Hodge Numbers of O'Grady 6 via Ngô Strings

A Dissertation presented

by

Ben Wu

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2022

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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The Betti and Hodge numbers of hyper-Kähler manifolds of $OG6$ type were first computed by Mongardi, Rapagnetta, and Saccá. We provide an alternative computation of the Betti and Hodge numbers of these manifolds using the method of Ngô Strings introduced by de Cataldo, Rapagnetta, and Saccá in their study of hyper-Kähler manifolds of $OG10$ type. More precisely, we study the geometry of a special member of the deformation class, one which admits a Lagrangian fibration, and we relate it to the geometry of several other fibrations. We then use a refinement of the Ngô Support Theorem to compute the Betti and Hodge numbers of manifolds of $OG6$ type. Our computation will also lead to a description of the Hodge structure of some special members of the deformation class in terms of the Hodge structure of a related Abelian surface.

Dedication Page

To my fiancé Lisa and to my parents Shunxiang Wu and Yang Han.

Contents

Acknowledgements

I would first like to greatly thank my advisor Prof. Mark Andrea de Cataldo for the many enlightening discussions and his continued support throughout my PhD. I am extremely greatful for all of the time and energy he has invested into me and for all that I have learned from him. I would next like to give a special thanks to Prof. Antonio Rapagnetta for being a part of my defense committee and for an extremely useful discussion on monodromy (in particular, see Lemma 3.2.5 and Section 4.2). I would also like to thank Prof. Giulia Saccà for reading a preliminary version of the paper on which this thesis is based and for her useful comments. I would also like to thank Prof. Radu Laza for being on my defense committee, Distinguished Prof. Robert Lazarsfeld for being the chairperson of my defense and for their wonderful classes and for all that I have learned from them.

I would next like to greatly thank my fiancé Lisa Marquand for her unwavering love and support through the good times and the bad, for the many stimulating mathematical discussions over the years, and for reading several prelimary versions of this thesis. I would also like to thank Mads Bach Villadsen for his friendship and compassion, for always being willing to discuss math, and for reading several prelimary versions of this thesis. I would also like to thank Yoonjoo Kim for an extremely useful discussion on the LLV decomposition. I would also like to thank the entire mathematics graduate community at Stony Brook for being welcoming and collaborative and for best years of my life so far.

I would finally like to thank my parents, Shunxiang Wu and Yang Han for all of their love and support over the course of my PhD.

Chapter 1

Introduction

Irreducible holomorphic symplectic (IHS) manifolds can be characterized as the simply connected compact Kähler manifolds equipped with a unique, up to scaling, holomorphic symplectic two-form. They are hyper-Kähler manifolds by the Calabi-Yau theorem [Yau78]. IHS manifolds are, along with complex tori and Calabi-Yau manifolds, one of the three building blocks of smooth complex algebraic varieties with trivial first Chern class by the Beauville-Bogomolov decomposition theorem [Bea83; Bog74]. There are few known examples of IHS manifolds. There are two deformation classes appearing in every even complex dimension, namely manifolds of $K3^{[n]}$ type and of Kum_n type [Bea83; Fuj83]. Manifolds of $K3^{[n]}$ type are deformation equivalent to the Hilbert scheme of n points on a $K3$ surface and manifolds of Kum_n type are deformation equivalent to the generalized Kummer variety of dimension $2n$. There are two other known exceptional examples occurring in dimensions six and ten, and they were constructed by O'Grady by symplectically resolving certain singular moduli spaces of sheaves on Abelian surfaces and $K3$ surfaces respectively [O'Gr03; O'Gr99]. Kaledin, Lehn, and Sorger proved that no new examples can be obtained by using O'Grady's methods [KLS06]. We denote O'Grady's exceptional examples by $OG6$ and $OG10$. Manifolds deformation equivalent to $OG6$ (resp. $OG10$) are said to be of $OG6$ type (resp. $OG10$ type). All other known examples are deformation equivalent to one of the above examples.

The topology of the known examples of IHS manifolds has been studied intensely. The Betti numbers of the Hilbert schemes of n points on a $K3$ surface and of the generalized Kummer varieties were computed by Göttsche using the Weil conjectures in [Göt90; Göt93] and were recomputed by Göttsche and Soergel in [GS93] using the theory of perverse sheaves and the Beilinson-Bernstein-Deligne-Gabber decomposition theorem [BBD82]. Using Saito's refinement of the BBDG decomposition theorem for mixed Hodge modules [Sai90], Göttsche and Soergel were also able to compute the Hodge numbers for the Hilbert schemes of n points on a $K3$ surface and of the generalized Kummer varieties.

The second Betti number for manifolds of $OG6$ type was computed by O'Grady in [O'Gr03] and the Euler characteristic was determined shortly after by Rapagnetta in [Rap07]. The Betti and Hodge numbers were then completely determined by Mongardi, Rapagnetta, and Saccá in [MRS18] by looking at a specific manifold of $OG6$ type and relating it to the geometry of a manifold of $K3^{[3]}$ type. Mongardi, Rapagnetta, and Saccà note that their method cannot be applied to the $OG10$ case since a certain class is not divisible by two in the integral cohomology.

The second Betti number for manifolds of $OG10$ type was first computed by Rapagnetta in [Rap08] and the Euler characteristic was determined by Mozgovyy in [Moz07]. The Betti and Hodge numbers were then completely determined by de Cataldo, Rapagnetta, and Saccà in [dCRS21] using the method of Ngô strings.

More recently, the cohomology algebra of IHS manifolds has been studied by Green, Kim, Laza, and Robles in [GKLR22]. Using knowledge of the Betti and Hodge numbers, they compute the Looijenga-Lunts-Verbitsky (LLV) decomposition of the cohomology algebra for all of the known examples. In particular, this gives a description of the Hodge structure of all known examples in terms of the Hodge structure on its second cohomology.

The purpose of this thesis is to apply the method of Ngô strings to give a new computation of the Betti and Hodge numbers for manifolds of $OG6$ type. This method was first introduced by de Cataldo, Rapagnetta, and Saccà in [dCRS21] to compute the Betti and Hodge numbers of manifolds of $OG10$ type. We also give a description of the Hodge structure of some special manifolds of $OG6$ type in terms of the Hodge structure of a related Abelian surface. Although we do not describe the Hodge structure of all manifolds of $OG6$ type, as is done in [GKLR22] using the LLV decomposition, we note that our computation of the Hodge structure using Ngô strings does not require prior knowledge of the Hodge numbers as an input.

Theorem 1. *The odd Betti numbers of the six dimensional complex projective manifolds of $OG6$ type are zero and the even ones are:*

$$b_0 = 1, b_2 = 8, b_4 = 199, b_6 = 1504.$$

The relevant part of the Hodge diamond listing the Hodge numbers is given *below:*

$$\begin{array}{cccc} & & & h^{0,0}=1 \\ & & h^{2,0}=1 & h^{1,1}=6 \\ & h^{4,0}=1 & h^{3,1}=12 & h^{2,2}=173 \\ h^{6,0}=1 & h^{5,1}=6 & h^{4,2}=173 & h^{3,3}=1144. \end{array}$$

The rows corresponding to the odd weight cohomology are omitted since those entries vanish and all other Hodge numbers can be recovered via the usual symmetries satisfied by compact Kähler manifolds.

The starting point for both the original method of Mongardi, Rapagnetta, and Saccà in [MRS18] and the new method of Ngô Strings is the same. Consider an Abelian surface $J = J(C_0)$, which is the Jacobian of a general genus two curve C_0, with $NS(J) = \mathbb{Z}c_1(\theta)$, where θ is a symmetric theta divisor. Consider the moduli space of pure dimension one sheaves on J with Mukai vector $v = (0, 2\theta, -2)$. There is a natural morphism to the Abelian fourfold $J^\vee \times J$, where $J^\vee$ is the dual Abelian variety of J. Denote the fiber of this map over $(\mathcal{O}_J, 0) \in J^\vee \times J$ by M. The variety M is singular but admits a symplectic resolution $\widetilde{M}$ of $OG6$ type. The morphism sending a sheaf to its Fitting support then realizes $\widetilde{M}$ as a Lagrangian fibration over the linear system $B = |2\theta| \simeq \mathbb{P}^3$.

The original method of Mongardi, Rapagnetta, and Saccà then relates the geometry of $\widetilde{M}$ to the geometry of a manifold of $K3^{[3]}$ type. Key to their argument is the fact that the exceptional divisor, $\widetilde{\Sigma}$, of the symplectic resolution $\widetilde{M} \to M$ is divisible by two in $H^2(\widetilde{M}, \mathbb{Z})$ (cf. [MRS18, Theorem 4.1] and [Rap07, Theorem 3.3.1]). This fact allows them to construct a manifold $\underline{Y}$ of $K3^{[3]}$ type [MRS18, Proposition 5.3] which is related to $\widetilde{M}$ using only birational modifications of smooth projective varieties and a double cover [MRS18, Section 6]. From this description of $\widetilde{M}$, they are able to deduce the Betti and Hodge numbers of $\widetilde{M}$. Mongardi, Rapagnetta, and Saccà remark in the end of their introduction that their method cannot be applied to manifolds of $OG10$ type since the divisor analogous to $\widetilde{\Sigma}$ is not divisible by two in the second integral cohomology in that case and point to [Rap08] for a proof.

In contrast, the method of Ngô strings, introduced by de Cataldo, Rapagnetta, and Saccà in [dCRS21] is a more general approach and can be adapted to the $OG6$ case, as shown in this thesis. Due to the generality of the approach, the method of Ngô strings may be useful in other similar situations, such as studying the Hitchin fibration.

We now summarize our approach using the method of Ngô strings, which follows the strategy introduced in [dCRS21]. Similar to the original approach used in [MRS18], we relate the geometry of $\widetilde{M}$ to another better understood manifold. However, we relate $\widetilde{M}$ to a manifold N of Kum_3 type rather than the manifold $\underline{Y}$ of $K3^{[3]}$ type. The manifold N is constructed by considering the moduli space of sheaves on J with Mukai vector $w = (0, 2\theta, -3)$. This moduli spaces also admit a natural morphism to the Abelian fourfold $J^\vee \times J$, and N is defined as the fiber of this morphism over $(\mathcal{O}_J, 0)$. It is known that N is smooth and is of Kum_3 type [Yos01]. The morphism sending a sheaf to its Fitting support then realizes N as a Lagrangian fibration over the linear system $B = |2\theta| \simeq \mathbb{P}^3$.

The two fibrations $\widetilde{M} \to B$ and $N \to B$ are closely linked via a group scheme $G \to B$ which acts fiberwise on both $\widetilde{M}$ and N. More precisely, $\widetilde{M}$ and N both contain open dense subvarieties which are torsors under the action of G. We note that the subvariety of $\widetilde{M}$ is a G-torsor over the entire base B, while the subvariety of N is a G-torsor over an open subset strictly contained in the base B. Using this relationship between $\widetilde{M}$ and N, we are able to use the known cohomology of generalized Kummer varieties, along with some related smaller dimensional fibrations, to deduce the Betti and Hodge numbers of $\widetilde{M}$ by working in an appropriate Grothendieck group.

As in the $OG10$ case, the main technical tool that makes this comparison of the cohomology of $\widetilde{M}$ and N possible is the Ngô Support Theorem (see [Ngô10, Théorèm 7.2.1] for the original statement, [dCRS21, Theorem A.0.3] for the Hodge theoretic refinement, and Section 2.2.1 for a discussion in this thesis). The Ngô Support Theorem is a refinement of the Decomposition Theorem [BBD82] for special types of fibrations called *δ-regular weak Abelian fibrations*, which encompass many Lagrangian fibrations, including $\widetilde{M} \to B$, $N \to B$, and the Hitchin fibration. The Decomposition Theorem is a general theorem which states that the singular cohomology, or more generally the intersection cohomology, of the domain of a fibration is a direct sum of pieces which are governed by uniquely determined subvarieties, called supports, of the target. In practice, it is extremely difficult to determine exactly which direct summands and supports appear in the Decomposition Theorem. However, if a fibration is a δ-regular weak Abelian fibration, the Ngô Support Theorem reduces the question of determining the supports to questions about the top degree direct image sheaf and completely determines the direct summands, which are called *Ngô strings*, appearing in the Decomposition Theorem corresponding to a given support. This is extremely powerful as questions about the top degree direct image sheaf are often easier to answer and can be tackled by studying the irreducible components of the fibers of the fibration.

The Hodge theoretic refinement of the Ngô support Theorem [dCRS21, Theorem A.0.3] allows us to describe rational Hodge structure of $\widetilde{M}$ in terms of the Hodge structures of N and some smaller dimensional fibrations. However, more can be said since the rational pure Hodge structure of N is known. More precisely, the pure Hodge structure of N can be described in terms of the Hodge structure of the Abelian surface J from which N is constructed. Using this description, we are able to deduce our second main result, which describes the pure Hodge structure of $\widetilde{M}$ in terms of the pure Hodge structure of J. We include a statement about the known Hodge structure of N for completeness.

Theorem 2. *Let J be the Jacobian of a general genus two curve with $NS(J) = \mathbb{Z}c_1(\theta)$, where θ is a symmetric theta divisor. Let $\widetilde{M}$ be the manifold of $OG6$ type and N be manifold of Kum_3 type constructed from J as described above. Let $U = H^{even}(J, \mathbb{Q})$, $W = H^{odd}(J, \mathbb{Q})$, and $\langle \bullet \rangle := [-2\bullet](-\bullet)$. Then*

$$H^*(\widetilde{M}) = \mathrm{Sym}^3 U \oplus \left((U^{\otimes 2})^{\oplus 2} \oplus W^{\otimes 2}\right)\langle 1\rangle \oplus U^{\oplus 137}\langle 2\rangle \oplus \mathbb{Q}^{\oplus 512}\langle 3\rangle.$$

$$H^*(N) = \mathrm{Sym}^3 U \oplus \left(U^{\otimes 2} \oplus (U \otimes W)^{\oplus 2}\right)\langle 1\rangle \oplus U^{\oplus 16}\langle 2\rangle \oplus \mathbb{Q}^{\oplus 256}\langle 3\rangle,$$

We note that using the language of Schur functors, one can rewrite the cohomology of $\widetilde{M}$ purely in terms of the even cohomology U of J. If one does this, one recovers the description of the cohomology of $\widetilde{M}$ coming from the LLV decomposition given in [GKLR22]. See Section 5.6.1 for more details.

We now summarize the structure of this thesis. In Chapter 2, we review the necessary preliminaries required for the thesis. In Section 2.1, we establish our notation. In Section 2.2 we define the notion of a δ-regular weak Abelian fibration and discuss the Ngô Support Theorem. In Sections 2.3 and 2.4, we review the construction of the manifolds $\widetilde{M}$ and N in more detail. In Section 2.5, we discuss the construction of the group scheme $G \to B$ over B. In Section 2.6, we show that the group scheme G acts fiberwise on $\widetilde{M}$ and N and we then realize the triples $(\widetilde{M}, B, G)$ and (N, B, G) as δ-regular weak Abelian fibrations. We remark that unlike the $OG10$ case, the fibers of our group scheme G are not necessarily connected.

We then turn our attention to the study of the top degree direct image sheaves for the fibrations $\widetilde{M} \to B$ and $N \to B$. In Chapter 3, we first recall Rapagnetta's stratification of the linear system $|2\theta|$ by analytic type of singularity as described in [Rap07]. We then explicitly determine the number of irreducible components of the fibers of the Lagrangian fibrations over each stratum. The results are summarized in Proposition 3.1.3.

With this detailed description of the irreducible components of the fibers of the Lagrangian fibrations, we compute the direct image sheaves in top degree for both Lagrangian fibrations over various strata in Chapter 4. For precise statements, see Proposition 4.1.1 and Proposition 4.2.6. The key to determining the direct image sheaves is a precise understanding of the monodromy of the irreducible components, which is more subtle than the $OG10$ case due to the fact that the fibers of our group scheme G are not necessarily connected.

Finally in Section 5, we describe how the knowledge of the top degree direct image sheaf along with the Ngô Support Theorem can be used to determine the shape of the Decomposition Theorems for both Lagrangian fibrations. For a precise statement, see Proposition 5.4.2. Similar to the $OG10$ case, we fall short of determining the exact shape of the Decomposition Theorems for $\widetilde{M} \to B$ and $N \to B$. There is also an indeterminacy in both of these fibrations which remarkably cancels out when comparing the shapes of the Decomposition Theorems in the appropriate Grothendieck group. Using this comparison, we deduce Theorem 1 about the Betti and Hodge numbers in Section 5.5. Crucial to our proof are the following two facts: the first is that the Hodge numbers and Hodge structures of the generalized Kummer type manifold N are known. The second is that some of the Ngô strings which appear in the Decomposition Theorems for $\widetilde{M} \to B$ and $N \to B$ also appear in other lower dimensional fibrations related to the Abelian surface J. The geometry of these lower dimensional fibrations are much simpler and their Hodge theory is completely understood. For a more detailed discussion of these Ngô strings, see Sections 5.2 and 5.3.

We conclude with following remark. If one is only interested in the statements about the Betti and Hodge numbers of $OG6$, a result by Shen and Yin [SY22, Theorem 0.2] implies that it suffices to work only in the constructible derived category. They prove that for any compact hyper-Kähler manifold M admitting a Lagrangian fibration $M \to B$, the Hodge numbers can be recovered from the perverse filtration associated to the Lagrangian fibration. However, if one is interested in statements about the Hodge structure, then it is necessary to work in the category of mixed Hodge modules.

Chapter 2

Preliminaries

2.1 Notation

We work over the field of complex numbers, $\mathbb{C}$. In this thesis, a variety is a separated scheme of finite type over $\mathbb{C}$. Unless otherwise stated, by point we mean closed point. Given a morphism of varieties $f\colon X \to Y$ and a subvariety $Z \subseteq Y$, we set $X_Z := f^{-1}(Z)$. In the special case that $Z = \{y\}$ is a point, we will denote the fiber by X_y. Given a coherent sheaf $\mathscr{F}$ on a smooth projective variety X, the i^{th} Chern class of $\mathscr{F}$, denoted by $c_i(\mathscr{F})$, takes values in $CH^i(X)$, where $CH^i(X)$ is the Chow group of cycles of codimension i on X modulo rational equivalence. If X has dimension n, the Chow group of cycles of dimension i (or i-cycles) is denoted by $CH_i(X) := CH^{n-i}(X)$. By abuse of notation, we will denote the image of $c_i(\mathscr{F})$ under the cycle class map $cl\colon CH^i(X) \to H^{2i}(X, \mathbb{Z})$ by the same symbol.

Given a variety X, there are two categories which we will work with. The first is the bounded constructible derived category $D^b_c(X, \mathbb{Q})$, or simply $D^b(X, \mathbb{Q})$, whose objects are complexes of sheaves on X of $\mathbb{Q}$-vector spaces whose cohomology complexes are constructible with respect to some finite algebraic stratification. Note that here, we are implicitly using the analytic topology on X. For basics and references, see [dCM09]. The second is Saito's bounded derived category $D^bMHM_{alg}(X)$ of algebraic mixed Hodge modules which rational structure, which is endowed with the formalism of weights. See [Sai90] for the foundations and [Sch14] for the basics and references. Given an object $K \in D^bMHM_{alg}(X)$, we denote by $K(k)$ the Tate twist of K by an integer k.

There is a natural exact functor $\mathrm{rat}\colon D^bMHM_{alg}(X) \to D^b(X, \mathbb{Q})$, which is neither essentially surjective nor fully faithful. However, via the functor rat, the standard t-structure on $D^bMHM_{alg}(X)$ corresponds to the middle perversity t-structure on $D^b(X, \mathbb{Q})$. In particular, this implies that if $K \in MHM_{alg}(X)$, then $\mathrm{rat}(K)$ is a perverse sheaf on X. Moreover, a splitting of the object K induces a splitting of the object $\mathrm{rat}(K)$. Given an object $K \in D^bMHM_{alg}(X)$, we will often abuse notation and denote $\mathrm{rat}(K)$ by K.

If $Z \subseteq X$ is a closed irreducible subvariety and $\mathscr{L}$ is a polarizable variation of rational pure Hodge structures of weight $w(\mathscr{L})$ on some Zariski dense open subset $V \subseteq Z^{reg}$, then the intersection cohomology object $IC_Z(\mathscr{L}) \in MHM_{alg}(X)$ yields, via rat, the usual intersection cohomology complex of Z with coefficients in the local system underlying $\mathscr{L}$. Such a local system is necessarily self dual, i.e. $\mathscr{L} \simeq \mathscr{L}^\vee$, and is semi-simple. We define

$$\mathscr{I}\mathscr{C}_Z(\mathscr{L}) \coloneqq IC_Z(\mathscr{L})[-\dim(Z)]$$

and note that the object $\mathrm{rat}(\mathscr{I}\mathscr{C}_Z(\mathscr{L}))$ is not Verdier self-dual and the cohomology sheaves of $\mathscr{I}\mathscr{C}_Z(\mathscr{L})$ are concentrated in non-negative degrees beginning in degree zero. When making geometric statements, we use the non self-dual complexes $\mathscr{I}\mathscr{C}$. However in proofs, the perverse object IC is sometimes used to exploit the simplified book-keeping when considering the Poincaré-Verdier and Relative Hard Lefschetz dualities.

If the object $\mathscr{I}\mathscr{C}_Z(\mathscr{L})$, with $\mathscr{L}$ pure of weight of $w(\mathscr{L})$, is considered in $MHM_{alg}(X)[-\dim(Z)] \subset D^bMHM_{alg}(X)$, then the (hyper)cohomology group $IH^\bullet(Z, \mathscr{L}) \coloneqq H^\bullet(Z, \mathscr{I}\mathscr{C}(\mathscr{L}))$ is a pure polarizable Hodge structure of weight $w(\mathscr{L}) + \bullet$.

2.2 δ-Regular Weak Abelian Fibrations

The notion of a δ-regular weak Abelian fibration was introduced by B.C. Ngô in [Ngô10]. We follow the discussion in [dCRS21, §2.2 and §2.3] and summarize the notion below.

Let $g\colon G \to B$ be a smooth commutative group scheme over B, and let $g^0\colon G^0 \to B$ be the identity component. Given a closed point of B, there is the canonical Chevalley devissage of the fiber G_b^0 of G at b, i.e. there is a canonical short exact sequence

$$0 \to R_b \to G_b^0 \to A_b \to 0 \tag{2.1}$$

over the perfect field $k(b)$, where A_b is an Abelian variety and R_b is affine, connected, and maximal with respect to these properties.

Remark 2.2.1. *Since we are working over $\mathbb{C}$, given any integral locally closed subvariety $Z \subset B$, there exists an open dense subvariety $V \subset Z$ and a short exact sequence of smooth commutative algebraic group schemes over V with connected fibers*

$$0 \to R_V \to G^0|_V \to A_V \to 0 \tag{2.2}$$

which realizes the Chevalley devissage point-by-point on V (cf. [Ngô10, §7.4.8]).

The function $\delta \colon B \to \mathbb{Z}$ sending $b \in B$ to $\dim_{k(b)} R_b$ is upper-semicontinuous and we define the δ-loci B_i by $B_i \coloneqq \{b \in B \mid \delta(b) = i\}$. Now suppose that $d = \dim_{k(b)} G_b$ is constant on B (it is so on the connected components of B by smoothness). The **Tate module** of G/B is defined to be the object

$$T(G) = T(G/B) \coloneqq R^{2d-1} g^0_! \mathbb{Q}_{G^0}(d)$$

in $D^b MHM(B)_{alg}$. Given any $b \in B$, the short exact sequence in Equation 2.1 induces a short exact sequence

$$0 \to T(R_b) \to T(G^0_b) \to T(A_b) \to 0$$

of rational mixed Hodge structures.

For any fixed prime ℓ, let $T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(G/B)$ be the $\overline{\mathbb{Q}}_\ell$-adic counterpart of $T(G/B)$, i.e. it is defined by the same formula using the étale topology/cohomology formalism. The Tate module $T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(G/B)$ is defined to be polarizable if étale locally, there is a pairing

$$T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(G/B) \otimes T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(G/B) \to \overline{\mathbb{Q}}_\ell(1)$$

such that for every $b \in B$, the kernel of the pairing at b is $T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(R_b)$. With these preliminaries out of the way, we can define the notion of a δ-regular weak Abelian fibration.

Definition 2.2.2. *A **weak Abelian fibration**, denoted as a triple (M, B, G) is a pair of of morphisms $M \xrightarrow{f} B \xleftarrow{g} G$ such that*

1. *f is proper,*
2. *G is a smooth commutative group scheme over B,*
3. *f and g have the same pure relative dimension d,*
4. *there is an action $a \colon G \times_B M \to M$ of G on M over B,*

5. *the action has affine stabilizers at every point $m \in M$,*

6. *the Tate module $T_{\mathrm{et},\overline{\mathbb{Q}}_\ell}(G/B)$ is polarizable.*

If in addition the δ-loci, B_i, satisfy the inequality

$$\operatorname{codim} B_i \geq i \tag{2.3}$$

*for every non-negative integer i, then (M, B, G) is said to be a **δ-regular weak Abelian fibration**. In particular, this inequality implies that the general fiber G_b^0 is an Abelian variety.*

Given a weak Abelian fibration (M, B, G), if M is a quasi-projective holomorphic symplectic manifold of complex dimension $2d$ and the morphism $M \to B$ is a Lagrangian fibration, i.e. a proper surjective morphism with connected fibers and with general fiber a Lagrangian subvariety, then de Cataldo, Rapagnetta, and Saccá give the following criterion for determining when (M, B, G) is δ-regular.

Proposition 2.2.3. *([dCRS21, Proposition 2.3.2]) Let (M, B, G) be a weak Abelian fibration such that*

1. *M is a quasi-projective holomorphic symplectic manifold and B is smooth,*

2. *$M \to B$ is a proper Lagrangian fibration and $G \to B$ has connected fibers,*

3. *there is an open subset $M^0 \subset M$ such that M^0/B is a G/B-torsor.*

Then (M, B, G) is δ-regular.

2.2.1 The Ngô Support Theorem

In this section, we recall the definition of a support and state the refinement of the Ngô Support Theorem given by de Cataldo, Rapagnetta, and Saccá in [dCRS21].

Definition 2.2.4 (Supports)**.** *Let $f\colon X \to Y$ be a proper map of complex algebraic varieties. By the Decomposition and Semisimplicity Theorems ([BBD82, §6]), there exists finitely many triples $(V_\alpha, \mathscr{L}_\alpha, d_\alpha)$, where $V_\alpha \subset Y$ is a locally closed smooth irreducible algebraic subvariety of Y, $\mathscr{L}_\alpha$ is a local system on V_α, and d_α is an integer, such that there is a canonical decomposition*

$$Rf_*IC_X \simeq \bigoplus_\alpha IC_{\overline{V_\alpha}}(\mathscr{L}_\alpha)[\dim X - \dim V_\alpha - d_\alpha]$$

*in the bounded constructible derived category $D^b(Y)$. The closed subvarieties $Z_\alpha \coloneqq \overline{V_\alpha}$ are called the **supports** of $Rf_* IC_X$.*

In the remainder of this section, we work with the following set up.

Set-up 2.2.5. *Let (M, B, G) be a δ-regular weak Abelian fibration. Suppose that $d = \dim(M/B) = \dim(G/B)$, M/B is projective, B is irreducible, and M is a rational homology manifold, i.e. $\mathscr{IC}_M \simeq \mathbb{Q}_M$.*

We note that the assumption on the equality of relative dimensions is made only to simplify the numerology. Let $\mathscr{A}$ be a finite set enumerating the supports of $Rf_*\mathbb{Q}_M$, so that the supports are denoted Z_α for $\alpha \in \mathscr{A}$. In view of Remark 2.2.1, there is an open dense subvariety $V_\alpha \subset Z_\alpha$ and a short exact sequence of smooth commutative algebraic group schemes over V_α with connected fibers

$$0 \to R_{V_\alpha} \to G^0|_{V_\alpha} \to A_{V_\alpha} \to 0$$

realizing the Chevalley devissge point-by-point on V_α. Denote the natural map $A_\alpha \to V_\alpha$ by g_α and let δ_α^{ab} denote the relative dimension of A_α/V_α. Let $\Lambda_\alpha^\bullet \coloneqq R^\bullet g_{\alpha *}\mathbb{Q}_{A_\alpha}$ where $0 \le \bullet \le \delta_\alpha^{ab}$.

Theorem 3. *With the set-up above, there is an isomorphism in $D^b MHM_{alg}(B)$ of pure objects of weight 0.*

$$Rf_*\mathbb{Q}_M \simeq \bigoplus_{\alpha \in \mathscr{A}} \mathscr{I}_\alpha[-2(d - \delta_\alpha^{ab})](\delta_\alpha^{ab} - d) \tag{2.4}$$

where

$$\mathscr{I}_\alpha \coloneqq \bigoplus_{\bullet=0}^{2\delta_\alpha^{ab}} \mathscr{IC}_{Z_\alpha}(\Lambda_\alpha^\bullet \otimes \mathscr{L}_\alpha)[-\bullet] \tag{2.5}$$

and $\mathscr{L}_\alpha$ is a polarizable variation of pure Hodge structures of weight 0.

