Singular cubic threefolds, and their cohomology

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Abstract of the Dissertation

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Cubic hypersurfaces are important examples in algebraic geometry and a lot of research, both classical and recent, is devoted to them. In this dissertation, we focus on complex cubic threefolds. First, we describe the combinations of isolated singularities appearing on cubic threefolds. The analogous question for cubic surfaces has a beautiful answer, originally due to Schläfli [Sch63] and later reworked by Bruce and Wall [BW79] in a modern way. We use their methods together with deformation theory results to get a complete classification in the three-dimensional case. Additionally, we give a concise combinatorial description of the configurations of simple singularities that can occur. Essentially, such configurations correspond to subgraphs of a certain graph. We also give a lattice theoretic criterion for the possible combinations of isolated singularities on cubic threefolds. Finally, we show how our classification and techniques can help in understanding the cohomology of a singular cubic threefold.

Моему дедушке

Владимиру Алексеевичу Давыдову

Table of Contents

List of Figures

List of Tables

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Introduction

Singular cubic threefolds

Historically, cubic hypersurfaces are among the first non-trivial examples in algebraic geometry. Cubic hypersurfaces of dimension one, or elliptic curves, are a cornerstone of algebraic, arithmetic, and complex geometry. The two-dimensional case was studied classically by Cayley, Salmon and many others. Smooth cubic surfaces have many beautiful properties including the celebrated result that they contain 27 lines ([Cay49], [Sal49]). Cubic surfaces with isolated singularities were classified by Schläfli in [Sch63] (and partially by Salmon in [Sal49]). Subsequently, this classification was reworked in a modern way by Bruce and Wall [BW79].

The study of cubic threefolds goes back to the Italian school, and Fano [Fan04] in particular. The study of singular cubic threefolds can be traced back to Segre [Seg87] who showed that the maximal number of nodes on a cubic threefold is 10, and there is a unique such cubic (up to projective equivalence). Further steps towards classification of singular cubic threefold are the description of cubic threefolds admitting a $\mathbb{C}$ or $\mathbb{C}^*$-action by du Plessis and Wall [PW08], and Allcock and Yokoyama's list of several maximal combinations of A_n singularities in the context of GIT analysis ([All03], [Yok02]).

In Chapters 3, 4 and 5 of this dissertation, we give a complete classifications of possible combinations of isolated singularities on cubic threefolds. This part of our work is a direct higher dimensional analogue of the paper of Bruce and Wall [BW79]. They present the classification of isolated singularities on cubic surfaces in a beautiful and concise way. Namely, *a combination of ADE singularities appears on a cubic surface if and only if the union of the corresponding Dynkin graphs can be obtained by removing a number of vertices of the extended* $\tilde{E}_6$ *graph together with all the edges that terminate at them.* This type of results is reminiscent of the Brieskorn–Grothendieck description of adjacencies of ADE singularities (see [Arn+98]): *a simple singularity of type* L *is adjacent to a simple singularity of type* K *if and only if the Dynkin graph of type* K *embeds in the Dynkin graph of type* L. There are similar classification results for singularities of low degree $K3$ surfaces due to Urabe [Ura87; Ura] and Yang [Yan94].

Our main approach to the classification problem is the projection method of [BW79]. Specifically, a singular cubic $X \subset \mathbb{P}^4$ with isolated singularities is rational via a projection π_p from a singular point p. The inverse map π_p^{-1} has a base locus which is a $(2, 3)$ complete intersection curve $C \subset \mathbb{P}^3$. One sees that there is a close connection between the singularities of X and the singularities of C. Moreover, there is an important technical point that allow us to complete the analysis efficiently. Concretely, the global deformations of a cubic with isolated singularities which is not a cone give independent versal deformations of the local singularities [PW00a], [Du 07]. In particular, there is a well-defined notion of maximal configurations (i.e. the configurations that are not determined by adjacencies of other singularities), and all the other configurations are deformations of the maximal ones.

Theorem I.

1. *The maximal combinations of isolated singularities on a cubic threefold which is not a cone are* U_{12}, T_{266}, $J_{10} + A_2$, $D_5 + 2A_3$, $3D_4$, $5A_2$, *and* $10A_1$.

2. *The maximal ADE combinations of singularities on a cubic threefold are* $E_8 + A_2$, $E_7 + A_2 + A_1$, $E_6 + 2A_2$, $D_8 + A_3$, $D_6 + A_3 + 2A_1$, $D_5 + 2A_3$, $3D_4$, A_{11}, $A_7 + A_4$, $2A_5 + A_1$, $5A_2$, $10A_1$.

Inspired by the results of Brieskorn–Grothendieck and Bruce–Wall mentioned earlier in this chapter, we find a compact way to encode the ADE combinations on cubic threefolds:

Theorem II. *A combination of ADE singularities appears on a cubic threefolds if and only if the union of the corresponding Dynkin graphs is $10A_1$, $5A_2$ or an ADE subgraph of graph Γ (Figure 0.1) obtained by removing a number of vertices of Γ together with all the edges that terminate at them.*

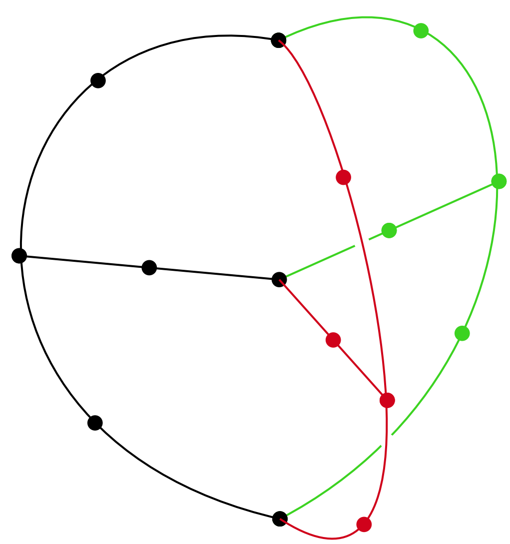

Figure 0.1: Graph Γ

One can see that the classification problem is related to the adjacencies of the O_{16} singularity (a cone over a cubic surface) since any combinations of isolated singularities on X degenerates to O_{16}. The graphs Γ, $10A_1$, and $5A_2$ encode partial bases of vanishing cycles for O_{16}, and the fact that one obtains multiple graphs has to do with the signature of the quadratic form associated to O_{16}. The quadratic form of O_{16} is isomorphic to $D_4^3 \oplus U^2$ and thus is indefinite in contrast to semidefinite for $\tilde{E}_6$ or definite for ADE. Similar phenomenon for

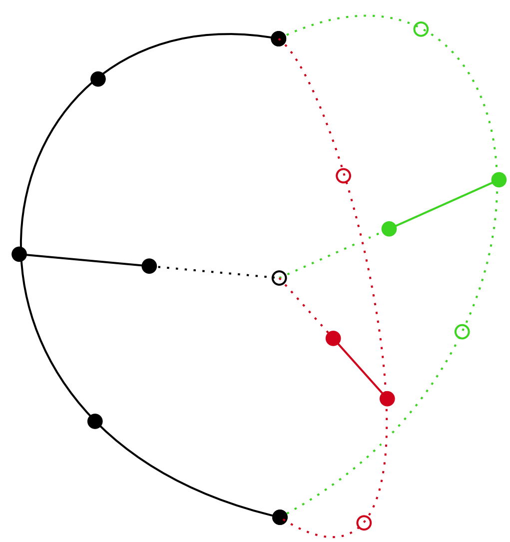

Figure 0.2: $E_6 + 2A_2$ as a subdiagram of Γ

$K3$ surfaces was observed by Urabe in [Ura87; Ura]. We notice that there is a correspondence between certain sublattices of $D_4^3 \oplus U^2$ and the possible configurations of isolated singularities on cubic threefolds. We can connect our problem to a classification problem for cubic fourfolds and use an adaptation of results of [LPZ18] to get the following:

Theorem III. *A root sublattice $R \subset D_4^3 \oplus U^2$ corresponds to a combination of ADE singularities of a cubic threefold if and only if $Sat(R)$ does not contain a primitive vector v such that $v^2 = 4$ and $div(v) = 2$.*

Finally, we would like to remark that while there is no classification of singularities of cubic fourfolds, there are some partial results in this direction in the thesis of Stegmann [Ste20].

Cohomology of cubic threefolds

In the last Chapter 6, we turn our attention to understanding the cohomology of singular cubics. This type of question has a long history. In their landmark paper [CG72], Clemens and Griffiths use the intermediate Jacobian construction as a principal tool to prove irrationality of all smooth cubic threefolds over a field of characteristic zero. In this case, the intermediate Jacobian is built from the third cohomology of a cubic threefold, and is a principally polarized abelian variety. Clemens and Griffiths show that it is not isomorphic to the Jacobian of a curve which turns out to be essential for rationality. In contrast, it is not hard to see that cubic threefolds with isolated singularities are rational. Collino [Col79], van der Geer and Kouvidakis [GK10], and Gwena [Gwe05] use specializations to mildly singular cubic threefolds to recover the result of [CG72] for very general cubic threefolds. Namely, they study the degenerations of intermediate Jacobians induced by specializations to cubics with one ordinary double point.

More recently, degenerations of intermediate Jacobians were studied in [GH12], [Cas+17], [Cas+21]. The common theme of these papers is the study of the singular cubics and their cohomology via Prym varieties. The approach we choose in this dissertation is more Hodge theoretical and directly in terms of cubic threefolds and their singularities. For instance, this point of view appers in the recent work of Kerr, Laza and Saito ([KLS21]) on generalizations of Clemens-Schmidt exact sequence to the non-normal crossings. Our approach also ties in nicely with the deformation theory of Fano threefolds by Friedman [Fri86], Namikawa-Steenbrink [NS95], and Namikawa [Nam97]. Applying their methods and the results of [FL22], we expect to obtain a new proof of the simultaneous versal deformation theorem for cubic threefolds (originally due to du Plessis and Wall [PW00a]). In [Nam97], Namikawa shows that the map from global to local first order deformations of a Fano threefold X

$$\alpha : \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega^1(X), \mathcal{O}_X))$$

is surjective if X has only ordinary double points. The surjectivity of α for all cubic threefolds will imply the theorem of [PW00a]. Notice that the same statement is not true for some singular Fano threefolds ([Nam97]).

Essentially, the cohomology groups of a cubic threefolds X with isolated singularities $p_1, \ldots, p_k$ are controlled by two local invariants and a global invariant. The local ones are related to du Bois invariants (see [Ste97]) and control the mixed Hodge structure on the vanishing cohomology $H^3_{van}(X, p_i)$. They can be computed in terms of the spectrum of the singularities of X. The global invariant is called the *defect*, and can be defined as

$$\sigma(X) = h_4(X) - h_2(X).$$

For a generic smoothing of X there is the following exact sequence (see [KLS21]):

$$\ldots \to \oplus_{i=1}^k H^2_{van}(X, p_i) \to H^3(X) \to H^3_{lim} \to \oplus_{i=1}^k H^3_{van}(X, p_i) \to H^4(X) \to H^4_{lim} \to \ldots$$

Since the singularities $p_1, \ldots, p_k$ are isolated, $H^2_{van}(X, p_i) = 0$ for each i. We also have that the mixed Hodge structure on $H^4(X)$ is pure of type $(2,2)$ (see [CM07]). The only remaining information that we need to compute $H^3(X)$ and $H^4(X)$ is the defect $\sigma(X)$. One can use the projection method of [BW79] to relate the defect of a cubic threefold with the number of components of the (2,3) complete intersection curve parameterizing the lines passing through a singular point. In the nodal case, we get

Theorem IV. *If X is a cubic threefold with only A_1 singuarities, and the (2,3) complete intersection curve C corresponding to one of the singularities of X has k irreducible components, then $\sigma(X) = k - 1$.*

Using Namikawa's theory [Nam97], one can define minimal cases for which the defect is not zero. This is relevant in the context of study of Hyperkähler manifolds (see [Bro16]).

Structure of the dissertation

In Chapter 1, we give an overview of the singularity theory results we need and introduce Varchenko's theorem on semicontinuity of spectrum (see [Var83]) which gives some restrictions for the possible singularities on a cubic threefold.

In Chapter 2, we introduce the projection method of Bruce and Wall [BW79] and give a brief overview of the results of [BW79] on singular cubic surfaces and [PW08] on symmetric singular cubic threefolds.

In Chapter 3, we classify the possible combinations of isolated singularities on cubic threefolds by corank. We analyze the cases systematically and obtain a list of maximal configurations of isolated singularities (Theorem I). The rest of the configurations can be determined by the maximal ones.

In Chapter 4, we study the Milnor lattice of O_{16} and give the lattice theoretic criterion (Theorem III) for ADE combinations on cubic threefolds.

In Chapter 5, we prove Theorem II.

In Chapter 6, we study the cohomology of cubic threefolds and prove Theorem IV.

In Appendix A, we give the complete list of possible combinations of isolated singularities on cubic threefolds.

Chapter 1

Singularity theory

1.1 Preliminaries

Definition 1.1.1 ([Arn+98]). The *corank* of a critical point of a function is the dimension of the kernel of its second differential at the critical point.

Definition 1.1.2 ([Arn+98]). Two function-germs at $p \in \mathbb{C}^n$ are said to be *equivalent* if one is taken into the other by a biholomorphic change of coordinates that keeps the point p fixed. Two critical points are said to be *equivalent* if the function-germs that define them are equivalent.

Definition 1.1.3 ([Arn+98]). The equivalence class of a function-germ at a critical point is called a *singularity*.

Theorem 1.1.1 ([Arn+98]). *In a neighborhood of a critical point p of corank k, a holomorphic function $f(x_1, \ldots, x_n)$ is equivalent to a function of the form*

$$\varphi(x_1, \ldots, x_k) + x_{k+1}^2 + \ldots + x_n^2,$$

where the second differential of φ at p is equal to zero.

Definition 1.1.4 ([Arn+98]). A *stabilization* of $f : \mathbb{C}^n \to \mathbb{C}$ is a function $\tilde{f} : \mathbb{C}^{n+k} \to \mathbb{C}$ of the form

$$\tilde{f}(x_1, \ldots, x_{n+r}) = f(x_1, \ldots, x_n) + x_{n+1}^2 + \ldots + x_{n+r}^2.$$

Two function-germs in different number of variables are said to be *stably equivalent* if they admit equivalent stabilizations. Two critical points (or singularities) are said to be *stably equivalent* if the function-germs that define them are stably equivalent.

In this dissertation, we will be concerned with the following series of singularities:

- simple (ADE) singularities: A_n, D_n ($n \geq 4$), E_6, E_7 and E_8;
- parabolic (or simply-elliptic) singularities: $P_8 = \tilde{E}_6$, $X_9 = \tilde{E}_7$ and $J_{10} = \tilde{E}_8$;
- hyperbolic singularities: T_{pqr} ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$);
- exceptional unimodal: U_{12}, S_{11}, Q_{10}.

For normal forms we refer to [AGV12, p. 246]. These types of singularities are defined in any dimension. For example, simple singularities are usually defined on surfaces, and the normal form for A_n in dimension 2 is

$$x_1^{n+1} + x_2^2 + x_3^2 = 0.$$

However, we can consider singularity types up to stable equivalence, and have A_n defined in dimension $k + 1$ as

$$x_1^{n+1} + x_2^2 + x_3^2 + \ldots + x_k^2 = 0.$$

Additionally, we need to define the following singularity type:

Definition 1.1.5. We will denote by O_{16} any singularity which is stably equivalent to a singularity defined by a homogeneous cubic polynomial $f_3(x_1, \ldots, x_4)$ that has no critical points except for $(0, 0, 0, 0)$.

