\mathcal{T}_{\mathcal{O}_i} \cong [\frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)} \times \frac{\mathrm{GL}(1,\mathbb{R})}{\mathrm{O}(1)}]/\mathbb{R}_+ \cong \mathbb{R}$.

3.6 Period map for metrics

We now describe the period map picture for the flat torus T^4. Let $\mathbb{R}^4$ be its universal cover and let $g_0 = \sum_{i=1}^{i=4} dx_i \otimes dx_i$ be the standard flat metric. First, the second cohomology can be easily computed $H^2(T^4, \mathbb{R}) = \Lambda^2 H^1(T^4, \mathbb{R}) = \mathbb{R}^6$ using the Künneth formula. We claim that $b_+ = b_- = 3$. To see this, observe that

$$\theta_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4, \quad \theta_2 = dx_1 \wedge dx_3 + dx_4 \wedge dx_2, \quad \theta_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3$$

are self-dual harmonic two-forms on $\mathbb{R}^4$ which descend to T^4. Likewise we define anti-self-dual harmonic two-forms on T^4

$$\theta_4 = dx_1 \wedge dx_2 + dx_4 \wedge dx_3, \quad \theta_5 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4, \quad \theta_6 = dx_1 \wedge dx_4 + dx_3 \wedge dx_2.$$

Alternately, one could start with two-forms defined at a point as above then parallel transport to the whole torus T^4. This is well-defined as the holonomy group of T_4 is trivial. It follows that $b_+ = b_- = 3$.

Fix a flat metric g on T^4, the corresponding self-dual harmonic two-forms $\mathbb{H}^+(g)$ in $H^2(X, \mathbb{R})$ is an oriented and maximal positive three-dimensional subspace. There is then a natural period map from the Teichmüller space $\mathcal{P}_{\mathcal{T}} : \mathcal{T}(T^4) \to \mathrm{Gr}_3^+(H^2(T^4, \mathbb{R}))$ to the Grassmannian of oriented positive definite three-planes. Furthermore, the group of self-diffeomorphisms $\mathcal{D}(T^4)$ acts on the Teichmüller space $\mathcal{T}(T^4)$, so we have the following diagram

$$\begin{tikzcd}
\mathcal{T}(T^4) \arrow[r, "\mathcal{P}_{\mathcal{T}}"] \arrow[d, "\pi"'] & \mathrm{Gr}_3^+(H^2(T^4, \mathbb{R})) \arrow[d] \\
\mathcal{E}(T^4) \arrow[r, "\mathcal{P}_{\mathcal{E}}"] & \Gamma_{T^4} \setminus \mathrm{SO}^+(3,3)/\,\mathrm{SO}(3) \times \mathrm{SO}(3),
\end{tikzcd}$$

where Γ_{T^4} is the isometry group of $H^2(T^4, \mathbb{Z})$ with its intersection form.

Analogs of the above period maps will be used extensively in the following sections.

3.7 Relation with Kähler geometry

Remark 3.16. Let M be a compact oriented flat four-manifold. The fact that Euler characteristic vanishes $\chi(M) = 0$ follows from the Gauss-Bonnet formula 2.13. This means that b_1 is greater than zero. On the other hand, the Bochner theorem gives an upper bound for $b_1 = \dim H^1(M, \mathbb{R}) \leq \dim M = 4$. We further note that b_1 cannot be three. If there are three linearly independent parallel harmonic one-forms μ_1, μ_2, μ_3 on M, we can define another one-form ν using the metric by requiring ν to be perpendicular to all the μ_i's. This ν is also parallel and harmonic, so $b_1 = 4$ in this case. To sum up, there are exactly three possible values for the first Betti number b_1 on a compact flat oriented four-manifold.

Theorem 3.17. *A compact flat oriented four-manifold M is Kähler if and only if $b_1 = 2$ or 4.*

Proof. If M is Kähler, there is a decomposition $H^1(M, \mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M)$, with $\overline{H^{1,0}}(M) = H^{0,1}(M)$. Putting together this observation with

Remark 3.16, we see that $b_1 = 2$ or 4. Conversely, let M be a compact oriented flat manifold with $b_1 = 4$. The holonomy group $\mathrm{Hol}(M)$ is then trivial. If $b_1 = 2$, then $\mathrm{Hol}(M)$ is reduced to $\mathrm{SO}(2)$. In both cases, the holonomy group is contained in the unitary group $\mathrm{U}(2)$, so M is Kähler. $\square$

We will illustrate the compact flat four-manifolds in Theorem 3.17 with $b_1 = 2$ using seven classes of compact complex surfaces known as the hyperelliptic surfaces. A hyperelliptic surface is a complex surface $(E \times F)/G$, where E, F are elliptic curves, G is a finite group acting freely on $E \times F$ such that G belongs to the group of translations of E and $F/G \cong \mathbb{P}^1$. It is immediate from definition that every hyperelliptic surface is finitely covered by an Abelian variety. A hyperelliptic surface is also called bi-elliptic since its Albanese morphism is an elliptic fibration to an elliptic curve. There is a classification result due to Bagnera-de Franchis. For $E \cong \mathbb{C}/\Gamma$, every hyperelliptic surface is one of the following.

Types	Γ	G	Action of G on E
$\mathrm{I_a}$	Any	$\mathbb{Z}_2$	$e \mapsto -e$
$\mathrm{I_b}$	Any	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$e \mapsto -e$, $e \mapsto e + e_1$, $-e_1 = e_1$
$\mathrm{III_a}$	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_3$	$e \mapsto \omega e$
$\mathrm{III_b}$	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_3 \oplus \mathbb{Z}_3$	$e \mapsto \omega e$, $e \mapsto e + e_1$, $\omega e_1 = e_1$
$\mathrm{II_a}$	$\mathbb{Z} \oplus \mathbb{Z}i$	$\mathbb{Z}_4$	$e \mapsto ie$
$\mathrm{II_b}$	$\mathbb{Z} \oplus \mathbb{Z}i$	$\mathbb{Z}_4 \oplus \mathbb{Z}_2$	$e \mapsto ie$, $e \mapsto e + e_1$, $ie_1 = e_1$
$\mathrm{III_c}$	$\mathbb{Z} \oplus \mathbb{Z}\omega$	$\mathbb{Z}_6$	$e \mapsto -\omega e$

The above table is taken from [6], with ω denoting a primitive cube root of unity. The holonomy groups for hyperelliptic surfaces of types $\{\mathrm{I_a}, \mathrm{I_b}\}$, $\{\mathrm{III_a}, \mathrm{III_b}\}$, $\{\mathrm{II_a}, \mathrm{II_b}\}$, $\{\mathrm{III_c}\}$ are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ respectively. All these holonomy groups are subgroups of $\mathrm{SO}(2) \subset \mathrm{U}(2)$, so all such complex surfaces are Kähler. Nevertheless, they are not contained in $\mathrm{SU}(2)$. Hence none of them is hyperKähler. One could also verify this fact by considering the canonical bundle. In fact, the order of the canonical bundle is exactly the size of the holonomy group. These seven families of hyperelliptic surfaces correspond to the compact flat oriented manifolds $\mathcal{O}_2, \cdots, \mathcal{O}_8$ in the proof of Theorem 3.14.

4 Almost complex structures and semi-complex structure on flat manifolds

In this section, we digress briefly to discuss two generalizations of integrable complex structures. Let M be a compact oriented smooth manifold. An almost complex structure J on M is a section of the bundle $\Gamma(TM \otimes T^*M)$ such that $J^2 = -1$ on T_xM for every $x \in M$.

Proposition 4.1. *For a Bieberbach manifold M, $b_1 = 1$ if and only if $b_2^+ = 0$.*

Proof. From the formulas for Euler characteristic 2.13 and signature 2.14, we obtain $b_2^+ = b_1 - 1$. $\square$

To prove the next theorem, we will need the following result.

Proposition 4.2 (Wu)**.** *A compact oriented four-manifold W admits an almost complex structure J with $c_1(W, J) = h$ if and only if*

1. $h^2 = 2\chi + 3\tau$
2. $h \equiv w_2$ *mod* 2

Lemma 4.3. *A Bieberbach manifold M with $b_1 = 1$ admits an almost complex structure.*

Proof. It is sufficient to check the existence of an $h \in H^2(M, \mathbb{Z})$ satisfying the two criteria in Wu's theorem. Our assumption $b_1 = 1$ implies that $b_2 = 0$. Hence $H^2(M, \mathbb{Z})$ consists only of torsion elements, $h^2 = 0$ is then obvious. The existence of a lift of w_2 is automatic since every oriented four manifold admits a spin^c structure. $\square$

Theorem 4.4. *All Bieberbach manifolds admit almost complex structure.*

Proof. From Remark 3.16, the first Betti number of M must have value either 1, 2 or 4. The Bieberbach manifolds with b_1 even were shown to be Kähler in Theorem 3.17. The remaining case $b_1 = 1$ follows from the above lemma. $\square$

A semi-complex structure on M is a one-dimensional subbundle $L \subset TM \otimes T^*M$ such that in a neighborhood of any point $p \in M$, L is spanned by an integrable complex structure, see [61]. Suppose M is equipped with a semi-complex structure, then locally near any point, the subbundle L is generated by two complex structures J and $-J$. M therefore naturally has an atlas of charts whose transition functions are either holomorphic or anti-holomorphic. There is an alternative way to describe a semi-complex structure which is very convenient for our purpose. Take the line bundle L in the definition and define M' to be the sphere bundle of L, which is evidently a complex manifold. Interchanging the two sheets gives rise to a free anti-holomorphic action on M'. Conversely, a complex manifold with a free anti-holomorphic involution yields a semi-complex structure. Thus, a semi-complex manifold is equivalently a complex manifold together with a free anti-holomorpic involution.

Example 4.5. A non-orientable surface $(X, [g])$ with a fixed conformal class of metrics does not admit any compatible almost complex structure. But there exists a unique semi-complex structure on X compatible with $[g]$.

Proposition 4.6. *An oriented Bieberbach manifold M whose holonomy group is a dihedral group has a semi-complex structure.*

Proof. We will use the notation employed in the preceding section. The possible holonomy groups Φ are D_2, D_3, D_4 and D_6. Choosing an appropriate set of basis, the rotation r and reflection s has the following form.