The summands $\mathscr{I}_\alpha$ appearing in Theorem 3 are called **Ngô strings**.

Remark 2.2.6. *We note that there is a stronger version of the Ngô Support Theorem (cf. [Ngô10, Proposition 7.2.3]) which keeps track of the action of the group of connected components of the group scheme G on the complex $Rf_*\mathbb{Q}_M$. As this stronger version is not needed in this thesis, we do not state it precisely.*

2.3 Moduli Spaces of Pure Dimension One Sheaves on Surfaces

In this section, we recall some basic definitions about pure dimension one sheaves on surfaces. Let X be a smooth projective surface. A coherent sheaf $\mathscr{F}$ on X is **pure of dimension one** if for every non-trivial subsheaf $\mathscr{G} \subset \mathscr{F}$, $\dim(\mathscr{G}) = 1$, where $\dim(\mathscr{G})$ is the dimension of the schematic support of $\mathscr{G}$. If $\mathscr{F}$ is a pure dimension one sheaf, a key property is that there exists a length one locally free resolution

$$0 \to \mathscr{F}_1 \xrightarrow{\phi} \mathscr{F}_0 \to \mathscr{F} \to 0 \tag{2.6}$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ have the same rank (cf. [HL97, §1.1]).

Using this resolution, the **Fitting support** of $\mathscr{F}$, denoted by $\mathrm{FittSupp}(\mathscr{F})$, as the vanishing subscheme induced by the morphism $\det(\mathscr{F}_1) \to \det(\mathscr{F}_0)$ between determinant bundles (cf. [Eis95, Corollary-Definition 20.4]). We note that this definition is independent of the resolution.

One can also verify from the resolution in Equation 2.6 that stability with respect to the reduced Hilbert polynomial amounts to the following notion of stability.

Definition 2.3.1. *(Semi-stability for Pure Dimension One Sheaves) A pure dimension one sheaf $\mathscr{F}$ on a smooth projective polarized surface (X, H) is* ***Gieseker (semi-)stable*** *with respect to H if for all proper pure dimension one quotients $\mathscr{F} \to \mathscr{G}$, the following inequality holds,*

$$\frac{\chi(\mathscr{F})}{c_1(\mathscr{F}) \cdot H} \underset{(}{\leq}{}_{)} \frac{\chi(\mathscr{G})}{c_1(\mathscr{G}) \cdot H} \tag{2.7}$$

Note that Remark 2.3.2 below implies that the denominators in Equation 2.7 are always non-zero by the Nakai-Moishezon criterion since H is ample. In the remainder of this thesis, semi-stability will always mean Gieseker semi-stability as it is the only type of semi-stability considered.

We end this section with two remarks on the Fitting support.

Remark 2.3.2. *The Fitting support of a pure dimension one sheaf $\mathscr{F}$ on X contains the schematic support of $\mathscr{F}$ [Eis95, Proposition 20.7] and Equation 2.6 implies that the Fitting support represents $c_1(\mathscr{F})$ in the $CH^1(X)$. In particular, two pure dimension one sheaves $\mathscr{F}$ and $\mathscr{G}$ have isomorphic determinant bundles if and only if their Fitting supports are linearly equivalent.*

Remark 2.3.3. *If the Fitting support of $\mathscr{F}$ is an integral curve, then $\mathscr{F}$ is the pushforward of a rank one torsion free sheaf on the curve. It follows that $\mathscr{F}$ has no pure dimension quotient and is automatically stable. If $\mathscr{F}$ is the pushforward of a line bundle on a curve C which is possibly both reducible and non-reduced, then the only proper pure dimension one quotients of $\mathscr{F}$ are restrictions to the pure dimension one subschemes of C. It follows that $\mathscr{F}$ is (semi-)stable if and only if*

$$\frac{\chi(\mathscr{F})}{C \cdot H} \underset{(}{\leq}{}_{)} \frac{\chi(\mathscr{F}|_D)}{D \cdot H} \tag{2.8}$$

for every proper subcurve $D \subset C$.

2.4 The Varieties $\widetilde{M}$, M, N and B

In this section, we review the construction of the Kummer and O'Grady type manifolds. Throughout the rest of this thesis, we work with a fixed curve C_0 of genus two. Fix a Weierstrass point $w_0 \in C_0$ and let $J := H^0(C_0, \Omega^1_{C_0})^\vee / H_1(C_0, \mathbb{Z})$ be the Jacobian of C_0. The Abel-Jacobi Theorem gives a canonical isomorphism $\mathrm{Pic}^0(C_0) \xrightarrow{\simeq} J$ and under this identification, the Abel-Jacobi embedding can be written as

$$C_0 \overset{u}{\hookrightarrow} J; \quad x \mapsto [x - w_0]. \tag{2.9}$$

The image $\theta := u(C_0)$ is a symmetric theta divisor, i.e. θ is invariant under pullback by the natural (-1)-involution on J sending x to $-x$. We will work under the additional assumption that

$$NS(J) \simeq \mathbb{Z}c_1(\theta), \tag{2.10}$$

which holds if the curve C_0 fixed at the beginning is general.

Using the theta divisor, we can also identify J with its dual Abelian variety, $J^\vee$, via the map

$$J \to J^\vee; \quad x \mapsto [\theta_x - \theta], \tag{2.11}$$

where $\theta_x := \theta + x$.

We now recall the notion of the Mukai lattice and Mukai vectors (cf. [HL97, Section 6.1]). Let $H^{even}(J, \mathbb{Z}) := H^0(J, \mathbb{Z}) \oplus H^2(J, \mathbb{Z}) \oplus H^4(J, \mathbb{Z})$ denote the even integral cohomology of J. Given a vector $v = (v_0, v_2, v_4) \in H^{even}(J, \mathbb{Z})$, define $\overline{v} := (v_0, -v_2, v_4)$. For vectors $v, w \in H^{even}(J, \mathbb{Z})$, the Mukai pairing is given by

$$(v, w) := -\int_J (\overline{v}, w), \tag{2.12}$$

and $(H^{even}(J,\mathbb{Z}),(\cdot,\cdot))$ is called the **Mukai lattice**.

Given a coherent sheaf $\mathscr{F}$ on J, the **Mukai vector of $\mathscr{F}$** is defined to be $v(\mathscr{F}) := ch(\mathscr{F}) \cup \sqrt{Td(J)}$, where $ch(\mathscr{F})$ is the Chern character of $\mathscr{F}$ and $Td(J)$ is the Todd class of J. Note that since J is an Abelian surface, $Td(J) = 1$, which implies that $v(\mathscr{F}) = ch(\mathscr{F})$. In particular,

$$v(\mathscr{F}) = (r, c_1, \chi) \tag{2.13}$$

where $r = rk(\mathscr{F})$, $c_1 = c_1(\mathscr{F})$ and $\chi = ch_2(\mathscr{F})$. Note that by Hirzebruch-Riemann-Roch, $\chi = ch_2(\mathscr{F}) = \chi(\mathscr{F})$ is the Euler characteristic of $\mathscr{F}$.

Equation 2.13 implies that the Mukai vector of a coherent sheaf on J takes values in

$$H^{even}_{alg}(J) := H^0(J,\mathbb{Z}) \oplus NS(J) \oplus H^4(J,\mathbb{Z}) \subset H^{even}(J,\mathbb{Z}). \tag{2.14}$$

Since we are working under the assumption $NS(J) \simeq \mathbb{Z}c_1(\theta)$, we can identify $H^{even}_{alg}(J)$ with $\mathbb{Z}^3$ in the natural way. Under this identification, we will denote vectors in $H^{even}_{alg}(J)$ by $v = (0, 2\theta, \chi)$.

Consider a vector of the form $v = (0, 2\theta, \chi) \in H^{even}_{alg}(J)$ with $\chi \neq 0$ and let $\mathbf{M}_v$ be the moduli space of semi-stable sheaves on J with Mukai vector v. Points of $\mathbf{M}_v$ are in bijection with S-equivalence classes of sheaves, or equivalently isomorphism classes of polystable sheaves (see [HL97, Definitions 1.5.3, 1.5.4]). Sheaves with such a Mukai vector v are pure of dimension one. By [Muk84], $\mathbf{M}_v$ is a normal irreducible projective variety of dimension $v^2 + 2 = 10$ and the smooth locus, which is precisely the locus parameterizing strictly stable sheaves, admits a symplectic form.

There is a natural morphism

$$\mathbf{a}_v \colon \mathbf{M}_v \to J^\vee \times J; \quad [\mathscr{F}] \mapsto \big(\det(\mathscr{F}) \otimes \mathcal{O}_J(2\theta)^\vee, \textstyle\sum c_2(\mathscr{F})\big), \tag{2.15}$$

where $[\mathscr{F}]$ is the S-equivalence class of $\mathscr{F}$, $\det(\mathscr{F})$ is the determinant bundle of $\mathscr{F}$, and $\sum c_2(\mathscr{F})$ is the sum in J of any representative of $c_2(\mathscr{F})$ in $CH_0(J)$, the Chow group of zero cycles on J (cf. [HL97, §4.5 and Proposition 10.3.6]). The fiber of this morphism

$$M_v := \mathbf{a}_v^{-1}(\mathcal{O}_J, 0), \tag{2.16}$$

as well as its resolution, will be our main objects of interest in this paper. The points of M_v parameterize S-equivalence classes of semi-stable sheaves, on J having determinant bundle isomorphic to $\mathcal{O}_J(2\theta)$ and second Chern class summing up to 0.

The moduli spaces $\mathbf{M}_v$ and the fibers M_v depend on the parity of the Euler characteristic χ in the Mukai vector $v = (0, 2\theta, \chi)$.

If χ is odd, then the Mukai vector v is primitive and Yoshioka shows in [Yos01, Theorem 0.1 and Theorem 0.2] $\mathbf{M}_v$ is smooth of dimension 10 and that the fiber M_v is deformation equivalent to a generalized Kummer variety of dimension 6.

If χ is even, then the Mukai vector $v = 2v'$ is twice a primitive Mukai vector and Rapagnetta shows in [Rap07] that the moduli space $\mathbf{M}_v$ and the fiber M_v are both reduced, but singular. The singular locus $\Sigma_v \subset M_v$ parameterizes polystable sheaves of the form

$$\mathscr{F}_1 \oplus \mathscr{F}_2,$$

with $v(\mathscr{F}_i) = v'$, determinant bundle isomorphic to $\mathscr{O}_J(2\theta)$ and second Chern class summing up to 0. The locus Σ_v is itself singular and its singular locus, $\Omega_v \subset \Sigma_v$, parameterizes polystable sheaves of the form

$$\mathscr{F} \oplus \mathscr{F}.$$

Both the moduli space $\mathbf{M}_v$ and the fiber M_v admit symplectic resolutions

$$\pi_v \colon \widetilde{M_v} \to M_v,$$

which can be realized by first blowing up Ω_v in M_v, then blowing up $Bl_{\Omega_v} M_v$ along the strict transform of Σ_v, and finally contracting the inverse image of Ω_v via the two blowups. The resolution $\widetilde{M_v}$ is an irreducible holomorphic symplectic manifold which is birational, and hence deformation equivalent, to O'Grady's six dimensional exceptional example ([Rap07, Proposition 2.2.1]).

Remark 2.4.1. *The fibers of the resolution $\pi \colon \widetilde{M} \to M$ are described by Rapagnetta in [Rap07, Remark 1.1.5]. In particular, if $[\mathscr{F}] \in \Sigma \subset M$, then $\pi^{-1}([\mathscr{F}])$ is a $\mathbb{P}^1$. If $[\mathscr{F}] \in \Omega \subset M$, then $\pi^{-1}([\mathscr{F}])$ is a smooth 3-dimensional quadric.*

We also record the following fact about the Decomposition Theorem for the symplectic resolution.

Proposition 2.4.2. *There is a canonical isomorphism in $D^b MHM_{alg}(M)$ and in $D^b_c(M, \mathbb{Q})$ (turn off the Tate twists)*

$$R\pi_* \mathbb{Q}_{\widetilde{M}} \simeq \mathscr{IC}_M \oplus \mathbb{Q}_\Sigma[-2](-1) \oplus \mathbb{Q}_\Omega[-6](-3). \tag{2.17}$$

Proof. By a general result of Kaledin (see [Kal06, Lemma 2.11]), the symplectic resolution $\pi\colon \widetilde{M} \to M$ is semismall. The Decomposition Theorem for semismall morphisms [dCM09, Theorem 4.2.7] implies that

$$R\pi_* \mathbb{Q}_{\widetilde{M}} \simeq \mathscr{I}\mathscr{C}_M \oplus \mathscr{I}\mathscr{C}_\Sigma[-2](-1) \oplus \mathscr{I}\mathscr{C}_\Omega[-6](-3). \tag{2.18}$$

Rapagnetta, in the proof of [Rap07, Theorem 2.1.7], shows that the singular locus $\Sigma \subset M$ is isomorphic to $(J^\vee \times J)/\pm 1$. In particular, Σ has finite quotient singularities, which implies that the intersection complex $\mathscr{I}\mathscr{C}_\Sigma$ is the constant sheaf. Under the isomorphism $\Sigma \simeq (J^\vee \times J)/\pm 1$, the singular locus Ω of Σ is identified with the 256 nodes. Since Ω is a finite set, the intersection complex $\mathscr{I}\mathscr{C}_\Omega$ is also the constant sheaf. $\square$

For Mukai vectors of the form $v = (0, 2\theta, \chi)$, analysis of the spaces M_v is simplified by the existence of the additional structure of a Lagrangian fibration, which we now describe. There is a Le Potier support morphism from the moduli space $\mathbf{M}_v$ to the Hilbert scheme parameterizing closed subschemes of J with cohomology class 2θ sending a sheaf to its Fitting support ([LeP93, p. 24]). The image of the support morphism restricted to $M_v \subset \mathbf{M}_v$ is the linear system $|2\theta|$ since the Fitting support of sheaves in M_v are all linearly equivalent.

If $C \subset J$ is an integral curve with cohomology class 2θ, then a pure dimension one sheaf $\mathscr{F}$ on J with Fitting support C is the pushforward of a torsion free sheaf F on C. Using this fact and adjunction, one can conclude that the rank of F must be one by looking at the Hilbert polynomial of $\mathscr{F}$. In particular, the fiber of this morphism over an integral curve C is precisely the degree $\chi + 4$ compactified Jacobian of C (see [LeP93, p. 24] and [Ale04, §1]).

In the remainder of the paper, we fix some Mukai vectors and will work with the following varieties.

Definition 2.4.3. *Set*

$$\widetilde{\mathbf{M}} \coloneqq \widetilde{\mathbf{M}}_{(0,2\theta,-2)}, \quad \widetilde{M} \coloneqq \widetilde{M}_{(0,2\theta,-2)}, \tag{2.19}$$

$$\mathbf{M} \coloneqq \mathbf{M}_{(0,2\theta,-2)}, \quad M \coloneqq M_{(0,2\theta,-2)} \tag{2.20}$$

$$\mathbf{N} \coloneqq \mathbf{M}_{(0,2\theta,-3)}, \quad N \coloneqq M_{(0,2\theta,-3)}, \tag{2.21}$$

Set $B \coloneqq |2\theta| \simeq \mathbb{P}^3$ and let

$$m\colon M \to B, \tag{2.22}$$

$$n\colon N \to B, \tag{2.23}$$

denote the respective support morphisms and let

$$\widetilde{m}\colon \widetilde{M} \xrightarrow{\pi} M \xrightarrow{m} B \tag{2.24}$$

denote the composition of the support morphism with the symplectic resolution.

Remark 2.4.4. *The varieties $\widetilde{M}$ and N are holomorphic symplectic manifolds. By [Mat01, Theorem 1], the support morphisms are Lagrangian fibrations. By [Mat00, Theorem 1], the Lagrangian fibrations $m\colon \widetilde{M} \to B$ and $n\colon N \to B$ are equidimensional and hence flat by the miracle flatness theorem. Moreover, we can say that*

1. *since M has a symplectic resolution, it has canonical singularities and is thus Cohen-Macaulay (see [KM98, Corollary 5.24 and Lemma 5.12]);*

2. *since the fibers of m are dominated by the corresponding fibers of $\widetilde{m}$, they all have the same dimension 3 by upper semi-continuity of fiber dimension for the proper morphism m.*

Since the base B is regular, it again follows again from the miracle flatness theorem that the morphism $m\colon M \to B$ is flat.

2.5 The Group Scheme G

In this section, we introduce the relevant group scheme G as an open subset

$$G \subset M_{(0,2\theta,-4)}.$$

We follow the discussion in Section 3.3 in [dCRS21]. Although their statements are formulated in terms of moduli spaces of sheaves on a $K3$ surface, many of their proofs carry over to our situation. We begin with the following two lemmas which are the analogues of Lemmas 3.3.1 and 3.3.2 in [dCRS21].

Lemma 2.5.1. *For any Mukai vector of the form $v = (0, 2\theta, \chi)$, the locus $\mathbf{G}_v \subset \mathbf{M}_v$ parameterizing stable sheaves that are pushforwards of line bundles on their schematic supports is a nonempty Zariski open subset*

Proof. The proof of Lemma 3.3.1 in [dCRS21] only uses the fact that $\mathbf{M}_v$ is a moduli space of pure dimension one sheaves on a surface and applies in our case. ☐

Lemma 2.5.2. *Let $\mathscr{F}$ be a coherent sheaf on J with $v(\mathscr{F}) = (0, 2, \chi)$ and assume that $\mathscr{F}$ is the pushforward of a line bundle on a curve $C \in |2\theta|$.*

1. *If $C = 2C_{red}$, then χ is even, $\mathscr{F}$ is stable, and the degree of the restriction of $\mathscr{F}$ to C_{red} is* $\deg(\mathscr{F}_{C_{red}}) = (\chi/2) + 2$.

2. (a) *If $C = C_1 + C_2$ for $C_1 \neq C_2$ and χ is even, then the sheaf $\mathscr{F}$ is stable if and only if* $\deg(\mathscr{F}_{C_1}) = \deg(\mathscr{F}_{C_2}) = (\chi/2) + 2$.

   (b) *If $C = C_1 + C_2$ for $C_1 \neq C_2$ and χ is odd, then the sheaf $\mathscr{F}$ is stable if and only if* $\deg(\mathscr{F}_{C_1}) = 2+(\chi+1)/2$ *and* $\deg(\mathscr{F}_{C_2}) = 2+(\chi-1)/2$ *or vice-versa.*

Proof. Noting that J has trivial canonical bundle, $p_a(C) = 5$, and $p_a(C_{\mathrm{red}}) = 2$, the computation done in the proof of Lemma 3.3.2 in [dCRS21] applies in our case. □

We now discuss the construction of the group scheme G. Recall from Equation 2.15 that there is a map $\mathbf{a}_{(0,2\theta,-4)} \colon \mathbf{M}_{(0,2\theta,-4)} \to J^\vee \times J$ from the moduli space of sheaves with Mukai vector $(0, 2\theta, 4)$ to the Abelian fourfold $J^\vee \times J$ and recall that $M_{(0,2\theta,-4)} \coloneqq \mathbf{a}^{-1}_{(0,2\theta,-4)}(\mathcal{O}_J, 0)$ is the corresponding fiber over $(\mathcal{O}_J, 0) \in J^\vee \times J$.

Let $\mathbf{M}^B_{(0,2\theta,-4)} \coloneqq \mathbf{a}^{-1}_{(0,2\theta,-4)}(\{\mathcal{O}_J\} \times J)$ be the pre-image of the slice $\{\mathcal{O}_J\} \times J \subset J^\vee \times J$. Note that since the sheaves parameterized by $\mathbf{M}^B_{(0,2\theta,-4)}$ have isomorphic determinant bundles, the Fitting support of these sheaves are all in the linear system $B = |2\theta|$ by Remark 2.3.2. Set

$$\begin{aligned} \mathbf{G}^B &\coloneqq \mathbf{G}_{(0,2\theta,-4)} \cap \mathbf{M}^B_{(0,2\theta,-4)}, \\ G &\coloneqq \mathbf{G}_{(0,2\theta,-4)} \cap M_{(0,2\theta,-4)}. \end{aligned} \tag{2.25}$$

We summarize the relationship between these spaces in the following diagram:

$$\begin{tikzcd} {}^6G \arrow[r, hook] \arrow[d, hook] & {}^8\mathbf{G}^B \arrow[r, hook] \arrow[d, hook] & {}^{10}\mathbf{G}_{(0,2\theta,-4)} \arrow[d, hook] \\ {}^6M_{(0,2\theta,-4)} \arrow[r, hook] \arrow[dr, two heads] & {}^8\mathbf{M}^B_{(0,2\theta,-4)} \arrow[r, hook] \arrow[d, two heads] & {}^{10}\mathbf{M}_{(0,2\theta,-4)} \\ & B & \end{tikzcd}$$

where the superscripts on the left denote the respective dimensions of the spaces, the hooked arrows are inclusions, and the arrows with two heads are the support morphisms (which are surjective).

Corollary 2.5.3. *The open subset $\mathbf{G}^B_{(0,2\theta,-4)} \subset \mathbf{M}^B_{(0,2\theta,-4)}$ can be identified with the relative degree-0 Picard scheme $\mathrm{Pic}^0_{\mathcal{C}/B}$ of the family $\mathcal{C}/B$ of curves in the linear system B.*

Proof. The proof of Corollary 3.3.3 in [dCRS21] applies. $\square$

Using this identification, the morphism $\mathbf{M}_{(0,2\theta,-4)} \xrightarrow{\mathbf{a}_{(0,2\theta,-4)}} J^\vee \times J \xrightarrow{pr_2} J$ restricted to $\mathbf{G}^B_{(0,2\theta,-4)}$ induces a morphism

$$\mathbf{a}\colon \mathrm{Pic}^0_{\mathcal{C}/B} \to J \times B. \tag{2.26}$$

where the map to B is just the map $\mathrm{Pic}^0_{\mathcal{C}/B} \to B$. In view of the following lemma, the morphism $\mathbf{a}$ is a morphism of B-group schemes.

Lemma 2.5.4. *Let $C \in B$ and $i\colon C \to J$ be the inclusion. Then the map*

$$a\colon \mathrm{Pic}^0(C) \to J; \quad L \mapsto \sum c_2(i_*L). \tag{2.27}$$

is a group homomorphism.

Proof. Although $C \in B$ can be singular, the inclusion $i\colon C \to J$ is a regular embedding with normal bundle $i^*\mathcal{O}_J(C)$. Riemann-Roch without denominators (cf. [FH91, Theorem 15.3]) holds for regular embeddings (cf. [Ful98, Example 15.3.6]) and implies that the Chern polynomial, $c_t(i_*L)$, is given by

$$c_t(i_*L) = 1 + [C]t + ([C]^2 - i_*c_1(L))t^2. \tag{2.28}$$

We now claim that the sum of a representative of $[C]^2$ in $CH_0(J)$ is zero, i.e. $\sum [C]^2 = 0 \in J$. To see this, choose a smooth curve $C' \in B$ which does not contain any 2-torsion points in $J[2]$ and intersects C transversely. Such a choice is always possible by Rapagnetta's description of the linear system B (see the upcoming Section 3.1). Then, a representative of $[C]^2$ in $CH_0(J)$ is given by $C \cap C'$. Since $B = |2\theta|$ where θ is a symmetric theta divisor, C and C' are symmetric. This implies that if $x \in C \cap C'$, then $-x \in C \cap C'$. Since C' does not contain any 2-torsion points, x and $-x$ must be distinct points in $C \cap C'$ and we conclude that $\sum [C]^2 = 0$.

Equation 2.28 then implies that

$$a(L) = \sum c_2(i_*L) = \sum [C]^2 - \sum i_*c_1(L) = -\sum i_*c_1(L).$$

Since the first Chern class is a group homomorphism from $\mathrm{Pic}^0(C)$ to $CH_0(C)$, we conclude that a is a group homomorphism. $\square$

An immediate corollary is the following.

Corollary 2.5.5. *The open subset $G \coloneqq \mathbf{G}_{(0,2\theta,-4)} \cap M_{(0,2\theta,-4)} \subset M_{(0,2\theta,-4)}$ can be identified with the kernel of the map $\mathbf{a}\colon \mathrm{Pic}^0_{\mathcal{C}/B} \to J \times B$ and thus inherits the structure of a B-group scheme.*

2.6 $\widetilde{M}$ and N as δ-regular Weak Abelian Fibrations

In this section, we show that the triples $(\widetilde{M}, B, G)$ and (N, B, G) are δ-regular weak Abelian fibrations satisfying the assumptions of Theorem 3 (cf. Section 2.2.1). As in the previous section, we follow the discussion in [dCRS21, Section 3.4]. We begin with a series of lemmas.

Lemma 2.6.1. *The fiber product $G \times_B M$ is irreducible.*

Proof. Let $S \subset B$ be the locus parameterizing smooth curves. For any $b \in S$, the universal property of the fiber product implies that

$$(G \times_B M)_b \simeq G_b \times M_b$$

where $(G \times_B M)_b$ is the fiber of $(G \times_B M)$ over b. Since $b \in S$ corresponds to a smooth curve in the linear system $|2\theta|$, the fibers G_b and M_b are irreducible and it follows that $(G \times_B M)_b$ is irreducible. Now consider the fiber product diagram

$$\begin{tikzcd}
(G \times_B M)_S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \times_B M \arrow[d] \\
S \arrow[r] & B
\end{tikzcd}$$

By the discussion above, every fiber of the map $(G \times_B M)_S \to S$ is irreducible. Since $S \subset B$ is also irreducible, it follows that $(G \times_B M)_S$ is irreducible. However, since S is a Zariski open subset of B, $(G \times_B M)_S$ is a Zariski open subset of $G \times_B M$ and hence is dense in $G \times_B M$. Since the closure of an irreducible space is irreducible, we conclude that $G \times_B M$ is irreducible. $\square$

Lemma 2.6.2. *Tensoring a sheaf by a line bundle with the same Fitting support induces (algebraic) actions $a_M \colon G \times_B M \to M$ and $a_N \colon G \times_B N \to N$.*

Proof. By Lemma 3.4.1 in [dCRS21], there is an algebraic action

$$\mathbf{a_M} \colon \mathbf{G} \times_B \mathbf{M} \to \mathbf{M}.$$

The universal property of fiber products gives a morphism

$$G \times_B M \to \mathbf{G} \times_B \mathbf{M} \to \mathbf{M}$$

To show that there is an algebraic action $a_M \colon G \times_B M \to M$, it suffices to show that the image of the above map is contained in M. In particular, we must show that given $(\mathscr{L}, \mathscr{F}) \in G_C \times M_C$, $\det(\mathscr{F} \otimes \mathscr{L}) \otimes \mathcal{O}_J(2\theta)^\vee \simeq \mathcal{O}_J$ and $\sum c_2(\mathscr{F} \otimes \mathscr{L}) = 0$.

We first show that $\det(\mathscr{F} \otimes \mathscr{L}) \otimes \mathcal{O}_J(2\theta)^\vee \simeq \mathcal{O}_J$. If the Fitting support of both $\mathscr{F}$ and $\mathscr{L}$ is a smooth curve $C \in B$, then the Fitting support of $\mathscr{F} \otimes \mathscr{L}$ is also C since the Fitting support is equal to the schematic support in this case. Since the Fitting support is a representative of the first Chern class in $CH^1(J)$, it follows that $\det(\mathscr{F} \otimes \mathscr{L}) = \det(\mathscr{F})$. It follows that the restriction of the composition

$$G \times_B M \to \mathbf{M} \to J^\vee,$$

to the open subset $(G \times_B M)_S$ is the zero map, where the map $\mathbf{M} \to J^\vee$ is the morphism taking a sheaf $\mathscr{F}$ to $\det(\mathscr{F}) \otimes \mathcal{O}_J(2\theta)^\vee$. Since $G \times_B M$ is irreducible by Lemma 2.6.1, we conclude that the composition is the zero map.

We next show that $\sum c_2(\mathscr{F} \otimes \mathscr{L}) = 0$. If $C \in |2\theta|$ is a smooth curve and $i\colon C \to J$ denotes the inclusion, then any $(\mathscr{L}, \mathscr{F}) \in G_C \times M_C$ can be written as (i_*L, i_*F) where L and F are line bundles of degrees 0 and 2 on C respectively satisfying $\sum c_2(i_*L) = \sum c_2(i_*F) = 0$. It follows from the proof of Lemma 2.5.4

$$\sum c_2(\mathscr{F} \otimes \mathscr{L}) = \sum c_2(i_*F) + \sum c_2(i_*L) = 0.$$

It follows that the restriction of the composition

$$G \times_B M \to \mathbf{M} \to J,$$

to the open subset $(G \times_B M)_S$ is the zero map, where the map $\mathbf{M} \to J$ is the morphism taking a sheaf $\mathscr{F}$ to $\sum c_2(\mathscr{F})$. Since $G \times_B M$ is irreducible by Lemma 2.6.1, we conclude that the composition is the zero map.