Definition 1.1.6 ([Arn+98])**.** A class of singularities L is said to be *adjacent* to a class K, and one writes $L \to K$, if every function $f \in L$ can be deformed to a function of class K by an arbitrarily small perturbation.

In Tables 1.1 and 1.2 (from [Arn+98]), we show some adjacencies of simple, parabolic, and exceptional unimodal singularities.

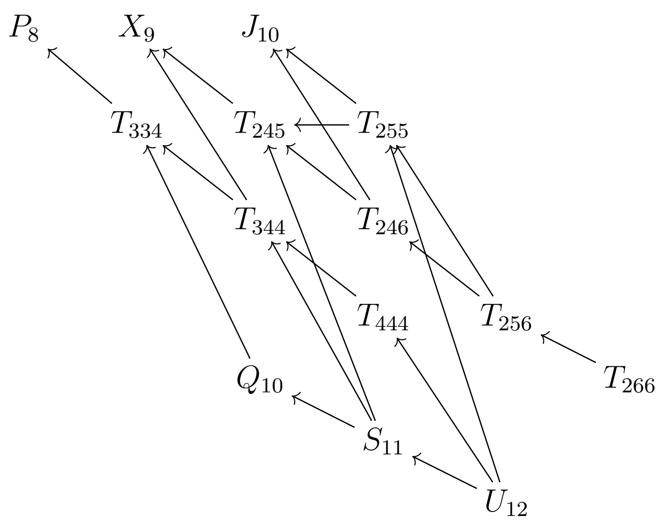

Table 1.1: Some adjacencies of unimodal singularities

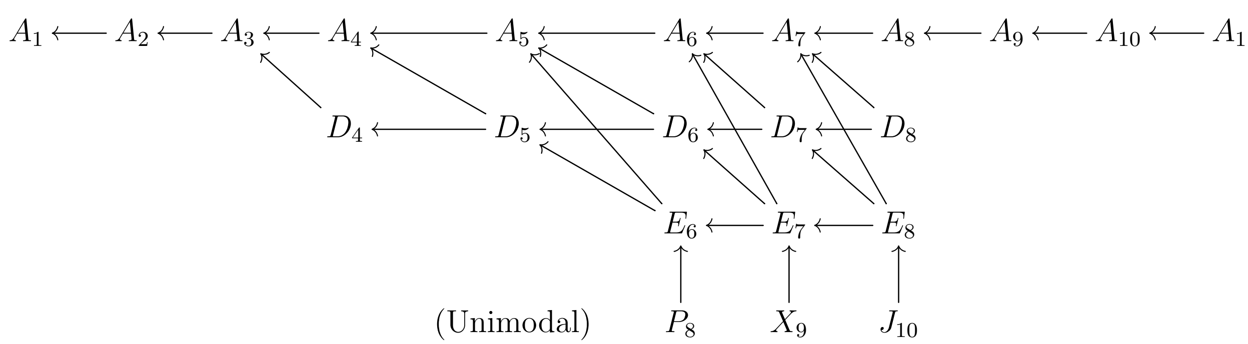

Table 1.2: Adjacencies of simple singularities

Definition 1.1.7. A class of singularities L is said to be *adjacent* to a *combination of singularities* $K_1 + \ldots + K_m$, and one writes $L \to K_1 + \ldots + K_m$, if every function $f \in L$

can be deformed to a function which has takes critical value 0 at critical points of types $K_1, \dots, K_m$ by an arbitrarily small perturbation.

Now we will prove some statements about singularities on cubic threefolds. First notice that if $X \subset \mathbb{P}^4$ is a cubic threefold with a singular point $p \in X$, we can choose coordinates in $\mathbb{P}^4$ in which $p = [1 : 0 : \dots : 0]$, and X is defined by the equation

$$f(x_0, \dots, x_4) = x_0 f_2(x_1, \dots, x_4) + f_3(x_1, \dots, x_4),$$

where $f_2(x_1, \dots, x_4)$ and $f_3(x_1, \dots, x_4)$ are homogeneous polynomials of degree 2 and 3 respectively.

Claim 1.1.2. *The corank of p is equal to the corank of f_2.*

Proposition 1.1.3. *The corank of f_2 is equal to one if and only if p is of A_r type with $r > 1$.*

Proof. Assume that the corank of f_2 is equal to one. By the previous theorem, $f_2 + f_3$ is equivalent to $\varphi(x_1) + x_2^2 + \dots x_n^2$ where $\varphi(x_1) = a_{r+1}x_1^{r+1} + \dots + a_s x_1^s$, $a_{r+1} \neq 0$ and $\deg \varphi \leq \mu(f_2 + f_3) + 1$. The Milnor number $\mu(f_2 + f_3)$ equals $\mu(\varphi) = r + 1$ which means that $r = \mu(f_2 + f_3)$ and $\varphi = a_{r+1}x_1^{r+1}$. Conversely, the corank of an A_r singularity is equal to one when $r > 1$. □

Remark 1.1.4. The corank of f_2 is equal to zero if and only if p is an A_1 singularity.

Proposition 1.1.5. *O_{16} is adjacent to any combination of singularities on a cubic threefold.*

Proof. A generic hyperplane section of X is smooth. Assume that this section is given by $x_4 = 0$.

$$F = f(x_0, x_1, x_2, x_3) + q(x_0, x_1, x_2, x_3)x_4 + l(x_0, x_1, x_2, x_3)x_4^2 + cx_4^3;$$

$$F_t = f(x_0, x_1, x_2, x_3) + q(x_0, x_1, x_2, x_3)tx_4 + l(x_0, x_1, x_2, x_3)t^2x_4^2 + ct^3x_4^3.$$

$V(F) \simeq V(F_t)$ for $t \neq 0$. When $t = 0$, we get a cone over a smooth cubic surface. □

We finish this section by stating the following claims about singularities of curves which we will need in the last section of Chapter 3 about constellations of A_n singularities:

Claim 1.1.6. *The blow up of an A_n singularity is an A_{n-2} singularity.*

Claim 1.1.7. *If an irreducible curve C has a combination of singularities $a_1A_1 + ... + a_nA_n$, then $g_a(C) \geq \sum a_i \lceil \frac{i}{2} \rceil$.*

1.2 Intersection matrices and Dynkin diagrams

Consider a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, which has an isolated critical point at $p = (0, \ldots, 0)$. If we pick sufficiently small neighborhoods $p \in U \subset \mathbb{C}^n$ and $f(p) \in T \subset \mathbb{C}$, we get a smooth hypersurface $X_t = f^{-1}(t) \cap U$ for each $t \in T$, $t \neq f(p)$. One can choose a *distinguished basis* $\delta_1, \ldots, \delta_\mu$ of *vanishing cycles* for $H_{n-1}(X_t, \mathbb{Z})$ (see Section 1 of [Ebe19] for precise definitions).

Definition 1.2.1. The dimension μ of $H_{n-1}(X_t, \mathbb{Z})$ is called the *Milnor number* of p.

Definition 1.2.2. Let $\langle , \rangle$ be the intersection form on $H_{n-1}(X_t, \mathbb{Z})$. The matrix $(\langle \delta_i, \delta_j \rangle)_{i,j=1,\ldots,\mu}$ is called the intersection matrix of the singularity of f at p with respect to the distinguished basis $\delta_1, \ldots, \delta_\mu$.

Proposition 1.2.1 ([Ebe19], Proposition 1)**.** *A vanishing cycle δ has the self-intersection number*

$$\langle \delta, \delta \rangle = (-1)^{n(n-1)/2}(1 - (-1)^n) = \begin{cases} 0 & \textit{for } n \textit{ even}, \\ 2 & \textit{for } n \equiv 1 \pmod 4, \\ -2 & \textit{for } n \equiv 3 \pmod 4. \end{cases}$$

The intersection matrices of stably equivalent singularities determine one another ([Ebe19], Theorem 13). In particular, there are exactly four distinct intersection forms in a class of stably equivalent singularities. It follows that, by taking suitable stabilization, one can assume that $n \equiv 1 \pmod 4$.

Remark 1.2.2. When n is even, the intersection matrix is a skew-symmetric bilinear form. When n is odd, it is symmetric. We choose n odd because it is easier to work with. For instance, the monodromy of ADE singularities is finite in this case.

Definition 1.2.3. The symmetric form $(\, , \,)$ on $H_{n-1}(X_t, \mathbb{Z})$ associated with the intersection matrix of the stabilization $\tilde{f}$ of f in $n+k$ variables such that $n+k \equiv 1 \pmod 4$ is called the quadratic form of the singularity.

For a distinguished basis $\delta_1, \ldots, \delta_\mu$, we construct the corresponding Dynkin diagram. A vertex labeled i is assigned to each root δ_i; two vertices i and j are connected by a (dashed) edge with index k if $(\delta_i, \delta_j) = k$ and $k < 0$ (resp. $k > 0$). Since the self-intersection $(\delta_i, \delta_i) = 2$ for any distinguished basis element δ_i, the Dynkin diagram completely determines the quadratic form.

Example 1.2.3. For a simple singularity, the ADE type of the corresponding Dynkin diagram coincides with the ADE type of the singularity. The Dynkin graph does not depend on the choice of a distinguished basis in this case.

We can describe all the adjacencies of ADE singularities using the following results by Brieskorn and Grothendieck:

Theorem 1.2.4 (Brieskorn–Grothendieck, [Arn+98], Section 5.9)**.** *A simple singularity of type L is adjacent to a simple singularity of type K if and only if the Dynkin diagram of the root system of K embeds in the Dynkin diagram of the root system of L.*

Definition 1.2.4. An induced subgraph of a graph G is a subset of the vertices of G together with any edges whose endpoints are both in this subset.

Theorem 1.2.5 (Grothendieck, [Arn+98], Section 5.9)**.** *A simple singularity of type L is adjacent to a combination of simple singularities $K_1, \ldots, K_m$ if and only if the disjoint union of the Dynkin diagrams of the root systems of $K_1, \ldots, K_m$ is an induced subgraph of the Dynkin diagram of the root system of L.*

In general, any partition of an isolated singularity corresponds to a partition of a corresponding Dynkin graph, however such a graph may not be unique.

1.3 Deformation theory

Consider the family of degree d hypersurfaces in $\mathbb{P}^n$ parametrized by the projective space $\mathbb{P}(V)$, $V = H^0(\mathbb{P}^n, \mathcal{O}(d))$. Let $X \in \mathbb{P}(V)$ be a hypersurface with isolated singular points $p_1, \dots, p_k \in X$. As every deformation of X in $\mathbb{P}(V)$ induces a deformation of its singularities, we have the morphism of functors

$$\mathrm{Def}(X) \to \prod_{i=1}^{k} \mathrm{Def}(X, p_i).$$

This natural global-to-local map is not always surjective, i.e. global deformations of X do not always induce all the possible unfoldings of each p_i. However it is true in some cases, in particular for cubic threefolds (also cubic surfaces and curves). It follows from a theorem by du Plessis and Wall:

Theorem 1.3.1 (du Plessis–Wall [PW00a], [Du 07])**.** *Given a complex hypersurface X of degree d in $\mathbb{P}^n$ with only isolated singularities, the family of hypersurfaces of degree d induces a simultaneous versal deformation of all the singularities of X, provided $\mu(X) < \delta(d)$, where $\delta(d) = 16$, 18 or $4(d-1)$, for $d = 3$, 4 or $d \geq 5$, respectively.*

Remark 1.3.2. In the original statement of the theorem, the required inequality is $\tau(X) < \delta(d)$ where $\tau(X)$ is the total Tjurina number of X. We do not define the Tjurina number in this text but it holds that $\tau(X) \leq \mu(X)$ for any X.

Example 1.3.3. In the most elementary case of cubic curves, the possible combinations of singularities are $D_4, A_3, A_2, 3A_1, 2A_1$ and A_1. By Theorem 1.2.5, D_4 is adjacent to all the listed combinations. Theorem 1.3.1 gives us a stronger statement that we can deform a plane cubic with a D_4 singularity (i.e. three concurrent lines) into a plane cubic with any of the listed combinations of singularities.

Definition 1.3.1. We call a combination of singularities on a cubic threefold *maximal* in a certain class of combinations of singularities, if we cannot get this combination of singularities after a (global) deformation of another combination of singularities in this class.

1.4 Spectrum of hypersurface singularities

For the exact definition of the spectrum, we refer the reader to [Var83]. In what follows, we use the that it is a deformation invariant of a singularity. In particular, it means that we can compute the spectrum of any O_{16} singularity using the local equation $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. In this subsection, we present two theorems of Varchenko and apply them to the case of cubic threefolds.

Theorem 1.4.1 (Varchenko [Var83])**.** *Let $Z \subset \mathbb{P}^n$ be a hypersurface of degree d with isolated singular points $p_1, p_2, \ldots, p_N$. Let the singularity (Z, p_i) be described by $f_i \in \mathbb{C}[x_1, x_2, \ldots, x_n]$. Then for each $\alpha \in \mathbb{R}$ one has an inequality*

$$\#(\alpha, \alpha+1) \cap sp(x_1^d + x_2^d + \cdots + x_n^d) \geq \sum_{i=1}^{N} \#(\alpha, \alpha+1) \cap sp(f_i).$$

The combined number of spectral numbers for all singularities on a degree d hypersurface in any open interval of length one is bounded above by the number of spectral numbers in the corresponding interval for the singularity described by $x_1^d + x_2^d + \cdots + x_n^d$.