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now let $\tilde{M}$ be the smooth manifold $T^4/\langle r+v\rangle$ and let $M = T^4/\langle r+v, s+w\rangle$, where v and w are translations so that $r+v$ and $s+w$ belong to the corresponding Bieberbach group. $\tilde{M}$ is naturally a complex surface with holonomy group $\langle r\rangle$. The quotient map $q : \tilde{M} \to M$ then exhibits $\tilde{M}$ as a double cover of M obtained via the action of an anti-holomorphic involution. □

Remark 4.7. The double cover $\tilde{M}$ is in fact a hyperelliptic surface. Our proof shows that oriented Bieberbach manifolds with holonomy groups D_2, D_3, D_4 and D_6 are covered by hyperelliptic surfaces of types $\{\mathrm{I_a}, \mathrm{I_b}\}$, $\{\mathrm{III_a}, \mathrm{III_b}\}$, $\{\mathrm{II_a}, \mathrm{II_b}\}$, $\{\mathrm{III_c}\}$, respectively.

Lemma 4.8. *If an oriented Bieberbach manifold M^4 is a complex manifold, then it is of Kähler type.*

Proof. Such a flat manifold M is finitely covered by a complex torus. Every complex torus is of Kähler type as the first Betti number is even. Using an averaging argument, one could show that any complex manifold finitely covered by a Kähler manifold is itself of Kähler type. $\square$

Proposition 4.9. *An oriented Bieberbach 4-manifold M does not admit any semi-complex structure if the holonomy group of M is the alternating group A_4.*

Proof. First, notice that $\mathrm{Hom}(\pi_1(M), \mathbb{Z}_2) = H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$. Thus, M has a non-trivial double cover. We can see this more explicitly. Since the first Betti number $b_1 = 1$, there exists a fibration $f : M \to S_1$ whose fiber is flat. Now S_1 has a natural double cover $p : S^1 \to S^1$, so the pullback bundle p^*M is a double cover of M. Assume the contrary that M has a semi-complex structure. Then there is a double cover $\tilde{M} \to M$ which is a complex manifold. Pulling back the standard metric on M, we see that $\tilde{M}$ is flat, so $M \cong \mathbb{R}^4/\Gamma$ and $\tilde{M} \cong \mathbb{R}^4/\tilde{\Gamma}$ for some torsion-free Bieberbach groups Γ and $\tilde{\Gamma}$. A complex Bieberbach manifold must be of Kähler type by Lemma 4.8. Theorem 3.17 in turn gives $b_1 = 2$ or 4. This implies $\tilde{M}$ is either a complex torus T^4 or a hyperelliptic surface. The holonomy group of $\tilde{\Gamma}$ is then one of $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$. The covering involution σ is naturally an isometry, so it belongs to Γ. Now we can write it as $\sigma = A + v$, where $A \in \mathrm{O}(4)$ and $v \in \mathbb{R}^4$. From our construction, A and the holonomy group $\mathrm{Hol}(\tilde{\Gamma})$ generate the holonomy group $\mathrm{Hol}(\Gamma)$ of M. We can eliminate the case that $\mathrm{Hol}(\tilde{\Gamma}) = \mathbb{Z}_6$ because A_4 has no subgroup of order 6. Finally, if $\mathrm{Hol}(\tilde{\Gamma})$ is either $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_4$, $|\langle \mathrm{Hol}(\tilde{\Gamma}), \sigma\rangle| \leq 8 < 12$. This gives the required contradiction.

$\square$

We will give another proof of the above theorem by analyzing our problem through the lens of affine structures, see section 3.3 for definition.

Take an oriented Bieberbach manifold whose holonomy group is A_4. Assume that M is equipped with a semi-complex structure, so there is a double cover $\pi : M' \to M$ which is a complex manifold. M' admits a finite cover $\theta : N \to M'$ diffeomorphic to the four-torus T^4. N inherits a complex structure from M' by pulling back. To endow N with an affine structure, we consider the Albanese morphism $\phi : N \to \mathrm{Alb}(N) \cong H^0(N, \Omega^1_N)^*/H_1(N, \mathbb{Z})$. The Albanese variety is itself a complex torus of complex dimension $h^{1,0} = 2$. By the universal property of the Albanese variety, we easily see that ϕ is a biholomorpshim. Hence we have the isomorpshim $N \cong \mathrm{Alb}(N) \cong \mathbb{C}^2/\Lambda$ for some full-rank lattice Λ.

Notice that $\mathbb{C}^2/\Lambda$ is a Kähler manifold with a torsion-free flat connection ∇, a complex structure J and a flat metric g, each compatible with one another.

$$\begin{tikzcd}
N \arrow[r, "\cong"] \arrow[d, "\theta"'] & \mathbb{C}^2/\Lambda \arrow[dl, "\alpha"'] \arrow[ddl, "\beta"] \\
M' \arrow[d, "\pi"'] & \\
M &
\end{tikzcd} \tag{4.1}$$

Lemma 4.10. *Biholomorphisms and anti-biholomorphisms of* $\mathbb{C}^2/\Lambda$ *are affine.*

We omit the proof of this lemma as it is shown in many texts in complex analysis.

Now the covering transformations for $\alpha : \mathbb{C}^2/\Lambda \to M'$ are biholomorphic, so must be affine by Lemma 4.10. This implies the affine structure on $\mathbb{C}^2/\Lambda$ descends to M'. Since the group of covering transformations is finite, we can define a new metric g' by averaging, i.e. $g' = \sum_\mu \mu^* g$. This new metric is invariant under the action of all covering transformations, so g' descends to a metric on M'. The triplet (∇, J, g') on M' is again pairwise compatible.

Next, we will apply similar arguments for the covering $\beta : \mathbb{C}^2/\Lambda \to M$. All covering transformations of β are affine by Lemma 4.10 as they are either biholomorphic or anti-biholomorphic map. Unlike the preceding case $\alpha : \mathbb{C}^2 \to M'$, the complex structure does not descend to M. But the compatibility between the semi-complex structure, the torsion-free affine connection and the metric persists. A semi-complex structure can be viewed as a section Φ of the bundle $\mathrm{End}(TM) \otimes \mathrm{End}(TM)$. Indeed, we simply define Φ to be $J \otimes J$ locally. Then $\nabla\Phi = 0$ follows from $\nabla J = 0$. We define a new metric $\tilde{g}$ on M by averaging over the covering transformations as before. This metric is locally Kähler on M. As a consequence, the holonomy group of M belongs to $(\mathrm{U}(2) \rtimes \mathbb{Z}_2) \cap \mathrm{SO}(3)$, giving a contradiction.

5 HyperKähler metrics on K3 surfaces

In this section, our goal is to review the moduli theory of complex structures and Einstein metrics for K3 surfaces.

A compact connected complex surface X is called a K3 surface if it is simply connected and has trivial canonical bundle. Since $c_1(X) = 0$ and $b_1 = 0$, by the Riemann-Roch theorem, we get $c_2(X) = 24$, so $H^2(X, \mathbb{Z})$ is torsion-free with rank 22. From the Hirzebruch signature formula, we see that the signature $\tau(X)$ equals -16. Hence $H^2(X, \mathbb{Z})$ is an indefinite unimodular lattice with signature (3,19). But such a lattice is unique up to isometry, so we must have $H^2(X, \mathbb{Z}) \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Here $E_8(-1)$ is the negative definite rank 8 lattice equipped with the inner product coming from the Cartan matrix of E_8 and U is the rank 2 hyperbolic lattice, with inner product given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. From now on, let L denote the K3 lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

5.1 Moduli space of complex structures

The material here is taken from [36]. The local structure of the moduli space of complex structures for K3 surfaces is known to be smooth from the Kodaira-Spencer theory as the obstruction space $H^2(X, \mathcal{T}_X) = 0$. However, the global structure is vastly more complicated and badly behaved. To probe the structure of this moduli space, we will appeal to the theory of period map and the period domain. The latter parametrizes the Hodge structures of the corresponding K3 surface.

Definition 5.1. The open subset of the following quadric in $\mathbb{P}^{21}$

$$\Omega := \{x \in \mathbb{P}(L_{\mathbb{C}}) : (x, x) = 0, (x, \overline{x}) > 0\}$$

is called the period domain associated with L.

By a famous theorem of Siu [75], every K3 surface X is Kähler. So the cohomology of X admits a decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

For a K3 surface X_0, a *marking* is a choice of lattice isomorphism $\varphi : H^2(X_0, \mathbb{Z}) \to L$. Let $f : \mathfrak{X} \to S$ be the Kuranishi family of X_0. Then a chosen marking φ

induces canonical markings for all fibers $H^2(X_s, \mathbb{Z}) \cong L$. In other words, the locally constant system $R^2 f_* \mathbb{Z} \cong \underline{H}^2(X_0, \mathbb{Z})$ is constant.

The local Torelli theorem, attributed to Andreotti and Weil by Kodaira, marks the beginnning of the study of moduli theory for K3 surfaces, see [42], [36] and [6].

Theorem 5.2 (Local Torelli Theorem)**.** *The local period map $\mathcal{P}_S : S \to \mathbb{P}(L_{\mathbb{C}})$ sending*

$$s \mapsto [\varphi_{\mathbb{C}}(H^{2,0}(X_s))], \tag{5.1}$$

is a local isomorphism, where $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$ and $\varphi_{\mathbb{C}}$ are both complexifications.

The *moduli space of marked K3 surfaces* is defined to be

$$\mathcal{N} = \{(X, \varphi)\}/\sim,$$

where $(X, \varphi) \sim (X', \varphi')$ if there is an isomorphism $f : X \to X'$ such that $\varphi \circ f^* = \varphi'$. $\mathcal{N}$ can be given a complex manifold structure as follows. For a marked K3 surface (X, φ), let $\mathfrak{X} \to S$ be its universal Kuranishi family. We take the collection of these Kuranishi families as complex charts on $\mathcal{N}$. By the local Torelli theorem, S can be embedded in Ω via the local period map. Suppose $\mathfrak{X}' \to S'$ is the Kuranishi family for some (X', φ'), then we can glue S and S' according to their images in the period domain if $\mathcal{P}_S(S) \cap \mathcal{P}_{S'}(S')$ is non-empty.