We have shown that for any $C \in B$, the composition

$$G \times_B M \to \mathbf{M} \xrightarrow{a} J^\vee \times J$$

is the zero map. It follows that the map $G \times_B M \to \mathbf{M}$ factors through M as desired. $\square$

Lemma 2.6.3. *The action $G \times_B M \to M$ lifts to a unique action $G \times_B \widetilde{M} \to \widetilde{M}$.*

Proof. This follows from the fact that the resolution $\widetilde{M} \to M$ is the blowing up of a locus which is invariant under the action of the group scheme. $\square$

Lemma 2.6.4. *The actions of G on $\widetilde{M}, M$, and N have affine stabilizers.*

Proof. The proof of Lemma 3.4.4 in [dCRS21] applies to our case and shows that the action of $\mathbf{G}$ on $\mathbf{M}, \widetilde{\mathbf{M}}$, and $\mathbf{N}$ have affine stabilizers. Given a sheaf $\mathscr{F} \in M, \widetilde{M}$, or N, the stabilizer of $\mathscr{F}$ in G, is contained in the stabilizer of $\mathscr{F}$ in $\mathbf{G}$. Since the stabilizer of $\mathscr{F}$ in $\mathbf{G}$ is affine, the stabilizer of $\mathscr{F}$ in G is also affine. $\square$

Lemma 2.6.5. *(Polarizability of the Tate module) The $\overline{\mathbb{Q}_\ell}$-adic counterparts $T_{et,\overline{\mathbb{Q}_\ell}}(-)$ of the Tate module $T(-)$ associated with $G \to B$ are polarizable. The same is true if we restrict the family of curves to any subfamily of curves over a locally closed subvariety of B.*

Proof. Although stated for the family of spectral curves in the GL_n and SL_n Hitchin systems, the polarizability results in [dC17a, Theorem 3.3.1 and Theorem 4.7.2] for the relative Pic^0 and the relative Prym varieties are proved for any family of curves obtained via base change from a linear system of curves on a nonsingular surface.

We note that de Cataldo's proof of polarizability for the relative Prym variety can also be adapted to give polarizatibility for our group scheme G. The key observation is the following. For any $b \in B$, let $C_b = \sum m_{b_j} C_{b_j}^{\mathrm{red}} \subset J$ be the corresponding curve in J and let $\widetilde{C}_{b_j} \to C_{b_j}^{\mathrm{red}}$ be the normalization of the irreducible components of C_b. Let $\widetilde{\mathrm{alb}}_{b_j} \colon \mathrm{Pic}^0(\widetilde{C}_{b_j}) \to J$ be the morphism on Albanese varieties induced by the map $\widetilde{i}_{b_j} \colon \widetilde{C}_{b_j} \to J$ and let $\widetilde{a}_b := \sum m_{b_j} \widetilde{\mathrm{alb}}_{b_j}$. In view of the upcoming Lemmas 3.2.1, 3.3.3, and 3.4.11, the diagram

$$\begin{tikzcd} \mathrm{Pic}^0(C_b) \arrow[r, "\nu_b^*"] \arrow[d, "a_b"] & \mathrm{Pic}^0(\widetilde{C}_b) \arrow[dl, "\widetilde{a}_b"] \\ J & \end{tikzcd}$$

commutes. One then checks that the morphisms $\widetilde{a}_b$ and $\widetilde{i}_b^*$ play the same role as the morphisms $N_{p_a}^{ab}$ and $\widetilde{p}_a^*$ in Lemma 4.7.1 of [dC17a]. One then follows the proof of Theorem 4.7.2 in [dC17a] to conclude polarizability of the Tate module. $\square$

Lemma 2.6.6. *Let $\widetilde{M}^{lf} \subset \widetilde{M}$ and $N^{lf} \subset N$ be open subsets parameterizing stable sheaves which are pushforwards of line bundles on their schematic supports. Then*

1. *$\widetilde{M}^{lf}$ surjects onto B and is a torsor under the group scheme G,*

2. *N^{lf} surjects onto $B \setminus NR$ where NR is the locus of non-reduced curves in B and is a torsor under the restriction of G to $B \setminus NR$. Moreover, there is no open set of N which is a G-torsor over the entire base B.*

Proof. The claims follow from the upcoming Propositions 3.2.10, 3.3.8, and 3.4.14. □

Proposition 2.6.7. *The triples $(\widetilde{M}, B, G)$ and (N, B, G) are δ-regular weak Abelian fibrations which satisfy the assumptions of Set-up 2.2.5.*

Proof. By Lemma 2.6.2, Lemma 2.6.3, Lemma 2.6.4, and Lemma 2.6.5, the triples $(\widetilde{M}, B, G)$ and (N, B, G) are weak Abelian fibrations. Lemma 2.6.6 and Proposition 2.2.3 imply that the triple (M, B, G) is a δ-regular weak Abelian fibration. Since δ-regularity is a property of the group scheme G, the triple (N, B, G) is also a δ-regular weak Abelian fibration although N does not contain an open subset which is a G-torsor over the entire base B. To see that the assumptions of Set-up 2.2.5 are satisfied, note that $\widetilde{M}$, N are both smooth and projective, $B \simeq \mathbb{P}^3$ is irreducible, and $\widetilde{M}/B$, N/B, G/B all have relative dimension three. □

Chapter 3

Irreducible Components

In this chapter, we first describe Rapagnetta's stratification of the linear system $B = |2\theta|$. We then study the fibers of the Lagrangian fibrations $\widetilde{m}\colon \widetilde{M} \to B$ and $n\colon N \to B$ over each stratum, with an emphasis on describing the irreducible components. Our main results are summarized in Section 3.1 and the proofs will be carried in Sections 3.2, 3.3, and 3.4. The results in this section are the key input to the proofs of Propositions 4.1.1 and 4.2.6, which describe the top degree direct image sheaves $R^6\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$ and $R^6 n_*\mathbb{Q}_N$.

3.1 Rapagnetta's Stratification of the Linear System $|2\theta|$

Recall that $J = J(C_0)$ is the Jacobian of a genus two curve with $NS(J) \simeq \mathbb{Z}c_1(\theta)$ where θ is a symmetric theta divisor. In [Rap07], Rapagnetta stratifies the linear system $|2\theta|$ according to the analytic type of singularity by looking at the morphism

$$\phi\colon J \to |2\theta|^\vee$$

induced by the linear system $|2\theta|$, whose image, K, is a singular Kummer quartic surface in $|2\theta|^\vee$.

Proposition 3.1.1. *[Rap07, Proposition 2.1.3] Rapagnetta's stratification of the linear system $|2\theta|$ according to analytic type of singularity is the following:*

- *Stratum S: the open dense locus parameterizing smooth curves of genus 5.*

- *Stratum N(1): the locus parameterizing irreducible nodal curves singular in a unique 2-torsion point. We have* $\dim N(1) = 2$ *and* $\overline{N(1)} = \cup_{p \in J[2]} N_p$ *where* $N_p \simeq \mathbb{P}^2$ *parameterizes curves singular at* p.

- *Stratum N(2): the locus parameterizing irreducible nodal curves singular in exactly two distinct 2-torsion points. We have* $\dim N(2) = 1$ *and* $\overline{N(2)} = \cup_{p,q \in J[2], p \neq q} N_{pq}$ *where* $N_{pq} \coloneqq N_p \cap N_q \simeq \mathbb{P}^1$. *Note that there are* $\binom{16}{2} = 120$ *lines* N_{pq}.

- *Stratum N(3): the locus parameterizing irreducible nodal curves singular in exactly three distinct 2-torsion points. We have* $\dim N(3) = 0$ *and* $|N(3)| = 240$.

- *Stratum R(1): the locus parameterizing reducible curves of the form* $\theta_x + \theta_{-x}$ *where* θ_x *and* θ_{-x} *are translates of the theta divisor* θ *by* $x \in J \setminus J[2]$. *Such curves necessarily have two distinct singular points. We have* $\dim R(1) = 2$ *and* $R \coloneqq \overline{R(1)}$ *is isomorphic to the Kummer quartic* K *in* B.

- *Stratum R(2): the locus parameterizing reducible curves of the form* $\theta_x + \theta_{-x}$. *Such curves necessarily have a unique singular point belonging to* $J[2]$. *We have* $\dim R(2) = 1$ *and* $R(2) = \cup_{p \in J[2]} Q_p$, *where* Q_p *is a conic in* N_p.

- *Stratum NR: the locus parameterizing non-reduced curves of the form* $C = 2C_{\mathrm{red}}$ *where* $C_{\mathrm{red}} \simeq \theta_p$ *is the translate of the theta divisor by a 2-torsion point* $p \in J[2]$. *We have* $\dim NR = 0$ *and* $|NR| = 16$.

In particular, the locus parameterizing singular curves in B *consists of* 17 *irreducible divisors, namely the divisor* R *parameterizing reducible curves and the divisors* N_p *for* $p \in J[2]$ *parameterizing curves which are singular at* p.

Remark 3.1.2. *The following poset structure of Rapagnetta's stratification is implicit in the description given by Rapagnetta in [Rap07].*

$$\begin{tikzcd}
{}^0NR \arrow[r, hook] \arrow[dr, hook] & {}^1\overline{R(2)} \arrow[r, hook] \arrow[dr, hook] & {}^2R \arrow[r, hook] & {}^3B \\
{}^0N(3) \arrow[r, hook] & {}^1\overline{N(2)} \arrow[r, hook] & {}^2\overline{N(1)} \arrow[ur, hook] &
\end{tikzcd} \tag{3.1}$$

where the superscripts on the left denote the respective dimensions.

We end this section by stating our main results on the irreducible components of the fibers of the Lagrangian fibrations $\widetilde{m}$ and n.

Proposition 3.1.3. *The number of irreducible components of the fibers of* $\widetilde{m}\colon \widetilde{M} \to B$ *and* $n\colon N \to B$ *over each stratum are summarized by the following table. The entries in the table are the number of irreducible components in the respective fibers.*

Strata	${}^3S$	${}^2N(1)$	${}^1N(2)$	${}^0N(3)$	${}^2R(1)$	${}^1R(2)$	${}^0NR$
$\widetilde{M}_b$	*1*	*1*	*2*	*4*	*2*	*2*	*34*
N_b	*1*	*1*	*2*	*4*	*2*	*2*	*2*

where the superscripts on the left denote the respective dimensions.

Proof. This follows from the upcoming Propositions 3.2.10, 3.3.8, and 3.4.14. □

Our computation of the irreducible components of the fibrations $\widetilde{m}$ and n will also lead us to the following description of the fibers of the group scheme $g\colon G \to B$.

Proposition 3.1.4. *The fibers of the identity component of the group scheme* $g\colon G \to B$ *over the various strata, the group of connected components of the group scheme, and the* δ *values (see Section 2.6) are summarized by the following table:*

Strata	*Isomorphism Type of* G_b^0	$\pi_0(G_b)$	δ
${}^3S$	*3-dimensional Abelian variety*	*{1}*	*0*
${}^2N(1)$	$\mathbb{C}^*$*-bundle over an Abelian surface*	*{1}*	*1*
${}^1N(2)$	$(\mathbb{C}^*)^2$*-bundle over an elliptic curve*	$\mathbb{Z}/2\mathbb{Z}$	*2*
${}^0N(3)$	$(\mathbb{C}^*)^3$	$(\mathbb{Z}/2\mathbb{Z})^2$	*3*
${}^2R(1)$	$\mathbb{C}^*$*-bundle over an Abelian surface*	*{1}*	*1*
${}^1R(2)$	$\mathbb{C}$*-bundle over an Abelian surface*	*{1}*	*1*
${}^0NR$	$\mathbb{C}^3$	$(\mathbb{Z}/2\mathbb{Z})^4$	*3*

where the superscripts on the left denote the respective dimensions.

3.2 Irreducible Components Over Integral Curves

In this section, we study the irreducible components of the fibers of the Lagrangian fibrations $\widetilde{m}\colon \widetilde{M} \to B$ and $n\colon N \to B$ over the locus of integral curves, i.e. over the strata $S, N(1), N(2)$, and $N(3)$ (cf. Diagram 3.1). Since any curve C in these strata is integral, the sheaves parameterized by the fiber M_C are all stable by Remark 2.3.3. In particular, the fibers M_C and $\widetilde{M}_C$ are isomorphic, which implies that studying the fibers of $\widetilde{m}\colon \widetilde{M} \to B$ over the locus of integral curves is equivalent to studying the fibers of $m\colon M \to B$ over the locus of integral curves. The main result of this section is Proposition 3.2.10.

Throughout this section, we will work with an arbitrary Mukai vector $v = (0, 2\theta, \chi)$ with $\chi \neq 0$ to deal with the cases M and N simultaneously. Recall that $\mathbf{M}_v$ denotes the moduli space of semi-stable sheaves on J with Mukai vector v and $M_v := \mathbf{M}_v \cap \mathbf{a}_v^{-1}(\mathcal{O}_J, 0)$ where $\mathbf{a}_v\colon \mathbf{M}_v \to J^\vee \times J$ is the morphism defined in Equation 2.15.

A description of the fibers $M_{v,C}$ over the locus of integral curves was previously given by Rapagnetta in the proof of Proposition 2.1.4 in [Rap07], which we record in the following lemma. We summarize Rapagnetta's proof as it introduces ideas and notation that will be used later in the section.

Lemma 3.2.1. *Suppose $C \in B$ is an integral curve with k nodes. Let $i\colon C \to J$ be the inclusion map and $\nu\colon \widetilde{C} \to C$ be the normalization of C. Let $\widetilde{a}\colon \mathrm{Pic}^0(\widetilde{C}) \to J$ be the morphism on Albanese varieties induced by the composition $\widetilde{C} \xrightarrow{\nu} C \xrightarrow{i} J$. Then*

1. *the fiber $M_{v,C}$ is stratified by subvarieties $U(C')$ where $C' \to C$ is a partial normalization of C at r nodes,*
2. *$U(C')$ is isomorphic to a $(\mathbb{C}^*)^{(k-r)}$-bundle over some fiber of $\widetilde{a}$.*

Proof. Since C is integral, points of $\mathbf{M}_{v,C}$ are parameterized by rank one torsion free sheaves on C with Euler characteristic $\chi \neq 0$. In particular, since C is a nodal curve, it is known (see [Bea99, Section 2]) that such a sheaf on C is either a degree-$(\chi + 4)$ line bundle on C or the pushforward of a degree-$(\chi + 4 - r)$ line bundle on a partial normalization $\nu'\colon C' \to C$ desingularizing exactly r-nodes.

Let $\mathbf{U}(C') \subset \mathbf{M}_{v,C}$ be the subset parameterizing sheaves coming from a partial normalization C' and let $U(C') := \mathbf{U}(C') \cap M_{v,C}$. By the discussion above, the sub-varieties $\mathbf{U}(C')$ give a stratification of $\mathbf{M}_{v,C}$. This implies that the subvarieties $U(C')$ give a stratification of $M_{v,C}$ which proves the first claim.

To see the second claim, let $a'\colon \mathbf{U}(C') \to J$ be the restriction of the morphism $pr_2 \circ \mathbf{a}_v\colon \mathbf{M}_v \to J^\vee \times J \to J$ to $\mathbf{U}(C')$. The restriction a' associates to a line bundle L' on C' the point $\sum n_i p_i + \sum q_k$ where $\sum n_i p_i$ is the pushforward to J of a representative, in the $CH_0(\widetilde{C})$, of the first Chern class of the pullback of L' to $\widetilde{C}$ and $q_k \in J[2] \cap C$ are the points having two distinct pre-images in C'. In particular, we see that the map a' descends to a map $\widetilde{a}\colon \mathrm{Pic}^{\chi+4-r}(\widetilde{C}) \to J$.

If we fix any $[\mathscr{F}_0] \in \mathbf{U}(C')$, we can identify $\mathbf{U}(C')$ with $\mathrm{Pic}^0(C')$ and $\mathrm{Pic}^{\chi+4-r}(\widetilde{C})$ with $\mathrm{Pic}^0(\widetilde{C})$. Under this identification, the discussion above implies that there is a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}^0(C') \arrow[r, two heads, "\mu^*"] \arrow[d, two heads, "a'"] & \mathrm{Pic}^0(\widetilde{C}) \arrow[d, two heads, "\widetilde{a}"] \\ J \arrow[r, "t'"] & J \end{tikzcd} \tag{3.2}$$

where $\mu\colon \widetilde{C} \to C'$ is the normalization of C' and $t'\colon J \to J$ is translation by the point $\sum q_k \in J$. Moreover, one can check that the morphism $\widetilde{a}$ which sends, a line bundle L to the point $\sum (i \circ \nu)_* c_1(L) \in J$, can be identified with the map on Albanese varieties induced by the composition $\widetilde{C} \xrightarrow{\nu} C \xrightarrow{i} J$.

Since C' is a nodal curve with $(k-r)$-nodes, it is known that $\mathrm{Pic}^0(C')$ is a $(\mathbb{C}^*)^{(k-r)}$-bundle over $\mathrm{Pic}^0(\widetilde{C})$ (cf. [HM98, Chapter 5.B]). Diagram 3.2 then implies that $U(C') \simeq \ker(a')$ is a $(\mathbb{C}^*)^{(k-r)}$-bundle over the fiber of $\widetilde{a}$. $\square$

Corollary 3.2.2. *If $C \in B$ is an integral curve, $i\colon C \to J$ is the inclusion map, and $\nu\colon \widetilde{C} \to C$ is the normalization of C, then there are bijections between the sets of*

1. *irreducible components of $M_{v,C}$,*

2. *connected components of* $\ker(a)$*, where $a\colon \mathrm{Pic}^0(C) \to J$ is the group homomorphism sending $L \in \mathrm{Pic}^0(C)$ to $\sum c_2(i_*L)$ described in Equation 2.27,*

3. *connected components of* $\ker(\widetilde{a})$*, where $\widetilde{a}\colon \mathrm{Pic}^0(\widetilde{C}) \to J$ is the morphism on Albanese varieties induced by the composition $\widetilde{i} = i \circ \nu$.*

In particular, the number of irreducible components in the fiber $M_{v,C}$ over the locus of irreducible nodal curves is independent of the nonzero integer χ appearing in the Mukai vector v.

Proof. There is a bijection between the set of irreducible components of $M_{v,C}$ and the set of top dimensional strata. The top dimensional strata are precisely the connected components of $\ker(a)$.

The bijection between the set of connected components of $\ker(a)$ and the set of connected components of $\ker(\widetilde{a})$ follows from Diagram 3.2 with $C' = \widetilde{C}$. $\square$

In the case that $C \in S \cup N(1)$, the kernel of the map $a\colon \mathrm{Pic}^0(C) \to J$ has been studied and $\ker(a)$ is shown to be connected (cf. [MRS18, Remark 5.1 (5)]). We give a slightly more general proof of this fact which allows us to describe the fiber $\ker(a)$ when over the entire locus of integral curves.

In the remainder of this section, we will work with a fixed integral curve $C \in B$ with k nodes. Let $i\colon C \to J$ be the inclusion map and $\nu : \widetilde{C} \to C$ be the normalization of C. Note that C has arithmetic genus 5 and $\widetilde{C}$ is a smooth curve of genus $5-k$. Let $D = \phi(C)$ be the image of C in K and $\tau\colon \widetilde{D} \to D$ be the normalization of D. Note that D is an integral curve of arithmetic genus 3 with k nodes and $\widetilde{D}$ is a smooth curve of genus $3-k$.

We abuse notation and denote by $\phi\colon C \to D$ the restriction of ϕ to C. Note that the map ϕ is double cover ramified at the k nodes p_i of C. There is a commutative diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\nu"] & \widetilde{D} \arrow[d, "\tau"] \\ C \arrow[r, "\phi"] & D \end{tikzcd}$$

where the map $\widetilde{\phi}$ is exists by the universal property of normalizations and is a double cover which is ramified at the $2k$ points in $\widetilde{C}$ which lie over the nodes p_i. There are natural involutions ι on C and $\widetilde{\iota}$ on $\widetilde{C}$ which interchange the fibers of the coverings. These give rise to involutions $\sigma := \iota^*$ on $\mathrm{Pic}^0(C)$ and $\widetilde{\sigma} := \widetilde{\iota}^*$ on $\mathrm{Pic}^0(\widetilde{C})$.

We have the following lemmas which will help us describe $\ker(a)$. We note that the observation in Lemma 3.2.4 is due to Rapagnetta.

Lemma 3.2.3. *The identity component of* $\ker(\widetilde{a})$ *can be identified with the image* $\widetilde{\phi}^*(\mathrm{Pic}^0(\widetilde{D})) \subset \mathrm{Pic}^0(\widetilde{C})$*.*

Proof. If $L \in \mathrm{Pic}^0(\widetilde{D})$, then ϕ^*L is $\widetilde{\sigma}$-invariant. This implies that $\widetilde{a}(\widetilde{\phi}^*L) = -\widetilde{a}(\widetilde{\phi}^*L)$ and in particular, is a 2-torsion point of J. It follows that the image of the subset $\widetilde{\phi}^*(\mathrm{Pic}^0(\widetilde{D})) \subset \mathrm{Pic}^0(\widetilde{C})$ under $\widetilde{a}$ is contained in the finite set $J[2]$. Since $\widetilde{a}$ is continuous and $\widetilde{a}(\widetilde{\phi}^*\mathcal{O}_{\widetilde{D}}) = \widetilde{a}(\mathcal{O}_{\widetilde{C}}) = 0$, we conclude that $\widetilde{\phi}^*(\mathrm{Pic}^0(\widetilde{D})) \subset \ker(\widetilde{a})$. Since $\widetilde{\phi}^*(\mathrm{Pic}^0(\widetilde{D}))$ and $\ker(\widetilde{a})$ are both Abelian varieties of the same dimension, we must have equality. $\square$

Lemma 3.2.4. *Consider $C \in N(2)$ with nodes at p, q. Denote by p_1, p_2 the points in $\widetilde{C}$ lying over p. Then the degree 0 line bundle $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ is not the pullback of a line bundle from $\widetilde{D}$.*

Proof. The involution $\widetilde{\iota}$ defines a $\mathbb{Z}/2\mathbb{Z}$ action on $\widetilde{C}$ whose quotient is $\widetilde{D}$. Since p_1, p_2 are in the fixed locus of the involution ι on $\widetilde{C}$, the line bundle $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ is $\mathbb{Z}/2\mathbb{Z}$ invariant. This line bundle is also $\mathbb{Z}/2\mathbb{Z}$-linearizable since any $\mathbb{Z}/2\mathbb{Z}$-invariant line bundle is $\mathbb{Z}/2\mathbb{Z}$-linearizable (cf. [Dol03, Remark 7.2]). By the Kempf Descent Lemma, $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ descends to a line bundle on $\widetilde{D}$ if and only if there exists a $\mathbb{Z}/2\mathbb{Z}$-linearization of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ such that the linearized $\mathbb{Z}/2\mathbb{Z}$ action on the fibers of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ over the fixed points p_1, p_2, q_1, q_2 of ι is trivial (cf. [HL97, Proposition 4.2.15]).

One linearization of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ can be described explicitly as follows. Consider $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ as the subsheaf of the sheaf of rational functions consisting of rational functions on $\widetilde{C}$ vanishing at p_2 and having at most a simple pole at p_1. If x_i, y_i are local coordinates at p_i, q_i respectively for which the $\mathbb{Z}/2\mathbb{Z}$ action on $\widetilde{C}$ is linear, then the stalk of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ at p_1 is generated by the function $\frac{1}{x_1}$, the stalk at p_2 is generated by the function x_2 and the stalks at q_i are generated by the function 1. Since the involution ι acts by -1 in each coordinate chart, the induced action on the stalks of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ at p_i are non-trivial while the induced action on the stalks at q_i are trivial. By [KKV89, §2], the only other possible $\mathbb{Z}/2\mathbb{Z}$-linearization of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ differs from the described one by the $\mathbb{Z}/2\mathbb{Z}$ character -1. It follows that there is no $\mathbb{Z}/2\mathbb{Z}$-linearization of $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ satisfying the conditions of Kempf's Descent Lemma and we conclude that $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ is not the pullback of a line bundle from $\widetilde{D}$. □

We have the following useful corollary.

Corollary 3.2.5. *The line bundle $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ does not lie in the identity component of* $\ker(\widetilde{a})$.

Proof. By Lemma 3.2.3, the identity component of $\widetilde{a}^{-1}(0)$ can be identified with $\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D})$. Since the line bundle $\mathcal{O}_{\widetilde{C}}(p_1 - p_2)$ does not descend to a line bundle on $\widetilde{D}_b$ by Lemma 3.2.4, it does not lie in the identity component of $\ker(\widetilde{a})$. □

We now observe that there is an involution, σ, on the moduli space $\mathbf{M}_v$. This involution is induced by pullback via the -1 involution on the Abelian surface J sending x to $-x$, using the fact that θ is symmetric. In particular, we have

$$\sigma\colon \mathbf{M}_v \to \mathbf{M}_v; \quad [\mathscr{F}] \mapsto [(-1)^*\mathscr{F}] \tag{3.3}$$

When the Mukai vector $v = (0, 2\theta, 2\chi')$ with $\chi' \neq 0$ is even, we have the following lemma.

Lemma 3.2.6. *Let $v = (0, 2\theta, 2\chi')$ be a Mukai vector with $\chi' \neq 0$ and let $\sigma\colon \mathbf{M}_v \to \mathbf{M}_v$ be the involution described above. The fixed locus of σ contains M_v.*

Proof. If $C \in S \subset B$ is a smooth curve in J, then it is an étale double cover of its image $D := \phi(C)$ in K. Let L be a degree-$(\chi'+2)$ line bundle on D. Pulling back this line bundle to C via the étale double cover gives a line bundle of degree $2\chi' + 4$ on C which is -1^*-invariant. Pushing forward this line bundle on C to J determines a stable sheaf in $M_v \subset \mathbf{M}_v$ which is σ-invariant. Since, $\mathrm{Pic}^{\chi'+2}(D)$ is three-dimensional, it follows that a six-dimensional algebraic subset of M_v is fixed by σ. Since $\dim M_v = 6$, all of M_v is fixed by σ by closure of the fixed locus. $\square$

Remark 3.2.7. *An important observation is that the involution σ on $\mathrm{Pic}^0(C)$ described above is precisely the restriction of the involution σ on the moduli space of semi-stable sheaves described in Equation 3.3 to the locus parameterizing line bundles in a fiber over C. In particular, Lemma 3.2.6 implies that $a^{-1}(0) \subset \mathrm{Fix}(\sigma)$.*

We now study the fixed locus of σ. The main tool is the theory of Prym varieties for the branched double covers $C \to D$ and $\widetilde{C} \to \widetilde{D}$. The theory in the smooth case was developed by Mumford in [Mum74] and was extended to the nodal case by Beauville in [Bea77]. In the nodal case, there is still a norm map $\mathrm{Nm}\colon \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(D)$ (see [EGA II, §6.5]) and Beauville defines the Prym variety P as identity component of the kernel of the norm map. Beauville shows that P is an Abelian variety of dimension $p_a(D) - 1 = 2$. As in the classical case, the Prym variety can alternatively be defined as the image of the map $1 - \sigma\colon \mathrm{Pic}^0(\widetilde{C}) \to \mathrm{Pic}^0(\widetilde{C})$.

Lemma 3.2.8. $\mathrm{Fix}(\sigma) = \ker(1 - \sigma)$ *is connected if $k = 0, 1$ and has 2^{k-1} connected components if $k = 2, 3$. Each connected component is isomorphic to a $(\mathbb{C}^*)^k$-bundle over* $\mathrm{Pic}^0(\widetilde{D})$.

Proof. Consider the commutative diagram of short exact sequences (note that the ones have been omitted from the columns)

$$\begin{tikzcd}
 & & \ker(1-\sigma) \arrow[d, hook] & \ker(1-\widetilde{\sigma}) \arrow[d, hook] & \\
1 \arrow[r] & (\mathbb{C}^*)^k \arrow[r] \arrow[d] & \mathrm{Pic}^0(C) \arrow[r, "\nu^*"] \arrow[d, two heads, "1-\sigma"] & \mathrm{Pic}^0(\widetilde{C}) \arrow[r] \arrow[d, two heads, "1-\widetilde{\sigma}"] & 1 \\
1 \arrow[r] & \ker(\alpha) \arrow[r] & (1-\sigma)\mathrm{Pic}^0(C) \arrow[r, "\alpha"] & (1-\widetilde{\sigma})\mathrm{Pic}^0(\widetilde{C}) \arrow[r] & 1,
\end{tikzcd}$$

where α is the restriction of ν^* to the subgroup $(1-\sigma)\mathrm{Pic}^0(C)$. As discussed above, $(1-\widetilde{\sigma})\mathrm{Pic}^0(\widetilde{C})$ is the Prym variety $\widetilde{P}$ and $(1-\sigma)\mathrm{Pic}^0(C)$ is the Prym variety P. The Prym varieties P and $\widetilde{P}$ are both Abelian varieties of dimension 2.