Corollary 1.4.2. *If a cubic threefold X has only ADE singularities, then $\mu(X) \leq 14$.*

Proof. Spectra of ADE singularities:

$$sp(A_k) = \left\{ \frac{3}{2} + \frac{1}{k+1}, \frac{3}{2} + \frac{2}{k+1}, \ldots, \frac{3}{2} + \frac{k}{k+1} \right\},$$

$$sp(D_k) = \left\{ \frac{3}{2} + \frac{1}{2k-2}, \frac{3}{2} + \frac{3}{2k-2}, \ldots, \frac{3}{2} + \frac{2k-3}{2k-2} \right\} \cup \{2\},$$

$$sp(E_6) = \left\{ \frac{3}{2} + \frac{1}{12}, \frac{3}{2} + \frac{4}{12}, \frac{3}{2} + \frac{5}{12}, \frac{3}{2} + \frac{7}{12}, \frac{3}{2} + \frac{8}{12}, \frac{3}{2} + \frac{11}{12} \right\},$$

$$sp(E_7) = \left\{ \frac{3}{2} + \frac{1}{18}, \frac{3}{2} + \frac{5}{18}, \frac{3}{2} + \frac{7}{18}, \frac{3}{2} + \frac{9}{18}, \frac{3}{2} + \frac{11}{18}, \frac{3}{2} + \frac{13}{18}, \frac{3}{2} + \frac{17}{18} \right\},$$

$$sp(E_8) = \left\{ \frac{3}{2} + \frac{1}{30}, \frac{3}{2} + \frac{7}{30}, \frac{3}{2} + \frac{11}{30}, \frac{3}{2} + \frac{13}{30}, \frac{3}{2} + \frac{17}{30}, \frac{3}{2} + \frac{19}{30}, \frac{3}{2} + \frac{23}{30}, \frac{3}{2} + \frac{29}{30} \right\}.$$

Varchenko's theorem allows us to compare these spectra to the spectrum of the O_{16} singularity described by the local equation $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$. We can apply the Thom-Sebastiani theorem (see [Var83]) to compute its spectrum. The result we get is as follows:

$$sp(O_{16}) = \left\{ \frac{4}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, 2, 2, 2, 2, 2, 2, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{8}{3} \right\}.$$

Now one can apply the theorem for $\alpha = \frac{3}{2}$. Notice that all the ADE spectra lie within the interval $(\frac{3}{2}, \frac{5}{2})$ and that $\#(\frac{3}{2}, \frac{5}{2}) \cap sp(O_{16}) = 14$. It follows that the total number of the spectral numbers of all the ADE singularities on X is less or equal to 14. □

Corollary 1.4.3. *A cubic threefold has at most* 10 *nodes.*

Proof. A node is an A_1 singularity, $sp(A_1) = \{2\}$. Assume X has m nodes. Applying Varchenko's theorem for $\alpha = \frac{4}{3}$, we get $\#(\frac{4}{3}, \frac{7}{3}) \cap sp(O_{16}) = 10 \geq m$. □

Theorem 1.4.4 (Varchenko [Var83])**.** *A necessary condition for adjacency $T \xrightarrow{\sim} T'$ is that the spectra are adjacent in the sense that $\alpha_i \leq \alpha_i'$.*

Corollary 1.4.5. *If a cubic threefold has an A_k singularity, then $k \leq 11$.*

Proof. By Proposition 1.1.5, O_{16} is adjacent to any singularity appearing on a cubic threefold. By Theorem 1.4.4 above, $\alpha_2(O_{16}) \leq \alpha_2(A_k)$. Solving $\frac{5}{3} \leq \frac{3}{2} + \frac{2}{k+1}$, we get $k \leq 11$. □

Chapter 2

Singularities of cubic hypersurfaces

2.1 Projection method

2.1.1 Notation

Let $X \subset \mathbb{P}^n$ be a cubic hypersurface ($n \geq 3$). We fix a singular point $p \in X$ and choose coordinates in which $p = [1 : 0 : \ldots : 0]$. In these coordinates X is defined by the equation

$$f(x_0, \ldots, x_n) = x_0 f_2(x_1, \ldots, x_n) + f_3(x_1, \ldots, x_n)$$

where $f_2(x_1, \ldots, x_n)$ and $f_3(x_1, \ldots, x_n)$ are homogeneous polynomials of degree 2 and 3 respectively.

Let N be the hyperplane at infinity defined by $x_0 = 0$, $Q \subset N$ be the quadric hypersurface defined by $f_2 = 0$, $S \subset N$ be the cubic hypersurface defined by $f_3 = 0$, and C be the intersection of Q and S.

Remark 2.1.1. While Q and C are uniquely determined by the singular point p, the cubic hypersurface S is only defined modulo Q. If we choose a hyperplane N' with the equation $x'_0 = x_0 - \sum_{i=1}^{n} a_i x_i$ then $f'_3(x_1, \ldots, x_n) = f_3 + (\sum_{i=1}^{n} a_i x_i) f_2$. Thus X can be defined by the equation $f'(x'_0, x_1, \ldots, x_n) = x'_0 f_2 + f'_3$.

2.1.2 Projection method

Notice that if $f_3 \equiv 0$ then X is reducible and if $f_2 \equiv 0$ then X is a cone. For the rest of this subsection, we will assume that $f_2, f_3 \not\equiv 0$ and X has an isolated singularity at p.

Proposition 2.1.2.

a) Q and S do not have common components.

b) The lines $L \subset X$ passing through p are in $1:1$ correspondence with the points of C.

c) Let $q \in X$ be a singular point other than p. Then the line $L = < p, q >$ is contained in X and the only singular points of X on L are p and q.

Proof. If Q and S have a common component, then X is reducible and singular along the intersection of its irreducible components.

A line passing through p intersects X at p with multiplicity at least 2. In particular, such a line either meets X at exactly one point other than p, intersects X at p with multiplicity 3 or is contained in X. Now, b) follows from the fact that Q is the projectivized tangent cone to X at p and can be interpreted as the locus of lines intersecting X at p with multiplicity at least 3. Thus if $x \in C (= Q \cap S \subset X)$, the intersection number of the line $L = < p, x >$ and X is at least 4 (multiplicity ≥ 3 at p and ≥ 1 at x), which means L is contained in X. The converse holds by a similar argument.

The first part of c) is immediate. For the second part, we note that $Sing(X)$ is cut by quadric hypersurfaces (the partial derivatives of f). If there are three points $p, q, r \in Sing(X)$ on a line L then L has to be contained in each of the quadrics. Thus $L \subset Sing(X)$ and the singularity at p is not isolated. □

Let $\epsilon : \tilde{X} \to X$ be the blow-up of X at p with the exceptional divisor E. Let $\pi : X \dashrightarrow N \cong \mathbb{P}^{n-1}$ be the projection from p onto N.

Corollary 2.1.3. *The projection $\pi : X \dashrightarrow N$ is a birational map, and there is a unique birational morphism $\phi : \tilde{X} \to N$ that fits into the diagram*

$$\begin{array}{ccc} & \tilde{X} & \\ {}^{\epsilon}\swarrow & & \searrow^{\phi} \\ X & \overset{\pi}{\dashrightarrow} & N. \end{array}$$

Furthermore, the restriction of ϕ to the exceptional divisor E gives an isomorphism $\phi|_E : E \to Q \subset N$.

The following theorem, adapted from [Wal99, Theorem 2.1], shows how the singularities of C determine the singularities of X away from p:

Theorem 2.1.4 ([Wal99])**.** *Consider a point $q \in C$. If Q and S are both singular at q, then X is singular along the line $< p, q >$. Otherwise write T for the type of the singularity of C at q in the (locally) smooth variety Q (or S).*

(i) If Q is smooth at q, X has a unique singular point on the line $< p, q >$ other than p, and the singularity there has type T.

(ii) If Q is singular at q, the only singular point of X on $< p, q >$ is p, and the blow-up $\tilde{X}$ of X at p has a singular point of type T at $\phi|_E^{-1}(q)$ where ϕ is as in Corollary 2.1.3.

2.2 Singular cubic surfaces

The classification of cubic surfaces by their singularities was given by Schläfli over a century ago. In [BW79] Bruce and Wall present this classification in a modern way:

Theorem 2.2.1 ([BW79], Section 4)**.** *A cubic surface $X \subset \mathbb{P}^3$ with only isolated singularities has either one $\tilde{E}_6$ singularity or a combination of ADE singularities. A combination of ADE singularities can occur on X if and only if its corresponding Dynkin diagram is a subdiagram of $\tilde{E}_6$ (Figure 2.1).*

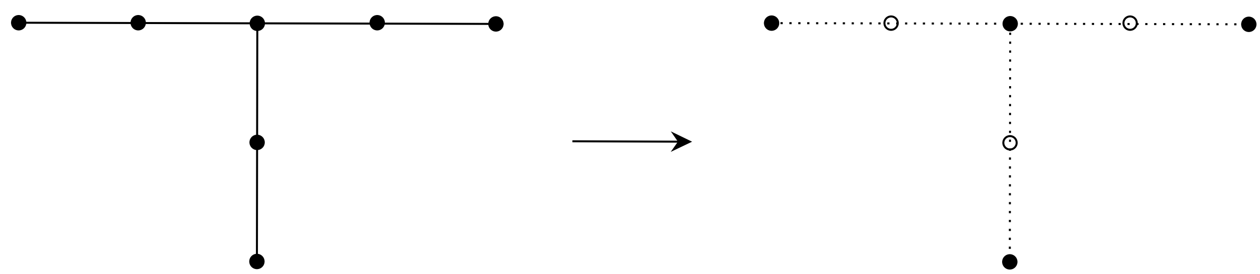

Figure 2.1: $\tilde{E}_6$ deforms to $4A_1$

Remark 2.2.2. The $\tilde{E}_6$ graph on Figure 2.1 is not a full Dynkin diagram of $\tilde{E}_6$ (the singularity $\tilde{E}_6$ has Milnor number 8, and the graph has 7 vertices). This graph corresponds to a partial basis of vanishing cycles of $\tilde{E}_6$.

From the point of view of the projection method 2.1.2, the theorem above can be reformulated as follows:

Theorem 2.2.3. *[BW79] Let X be a cubic surface with only isolated singularities and $p \in X$ a singular point. Let Q, S, C be as in Section 2.1.1. Then we have one of the following possibilities*

1) if $Q \equiv 0$ then the singularity at p is of type $\tilde{E}_6$ (the cone over a smooth elliptic curve).

2) if Q is a double line then the singularity at p is of type D_4, D_5 or E_6 depending on the type of intersection of Q and S (three simple points, one simple and one double and respectively one triple point).

3) if Q is reducible, but reduced, that is two distinct lines meeting in v. Then the singularity at p is of type A_n for $2 \leq n \leq 5$, where n is determined by the intersection multiplicity of Q and S at v (which can be 0, 2, 3 and 4 respectively).

4) if Q is an irreducible (thus smooth) conic, then the singularity at p is A_1.

Note that in this case C is just a collection of 6 points (taken with multiplicities). And a point with multiplicity n can be said to be a singularity of type A_{n-1}. In other words, we

Singularity at p	Singularities of Q	$Sing(C) \cap Sing(Q)$
E_6	a line	A_2
D_5	a line	A_1
D_4	a line	$\emptyset$
A_5	a point	A_3
A_4	a point	A_2
A_3	a point	A_1
A_2	a point	$\emptyset$
A_1	empty	$\emptyset$

Table 2.1: Singularities of a cubic surface

can interpret the theorem as giving first a stratification of the singularities of a cubic surface by the type of conic Q, and then the analytic type will be determined by the singularities of C along the singular locus of Q (see Table 2.1).

Once we have projected away from p, from the distribution of the points of C (away from $Sing(Q)$), we can determine all the other singularities. For instance, once we projected from an A_1 singularity, the points of C can be distributed in any partition of 6, for instance the partition $2+2+2$ corresponds to the cubic surface with $3A_1 + A_1$, thus 4 ordinary double points.

2.3 Cubic threefolds with one-parameter symmetry groups

There are several papers by bu Plessis and Wall ([PW99], [PW00b], [PW08], [PW10]) where they study quasi-smooth projective hypersurfaces with symmetry. In particular, they give a complete classification of singularities on quasi-smooth 1-symmetric cubic threefolds in [PW08].

Definition 2.3.1. A variety $X \subset \mathbb{P}^n$ is quasi-smooth if it has only isolated singularities and is not a cone.

For the rest of this section, we will assume that $X \subset \mathbb{P}^n$ is a quasi-smooth hypersurface of degree d.

Definition 2.3.2. We say that X is k-symmetric if it admits a k-dimensional algebraic subgroup G of $PGL_n(\mathbb{C})$ as automorphism group.

In [PW08], du Plessis and Wall describe singularities of 1-symmetric quasi-smooth hypersurfaces. If X is 1-symmetric, there are two possibilities for G: a linear algebraic one-parameter group is isomorphic either to the multiplicative group $\mathbb{C}^*$ (semisimple case) or to the additive group $\mathbb{C}$ (unipotent case).

In the semisimple case, singularities of X are determined by the weights of the corresponding $\mathbb{C}^*$-action. General methods are introduced in Sections 3 and 5 of [PW00b]. The cubic threefold case is considered in Section 5 of [PW08].

Theorem 2.3.1 ([PW08], Section 5). *Let $X \subset \mathbb{P}^4$ be a 1-symmetric quasi-smooth cubic threefold with G semisimple. The following table has a list of possible combinations of singularities on such a threefold together with the weights of the corresponding $\mathbb{C}^*$-action. In the two cases marked with $*$ additional singularities can appear: A_1 for $[-2,-1,0,1,2]$ and A_1, $2A_1$, A_2, $3A_1$, A_3 or D_4 for $[-1,0,0,0,1]$.*

The unipotent case is handled in Section 4 of [PW08]. In the cubic threefold case, the classification is given in the theorem below:

Theorem 2.3.2 ([PW08], Section 5). *Let $X \subset \mathbb{P}^4$ be a 1-symmetric quasi-smooth cubic threefold with G unipotent. Then the possible combinations of singularities on X are A_{11}, U_{12}, T_{266}, T_{256}, $T_{246} + A_1$, $J_{10} + A_2$ and $J_{10} + A_1$.*

Corollary 2.3.3. *Let $X \subset \mathbb{P}^4$ be a 2-symmetric quasi-smooth cubic threefold. Then $\tau(X) = 12$ and X has one of the following combinations of singularities:Then the possible combinations of singularities on X are U_{12}, T_{266}, $J_{10} + A_2$, $3D_4$.*

μ	Weights	Singularities	μ	Weights	Singularities
11	$[-8,-2,1,4,16]$	S_{11}	11	$[-10,-4,2,5,8]$	A_7+A_4
11	$[-8,-2,1,4,4]$	D_5+2A_3	11	$[-8,-2,1,4,7]$	D_8+A_3
11	$[-5,-2,1,1,4]$	$D_5+A_3+2A_1$	11	$[-2,-2,1,1,4]$	X_9+2A_1
10	$[-4,-1,0,2,8]$	Q_{10}	10	$[-4,-1,0,2,5]$	E_8+A_2
10	$[-4,-1,0,2,4]$	D_7+A_3	10	$[-2,-2,0,1,4]$	E_6+2A_2
10	$[-2,-1,0,1,3]$	$E_7+A_2+A_1$	$10*$	$[-2,-1,0,1,2]$	$2A_5$
12	$[-2,0,0,1,4]$	U_{12}	12	$[-2,0,0,1,2]$	$J_{10}+A_2$
8	$[-1,0,0,0,2]$	P_8	$8*$	$[-1,0,0,0,1]$	$2D_4$

Table 2.2: Singularities of 1-symmetric cubic threefolds (semisimple case)

Remark 2.3.4. A cubic threefold with a U_{12} singularity can be 1 or 2-symmetric.

Chapter 3

Isolated singularities of cubic threefolds

3.1 Singularities of corank 3

Let $X \subset \mathbb{P}^4$ be a cubic threefold with an isolated singular point $p = [1 : 0 : 0 : 0 : 0]$ of corank 3. Let Q and S be as in Section 2.1.1. In this case $Q \subset \mathbb{P}^3$ is a double plane and can be defined by the equation $x_1^2 = 0$. Let C be the intersection of S and the plane $x_1 = 0$.

We will use the geometry of C to describe possible singularities on X. The plane cubic C is one of the following: three concurrent lines, a conic and its tangent, a triangle, a cuspidal cubic, a conic and its secant, a nodal cubic or a smooth cubic.

Remark 3.1.1. Notice that if C contains a double line then $Sing(Q) \cap Sing(S) = Q \cap Sing(S) \neq \emptyset$ and p is not isolated by 2.1.4.