Remark 5.3. $\mathcal{N}$ is non-Hausdorff, as shown by Atiyah in [3]. Consider the quadric cone Q in $\mathbb{C}^4$ defined by $z_1 z_4 - z_2 z_3 = 0$. Blowing up Q gives a birational morphism $\tilde{Q} \to Q$ with exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. $\tilde{Q}$ is isomorphic to the bundle $\mathcal{O}(-1, -1) \to \mathbb{P}^1 \times \mathbb{P}^1$. Denote the homogeneous coordinates of the first $\mathbb{P}^1$ by $[\alpha : \beta]$, the second one by $[\mu : \nu]$. Then elements of $\mathcal{O}(-1, -1)$ can be represented by the matrix $\begin{pmatrix} \alpha\mu & \alpha\nu \\ \beta\mu & \beta\nu \end{pmatrix}$. If we project onto the first $\mathbb{P}^1$, the resulting threefold W_1 is isomorphic to the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. Similarly, projecting onto the second factor yields a threefold W_2 that is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. This construction produces a birational map $W_1 \dashrightarrow W_2$ which cannot be extended to a morphism. This example is known as a flop in birational geometry. Atiyah's construction shows we can blow up a node in two different ways and get two distinct families of K3 surfaces. However, these two families are related by a flop and agree outside one point on the base. Hence $\mathcal{N}$ cannot be Hausdorff.

The two foundational results that form the backbone of the moduli theory for K3 surfaces are the global Torelli Theorem and the surjectivity of the period map. We state them here for they will be used repeatedly in the following sections.

Theorem 5.4 (Global Torelli Theorem)**.** *Two K3 surfaces X and X' are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \cong H^2(X',\mathbb{Z})$. If $\phi : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an effective Hodge isometry then ϕ is induced by a unique isomorphism $f : X \to X'$.*

The above theorem implies that the global period map $\mathcal{P}$ is generically injective, which means $\mathcal{P}$ is injective on a Zariski open subset on each component of $\mathcal{N}$. The Torelli Theorem can be deduced from the fact that the monodromy group of a fixed K3 surface has index 2 in the group of isometries $\mathrm{O}(H^2(X,\mathbb{Z}))$, which in turn shows that $\mathcal{N}$ has exactly two components. The Torelli theorem for algebraic K3 surfaces was discovered by Piatetski-Shapiro and Shafarevich [72]. The case for general complex K3 surfaces is due to Burns and Rapoport [16].

Theorem 5.5 (Surjectivity of the period map)**.** *The global period map*

$$\mathcal{P} : \mathcal{N}_i \to \Omega, \tag{5.2}$$

is surjective, where $\mathcal{N}_i$ is any connected components of $\mathcal{N}$.

The Calabi-Yau theorem is important in the theory of K3 surfaces for it implies any K3 surface can be given a hyperKähler structure. This is used by Todorov in [80] to attain a proof of the surjectivity theorem. Subsequent proofs are given by Looijenga [62] and Huybretchs [37].

5.2 Einstein metrics on K3 surfaces

First, we note that, by Theorem 2.2, any Einstein metric on a K3 surface is necessarily hyperKähler. Second, any K3 surface was shown by Kodaira to be deformation equivalent to a quartic in $\mathbb{P}^3$, so in particular there is a unique diffeomorpshim type for K3 surfaces. Thus, to study the Einstein moduli space on K3 surfaces, there is no loss of generality by considering the space of hyperKähler metrics on a fixed K3 surface X.

We will go along with the convention in [16], [40] and [63].

Definition 5.6. Take an $[x] \in \Omega$, we define

$$\Delta(x) = \{\delta \in L : (\delta, x) = 0, (\delta, \delta) = -2\},$$

$$V(x) = \{\omega \in L_{\mathbb{R}} : (\omega, x) = 0, (\omega, \omega) = 1\},$$

$$V_\Delta(x) = \{\omega \in V(x) : (\omega, \delta) \neq 0 \text{ for every } \delta \in \Delta\}.$$

$\Delta(x)$ is called the set of *roots* associated with x. Geometrically, if we identify L with $H^2(X, \mathbb{Z})$ for some K3 surface X, then either δ or $-\delta$ is effective by the Riemann-Roch theorem. $V(x)$ and $V_\Delta(x)$ contain the Kähler classes with unit norm.

Next, we introduce marked polarized space and the polarized period domain.

Definition 5.7. The polarized period domain is defined to be

$$K\Omega = \{(\omega, [x]) \in L_{\mathbb{R}} \times \Omega : \omega \in V(x)\},$$

$$K\Omega^0 = \{(\omega, [x]) \in L_{\mathbb{R}} \times \Omega : \omega \in V_\Delta(x)\}.$$

$K\Omega^0$ is an open subset in $K\Omega$.

Definition 5.8. The moduli space of marked polarized K3 surfaces is

$$\mathfrak{M} = \{(X, \omega, \varphi) : \omega \text{ a unit-norm Kähler class on } X, \varphi \text{ a marking}\}/\sim,$$

where $(X, \omega, \varphi) \sim (X', \omega', \varphi')$ if there exists an isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$ and $\varphi \circ f^* = \varphi'$. This allows us to define the polarized period map $\mathcal{P}_{\mathfrak{M}} : \mathfrak{M} \to K\Omega$ by sending $(X, \omega, \varphi) \mapsto (\varphi_{\mathbb{R}}(\omega), [\varphi_{\mathbb{C}}(H^{2,0}(X))])$.

Definition 5.9. The Teichmüller space of marked K3 manifolds with smooth hyperKähler metrics is defined as

$$\mathfrak{N} = \{(X, g, \varphi) : g \text{ a unit-volume hyperKähler metric on } X\}/\sim,$$

where $(X, g, \varphi) \sim (X', g', \varphi')$ if there exists a diffeomorphism $f : X \to X'$ such that $f^*g' = g$ and $\varphi \circ f^* = \varphi'$. We can define a period map $\mathcal{P}_{\mathfrak{N}} : \mathfrak{N} \to \mathrm{Gr}_3^+(L_{\mathbb{R}})$ by sending $(X, g, \varphi) \mapsto \varphi(\mathcal{H}^+(g))$, where $\mathcal{H}^+(g)$ is the space of self-dual harmonic two-forms on X.

Remark 5.10. Definition 5.8 admits a generalization to incorporate K3 surface with ADE singularities, i.e. compact connected complex surface with simple singularities whose minimal resolution is a K3 surface. Denote the moduli space of marked polarized ADE K3 surfaces as $\tilde{\mathfrak{M}}$ and the corresponding polarized period map as $\mathcal{P}_{\tilde{\mathfrak{M}}} : \tilde{\mathfrak{M}} \to K\Omega$. Similarly, we can extend definition 6.14 to include orbifold hyperKähler metrics. Denote the Teichmüller space of marked ADE K3 surfaces with orbifold Ricci-flat metrics by $\tilde{\mathfrak{N}}$ and the period map as $\mathcal{P}_{\tilde{\mathfrak{N}}} : \tilde{\mathfrak{N}} \to \mathrm{Gr}_3^+(L_{\mathbb{R}})$, see [40] for details.

The ensuing result combines the Torelli theorem and Surjectivity of period map for the marked polarized K3 surfaces.

Theorem 5.11 (Looijenga)**.** *The polarized period map $\mathcal{P}_{\mathfrak{M}} : \mathfrak{M} \to K\Omega^0$ is a bijection.*

If we include the orbifold K3 surfaces with ADE singularities $\tilde{\mathfrak{M}}$, the above bijection also extends. This $\tilde{\mathfrak{M}}$ is the completion of $\mathfrak{M}$ with respect to the Gromov-Hausdorff topology or the L^2-metric of Anderson. The polarized period map $\mathcal{P}_{\tilde{\mathfrak{M}}}$ is an isometry between the L^2-metric on $\tilde{\mathfrak{M}}$ and the locally symmetric metric on $K\Omega$.

Theorem 5.12 (Morrison)**.** *The polarized period map $\mathcal{P}_{\tilde{\mathfrak{M}}} : \tilde{\mathfrak{M}} \to K\Omega$ is a bijection.*

Building on Morrison's work, Kobayashi and Todorov gave an explicit description of the moduli space of orbifold Einstein metrics on a K3 surface X.

Theorem 5.13 (Kobayashi,Todorov)**.** *$\mathcal{P}_{\tilde{\mathfrak{N}}} : \tilde{\mathfrak{N}} \to \mathrm{Gr}_3^+(L_{\mathbb{R}})$ is a bijection. The orbifold Einstein moduli space*

$$\tilde{\mathcal{E}} = \{(\tilde{X}, \tilde{g}) : \tilde{X} \textit{ an ADE K3 surface with orbifold Ricci-flat metric } \tilde{g}\}/\sim$$

is therefore diffeomorphic to

$$\Gamma \setminus \mathrm{SO}^+(3,19)/\,\mathrm{SO}(3) \times \mathrm{SO}(19),$$

where Γ is the isometry group of the K3 lattice.

The set of orbifold metrics with simple singularities is a countable union of submanifolds of codimension 3 [40]. Removing this set we see that the Einstein moduli space for a K3 surface is connected.