When $k=0$, $C=\widetilde{C}$ and the double cover $\phi\colon C\to D$ is unramified. In this case, the pullback $\phi^*\colon \mathrm{Pic}^0(D)\to\mathrm{Pic}^0(C)$ is not injective and $\ker(\phi^*)\simeq\mathbb{Z}/2\mathbb{Z}$. Mumford proves that

$$\ker(1-\sigma)\simeq\phi^*\mathrm{Pic}^0(D)$$

and it follows that $\ker(1-\sigma)$ is connected (see [Mum74, §2 Data II (vi)]).

When $k=1,2,3$, the double cover $\widetilde{\phi}\colon\widetilde{C}\to\widetilde{D}$ is ramified and the pullback $\widetilde{\phi}^*\colon\mathrm{Pic}^0(\widetilde{D})\to\mathrm{Pic}^0(\widetilde{C})$ is injective. Mumford proves that (loc. cit.)

$$\ker(1-\widetilde{\sigma})\simeq\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D})\times(\mathbb{Z}/2\mathbb{Z})^{2k-2}.$$

Beauville proves that the induced map $\alpha\colon P\to\widetilde{P}$ is an isogeny of degree 2^{k-1} (see [Bea77, Remark 3.6]). In particular, $\ker(\alpha)$ is a finite group of order 2^{k-1}.

Since $\ker(\alpha)$ is a finite group, there is no nontrivial group homomorphism $\mathbb{C}^*\to\ker(\alpha)$ and the snake lemma implies that there is an exact sequence

$$1\to(\mathbb{C}^*)^k\to\ker(1-\sigma)\to\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D})\times(\mathbb{Z}/2\mathbb{Z})^{2k-2}\to\ker(\alpha)\to 1.$$

Since $\ker(\alpha)$ is a quotient of $(\mathbb{Z}/2\mathbb{Z})^{2k-2}$ of order 2^{k-1}, there is a short exact sequence

$$1\to(\mathbb{C}^*)^k\to\ker(1-\sigma)\to\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D})\times(\mathbb{Z}/2\mathbb{Z})^{k-1}\to 1,$$

and the lemma is shown. □

Lemma 3.2.9. *For $k=1,2,3$, $\ker(a)=\mathrm{Fix}(\sigma)$.*

Proof. For $k=1,2,3$, the inclusion $\ker(a)\subseteq\mathrm{Fix}(\sigma)$ was discussed in Remark 3.2.7. We will show the reverse inclusion case by case.

If $k=1$, then $\mathrm{Fix}(\sigma)$ is connected by Lemma 3.2.8 which implies that $\ker(a)=\mathrm{Fix}(\sigma)$.

If $k=2$, note that by Equation 3.2, $\ker(a)$ is a $(\mathbb{C}^*)^k$-bundle over $\ker(\widetilde{a})$. Lemma 3.2.5 implies that $\ker(\widetilde{a})$, and hence $\ker(a)$, is not connected. Since $\mathrm{Fix}(\sigma)$ has exactly two connected components by Lemma 3.2.8, we conclude that $\ker(a)=\mathrm{Fix}(\sigma)$.

If $k=3$, then Rapagnetta shows in [Rap07, Proposition 2.1.4] that $\ker(a)$ has four connected components. Since $\mathrm{Fix}(\sigma)$ also has four connected components by Lemma 3.2.8, we conclude that $\ker(a)=\mathrm{Fix}(\sigma)$. □

We summarize the results of this section in the following Proposition.

Proposition 3.2.10. *If $C \in B$ is an integral curve with k nodes, then:*

1. *the fiber G_C is isomorphic to a $(\mathbb{C}^*)^k$-bundle over $\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D})$ if $k = 0, 1$ and is isomorphic to a $(\mathbb{C}^*)^k$-bundle over $\widetilde{\phi}^*\mathrm{Pic}^0(\widetilde{D}) \times (\mathbb{Z}/2\mathbb{Z})^{k-1}$ if $k = 2, 3$,*

2. *the fiber $\widetilde{M_C}$ is irreducible of dimension three if $k = 0, 1$, has 2^{k-1} irreducible components of dimension three if $k = 2, 3$, is reduced, Cohen-Macaulay, and has an open dense subset parameterizing line bundles of degree 2,*

3. *the fiber N_C is irreducible of dimension three if $k = 0, 1$, has 2^{k-1} irreducible components of dimension three if $k = 2, 3$, is reduced, Cohen-Macaulay, and has an open dense subset parameterizing line bundles of degree 1.*

Proof. The statements about G_C and the irreducible components follow from Lemma 3.2.8, and Lemma 3.2.9. We now argue that $\widetilde{M_C}$ is reduced. First recall that $\widetilde{M_C} \simeq M_C$ since every point in M_C is a stable sheaf. Since the Le Potier support morphism is smooth at every point corresponding to a line bundle [LeP93, p. 24], the fiber M_C is reduced on the open dense subset M_C^{lf}. By Remark 2.4.4, the fiber M_C is Cohen-Macaulay and thus, is reduced everywhere. The same argument shows that N_C is reduced. $\square$

3.3 Irreducible Components Over Reducible Curves

In this section, we study the irreducible components of the fibers of the Lagrangian fibrations $\widetilde{m}\colon \widetilde{M} \to B$ and $n\colon N \to B$ over the locus of reduced, but reducible curves, i.e. over $R(1) \cup R(2)$ (cf. Diagram 3.1). Unlike the case of integral curves, the fibers of $\widetilde{M}$ and N will differ over this locus. The fibers of $\widetilde{M}$ over this locus have already been described by Rapagnetta in [Rap07, Proposition 2.1.4]. We give a slight generalization of Rapagnetta's proof which will allow us to also describe the fibers of N. We summarize the results of this section in the following proposition.

Proposition 3.3.1. *If $C \in R(1) \cup R(2)$, then the fiber*

1. *G_C is isomorphic to a $\mathbb{C}^*$-bundle over the Abelian surface J if $C \in R(1)$ and is isomorphic to a $\mathbb{C}$-bundle over J if $C \in R(2)$;*

2. *$\widetilde{M}_C$ has two irreducible components of dimension three and has a non-dense open subset parameterizing line bundles on C of bidegree $(1,1)$;*

3. *N_C has two irreducible components of dimension three, is reduced, and has an open dense subset parameterizing line bundles on C of bidegree $(0,1)$ and $(1,0)$.*

Proof. The first two statements are proved in Proposition 3.3.8 and the third statement is proved in Corollary 3.3.10. □

We initially work with an arbitrary Mukai vector $v = (0, 2\theta, \chi)$ with $\chi \neq 0$ and will later deal with the cases $\widetilde{M}$ and N separately. Again, recall that $\mathbf{M}_v$ denotes the moduli space of semi-stable sheaves on J with Mukai vector v, $M_v := \mathbf{M}_v \cap \mathbf{a}^{-1}(\mathcal{O}_J, 0)$ where $\mathbf{a}\colon \mathbf{M}_v \to J^\vee \times J$ is the morphism defined in Equation 2.15.

Fix $C \in R(1) \cup R(2)$ and recall that C is reduced, but reducible and can be expressed as $C = \theta_x + \theta_{-x}$ for some $x \in J \setminus J[2]$. Let $i_x\colon C_0 \to J$ and $i_{-x}\colon C_0 \to J$ denote the respective embeddings of the fixed genus 2 curve C_0 into J with images θ_x and θ_{-x}. We begin with the following lemma.

Lemma 3.3.2. *Suppose $\mathscr{F}$ is a stable sheaf on J with Fitting support $\theta_x + \theta_{-x}$. Let L_1 and L_2 be the torsion free parts of $\mathscr{F}|_{\theta_x}$ and $\mathscr{F}|_{\theta_{-x}}$ respectively. Then there is an exact sequence*

$$0 \to \mathscr{F} \xrightarrow{\alpha} i_{x*}L_1 \oplus i_{-x*}L_2 \xrightarrow{\beta} Q \to 0. \tag{3.4}$$

where Q is the defined as the cokernel of α and the restriction of β to each summand is surjective. Moreover,

1. *if $\chi(\mathscr{F}) = 2\chi'$ is even, then $\chi(Q) = 2$ and $\deg(L_1) = \deg(L_2) = \chi' + 2$;*

2. *if $\chi(\mathscr{F}) = 2\chi' + 1$ is odd, then either*

   (a) $\chi(Q) = 2$, $\deg(L_1) = \chi' + 2$, $\deg(L_2) = \chi' + 3$, *or*

   (b) $\chi(Q) = 2$, $\deg(L_1) = \chi' + 3$, $\deg(L_2) = \chi' + 2$, *or*

   (c) $\chi(Q) = 1$, $\deg(L_1) = \deg(L_2) = \chi' + 2$.

Proof. There are natural surjective maps from $\mathscr{F}$ to $i_{x*}L_1$ and $i_{-x*}L_2$. The map α is the direct sum of these two maps. Away from $\theta_x \cap \theta_{-x}$, α is an isomorphism. Since both components of α are surjective, Q is a quotient of both $i_{x*}L_1$ and $i_{-x*}L_2$. In particular, Q must be a quotient of $\mathcal{O}_{\theta_x \cap \theta_{-x}}$. Since

$\mathscr{F}$ is stable, $\chi(\mathscr{F}) < 2\chi(L_i)$ for $i = 1, 2$ by Equation 2.8. In particular, the Riemann-Roch formula says that

$$\deg(L_i) > \chi(\mathscr{F})/2 + 1. \tag{3.5}$$

Taking Euler characteristics in Equation 3.4 gives $\chi(Q) = \deg(L_1) + \deg(L_2) - 2 - \chi(\mathscr{F})$. Since $\chi(Q) \leq \chi(\mathcal{O}_{\theta_x \cap \theta_{-x}}) = 2$,

$$\deg(L_1) + \deg(L_2) \leq 4 + \chi(\mathscr{F}). \tag{3.6}$$

Combining Equations 3.5 and 3.6 then gives the result. □

Lemma 3.3.3. *Suppose $\mathscr{F}$ is a stable sheaf on J with Fitting support $\theta_x + \theta_{-x}$ which sits in a short exact sequence*

$$0 \to \mathscr{F} \xrightarrow{\alpha} i_{x*}L_1 \oplus i_{-x*}L_2 \to Q \to 0.$$

for some line bundles L_1, L_2 on C_0 of degrees d_1, d_2 respectively. If $\sum c_2(\mathscr{F}) = 0$, then L_2 is uniquely determined by L_1 and x.

Proof. Using multiplicativity of the Chern polynomial and the description of the intersection points of $\theta_x + \theta_{-x}$ given in Proposition 3.1.1, we compute that

$$\sum c_2(i_{x*}L_1 \oplus i_{-x*}L_2) = \sum c_2(i_{x*}L_1) + \sum c_2(i_{-x*}L_2)$$

Using multiplicativity of the Chern polynomial and the fact that $c_1(Q) = 0$ since Q is a skyscraper sheaf, we see that if $\sum c_2(\mathscr{F}) = 0$, then

$$\sum c_2(i_{x*}L_1) + \sum c_2(i_{-x*}L_2) = \sum c_2(Q). \tag{3.7}$$

Now note that for any $x \in J$, the map $\mathrm{Pic}^0(C_0) \to J$ sending a degree zero line bundle L to the point $\sum c_2(i_{x*}L)$ can be identified with the map on Albanese varieties induced by the inclusion $i_x \colon C_0 \to J$ and thus, is an isomorphism. After identifying $\mathrm{Pic}^{d_i}(C_0)$ with $\mathrm{Pic}^0(C_0)$ via the fixed Weierstrass point w_0, we can use this isomorphism to see that for a given $L_1 \in \mathrm{Pic}^{d_1}(C_0)$, there is a unique $L_2 \in \mathrm{Pic}^{d_2}(C_0)$ satisfying Equation 3.7. □

With these lemmas, we can now describe the fibers $M_{v,C}$ over $C \in R(1) \cup R(2)$. As mentioned at the beginning of the section, the fibers $M_{v,C}$ behave differently depending on the parity of the Mukai vector. We begin with the case when the Mukai vector is even.

3.3.1 The Even Mukai Vector Case

Fix an even Mukai vector of the form $v = 2v'$ with $v' = (0, \theta, \chi')$ with $\chi' \neq 0$. The description of the singular locus $\Sigma_v \subset M_v$ implies that the fiber $M_{v,C}$ contains sheaves which are strictly semi-stable since $C = \theta_x + \theta_{-x}$ is reducible. We begin with a description of the strictly semi-stable locus $M^{ss}_{v,C} \subset M_{v,C}$.

Lemma 3.3.4. *Fix an even Mukai vector of the form $v = 2v'$ with $v' = (0, \theta, \chi')$ with $\chi' \neq 0$. Then the strictly semi-stable locus $M^{ss}_{v,C} \subset M_{v,C}$ is isomorphic to the Abelian surface J.*

Proof. Recall that the strictly semi-stable sheaves in $M_{v,C}$ are of the form

$$\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2,$$

where $\mathscr{F}_1$ and $\mathscr{F}_2$ are stable sheaves with Mukai vector $(0, \theta, \chi')$, satisfying $\sum c_2(\mathscr{F}) = 0$. In particular, this implies that $\mathscr{F}_1 \simeq i_{x*}L_1$ and $\mathscr{F}_2 \simeq i_{-x*}L_2$ where L_1 and L_2 are degree-$(\chi' + 1)$ line bundles on C_0. Lemma 3.3.3, with $Q = 0$, implies that for a fixed line bundle L_1 on C_0, there is a unique line bundle L_2 such that $\sum c_2(\mathscr{F}) = 0$. This implies that the strictly semi-stable locus $M^{ss}_{v,C}$ is isomorphic to J. $\square$

We now describe the strictly stable locus $M^s_{v,C} \subset M_{v,C}$.

Lemma 3.3.5. *The locus of strictly stable sheaves $M^s_{v,C}$ is parameterized by isomorphism classes of kernels of surjective maps $\beta\colon i_{x*}L_1 \oplus i_{-x*}L_2 \to \mathcal{O}_{\theta_x \cap \theta_{-x}}$ where L_1 and L_2 are degree-$(\chi'+2)$ line bundles on C_0 satisfying $\sum c_2(i_{x*}L_1) + c_2(i_{-x*}L_2) = 0$ and the restriction of β to each summand is already surjective.*

Proof. If $\mathscr{F} \in M_{v,C}$ is stable, then by Lemma 3.3.2, $\mathscr{F}$ sits in a short exact sequence

$$0 \to \mathscr{F} \xrightarrow{\alpha} i_{x*}L_1 \oplus i_{-x*}L_2 \xrightarrow{\beta} \mathcal{O}_{\theta_x \cap \theta_{-x}} \to 0,$$

with $\deg(L_1) = \deg(L_2) = \chi' + 2$ and the restriction of β to each summand is already surjective. Lemma 3.3.3 implies that given $L_1 \in \mathrm{Pic}^{\chi'+2}(C_0))$, there exists a unique $L_2 \in \mathrm{Pic}^{\chi'+2}(C_0)$ such that the kernel $\mathscr{F}$ is in $M^s_{v,C}$.

Conversely, given such an $L_1, L_2 \in \mathrm{Pic}^{\chi'+2}(C_0)$, any kernel $\mathscr{F}$ of a surjective map $i_{x*}L_1 \oplus i_{-x*}L_2 \to \mathcal{O}_{\theta_x \cap \theta_{-x}}$ is seen to be stable using Equation 2.7. Moreover, given two such kernels, they are isomorphic if and only if they differ by an automorphism of $i_{x*}L_1 \oplus i_{-x*}L_2$. $\square$

Remark 3.3.6. *As mentioned by Rapagnetta in the proof of Proposition 2.1.4 in [Rap07], for a fixed pair of line bundles L_1, L_2, the isomorphism classes of kernels are parameterized by $\mathbb{C}^*$ if $C \in R(1)$ and by $\mathbb{C}$ if $C \in R(2)$.*

Lemma 3.3.7. *Fix an even Mukai vector of the form $v = 2v'$ with $v' = (0, \theta, \chi')$ with $\chi' \neq 0$.*

1. *If $C \in R(1)$, then the strictly stable locus $M^s_{v,C} \subset M_{v,C}$ is isomorphic to a $\mathbb{C}^*$-bundle over the Abelian surface J.*

2. *If $C \in R(2)$, then the strictly stable locus $M^s_{v,C} \subset M_{v,C}$ is isomorphic to a $\mathbb{C}$-bundle over the Abelian surface J.*

Proof. By Lemma 3.3.5, the strictly stable locus $M^s_{v,C} \subset M_{v,C}$ is parameterized by isomorphism classes of kernels of of surjective maps $\beta\colon i_{x*}L_1 \oplus i_{-x*}L_2 \to \mathcal{O}_{\theta_x \cap \theta_{-x}}$ where L_1 and L_2 are degree-$(\chi' + 2)$ line bundles on C_0, L_2 is uniquely determined by L_1, and the restriction of β to each summand is already surjective. By Remark 3.3.6, if $C \in R(1)$, isomorphism classes of such kernels are parameterized by $\mathbb{C}^*$ and if $C \in R(2)$, isomorphism classes of such kernels are parameterized by $\mathbb{C}$. $\square$

We conclude this section by describing the fibers G_C, M_C, and $\widetilde{M}_C$.

Proposition 3.3.8. *If $C \in R(1) \cup R(2)$, then*

1. *the fiber G_C is isomorphic to a $\mathbb{C}^*$-bundle over J if $C \in R(1)$ and G_C is isomorphic to a $\mathbb{C}$-bundle over J if $C \in R(2)$,*

2. *the fiber M_C is irreducible, reduced, Cohen-Macaulay of dimension three, and has an open dense subset parameterizing line bundles whose restriction to each component has degree 1.*

3. *the fiber $\widetilde{M}_C$ has two irreducible components.*

Proof. By definition, the fibers G_C is contained in the stable locus $M^s_{v,C}$ where $v = (0, 2\theta, -4)$ is an even Mukai vector. Conversely, given a stable sheaf $\mathscr{F} \in M^s_{v,C}$, $\mathscr{F}$ is the kernel of a surjective map $i_{x*}L_1 \oplus i_{-x*}L_2 \to i_*\mathcal{O}_{\theta_x \cap \theta_{-x}}$ by Lemma 3.3.5. It follows that every fiber of $\mathscr{F}$ has rank one, which implies that $\mathscr{F}$ is the pushforward of a line bundle on its support. In particular, we see that G_C can be identified with the stable locus $M_{v,C}$. Lemma 3.3.7 then gives the first claim.

Noting that M_C is a fiber of the form $M_{v,C}$ where $v = (0, 2\theta, -2)$ is an even Mukai vector, following the same argument as above shows that the stable locus M^s_C can be identified with the locus parameterizing sheaves which are the pushforwards of line bundles on C. Lemma 2.5.2 implies that the restriction of the line bundle to either component of C has degree one. Lemmas 3.3.7 and

3.3.4 together then imply that the stable locus M_C^s is connected and dense in M_C. The proof of reducedness and Cohen-Macaulayness follow from the same argument as in Proposition 3.2.10.

To determine the number of irreducible components for $\widetilde{M}_C$, recall from Remark 2.4.1 that the restriction of the symplectic desingularization map $\pi\colon \widetilde{M} \to M$ to Σ is a $\mathbb{P}^1$-bundle outside of the 256 points in Ω where the fibers are smooth three-dimensional quadrics. Moreover, recall that the strictly semi-stable locus $M_C^{ss} = M_C \cap \Sigma \simeq J$ by Lemma 3.3.4. It follows that $\widetilde{M}_C$ consists of two irreducible components, namely the strict transform of M_C under the symplectic resolution and the exceptional divisor restricted to the fiber over C, which is isomorphic to a $\mathbb{P}^1$-bundle over J. □

3.3.2 The Odd Mukai Vector Case

When the Mukai vector v is odd, i.e. $v = (0, 2\theta, \chi)$ with $\chi = 2\chi' + 1$, every sheaf in $M_{v,C}$ is stable. As in the even Mukai vector case, the stable locus can be characterized in terms of kernels of certain surjective maps. The proof of the following lemma is analogous to the proof of Lemma 3.3.5 and will be omitted.

Lemma 3.3.9. *The locus of strictly stable sheaves $M_{v,C}^s$ is parameterized by isomorphism classes of kernels of surjective maps $\beta\colon i_{x*}L_1 \oplus i_{-x*}L_2 \to Q$ where L_1 and L_2 line bundles on C_0 with either*

1. $Q = \mathcal{O}_{\theta_x \cap \theta_{-x}}$, $\deg(L_1) = \chi' + 2$, *or* $\deg(L_2) = \chi' + 3$,
2. $Q = \mathcal{O}_{\theta_x \cap \theta_{-x}}$, $\deg(L_1) = \chi' + 3$, *or* $\deg(L_2) = \chi' + 2$,
3. $Q = \mathcal{O}_p$ *for some* $p \in J$, $\deg(L_1) = \deg(L_2) = \chi' + 2$.

Moreover, L_2 is uniquely determined by L_1 and the restriction of β to each summand is already surjective.

Proposition 3.3.10. *If $C \in R(1) \cup R(2)$, then the fiber N_C has two irreducible components, is reduced, Cohen-Macaulay of dimension three, and has an open dense subset parameterizing line bundles.*

Proof. Since N parameterizes sheaves with odd Euler characteristic, Lemma 3.3.9 implies that a sheaf in the fiber N_C corresponds to an isomorphism class of a kernel of a surjective map $\beta\colon i_{x*}L_1 \oplus i_{-x*}L_2 \to Q$. By Remark 3.3.6, isomorphism classes of such kernels in cases (1) and (2) are parameterized by a $\mathbb{C}^*$-bundle over the Abelian surface J in $C \in R(1)$ and by a $\mathbb{C}$-bundle over the Abelian surface J if $C \in R(2)$.Finally, if $C \in R(1) \cup R(2)$, the isomorphism classes of kernels in case (3) of the lemma are parameterized by just the Abelian surface J.

If $\mathscr{F} \in N$ is a kernel of type (1) or (2), then every fiber of $\mathscr{F}$ has rank one. This implies that $\mathscr{F}$ is the pushforward of a line bundle on its support. Conversely, Lemmas 3.3.9 and 2.5.2 imply that the every sheaf which is the pushforward of a line bundle is either a kernel of type (1) or (2). It follows that the locus in N_C parameterizing pushforwards of line bundles consists of two connected components and is dense in N_C. The proof of reducedness and Cohen-Macaulayness follow from the same argument as in the proof of Proposition 3.2.10. □

3.4 Irreducible Components over Non-Reduced Curves

In this section, we study the irreducible components of the fibers of the Lagrangian fibrations $\widetilde{m}\colon \widetilde{M} \to B$ and $n\colon N \to B$ over the locus of non-reduced curves, i.e. over the stratum NR (cf. Diagram 3.1). We remark that the analysis in this section is simpler than the analysis over the non-reduced locus in the $OG10$ case since the underlying reduced curves in our case are always smooth.

As in the case of reduced, but reducible curves, the fibers of $\widetilde{M}$ and N will differ over the locus of non-reduced curves. The fibers of $\widetilde{M}$ over this locus have already been described by Rapagnetta in [Rap07, Proposition 2.1.4]. We give a slight generalization of Rapagnetta's proof which will allow us to also describe the fibers of N over this locus. The main result of this section is Proposition 3.4.14.

We initially work with an arbitrary Mukai vector $v = (0, 2\theta, \chi)$ with $\chi \neq 0$ and will later deal with the cases $\widetilde{M}$ and N separately. Again, recall that $\mathbf{M}_v$ denotes the moduli space of semi-stable sheaves on J with Mukai vector v, $M_v \coloneqq \mathbf{M}_v \cap \mathbf{a}_v^{-1}(\mathcal{O}_J, 0)$ where $\mathbf{a}_v\colon \mathbf{M}_v \to J^\vee \times J$ is the morphism defined in Equation 2.15.

Fix $C \in NR$ and recall that $C = 2C_{\text{red}}$ where $C_{\text{red}} = \theta_p$ for some $p \in J[2]$. Let $\mathcal{I} \subset \mathcal{O}_C$ be the ideal sheaf C_{red} in C. Then $\mathcal{I}^2 = 0$, $\mathcal{I}$ is an $\mathcal{O}_{C_{\text{red}}}$-module, $\mathcal{I} \simeq \mathcal{O}_{C_{\text{red}}}(-C_{\text{red}}) \simeq \omega^\vee_{C_{\text{red}}}$, and $\deg(\mathcal{I}) = -2$.

Remark 3.4.1. *Consider two curves $C = 2\theta_p, C' = 2\theta_{p'} \in NR$ for some $p, p' \in J[2]$. Translation by the 2-torsion point $p+p'$ on J induces an automorphism of $B = |2\theta|$ which takes C to C' (cf. [Keu99, §5]). This induces an isomorphism* *between the fibers $M_{v,C}$ and $M_{v,C'}$ for any Mukai vector of the form $(0, 2\theta, \chi)$ with $\chi \neq 0$.*

As in the case of O'Grady 10, the sheaves parameterized by the fibers $M_{v,C}$ can be divided into two types, namely sheaves of type I and type II (see [dCRS21, §4.2.3]). We recall the definitions below.

Definition 3.4.2. *A $\mathscr{F}$ be sheaf on J with Fitting support $C = 2\theta_p$ is said to be of type I if the composition of the natural morphisms $\mathcal{O}_J \to \mathcal{O}_C \to \mathcal{E}nd_J(\mathscr{F})$ factors via the natural surjection $\mathcal{O}_J \to \mathcal{O}_{C_{\mathrm{red}}}$. If this does not hold, the sheaf $\mathscr{F}$ is said to be of type II. The locus of sheaves in the fiber $M_{v,C}$ of type I will be denoted by $M^I_{v,C}$ and the locus of sheaves in the fiber $M_{v,C}$ of type II will be denoted by $M^{II}_{v,C}$. Note that by definition, $M_{v,C} = M^I_{v,C} \sqcup M^{II}_{v,C} = M^I_{v.C} \cup \overline{M^{II}_{v,C}}$.*

In particular, sheaves of type I are sheaves on J with schematic support C_{red} and sheaves of type II are sheaves on J with schematic support C.

In what follows, we will need the following proposition from [dCRS21].

Lemma 3.4.3. *[dCRS21, Proposition 4.3.8] Let $C = 2C_{\mathrm{red}} \in NR$ and suppose $\mathscr{F} \in M_{v,C}$ is a stable sheaf of type II. Let $F := \mathscr{F}|_{C_{\mathrm{red}}}/T$ where $T := Tors(\mathscr{F}|_{C_{\mathrm{red}}})$ is the torsion subsheaf of $\mathscr{F}|_{C_{\mathrm{red}}}$. Then there is a commutative diagram of short exact sequences (note that the zeros have been omitted from the vertical ones):*

$$\begin{tikzcd}
 & & & T \arrow[d, hook] & \\
0 \arrow[r] & F \otimes_{C_{\mathrm{red}}} \mathcal{I} \arrow[r] \arrow[d, hook] & \mathscr{F} \arrow[r] \arrow[d, "id"] & \mathscr{F}|_{C_{\mathrm{red}}} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d, two heads] & \mathscr{F} \arrow[r] & i_{\mathrm{red}*}F \arrow[r] & 0 \\
 & T & & &
\end{tikzcd} \tag{3.8}$$

where $T = 0$ if $\chi(\mathscr{F})$ is even, or $T = \mathcal{O}_x$ for a point $x \in C_{\mathrm{red}}$ if $\chi(F)$ is odd.

Proof. Although stated for double curves in a K3 surface, the proof of Proposition 4.3.8 in [dCRS21] applies in our case since it only relies on the notion of stability for sheaves of type II and the double structure of the curve. □

In particular, this implies that N_C does not contain any sheaves which are the pushforwards of line bundles on C.

Corollary 3.4.4. *Let $\mathscr{F}$ be a sheaf with Fitting support $C = 2C_{\mathrm{red}} \in NR$ with odd Euler characteristic. Then $\mathscr{F}$ is not the pushforward of a line bundle on C.*

Proof. If $\mathscr{F}$ is of type I, then $\mathscr{F}$ cannot be locally free on its schematic support. If $\mathscr{F}$ is of type II, then by Lemma 3.4.3, $\mathscr{F}|_{C_{red}}$ is not locally free since $\chi(\mathscr{F})$ is odd. Thus, $\mathscr{F}$ cannot be the pushforward of a line bundle on C. $\square$

The sheaf K appearing in Equation 3.8 can be described more explicitly as follows.