Proposition 3.1.2. *If C is the union of three concurrent lines then p is a U_{12} singularity.*

Proof. In the affine chart given by $x_0 \neq 0$, X is defined by the equation

$$x_1^2 + f_3(x_1, x_2, x_3, x_4) = 0.$$

Let $h_3(x_2, x_3, x_4) = f_3(0, x_2, x_3, x_4)$. Since C is the union of three concurrent lines, we can choose coordinates in which $h_3(x_2, x_3, x_4) = x_2^3 + x_2 x_3^2$. Then

$$f_3(x_1, x_2, x_3, x_4) = x_2^3 + x_2 x_3^2 + x_1 h_2(x_2, x_3, x_4) + x_1^2 h_1(x_2, x_3, x_4) + c x_1^3$$

where h_1 and h_2 are homogeneous polynomials of degree 1 and 2 respectively and c is a complex number.

The singularity at p is isolated, thus Q and S do not share any singularities by 2.1.4.

$$df_2 = 2x_1 dx_1,$$

$$df_3 = (3x_2^2 + x_3^2)dx_2 + 2x_2 x_3 dx_3 + h_2 dx_1 + x_1 dh_2 + 2x_1 h_1 dx_1 + x_1^2 dh_1 + 3c x_1^2 dx_1.$$

If $df_2(x_1, x_2, x_3, x_4) = 0$ then $x_1 = 0$, if $df_3(0, x_2, x_3, x_4) = 0$ then $x_2 = x_3 = 0$. We have

$$df_3(0, 0, 0, x_4) = h_2(0, 0, x_4)dx_1 = b x_4^2 dx_1.$$

The coefficient b is not equal to 0 because otherwise $[0 : 0 : 0 : 1]$ is a singular point for both Q and S.

We will use theorems from [AGV12] to determine the singularity type of p. In order to apply them, we need to do a sequence of coordinate changes. The first coordinate change is as follows:

$$x_1 = y_1 - \frac{1}{2}h_2 - \frac{1}{2}y_1 h_1,$$

$$f_2 + f_3 = y_1^2 + x_2^3 + x_2 x_3^2 - \frac{1}{4}h_2^2 - y_1 h_1 h_2 - \frac{3}{4}y_1^2 h_1^2 + O_5(x_1, x_2, x_3, x_4)$$

where O_k stands for any series with terms of degree k and higher.

Then we can set $y_1 = z_1 + \psi_3$ for a certain series ψ_3 with terms of degree 3 and higher so that

$$f_2 + f_3 = z_1^2 + x_2^3 + x_2 x_3^2 - \frac{1}{4}h_2^2 + O_5(x_2, x_3, x_4).$$

Finally we set $x_2 = z_2$, $x_3 = z_3$, $x_4 = \frac{2\sqrt{-1}}{b}z_4$ and get

$$f_2 + f_3 = z_1^2 + z_2^3 + z_2 z_3^2 + z_4^4 + z_4^3 g_1(z_2, z_3) + z_4^2 g_2(z_2, z_3) + z_4 g_3(z_2, z_3) + g_4(z_2, z_3) + O_5(z_2, z_3, z_4)$$

where each g_i is a homogeneous polynomial of degree i.

Now we will use the determinator of singularities from Chapter 16 of [AGV12]. The third jet of f_2+f_3 equals $z_2^3+z_2z_3^2$ which leads us to case 83_1 on page 265. The quasijet $j^*_{z_4^4}(f_2+f_3)$ equals $z_2^3+z_2z_3^2+z_4^4$ where $j^*_{z_4^4}(z_2^{m_2}z_3^{m_3}z_4^{m_4})$ is defined to be $z_2^{m_2}z_3^{m_3}z_4^{m_4}$ if $\frac{m_2}{3}+\frac{m_3}{3}+\frac{m_4}{4}\leq 1$ and 0 otherwise. Thus we are directed to case 84_1 on page 265 of [AGV12] which means that p is a U_{12} singularity. □

Proposition 3.1.3. *A singular point $p \in X$ as above has one of the following types: U_{12}, S_{11}, Q_{10}, T_{444}, T_{344}, T_{334} or P_8. Furthermore p is the unique singularity of X.*

μ	Singularity at p	Geometry of C	$Sing(C)$
12	U_{12}	three concurrent lines	D_4
11	S_{11}	conic and tangent	A_3
11	T_{444}	triangle	$3A_1$
10	Q_{10}	cuspidal cubic	A_2
10	T_{344}	conic and secant	$2A_1$
9	T_{334}	nodal cubic	A_1
8	$P_8=T_{333}$	smooth cubic	-

Table 3.1: Singularities of corank 3

Proof. On the one hand, U_{12} deforms to S_{11}, Q_{10}, T_{444}, T_{344}, T_{334} and P_8 (see Table 1.1), and thus these singularities can appear on a cubic threefold by Theorem 1.3.1. On the other hand, there are 7 possible diffeomorphism types for C which by Theorem 2.1.4 means that there are at most 7 possible singularity types for p. Thus U_{12}, S_{11}, Q_{10}, T_{444}, T_{344}, T_{334} and P_8 are all the possibilities.

By Theorem 2.1.4, singularities of X other than p correspond to singularities of $Q\cap S$ contained in the smooth locus of Q, and when p is a corank 3 singularity Q is singular everywhere. □

Remark 3.1.4. The correspondence between the singularity type of p and the geometry of C from Table 3.1 can be established by comparing the adjacencies in Table 1.1 and the way different types of plane cubics deform to each other.

3.2 Singularities of corank 2

Let $X \subset \mathbb{P}^4$ be a cubic threefold with only isolated singularities and assume $p \in X$ is a singularity of corank 2. Let Q, S, C be as in Section 2.1.1.

If p is of corank 2, $Q = P_1 \cup P_2$ where P_i are distinct planes. Let $L = Sing(Q) = P_1 \cap P_2$. By Theorem 2.1.4, the singularities of X away from p correspond to the singularities of C away from L. The singularity type of p depends on the singularities of C along L.

Claim 3.2.1. *The curve C contains L if and only if the third jet of $f_2 + f_3$ vanishes.*

Proof. We can choose coordinates in which Q is defined by the equation $x_1x_2 = 0$. The third jet of $x_1x_2 + f_3$ vanishes if and only if $f_3 = x_1g_2(x_1, x_2, x_3, x_4) + x_2h_2(x_1, x_2, x_3, x_4)$ where g_2 are h_2 homogeneous polynomials of degree 2. The polynomial f_3 is of the form $x_1g_2 + x_2h_2$ if and only if $L \subset C$. □

3.2.1 Singularities with vanishing third jet

By Claim 3.2.1, if the third jet of $f_2 + f_3$ vanishes then $C = L \cup C_1 \cup C_2$ where $C_1 \subset P_1$ and $C_2 \subset P_2$ are plane conics. We can choose coordinates in which P_1 and P_2 are defined by $x_1 = 0$ and $x_2 = 0$ respectively.

Claim 3.2.2.

a) If C_1 or C_2 is a double line then X has a non-isolated singularity.

b) If $C_1 \cap C_2 \neq \emptyset$ then X has a non-isolated singularity.

Proof.

a) Assume C_1 is a double line defined by $x_3^2 = 0$. Then $f_3 = x_2x_3^2 + x_1g_2(x_1, x_2, x_3, x_4)$ where g_2 is a homogeneous polynomial of degree 2. In the affine chart given by $x_0 \neq 0$, X is defined by the equation $f_2 + f_3 = x_1x_2 + x_2x_3^2 + x_1g_2 = 0$. We have

$$d(f_2 + f_3) = (x_2 + g_2)dx_1 + (x_1 + x_3^2)dx_2 + 2x_2x_3dx_3 + x_1dg_2.$$

Both $f_2 + f_3$ and $d(f_2 + f_3)$ vanish along the curve $K \subset X \subset \mathbb{P}^4$ defined by $x_1 = 0$, $x_3 = 0$, $x_2 + g_2 = 0$. Thus the singular point $p \in K \subset X$ is not isolated.

b) Assume there exists a point $q \in C_1 \cap C_2 \subset L$. Then $f_3 = x_1g_2(x_1, x_2, x_3, x_4) + x_2h_2(x_1, x_2, x_3, x_4)$ where g_2 and h_2 are homogeneous polynomials of degree 2 such that $g_2(q) = h_2(q) = 0$. The differential $df_3 = g_2dx_1 + h_2dx_2 + x_1dg_2 + x_2dh_2$ vanishes at q which means that $q \in Sing(S)$. Thus $q \in Sing(Q) \cap Sing(S)$ and the singularity at p is not isolated by 2.1.4.

□

There are 10 possible geometric configurations of C. Typical pictures are shown in Figure 3.1. We will see that these pictures correspond to $X_9 + 2A_1$ (on the left) and X_9 (on the right) combinations.

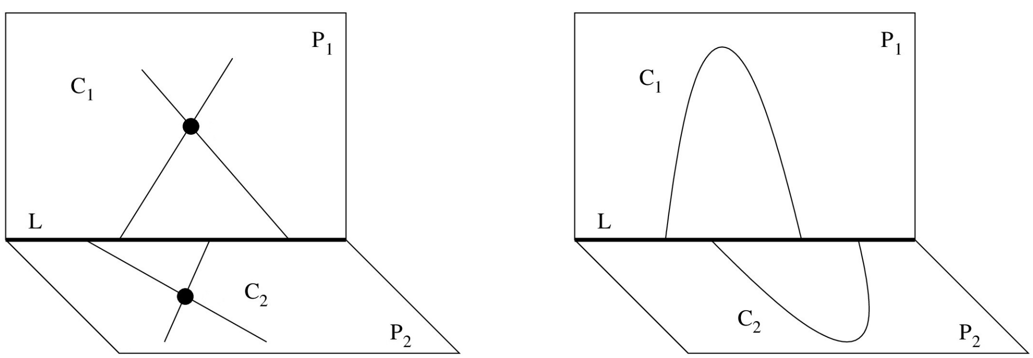

Figure 3.1: The curve C for cubic threefolds with $X_9 + 2A_1$ and X_9 singularities

All the configurations of C together with the corresponding combinations of singularities are given schematically in Figure 3.2. This correspondence is going to be explained in

Proposition 3.2.5. Two configurations in Figure 3.2 are connected by an arrow if one be can deformed to the other by a small perturbation.

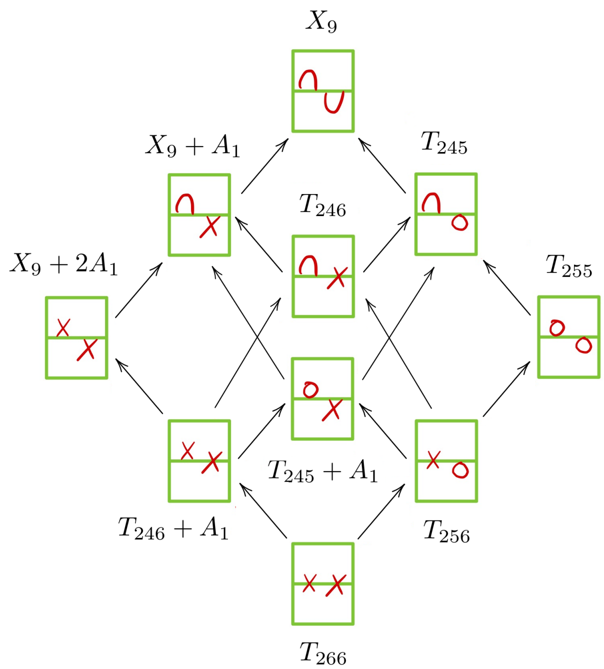

Figure 3.2: Adjancencies of corank 2 singularities with vanishing third jet

We will first consider the case where each C_i is a pair of distinct lines with the intersection point on L. We can choose coordinates such that $f_2 = x_1x_2$, $f_3(0, x_2, x_3, x_4) = ax_2x_3(x_2 - x_3)$ and $f_3(x_1, 0, x_3, x_4) = bx_1x_4(x_1 - x_4)$ where $a, b \in \mathbb{C}$, $ab \neq 0$. In these coordinates $f_3 = ax_2x_3(x_2 - x_3) + bx_1x_4(x_1 - x_4) + x_1x_2g_1(x_1, x_2, x_3, x_4)$ where g_1 is a homogeneous polynomial of degree 1. We can get rid of the $x_1x_2g_1$ term by a linear coordinate change in $\mathbb{P}^5$ (see Remark 2.1.1). After another linear change we can assume that $a = b$. Thus we have $f_3 = ax_2x_3(x_2 - x_3) + ax_1x_4(x_1 - x_4)$. In the affine chart given by $x_0 \neq 0$, X is defined by $x_1x_2 + a(x_2^2x_3 - x_2x_3^2 + x_1^2x_4 - x_1x_4^2) = 0$.

Proposition 3.2.3. *If $f_2 + f_3 = x_1x_2 + a(x_2^2x_3 - x_2x_3^2 + x_1^2x_4 - x_1x_4^2)$ then the singularity at p is of type T_{266}. Moreover p is the unique singularity of X.*

Proof. We will do a sequence of coordinate changes. First we will set $x_1 = y_1 - ay_2x_3 + ax_3^2$,

$x_2 = y_2 - ay_1x_4 + ax_4^2$. After some term cancellation the function $f_2 + f_3$ has the form

$$y_1y_2 - a^2x_3^2x_4^2 + a^3(x_3x_4^4 + x_3^4x_4) + 2a^2(y_1x_3^2x_4 + y_2x_3x_4^2)$$
$$- 2a^3(y_1x_3x_4^3 + y_2x_3^3x_4) - 3y_1y_2x_3x_4 + a^3(y_1^2x_3x_4^2 + y_2^2x_3^2x_4).$$

The next coordinate change is as follows:

$$y_1 = z_1 - 2a^2x_3x_4^2 + 2a^3x_3^3x_4 + \frac{3}{2}z_1x_3x_4 - a^3z_2x_3^2x_4,$$

$$y_2 = z_2 - 2a^2x_3^2x_4 + 2a^3x_3x_4^3 + \frac{3}{2}z_2x_3x_4 - a^3z_1x_3x_4^2;$$

$$f_2 + f_3 = z_1z_2 - a^2x_3^2x_4^2 + a^3(x_3x_4^4 + x_3^4x_4) - 8a^4x_3^3x_4^3 + z_1O_5 + z_2O_5 + O_7$$

where O_k stands for any series with terms of degree k and higher. Now we set

$$x_3 = y_3 + \frac{a}{2}y_4^2,\ x_4 = y_4 + \frac{a}{2}y_3^2.$$

We get

$$f_2 + f_3 = z_1z_2 - a^2(y_3^2y_4^2 + \frac{a^2}{4}y_3^6 + \frac{a^2}{4}y_4^6) + y_3^2y_4O_3 + y_3y_4^2O_3 + z_1O_5 + z_2O_5 + O_7.$$

We can make a final coordinate change such that

$$f_2 + f_3 = z_1z_2 + bz_3^2z_4^2 + z_3^6 + z_4^6 + O(7),\ b \neq 0$$

which is a local description of a T_{266} singularity (see case 16 in Chapter 16 of [AGV12]).