Theorem 5.14. *The Einstein moduli space $\mathcal{E}(X)$ for a K3 surface X is path-connected.*

Proof. First, we note that the above period maps fit into the following diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\mathcal{P}_{\mathfrak{M}}"] \arrow[d, "\alpha"'] & K\Omega \arrow[d, "\pi"] \\ \mathfrak{N} \arrow[r, "\mathcal{P}_{\mathfrak{N}}"] & \mathrm{Gr}_3^+(L_{\mathbb{R}}), \end{tikzcd} \tag{5.3}$$

where $\alpha : (X, \omega, \varphi) \mapsto (X, g, \varphi)$ is obtained via the Calabi-Yau theorem and $\pi : (\omega, [x]) \mapsto (\omega, \mathrm{Re}(x), \mathrm{Im}(x))$. By Theorem 5.11, Theorem 5.12 and Theorem 5.13, it suffices to show that the image of $K\Omega \setminus K\Omega^0$ under π has codimension three in $\mathrm{Gr}_3^+(L_{\mathbb{R}})$. But

$$K\Omega^0 = K\Omega \setminus \bigcup_{\delta^2=-2} \delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp].$$

So for each $\delta \in L$ with $\delta^2 = -2$, $\pi(\delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp] \cap K\Omega) = \mathrm{Gr}_3^+(\delta_{\mathbb{R}}^\perp) \subset \mathrm{Gr}_3^+(L_{\mathbb{R}})$ is a submanifold in $\mathrm{Gr}_3^+(L_{\mathbb{R}})$ of codimension three. Since $\mathcal{P}_{\mathfrak{N}} : \mathfrak{N} \to \mathrm{Gr}_3^+(L_{\mathbb{R}})$ is a bijection onto its image, from the diagram (5.3),

$$\mathfrak{N} \cong \mathrm{Gr}_3^+(L_{\mathbb{R}}) \setminus \bigcup_{\delta^2=-2} \mathrm{Gr}_3^+(\delta_{\mathbb{R}}^\perp),$$

so $\mathfrak{N}$ is path-connected. Now observe that there is a natural action of Γ on $\mathfrak{N}$ so that the quotient forgets the markings. Hence $\mathfrak{N}/\Gamma \cong \mathcal{E}(X)$ is path-connected as well. $\square$

Remark 5.15. One could define a period map for metric without using the markings as follows. Fix a hyperKähler metric g on a K3 surface X, the corresponding self-dual harmonic two-forms $\mathcal{H}^+(g)$ in $H^2(X, \mathbb{R})$ is an oriented and maximal positive three-dimensional subspace. This yields a well-defined map from the Teichmüller space $\mathcal{P} : \mathcal{T}(X) \to \mathrm{Gr}_3^+(H^2(X, \mathbb{R}))$ to the Grassmannian of oriented positive definite three-planes. Considering the action of $\mathcal{D}(X)$, we get another map and the following diagram

$$\begin{tikzcd} \mathcal{T}(X) \arrow[r, "\mathcal{P}_{\mathcal{T}}"] \arrow[d, "\pi"'] & \mathrm{Gr}_3^+(H^2(X, \mathbb{R})) \arrow[d] \\ \mathcal{E}(X) \arrow[r, "\mathcal{P}_{\mathcal{E}}"] & \Gamma \setminus \mathrm{SO}^+(3,19)/\mathrm{SO}(3) \times \mathrm{SO}(19), \end{tikzcd}$$

where Γ is the isometry group of the K3 lattice, $H^2(X, \mathbb{Z})$ with its intersection form.

6 Locally hyperKähler metrics on Enriques surfaces

An Enriques surface Y is a compact connected surface such that the irregularity $q(Y) = 0$ and the canonical bundle K_Y has square $K_Y^{\otimes 2} = \mathcal{O}_Y$ but $K_Y \neq \mathcal{O}_Y$. Every Enriques surfaces is projective and admits an elliptic fibration over $\mathbb{P}_1$.

By the condition $K_Y^{\otimes 2} = \mathcal{O}_Y$, Y does not admit non-zero global holomorphic two-form and the first Chern class satisfies $c_1(Y)^2 = 0$. Hence the geometric genus $p_g(Y)$ vanishes and $\chi(\mathcal{O}_Y) = 1$. Using the Noether's formula, we obtain the Euler characteristic $\chi(Y) = 12$. On the other hand, $H_1(Y, \mathbb{Z})$ contains a non-trivial two-torsion element so Y admits an unramified double cover $\varpi : X \to Y$ such that $K_X = \varpi^* K_Y = \mathcal{O}_X$. From $\chi(X) = 2\chi(Y) = 24$, the Noether's formula yields $q(X) = 0$, i.e. $\dim H^1(X, \mathcal{O}_X) = 0$. X is therefore a K3 surface. The above argument shows that every Enriques surface Y arises as the quotient of a K3 surface by a $\mathbb{Z}_2$ action generated by a holomorphic involution ρ on X. The signature of Y is $\tau(Y) = 1/2 \cdot \tau(X) = -8$. $H^2(Y, \mathbb{Z})$ is then a unimodular indefinite lattice of signature $(1, 9)$, so $H^2(Y, \mathbb{Z}) = E_8(-1) \oplus U$.

6.1 Moduli space of complex structure

The theory of Enriques surfaces can be subsumed under the theory of algebraic K3 surfaces. For an Enriques surface Y, the Hodge number $H^{2,0}(Y) = 0$, so one has to pass to its universal cover X to define a period map for the complex structure. For all the universal covers X are algebraic, we expect the period domain for Enriques surface to be a proper subset of the period domain Ω for K3 surfaces that is invariant under the action of the involution. Take any holomorphic two-form α on X, then $\rho^*\alpha = -\alpha$ since Y has no non-zero global holomorphic two-form. Suppose $\gamma \in H_2(X, \mathbb{Z})$ is a two-cycle such that $\rho^*\gamma = \gamma$, then

$$\int_\gamma \alpha = \int_{\rho^*\gamma} \rho^*\alpha, \tag{6.1}$$

implying the integral must vanish. The complex structure on Y is captured by the period over cycles with $\rho^*\gamma = -\gamma$.

Theorem 6.1 (Horikawa)**.** *Any two Enriques surfaces are deformation equivalent.*

Deformation equivalence is important for the theory of Enriques surfaces as it allows one to have a “canonical ”involution for the induced action ρ on $H^2(X,\mathbb{Z})$ after choosing a suitable marking. This was observed by Horikawa in [34]. Recall the K3 lattice is

$$L = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U.$$

Let $\zeta : L \to L$ be the involution sending

$$(x, y, u, v, w) \longmapsto (y, x, v, u, -w),$$

then the $+1$-eigenlattice L^+ and the -1-eigenlattice L^- are

$$\begin{cases} L^+ = \{(x, x, y, y, 0) \in L\} = E_8(-2) \oplus U(2), \\ L^- = \{(x, -x, y, -y, z) \in L\} = E_8(-2) \oplus U(2) \oplus U. \end{cases} \tag{6.2}$$

Proposition 6.2 (Horikawa). *For each Enriques surface Y, let X be its covering K3 surface X and ρ be the covering involution. Then there exists a lattice isomorphism $\varphi : H^2(X,\mathbb{Z}) \to L$ such that*

$$\varphi \circ \rho^* = \zeta \circ \varphi. \tag{6.3}$$

We will now introduce the appropriate marked moduli spaces and period domains for Enriques surfaces.

Definition 6.3. A marked Enriques surface is a pair (Y, φ), where Y is an Enriques surface and $\varphi : H^2(X,\mathbb{Z}) \to L$ an isomorphism such that $\varphi\circ\rho^* = \zeta\circ\varphi$.

Definition 6.4. The moduli space of marked Enriques surfaces is defined to be

$$\mathcal{N}_E = \{(Y, \varphi) : \varphi : H^2(X,\mathbb{Z}) \to L \text{ an isomorphism such that } \varphi\circ\rho^* = \zeta\circ\varphi\}/\sim,$$

where $(Y, \varphi) \sim (Y', \varphi')$ if there is a biholomorphic $f : X \to X'$ satisfying $f \circ \rho = \rho' \circ f$ and $\varphi \circ f^* = \varphi'$.

Definition 6.5. The period domain of Enriques surfaces is defined to be

$$\Omega_E := \{x \in \mathbb{P}(L^- \otimes_{\mathbb{Z}} \mathbb{C}) : (x, x) = 0, (x, \overline{x}) > 0\}.$$

We have a natural global period map $\mathcal{P}_{\mathcal{N}_E} : \mathcal{N}_E \to \Omega_E$ sending $(Y, \varphi) \mapsto \varphi(H^{2,0}(X))$.

Theorem 6.6 (Global Torelli Theorem)**.** *The isomorphism class of an Enriques surface is uniquely determined by its period point.*

Curiously enough, the period map $\mathcal{P}_{\mathcal{N}_E} : \mathcal{N}_E \to \Omega_E$ is not surjective. One can easily see this by taking a class $d \in \delta^{\perp} \cap L^-$, where $\delta \in L^-$ and $\delta^2 = -2$. Such a d then lies in $\mathrm{NS}(X)$. The Riemann-Roch theorem then shows that either d or $-d$ is effective. But no effective class belongs to L^-. Removing these hyperplanes gives a surjectivity theorem as follows.

Theorem 6.7 (Surjectivity of the period map)**.** *The period map*

$$\mathcal{P}_{\mathcal{N}_E} : \mathcal{N}_E \to \Omega_E^0 = \Omega_E \setminus \bigcup_{\delta \in L^-, \delta^2 = -2} \delta^{\perp}$$

is surjective.

6.2 Polarized period domain

First, we introduce the corresponding marked polarized space and polarized period domain for Enriques surfaces.

Definition 6.8. The moduli space of marked polarized Enriques surfaces is

$$\mathfrak{M}_E = \{(Y, [\omega], \varphi) : \omega \text{ a Kähler form on } X \text{ with } [\omega]^2 = 1, \varphi \circ \rho^* = \zeta \circ \varphi.\}/\sim,$$

where $(Y, [\omega], \varphi) \sim (Y', [\omega'], \varphi')$ if there exists an isomorphism $f : X \to X'$ such that $f^*[\omega]' = [\omega]$, $f \circ \rho = \rho' \circ f$ and $\varphi \circ f^* = \varphi'$.

Fix a Kähler form ω_Y on Y, then its pull-back to X, ω satisfies $\rho^*\omega = \omega$. For a global holomorphic two-form α, we know from section 6.1 that $\rho^*\alpha = -\alpha$, so $\omega \in L^+_{\mathbb{R}}$, $\mathrm{Re}(\alpha) \in L^-_{\mathbb{R}}$ and $\mathrm{Im}(\alpha) \in L^-_{\mathbb{R}}$.

Definition 6.9. The polarized period domain is defined to be

$$K\Omega_E = \{(\sigma, [x]) \in L^+_{\mathbb{R}} \times \Omega_E : \sigma \in V(x)\},$$

$$K\Omega_E^0 = \{(\sigma, [x]) \in L^+_{\mathbb{R}} \times \Omega_E^0 : \sigma \in V_{\Delta}(x)\}.$$

$K\Omega^0$ is an open subset in $K\Omega$.