Lemma 3.4.5. *Let $\mathscr{F}$ be a stable sheaf in $M_{v,C}$ of type II. Let F, T, and K be as in Lemma 3.4.3.*

1. *If $\chi(\mathscr{F}) = 2\chi'$ is even, then F is a line bundle of degree $\chi' + 2$ on C_{red} and $K \simeq i_{\mathrm{red}*}(F \otimes \mathcal{I})$.*

2. *If $\chi(\mathscr{F}) = 2\chi' + 1$ is odd, then F is a line bundle of degree $\chi' + 2$ on C_{red} and $K \simeq i_{\mathrm{red}*}(F \otimes \mathcal{I} \otimes \mathcal{O}_{C_{\mathrm{red}}}(x))$ where $x \in C_{\mathrm{red}}$ is a point.*

Proof. Since K is a subsheaf of $\mathscr{F}$, K is pure of dimension one. The second row of Equation 3.8 implies that $c_1(K) = c_1(\mathscr{F}) - c_1(i_{\mathrm{red}*}F) = C_{\mathrm{red}}$. Remark 2.3.2 implies that the Fitting support of K is C_{red}. Remark 2.3.3 implies that K is the pushforward of a rank one torsion free sheaf on C_{red}. Since C_{red} is smooth, K must be the pushforward of a line bundle on C_{red}.

If $\chi(\mathscr{F}) = 2\chi'$ is even, Lemma 3.4.3 implies that $T = 0$. The left column of Equation 3.8 then implies that $K \simeq F \otimes_{C_{\mathrm{red}}} \mathcal{I}$.

If $\chi(\mathscr{F}) = 2\chi' + 1$ is odd, Lemma 3.4.3 implies that $T = \mathcal{O}_x$ for some $x \in C_{\mathrm{red}}$. Twisting the ideal sheaf sequence for x by the line bundle $i^*_{\mathrm{red}}K$ give the short exact sequence

$$0 \to \mathcal{O}_{C_{\mathrm{red}}}(-x) \otimes i^*_{\mathrm{red}}K \to i^*_{\mathrm{red}}K \to \mathcal{O}_x \to 0.$$

Pushing forward by the closed embedding i_{red} and using left column of Equation 3.8, we see that $K \simeq i_{\mathrm{red}*}(F \otimes \mathcal{I} \otimes \mathcal{O}_{C_{\mathrm{red}}}(x))$ as desired. $\square$

Lemma 3.4.6. *Let $\mathscr{F}$ be a sheaf on J with Fitting support C. Suppose $\mathscr{F}$ sits in an exact sequence of $\mathcal{O}_C$ modules*

$$0 \to K \to \mathscr{F} \to F \to 0$$

where K, F are line bundles on C_{red}. The $\mathscr{F}$ is a sheaf of type II if and only if the short exact sequence determines a point in $\mathbb{P}(\mathrm{Ext}^1_C(F, K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F, K))$.

Proof. The spectral sequence for the change of coefficient in the Ext groups [Moz07, Corollary 3.2.2] gives the exact sequence

$$0 \to \mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K) \to \mathrm{Ext}^1_C(F,K) \to \mathrm{Hom}_{C_{\mathrm{red}}}(F \otimes \mathcal{I}, K) \to \mathrm{Ext}^2_{C_{\mathrm{red}}}(F,K). \tag{3.9}$$

If $\mathscr{F}$ is a type II sheaf, then $\mathscr{F}$ is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module which implies that the extension must be nontrivial. In particular, the extension determines a point in $\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K))$.

Conversely, a point in $\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K))$ gives an extension $\mathscr{F}$ which is not an $\mathcal{O}_{C_{\mathrm{red}}}$-module. In particular, $\mathscr{F}$ must be of type II. $\square$

Lemma 3.4.7. *Let $\mathscr{F}$ be a stable sheaf with Fitting support C of type II. Let F, T, and K be as in Lemma 3.4.3. Then*

1. *$\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K)) \simeq \mathbb{C}^3$ if $\chi(\mathscr{F})$ is even, and*

2. *$\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K)) \simeq \mathbb{C}^2$ if $\chi(\mathscr{F})$ is odd.*

Proof. If $\chi(\mathscr{F})$ is even, then $K \simeq F \otimes \mathcal{I}$ by Lemma 3.4.5. Since F, K are line bundles, $\mathrm{Ext}^2_{C_{\mathrm{red}}}(F,K) = H^2(C_{\mathrm{red}}, F^\vee \otimes K) = 0$ and $\mathrm{Hom}_{C_{\mathrm{red}}}(F \otimes \mathcal{I}, K) = \mathbb{C}$. Since F is line bundle on C_{red}, $\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K) = H^1(C_{\mathrm{red}}, \mathcal{I}) = \mathbb{C}^3$. It follows from the exact sequence described in Equation 3.9 coming from the spectral sequence for the change of coefficient in the Ext groups that $\mathrm{Ext}^1_C(F,K) \simeq \mathbb{C}^4$ and $\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K)) \simeq \mathbb{C}^3$.

If $\chi(\mathscr{F})$ is odd, then $K \simeq F \otimes \mathcal{I} \otimes \mathcal{O}_{C_{\mathrm{red}}}(x)$ for some $x \in C_{\mathrm{red}}$ by Lemma 3.4.5. Since F and K are still line bundles on C_{red}, we have $\mathrm{Ext}^2_{C_{\mathrm{red}}}(F,K) = 0$. The description of K also implies that

$$\mathrm{Hom}_{C_{\mathrm{red}}}(F \otimes \mathcal{I}, K) = H^0(C_{\mathrm{red}}, \mathcal{O}_{C_{\mathrm{red}}}(x)) = \mathbb{C}$$

Since F is line bundle on C_{red}, $\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K) = H^1(C_{\mathrm{red}}, \mathcal{I} \otimes \mathcal{O}_{C_{\mathrm{red}}}(p)) = \mathbb{C}^2$. It follows from the exact sequence described in Equation 3.9 coming from the spectral sequence for the change of coefficient in the Ext groups that $\mathrm{Ext}^1_C(F,K) \simeq \mathbb{C}^3$ and $\mathbb{P}(\mathrm{Ext}^1_C(F,K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F,K)) \simeq \mathbb{C}^2$. $\square$

With these lemmas, we can now describe the fibers $M_{v,C}$ over $C \in NR$. We begin by discussing sheaves of type I.

3.4.1 Sheaves of Type I

Lemma 3.4.8. *Let $C = 2C_{\mathrm{red}} \in NR$ and let $i_{\mathrm{red}}\colon C_{\mathrm{red}} \to J$ be the inclusion map. If $\mathscr{F}$ is a sheaf with Fitting support C of type I with Euler characteristic $\chi \neq 0$ and $\sum c_2(\mathscr{F}) = 0$, then $F = i_{\mathrm{red}*}V$ where V is a semi-stable rank two vector bundle on C_{red} of degree $\chi + 2$ and fixed determinant.*

Proof. Since C_{red} is a smooth curve in J, Grothendieck-Riemann-Roch applied to the inclusion $i_{\text{red}}\colon C_{\text{red}} \to C$ implies that

$$0 = \sum c_2(i_{\text{red}*}V) = -\sum (i_{\text{red}*}c_1(V) + [C_{\text{red}}]^2) = \sum c_2(i_{\text{red}*}\det(V)).$$

where the last equality uses the fact that $c_1(V) = c_1(\det(V))$. After fixing an isomorphism $\operatorname{Pic}^{\chi+2}(C_{\text{red}}) \simeq \operatorname{Pic}^0(C_{\text{red}})$ and using the isomorphism $\operatorname{Pic}^0(C_{\text{red}}) \simeq J$ given by sending the degree 0 line bundle L on C_0 to the point $\sum c_2(i_{\text{red}*}L)$ in J, we see that $\det(V)$ is fixed. □

Using this lemma, we can describe the sheaves of type I.

Lemma 3.4.9. *Fix an even Mukai vector of the form $v = 2v'$ with $v' = (0, \theta, \chi')$ with $\chi' \neq 0$. Then $M^I_{v,C}$, with its reduced induced structure, is irreducible and isomorphic to $\mathbb{P}^3$. Moreover, the locus of strictly semi-stable bundles in $M^I_{v,C}$, with its reduced induced structure, is isomorphic to a Kummer quartic surface $K \subset \mathbb{P}^3$.*

Proof. Narasimhan and Seshadri prove in [NR69] that the moduli space of semi-stable rank two vector bundles of even degree on a genus two curve with fixed determinant is isomorphic to $\mathbb{P}^3$ and show that the strictly semi-stable locus is isomorphic to a Kummer quartic surface $K \subset \mathbb{P}^3$. In fact, they prove that this moduli space is naturally isomorphic to the linear system $|2\theta|$. □

Lemma 3.4.10. *Fix an odd Mukai vector of the form $v = (0, 2\theta, 2\chi' + 1)$ with $\chi' \neq 0$. Then $M^I_{v,C}$, with its reduced, induced structure, is irreducible and is the intersection of two quadrics in $\mathbb{P}^5$.*

Proof. Narasimhan and Seshadri prove in [NR69] that the moduli space of semi-stable rank two vector bundles of odd degree on a genus two curve with fixed determinant is irreducible and is isomorphic to the intersection of two quadrics in $\mathbb{P}^5$. □

3.4.2 Sheaves of Type II

We now describe sheaves of type II. We begin by discussing the even Mukai vector case. For notational simplicity, we will only consider the Mukai vector $v = (0, 2\theta, -2)$ corresponding to M as the other cases are analogous.

Lemma 3.4.11. *Let $C = 2C_{\text{red}} \in NR$ and let $i_{\text{red}}\colon C_{\text{red}} \to J$ be the inclusion map. If $\mathscr{F}$ is any type II sheaf on J with Fitting support C, $\chi(\mathscr{F}) = -2$, and $\sum c_2(\mathscr{F}) = 0$, then*

$$F^{\otimes 2} \otimes \mathcal{I} \simeq \mathcal{O}_{C_{\text{red}}} \tag{3.10}$$

where F is the degree-1 line bundle $\mathscr{F}|_{C_{\text{red}}}$ on C_{red}.

Proof. Recall that the map $\mathrm{Pic}(C_{\mathrm{red}}) \to J$ sending a line bundle L on C_{red} to the point $\sum c_2(i_{\mathrm{red}*}L) \in J$ is a group homomorphism and induces an isomorphism of Abelian varieties when restricted to $\mathrm{Pic}^0(C_{\mathrm{red}})$.

By Lemma 3.4.5, there is a short exact sequence, viewed as sheaves on J,

$$0 \to i_{\mathrm{red}*}(F \otimes \mathcal{I}) \to \mathscr{F} \to i_{\mathrm{red}*}F \to 0.$$

where $F = \mathscr{F}|_{C_{\mathrm{red}}}$ is a degree-1 line bundle on C_{red}. Multiplicativity of the Chern character implies that $0 = \sum c_2(\mathscr{F}) = \sum c_2(i_{\mathrm{red}*}(F^{\otimes 2} \otimes \mathcal{I}))$. Under the isomorphism $\mathrm{Pic}^0(C_{\mathrm{red}}) \simeq J$ described above, this implies that $F^{\otimes 2} \otimes I \simeq \mathcal{O}_{C_{\mathrm{red}}}$ as desired. □

Lemma 3.4.12. *If $C = 2C_{\mathrm{red}} \in NR$, then M_C^{II} is reduced and consists of 16 connected components, each of which is isomorphic to $\mathbb{C}^3$.*

Proof. By Lemma 3.4.11, sheaves $\mathscr{F}$ in M_C^{II} are parameterized by non-trivial extensions of the form

$$0 \to i_{\mathrm{red}*}(F \otimes I) \to \mathscr{F} \to i_{\mathrm{red}*}F \to 0$$

where $F = \mathscr{F}|_{C_{\mathrm{red}}}$ is a degree 1 line bundle on C_{red} satisfying $F^{\otimes 2} \otimes I \simeq \mathcal{O}_{C_{\mathrm{red}}}$. In particular, we see that there are 16 possibilities for F. For a fixed such line bundle F, Lemma 3.4.7 implies that the possible extensions are parameterized by $\mathbb{P}(\mathrm{Ext}^1_C(F, K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F, K)) \simeq \mathbb{C}^3$. □

We now discuss sheaves of type II in the odd Mukai vector case. Again for notational simplicity, we only consider the Mukai vector $(0, 2\theta, -1)$.

Lemma 3.4.13. *If $C = 2C_{\mathrm{red}} \in NR$, then N_C^{II} is connected and has dimension three.*

Proof. By Lemma 3.4.5, sheaves $\mathscr{F}$ in N_C^{II} are parameterized by non-trivial extensions of the form

$$0 \to i_{\mathrm{red}*}(F \otimes \mathcal{I} \otimes \mathcal{O}_{C_{\mathrm{red}}}(x)) \to \mathscr{F} \to i_{\mathrm{red}*}(F) \to 0$$

where $F = \mathscr{F}|_{C_{\mathrm{red}}}$ is a degree 1 line bundle on C_{red} and $\sum c_2(\mathscr{F}) = 0$. Noting that the map $\mathrm{Pic}(C_{\mathrm{red}}) \to J$ sending a line bundle L on C_{red} to the point $\sum c_2(i_{\mathrm{red}*}L) \in J$ is a group homomorphism, multiplicativity of the Chern character implies that

$$0 = \sum c_2(\mathscr{F}) = \sum c_2(i_{\mathrm{red}*}(F^{\otimes 2} \otimes \mathcal{I})) - i_{\mathrm{red}}(x). \tag{3.11}$$

The collection of line bundles F on C_{red} which satisfy Equation 3.11 is precisely the fiber over 0 of the map

$$g\colon C_{\mathrm{red}} \times \mathrm{Pic}^1(C_{\mathrm{red}}) \to J; \quad (x, F) \mapsto \sum c_2(i_{\mathrm{red}*}(F^{\otimes 2} \otimes \mathcal{I}) - i_{\mathrm{red}}(x),$$

Notice that for any $(x, F) \in C_{\mathrm{red}} \times \mathrm{Pic}^0(C_{\mathrm{red}})$, $g(x, F) = h(F) - i_{\mathrm{red}}(x)$ where h is the morphism

$$h\colon \mathrm{Pic}^1(C_{\mathrm{red}}) \to J; \quad F \mapsto \sum c_2(i_{\mathrm{red}*}(F^{\otimes 2} \otimes \mathcal{I}).$$

It follows that the fiber of g over $0 \in J$ can be described by the fiber product diagram

$$\begin{tikzcd} g^{-1}(0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Pic}^1(C_{\mathrm{red}}) \arrow[d, "h"] \\ C_{\mathrm{red}} \arrow[r, "i_{\mathrm{red}}"] & J \end{tikzcd}$$

Since the map $\mathrm{Pic}^0(C_{\mathrm{red}}) \to J$ sending a line bundle L to the point $\sum c_2(i_* L)$ is an isomorphism of Abelian varieties, the map h is seen to be a degree 16 isogeny of J. Moreover, since $C_{\mathrm{red}} = \theta_p$, $i_{\mathrm{red}}\colon C_{\mathrm{red}} \to J$ can be identified with the Albanese morphism. By [Bea96, Remark V.14 (5)], the pullback of a connected étale cover of the Albanese variety by the Albanese map is connected and we conclude that $g^{-1}(0)$ is connected. For a fixed line bundle F in $g^{-1}(0)$, Lemma 3.4.7 implies that the possible extensions are parameterized by $\mathbb{P}(\mathrm{Ext}^1_C(F, K)) \setminus \mathbb{P}(\mathrm{Ext}^1_{C_{\mathrm{red}}}(F, K)) \simeq \mathbb{C}^2$. It follows that N_C^{II}, with its reduced, induced structure, is isomorphic to a $(\mathbb{C}^2)$-bundle over $g^{-1}(0)$. □

3.4.3 The Description of the Fibers

In this section, we combine our results about the sheaves of type I and type II to give a description of the fibers $M_{v,C}$ when $C \in NR$. Since irreducibility is a topological notion, it suffices to study $M_{v,C}$ with its reduced induced structure.

Proposition 3.4.14. *Let $C \in NR$. Then*

1. *the fiber G_C is isomorphic to $\mathbb{C}^3 \times J[2] \simeq \mathbb{C}^3 \times (\mathbb{Z}/2\mathbb{Z})^4$,*

2. *the fiber M_C consists of 17 irreducible components and contains a non-dense open subset parameterizing pushforwards of line bundles from C,*

3. *the fiber $\widetilde{M}_C$ consists of 34 irreducible components and contains a non-dense open subset parameterizing pushforwards of line bundles from C,*

4. the fiber N_C has two irreducible components. Moreover, no sheaf in N_C is the pushforward of a line bundle on C.

Proof. The fibers G_C and M_C are of the form $M_{v,C}$ where v is an even Mukai vector. Recall that by definition, $M_{v,C} = M^{I}_{v,C} \sqcup M^{II}_{v,C}$. By Lemma 3.4.9, $M^{I}_{v,C}$, with its reduced induced structure, is isomorphic to $\mathbb{P}^3$. By Lemma 3.4.12, $M^{II}_{v,C}$, which is actually reduced, consists of 16 connected components which parameterize the pushforwards of line bundles from C. It follows that M_C has 17 irreducible components and that G_C is isomorphic to $\mathbb{C}^3 \times J[2]$.

To see that the fiber $\widetilde{M}_C$ has 34 irreducible components, note that by Lemma 3.4.9, the strictly semi-stable locus $M^{ss}_{v,C} = M_{v,C} \cap \Sigma_v$, with its reduced induced structure, is isomorphic to the Kummer quartic surface $K \subset |2\theta|$. It follows that $M_{v,C} \cap \Omega_v$ is isomorphic to a finite set consisting of the 16 nodes of K. Recall that by Remark 2.4.1, the restriction of the symplectic desingularization map to Σ is a $\mathbb{P}^1$-bundle outside of the 256 points in Ω where the fibers are smooth three-dimensional quadrics. In particular, the exceptional divisor of the resolution, restricted to the fiber over C, consists of 17 irreducible components, namely the closure of the $\mathbb{P}^1$-bundle over K^{reg} in $\widetilde{M}_{v,C}$ and the 16 smooth 3-dimensional quadrics over the nodes of K. It follows that $\widetilde{M}_{v,C}$ consists of 34 irreducible components, namely the strict transforms of the 17 components in $M_{v,C}$ and the 17 irreducible components which constitute the exceptional divisor of the resolution restricted to the fiber over C.

Finally, to see that the fiber N_C hs 2 irreducible components, note that the fiber N_C is a fiber of the form $M_{v,C}$ where v is an odd Mukai vector and recall that by definition, $M_{v,C} = M^{I}_{v,C} \sqcup M^{II}_{v,C}$. By Lemma 3.4.10, $M^{I}_{v,C}$, with its reduced induced structure, is irreducible and is isomorphic to the intersection of two quadrics in $\mathbb{P}^5$. By Lemma 3.4.13, $M^{II}_{v,C}$, with its reduced induced structure, is irreducible of dimension three. It follows that N_C has 2 irreducible components. The statement that no sheaf in N_C is the pushforward of a line bundle on C is proved in Corollary 3.4.4. $\square$

Chapter 4

The Top Degree Direct Image Sheaves

The goal of this chapter is to describe the restriction of the top degree direct image sheaves to various the strata appearing in Rapagnetta's stratification of the linear system $B = |2\theta|$. This description will be the key input into the proof of Proposition 5.4.2 on the Decomposition Theorems for the fibrations $\widetilde{m}\colon \widetilde{M} \to B$ and $n\colon N \to B$. As in the O'Grady 10 case, we will use some properties of the trace morphism and the sheaf of irreducible components.

4.1 The Top Degree Direct Image Sheaves Away from $N(2)$

We begin by recalling the following facts.

Fact. *Let $f\colon X \to T$ be a flat morphism of relative dimension d. Then*

1. *there is a trace morphism $\mathrm{Tr}_f\colon R^{2d} f_! \mathbb{Q}_X(d) \to \mathbb{Q}_T$ which is an isomorphism if and only if all of the fibers of f have a unique irreducible component of dimension d (cf. [SGA 4.3, Théorème 2.9 and Remarque 2.10.1]),*

2. *if f has reduced fibers, then the sheaf $R^{2d} f_! \mathbb{Q}_X(d)$ is the $\mathbb{Q}$-linearization of the sheaf of sets of irreducible components of the fibers of f (cf. [Ngô08, Lemme 7.1.8]).*

We now describe the top degree direct image sheaves for the maps $\widetilde{m}\colon \widetilde{M} \to B$, $m\colon M \to B$, and $n\colon N \to B$ over loci away from the stratum $N(2)$.

Proposition 4.1.1. *Define* $R^6_M := R^6 m_* \mathbb{Q}_M, R^6_{\widetilde{M}} := R^6 \widetilde{m}_* \mathbb{Q}_{\widetilde{M}}$ *and* $R^6_N := R^6 n_* \mathbb{Q}_N$*. There are canonical isomorphisms of constructible sheaves:*

$$R^6_M|_{S\cup N(1)} \simeq R^6_{\widetilde{M}}|_{S\cup N(1)} \simeq R^6_N|_{S\cup N(1)} \simeq \mathbb{Q}_{S\cup N(1)} \tag{4.1}$$

$$R^6_M|_{N(3)} \simeq R^6_{\widetilde{M}}|_{N(3)} \simeq R^6_N|_{N(3)} \simeq \mathbb{Q}^{\oplus 4}_{N(3)} \tag{4.2}$$

$$R^6_M|_{NR} \simeq \mathbb{Q}^{\oplus 17}_{NR}, R^6_{\widetilde{M}}|_{NR} \simeq \mathbb{Q}^{\oplus 34}_{NR}, R^6_N|_{NR} \simeq \mathbb{Q}^{\oplus 2}_{NR} \tag{4.3}$$

$$R^6_{\widetilde{M}}|_{R^0} \simeq \mathbb{Q}^{\oplus 2}_{R^0} \tag{4.4}$$

$$R^6_N|_{R^0} \simeq \mathbb{Q}_{R^0} \oplus \mathscr{L}_{R^0} \tag{4.5}$$

where $\mathscr{L}_{R^0}$ *is the rank one local system on* $R^0 := R \setminus NR$ *corresponding to the étale double cover* $J \setminus J[2] \to R^0$ *sending a point* x *to the curve* $\theta_x + \theta_{-x}$*.*

Proof. Recall that by Remark 2.4.4, the morphisms $m, \widetilde{m}$ and n are all flat.

We begin by proving Equation 4.1. Since the fibers of $M, \widetilde{M}$, and N over the loci $S \cup N(1) \subset B$ are integral by Proposition 3.2.10, Fact 1 implies the isomorphisms.

We next prove Equation 4.2. Since $N(3)$ consists of only points, the direct image in top degree for the morphisms $m, \widetilde{m}$ and n are direct sums of the stalks at points in $N(3)$. Since the fibers of $m, \widetilde{m}$, and n all have four irreducible components by Corollary 3.2.10, the isomorphisms follow.

We next prove Equation 4.3. Since NR consists of only points, the direct image in top degree for the morphisms $m, \widetilde{m}$ and n are direct sums of the stalks at points in NR. The isomorphisms then follow from Proposition 3.4.14.

We next prove Equation 4.4. Recall that by Lemma 2.4.2, the Decomposition Theorem for the symplectic resolution $\pi\colon \widetilde{M} \to M$ is given by

$$R\pi_* \mathbb{Q}_{\widetilde{M}} \simeq \mathscr{I}\mathscr{C}_M \oplus \mathbb{Q}_\Sigma[-2](-1) \oplus \mathbb{Q}_\Omega[-6](-3).$$

Proper base change implies that $R\widetilde{m}_* \mathbb{Q}_{\widetilde{M}_{R^0}} = Rm_* \mathbb{Q}_{M_{R^0}} \oplus Rr_* \mathbb{Q}_{\Sigma_{R^0}}[-2](-1)$ where r denotes the restriction of the support morphism $m\colon M \to B$ to Σ_{R^0}. Thus,

$$R^6 \widetilde{m}_* \mathbb{Q}_{\widetilde{M}_{R^0}} \simeq R^6 m_* \mathbb{Q}_{M_{R^0}} \oplus R^4 r_* \mathbb{Q}_{\Sigma_{R^0}}(-1). \tag{4.6}$$

Since the fiber M_C for any $C \in R^0$ is irreducible by Proposition 3.3.1, Fact 1 implies that $R^6 m_* \mathbb{Q}_{M_{R^0}} \simeq \mathbb{Q}_{R^0}$. By Lemma 3.3.4, the fibers of the map $\Sigma_{R^0} \to R^0$ are isomorphic to the Abelian surface J and hence irreducible. Remark 1 then implies that $R^4 r_* \mathbb{Q}_{\Sigma_{R^0}} \simeq \mathbb{Q}_{R^0}$. Equation 4.4 then follows from these isomorphisms and Equation 4.6.

We finally prove Equation 4.5. Recall that the morphism $N_{R^0} \to R^0$ is flat and by Proposition 3.3.1, has reduced fibers each having two irreducible components. Fact 2 implies that $R^6 \mathbb{Q}_{N_{R^0}}$ is the $\mathbb{Q}$-linearization of the sheaf of sets $\mathrm{Irr}(N_{R^0})$ of irreducible components of N_{R^0} which is locally constant with stalks of cardinality two.

Let $N^{lf}_{R^0} \to R^0$ be the locus parameterizing sheaves which are the pushforwards of line bundles. Again by Proposition 3.3.1, the morphism $N^{lf}_{R^0} \to R^0$ is smooth, dense in every fiber, and surjective with two connected components which parameterize line bundles of bidegree $(1, 0)$ and bidegree $(0, 1)$. It follows that the sheaf of sets $\mathrm{Irr}(N_{R^0})$ can be identified with the sheaf of connected components of $N^{lf}_{R^0}$.

Using this interpretation, we now examine the monodromy of $\mathrm{Irr}(N_{R^0})$. There is an étale double cover

$$J \setminus J[2] \to R^0; \quad x \mapsto \theta_x + \theta_{-x} \tag{4.7}$$

There is a monodromy action of $\pi_1(R^0)$ on $J \setminus J[2]$ which interchanges the fibers of the covering map and there is a natural surjective group homomorphism $\pi_1(R^0) \to \mathbb{Z}/2\mathbb{Z}$. It follows that the monodromy action on the family of curves over R^0 swaps the components of the broken curves $\theta_x + \theta_{-x}$. Thus, the bidegrees of the line bundles are swapped as well and it follows that the components of $N^{lf}_{R^0,C}$ are swapped. Since $(R^6 n_* \mathbb{Q}_N)|_{R^0}$ is the $\mathbb{Q}$-linearization of the sheaf of sets $\mathrm{Irr}(N_{R^0})$, we conclude that $(R^6 n_* \mathbb{Q}_N)|_{R^0} \simeq \mathbb{Q}_{R^0} \oplus \mathscr{L}_{R^0}$ where $\mathscr{L}_{R^0}$ is the rank one local system on R^0 corresponding to the étale double described in Equation 4.7. $\square$

4.2 The Top Degree Direct Image Sheaves Over $N(2)$

In this section, we study the top degree direct image sheaves for the maps $\widetilde{m}$, and n over the loci $N(2)$. The analysis of these direct image sheaves over the loci $N(2)$, particularly of the monodromy of the irreducible components of $\widetilde{M}$ and N, is more subtle than the cases above. Our analysis of the direct image sheaves will follow a strategy communicated to us by A. Rapagnetta. We first note that it suffices to study the direct image sheaves in top degree for the maps m and n since $\widetilde{M}|_{N(2)} \simeq M_{N(2)}$.

Recall that $\overline{N(2)} = \cup_{p,q\in J[2],p\neq q} N_{pq}$ where $N_{pq} \simeq \mathbb{P}^1$ and there are $\binom{16}{2} = 120$ choices for p and q. Let $U_{pq} = N_{pq} \setminus NR \simeq \mathbb{C}^*$, and $V_{pq} = N_{pq} \setminus (NR \cup N(3)) \simeq \mathbb{P}^1 \setminus \{8 \text{ points}\}$. In the remainder of the section, we work over a fixed line N_{pq}. Recall that the curves $C \in V_{pq} \subset N_{pq}$ are integral of arithmetic genus 5 with nodes at p and q and smooth otherwise.

We begin with some notation. Consider the commutative diagram

$$\begin{tikzcd} \widetilde{J} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\varepsilon"] & \widetilde{K} \arrow[d, "\tau"] \\ J \arrow[r, "\phi"] & K \end{tikzcd} \tag{4.8}$$

where $\widetilde{J}$ is the blow up of J along the locus of 2-torsion points $J[2]$, $K \simeq J/\pm 1$, and $\widetilde{K}$ is the Kummer K3 surface associated to the Abelian surface J. Denote the exceptional divisors of ε and τ by F_r and E_r respectively for $r \in J[2]$.