By Theorem 2.1.4, singularities of X other than p correspond to singularities of C contained in the smooth locus of Q. In our case C is smooth outside of $Sing(Q) = L$. □

Corollary 3.2.4. *Any T_{2pq} singularity with $4 \leq p, q \leq 6$ can appear on a cubic threefold.*

Proof. A T_{266} singularity is possible by Proposition 3.2.3 and is adjacent to T_{2pq} singularities with $p, q \leq 6$ (see Table 1.1). Thus a T_{2pq} with $4 \leq p, q \leq 6$ singularity can appear on a cubic threefold by Theorem 1.3.1. □

Proposition 3.2.5. *The possible combinations of singularities on a cubic threefold X containing a singular point p of corank 2 with vanishing third jet are T_{266}, T_{256}, $T_{246}+A_1$, $T_{245}+A_1$, T_{255}, $X_9+2A_1=T_{244}+2A_1$, T_{246}, T_{245}, X_9+A_1 and X_9. The adjacencies of these combinations are shown in Figure 3.2.*

Proof. By Theorem 2.1.4, the geometry of C determines the type of singularity at p. Thus we need to assign a combination of singularities to each configuration in Figure 3.2. The singularity at p is unique if and only if the corresponding curve C does not contain a pair of distinct lines with the intersection point outside of L, otherwise we get additional A_1 singularities. By Corollary 3.2.4, the six configurations without such pairs of lines should correspond to T_{2pq} with $4 \leq p, q \leq 6$. Taking into account the fact that $T_{246}+A_1$ can appear on a cubic threefold by 2.3.2, there is only one way to assign combinations of singularities to the configurations of C in Figure 3.2. □

3.2.2 Singularities with nonvanishing third jet

By Claim 3.2.1, if $p \in X$ has a nonvanishing third jet then $L \not\subset C$, $C = C_1 \cup C_2$ where $C_1 \subset P_1$ and $C_2 \subset P_2$ are plane cubics. We can choose coordinates in which P_1 and P_2 are defined by $x_1 = 0$ and $x_2 = 0$ respectively.

Claim 3.2.6. *If C_1 or C_2 contains a double line then X has a non-isolated singularity.*

Proof. Analogous to Claim 3.2.2. □

Claim 3.2.7. *The intersection $C \cap L$ is either a triple point, a double point and a simple point or three simple points. In these cases, the third jet of $f_2 + f_3$ is equal to x_3^3, $x_3^2x_4$ or $x_3^2x_4 + x_4^3$ respectively.*

Proof. Follows from case 3 in Chapter 16 of [AGV12]. □

We will consider these three cases separately starting with the triple point case.

Claim 3.2.8. *Let $C \cap L$ be a triple point q. We can assume that C_2 is non-singular at q and C_1 has D_4, A_2, A_1 or no singularity at q.*

Proof. A cubic curve can only have D_4, A_3, A_2 and A_1 singularities. If $q \in C_1$ is an A_3 singularity then C_1 is a union of a conic and its tangent line. Since q is a triple point, this tangent line should coincide with L but $L \not\subset C_1$.

If C_1 and C_2 are both singular at q then S is singular at q and p is not isolated by 2.1.4. □

Claim 3.2.9. *Let $C \cap L$ be a triple point. If C_i is smooth at q then it can only have one A_2 or A_1 singularity away from q. If C_i has a D_4 or A_2 singularity at q then it does not have any other singularities. If C_i has an A_1 singularity at q then it can have another A_1 singularity.*

Proof. If C_i is smooth at q then it should be irreducible because otherwise it will have more than one intersection point with L. An irreducible cubic curve can have at most one A_2 or A_1 singularity. If C_i has an A_1 singularity at q and two additional A_1 singularities then it is a union of three lines and thus intersects L at two points. □

Claim 3.2.10. *Singularities of types J_{10}, E_8, E_7 and E_6 can appear on a cubic threefold.*

Proof. A T_{246} singularity is possible by Corollary 3.2.4 and is adjacent to J_{10}. A J_{10} singularity is in turn adjacent to E_8, E_7 and E_6 singularities (see tables 1.1 and 1.2). The claim now follows from Theorem 1.3.1. □

Proposition 3.2.11. *All the possible singularities on X with the third jet equal to x_3^3 are J_{10}, E_8, E_7 and E_6. The maximal configurations among the ones containing these singularities are $J_{10} + A_2$, $E_7 + A_2 + A_1$ and $E_6 + 2A_2$.*

Proof. By Theorem 2.1.4, the singularity type of p depends on the geometry of C_1 and C_2. To prove the proposition, we can combine claims 3.2.8 and 3.2.10: if C_2 is smooth then there are 4 possibilities for the geometry of C_1 at q, and we already know 4 singularity types with the third jet equal to x_3^3 that can appear on a cubic threefold.

The second statement of the proposition follows from Claim 3.2.9 and Theorem 1.3.1. □

μ	Singularity at p	Other singularities of C_1	Singularity of C_1 along L
10	J_{10}	-	D_4
8	E_8	-	A_2
7	E_7	A_1 or none	A_1
6	E_6	A_2, A_1 or none	-
8	D_8	-	A_3
7	D_7	-	A_2
6	D_6	$2A_1$, A_1 or none	A_1
5	D_5	A_3, A_2, $2A_1$, A_1 or none	-
4	D_4	D_4, A_3, A_2, $3A_1$, $2A_1$, A_1 or none	-

Table 3.2: Singularities of corank 2 with nonvanishing first jet

Claim 3.2.12. *Let $C \cap L$ be a simple point q_1 and a double point q_2. Then both C_1 and C_2 are smooth at q_1. We can assume that C_2 is non-singular at q_2 and C_1 has an A_3, A_2, A_1 or no singularity at q_2.*

Proof. Analogous to Claim 3.2.8. □

Claim 3.2.13. *Let $C \cap L$ be a simple point q_1 and a double point q_2. If C_i is smooth at q_2 then it can have A_3, A_2, $2A_1$ or A_1 combinations of singularities away from q_2. If C_i has an A_3 or A_2 singularity at q_2 then it does not have any other singularities. If C_i has an A_1 singularity at q_2 then it can have two more A_1 singularities.*

Proof. Analogous to Claim 3.2.9. □

Proposition 3.2.14. *All the possible singularities on X with the third jet equal to $x_3^2x_4$ are D_k, $5 \leq k \leq 8$. The maximal cases among the ones containing such D_k are $D_8 + A_3$, $D_6 + A_3 + 2A_1$ and $D_5 + 2A_3$.*

Proof. By case 5 in Chapter 16 of [AGV12], if the third jet of p is equal to $x_3^2x_4 + x_4^3$ then p is a D_k singularity with $k \geq 5$.

By Theorem 2.1.4, the singularity type of p depends on the geometry of C_1 and C_2. Claim 3.2.12 says that if C_2 is smooth then there are 4 possibilities for the geometry of C_1 at q_2. It now follows from Theorem 1.3.1 that p is of type D_k such that $5 \leq k \leq 8$.

The second statement of the proposition follows from Claim 3.2.13 and Theorem 1.3.1. □

By case 4 in Chapter 16 of [AGV12], if the third jet of p is equal to $x_3^2x_4 + x_4^3$ then p is a D_4 singularity.

Proposition 3.2.15. *The maximal configuration of singularities on X among the ones containing D_4 is $3D_4$.*

Proof. If p is a D_4 singularity then $C \cap L$ is three simple points and both C_1 and C_2 are smooth at these points. Each C_i can have any of the possible plane cubic combinations of singularities: D_4, A_3, A_2, $3A_1$, $2A_1$, A_1 or no singularities. By Theorem 1.3.1, $3D_4$ is the maximal combination containing D_4. □

The list of all the possible combinations of singularities of corank 2 with nonvanishing third jet is given in Table A.3.

3.3 Constellations of A_n singularities

Let $X \subset \mathbb{P}^4$ be a cubic threefold with A_n singularities only. Let Q, S, C be as in Section 2.1.1. We have $\deg C = 6$ and $g_a(C) = 4$ since C is a complete intersection curve of multidegree $(2, 3)$.

In this section, we will describe possible constellations of A_n singularities on X using the projection method introduced in Section 2.1.2. We will be projecting from the worst singular point $p \in X$.

If p is of A_n type with $n > 1$ then its corank is equal to 1 by Proposition 1.1.3 and $Q \subset \mathbb{P}^3$ can be defined by the equation $x_1^2 + x_2^2 + x_3^2 = 0$. The blow-up $\tilde{Q}$ of Q at $v = [0 : 0 : 0 : 1]$ is the Hirzebruch surface $\mathbb{F}_2$. Denote by $\tilde{C} \in \mathbb{F}_2$ the strict transform of $C \in Q$.

Proposition 3.3.1 ([Bea96], Proposition IV.1)**.** *The Picard group* $\operatorname{Pic} \mathbb{F}_2$ *is isomorphic to* $\mathbb{Z}\sigma \oplus \mathbb{Z}f$ *where* σ *is the class of the unique irreducible curve with negative self-intersection and* f *is the class of a fibre. We have* $\sigma^2 = -2$, $\sigma.f = 1$, $f^2 = 0$, $K_{\mathbb{F}_2} = -2\sigma - 4f$.

Remark 3.3.2. The pullback $H \in \operatorname{Pic} \mathbb{F}_2$ of the hyperplane section class in $\operatorname{Pic} Q$ is equal to $\sigma + 2f$.

Proposition 3.3.3. *If* p *is of* A_n *type with* $n > 2$ *then* C *passes through* v, $g_a(\tilde{C}) = 3$ *and* $[\tilde{C}] = 2\sigma + 6f$.

Proof. By Theorem 2.1.4 and Proposition 1.1.6, v is a singularity of A_{n-2} type on C and in particular C passes through v since $n - 2 > 0$. After blowing up, we get a singularity of A_{n-4} type on $\tilde{C}$ and thus $g_a(\tilde{C}) = g_a(C) - 1 = 3$ by Proposition 1.1.7.

Let $[\tilde{C}] = a\sigma + bf$. Since $\deg C = 6$, $[\tilde{C}].H = 6$ and $b = 6$ where H is as in Remark 3.3.2.

By the genus formula, $g_a(\tilde{C}) = \frac{1}{2}[\tilde{C}].([\tilde{C}] + K_{\mathbb{F}_2}) + 1 = \frac{1}{2}(a\sigma + 6f).(a\sigma + 6f - 2\sigma - 4f) + 1 = -a^2 + 6a - 5$.

Solving $-a^2 + 6a - 5 = 3$, we get $a = 2$ or $a = 4$. If $a = 4$ then $[\tilde{C}].\sigma = -2 < 0$ which is impossible because $\tilde{C}$ is the strict transform of C and σ is the class of the exceptional divisor of the blow-up. Thus $a = 2$ and $[\tilde{C}] = 2\sigma + 6f$. □

Proposition 3.3.4. *If* p *is of* A_2 *type then* C *does not pass through* v, $g_a(\tilde{C}) = 4$ *and* $[\tilde{C}] = 3\sigma + 6f$.

Proof. Assume that C passes through v. Then C is singular at v and the singularity at p is worse than A_2 by Theorem 2.1.4. Contradiction.

The rest of the proof is analogous to Proposition 3.3.3. □

Proposition 3.3.5.

(12) A cubic threefold cannot have singularities of A_n type with $n \geq 12$.

(10) If p is of A_{10} type then it is the only singularity on X.

(8) If p is of A_8 type then the corresponding maximal configuration on X is $A_8 + A_2$.

(6) If p is of A_6 type then the corresponding maximal configuration on X is $A_6 + A_4$.

(4) If p is of A_4 type then the corresponding maximal configurations on X are $2A_4 + A_2$ and $A_4 + 2A_3$.

Proof. We will first prove part (12). By Theorem 2.1.4, an A_{12} singularity at p on X gives an A_{10} singularity at v on C. Denote the irreducible component of C containing v by A. By Proposition 1.1.7, the arithmetic genus of A should be at least 5.

If A is the only irreducible component of C then we get a contradiction since $g_a(C) = 4$.

If $C = A \cup B$ then we have $g_a(C) = g_a(A) + g_a(B) + A.B - 1$. Consider the strict transforms $\tilde{A}$ and $\tilde{B}$ of A and B. By Proposition 3.3.3, $\tilde{C}.\sigma = (2\sigma + 6f).\sigma = 2$ and we get $\tilde{A}.\sigma = 2$, $\tilde{B}.\sigma = 0$ since $v \notin B$. There are two possibilities: either $\tilde{A} = 2f$ and $\tilde{B} = 2\sigma + 4f$ or $\tilde{A} = \sigma + 4f$ and $\tilde{B} = \sigma + 2f$. In the first case, $\tilde{A}$ is reducible. In the second case, $g_a(\tilde{A}) = 0$ and $g_a(A) = g_a(\tilde{A}) + 1 \leq 5$. Contradiction.

If a cubic threefold has an A_n singularity with $n > 12$ then it deforms to a cubic threefold with an A_{12} singularity by Theorems 1.2.4 and 1.3.1. Thus such a threefold is not possible.

We can use exactly the same reasoning to show that C cannot be reducible in parts (10), (8) and (6) of the proposition. By Theorem 2.1.4 and Proposition 1.1.7, we get that the corresponding maximal configurations on X are A_{10}, $A_8 + A_2$, $A_6 + A_4$ and $A_6 + 2A_2$. However a constellation of $A_6 + 2A_2$ type is not possible because it deforms to $A_3 + 3A_2$ by Theorem 1.2.4, and $A_3 + 3A_2$ is not possible by Proposition 3.3.10.

Similarly, if p is of A_4 type and C is irreducible, we get that the corresponding maximal configuration on X is $2A_4 + A_2$.

If p is of A_4 type and C is reducible, we have $\tilde{A} = \sigma + 4f$, $\tilde{B} = \sigma + 2f$, $g_a(A) = g_a(\tilde{A}) + 1 = 1$, $g_a(B) = 1$. In this case, both A and B are irreducible, A has one A_2

singularity and B can possibly have an A_2 or an A_1 singularity by Proposition 1.1.7. Since $A.B = (\sigma + 4f).(\sigma + 2f) = 4$, we can get $2A_3$, $A_3 + 2A_1$ or $4A_1$ singularities by intersecting A and B. The corresponding maximal case on X is $A_4 + 2A_3$. □

Proposition 3.3.6.

(11) If p is of A_{11} type then it is the only singularity on X.

(9) If p is of A_9 type then the corresponding maximal configuration on X is $A_9 + A_1$.

(7) If p is of A_7 type then the corresponding maximal configuration on X is $A_7 + A_4$.

(5) If p is of A_5 type then the corresponding maximal configuration on X is $2A_5 + A_1$.

Proof.

(11) By Theorem 2.1.4, an A_{11} singularity at p on X gives an A_9 singularity at v on C. A singularity of A_9 type can either come from one irreducible component of C or from intersecting two components.

Using the same argument as in the A_{12} case in Proposition 3.3.5, we can show that A_9 cannot come from one component.

If A_9 comes from the intersection of two components A and B, we have $\tilde{A}.\sigma = 1$ and $\tilde{B}.\sigma = 1$. Then either $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ or $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$. In the first case we have $\tilde{A}.\tilde{B} = 2$, in the second case we have $\tilde{A}.\tilde{B} = 4$. However $\tilde{A}.\tilde{B}$ should be at least 4 since we have an A_7 singularity on $\tilde{C}$ and thus the first case is not possible.

If $\tilde{A}$ is reducible then $\tilde{A} = f + (\sigma + 2f)$. This situation is impossible because it is the same as the first case from the previous paragraph after relabeling the components. For the same reason, $\tilde{B}$ cannot be reducible. By the genus formula, $g_a(\sigma + 3f) = 0$. Since $g_a(A) = g_a(\tilde{A}) = 0$ and $g_a(B) = g_a(\tilde{B}) = 0$, both A and B are smooth and v is the only singularity on C.

(9) By Theorem 2.1.4, an A_9 singularity at p on X gives an A_7 singularity at v on C. A singularity of A_7 type can either come from one irreducible component of C or from intersecting two components.