This allows us to define the polarized period map $\mathcal{P}_{\mathfrak{M}_E} : \mathfrak{M}_E \to K\Omega_E$ by sending $(Y, [\omega], \varphi) \mapsto (\varphi_{\mathbb{R}}([\omega]), [\varphi_{\mathbb{C}}(H^{2,0}(X))])$.

Proposition 6.10. *For any $\alpha \in \mathrm{NS}(X)$ with $\rho^*\alpha = \alpha$, $\alpha^2 > 0$ and $(\alpha, \delta) \neq 0$ for every $\delta \in \mathrm{NS}(X)$ with $\delta^2 = -2$, there exists $w \in W(X)$ such that $\pm w(\alpha)$ is an ample class and w commutes with ρ^*.*

Recall that $\mathrm{Nef}(X) \cap \mathcal{C}_X$ is a fundamental domain for $W(X)$ acting on $\mathcal{C}_X$, where $\mathcal{C}_X$ is the connected component of the positive cone of $\mathrm{NS}(X)_{\mathbb{R}}$ that contains an ample class. The above proposition shows that, on top of the transitivity of the Weyl group action, we enjoy an extra symmetry so that element w can be chosen to be compatible with the involution ρ^*.

Theorem 6.11 (Torelli Theorem)**.** *The polarized period map $\mathcal{P}_{\mathfrak{M}_E} : \mathfrak{M}_E \to K\Omega^0_E$ is an injection.*

Proof. To show injectivity, assume $(Y, [\omega], \varphi)$ and $(Y', [\omega'], \varphi')$ are mapped to the same image $(\sigma, [x])$ under $\mathcal{P}_{\mathfrak{M}_E}$. Explicitly, we have

$$\varphi_{\mathbb{C}}(H^{2,0}(X) = \varphi'_{\mathbb{C}}(H^{2,0}(X'))),$$
$$\varphi_{\mathbb{C}}(\omega) = \varphi'_{\mathbb{C}}(\omega').$$

This means $\phi = \varphi^{-1} \circ \varphi'$ is an effective Hodge isometry satisfying

$$\phi \circ (\rho')^* = \rho^* \circ \phi. \tag{6.4}$$

By the global Torelli theorem 5.4, such an isometry is induced by a unique isomorphism $f : X \to X'$. Moreover, $f \circ \rho \circ f^{-1} \circ (\rho')^{-1}$ induces identity on $H^2(X', \mathbb{Z})$ by equation 6.4, which implies $f \circ \rho = \rho' \circ f$. Hence, $(Y, [\omega], \varphi) \sim (Y', [\omega'], \varphi')$ as defined in definition 6.8. $\square$

Theorem 6.12 (Surjectivity)**.** *The polarized period map $\mathcal{P}_{\mathfrak{M}_E} : \mathfrak{M}_E \to K\Omega^0_E$ is a surjection.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathfrak{M}_E \arrow[r, "\mathcal{P}_{\mathfrak{M}_E}"] \arrow[d, "\nu"'] & K\Omega^0_E \arrow[d, "\kappa"] \\ \mathcal{N}_E \arrow[r, "\mathcal{P}_{\mathcal{N}_E}"] & \mathrm{Gr}^+_2(L^-_{\mathbb{R}}), \end{tikzcd} \tag{6.5}$$

where $\nu : \mathfrak{M}_E \to \mathcal{N}_E$ forgets the polarization and κ is the projection $(\sigma, [x]) \mapsto [x]$. First, we show surjectivity. Take $(\sigma, [x])$ in $K\Omega_E^0$, by Theorem 6.7 there exists a marked Enriques surface (Y, φ) whose period point is $[x]$. Let X be its universal cover and let ρ the covering involution. Since $H^{2,0}(Y) = 0$, the Néron-Severi lattice $\mathrm{NS}(Y) = H^2(Y, \mathbb{Z})_f$. By definition of σ, $\varsigma = \varphi^{-1}(\sigma)$ satisfies $\rho^*\varsigma = \varsigma$, $\varsigma^2 > 0$ and $(\varsigma, \delta) \neq 0$ for all $\delta \in \mathrm{NS}(X)$ with $\delta^2 = -2$. From $\rho^*\varsigma = \varsigma$, there exists $\theta \in H^2(Y, \mathbb{R})$ such that $\pi^*(\theta) = \varsigma$, so $\varsigma \in \mathrm{NS}(X)_{\mathbb{R}}$. Notice that each chamber of $\mathrm{NS}(X)_{\mathbb{R}}$ is an open convex cone, hence the chamber in which ς lives must contain an element of $\mathrm{NS}(X)$. Now using Proposition 6.10, we can find an element w in the Weyl group such that $\pm w(\varsigma)$ lies inside the ample cone, so is naturally a Kähler class.

Next we show surjectivity directly by mimicking the proof of Proposition 6.10. Consider the action of the Weyl group on the positive cone $\mathcal{C}_X$ of $H^{1,1}(X, \mathbb{R})$. Composing φ with -1 if necessary, we can assume $\varsigma = \varphi^{-1}(\sigma)$ is contained in the positive cone. The Kähler cone for X is

$$\mathcal{K}_X = \{\alpha \in H^{1,1}(X, \mathbb{R}) : \alpha \in \mathcal{C}_X \text{ and } (\alpha, C) > 0 \text{ for all rational curve } C\}. \tag{6.6}$$

If ς does not lie inside the Kähler cone, then there exists a $C_1 \cong \mathbb{P}^1$ such that $(\varsigma, C_1) < 0$. Let $\varsigma_1 = \varsigma$. Suppose we have found n rational curves $C_1, \ldots, C_n$ such that $(\varsigma_k, C_k) < 0$ for each k with

$$\varsigma_k = w_{k-1}(\varsigma_{k-1}) = s_{\rho^* C_{k-1}} \circ s_{C_{k-1}}(\varsigma_{k-1}),$$

where s_{C_i} denotes the Picard-Lefschetz reflection with respect to C_i.

We claim that this procedure must stop after a finite number of steps. Since $C_i + \rho^* C_i$ is invariant under ρ^*, it is the preimage of a (-2)-curve D_i on Y, $\pi^*(D_i) = C_i + \rho^* C_i$. Recall every Enriques surface Y admits an elliptic fibration over $\mathbb{P}^1$. Take any smooth fiber F_0, then $(F_0, D_i) \geq 0$ for all i. So there can only be finitely many D_i's not lying in a fiber, else we would have $(\varsigma_n, \pi^* F_0) < 0$ for large n, contradicting the fact that ς_n is contained in the positive cone $\mathcal{C}_X$.

Now it suffices to rule out the case that some (-2)-curve D appears infinitely many times as D_i in a singular fiber. Every elliptic pencil of an Enriques surface has exactly two multiple fibers $2F_1$ and $2F_2$. This fibration admits a two-section, an irreducible curve E whose intersection $(E, F) = 2$ for all fibers F. If E is not a smooth rational curve, then it is a half pencil for another elliptic fibration. By considering this second fibration and repeating the above argument, one sees that D is allowed to appear only finitely many times in a singular fiber. So we may assume $E^2 = -2$. Suppose D lies in the singular fiber F_1. Then F_1 contains more than one component, say $D = G_1, \dots, G_k$, and each component is a smooth rational curve. From the preceding paragraph, only finitely many D_j's coincide with E. E must intersect a component G_i. This implies G_i appears finitely many times as D_i. Thus there exists a (-2)-curve G that intersects D and is allowed as D_i for finitely many i's. But an immediate consequence is $(\varsigma_n, \pi^* G) \to -\infty$ as $n \to \infty$, a contradiction. Finally, w_k indeed commutes with ρ^* as C_k and $\rho^* C_k$ are disjoint. This completes the proof for surjectivity of $\mathcal{P}_{\mathfrak{M}_E}$. □

Corollary 6.13. *The polarized period map $\mathcal{P}_{\mathfrak{M}_E} : \mathfrak{M}_E \to K\Omega_E^0$ is a bijection.*

6.3 Einstein metrics on Enriques surfaces

Now we are ready to show that $\mathcal{E}(Y)$ is path-connected. First, by Theorem 2.2, any Einstein metric on an Enriques surface must be locally hyperKähler.

Definition 6.14. The Teichmüller space of marked Enriques surfaces $\mathfrak{N}_E$ with a unit-volume Einstein metric g is defined as

$$\{(Y, g, \varphi) : g \text{ unit-volume Einstein on } X \text{ with } \rho^* g = g\} / \sim,$$

where (Y, φ) is a marked Enriques surface and $(Y, g, \varphi) \sim (Y', g', \varphi')$ if there exists a diffeomorphism $f : X \to X'$ such that $f^* g' = g$, $f \circ \rho = \rho' \circ f$ and $\varphi \circ f^* = \varphi'$.

The period map for Einstein metrics $\mathcal{P}_{\mathfrak{N}_E} : \mathfrak{N}_E \to \mathrm{Gr}_3^+(L_{\mathbb{R}})$ is defined by sending $(X, g, \varphi) \mapsto \varphi(\mathcal{H}^+(g))$, where $\mathcal{H}^+(g)$ is the space of self-dual harmonic two-forms on X.