Let $\mathcal{C}_{pq} \subset J \times V_{pq}$ and $\widetilde{\mathcal{C}}_{pq} \subset \widetilde{J} \times V_{pq}$ be the families of curves given by the incidence varieties

$$\mathcal{C}_{pq} := \{(x,b) \in J \times V_{pq} \mid x \in C_b\},$$
$$\widetilde{\mathcal{C}}_{pq} := \{(x,b) \in \widetilde{J} \times V_{pq} \mid x \in \widetilde{C}_b\},$$

where $\widetilde{C}_b$ is the strict transform of C_b. Let

$$\widetilde{\mathcal{D}}_{pq} := (\widetilde{\phi} \times id)(\widetilde{\mathcal{C}}_{pq}) \subset \widetilde{K} \times V_{pq},$$

denote the corresponding image. Note that $\mathcal{C}_{pq}$ is the family of curves in V_{pq} and $\widetilde{\mathcal{C}}_{pq}$ is the family of normalizations. We will denote the fibers of $\mathcal{C}_{pq}, \widetilde{\mathcal{C}}_{pq}$, and $\widetilde{\mathcal{D}}_{pq}$ over $b \in V_{pq}$ by C_b, $\widetilde{C}_b$, and $\widetilde{D}_b$ respectively and will think of them as curves in their respective surfaces.

For any $b \in V_{pq}$, let $p_{1,b}, p_{2,b}$ denote the two points in $\widetilde{C}_b \cap F_p$ lying over the node $p \in C_b$. In what follows, we will often abuse notation and omit the dependence of these points on b and simply refer to the points lying over the node $p \in C_b$ by p_1 and p_2. Similarly, we will denote the two points in $\widetilde{C}_b$ lying over the node q by q_1 and q_2.

Recall that by Lemma 3.2.1 the loci $N^{lf}_{V_{pq}}$ and $M^{lf}_{V_{pq}}$ parameterizing sheaves which are the pushforwards of line bundles can be identified with the fiber over 0 of the map $\mathbf{a}\colon \mathrm{Pic}^d_{\mathcal{C}_{pq}/V_{pq}} \to J$ for $d = 1, 2$ respectively.

There is a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}^d_{\mathcal{C}_{pq}/V_{pq}} \arrow[r, two heads] \arrow[d, two heads, "\mathbf{a}"] & \mathrm{Pic}^d_{\widetilde{\mathcal{C}}_{pq}/V_{pq}} \arrow[dl, two heads, "\widetilde{\mathbf{a}}"] \\ J & \end{tikzcd} \tag{4.9}$$

where for any $b \in V_{pq}$, the short exact sequence

$$0 \to (\mathbb{C}^*)^2 \to \mathrm{Pic}^d(C_b) \to \mathrm{Pic}(\widetilde{C}_b) \to 0$$

implies that the vertical map in Equation 4.9 is surjective with connected fibers. It follows that the induced map $\mathbf{a}^{-1}(0) \to \widetilde{\mathbf{a}}^{-1}(0)$ is surjective with connected fibers. This implies that $\mathbf{a}^{-1}(0)$ is irreducible if and only if $\widetilde{\mathbf{a}}^{-1}(0)$ is irreducible.

The families $\widetilde{\mathcal{C}}_{pq}$ and $\widetilde{\mathcal{D}}_{pq}$ both have natural 2-sections. Let

$$S_p := (F_p \times V_{pq}) \cap \widetilde{\mathcal{C}}_{pq} \subset \widetilde{J} \times V_{pq} \tag{4.10}$$

$$T_p := (E_p \times V_{pq}) \cap \widetilde{\mathcal{D}}_{pq} \subset \widetilde{K} \times V_{pq} \tag{4.11}$$

and notice that the respective projection maps to V_{pq} are surjective of degree 2.

Lemma 4.2.1. *Given $(x, b) \in T_p$, there exists a unique line $\ell_x \subset |2\theta|^\vee$ and a unique plane $H_b \subset |2\theta|^\vee$ such that $\ell_x \subset H_b$ and $\phi(q) \notin \ell_x$.*

Proof. If $b \in V_{pq}$, let $C_b \subset J$ be the corresponding curve. Then $D_b = \phi(C_b) = H_b \cap K$ for some plane $H_b \subset |2\theta|^\vee$ containing $\phi(p)$ and $\phi(q)$. Now given $(x, b) \in T_p$, we see that $x \in \widetilde{D}_b \cap E_p$. In particular, x corresponds to a tangent direction to D_b at the node $\phi(p)$. Let ℓ_x be the unique line in $|2\theta|^\vee$ corresponding to this tangent direction. Since $D_b = H_b \cap K$, we see that $\ell_x \subset H_b$.

To see that $\phi(q) \notin \ell_x$, note that $D_b \subset H_b$ is a quartic curve which is nodal at $\phi(p)$ and $\phi(q)$. Since ℓ_x is tangent to D_b at the node $\phi(p)$, the intersection multiplicity of $\ell_x \cap D_b$ at $\phi(p)$ is 3. If $\phi(q) \in \ell_x$, then the intersection multiplicity of $\ell_x \cap D_b$ at $\phi(p)$ would be strictly larger than 1 since D_b is singular at $\phi(p)$. This would imply that the intersection number $\ell_x \cdot D_b$ would be larger than 5 which contradicts the fact that D_b is a quartic curve in H_b. $\square$

Lemma 4.2.2. *The natural projection $f : T_p \to E_p$ is injective.*

Proof. Fix a hyperplane H containing $\phi(q)$ but not $\phi(p)$. Given any $(x, b) \in T_p$, let ℓ_x, H_b be as in Lemma 4.2.1. Let $\ell_b \subset H$ be the line $H_b \cap H$ and let y be the point $\ell_x \cap H$. Since $\phi(q) \in H_b$ for all b and $\phi(q) \in H$ by choice of H, $\phi(q) \in \ell_b$. Since y and $\phi(q)$ are distinct by Lemma 4.2.1, ℓ_b is the unique line through y and $\phi(q)$. Now if $(x, b), (x', b') \in T_p$ with $x = x'$, then $y = y'$. It follows that the lines ℓ_b and ℓ'_b coincide which implies that $b = b'$. $\square$

Corollary 4.2.3. *The 2-section $S_p \subset \widetilde{\mathcal{C}}_{pq}$ is irreducible.*

Proof. Since the map $\widetilde{\phi}\colon \widetilde{J} \to \widetilde{K}$ restricted to the exceptional divisor $F_p \subset \widetilde{J}$ is an isomorphism onto the exceptional divisor $E_p \subset \widetilde{K}$, the map $\widetilde{\phi} \times id\colon \widetilde{\mathcal{C}}_{pq} \to \widetilde{\mathcal{D}}_{pq}$ restricted to S_p is an isomorphism onto T_p. Since T_p is isomorphic to a Zariski open subset of the irreducible conic E_p by Lemma 4.2.2, T_p is irreducible. It follows that S_p is also irreducible. $\square$

Lemma 4.2.4. *The fiber of the map* $\widetilde{\mathbf{a}}\colon \mathrm{Pic}^1_{\widetilde{C}/V_{pq}} \to J$ *over* 0 *is connected.*

Proof. We first consider the case where $p = 0 \in J[2]$. Let $b \in V_{pq}$ and $\widetilde{C}_b$ be the fiber of the family of curves $\widetilde{\mathcal{C}}_{pq}$ at b. Let $\widetilde{\mathbf{a}}_b\colon \mathrm{Pic}^1(\widetilde{C}_b) \to J$ be the restriction of $\widetilde{\mathbf{a}}$ to $\widetilde{C}_b$. Let p_1, p_2 be the two points in $\widetilde{C}_b$ lying over the node $p \in C_b$. Since p_1, p_2 lie over $p = 0$, Grothendieck-Riemann-Roch for the composition $\widetilde{C}_b \to C_b \to J$ implies that $\widetilde{\mathbf{a}}(\mathcal{O}_{\widetilde{C}_b}(p_i)) = 0$ for $i = 1, 2$.

Recall that by Corollary 3.2.5, the degree 0 line bundle $\mathcal{O}_{\widetilde{C}_b}(p_1 - p_2)$ is not in the identity component of kernel of the map $a\colon \mathrm{Pic}^0(\widetilde{C}_b) \to J$. This implies that the degree 1 line bundles $\mathcal{O}_{\widetilde{C}_b}(p_1)$ and $\mathcal{O}_{\widetilde{C}_b}(p_2)$ must lie in different connected components of $\widetilde{\mathbf{a}}_b^{-1}(0)$.

Now let $S_p \subset \widetilde{\mathcal{C}}_{pq}$ be the 2-section described in Equation 4.11. Consider the base change diagram

$$\begin{tikzcd} \widehat{\mathcal{C}}_{pq} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}_{pq} \arrow[d] \\ S_p \arrow[r] & V_{pq} \end{tikzcd} \tag{4.12}$$

The preimage of the 2-section $S_p \subset \mathcal{C}_{pq}$ in $\widetilde{\mathcal{C}}_{pq}$ splits into two irreducible components S_1 and S_2. Using representability of the Picard functor, the line bundle $\mathcal{O}_{\widehat{\mathcal{C}}_{pq}}(S_1)$ on $\mathcal{C}_{S_p}$ defines a section $s\colon S_p \to \mathrm{Pic}^1_{\widehat{\mathcal{C}}_{pq}/V_{pq}}$ such that $s(p_i, b) = \mathcal{O}_{\widetilde{C}_b}(p_i)$. Since $s(p_1, b)$ and $s(p_2, b)$ are in different connected components of $\mathbf{a}^{-1}(0)$ and S_p is irreducible, $\widetilde{\mathbf{a}}^{-1}(0)$ is connected.

If $p \neq 0 \in J[2]$, consider the translation by p morphism $t_p\colon J \to J$. Pullback by t_p determines an automorphism of the linear system $|2\theta|$ sending the line N_{pq} to the line N_{0r} where $r = p + q$. The connectivity of $\widetilde{\mathbf{a}}^{-1}(0)$ then follows from the $p = 0$ case. $\square$

Lemma 4.2.5. *The fiber of the map* $\widetilde{\mathbf{a}}\colon \mathrm{Pic}^2_{\widetilde{\mathcal{C}}_{pq}/V_{pq}} \to J$ *over* 0 *is disconnected.*

Proof. Consider the projection $\widetilde{\pi}_{pq}\colon \widetilde{\mathcal{C}}_{pq} \to \widetilde{J}$ and the line bundle $\widetilde{\pi}^*_{pq}\mathcal{O}_{\widehat{J}}(F_p)$ on $\widetilde{\mathcal{C}}_{pq}$. Using representability of the Picard functor, this line bundle defines a section $s\colon V_{pq} \to \mathrm{Pic}^2_{\widetilde{\mathcal{C}}_{pq}]/V_{pq}}$ such that $s(p_i, b) = \mathcal{O}_{\widetilde{C}_b}(p_1 + p_2)$. Since p_1, p_2

lie over the 2-torsion point $p \in J[2]$, we see that this section factors through $\widetilde{\mathbf{a}}^{-1}(0)$. $\square$

Using these lemmas, we will be able to give the following description of the top degree direct image sheaves over V_{pq}.

Proposition 4.2.6. *There are canonical isomorphisms of constructible sheaves:*

$$R^6_M|_{V_{pq}} \simeq R^6_{\widetilde{M}}|_{V_{pq}} \simeq \mathbb{Q}^{\oplus 2}_{V_{pq}}, \tag{4.13}$$

$$R^6_N|_{V_{pq}} \simeq \mathbb{Q}_{V_{pq}} \oplus \mathscr{L}_{V_{pq}}, \tag{4.14}$$

where $\mathscr{L}_{V_{pq}}$ is a rank one local system on V_{pq} corresponding to $\pi_1(V_{pq})$ representation for which loops around $N(3)$-points act by the identity and loops around NR-points act by -1.

Proof. We begin by proving Equation 4.13. The isomorphism $R^6_M|_{V_{pq}} \simeq R^6_{\widetilde{M}}|_{V_{pq}}$ in Equation 4.13 follows from the isomorphism $M_C \simeq \widetilde{M}_C$ between fibers for $C \in V_{pq}$. For the other isomorphism, we will use argue using the sheaf of irreducible components as in the proof of Equation 4.5.

Recall that the morphism $M_{V_{pq}} \to V_{pq}$ is flat, and by Proposition 3.2.10, has reduced fibers each having two irreducible components. Fact 2 from Section 4.1 implies that $R^6\mathbb{Q}_{M_{V_{pq}}}$ is the $\mathbb{Q}$-linearization of the sheaf of sets $\mathrm{Irr}(M_{V_{pq}})$ of irreducible components of $M_{V_{pq}}$. Lemma 4.2.5 implies that the sheaf $\mathrm{Irr}(M_{V_{pq}})$ is the constant sheaf with stalks of cardinality 2 and the second isomorphism in Equation 4.13 follows.

We now prove Equation 4.14. Recall that the morphism $N_{V_{pq}} \to V_{pq}$ is flat and by Proposition 3.2.10, has reduced fibers each having two irreducible components. Fact 2 from Section 4.1 implies that $R^6\mathbb{Q}_{N_{V_{pq}}}$ is the $\mathbb{Q}$-linearization of the sheaf of sets $\mathrm{Irr}(N_{V_{pq}})$ of irreducible components of $N_{V_{pq}}$. Lemma 4.2.4 implies that $\mathrm{Irr}(N_{V_{pq}})$ must be a non-trivial locally constant sheaf with stalks of cardinality two.

We now claim that loops around $N(3)$ points do not interchange the two irreducible components of a fiber N_C. To see this, note that Proposition 3.2.10 implies that the morphism $N_{V_{pq}} \to V_{pq}$ can be extended to a morphism $N_{U_{pq}} \to U_{pq}$ which is also flat with reduced fibers. Moreover, the locus $N^{lf}_{U_{pq}}$ parameterizing sheaves which are pushforwards of line bundles is dense in every fiber. Now fix any $\mathscr{F} \in N^{lf}_C$ where $C \in N(3) \cap N_{pq} = U_{pq} \setminus V_{pq}$. Since the Le Potier support morphism is smooth at every point of N^{lf}_C, the map n, viewed in the analytic topology, is a submersion at $\mathscr{F}$. It follows that there exists a small disk $\Delta_{pq} \subset U_{pq}$ about $C \in N(3)$ and a local section

$s\colon \Delta_{pq} \to N^{lf}_{\Delta_{pq}}$ such that $s(0) = \mathscr{F}$. Restricting this section to Δ^*_{pq} gives a local section of $N^{lf}_{\Delta^*_{pq}}$ which implies that $N^{lf}_{\Delta pq^*}$ is disconnected. In particular, loops around $N(3)$ points do not interchange the two irreducible components of a fiber.

Since $\mathrm{Irr}(M_{V_{pq}})$ is a non-trivial locally constant sheaf and the loops around $N(3)$ points act trivially on the irreducible components, it extends to a non-trivial locally constant sheaf on $V_{pq} = N_{pq} \setminus NR \simeq \mathbb{C}^*$. We conclude that loops around both NR points must interchange the two irreducible components of a fiber and the isomorphism in Equation 4.14 follows. $\square$

Chapter 5

Ngô Strings

In this chapter, we will use our knowledge of the top degree direct image sheaves for the fibrations $\widetilde{m}$ and n and the Ngô Support Theorem to determine the Decomposition Theorems for $R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$ and $Rn_*\mathbb{Q}_N$, which is recorded in Proposition 5.4.2. We first introduce several relevant strings that will appear in the Decomposition Theorems for $R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$ and $Rn_*\mathbb{Q}_N$.

5.1 The Relevant String over B

In this section, we will introduce the relevant string over the entire base $B = |2\theta|$ and will study some properties of the string.

Consider the group scheme $g\colon G \to B$ described in Section 2.5. Over the locus $S \subset B$ of smooth curves, the map $g\colon G_S \to S$ is smooth and the fibers of G are 3-dimensional Abelian varieties by Proposition 3.2.10. Consider the higher direct image sheaves $\Lambda^i_B := R^i g_*\mathbb{Q}_{G_S}$. We introduce the following complex, viewable in $D^bMHM_{alg}(B)$ or $D^b(B,\mathbb{Q})$, which we will call a string.

$$\mathscr{I}_B := \oplus_{i=0}^{6}\mathscr{I}\mathscr{C}_B(\Lambda^i_B)[-i]. \tag{5.1}$$

We will see in Section 5.4 that the string $\mathscr{I}_B$ appears in the Decomposition Theorems for $\widetilde{M}$ and N.

In the remainder of this section, we will study some properties of the string $\mathscr{I}_B$. It will be useful to first consider the universal family of curves $\gamma\colon \mathcal{C} \to B$ in the linear system B, where the superscripts denote the dimensions. Note that the total space $\mathcal{C}$ is smooth. Set $\Lambda^1_\gamma := R^1\gamma_*\mathbb{Q}_{\mathcal{C}_S}$.

Lemma 5.1.1. *The Decomposition Theorem for $R\gamma_*\mathbb{Q}_{\mathcal{C}}$ takes the following form:*

$$R\gamma_*\mathbb{Q}_{\mathcal{C}} \simeq \mathbb{Q}_B \oplus \left(i_{S*}\Lambda^1_\gamma)[-1] \oplus i_{R^0*}\mathscr{L}_{R^0}[-2]\right) \oplus \mathbb{Q}_B[-2](-1).$$

Proof. We prove the equivalent

$$R\gamma_*\mathbb{Q}_{\mathcal{C}}[4] \simeq \mathbb{Q}_B[3][1] \oplus \big(i_{S*}\Lambda^1_\gamma)[3] \oplus i_{R^0*}\mathscr{L}_{R^0}[2]\big) \oplus \mathbb{Q}_B 3][-1](-1).$$

By looking at the regular part of γ, the following summands appear in the Decomposition Theorem for $R\gamma_*\mathbb{Q}_{\mathcal{C}}[4]$:

$$\mathbb{Q}_B[3][1] \oplus IC_B(\Lambda^1_\gamma)[0] \oplus \mathbb{Q}[3][-1](-1).$$

The cohomology sheaf $\mathcal{H}^{-2}(IC_B(\Lambda^1_\gamma))$ contributes to $R^2\gamma_*\mathbb{Q}_{\mathcal{C}}$. By the support conditions for intersection complexes, this sheaf is supported in dimension ≤ 1. Since curves over the two dimensional locus R^0 have two irreducible components, there must be a contribution of the form $IC_R(\mathfrak{L})$ where $\mathfrak{L}$ is a rank one local system on some open dense subset R' of R. The intersection complex $IC_R(\mathfrak{L})$ can only have nonzero cohomology sheaves in degrees -2 and -1. However, since $\mathcal{H}^{-1}(IC_R(\mathfrak{L}))$ would contribute to $R^3\gamma_*\mathbb{Q}_{\mathcal{C}}$, it must be zero. It follows that $IC_R(\mathfrak{L}) = i_{R'*}\mathfrak{L}[2]$ where $i_{R'}\colon R' \to B$ is the locally closed embedding. Since the stalks of $R^2\gamma_*\mathbb{Q}_{\mathcal{C}}$ are one dimensional over NR, we conclude that the cohomology sheaves of $IC_B(\Lambda^1_\gamma)$ must vanish in degrees -2 and -1. It follows that $IC_B(\Lambda^1_\gamma) = i_{S*}\Lambda^1_\gamma[3]$. □

Using this lemma, we make the following observation about the cohomology sheaves of the string $\mathscr{I}_B$.

Lemma 5.1.2. *The cohomology sheaf $\mathcal{H}^1(\mathscr{I}\mathscr{C}_B(\Lambda^5_B))$ vanishes. In particular,*

$$\mathcal{H}^6(\mathscr{I}_B) = \mathbb{Q}_B \oplus \mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda^4_B)).$$

where $\mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda^4_B))$ is a skyscraper sheaf.

Proof. Let $f\colon \mathrm{Pic}^0_{\mathcal{C}/S} \to S$ be the relative Picard scheme of the universal family $\mathcal{C}$ over the locus of smooth curves $S \subset B$. Recall that by definition, G_S is a subgroup scheme of $\mathrm{Pic}^0_{\mathcal{C}/B}$. The natural closed embedding $G_S \to \mathrm{Pic}^0_{\mathcal{C}/S}$ induces a morphism of semi-simple local systems

$$R^1f_*\mathbb{Q}_{\mathrm{Pic}^0_{\mathcal{C}/S}} \to R^1g_*\mathbb{Q}_{G_S}. \tag{5.2}$$

The map on stalks over a point $C \in S$ can be identified with the restriction map $H^1(\mathrm{Pic}^0(C), \mathbb{Q}) \to H^1(G_C, \mathbb{Q})$, which is surjective since G_C is an Abelian subvariety of $\mathrm{Pic}^0(C)$. It follows that the map in Equation 5.2 is surjective and thus by semi-simplicity, $\Lambda^1_B \coloneqq R^1g_*\mathbb{Q}_{G_S}$ is a direct summand of $R^1f_*\mathbb{Q}_{\mathrm{Pic}^0_{\mathcal{C}/S}}$. Noting that $R^1f_*\mathbb{Q}_{\mathrm{Pic}^0_{\mathcal{C}/S}} \simeq R^1\gamma_*\mathbb{Q}_{\mathcal{C}} = \Lambda^1_\gamma$, we see that Λ^1_B is a direct summand of $R^1\gamma_*\mathbb{Q}_{\mathcal{C}}$. Since the intermediate extension functor preserves direct sums, $\mathscr{I}\mathscr{C}_B(\Lambda^1_B)$ is a direct summand of $\mathscr{I}\mathscr{C}_B(\Lambda^1_\gamma)$. Lemma 5.1.1 then implies that $\mathcal{H}^1(\mathscr{I}\mathscr{C}_B(\Lambda^1_B)) = 0$. The claim in the lemma then follows from the Relative Hard Lefschetz isomorphism $\Lambda^1_B \simeq \Lambda^5_B$. □

5.2 The Relevant Strings over R

In this section, we will describe the relevant strings over the loci R. Throughout this section, we use the canonical identification of $J^\vee \simeq \mathrm{Pic}^0(C_0)$ where C_0 is our fixed genus 2 curve.

Consider the support morphism $m\colon M \to B$. Denote the restriction of m to the singular locus $\Sigma \subset M$ of M by r. The image of r is precisely the locus $R \subset B$. According to Lemmas 3.3.4 and 3.4.9, the map

$$J^\vee \times J \to \Sigma; \quad (L, x) \mapsto [i_{x*}L \oplus i_{-x*}L^\vee]$$

is a double cover branched along the 256 2-torsion points of $J^\vee \times J$. Similarly, the map

$$J \to R; \quad x \mapsto \theta_x + \theta_{-x}$$

a double cover branched along the 16 2-torsion points of J.

Letting $A \coloneqq J^\vee \times J$, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "pr_2"] \arrow[d, "q"] \arrow[dr, "c"] & J \arrow[d, "q"] \\ \Sigma \arrow[r, "r"] & R \end{tikzcd} \tag{5.3}$$

where $pr_2\colon A \to J$ is projection onto the second factor, both q maps are the respective branched covers described above, and r is restriction of the support morphism $m\colon M \to B$ to Σ.

We introduce the following complexes, viewable in either $D^bMHM_{alg}(R)$ or $D^b(R, \mathbb{Q})$, which we will again call strings. Let $\Lambda^i_{R^0} \coloneqq R^i r_* \mathbb{Q}_{\Sigma_{R^0}}$,

$$\mathscr{I}^+_R \coloneqq \oplus^4_{i=0} \mathscr{IC}_R(\Lambda^i_{R^0})[-i], \tag{5.4}$$

$$\mathscr{I}^-_R \coloneqq \oplus^4_{i=0} \mathscr{IC}_R(\Lambda^i_{R^0} \otimes \mathscr{L}_{R^0})[-i], \tag{5.5}$$

where $\mathscr{L}_{R^0}$ is the rank one local system on R^0 associated to the étale double cover $J_{R^0} \to R^0$.

In the remainder of this section, we will study the Decomposition Theorems for morphisms $r\colon \Sigma \to R$ and $c\colon A \to R$.

Lemma 5.2.1. *There is an isomorphism*

$$Rr_*\mathbb{Q}_\Sigma \simeq \mathscr{I}^+_R.$$

Moreover, the intersection complexes $\mathscr{IC}_R(\Lambda^1_R)$ and $\mathscr{IC}_R(\Lambda^3_R)$ are sheaves, i.e.

$$\mathscr{IC}_R(\Lambda^1_R) = i_{R^0*}\Lambda^1_R \tag{5.6}$$

$$\mathscr{IC}_R(\Lambda^3_R) = i_{R^0*}\Lambda^1_R(-1). \tag{5.7}$$

Proof. For reasons of bookkeeping, we prove the equivalent:

$$Rr_*\mathbb{Q}_\Sigma[4] \simeq \mathbb{Q}_R[2][2] \oplus \bigoplus_{i=-1}^{1} IC_R(\Lambda^{i+2}_{R^0})[-i] \oplus \mathbb{Q}_R[2][-2](-2).$$

By looking at the regular part of r, the summands

$$\mathbb{Q}_R[2][2] \oplus \bigoplus_{i=-1}^{1} IC_R(\Lambda^{i+2}_{R^0})[-i] \oplus \mathbb{Q}_R[2][-2](-2)$$

must appear in the Decomposition Theorem for $Rr_*\mathbb{Q}_{A/\pm 1}[4]$. Since all fibers of r are irreducible, there can be no additional summands in perverse degrees 1 and 2. Relative Hard Lefschetz then implies that the only additional summands can appear in perverse degree 0. Again using the fact that all fibers of r are irreducible, the only additional summands can be intersection complexes of supported on one dimensional subvarieties of R placed in perverse degree 0. Such a summand would contribute non-trivially to $R^3r_*\mathbb{Q}_{A/\pm 1}$ in codimension one. However, since there are only finitely many points in R for which the fibers of r are not Abelian surfaces, no such summand can exist. We conclude that there are no additional summands which appear in the Decomposition Theorem for r.

To see the claim about the intersection complexes being sheaves, notice that since $R \subset B$ is a surface, the intersection complexes $\mathscr{IC}_R(\Lambda^i_R)$ can only have nonzero cohomology sheaves in degrees 0 and 1. The support condition for intersection complexes implies that the cohomology sheaves $\mathcal{H}^1(\mathscr{IC}_R(\Lambda^i_R))$ are supported in dimension ≤ 0. Since the fibers of r are all irreducible, we conclude that $\mathcal{H}^1(\mathscr{IC}_R(\Lambda^3_R)) = 0$. Equation 5.6 and Equation 5.7 then follow from the Relative Hard Lefschetz isomorphism $\Lambda^1_R \simeq \Lambda^3_R(1)$. □

We now study the Decomposition Theorem for the map $c = r \circ q\colon A \to R$ described in Equation 5.3. We note that the fibers of c are straightforward to describe. Using the top right triangle in Equation 5.3, we see that fiber over a point in R^0 consists of two irreducible components, while the fiber over a point in NR is irreducible.

Lemma 5.2.2. *There is an isomorphism*

$$Rc_*\mathbb{Q}_A \simeq \mathscr{I}^+_R \oplus \mathscr{I}^-_R$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
A_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[r, "pr_2"] \arrow[d, "q"] \arrow[dr, "c"] & J \arrow[d, "q"] \\
U \arrow[r, hook, "j_U"] & \Sigma \arrow[r, "r"] \arrow[dr, phantom, "\square"] & R \\
 & \Sigma_{R^0} \arrow[r, "r"] \arrow[u, hook, "j_\Sigma"] \arrow[ul, hook] & R^0 \arrow[u, "j_{R^0}"']
\end{tikzcd}$$

where $U \subset \Sigma$ is the smooth part of Σ. Note that the inclusion $\Sigma_{R^0} \subset U$ holds since each fiber of r over NR contains precisely 16 nodes.