By Proposition 1.1.7, if C is irreducible then v is its only singularity.

Now assume that C is reducible and A_7 comes from an irreducible component A of C. In this case we have $\tilde{A} = \sigma + 4f$ and $g_a(\tilde{A}) = 0$. Then $g_a(A) = g_a(\tilde{A}) + 1 = 1$ and A cannot have an A_7 singularity by Proposition 1.1.7.

If A_7 comes from the intersection of two components A and B then, similarly to part (11), $\tilde{A} = \tilde{B} = \sigma + 3f$ and both A and B are smooth. Since $\tilde{A}.\tilde{B} = 4$ and an A_7 singularity at v on C corresponds to a triple intersection on $\tilde{C}$, we have an extra A_1 singularity on C. Thus we get an $A_7 + A_1$ configuration of singularities on C and an $A_9 + A_1$ configuration on X by Theorem 2.1.4.

(7) By Theorem 2.1.4, an A_7 singularity at p on X gives an A_5 singularity at v on C. A singularity of A_5 type can either come from one irreducible component of C or from intersecting two components.

By Proposition 1.1.7, if C is irreducible then we have an A_7 and possibly one A_2 or A_1 singularity on X.

Similarly to part (9), A_5 cannot come from one irreducible component if C is reducible.

Now assume that A_5 comes from the intersection of two components A and B. Then either $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ or $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$.

If $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ then $g_a(\tilde{A}) = 0$, $g_a(\tilde{B}) = 2$. If $\tilde{B}$ is irreducible then we can get an additional A_i singularity with $i \leq 4$ on $\tilde{B}$ or one of the configurations $2A_2$, $A_2 + A_1$, $2A_1$. However an $A_7 + 2A_2$ configuration cannot appear on X because it deforms to $A_6 + 2A_2$ by Theorem 1.2.4, and $A_6 + 2A_2$ is not possible by Proposition 3.3.5.

If $\tilde{B}$ is reducible then $\tilde{B} = (\sigma + 3f) + (\sigma + 2f)$ or $\tilde{B} = f + (2\sigma + 4f)$. In the first case we get $\tilde{A}.(\sigma + 3f) = 1$, in the second case we get $\tilde{A}.f = 0$. Since an A_5 singularity at v on C corresponds to a double intersection on $\tilde{C}$, we get a contradiction.

If $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$ and both A and B are irreducible then, analogously to parts (11) and (9), we have either an $A_7 + A_3$ or an $A_7 + 2A_1$ configuration on X. If $\tilde{A}$ is reducible then we can relabel the components and get the $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ case which we have already considered.

(5) By Theorem 2.1.4, an A_5 singularity at p on X gives an A_3 singularity at v on C. A singularity of A_3 type can either come from one irreducible component of C or from intersecting two components.

By Proposition 1.1.7, if C is irreducible then the corresponding maximal configurations on X are $A_5 + A_4$ and $A_5 + 2A_2$.

Similarly to parts (7) and (9), A_3 cannot come from one irreducible component if C is reducible.

Now assume that A_3 comes from the intersection of two components A and B. Then either $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ or $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$.

If $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$, $g_a(\tilde{A}) = 0$, $g_a(\tilde{B}) = 2$ and $\tilde{B}$ is irreducible then we can get and additional A_i singularity with $i \leq 4$ on $\tilde{B}$ or one of the configurations $2A_2$, $A_2 + A_1$, $2A_1$. We also get an A_1 singularity from intersecting A and B. Thus the corresponding maximal configurations on X are $A_5 + A_4 + A_1$ and $A_5 + 2A_2 + A_1$.

If $\tilde{B}$ is reducible then $\tilde{B} = (\sigma+3f)+(\sigma+2f)$ or $\tilde{B} = f+(2\sigma+4f)$. In the first case we have $\tilde{A}.(\sigma + 3f) = 1$, in the second case we have $\tilde{A}.f = 0$. Since an A_3 singularity on C corresponds to an A_1 singularity on $\tilde{C}$, $\tilde{B} = (\sigma+3f)+(\sigma+2f)$. We get an A_3 singularity on C from intersecting $\tilde{A}$ and $\sigma + 3f$ and an A_1 singularity from intersecting $\tilde{A}$ and $\sigma + 2f$. The intersection number of the two components of $\tilde{B}$ is $(\sigma + 3f).(\sigma + 2f) = 3$

which means that there is an additional A_5 singularity on $\tilde{B}$ or one of the configurations $A_3 + A_1$, $3A_1$. Thus the corresponding maximal configuration on X is $2A_5 + A_1$.

If $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$ and both A and B are irreducible then, analogously to the previous parts, we have one of the configurations $2A_5$, $A_5 + A_3 + A_1$, $A_5 + 3A_1$ on X. If $\tilde{A}$ is reducible then we can relabel the components and get the $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ case which we have already considered. □

Proposition 3.3.7. *If p is of A_3 type then the corresponding maximal configurations on X are $3A_3 + A_1$, $2A_3 + A_2 + 2A_1$ and $2A_3 + 4A_1$.*

Proof. By Theorem 2.1.4, an A_3 singularity at p on X gives an A_1 singularity at v on C.

By Proposition 1.1.7, if C is irreducible then the possible maximal combinations on X are $A_3 + 3A_2$ and $2A_3 + A_2$. However an $A_3 + 3A_2$ configuration cannot appear on X by Proposition 3.3.10. Thus the the corresponding maximal combinations on X are $A_3+2A_2+A_1$ and $2A_3 + A_2$.

Now assume C is reducible. If there is an irreducible component A such that $\tilde{A}.\sigma = 2$ then $\tilde{A} = \sigma + 4f$, $\tilde{B} = \sigma + 2f$, $g_a(A) = g_a(\tilde{A}) + 1 = 1$, $g_a(B) = g_a(\tilde{B}) = 0$ and B is irreducible. We can get $2A_3$, $A_3 + 2A_1$ or $4A_1$ singularities from intersecting A and B. The corresponding maximal configuration on X is $3A_3$.

If C is a union of two components A and B such that A is irreducible, $\tilde{A}.\sigma = 1$ and $\tilde{B}.\sigma = 1$ then either $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ or $\tilde{A} = \sigma + 3f$, $\tilde{B} = \sigma + 3f$.

If $\tilde{A} = f$, $\tilde{B} = 2\sigma+5f$ and B is irreducible, we get an A_1 singularity at v and an additional A_3 or $2A_1$ combination of singularities from intersecting A and B. By Proposition 1.1.7, the possible maximal combination on B is $2A_2$. This gives us a $2A_3 + 2A_2$ configuration on X. However $2A_3 + 2A_2$ deforms to $A_3 + 3A_2$ which cannot appear on X by Proposition 3.3.10. Thus the maximal combinations on X we get in this case are $2A_3 + A_2 + A_1$ and $A_3 + 2A_2 + 2A_1$.

If B is reducible then we have the following options:

(i) $\tilde{A} = f$, $\tilde{B} = f + (2\sigma + 4f)$;

(ii) $\tilde{A} = f$, $\tilde{B} = (\sigma + 2f) + (\sigma + 3f)$;

(iii) $\tilde{A} = f$, $\tilde{B} = f + (\sigma + 2f) + (\sigma + 2f)$.

The maximal configuration on X corresponding to case (i) is $2A_3 + 4A_1$. The possible maximal configuration corresponding to case (ii) is $3A_3 + A_2$. However $2A_3 + 2A_2$ deforms to $A_3 + 3A_2$ which cannot appear on X by Proposition 3.3.10 and we get maximal cases $2A_3 + A_2 + 2A_1$ and $3A_3 + A_1$. In case (iii), we have an A_1 singularity from intersecting f and $\sigma + 3f$ components, an A_1 from intersecting f and $\sigma + 2f$ and an $A_3 + A_1$ or $3A_1$ configuration from intersecting $\sigma + 3f$ and $\sigma + 2f$. The corresponding maximal combination on X is $2A_3 + 3A_1$.

If $\tilde{A} = \sigma+3f$, $\tilde{B} = \sigma+3f$ and B is irreducible, we get a $2A_3$, A_3+2A_1 or $4A_1$ configuration from intersecting $\tilde{A}$ and $\tilde{B}$. The corresponding maximal combination on X is $3A_3$. If B is reducible then we can relabel the components and get the $\tilde{A} = f$, $\tilde{B} = 2\sigma + 5f$ case which we have already considered. □

Proposition 3.3.8. *If p is of A_2 type then the corresponding maximal configurations on X are $5A_2$, $2A_2 + 4A_1$ and $A_2 + 6A_1$.*

Proof. By Proposition 3.3.4, C does not pass through v.

By Proposition 1.1.7, if C is irreducible then the corresponding maximal configuration on X is $5A_2$.

Now assume that $C = A \cup B$ and A is irreducible. Then $\tilde{A} = \sigma + 2f = H$, $\tilde{B} = 2H$ where H is as in Remark 3.3.2. If B is irreducible, we have $g_a(\tilde{A}) = 0$, $g_a(\tilde{B}) = 1$, $\tilde{A}.\tilde{B} = 4$. Thus we get $4A_1$ singularities from intersecting $\tilde{A}$ and $\tilde{B}$ and possibly one A_2 or A_1 singularity on $\tilde{B}$. The corresponding maximal configuration on X is $2A_2 + 4A_1$.

If B is reducible, we have $\tilde{B} = H + H$, $g_a(H) = 0$. In this case all the components of C are smooth, and we get $6A_1$ singularities from intersecting these components. The corresponding maximal configuration on X is $A_2 + 6A_1$. □

Proposition 3.3.9. *If p is of A_1 type then the corresponding maximal configurations on X is $10A_1$.*

Proof. See Claim 1.4.3. □

Proposition 3.3.10. *A constellation of $A_3 + 3A_2$ type cannot appear on a cubic threefold.*

Proof. Assume it is possible and let $p \in X$ be the A_3 singularity. It follows from the proof of Proposition 3.3.7 that the corresponding curve C is irreducible. Consider a smooth hyperplane section L of the quadric cone Q. It is isomorphic to $\mathbb{P}^1$. Projecting from $v \in C$, we get a branched double cover $\eta : \tilde{C} \to L$. By Theorem 2.1.4, C has $3A_2 + A_1$ singularities and $\tilde{C}$ has $3A_2$ singularities. Consider the normalization $\nu : C^\nu \to \tilde{C}$. The curve C^ν is isomorphic to $\mathbb{P}^1$ by Propositions 3.3.3 and 1.1.7. The double cover $\eta \circ \nu : C^\nu \to L$ has at least 3 branch points corresponding to the A_2 singularities. Thus, by the Riemann-Hurwitz formula, $\chi(C^\nu) \leq 2\chi(L) - 3$ and we get $2 \leq 4 - 3 = 1$. Contradiction. □

Claim 3.3.11. *A constellation of $3A_3 + A_1$ type can appear on a cubic threefold.*

Proof. If p is an A_1 singularity then Q is a smooth quadric. If $f_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, $f_3 = x_3(x_1 - i\sqrt{2}x_2 - i\sqrt{2}x_3 - x_4)(x_1 + i\sqrt{2}x_2 - i\sqrt{2}x_3 - x_4)$ then C is a union of 3 smooth pairwise tangent curves on Q (and all the three intersection points are distinct). Thus C has $3A_3$ singularities and X has $3A_3 + A_1$ singularities. □

Combining all the results of this section, we get the following theorem:

Theorem 3.3.12. *Among the combinations of singularities only containing A_n singularities, the maximal ones are A_{11}, $A_7 + A_4$, $2A_5 + A_1$, $3A_3 + A_1$, $2A_3 + A_2 + 2A_1$, $2A_3 + 4A_1$, $5A_2$ and $10A_1$.*

The list of all the possible constellations of A_n singularities is given in Table A.4.

Chapter 4

The Milnor lattice of O_{16}

First we will review some results from [LPZ18] about cubic fourfolds with Eckardt points. Consider a pair (X, H) such that X is a cubic threefold and H is a hyperplane section: $X = V(f(x_0, \ldots, x_4))$, $H = V(x_0)$. We can construct a cubic fourfold $Y = V(f + x_0 x_5^2)$.

Theorem 4.0.1. *[LPZ18]*

(i) A cubic fourfold Y with an Eckardt point is the same as a pair (X, H).

(ii) The moduli of such cubic fourfolds is the period domain associated to $D_4^{\oplus 3} \oplus U^{\oplus 2}$.

If Y is smooth then X is smooth and H intersects X transversely. In this case $D_4^{\oplus 3} \oplus U^{\oplus 2}$ is the transcendental lattice in $H^4(Y, \mathbb{Z})$. If X has a node then Y has a node. If H is simply tangent to X then Y has a pair of nodes (there are two corresponding divisors in the moduli space).

Lemma 4.0.2. *[LPZ18] Let $v \in T$ be a primitive vector. Suppose that the reflection r_v preserves T and $\mathcal{D}_M \cap v^\perp \neq \emptyset$. Then there are two possibilities.*

(1) $v^2 = 2$ and $div(v) = 1$. In this case, $\hat{v} = 0$ in A_T.

(2) $v^2 = 4$ and $div(v) = 2$. In this case, $q_T(\hat{v}) \equiv 1 \mod 2\mathbb{Z}$.

Vectors of type (1) correspond to pairs (X, H) where X has a node, and vectors of type (2) correspond to pairs (X, H) where H is simply tangent to X.

Theorem 4.0.3. *[LPZ18] The period map $\mathcal{P}_0 : \mathcal{M}_0 \to \mathcal{D}_M/O^+(T)$ extends to a morphism $\mathcal{P} : \mathcal{M} \to \mathcal{D}_M/O^+(T)$ over the simple singularities with image the complement of the H arrangement. It is an isomorphism onto its image.*

The following propositions relates the study of cubic fourfolds with Eckardt points to the study of O_{16}:

Proposition 4.0.4. *[LPZ18] The Milnor lattice associated to the singularity O_{16} is isometric to $T = D_4^{\oplus 3} \oplus U^{\oplus 2}$.*

Corollary 4.0.5 (Theorem III)**.** *A root lattice R corresponds to a combination of ADE singularities on a cubic threefold if and only if $Sat(R)$ does not contain a primitive vector v such that $v^2 = 4$ and $div(v) = 2$.*

Proof. By Lemma 4.0.2, a vector $w \in T \otimes \mathbb{C}$ corresponds to a pair (X, H) with H tangent to X if and only it is contained in the Heegner divisor H_t. The Heegner divisor corresponds to hyperplanes in $T \otimes \mathbb{C}$ orthogonal to primitive vectors $v \in T$ such that $v^2 = 4$ and $div(v) = 2$. □

Chapter 5

Combinatorial description of the ADE configurations

In this section, we present a graph Γ whose induced subgraphs (see Definition 1.2.4) correspond to combinations of singularities on cubic threefolds. We constructed Γ from the $3D_4$ diagram. We can get the graph Δ on Figure 5.1 by adding one vertex to $3D_4$. However, Δ does not contain A_{11} or E_8 subgraphs, and the corresponding singularities occur on cubic threefolds. We obtained Γ by adding two more vertices to Δ. Table 5.1 shows which vertices we need to remove from Γ to get most of the maximal combinations. It is not possible to get $5A_2$ and $10A_1$ from Γ. We can get $4A_2 + A_1$ and $9A_1$, however these diagrams are induced subgraphs of $E_6 + 2A_2$ and $3D_4$ respectively.