Theorem 6.15. *The Teichmüller space $\mathcal{T}(Y)$ of Ricci-flat Kähler metrics on an Enriques surface is an open connected subset in $\{\mathrm{SO}^+(1,9)/\mathrm{SO}(9)\} \times \{\mathrm{SO}^+(2,10)/\mathrm{SO}(2) \times \mathrm{SO}(10)\}$. Hence the moduli space $\mathcal{E}(Y)$ is path-connected with dimension 29.*

Proof. Our strategy is the same as Theorem 5.14. First, consider the following digram

$$\begin{tikzcd} \mathfrak{M}_E \arrow[r, "\mathcal{P}_{\mathfrak{M}_E}"] \arrow[d, "\mu"'] & K\Omega_E^0 \arrow[d, "\pi"] \\ \mathfrak{N}_E \arrow[r, "\mathcal{P}_{\mathfrak{N}_E}"] & \mathrm{Gr}_3^+(L_{\mathbb{R}}), \end{tikzcd} \tag{6.7}$$

where $\mu : (Y, \omega, \varphi) \mapsto (Y, g, \varphi)$ is defined using the Calabi-Yau theorem and $\pi : (\omega, [x]) \mapsto (\omega, \mathrm{Re}(x), \mathrm{Im}(x))$. Using Corollary 6.13, we get

$$\mathcal{P}_{\mathfrak{N}_E}(\mathfrak{N}_E) \cong \pi(K\Omega_E^0).$$

Note that

$$K\Omega_E^0 = K\Omega_E \setminus \bigcup_{\delta^2=-2} \delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp].$$

The eigenlattices are $L^+ = E_8(-2) \oplus U(2)$ and $L^- = E_8(-2) \oplus U(2) \oplus U$ by 6.2. Let $\delta \in L = E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$ have self-intersection -2. Note that δ must belong to one of the above five summands. Even though $L^+ \oplus L^- \subsetneq L$ is a strict proper subset, we do have $L^-_{\mathbb{R}} \oplus L^+_{\mathbb{R}} = L_{\mathbb{R}}$. According to the decomposition in 6.2, δ can be written as $\delta^+ \oplus \delta^- = (a\delta, a\delta, 0, 0, 0) \oplus (b\delta, -b\delta, 0, 0, 0)$, $(0, 0, a\delta, a\delta, 0) \oplus (0, 0, b\delta, -b\delta, 0)$ or $(0, 0, 0, 0, 0) \oplus (0, 0, 0, 0, \delta)$ for appropriate $a, b \in \mathbb{R}$. In each of the 3 cases, the component in $L^-_{\mathbb{R}}$ is non-zero. The image of π is then

$$\pi(\delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp] \cap K\Omega_E) = \pi([(\delta^+)^\perp \cap L^+_{\mathbb{R}}] \oplus [(\delta^-)^\perp \cap L^-_{\mathbb{R}} \oplus i(\delta^-)^\perp \cap L^-_{\mathbb{R}}]),$$

which is a submanifold of codimension at least 2 in $\pi(K\Omega_E)$. Hence the Einstein moduli space $\mathcal{E}(Y)$ is isomorphic to a connected subset of dimension 29 in

$$\Gamma_E \setminus \{[\mathrm{SO}^+(1,9)/\,\mathrm{SO}(9)] \times [\mathrm{SO}^+(2,10)/\,\mathrm{SO}(2) \times \mathrm{SO}(10)]\}.$$

$\square$

7 Locally hyperKähler metrics on Hitchin manifolds

In this section, we consider the quotient of (X, g), a K3 surface together with a hyperKähler metric, by a free isometric $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. We will see below that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action is generated by a holomorphic involution ρ and an antiholomorphic involution τ. Thus the resulting 4-manifold is no longer a complex surface. Instead, it is the quotient of a real Enriques surface without real points by the conjugation. We call such a 4-manifold a *Hitchin manifold*.

Employing the identity 2.19, we can readily compute the signature $\tau(M) = 1/4 \cdot \tau(X) = -4$ and Euler characteristic $\chi(M) = 1/4 \cdot \chi(X) = 6$. The fundamental group of M is $\mathbb{Z}_2 \times \mathbb{Z}_2$, so the second integral cohomology $H^2(M, \mathbb{Z})$ is isomorphic to the lattice $\mathbb{Z}^4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with negative definite intersection form on the free part $H^2(M, \mathbb{Z})_f$. The only such unimodular form in dimension four is $(-1)^{\oplus 4}$. This can also be deduced from the powerful Donaldson diagonalization theorem [24].

7.1 Semi-complex structures

Consider the action of ρ on the space of harmonic self-dual 2-forms $\mathcal{H}_g^+$ on X. Since ρ is an involution, it is diagonalizable with eigenvalue ± 1. Using formula 2.20 in section 2.4, ρ has a 1-dimensional eigenspace for the eigenvalue 1 and a 2-dimensional eigenspace for the eigenvalue -1. The same argument carries over for the action of τ and $\rho \cdot \tau$ on $\mathcal{H}_g^+$. As ρ and τ commute by our assumption, they can be simultaneously diagonalized and they take the following form

$$\rho^* = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \quad \tau^* = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad (\rho \cdot \tau)^* = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

with respect to an orthonormal basis $\omega_1, \omega_2, \omega_3$ in $\mathcal{H}_g^+ \subset H^2(X, \mathbb{R})$ using the intersection form.

Let $\alpha_1 := \omega_2 + i\omega_3$, $\alpha_2 := \omega_3 + i\omega_1$ and $\alpha_3 := \omega_1 + i\omega_2$ and let $\beta_i = \omega_i$ for $i = 1, 2, 3$. Then

$$\rho^*\alpha_1 = -\alpha_1 \qquad \tau^*\alpha_1 = \overline{\alpha}_1, \qquad (\rho\tau)^*\alpha_1 = -\overline{\alpha}_1, \tag{7.1}$$

so indeed the action ρ is holomorphic, whereas τ and $\rho\tau$ are antiholomorphic with respect to the complex structure J_1 defined by the holomorphic (2,0)-form α_1. If we replace α_1 by α_2, then

$$\tau^*\alpha_2 = -\alpha_2 \qquad (\rho\tau)^*\alpha_2 = \overline{\alpha}_2, \qquad \rho^*\alpha_2 = -\overline{\alpha}_2, \tag{7.2}$$

so τ becomes holomorphic while ρ and $\rho\tau$ are antiholomorphic with respect the complex structures J_2 associated with α_2. Finally, $\rho\tau$ is holomorphic while ρ and τ are antiholomorphic with respect to the complex structure J_3 associated with α_3, for

$$(\rho\tau)^*\alpha_3 = -\alpha_3 \qquad \rho^*\alpha_3 = \overline{\alpha}_3, \qquad \tau^*\alpha_3 = -\overline{\alpha}_3. \tag{7.3}$$

We record our findings as follows.

Proposition 7.1. *We can perform hyperKähler rotations σ_1 and σ_2 on (X, J_1) such that*

1. *τ is holomorphic, whereas ρ and $\rho \cdot \tau$ are antiholomorphic with respect to the new complex structure J_2 obtained from $\sigma_1 : (\omega_1, \omega_2, \omega_3) \mapsto (\omega_3, \omega_1, \omega_2)$.*

2. *$\rho\tau$ is holomorphic, whereas ρ and τ are antiholomorphic with respect to the new complex structure J_2 obtained from $\sigma_2 : (\omega_1, \omega_2, \omega_3) \mapsto (\omega_2, \omega_3, \omega_1)$.*

The moduli theory for Hitchin manifolds is very much akin to the case of Enriques surfaces. But a Hitchin manifold M does not carry any complex structure, as we will see below. Thus there is no point of studying the complex structure. Instead, we will investigate the moduli space of semi-complex structures on M.

Real Enriques surfaces have been studied extensively by A. Degtyarev and V. Kharlamov in a series of papers [21], [22], [23], which culminates in a topological classification of the real parts of all real Enriques surfaces. The details are described carefully in their book [20] with I. Itenberg. Building on the previous works of V. Nikulin, the classification is achieved by an arduous analysis of the action of the two commuting antiholomorphic involutions on the lattice. A large part of their works boils down to dealing with the difficult algebraic problem of gluing the eigenlattices of the two actions.

Two Hitchin manifolds M and M' are said to be isomorphic if there exists a biholomorphic map $f : X \to X'$ such that $f \circ \rho = \rho' \circ f$ and $f \circ \tau =$

$\tau' \circ f$. In particular, this implies the semi-complex structures on M and M' are isomorphic.

In [22], Degtyarev and Kharlamov showed that any two Hitchin manifolds are deformation equivalent. This then enables a choice of a canonical action $\zeta : L \to L$ for the holomorphic involution ρ, and θ for the antiholomorphic involution τ.

Definition 7.2. The moduli space of marked Hitchin manifolds $\mathcal{N}_M$ is defined to be

$$\{(M, \varphi) : \varphi : H^2(X, \mathbb{Z}) \to L \text{ a marking with } \varphi \circ \rho^* = \zeta \circ \varphi,\ \varphi \circ \tau^* = \theta \circ \varphi\} / \sim,$$

where $(M, \varphi) \sim (M', \varphi')$ if there is a biholomorphic $f : X \to X'$ satisfying $f \circ \rho = \rho' \circ f$, $\quad f \circ \tau = \tau' \circ f$ and $\varphi \circ f^* = \varphi'$.

Next, we define a period map and period domain for Hitchin manifolds, as for K3 surfaces and Enriques surfaces.

The period domain of Hitchin manifolds Ω_M, is defined to be the Grassmanian of oriented 2-planes in $L^{-+} \oplus L^{--}$, which is given by a pair $(\omega_2, \omega_3) \in L^{-+} \oplus L^{--}$.

The associated period map $\mathcal{P}_M : \mathcal{N}_M \to \Omega_M$ for Hitchin manifolds is defined by sending $(M, \varphi) \mapsto \varphi_{\mathbb{C}}(H^{2,0}(X))$.