We begin by describing some local systems that will appear in the proof. Recall that $\mathscr{L}_{R^0}$ is the rank one local system on R^0 corresponding to the representation of $\pi_1(R^0)$ which factors through the $\mathbb{Z}/2\mathbb{Z}$ character -1. Let $\mathscr{L}_U$ be the rank one local system on U corresponding to the representation of $\pi_1(U)$ associated to the étale double cover $A_U \to U$, which factors through the $\mathbb{Z}/2\mathbb{Z}$ character -1. Let $\mathscr{L}_{\Sigma_{R^0}}$ be the restriction of the local system $\mathscr{L}_U$ to Σ_{R^0}. Since the complement of Σ_{R^0} in U has complex codimension 2, there is an isomorphism of fundamental groups $\pi_1(\Sigma_{R^0}) \simeq \pi_1(U)$. This implies that the local system $\mathscr{L}_{\Sigma_{R^0}}$ corresponds to the representation of $\pi_1(\Sigma_{R^0})$ which factors through the $\mathbb{Z}/2\mathbb{Z}$ character -1. We claim that there is an isomorphism of local systems $\mathscr{L}_{\Sigma_{R^0}} \simeq r^*\mathscr{L}_{R^0}$. To see this, consider the commutative diagram

$$\begin{tikzcd}
\pi_1(\Sigma_{R^0}) \arrow[r, two heads] \arrow[dr] & \pi_1(R^0) \arrow[r] \arrow[d, two heads] & \mathrm{Aut}(\mathbb{Q}) \\
 & \mathbb{Z}/2\mathbb{Z} \arrow[ur, "-1"] &
\end{tikzcd}$$

The right triangle corresponds to the local system $\mathscr{L}_{R^0}$ and the large outer triangle corresponds to the local system $r^*\mathscr{L}_{R^0}$. The left arrow in the top row is a surjection by the long exact sequence of homotopy groups for a fibration since the general fiber of r is connected. It follows that the arrow $\pi_1(\Sigma_{R^0}) \to \mathbb{Z}/2\mathbb{Z}$ is surjective and this gives the desired isomorphism of local systems $\mathscr{L}_{\Sigma_{R^0}} \simeq p^*\mathscr{L}_{R^0}$.

We now study the Decomposition Theorem for $c\colon A \to R$. Proper base change and the fact that q is a finite map imply that $Rq_*\mathbb{Q}_A = \mathbb{Q}_A \oplus j_{U*}\mathscr{L}_U$. It follows that $Rc_*\mathbb{Q}_A = Rr_*(q_*\mathbb{Q}_A) = Rr_*\mathbb{Q}_\Sigma \oplus Rr_*(j_{U*}\mathscr{L}_U)$. Lemma 5.2.1 implies that there is an isomorphism $Rr_*\mathbb{Q}_\Sigma \simeq \mathscr{I}_R^+$. To finish the proof, we must show that there is an isomorphism $Rr_*(j_{U*}\mathscr{L}_U) \simeq \mathscr{I}_R^-$. To see this, notice that by proper base change for r, $Rr_*(j_{U*}\mathscr{L}_U)|_R^0 = Rr_*j^*_{\Sigma_{R^0}}j_{U*}\mathscr{L}_U$.

Since j_U is an open embedding, there is an isomorphism $j_U^* j_{U*}\mathscr{L}_U \simeq \mathscr{L}_U$. This implies that there is an isomorphism $Rr_*(j_{U*}\mathscr{L}_U)|_{R^0} = Rr_*\mathscr{L}_{\Sigma_{R^0}}$. Using the isomorphism $\mathscr{L}_{\Sigma_{R^0}} \simeq r^*\mathscr{L}_{R^0}$ combined with the projection formula gives

$$Rr_*(j_{U*}\mathscr{L}_U)|_V = Rr_*r^*\mathscr{L}_{R^0} = (Rr_*\mathbb{Q}_\Sigma) \otimes \mathscr{L}_{R^0}.$$

Deligne's Theorem [dC17b, Theorem 1.5.3], i.e. the Decomposition Theorem for smooth proper morphisms, combined with the fact that the fibers of r over NR are irreducible then give the desired isomorphism $Rr_*(j_{U*}\mathscr{L}_U) \simeq \mathscr{I}_R^-$. $\square$

5.3 The Relevant Strings over N_{pq}

In this section, we will introduce the relevant strings over the loci N_{pq}. These strings will be direct summands appearing in the Decomposition Theorems for some special fibrations related to the Kummer $K3$ surface $\widetilde{K}$ associated to J.

Consider the commutative diagram

$$\begin{tikzcd} \widetilde{J} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\varepsilon"] & \widetilde{K} \arrow[d, "\tau"] \\ J \arrow[r, "\phi"] & K \end{tikzcd}$$

where $\widetilde{K}$ is the Kummer K3 surface associated to the Abelian surface J. Let H be a hyperplane section of K and let E_p and E_q be the exceptional divisors in $\widetilde{K}$ over nodes $p, q \in K$. Consider the linear system $B' = |\tau^*H - E_p - E_q|$ and note that $\dim B' = 1$ and B' is base point free.

Let $\widetilde{\mathcal{D}}_{B'} \subset \widetilde{K} \times B'$ be the universal family of curves associated to the linear system B'. Since B' is a base point free pencil, the projection morphism $\widetilde{\mathcal{D}}_{B'} \to \widetilde{K}$ is an isomorphism. Under this isomorphism, the family of curves $\widetilde{\mathcal{D}}_{B'} \to B'$ can be identified with the elliptic fibration $e\colon \widetilde{K} \to B'$ where e is the morphism determined by the linear system B'. We note that the base of this elliptic fibration can be identified with the locus N_{pq} parameterizing curves in the linear system $B = |2\theta|$ with nodes at p and q via the map

$$N_{pq} \to B'; \quad C \mapsto \tau^* f(C) - E_p - E_q.$$

In the remainder of this section, we will always identify N_{pq} with B'.

The elliptic fibration $e\colon \widetilde{K} \to B'$ given by the linear system B' has six I_2 fibers and two I_0^* fibers (in Kodaira's notation of singular fibers for elliptic surfaces) and admits a section (cf. [Kum14, §4]). Under the identification

$N_{pq} \simeq B'$, the six $N(3)$ points in N_{pq} are identified with the six points in B' having I_2 fibers and the two NR points in N_{pq} are identified with the two points in B' having I_0^* fibers. The section of e, coupled with the isomorphism $\widetilde{\mathcal{D}}_{B'} \simeq \widetilde{K}$, induces an identification between the elliptic fibration $e\colon \widetilde{K}_{V_{pq}} \to V_{pq}$ and the relative Picard scheme $\mathrm{Pic}^0_{\widetilde{\mathcal{D}}_{V_{pq}}/V_{pq}}$ over the open subset $V_{pq} \subset N_{pq}$.

We are now ready to introduce some relevant strings over the loci N_{pq}. In view of Proposition 3.2.10, there is a surjective morphism of smooth commutative group schemes

$$G^0|_{V_{pq}} \twoheadrightarrow \mathrm{Pic}^0_{\widetilde{\mathcal{D}}_{V_{pq}}/V_{pq}} \tag{5.8}$$

where $G^0|_{V_{pq}}$ is the identity component of $G|_{V_{pq}}$, which fiber-by-fiber realizes the Chevalley devissage.

Let $\Lambda^1_{V_{pq}} := R^1 e_* \mathbb{Q}_{\widetilde{K}_{V_{pq}}}$. We introduce the following complexes, viewable in $D^b MHM_{alg}(N_{pq})$ or $D^b(N_{pq}, \mathbb{Q})$, which we will again call strings.

$$\mathscr{I}^+_{N_{pq}} := \mathbb{Q}_{N_{pq}} \oplus j_{pq*}(\Lambda^1_{V_{pq}})[-1] \oplus \mathbb{Q}_{N_{pq}}[-2](-1) \tag{5.9}$$

$$\mathscr{I}^-_{N_{pq}} := j_{pq*}\mathscr{L}_{V_{pq}} \oplus j_{pq*}(\Lambda^1_{V_{pq}} \otimes \mathscr{L}_{V_{pq}})[-1] \oplus j_{pq*}\mathscr{L}_{V_{pq}}[-2](-1) \tag{5.10}$$

where $\mathscr{L}_{V_{pq}}$ is a rank one local system on V_{pq} with -1 monodromy around NR points and trivial monodromy around $N(3)$ points and $j_{pq}\colon U_{pq} \to N_{pq}$ is the inclusion.

In the remainder of the section, we study the strings $\mathscr{I}^+_{N_{pq}}$ and $\mathscr{I}^-_{N_{pq}}$. We begin by describing the Decomposition Theorem for the elliptic fibration $e\colon \widetilde{K} \to N_{pq}$.

Lemma 5.3.1. *Let $j_{pq}\colon V_{pq} \to N_{pq}$ be the inclusion map. There is an isomorphism*

$$Re_*\mathbb{Q}_{\widetilde{K}} \simeq \mathscr{I}^+_{N_{pq}} \oplus (\mathbb{Q}_{N_{pq}\cap N(3)} \oplus \mathbb{Q}^{\oplus 4}_{N_{pq}\cap NR})[-2](-1). \tag{5.11}$$

Proof. For reasons of bookkeeping, we prove the equivalent:

$$\begin{aligned} Re_*\mathbb{Q}_{\widetilde{K}}[2] \simeq \mathbb{Q}_{N_{pq}}[1][1] \oplus \big(&j_{pq*}\Lambda^1_{V_{pq}}[1] \oplus \mathbb{Q}_{N_{pq}\cap N(3)} \oplus \mathbb{Q}^{\oplus 4}_{N_{pq}\cap NR}\big)[0] \\ &\oplus \mathbb{Q}_{N_{pq}}[1][-1](-1) \end{aligned} \tag{5.12}$$

By looking at the regular part of p, the summands

$$\mathbb{Q}_{N_{pq}}[1][1] \oplus j_{pq*}\Lambda^1_{V_{pq}}[1][0] \oplus \mathbb{Q}_{N_{pq}}[1][-1](-1)$$

must appear in the Decomposition Theorem for $Re_*\mathbb{Q}_{\widetilde{K}}[2]$. Any additional summand in perverse degree 1 must be a skyscraper sheaf which would contribute non-trivially to $R^3 e_*\mathbb{Q}_{\widetilde{K}}$. Since the fibers are curves, no such summands can exist. Relative Hard Lefschetz then implies that the only additional summands that can appear are skyscrapers in perverse degree 0. With Kumar's description of the fibers of $e\colon \widetilde{K} \to B'$ and the identification of the base B' with N_{pq}, we see that the fiber of e over $C \in N(3)$ has two irreducible components and the fiber of e over $C \in NR$ has five irreducible components. Equation 5.12 then follows. $\square$

Corollary 5.3.2. *There is an isomorphism of rational polarizable Hodge structures*

$$H^*(N_{pq}, \mathscr{I}^+_{N_{pq}}) \simeq H^{even}(J) \oplus \mathbb{Q}^{\oplus 2}(-1). \tag{5.13}$$

Proof. Taking cohomology in Equation 5.11 gives $H^*(\widetilde{K}) = H^*(N_{pq}, \mathscr{I}^+_{N_{pq}}) \oplus \mathbb{Q}^{\oplus 14}(-1)$. Since $\widetilde{K}$ is the Kummer $K3$ surface associated to J, the Hodge structure of $\widetilde{K}$ is given by $H^*(\widetilde{K}) = H^{even}(J) \oplus \mathbb{Q}^{\oplus 16}(-1)$. The result then follows by combining these two equations. $\square$

To understand the string $\mathscr{I}^-_{N_{pq}}$, we first need to discuss the existence of a special double cover of the Kummer $K3$ surface $\widetilde{K}$, described by Mehran in [Meh11], which is "compatible" with the elliptic fibration $e\colon \widetilde{K} \to B'$. More precisely, consider the two singular fibers of type I_0^* in the elliptic fibration e. In each singular fiber of type I_0^*, there are four exceptional curves which appear with multiplicity one. Denote these exceptional curves by $E_1, \cdots, E_8$. One can show that $\Delta := E_1 + \cdots + E_8 \in 2NS(\widetilde{K})$. In particular, this means that we can find a double cover $q\colon Z \to \widetilde{K}$ branched along Δ. Notice if $F_i := q^{-1}(E_i)$ denotes the preimage of E_i, then the pullback of E_i is given by $q^*(E_i) = 2F_i$ and $F_i^2 = -1$. Thus, the F_i are exceptional curves and we can blow them down to obtain a surface $\tau\colon Z \to X$.

In our case, Mehran shows in [Meh11, Proposition 2.3] that the surface X is actually the Kummer $K3$ surface associated to some Abelian surface J'. Moreover, Mehran shows the Abelian surface J' admits an isogeny $J' \to J$ of degree 2. The 16 disjoint rational curves on X are easy to describe. Let $E_9, \cdots, E_{16}$ denote the eight rational curves in $\widetilde{K}$ which are not in the branch locus of q. Since q is an étale double cover outside of the branch locus, for $9 \leq i \leq 16$, the preimage of a curve $E_i \subset \widetilde{K}$ must be an étale double cover of $\mathbb{P}^1$ and thus consists of two disjoint rational curves $F_i, F_i' \subset Z$. The images of these curves in X under the blow down map $\tau\colon Z \to X$, which we will also denote by F_i and F_i', are the 16 disjoint rational curves on the Kummer $K3$ surface X. With this description of X, we make the following observation about the Hodge structures of X and Z.

Lemma 5.3.3. *There is an isomorphism of rational polarizable Hodge structures*

$$H^*(Z) \simeq H^{even}(J) \oplus \mathbb{Q}(-1)^{\oplus 24} \tag{5.14}$$

Proof. Since Z is the blow up of X at eight points, there is an isomorphism of rational Hodge structures $H^*(Z) \simeq H^*(X) \oplus \mathbb{Q}(-1)^{\oplus 8}$. Since X is the Kummer $K3$ associated to the Abelian surface J', there is an isomorphism of rational Hodge structures $H^*(X) \simeq H^{even}(J') \oplus \mathbb{Q}(-1)^{\oplus 16}$. The claim then follows by noticing that the Hodge structure of J' is isomorphic to the Hodge structure of J since J' is an étale cover of J. $\square$

We are now ready to study the string $\mathscr{I}^-_{N_{pq}}$. Let $q\colon Z \to \widetilde{K}$ be the double cover branched along Δ and $e\colon \widetilde{K} \to B'$ denote the elliptic fibration. Recall that by construction, Δ is contained within the two I_0^* fibers of e. Let $h \coloneqq e \circ q\colon Z \to B'$ denote the composition.

The number of irreducible components of h are straightforward to describe. The fiber of e over a point of V_{pq} is a smooth elliptic curve F which is disjoint from the branch locus. If the pre-image of this elliptic curve under q were connected, then F admits a degree two isogeny. Since $\widetilde{K}$ is is general Kummer $K3$ surface, this cannot happen. Thus, the pre-image of a smooth elliptic curve under the double cover q must split into two disjoint elliptic curves. In particular, the fiber of h over a point of V_{pq} has two irreducible components.

The fiber of e over an $N(3)$ point is disjoint from the branch locus and is a curve of type I_2, i.e. consists of two rational curves which meet transversely at two distinct points. Since q is étale over this fiber, each rational component must split into two disjoint rational components. It follows that a fiber of h over an $N(3)$ point has four irreducible components.

Finally, the fiber of e over a NR point intersects the branch locus and is a curve of type I_0^*. In particular, the fiber contains a non-reduced component which is a rational curve that intersects the branch locus in precisely four points. The preimage of this rational curve must be an elliptic curve since it is a double cover $\mathbb{P}^1$ branched at four points. In particular, it is connected and it follows that the fiber of h over an NR point consists of five irreducible components.

Using this description of the irreducible components of $h\colon Z \to B'$, we describe the Decomposition Theorem for h.

Lemma 5.3.4. *There is an isomorphism*

$$Rh_*\mathbb{Q}_Z \simeq \mathscr{I}^+_{N_{pq}} \oplus \mathscr{I}^-_{N_{pq}} \oplus \left(\mathbb{Q}^{\oplus 4}_{NR} \oplus \mathbb{Q}^{\oplus 2}_{N(3)}\right)[-2](-1) \tag{5.15}$$

Proof. Let $W = \widetilde{K} \setminus \Delta$ and consider the commutative diagram

$$
\begin{tikzcd}
Z_W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d, "q"] \arrow[dr, "h"] & \\
W \arrow[r, hook, "j_W"] & \widetilde{K} \arrow[r, "e"] \arrow[dr, phantom, "\square"] & B' \\
 & \widetilde{K}_{U_{pq}} \arrow[ul, hook] \arrow[u, hook, "j_{\widetilde{K}}"] \arrow[r, "e"] & U_{pq}. \arrow[u, hook, "j_{pq}"']
\end{tikzcd}
\tag{5.16}
$$

where the top left and bottom right squares are fiber product diagrams. Note that the inclusion $\widetilde{K}_{U_{pq}} \subset W$ holds since Δ is contained within the two I_0^* fibers of e. Since q is a branched double cover, $Rq_*\mathbb{Q}_Z = q_*\mathbb{Q}_Z = \mathbb{Q}_{\widetilde{K}} \oplus j_{W*}\mathscr{L}_W$, where $\mathscr{L}_W$ is the rank one local system on W corresponding to the representation

$$
\begin{tikzcd}
\pi_1(W) \arrow[r] \arrow[d, two heads] & \mathrm{Aut}(\mathbb{Q}) \\
\mathbb{Z}/2\mathbb{Z}. \arrow[ur, "-1"] &
\end{tikzcd}
$$

It follows that $Rh_*\mathbb{Q}_Z = Re_*\mathbb{Q}_{\widetilde{K}} \oplus Re_*(j_{W*}\mathscr{L}_W)$. By proper base change for e, there is an isomorphism $j_{pq}^* Re_*(j_{W*}\mathscr{L}_W) \simeq Re_*(j_{\widetilde{K}}^* j_{W*}\mathscr{L}_W)$.

Let $\mathscr{L}_{\widetilde{K}_{U_{pq}}}$ denote the pullback of the local system $\mathscr{L}_W$ to $\widetilde{K}_{U_{pq}}$. Since j_W is an open embedding, there is an isomorphism $j_W^* j_{W*}\mathscr{L}_W \simeq \mathscr{L}_W$ and this implies that there is an isomorphism $Re_*(j_{\widetilde{K}}^* j_{W*}\mathscr{L}_W) \simeq Re_*\mathscr{L}_{\widetilde{K}_{U_{pq}}}$.

We now claim that $\mathscr{L}_{\widetilde{K}_{U_{pq}}} \simeq e^*\mathscr{L}_{U_{pq}}$ where $\mathscr{L}_{U_{pq}}$ is the local system corresponding to the representation given by the bottom right triangle in the following commutative diagram.

$$
\begin{tikzcd}
\pi_1(\widetilde{K}_{V_{pq}}) \arrow[r, two heads] \arrow[d, two heads] & \pi_1(V_{pq}) \arrow[d, two heads] & \\
\pi_1(\widetilde{K}_{U_{pq}}) \arrow[r] \arrow[dr] & \pi_1(U_{pq}) \arrow[r] \arrow[d, two heads] & \mathrm{Aut}(\mathbb{Q}) \\
 & \mathbb{Z}/2\mathbb{Z} \arrow[ur, "-1"] &
\end{tikzcd}
$$

We note that the diagonal arrow in the bottom left triangle is defined to be the composition of the horizontal and vertical arrows. The top arrow in the square is a surjection by the long exact sequence of homotopy groups for a fibration. The left and right arrows in the square are surjections because the complements of the inclusions are both of complex codimension one. By commutativity of the diagram, the arrow $\pi_1(\widetilde{K}_{U_{pq}}) \to \mathbb{Z}/2\mathbb{Z}$ must be surjective as well. This implies that the large triangle on the bottom is precisely the representation which corresponds to the local system $e^*\mathscr{L}_{U_{pq}}$ which gives the desired isomorphism.

The isomorphism $\mathscr{L}_{\widetilde{K}_{U_{pq}}} \simeq e^*\mathscr{L}_{U_{pq}}$ implies that over V_{pq},

$$Re_*\mathscr{L}_{\widetilde{K}_{U_{pq}}}|_{V_{pq}} \simeq (Re_*\mathbb{Q}_{\widetilde{K}_{V_{pq}}}) \otimes \mathscr{L}_{V_{pq}} \simeq \bigoplus_{i=0}^{2}(R^i e_*\mathbb{Q}_{\widetilde{K}_{V_{pq}}} \otimes \mathscr{L}_{V_{pq}}).$$

where the first equality holds by the projection formula and the second by Deligne's Theorem. It follows that the strings $\mathscr{I}^+_{N_{pq}}$ and $\mathscr{I}^-_{N_{pq}}$ must be summands appearing in the Decomposition Theorem for $Rh_*\mathbb{Q}$. □

Corollary 5.3.5. *There is an isomorphism of rational Hodge structures*

$$H^*(N_{pq}, \mathscr{I}^-_{N_{pq}}) \simeq \mathbb{Q}^{\oplus 2}(-1). \tag{5.17}$$

Proof. Taking cohomology in Equation 5.15 gives and applying Corollary 5.3.2 gives $H^*(Z) \simeq H^{even}(J) \oplus H^*(N_{pq}, \mathscr{I}^-_{N_{pq}}) \oplus \mathbb{Q}(-1)^{\oplus 22}$. The result then follows from Equation 5.14. □

5.4 The Decomposition Theorem for the Lagrangian Fibrations $\widetilde{m}$ and n

In this section, we describe the Decomposition Theorems for the complexes $R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$ and $Rn_*\mathbb{Q}_N$. Similar to the O'Grady 10 case, we fall short of determining the exact shape of the Decomposition Theorem. However, the shortcoming is also measured by an integer r which appears in both expressions.

We begin with a lemma that restates some facts about the strings $\mathscr{I}_B$, $\mathscr{I}^+_R$, $\mathscr{I}^-_R$, $\mathscr{I}^+_{N_{pq}}$, and $\mathscr{I}^-_{N_{pq}}$ we have proved in previous sections.

Proposition 5.4.1.

1. *The cohomology sheaf of $\mathscr{I}_B$ in degree* 6 *takes the following shape:*

$$\mathcal{H}^6(\mathscr{I}_B) = \mathbb{Q}_B \oplus^{\leq 0} \mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda^4_B)).$$

2. *The cohomology sheaves of* $\mathscr{I}_R^+$ *and* $\mathscr{I}_R^-$ *in degree* 4 *take the following shape:*
$$\mathcal{H}^4(\mathscr{I}_R^+) = \mathbb{Q}_R, \quad \mathcal{H}^4(\mathscr{I}_R^-) = i_{R^0 *}\mathscr{L}_{R^0}.$$

3. *The cohomology sheaves of* $\mathscr{I}_{N_{pq}}^+$ *and* $\mathscr{I}_{N_{pq}}^-$ *in degree* 2 *take the following shape:*
$$\mathcal{H}^2(\mathscr{I}_{N_{pq}}^+) = \mathbb{Q}_{N_{pq}}, \quad \mathcal{H}^2(\mathscr{I}_{N_{pq}}^-) = i_{V_{pq}*}\mathscr{L}_{V_{pq}}.$$

Proof. The claim about the string $\mathscr{I}_B$ is proved in Lemma 5.1.2. The claims about the strings $\mathscr{I}_R^+$ and $\mathscr{I}_R^-$ are proved in Lemma 5.2.1 and Lemma 5.2.2 respectively. The claims about the strings $\mathscr{I}_{N_{pq}}^+$ and $\mathscr{I}_{N_{pq}}^-$ follows immediately from their definition. □

Proposition 5.4.2. *Let* $\langle \bullet \rangle := [-2\bullet](-\bullet)$. *The Decomposition Theorems for* $R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$, $Rn_*\mathbb{Q}_N$ *in* $D^bMHM_{alg}(B)$ *take the following form:*

$$R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}} \simeq \mathscr{I}_B \oplus \mathscr{I}_R^+\langle 1\rangle \oplus \bigoplus_{p,q\in J[2],p\neq q} \mathscr{I}_{N_{pq}}^+\langle 2\rangle \oplus \mathbb{Q}_{NR}^{\oplus r+16}\langle 3\rangle \tag{5.18}$$

$$Rn_*\mathbb{Q}_N \simeq \mathscr{I}_B \oplus \mathscr{I}_R^-\langle 1\rangle \oplus \bigoplus_{p,q\in J[2],p\neq q} \mathscr{I}_{N_{pq}}^-\langle 2\rangle \oplus \mathbb{Q}_{NR}^{\oplus r}\langle 3\rangle. \tag{5.19}$$

where $r = 0$ *or* 1.

Proof. Since the triples $(\widetilde{M}, B, G)$ and (N, B, G) are δ-regular weak Abelian fibrations satisfying the assumptions of the Ngô Support Theorem by Proposition 2.6.7, we can apply Theorem 3.

We first deal with $R\widetilde{m*}\mathbb{Q}_{\widetilde{M}}$. By Proposition 2.4.2, there is an isomorphism

$$R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}} \simeq Rm_*\mathscr{IC}_M \oplus R(m|_\Sigma)_*\mathbb{Q}_\Sigma\langle 1\rangle \oplus R(m|_\Omega)_*\mathbb{Q}_\Omega\langle 3\rangle,$$

where $\Sigma \simeq J^\vee \times J/\pm 1$ is the singular locus of M and $\Omega \simeq (J^\vee \times J)[2]$ is the singular locus of Σ (cf. Section 5.2). Since the restrictions of the support morphism m to Σ and Ω have images R and NR respectively, the subvarieties B, R, NR must be supports for $\widetilde{m}$ and the complexes $R(m|_\Sigma)_*\mathbb{Q}_\Sigma$, $R(m|_\Omega)_*\mathbb{Q}_\Omega$ are direct summands of $R\widetilde{m}_*\mathbb{Q}_{\widetilde{M}}$. We note that Lemma 5.2.1 implies that $R(m|_\Sigma)_*\mathbb{Q}_\Sigma \simeq \mathscr{I}_R^+$ and Lemma 3.4.9 implies that $R(m|_\Omega)_*\mathbb{Q}_\Omega \simeq \mathbb{Q}_{NR}^{\oplus 16}$.

By Proposition 4.1.1, the local systems appearing in Theorem 3 applied to each of these three supports are constant of some strictly positive ranks $r_{\widetilde{M},B}, r_{\widetilde{M},R}$, and $r_{\widetilde{M},NR}$. It follows that from Theorem 3 that the local systems appearing in the string $\mathscr{I}_R^+$ can be identified with the cohomology of the Abelian part of the identity component of the group scheme G over some

Zariski open subset of R and the direct sum of the Ngô strings associated with these three supports is:

$$\mathscr{I}_B^{\oplus r_{\widetilde{M},B}} \oplus \mathscr{I}_R^{+\oplus r_{\widetilde{M},R}}\langle 1\rangle \oplus \mathbb{Q}_{NR}^{r_{\widetilde{M},NR}}\langle 3\rangle.$$

According to Proposition 5.4.1, the combined contribution to the highest direct image $R^6_{\widetilde{M}}$ takes the form:

$$\big(\mathbb{Q}_B^{\oplus r_{\widetilde{M},B}} \oplus {}^{\leq 0}\mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda_B^4)) \oplus \mathbb{Q}_R^{\oplus r_{\widetilde{M},R}} \oplus \mathbb{Q}_{NR}^{\oplus r_{\widetilde{M},NR}}\big)(-3).$$

Proposition 4.1.1 then implies that $r_{\widetilde{M},|2\theta|} = r_{\widetilde{M},R} = 1$ and $16 \leq r_{\widetilde{M},NR} \leq 34$.

For any $p, q \in J[2]$ with $p \neq q$, this contribution restricted to the locus V_{pq} is simply $\mathbb{Q}_{V_{pq}}$. Since $R^6_{\widetilde{M}|_{V_{pq}}} \simeq \mathbb{Q}_{V_{pq}}^{\oplus 2}(-3)$ by Proposition 4.2.6, N_{pq} must be a support and the associated Ngô string is $\mathscr{I}^+_{N_{pq}}\langle 2\rangle$. Thus, we have shown that the direct sum of the Ngô strings appearing in the Decomposition Theorem for $\widetilde{M}$ associated with the supports S, R, N_{pq}, and NR is

$$\mathscr{I}_B \oplus \mathscr{I}_R^+\langle 1\rangle \oplus \bigoplus_{p,q\in J[2],p\neq q} \mathscr{I}^+_{N_{pq}}\langle 2\rangle \oplus \mathbb{Q}_{NR}^{\oplus r_{\widetilde{M},NR}}\langle 3\rangle \tag{5.20}$$

According to our description of $R^6_{\widetilde{M}}$, the only other possible support is $N(3)$. If $C \in N(3)$ has nodes at three distinct points $p, q, r \in J[2]$, then $\{C\} = N_{pq} \cap N_{pr} \cap N_{qr}$. It follows that the restriction of Equation 5.20 to C is $\mathbb{Q}^{\oplus 4}$. Since $R^6_{\widetilde{M}}|_C \simeq \mathbb{Q}^{\oplus 4}$ by Proposition 4.1.1, C is not a support. It follows that $N(3)$ is not a support and Equation 5.18 holds. Moreover, we can also say that the skyscraper sheaf ${}^{\leq 0}\mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda_B^4))$ is not supported on $N(3)$.