Remark 5.0.1. Notice that we remove vertex 3 in each of the cases in Table 5.1. It means that we can consider a graph Γ' that is obtained from Γ by removing 3. We choose to consider Γ because it is more symmetric.

Lemma 5.0.2. *If an induced subgraph of Δ is a union of ADE graphs, then the corresponding combination of ADE singularities appears on a cubic threefold.*

Proof. First notice that if we remove the central vertex of Δ, we get the $3D_4$ graph. There exists a cubic threefold with $3D_4$ singularities (5.1), and thus the induced subgraphs of $3D_4$

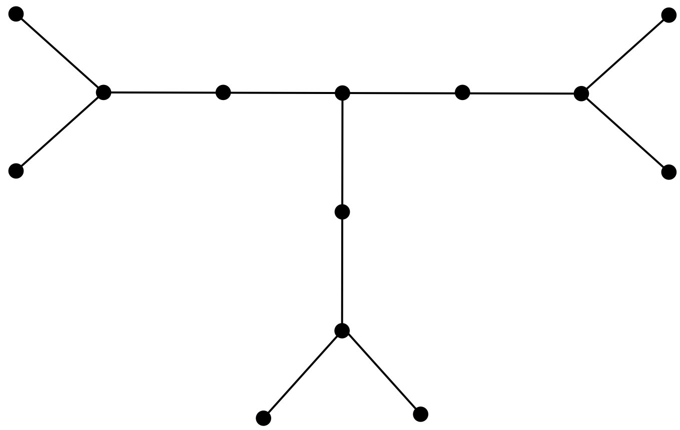

Figure 5.1: Graph Δ

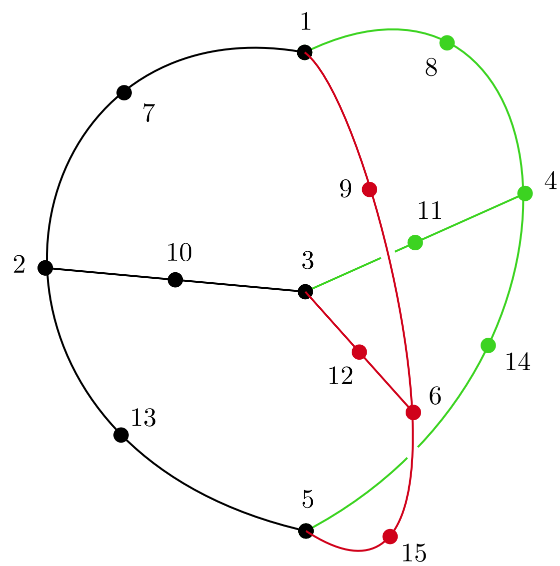

Figure 5.2: Graph Γ

correspond to configurations of singularities on cubic threefolds by Theorems 1.2.5 and 1.3.1. Now assume that we do not remove the center, and consider the following cases:

1. we remove a vertex that is one edge away from the center;

Maximal configuration	Points to remove from Γ
$E_8 + A_2$	3, 8, 9, 11, 15
$E_7 + A_2 + A_1$	3, 4, 8, 9, 15
$E_6 + 2A_2$	3, 8, 9, 14, 15
$D_8 + A_3$	1, 3, 11, 15
$D_6 + A_3 + 2A_1$	1, 3, 4, 15
$D_5 + 2A_3$	1, 3, 14, 15
$3D_4$	1, 3, 5
A_{11}	3, 8, 12, 13
$A_7 + A_4$	3, 9, 13, 14
$2A_5 + A_1$	2, 3, 9, 14

Table 5.1: Maximal ADE configurations and the corresponding induced subgraphs of Γ

2. we remove a vertex that is two edges away from the center and do not remove vertices that are one edge away;

3. we only remove vertices that are three edges away from the center.

In case (3), a resulting graph cannot be of ADE type because it contains the $\tilde{E}_6$ graph. The possibilities in case (1) are shown in Figure 5.3. We get the $D_5 + 2A_3$, $D_6 + A_3 + 2A_1$ and $D_8 + A_3$ subgraphs, and there are cubic threefolds with the corresponding combinations of singularities (3.2.14). The analysis in case (2) is similar. □

The six vertices 1, 2, 3, 4, 5, 6 have degree three in the graph Γ, and the remaining nine vertices $7, \dots, 15$ have degree two in Γ. Each of the degree three vertices can be moved to any other degree three vertex by a symmetry of Γ, and the same holds for the degree two vertices.

Theorem 5.0.3 (Theorem II)**.** *A combination of ADE singularities appears on a cubic threefolds if and only if the union of the corresponding Dynkin graphs is* $10A_1$*,* $5A_2$ *or can be realized as an induced ADE subgraph of* Γ*.*

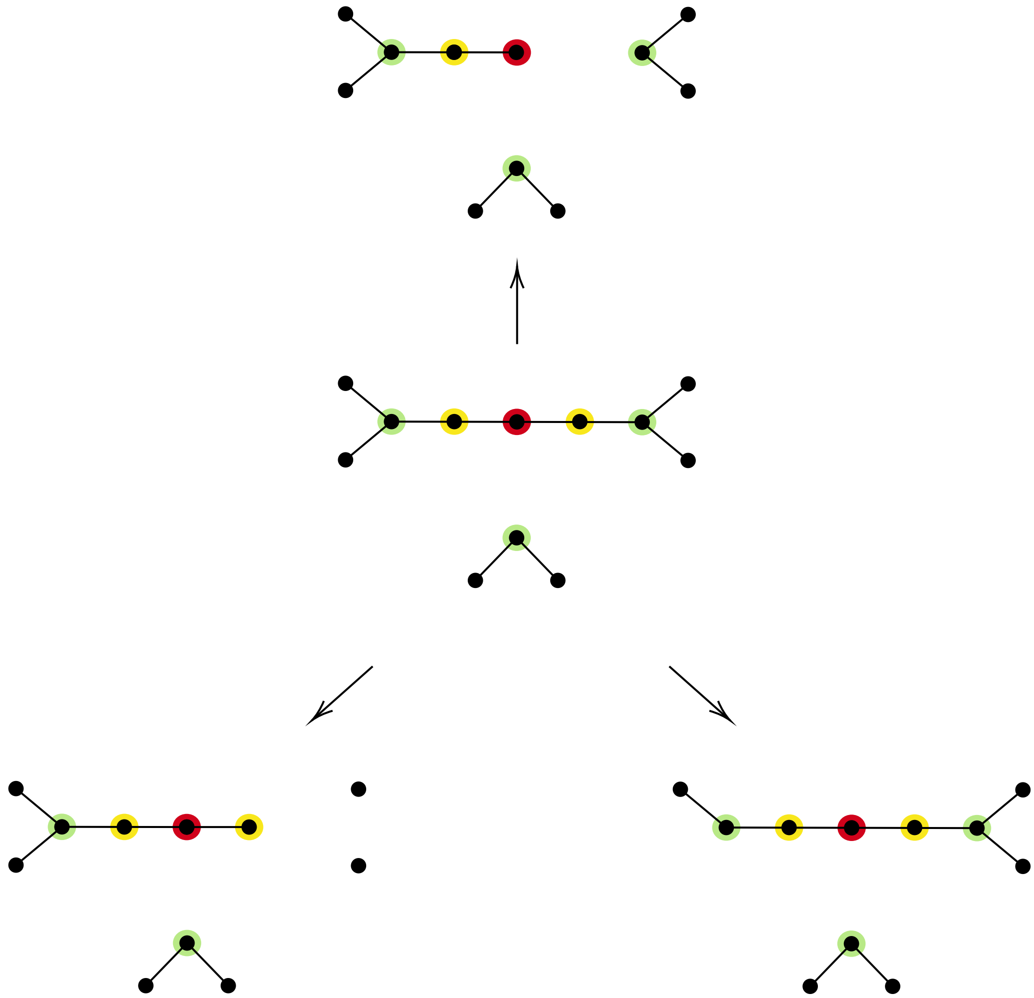

Figure 5.3: $D_5 + 2A_3$, $D_6 + A_3 + 2A_1$ and $D_8 + A_3$ as induced subgraphs of Δ

Proof. One direction follows from Table 5.1. For the other direction, we split the proof into three cases: when we remove at least 2 degree three points, exactly 1 degree three point or no degree three points. In this proof, we use Theorem 1.2.4, Theorem 1.3.1 and Theorem I. The labeling of vertices is from Figure 5.2.

(1) Suppose we remove 2 degree three vertices. Without loss of generality, we can say that these vertices are either 1 and 3 or 2 and 3. If we remove 1 and 3, we get the graph Δ which only contains induced ADE subdiagrams that come from combinations of singularities on cubic threefolds by Lemma 5.0.2.

Now assume that we remove 2 and 3. If we remove an additional degree three vertex, then the graph we get is a subgraph of Δ, and we can use Lemma 5.0.2 again. If we only remove 2 and 3, the resulting graph contains the cycle $(1, 8, 4, 14, 5, 15, 6, 9)$. Thus we need to remove $8, 9, 14$ or 15 which have degree two in Γ. Since the picture is symmetric, we can choose to remove 8. Figure 5.4 shows the graph we get (the points that have degree three in Γ are marked with green). The case by case analysis of this graph is straightforward and can be done similarly to the analysis in the proof of Lemma 5.0.2.

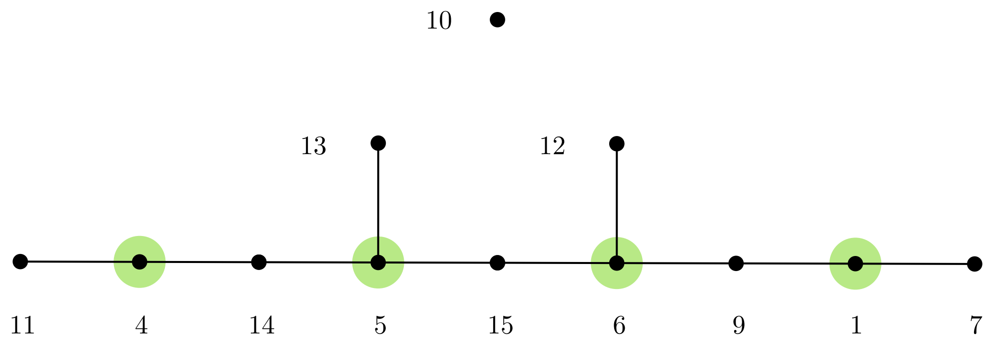

Figure 5.4: Graph Γ

(2) Suppose we remove exactly 1 vertex of degree three, say 3. The remaining graph contains three cycles: $(1, 7, 2, 13, 5, 14, 4, 8)$, $(1, 7, 2, 13, 5, 15, 6, 9)$ and $(1, 8, 4, 14, 5, 15, 6, 9)$. To get rid of them, we need to remove at least two vertices. By symmetry, we can assume these vertices are either 7 and 8 or 7 and 14. Both possibilities are shown in Figure 5.5, and their analysis is straightforward (notice that we are not allowed to remove the vertices marked with green here because they have degree three in Γ).

(3) First we make a few observations. Let γ be an induced subgraph of Γ which contains the vertices $1, \ldots, 6$. Notice that if 1 has degree three in γ (i.e. $7, 8, 9 \in \gamma$), then γ

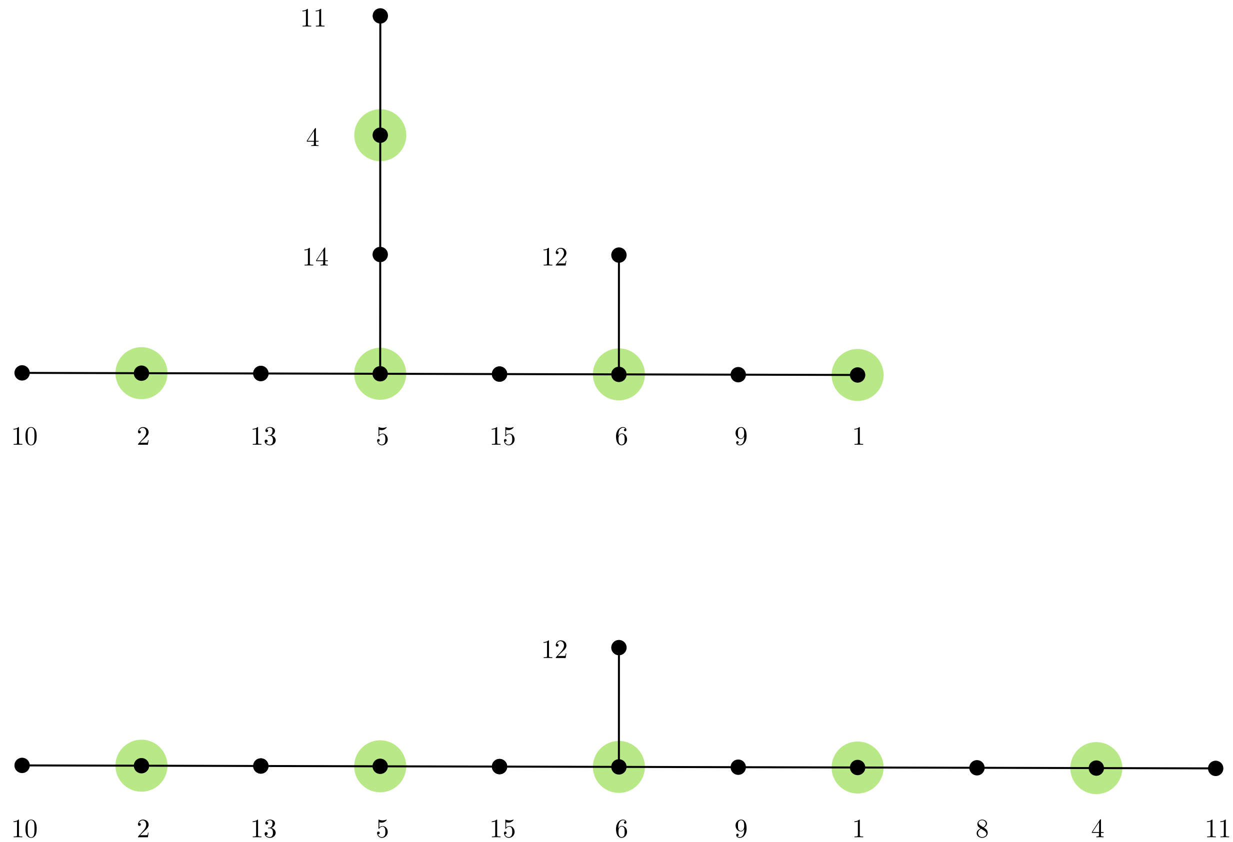

Figure 5.5: Graph Γ

contains an $\tilde{E}_6$ subgraph and thus γ is not a union of ADE graphs. Same holds if 2, 3, 4, 5 or 6 have degree three in γ. Thus γ contains only vertices of degree two or less. If one of the vertices $7, \ldots, 15$ is in γ, then it is of degree two in γ because $1, \ldots, 6$ are in γ. If $1, \ldots, 6$ are all of degree two in γ, then γ contains a cycle and is not a union of ADE graphs. Thus there is a vertex, say 1, of degree one. We can assume that $7 \in \gamma$ and $8, 9 \notin \gamma$.