Proposition 7.3. *For any $\alpha \in \mathrm{NS}(X)$ with $\rho^*\alpha = \alpha$, $\tau^*\alpha = -\alpha$ $\alpha^2 > 0$ and $(\alpha, \delta) \neq 0$ for every $\delta \in \mathrm{NS}(X)$ with $\delta^2 = -2$, there exists $w \in W(X)$ such that $\pm w(\alpha)$ is an ample class and w commutes with ρ^* and τ^*, where $W(X)$ is the Weyl group of the K3 surface.*

Proof. It is clear that we may assume α lies in the positive cone by considering $-\alpha$ if necessary. The ample cone of X is

$$\mathrm{Amp}(X) = \{\alpha \in \mathcal{C}_X : (\alpha, C) > 0 \text{ for all } C \simeq \mathbb{P}^1\}.$$

If $\alpha \in \mathrm{Amp}(X)$, we are done. If not, then there exists a $C_1 \simeq \mathbb{P}^1$ such that $(\alpha, d_1) < 0$, where d_1 denotes the class of C_1 in $\mathrm{NS}(X)$. Let d_2 be the class of $\rho(C_1)$. The image $\tau(C_1)$ is still a holomorphic curve with respect to the complex structure J_1. But since τ is antiholomorphic, τ sends the Kähler cone K_X to $-K_X$ and the set of positive roots Δ_+ to $-\Delta_+$, so the class of $\tau(C_1)$ with respect to J_1 is $d_3 = -\tau^* d_1$. The same argument carries over for $\rho\tau(C_1)$, so the class of $\rho\tau(C_1)$ with respect to J_1 is $d_4 = -\rho^*\tau^* d_1$. Now C_1, $\rho(C_1)$,

$\tau(C_1)$ and $\rho\tau(C_1)$ are pairwise disjoint, this implies that the composition of the Picard-Lefschetz reflections $w_1 = s_{\rho\tau(C_1)} \circ s_{\rho(C_1)} \circ s_{\tau(C_1)} \circ s_{C_1}$ commutes with ρ^* and τ^*. This can be easiy checked as follows. First, note that

$$w_1(x) = x + (x, d_1)d_1 + (x, d_2)d_2 + (x, d_3)d_3 + (x, d_4)d_4. \tag{7.4}$$

Applying ρ^* to $w_1(x)$ gives

$$\begin{aligned} \rho^* \circ w_1(x) &= \rho x + (x, C_1)\rho C_1 + (x, \rho C_1)C_1 + (x, \tau C_1)\rho\tau C_1 + (x, \rho\tau C_1)\tau C_1 \\ &= \rho x + (\rho x, C_1)C_1 + (\rho x, \rho C_1)\rho C_1 + (\rho x, \tau C_1)\tau C_1 + (\rho x, \rho\tau C_1)\rho\tau C_1 \\ &= w_1 \circ \rho^*(x). \end{aligned}$$

Similarly, τ^* commutes with w_1. The key feature of our choice is that $(\alpha, d_i) < 0$ for all i. Indeed,

$$\begin{aligned} (\alpha, d_2) = (\alpha, \rho d_1) = (\rho\alpha, \rho d_1) = (\alpha, d_1) < 0, \\ (\alpha, d_3) = (\alpha, -\tau d_1) = (-\tau\alpha, -\tau d_1) = (\alpha, d_1) < 0, \\ (\alpha, d_4) = (\alpha, -\rho\tau d_1) = (-\rho\tau\alpha, -\rho\tau d_1) = (\alpha, d_1) < 0. \end{aligned}$$

Now proceed by induction and assume we have found $C_1, \rho(C_1), \tau(C_1), \rho\tau(C_1), \cdots, C_k, \rho(C_k), \tau(C_k), \rho\tau(C_k)$. Denote their nodal classes by $d_1, \cdots, d_{4k}$. Let $w_i = s_{d_i} \circ \cdots \circ s_{d_1}$ and let $\alpha_i = w_i(\alpha)$, then for each i

$$\alpha_{i+1} = s_{d_{i+1}}(\alpha_i) = \alpha_i + (\alpha_i, d_{i+1})d_{i+1}. \tag{7.5}$$

But by our construction, $(\alpha_i, d_{i+1}) < 0$. Suppose D_i is a divisor lying in the class of d_i. Using the previous observation, we see that the dimension of the linear system is non-increasing

$$\dim|D_{i+1}| \leq \dim|D_i|. \tag{7.6}$$

Thus, the dimension stabilizes at some finite n and all the subsequent divisors $\sum_{j \geq n}(-\alpha_{j-1}, d_j)D_j$ belong to the fixed part of $|D_n|$. The procedure must therefore produce an ample class α_{4k} for some finite k. $\square$

The above proposition is an adaptation of Propostion 6.10, both of which share the same underlying philosophy. The proposition implies that, in addition to the transitivity of the Weyl group action on the chambers of $\mathrm{NS}(X)_{\mathbb{R}}$, one could impose extra symmetries by choosing an element in the Weyl group which commutes with both involutions.

Theorem 7.4 (Torelli Theorem). *The isomorphism class of a Hitchin manifold M is uniquely determined by its period point.*

Proof. Assume we have two marked Hitchin manifolds (M, φ) and (M', φ') mapping to the same period point. If (X, ρ, τ) and (X', ρ', τ') are the universal covers together with the commuting isometric involutions, then

$$\zeta \circ \varphi = \varphi \circ \rho^* \qquad \text{and} \qquad \theta \circ \varphi = \varphi \circ \tau^*, \tag{7.7}$$

$$\zeta \circ \varphi' = \varphi \circ (\rho')^* \qquad \text{and} \qquad \theta \circ \varphi' = \varphi' \circ (\tau')^*. \tag{7.8}$$

Our assumption means that $\psi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ defined by the composition $\psi := \phi^{-1} \circ \phi'$ is a Hodge isometry which is equivariant with the involutions

$$\rho^* \circ \psi = \psi \circ (\rho')^*, \qquad \tau^* \circ \psi = \psi \circ (\tau')^*. \tag{7.9}$$

Let $\alpha' \in H^2(X', \mathbb{Z})$ be an ample class invariant under ρ' and $\tau'\alpha' = -\alpha'$, with respect to the complex structure coming from J_1' in 7.1. We can compose ψ with an $\pm w$ where $w \in W(X)$ so that $\alpha = \Psi(\alpha')$ is an ample class on X, with respect to the complex structure J_1. By Proposition 7.3, this Ψ can be chosen so that it satisfies equation 7.9 in place of ψ. Now $\Psi = \pm w \circ \psi$ becomes an effective Hodge isometry. By the Torelli theorem for K3 surfaces, Ψ is induced by a biholomorphic map $f : X \to X'$. Moreover, $\rho \circ f \circ \rho^{-1} \circ f^{-1}$ induces the identity action on $H^2(X', \mathbb{Z})$, so it must be identity, which gives $\rho' \circ f = f \circ \rho$.

If we change the complex structures on both X and X' to the complex conjugates, $f : X \to X'$ will remain a biholomorphism. Thus, we can emulate our argument above and show $\tau' \circ f = f \circ \tau$. This implies M and M' are isomorphic. $\square$

Corollary 7.5. *The period point of a Hitchin manifold determines the semi-complex structure.*

Note that the involutions ρ and τ act naturally on the polarized period domain $K\Omega$ for K3 surface. The polarized period domain for Hitchin manifold $K\Omega_M$ in the next section can be defined as the invariant subset of both ρ and τ. $K\Omega_M^0$ is then the complement of the hyperplanes $\bigcup_{\delta^2=-2} \delta^\perp$ in $K\Omega_M$. Moreover, there is natural a forgetful map $\pi : K\Omega \to \Omega$. We define Ω_M^0 to be the image of $K\Omega^0(M)$ under such projection.

Theorem 7.6 (Surjectivity of the period map). *Each point in Ω_M^0 is the image under the period map for some Hitchin manifold.*

7.2 Polarized period domain

Let M be the underlying closed smooth manifold of a Hitchin manifold. Our main result in this section is that, roughly speaking, the Kähler structures on M are parametrized by the polarized period domain for M. In order to probe the structure of the space of marked polarized Hitchin manifolds, we emulate the story for K3 surfaces by defining a period map to a polarized period domain. Considering the action of the two involutions, we expect the polarized period domain for M to be the subset of the polarized period domain for K3 surfaces that is invariant under the actions. First, we give the following two definitions.

Definition 7.7. The moduli space of marked polarized Hitchin manifolds $\mathfrak{M}_M$ is

$$\{(M, [\omega], \varphi) : \omega \text{ unit-norm Kähler form on } X, \varphi \circ \rho^* = \zeta \circ \varphi, \varphi \circ \tau^* = \theta \circ \varphi\}/\sim,$$

where $(M, [\omega], \varphi) \sim (M', [\omega'], \varphi')$ if there exists an isomorphism $f : X \to X'$ such that $f^*[\omega]' = [\omega]$, $f \circ \rho = \rho' \circ f$, $f \circ \tau = \tau' \circ f$ and $\varphi \circ f^* = \varphi'$.

Definition 7.8. The polarized period domain is defined to be

$$K\Omega_M = \{(\sigma, [x]) \in L_{\mathbb{R}}^{+-} \times \Omega_M : \sigma \in V(x)\},$$

$$K\Omega_M^0 = \{(\sigma, [x]) \in L_{\mathbb{R}}^{+-} \times \Omega_M^0 : \sigma \in V_\Delta(x)\}.$$

Define the polarized period map $\mathcal{P}_{\mathfrak{M}_M} : \mathfrak{M}_M \to K\Omega_M^0$ as before, by mapping $(M, [\omega], \varphi)$ to $(\varphi_{\mathbb{C}}([\omega]), \varphi_{\mathbb{C}}(H^{2,0}(X)))$.

Theorem 7.9 (Torelli theorem)**.** *The polarized period map $\mathcal{P}_{\mathfrak{M}_M} : \mathfrak{M}_M \to K\Omega_M^0$ is injective.*

Proof. Our proof will emulate that of Theorem 6.11 and Theorem 7.4. Assume $(M, [\omega], \varphi)$ and $(M, [\omega'], \varphi')$ both map to the same period point. Let (X, ρ, τ) and (X', ρ', τ') be the universal covers with their corresponding involutions, where X and X' are equipped with the complex structure J_1 coming from 7.1. Our assumption means that

$$\varphi_{\mathbb{C}}(\omega) = \varphi'_{\mathbb{C}}(\omega') \quad \text{and} \quad \varphi_{\mathbb{C}}(H^{2,0}(X)) = \varphi'_{\mathbb{C}}(H^{2,0}(X')). \tag{7.10}$$

Together they imply $\psi = \varphi^{-1} \circ \varphi'$ is an effective Hodge isometry that is equivariant

$$\psi \circ (\rho')^* = \rho^* \circ \psi \quad \text{and} \quad \psi \circ (\tau')^* = \tau^* \circ \psi. \tag{7.11}$$

The Torelli theorem for K3 surfaces then gives a biholomorphic map $f : X \to X'$ which induces ψ. By the first condition in 7.11, $\rho' \circ f \circ \rho \circ f^{-1}$ induces the identity action on $H^2(C', \mathbb{Z})$, so it must be the identity map. Now if we change the complex structures on X and X' to the conjugate ones, the map f is still holomorphic with respect to the conjugate complex structures. Thus, one can repeat the above procedure and show that $\tau' \circ f = f \circ \tau$, so $(M, [\omega], \varphi) \sim (M, [\omega'], \varphi')$. ☐