The skyscraper sheaf ${}^{\leq 0}\mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda_B^4))$ can thus only be supported on NR. The stalks of this sheaf over different points of NR must all be isomorphic by Remark 3.4.1. It follows that there exists some integer $r_{2,4} \geq 0$ such that

$${}^{\leq 0}\mathcal{H}^2(\mathscr{I}\mathscr{C}_B(\Lambda_B^4)) = \mathbb{Q}_{NR}^{\oplus r_{2,4}}. \tag{5.21}$$

Although the presence of this skyscraper sheaf prohibits us from determining the exact shape of the Decomposition Theorem, we can say the following. Since any $C \in NR$ passes through exactly six 2-torsion points, C lies in $15 = \binom{6}{2}$ lines of the form N_{pq}. It follows that the restriction of Equation 5.20 to C is $\mathbb{Q}^{\oplus 17} \oplus \mathbb{Q}^{\oplus r_{2,4}}$. It follows that $r_{\widetilde{M},NR} = 34 - 17 - r_{2,4} = 17 - r_{2,4}$.

We now deal with $Rn_*\mathbb{Q}_N$. Proposition 4.1.1 implies that B must be a support and by the Ngô Support Theorem (see Theorem 3), the associated Ngô string is $\mathscr{I}_B$. Proposition 5.4.1 and the fact that the skyscraper sheaf

${}^{\leq 0}\mathcal{H}^2(\mathscr{IC}_B(\Lambda_B^4))$ is supported on NR imply that this contribution restricted to R^0 is $\mathbb{Q}_{R^0}$. Since $R_N^6|_{R^0} \simeq (\mathbb{Q}_{R^0} \oplus \mathscr{L}_{R^0})(-3)$ by Proposition 4.1.1, we see that R is must be a support and by Ngô Support Theorem, the associated Ngô string is $\mathscr{I}_R^-$.

Again by Proposition 5.4.1, for any $p, q \in J[2]$ with $p \neq q$, the contribution of the strings $\mathscr{I}_B$ and $\mathscr{I}_R^-$ restricted to V_{pq} is $\mathbb{Q}_{V_{pq}}$. Since $R_N^6|_{V_{pq}} \simeq (\mathbb{Q}_{V_{pq}} \oplus \mathscr{L}_{V_{pq}})(-3)$ by Proposition 4.2.6, we see that N_{pq} must be a support and by Theorem 3, the associated Ngô string is $\mathscr{I}_{N_{pq}}^-$. To summarize, we have so far shown that the direct sum of the Ngô strings appearing in the Decomposition Theorem for N associated with the supports S, R, N_{pq}, and NR is

$$\mathscr{I}_B \oplus \mathscr{I}_R^-\langle 1\rangle \oplus \bigoplus_{p,q\in J[2], p\neq q} \mathscr{I}_{N_{pq}}^-\langle 2\rangle. \tag{5.22}$$

According to our description of R_N^6, the only other possible supports are $N(3)$ and NR. If $C \in N(3)$ has nodes at three distinct points $p, q, r \in J[2]$, then $\{C\} = N_{pq} \cap N_{pr} \cap N_{qr}$. The description of the local system $\mathscr{L}_{V_{pq}}$ give in Proposition 4.2.6 implies that the restriction of Equation 5.22 to C is $\mathbb{Q}^{\oplus 4}$. Since $R_N^6|_C \simeq \mathbb{Q}^{\oplus 4}$ by Proposition 4.1.1, C is not a support. It follows that $N(3)$ is not a support.

If $C \in NR$, then the description of the local systems $\mathscr{L}_{R^0}$ and $\mathscr{L}_{V_{pq}}$ imply that restriction of Equation 5.22 to C is $\mathbb{Q}$. Since $R_N^6|_C \simeq \mathbb{Q}^{\oplus 2}$ by Proposition 4.1.1, C is potential support. Using the description of the skyscraper sheaf ${}^{\leq 0}\mathcal{H}^2(\mathscr{IC}_B(\Lambda_B^4))$ given in Equation 5.21, we conclude that

$$r_{N,NR} = 2 - 1 - r_{2,4} = 1 - r_{2,4}. \tag{5.23}$$

Letting $r = r_{N,NR} = 1 - r_{2,4}$, we see that $r_{\widetilde{M},NR} = r + 16$ as desired. We note that $r = 0$ or 1 with $r = 0$ if and only if the contribution from the skyscraper sheaf $r_{2,4} = 1$ and $r = 1$ if and only if the contribution from $r_{2,4}$ is 0. $\square$

5.5 Proof of Main Theorem 1

Using the description of the Decomposition theorem for $\widetilde{m}$ and n given in Proposition 5.4.2, we can prove our main Theorem 1 on the Hodge numbers of $OG6$ type manifolds. We begin by recalling the Fact from [dCRS21, Section 5.5].

Fact. *Let $\mathcal{A}$ be a semisimple Abelian category where every object has finite length and the isomorphism classes of simple objects form a set $\mathfrak{S}$. Every* *object $a \in \mathcal{A}$ is isomorphic to a unique finite direct sum of simple objects with multiplicities*

$$a \simeq \bigoplus_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{s}^{\oplus n_{\mathfrak{s}}(a)}$$

If we have an identity $[a] = [b] - [c]$ in the Grothendieck group $K(\mathcal{A})$ with $a, b, c \in \mathcal{A}$, then

$$n_{\mathfrak{s}}(a) = n_{\mathfrak{s}}(b) - n_{\mathfrak{s}}(c)$$

Let $\phi\colon \mathrm{Obj}(\mathcal{A}) \to \mathfrak{M}$ be an assignment into a commutative group which is additive in exact sequences. If $[a] = [b] - [c]$ as above, then

$$\phi(a) = \phi(b) - \phi(c) \tag{5.24}$$

In the remainder of this section, we let $\mathcal{A} = \mathbb{Q}GPPHS$ be the category of rational graded polarizable pure Hodge structures.

Proposition 5.5.1. *By abuse of notation, for a projective manifold X, we denote the graded rational polarizable pure Hodge structure $H^*(X, \mathbb{Q})$ simply by X. Recall that A is the Abelian four-fold $A \coloneqq J^\vee \times J$. Let $\langle \bullet \rangle \coloneqq [-2\bullet](-\bullet)$. Then we have*

$$\begin{aligned} b_*(\widetilde{M}) = b_*(N) + b_*((A/\pm 1)^{\oplus 2}[-2]) - b_*(A[-2]) \\ + b_*((J/\pm 1)^{\oplus 120}[-4]) + b_*(NR^{\oplus 16}[-6]) \end{aligned} \tag{5.25}$$

$$\begin{aligned} h^{\bullet,\star}(\widetilde{M}) = h^{\bullet,\star}(N) + h^{\bullet,\star}((A/\pm 1)^{\oplus 2}\langle 1\rangle) - h^{\bullet,\star}(A\langle 1\rangle) \\ + h^{\bullet,\star}((J/\pm 1)^{\oplus 120}\langle 2\rangle) + h^{\bullet,\star}(NR^{\oplus 16})\langle 3\rangle. \end{aligned} \tag{5.26}$$

Proof. By Proposition 5.4.2 and Lemma 5.2.1, we have isomorphisms of finite dimensional rational graded vector spaces, or in fact of rational polarizable graded pure Hodge structures

$$H^*(\widetilde{M}) \simeq H^*(\mathscr{I}_B) \oplus H^*(\mathscr{I}_R^+)\langle 1\rangle \oplus \bigoplus_{p,q \in J[2], p \neq q} H^*(\mathscr{I}_{N_{pq}}^+)\langle 2\rangle \oplus \mathbb{Q}_{NR}^{\oplus r+16}\langle 3\rangle,$$

$$H^*(N) \simeq H^*(\mathscr{I}_B) \oplus H^*(\mathscr{I}_R^-)\langle 1\rangle \oplus \bigoplus_{p,q \in J[2], p \neq q} H^*(\mathscr{I}_{N_{pq}}^-)\langle 2\rangle \oplus \mathbb{Q}_{NR}^{\oplus r}\langle 3\rangle,$$

$$H^*(A/\pm 1) \simeq H^*(\mathscr{I}_R^+),$$

$$H^*(A) \simeq H^*(\mathscr{I}_R^+) \oplus H^*(\mathscr{I}_R^-),$$

$$H^*(\mathscr{I}_{N_{pq}}^+) \simeq H^*(J/\pm 1) \oplus H^*(\mathscr{I}_{N_{pq}}^-).$$

Working in the Abelian category $\mathbb{Q}$PPHS, we obtain the the identity

$$\begin{aligned} H^*(\widetilde{M}) = H^*(N) &+ \big(H^*(A/\pm 1)^{\oplus 2} - H^*(A)\big)\langle 1\rangle \\ &+ H^*(J/\pm 1)^{\oplus 120}\langle 2\rangle \\ &+ H^*(NR)^{\oplus 16}\langle 3\rangle \end{aligned} \tag{5.27}$$

in the Grothendieck group $K(\mathcal{A})$. Applying Equation 5.24 gives the result. □

Proof of Main Theorem 1. Recall that the Betti and Hodge numbers of a manifold in the deformation class $OG6$ can be computed from any representative and in paticular, from the representative $\widetilde{M}$.

The Betti and Hodge numbers of all varieties appearing on the right hand side of Equations 5.25 and 5.26 are known. We record the Hodge diamond of N, which follows Göttsche-Soergel formula [GS93], below for convenience

$$\begin{array}{ccccccccccccc} &&&&&&1&&&&&& \\ &&&&&0&&0&&&&& \\ &&&&1&&5&&1&&&& \\ &&&0&&4&&4&&0&&& \\ &&1&&6&&37&&6&&1&& \\ &0&&4&&24&&24&&4&&0& \\ 1&&5&&37&&372&&37&&5&&1. \end{array}$$

We note that $b_0(NR) = h^{0,0}(NR) = 16$ since the finite set NR has cardinality 16. It then follows from Equation 5.26 that the Hodge diamond of $\widetilde{M}$ is given by

$$\begin{array}{ccccccc} &&&1&&& \\ &&1&6&1&& \\ &1&12&173&12&1& \\ 1&6&173&1144&173&6&1 \end{array} \tag{5.28}$$

as desired. □

5.6 Proof of Main Theorem 2

The graded pure polarizable Hodge structures of all varieties on the right hand side of Equation 5.27 are known and can be expressed in terms of of the Hodge structure of the underlying Abelian surface J. Let $V := H^*(J, \mathbb{Q})$ be the rational Hodge structure of the Abelian surface J. Let $U = H^{even}(J, \mathbb{Q}) = H^*(J/\pm 1, \mathbb{Q})$ and $W = H^{odd}(J, \mathbb{Q})$. Let $\langle \bullet \rangle := [-2\bullet](-\bullet)$.

In [GS93, §6], Göttsche and Soergel express the rational Hodge structure of $N \times J$ in terms of the rational Hodge structure of J. The rational Hodge structure of just N can also be described in terms of the rational Hodge structure of J and can be read off from the LLV decomposition of the cohomology of N described in Corollary 3.6 of [GKLR22]. More precisely, there is an action of the LLV algebra $\mathfrak{g} \simeq \mathfrak{so}(4,5)$ on the full cohomology $H^*(N)$. Since $\mathfrak{g}$ is semi-simple, $H^*(N)$ decomposes as a direct sum of irreducible $\mathfrak{g}$-modules of highest weight μ. This decomposition, called the LLV decomposition, for N is given as

$$H^*(N) = V_{(3)} \oplus V_{(1,1)} \oplus V^{\oplus 16} \oplus \mathbb{Q}^{\oplus 240} \oplus V_{(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})}, \tag{5.29}$$

where

$$V = \mathbb{Q} \oplus H^2(N)[-2] \oplus \mathbb{Q}\langle 2\rangle \tag{5.30}$$

is the standard representation of $\mathfrak{g}$. We note that the special Mumford-Tate algebra $\overline{\mathfrak{m}} \simeq \mathfrak{so}(2,3)$ is contained in the LLV algebra $\mathfrak{g}$ and that the Hodge structure of $H^*(N)$ is determined by the $\overline{\mathfrak{m}}$-module structure. In particular, this implies that the LLV decomposition can be interpreted as a decomposition of Hodge structures.

To express the Hodge structure of N in terms of the Hodge structure of J, we have the following Lemma.

Lemma 5.6.1. *There is an isomorphism of weight two $\mathbb{Q}$-Hodge structures*

$$H^2(N) = H^2(J) \oplus \mathbb{Q}(-1). \tag{5.31}$$

Proof. Recall that sheaves in N have Mukai vector $w = (0, 2\theta, -3)$. By [Yos01, Theorem 0.2], there is an isomorphism of weight two $\mathbb{Q}$-Hodge structures

$$(w^\perp, (\cdot,\cdot)) = (H^2(N), B_N), \tag{5.32}$$

where $w^\perp := \{x \in H^{even}(J) \mid (w, x) = 0\}$, $(\cdot,\cdot)$ is the Mukai pairing (cf. Section 2.4), and B_N is the Beauville-Bogomolov-Fujiki form. If $x = (x_0, x_2, x_4)$ is a vector in $H^{even}(J)$, then

$$(w, x) = -2\theta \cdot x_2 - 3x_0. \tag{5.33}$$

We note that the weight two Hodge structure on $w^\perp$ is inherited from the weight two Hodge structure on $H^{even}(J)$, which is defined to be

$$H^{2,0} = H^{2,0}(J)$$
$$H^{1,1} = H^0(J)(-1) \oplus H^{1,1}(J) \oplus H^4(J)(1)$$
$$H^{0,2} = H^{0,2}(J)$$

From Equation 5.33, we see that $x \in w^\perp$ if and only if either

1. $x_2 \in T \subset H^2(J)$ and $x_4 \in \mathbb{Q}$, where T is transcendental weight two Hodge structure of J, i.e. T is the subspace in $H^2(J)$ orthogonal to θ with respect to the intersection pairing.

2. $x_2 = k\theta$, $x_0 = \frac{-4k}{3}$ for $k \in \mathbb{Q}$, and $x_4 \in \mathbb{Q}$.

Let $\langle\theta\rangle \subset H^2(J)$ be the span of θ. It follows that the linear map

$$\langle\theta\rangle \oplus T \oplus \mathbb{Q}(-1) \to w^\perp \tag{5.34}$$

sending θ to $(\frac{-4}{3}, \theta, 0)$, $x \in T$ to $(0, x, 0)$ and $\lambda \in \mathbb{Q}(-1)$ to $(0, 0, \lambda)$ is an isomorphism of weight two $\mathbb{Q}$-Hodge structures. Since $H^2(J) = \langle\theta\rangle \oplus T$ by definition, we get the desired isomorphism. □

We now claim that we can view the right hand side of Equation 5.29 as an $\mathfrak{so}(4,4)$-module via restriction of scalars. From Lemma 5.6.1 and Equation 5.30, we see that $V = U \oplus \mathbb{Q}\langle 1\rangle$ as an $\mathfrak{so}(4,4)$ where, as above, $U = H^{even}(J)$. Using the branching rules described in [GKLR22, Appendix B.2.1] (one can also see [FH91, §25.3] for a more in depth discussion), we have that as $\mathfrak{so}(4,4)$-modules,

$$H^*(N) = \operatorname{Sym}^3 U \oplus U^{\otimes 2}\langle 1\rangle \oplus U^{\oplus 16}\langle 2\rangle \oplus \mathbb{Q}^{\oplus 256}\langle 3\rangle \oplus (U \otimes S^+ \oplus U \otimes S^-)\langle 1\rangle,$$

where S^+ and S^- are the two half spin representations. We note that $S^+ = W = H^{odd}(J)$. Again using the branching rules, we see that S^+ and S^- are isomorphic as $\overline{\mathfrak{m}}$-modules. In particular, this means that they are isomorphic as Hodge structures. It follows that the rational Hodge structure of N is given by

$$H^*(N) = \operatorname{Sym}^3 U \oplus U^{\otimes 2}\langle 1\rangle \oplus U^{\oplus 16}\langle 2\rangle \oplus \mathbb{Q}^{\oplus 256}\langle 3\rangle \oplus (U \otimes W)^{\oplus 2}\langle 1\rangle. \tag{5.35}$$

Using the identification of the rational Hodge structure $H^*(J^\vee)$ with $H^*(J)$ via the principal polarization θ, the rational Hodge structure of $A = J^\vee \times J$ is

$$H^*(A) = V^{\otimes 2} = U^{\otimes 2} \oplus \big(U \otimes W\big)^{\oplus 2} \oplus W^{\otimes 2}. \tag{5.36}$$

The rational Hodge structure of $A/\pm 1$ is the $\mathbb{Z}/2\mathbb{Z}$-invariant part of $H^*(A)$, which is given by

$$H^*(A/\pm 1) = U^{\otimes 2} \oplus W^{\otimes 2}. \tag{5.37}$$

Proof of Theorem 2. Substituting Equations 5.35, 5.36, 5.37 into Equation 5.27 expresses the Hodge structure of $\widetilde{M}$ in terms of U and W. Since the category of rational polarizable Hodge structures is semi-simple, we can make cancellations and find that

$$H^*(\widetilde{M}) = \operatorname{Sym}^3 U \oplus (U^{\otimes 2})^{\oplus 2}\langle 1\rangle \oplus W^{\otimes 2}\langle 1\rangle \oplus U^{\oplus 137}\langle 2\rangle \oplus \mathbb{Q}^{\oplus 512}\langle 3\rangle. \tag{5.38}$$

as desired. □

5.6.1 Relation to the LLV Decomposition

In this section, we show that our description of the Hodge structure of $\widetilde{M}$ in Theorem 2 agrees with the LLV decomposition of the Hodge structure given by Green, Kim, Laza, and Robles in [GKLR22] for manifolds of $OG6$ type.

We first express the Hodge structure of $\widetilde{M}$ purely in terms of $U = H^{even}(J)$. To do this, we will need the following lemmas, which use the language of Schur functors. In particular, for each partition λ of an integer k, let $\mathbb{S}_{(\lambda)}(-)$ denote the corresponding Schur functor. For more details about the notation and basic details, see [FH91, Ch. 6]. As discussed in [dCRS21, §6.2], if V is a rational polarizable Hodge structure, then each Schur module $\mathbb{S}_{(\lambda)}(V)$ is also a rational polarizable Hodge structure.

Lemma 5.6.2. *Let $H^1 = H^1(J,\mathbb{Q})$. There are isomorphisms of rational polarizable Hodge structures*

$$\begin{gathered}
\Lambda^3 H^1 \otimes H^1 = \mathbb{S}_{(2,1,1)}(H^1) \oplus \mathbb{Q}(-2),\\
\Lambda^2(\Lambda^2 H^1) = \mathbb{S}_{(2,1,1)}(H^1) \oplus \mathbb{Q}(-2),\\
\Lambda^2 H^1 \otimes \Lambda^4 H^1 = \mathbb{S}_{(2,2,1,1)}(H^1),\\
\Lambda^3(\Lambda^2 H^1) = \mathbb{S}_{(3,1,1,1)}(H^1) \oplus \mathbb{S}_{(2,2,2)}(H^1),\\
\Lambda^3 H^1 \otimes \Lambda^3 H^1 = \mathbb{S}_{(2,2,2)}(H^1) \oplus \mathbb{S}_{(2,2,1,1)}(H^1),\\
\mathbb{S}_{(2,2,2)}((H^1)^\vee) = \mathbb{S}_{(3,1,1,1)}(H^1)(6),\\
\mathbb{S}_{(2,2,1,1)}((H^1)^\vee) = \mathbb{S}_{(2,2,1,1)}(H^1)(6).
\end{gathered}$$

Proof. The first, third, and fifth statements follow from the Equation 6.9 on page 79 in [FH91]. The second statement follows from Exercise 6.16 on page 81 in [FH91]. To prove the fourth statement, notice that by Exercise 6.5 on page 78 in [FH91],

$$(\Lambda^2 H^1)^{\otimes 3} \simeq \mathrm{Sym}^3(\Lambda^2 H^1) \oplus \Lambda^3(\Lambda^2 H^1) \oplus \mathbb{S}_{(2,1)}(\Lambda^2 H^1)^{\oplus 2}. \tag{5.39}$$

On the other hand, by Exercise 6.9 on Page 82 in [FH91],

$$\begin{aligned}
(\Lambda^2 H^1)^{\otimes 3} = \mathbb{S}_{3,3}(H^1) \oplus \mathbb{S}_{(2,2,2)}(H^1) \oplus \mathbb{S}_{(3,1,1,1)}(H^1) \oplus \mathbb{S}_{(1,1,1,1,1,1)}(H^1)\\
\oplus \mathbb{S}_{(3,2,1)}(H^1)^{\oplus 2} \oplus \mathbb{S}_{(2,1,1,1,1)}(H^1)^{\oplus 2}\\
\oplus \mathbb{S}_{(2,2,1,1)}(H^1)^{\oplus 3}.
\end{aligned} \tag{5.40}$$

The dimensions of the Schur modules appearing in the right hand side of Equation 5.40 can be computed using Exercise 6.4 on page 78 in [FH91]. Comparing dimensions between Equations 5.39 and 5.40 then gives the last statement. The last two statements follow from Exercise 15.50 on page 233 in [FH91]. □

Lemma 5.6.3. *Let $U = H^{even}(J)$, $W = H^{odd}(J)$, and $W' := H^{odd}(J^\vee)$. There is an isomorphism of rational polarizable Hodge structures*

$$\Lambda^3 U \oplus U(-2) = (W \otimes W')(-1). \tag{5.41}$$

Proof. The idea is to express the right hand side and the left hand side of Equation 5.41 in terms of various Schur modules associated to $H^1(J, \mathbb{Q})$.

We begin with the right hand side of Equation 5.41. Writing $W = H^1_J \oplus H^3_J$ and $W' = H^1_{J^\vee} \oplus H^3_{J^\vee}$, there is an isomorphism

$$\begin{aligned}(W \otimes W')(-1) = &(H^1_J \otimes H^1_{J^\vee})(-1)\\ &\oplus \big((H^1_J \otimes H^3_{J^\vee}) \oplus (H^1_{J^\vee} \otimes H^3_J)\big)(-1)\\ &\oplus (H^3_J \otimes H^3_{J^\vee})(-1).\end{aligned}$$

Since $J^\vee$ is the dual Abelian surface of J, there is an isomorphism $H^1_{J^\vee} = (H^1_J)^\vee(-1)$ given by the principal polarization θ. Poincaré duality gives an isomorphism $(H^1_J)^\vee = H^3_J(2)$. Combining these gives an isomorphism

$$H^1_{J^\vee} = H^3_J(1). \tag{5.42}$$

Using this isomorphism, we see that

$$\begin{aligned}(W \otimes W')(-1) = &(H^1_J \otimes H^3_J)\\ &\oplus (H^3_{J^\vee})^{\otimes 2} \oplus (H^3_J)^{\otimes 2}\\ &\oplus (H^3_J \otimes H^1_J)(-2)\end{aligned}$$

Using the isomorphisms $H^3_J = \Lambda^3 H^1_J$ and $H^3_{J^\vee} = \Lambda^3 H^1_{J^\vee}$, we see that

$$\begin{aligned}(W \otimes W')(-1) = &(H^1_J \otimes \Lambda^3 H^1_J)\\ &\oplus (\Lambda^3 H^1_{J^\vee})^{\otimes 2} \oplus (\Lambda^3 H^1_J)^{\otimes 2}\\ &\oplus (H^1_J \otimes \Lambda^3 H^1_J)(-2)\end{aligned} \tag{5.43}$$

Using the isomorphism $H^1_{J^\vee} = (H^1_J)^\vee(-1)$, we can rewrite

$$(\Lambda^3 H^1_{J^\vee})^{\otimes 2} = (\Lambda^3 (H^1_J)^\vee)^{\otimes 2}(-6).$$

Using Lemma 5.6.2 to express Equation 5.43 in terms of Schur modules gives

$$\begin{aligned}(W \otimes W')(-1) = &\mathbb{S}_{(2,1,1)}(H^1_J) \oplus \mathbb{Q}(-2)\\ &\oplus \mathbb{S}_{(3,1,1,1)}(H^1_J) \oplus \mathbb{S}_{(2,2,2)}(H^1_J) \oplus \mathbb{S}_{(2,2,1,1)}(H^1_J)^{\oplus 2}\\ &\oplus (\mathbb{S}_{(2,1,1)}(H^1_J) \oplus \mathbb{Q}(-2))(-2)\end{aligned} \tag{5.44}$$

We now examine the left hand side of Equation 5.41. Since we only deal with the cohomology of J and not of $J^\vee$, we will denote H^i_J simply by H^i. Write $U = H^0 \oplus H^2 \oplus H^4$ and note that $H^2 = \Lambda^2 H^1$, and $H^4 = \Lambda^4 H^1$. There is an isomorphism

$$\Lambda^3 U = \Lambda^2 H^2 \oplus \left(H^2 \otimes H^4 \oplus \Lambda^3 H^2\right) \oplus \Lambda^2 H^2 \otimes H^4. \tag{5.45}$$

Moreover, we also have isomorphisms $\Lambda^2 H^2 \otimes H^4 = \Lambda^2 H^2(-2)$ and $\Lambda^2 H^1(-2) = \Lambda^2 H^1 \otimes H^4$. Using Equation 5.45 and these isomorphisms, the left hand side of Equation 5.41 can be rewritten as

$$\begin{aligned} \Lambda^3 U \oplus U(-2) = &\left(\Lambda^2(\Lambda^2 H^1) \oplus \mathbb{Q}(-2)\right) \\ &\oplus \left(\Lambda^3(\Lambda^2 H^1) \oplus (\Lambda^2 H^1 \otimes \Lambda^4 H^1)^{\oplus 2}\right) \\ &\oplus \left(\Lambda^2(\Lambda^2 H^1) \oplus \mathbb{Q}(-2)\right)(-2). \end{aligned} \tag{5.46}$$

Using Lemma 5.6.2 to express Equation 5.46 in terms of Schur modules gives

$$\begin{aligned} \Lambda^3 U \oplus U(-2) = &\,\mathbb{S}_{(2,1,1)}(H^1) \oplus \mathbb{Q}(-2) \\ &\oplus \mathbb{S}_{(3,1,1,1)}(H^1) \oplus \mathbb{S}_{(2,2,2)}(H^1) \oplus \mathbb{S}_{(2,2,1,1)}(H^1)^{\oplus 2} \\ &\oplus (\mathbb{S}_{(2,1,1)}(H^1) \oplus \mathbb{Q}(-2))(-2) \end{aligned} \tag{5.47}$$

Comparing Equations 5.44 and 5.47 and recalling that $H^1_J = H^1$ then gives the desired isomorphism. □

Combining Theorem 2 with Lemma 5.6.3 leads to the following Corollary.

Corollary 5.6.4. *There is an isomorphism of rational graded polarizable pure Hodge structures*

$$H^*(\widetilde{M}) = \operatorname{Sym}^3 U \oplus \Lambda^3 U \oplus (U^{\otimes 2})^{\oplus 2}(-1) \oplus U^{\oplus 138}(-2) \oplus \mathbb{Q}^{\oplus 512}(-3). \tag{5.48}$$

We now show that this description of the Hodge structure of $\widetilde{M}$ in terms of the even cohomology of J agrees with the LLV decomposition for manifolds of $OG6$ type given by Green, Kim, Laza, and Robles in [GKLR22]. More precisely, they show in Theorem 3.39 that

$$H^*(\widetilde{M}) = V_{(3)} \oplus V_{(1,1,1)} \oplus V^{\oplus 135} \oplus \mathbb{Q}^{\oplus 240} \tag{5.49}$$

as $\mathfrak{g}(4,6)$-modules, where $V = \mathbb{Q} \oplus H^2(\widetilde{M})[-2] \oplus \mathbb{Q}\langle 2\rangle$ is the standard representation of $\mathfrak{g}$. Lemma 5.6.1 and Theorem 2 imply that $H^2(\widetilde{M}) = H^2(J) \oplus \mathbb{Q}^{\oplus 2}$ as $\mathbb{Q}$-Hodge structures. This can also be seen more directly using [Rap08, Proposition 2.2.1, Corollary 3.5.13]) and using the fact that a birational map between two IHS manifolds induces a $\mathbb{Z}$-Hodge structure isomorphism on their second cohomology (cf. [GHJ03, Part III Section 27.1]). It follows that V can be identified with $U \oplus \mathbb{Q}^{\oplus 2}\langle 1\rangle$ where, as above, $U = H^{even}(J)$. Using this identification along with the formulas $\mathrm{Sym}^3(V) = V_{(3)} \oplus V$ and $V_{(1,1,1)} = \Lambda^3(V)$, we compute that

$$H^*(\widetilde{M}) = \mathrm{Sym}^3 U \oplus \Lambda^3 U \oplus (U^{\otimes 2})^{\oplus 2}(-1) \oplus U^{\oplus 138}(-2) \oplus \mathbb{Q}^{\oplus 512}(-3), \quad (5.50)$$

which agrees with Equation 5.48.

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