If 2 is of degree one, then 10 and 13 are not in γ. It implies that γ is an induced subgraph of the union of the A_3 diagram containing 1, 7, 2 and the cycle $(3, 11, 4, 14, 5, 15, 6, 12)$ of length 8. After breaking the cycle, we get an A_7 diagram. An $A_7 + A_3$ combination of singularities appears on cubic threefolds.

If 2 is of degree two, without loss of generality, $10 \in \gamma$. If 3 is of degree one, then γ is contained in the $2A_5$ diagram containing the vertices 1, 7, 2, 10, 3 and 1, 14, 5, 15, 6. A combination of two A_5 singularities is possible on cubic threefolds.

If 3 is of degree two, we can assume that $11 \in \gamma$. If 4 is of degree one, then γ is an induced subgraph of the $A_7 + A_3$ diagram containing the vertices 1, 7, 2, 10, 3, 11, 4 and 5, 15, 6.

If 4 is of degree two, then $14, 5 \in \gamma$. If $15 \notin \gamma$, then γ is a union of the A_9 diagram containing the vertices 1, 7, 2, 10, 3, 11, 4, 14, 5 and the A_1 diagram $\{6\}$. If $15 \in \gamma$, then γ is the A_{11} diagram containing the vertices 1, 7, 2, 10, 3, 11, 4, 14, 5, 15, 6. Both A_{11} and $A_9 + A_1$ combinations appear on cubic threefolds. □

Now we want to relate Γ to the Milnor lattice T of O_{16}. We can associate a matrix M to Γ as follows: we can set the self-intersection of each vertex of Γ to 2, and the intersection of two distinct vertices is minus the number of edges between them (see 5.0.1).

Claim 5.0.4. *The signature of the matrix M is* $(14, 1)$.

The following proposition shows another reason why Γ is a good choice of a graph to consider: its lattice embeds in the lattice of O_{16}.

Proposition 5.0.5. *Lattice M embeds into the Milnor lattice $T \cong D_4^{\oplus 3} \oplus U^{\oplus 2}$ of O_{16}.*

Proof. We choose a basis for T consisting of elements α_{ij}, $1 \leq i \leq 3$, $1 \leq j \leq 4$ which correspond to the $D_4^{\oplus 3}$ part of T, and u_i, v_i, $i = 1, 2$ which correspond to $U^{\oplus 2}$. We have $\alpha_{ij}^2 = 2$, $\alpha_{i,4}.\alpha_{i,j} = -1$ if $j \neq 4$; $u_i.v_i = -1$. All the other intersections are 0.

We can write an embedding explicitly:

$\beta_2 \to \alpha_{1,4}$, $\beta_4 \to \alpha_{2,4}$, $\beta_6 \to \alpha_{3,4}$;

$\beta_1 \to u_1 + v_1$, $\beta_3 \to u_2 + v_2$;

$\beta_7 \mapsto \alpha_{1,1} + u_1$, $\beta_8 \mapsto \alpha_{2,1} + u_1$, $\beta_9 \mapsto \alpha_{3,1} + u_1$;

$\beta_{10} \mapsto \alpha_{1,2} + u_2$, $\beta_{11} \mapsto \alpha_{2,2} + u_2$, $\beta_{12} \mapsto \alpha_{2,3} + u_2$;

$\beta_{13} \mapsto \alpha_{3,1},\ \beta_{14} \mapsto \alpha_{3,2},\ \beta_{15} \mapsto \alpha_{3,3};$

$\beta_5 \mapsto (u_1 - v_1) + (u_2 - v_2) + \Sigma_{i=1}^{3}(\alpha_{i1} + \alpha_{i2} + \alpha_{i4}).$ □

$$M = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix} \tag{5.0.1}$$

Chapter 6

Cohomology

In this section, we use the Mayer-Vietoris exact sequence to compute the defect of a nodal cubic threefold. Recall that if X is a cubic threefolds with only isolated singularities, and X_t is a smooth cubic threefold, then $H^k(X) = H^k(X_t)$ whenever $k \neq 3, 4$.

As discussed in the introduction, the cohomology groups of X are controlled by two local invariants and a global invariant. The local ones can be computed in terms of the spectrum of the singularities of X. The global invariant is the defect:

Definition 6.0.1. The defect σ of X is defined as $\sigma(X) = h_4(X) - h_2(X)$.

Theorem 6.0.1. *(adapted from [PS08]) Let $f : \tilde{X} \to X$ be the blow-up of X along a subvariety $D \subset X$. Put $E = f^{-1}(D)$. Let $g : E \to D$ and let $i : D \to X$ and $\tilde{i} : E \to \tilde{X}$ denote the inclusions. Then one has a long exact sequence of mixed Hodge structures:*

$$\dots \to H^k(X) \xrightarrow{(f^*, i^*)} H^k(\tilde{X}) \oplus H^k(D) \xrightarrow{\tilde{i}^* - g^*} H^k(E) \to H^{k+1}(X) \to \dots$$

It is called the Mayer-Vietoris sequence for the discriminant square associated to f.

Theorem 6.0.2. *(adapted from [PS08]) Weights of $H^k(Y)$ are $\leq k$ if Y is compact and $k \leq \dim Y$. If Y is smooth, then the Hodge structures on H^k are pure.*

Now let X be a cubic threefold with only A_1 singularities. Let p be one of the singularities, and C the corresponding $(2, 3)$-complete intersection curve. We have:

Claim 6.0.3. $Bl_p(X) = Bl_C(\mathbb{P}^3)$.

We can apply the Mayer-Vietoris theorem 6.0.1 to $Bl_p(X)$ and $Bl_C(\mathbb{P}^3)$, and use the observation above to relate the two sequences.

Proposition 6.0.4. *Let $h^1(C) = b_1$, $h^2(C) = b_2$. Then $h^3(Bl_C(\mathbb{P}^3)) = b_1$ and $h^4(Bl_C(\mathbb{P}^3)) = b_2 + 1$.*

Proof. We have $H^3(\mathbb{P}^3) = H^5(\mathbb{P}^3) = 0$, $H^3(C) = 0$. Thus there is a long exact sequence:

$$0 \to H^3(Bl_C(\mathbb{P}^3)) \to H^3(E) \to H^4(\mathbb{P}^3) \to H^4(Bl_C(\mathbb{P}^3)) \to H^4(E) \to 0.$$

Notice that E is a $\mathbb{P}^1$-bubdle over C. By Theorem 6.0.2, $H^3(E)$ is of weight ≤ 3. Also $h^3(E) = h^3(C \times \mathbb{P}^1) = b_1$, $h^4(E) = h^4(C \times \mathbb{P}^1) = b_2$. Since $H^4(\mathbb{P}^4)$ is pure of weight 4, we have $H^3(E) \cong H^3(Bl_C(\mathbb{P}^3))$, and the statement of the proposition immediately follows. □

Proposition 6.0.5 (Theorem IV)**.** *If X is a cubic threefold with only A_1 singuarities, and the (2,3) complete intersection curve C corresponding to one of the singularities of X has k irreducible components, then $h^4(X) - h^2(X) = k - 1$.*

Proof. We have $H^3(\mathbb{P}^3) = H^5(\mathbb{P}^3) = 0$, $H^3(C) = 0$. Thus there is a long exact sequence:

$$\ldots \to H^3(X) \to H^3(Bl_p(X)) \to H^3(Q) \to H^4(X) \to H^4(Bl_p(X)) \to H^4(Q) \to 0.$$

If p is an A_1 singularity, then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. In particular, $h^1(Q) = h^3(Q) = 0$, $h^2(Q) = 2$. Thus $h_4(X) = h^4(Bl_p(X)) - 2 = h^4(Bl_C(\mathbb{P}^3)) - 2 = b_2 + 1 - 2 = b_2 - 1$. □

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Appendix A

Tables of singularities

T	μ	type	T	μ	type
1	16	O_{16}	5	10	Q_{10}
2	12	U_{12}	6	10	T_{344}
3	11	S_{11}	7	9	T_{334}
4	11	T_{444}	8	8	$P_8 = T_{333}$

Table A.1: Configuration containing an O_{16} or a corank 3 singularity

T	μ	type	T	μ	type
9	13	T_{266}	14	11	$T_{245} + A_1$
10	12	T_{256}	15	10	T_{245}
11	11	T_{255}	16	11	$T_{244}+2A_1$
12	12	$T_{246} + A_1$	17	10	$T_{244} + A_1$
13	11	T_{246}	18	9	$X_9 = T_{244}$

Table A.2: Combinations containing a corank 2, $j_3 = 0$ singularity

T	μ	type	T	μ	type	T	μ	type
19	12	$J_{10}+A_2$	45	7	D_7	71	12	$3D_4$
20	11	$J_{10}+A_1$	46	11	$D_6+A_3+2A_1$	72	11	$2D_4+A_3$
21	10	J_{10}	47	10	$D_6+A_3+A_1$	73	10	$2D_4+A_2$
22	10	E_8+A_2	48	9	D_6+A_3	74	11	$2D_4+3A_1$
23	9	E_8+A_1	49	10	$D_6+A_2+2A_1$	75	10	$2D_4+2A_1$
24	8	E_8	50	9	$D_6+A_2+A_1$	76	9	$2D_4+A_1$
25	10	$E_7+A_2+A_1$	51	8	D_6+A_2	77	8	$2D_4$
26	9	E_7+A_2	52	10	D_6+4A_1	78	10	D_4+2A_3
27	9	E_7+2A_1	53	9	D_6+3A_1	79	9	$D_4+A_3+A_2$
28	8	E_7+A_1	54	8	D_6+2A_1	80	10	$D_4+A_3+3A_1$
29	7	E_7	55	7	D_6+A_1	81	9	$D_4+A_3+2A_1$
30	10	E_6+2A_2	56	6	D_6	82	8	$D_4+A_3+A_1$
31	9	$E_6+A_2+A_1$	57	11	D_5+2A_3	83	7	D_4+A_3
32	8	E_6+A_2	58	10	$D_5+A_3+A_2$	84	8	D_4+2A_2
33	8	E_6+2A_1	59	10	$D_5+A_3+2A_1$	85	9	$D_4+A_2+3A_1$
34	7	E_6+A_1	60	9	$D_5+A_3+A_1$	86	8	$D_4+A_2+2A_1$
35	6	E_6	61	8	D_5+A_3	87	7	$D_4+A_2+A_1$
36	11	D_8+A_3	62	9	D_5+2A_2	88	6	D_4+A_2
37	10	D_8+A_2	63	9	$D_5+A_2+2A_1$	89	10	D_4+6A_1
38	10	D_8+2A_1	64	8	$D_5+A_2+A_1$	90	9	D_4+5A_1
39	9	D_8+A_1	65	7	D_5+A_2	91	8	D_4+4A_1
40	8	D_8	66	9	D_5+4A_1	92	7	D_4+3A_1
41	10	D_7+A_3	67	8	D_5+3A_1	93	6	D_4+2A_1
42	9	D_7+A_2	68	7	D_5+2A_1	94	5	D_4+A_1
43	9	D_7+2A_1	69	6	D_5+A_1	95	4	D_4
44	8	D_7+A_1	70	5	D_5			

Table A.3: Combinations containing a corank 2, $j_3 \neq 0$ singularity

T	μ	type	T	μ	type	T	μ	type
96	11	A_{11}	133	5	A_5	170	6	$2A_3$
97	10	A_{10}	134	10	$2A_4 + A_2$	171	6	$A_3 + A_2 + A_1$
98	10	$A_9 + A_1$	135	10	$A_4 + 2A_3$	172	6	$A_3 + 3A_1$
99	9	A_9	136	9	$2A_4 + A_1$	173	5	$A_3 + A_2$
100	10	$A_8 + A_2$	137	9	$A_4 + A_3 + A_2$	174	5	$A_3 + 2A_1$
101	9	$A_8 + A_1$	138	9	$A_4 + 2A_2 + A_1$	175	4	$A_3 + A_1$
102	8	A_8	139	9	$A_4 + A_3 + 2A_1$	176	3	A_3
103	11	$A_7 + A_4$	140	8	$2A_4$	177	10	$5A_2$
104	10	$A_7 + A_3$	141	8	$A_4 + A_3 + A_1$	178	9	$4A_2 + A_1$
105	10	$A_7 + A_2 + A_1$	142	8	$A_4 + 2A_2$	179	8	$4A_2$
106	9	$A_7 + A_2$	143	8	$A_4 + A_2 + 2A_1$	180	8	$3A_2 + 2A_1$
107	9	$A_7 + 2A_1$	144	8	$A_4 + 4A_1$	181	8	$2A_2 + 4A_1$
108	8	$A_7 + A_1$	145	7	$A_4 + A_3$	182	8	$A_2 + 6A_1$
109	7	A_7	146	7	$A_4 + A_2 + A_1$	183	7	$3A_2 + A_1$
110	10	$A_6 + A_4$	147	7	$A_4 + 3A_1$	184	7	$2A_2 + 3A_1$
111	9	$A_6 + A_3$	148	6	$A_4 + A_2$	185	7	$A_2 + 5A_1$
112	9	$A_6 + A_2 + A_1$	149	6	$A_4 + 2A_1$	186	6	$3A_2$
113	8	$A_6 + A_2$	150	5	$A_4 + A_1$	187	6	$2A_2 + 2A_1$
114	8	$A_6 + 2A_1$	151	4	A_4	188	6	$A_2 + 4A_1$
115	7	$A_6 + A_1$	152	11	$3A_3 + A_1$	189	5	$2A_2 + A_1$
116	6	A_6	153	10	$2A_3 + A_2 + 2A_1$	190	5	$A_2 + 3A_1$
117	11	$2A_5 + A_1$	154	10	$2A_3 + 4A_1$	191	4	$2A_2$
118	10	$2A_5$	155	9	$3A_3$	192	4	$A_2 + 2A_1$
119	10	$A_5 + A_4 + A_1$	156	9	$2A_3 + A_2 + A_1$	193	3	$A_2 + A_1$
120	10	$A_5 + A_3 + 2A_1$	157	9	$A_3 + 2A_2 + 2A_1$	193	3	$A_2 + A_1$
121	10	$A_5 + 2A_2 + A_1$	158	9	$2A_3 + 3A_1$	194	2	A_2
122	9	$A_5 + A_4$	159	9	$A_3 + A_2 + 4A_1$	195	10	$10A_1$
123	9	$A_5 + A_3 + A_1$	160	9	$A_3 + 6A_1$	196	9	$9A_1$
124	9	$A_5 + 2A_2$	161	8	$2A_3 + A_2$	197	8	$8A_1$
125	9	$A_5 + A_2 + 2A_1$	162	8	$A_3 + 2A_2 + A_1$	198	7	$7A_1$
126	9	$A_5 + 4A_1$	163	8	$2A_3 + 2A_1$	199	6	$6A_1$
127	8	$A_5 + A_3$	164	8	$A_3 + A_2 + 3A_1$	200	5	$5A_1$
128	8	$A_5 + A_2 + A_1$	165	8	$A_3 + 5A_1$	201	4	$4A_1$
129	8	$A_5 + 3A_1$	166	7	$2A_3 + A_1$	202	3	$3A_1$
130	7	$A_5 + A_2$	167	7	$A_3 + 2A_2$	203	2	$2A_1$
131	7	$A_5 + 2A_1$	168	7	$A_3 + A_2 + 2A_1$	204	1	A_1
132	6	$A_5 + A_1$	169	7	$A_3 + 4A_1$			

Table A.4: Constellations of A_n singularities