Theorem 7.10 (Surjectivity)**.** *The polarized period map $\mathcal{P}_{\mathfrak{M}_M} : \mathfrak{M}_M \to K\Omega^0_M$ is surjective.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathfrak{M}_M \arrow[r, "\mathcal{P}_{\mathfrak{M}_M}"] \arrow[d, "\nu"'] & K\Omega^0_M \arrow[d, "\kappa"] \\
\mathcal{N}_M \arrow[r, "\mathcal{P}_{\mathcal{N}_M}"] & \mathrm{Gr}^+_2(L^{+-}_{\mathbb{R}}),
\end{tikzcd} \tag{7.12}$$

where ν and κ are the projection maps that forget polarizations. Fix a $(\sigma, [x]) \in K\Omega^0_M$. Since $[x]$ belongs to Ω^0_M, we can find a marked Hitchin manifold (M, φ) by the surjectivity theorem 7.6. From the definition of the polarized period domain, it is evident that $\alpha = \varphi^{-1}\sigma$ are the ±1-eigenvectors of ρ^* and σ^*

$$\rho^*\alpha = \alpha \quad \text{and} \quad \tau^*\alpha = -\alpha. \tag{7.13}$$

In addition, this preimage has non-vanishing pairing $(\alpha, \delta) \neq 0$ for every $\delta \in \mathrm{NS}(X)$ with $\delta^2 = -2$. Here X is assumed to be endowed with the complex structure J_1 in 7.1. Each chamber in $\mathrm{NS}(X)_{\mathbb{R}}$ is an open convex subset, so there is always an integral element in the same chamber as α satisfying the same non-vanishing pairing condition as above. Theorem 7.3 regarding the transitiviy of the Weyl group action then implies there is a $w \in W(X)$ such that $\alpha' = \pm w(\alpha)$ is lies in the ample cone, so it is a Kähler class on M with respect to the complex structure J_1. Moreover, $\varphi' = \varphi \circ \pm w^{-1}$ commutes with ρ and τ. So the conditions

$$\varphi' \circ \rho^* = \zeta \circ \varphi' \quad \text{and} \quad \varphi' \circ \tau^* = \theta \circ \varphi' \tag{7.14}$$

still hold. As a consequence, (M, α', φ') belongs to the marked polarized moduli space $\mathfrak{M}_M$. Finally, we check that indeed $\varphi'(\alpha') = \sigma$ and $\varphi'(H^{2,0}(X)) = \varphi \circ (\pm w^{-1})(H^{2,0}(X)) = [x]$ for $H^{2,0}(X)$ is preserved by any Picard-Lefschetz reflections. □

Corollary 7.11. *The polarized period map $\mathcal{P}_{\mathfrak{M}_M} : \mathfrak{M}_M \to K\Omega^0_M$ is a bijection.*

Remark 7.12. Theorem 7.9, Theorem 7.10 and Corollary 7.11 are also known to Degtyarev, Kharlamov and Itenberg, phrased in a modified but equivalent form, see [20]. Here we take a slightly different route by recovering the theorems for Kähler structures through the ones for complex structures. But the main idea is still to apply the corresponding theorems for K3 surface.

7.3 Einstein metrics on Hitchin manifolds

First, note that an Einstein metric on a Hitchin manifold must be locally hyperKähler by Theorem 2.2. Second, all Hitchin manifolds share a single diffeomorphism type. Hence, to consider the Einstein moduli space on a Hitchin manifold, we can instead pass to the hyperKähler metrics on a K3 surface satisfying the symmetries coming from the two commuting covering involutions.

Definition 7.13. The Teichmüller space of marked Hitchin manifolds $\mathfrak{N}_M$ with unit-volume Einstein metric g is defined as

$$\{(M, g, \varphi) : g \text{ Einstein with } \rho^* g = g, \tau^* g = g, \varphi \text{ a marking}\}/\sim,$$

where (M, φ) is a marked Hitchin manifold and $(M, g, \varphi) \sim (M', g', \varphi')$ if there exists a diffeomorpshim $f : X \to X'$ such that $f^* g' = g$, $f \circ \rho = \rho' \circ f$, $f \circ \tau = \tau' \circ f$ and $\varphi \circ f^* = \varphi'$.

Theorem 7.14. *The Teichmüller space of Ricci-flat Einstein metrics $\mathcal{T}(M)$ on a Hitchin manifold M is an open connected subset in $\{\mathrm{SO}^+(1,5)/\mathrm{SO}(5)\}^3$. So $\mathcal{E}(M)$ is connected with real dimension 15.*

Proof. Let $L^{ij} = \{x \in H^2(X, \mathbb{Z}) : \rho^* x = ix, \tau^* x = jx\}$ be the integral eigenlattice of ρ and τ. First, as ρ commutes with τ, $H^2(X, \mathbb{R})$ can be decomposed into simultaneous eigenspaces,

$$H^2(X, \mathbb{R}) = L^{++}_{\mathbb{R}} \oplus L^{+-}_{\mathbb{R}} \oplus L^{-+}_{\mathbb{R}} \oplus L^{--}_{\mathbb{R}}. \tag{7.15}$$

Using the Lefschetz fixed point theorem and the Hirzebruch signature formula, the dimensions of the eigenspaces can be computed to be

$$
\begin{aligned}
\dim L_{\mathbb{R}}^{++} &= 4 + \frac{1}{2}\chi, \\
\dim L_{\mathbb{R}}^{+-} &= 6 - \frac{1}{2}\chi, \\
\dim L_{\mathbb{R}}^{-+} &= 6 + \frac{1}{2}(\chi_\tau - \chi_{\rho\cdot\tau}), \\
\dim L_{\mathbb{R}}^{--} &= 6 + \frac{1}{2}(\chi_\tau - \chi_{\rho\cdot\tau}),
\end{aligned}
\tag{7.16}
$$

where χ is the Euler characteristic of the fixed point set of τ on $X/\langle\rho\rangle$, χ_τ and $\chi_{\rho\cdot\tau}$ denote the Euler characteristic of the fixed point set of τ and ρ on X respectively, see [23]. In our case, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action is free, so the fixed point sets are all empty and $\chi = \chi_\tau = \chi_{\rho\cdot\tau} = 0$. It follows that $L_{\mathbb{R}}^{++}$ has dimension 4 with signature -4, each of $L_{\mathbb{R}}^{+-}$, $L_{\mathbb{R}}^{-+}$ and $L_{\mathbb{R}}^{--}$ has dimension 6 and signature -4.

Next, we consider the diagram

$$
\begin{tikzcd}
\mathfrak{M}_M \arrow[r, "\mathcal{P}_{\mathfrak{M}_M}"] \arrow[d, "\mu"'] & K\Omega_M^0 \arrow[d, "\pi"] \\
\mathfrak{N}_M \arrow[r, "\mathcal{P}_{\mathfrak{N}_M}"] & \mathrm{Gr}_3^+(L_{\mathbb{R}}),
\end{tikzcd}
\tag{7.17}
$$

where $\mu : (Y, \omega, \varphi) \mapsto (Y, g, \varphi)$ is defined using the Calabi-Yau theorem and $\pi : (\omega, [x]) \mapsto (\omega, \mathrm{Re}(x), \mathrm{Im}(x))$. Corollary 7.11 gives the identification

$$\mathcal{P}_{\mathfrak{N}_M}(\mathfrak{N}_M) \cong \pi(K\Omega_M^0).$$

From the previous section, we know that the Kähler class ω_1 belongs to $L_{\mathbb{R}}^{+-}$, whereas the real part ω_2 lies in $L_{\mathbb{R}}^{-+}$ and the imaginary part ω_3 belongs to $L_{\mathbb{R}}^{--}$. Due to

$$K\Omega_M^0 = K\Omega_M \setminus \bigcup_{\delta^2=-2} \delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp],$$

it suffices to show that the image under π of each plane $\delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp]$ with $\delta^2 = -2$ has codimension at least two in $\pi(K\Omega_M)$.

Take such a $\delta \in L$ with self-intersection -2. We can split δ into a direct sum

$$\delta = \delta^{++} \oplus \delta^{+-} \oplus \delta^{-+} \oplus \delta^{--},$$

according to 7.15. Moreover, from the proof of Theorem 6.15, we have seen that $\delta^- = \delta^{-+} \oplus \delta^{--}$ is always non-zero. Applying Proposition 7.1, we note that the eigenlattices $L^{+-} \cong L^{-+} \cong L^{--}$ are isomorphic, for any pair of the triplet arise as the simultaneous eigenlattices of the holomorphic and antiholomorphic involutions on $H^2(X, \mathbb{Z})$ upon performing hyperKähler rotations by changing the underlying complex structure from J_1 to J_2 and J_3. This symmetry enables us to conclude that each of δ^{+-}, δ^{-+} and δ^{--} is non-zero. Now, the image of π, $\pi(\delta^\perp \times [\delta^\perp \oplus i \cdot \delta^\perp] \cap K\Omega_M)$ can be written as

$$\pi([(\delta^{+-})^\perp \cap L_{\mathbb{R}}^{+-}] \oplus [(\delta^{-+})^\perp \cap L_{\mathbb{R}}^{-+} \oplus i(\delta^{--})^\perp \cap L_{\mathbb{R}}^{--}]),$$

so it is submanifold of codimension 3 in $\pi(K\Omega_M)$. Thus indeed $\mathcal{T}(M)$ is an open path-connected subset in$\{\mathrm{SO}^+(1,5)/\,\mathrm{SO}(5)\}^3$. $\square$

Remark 7.15. It can be shown by pure lattice-theoretic method that the action of τ^* on $H^2(Y, \mathbb{Z})$ has $+1$-eigenlattice isomorphic to $D_4(-1)$ and -1-eigenlattice isomorphic to $D_4(-1) \oplus U$.

7.4 Almost complex structure

Let X be an oriented four-manifold with a given almost complex structure J. Then J induces a spin^c structure on X. The index of the Dirac operator $\not{D} = \bar{\partial} + \bar{\partial}^*$ with respect to this spin^c structure equals the Todd genus of X,

$$\operatorname{index}(\not{D}) = \operatorname{Todd}(X) = \frac{\chi(X) + \tau(X)}{4}. \tag{7.18}$$

In particular, $\chi(X) + \tau(X) \equiv 0 \mod 4$.

Theorem 7.16. *A Hitchin manifold M does not admit any almost complex structure.*

Proof. $\chi(M) + \tau(M) = 2$, which is not divisible by 4. $\square$

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