On the Category of Boundary Values in the extended Crane-Yetter TQFT

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Abstract of the Dissertation

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The Crane-Yetter state sum is an invariant of closed 4-manifolds, defined in terms of a triangulation, based on 15-j symbols associated to the category $\mathcal{A}$ of representations over quantum $\mathfrak{sl}_2$ (at a root of unity). In this thesis, we define the state sum in terms of a "PLCW decomposition", which generalizes triangulations, and generalize $\mathcal{A}$ to an arbitrary premodular category. We extend the state sum to 4-manifolds with corners, making it an extended TQFT. We also develop a parallel theory based on skeins, which are essentially $\mathcal{A}$-colored graphs, and we show that the two theories are equivalent.

Focusing on the 2-dimensional part, we prove several properties of skein categories, the most important of which is that they satisfy excision. We provide explicit algebraic descriptions of the category associated to the once-punctured torus and the annulus, giving rise to a new tensor product on the Drinfeld center of a premodular category.

As it is well-known that, when $\mathcal{A}$ is modular, the Crane-Yetter state sum computes the signature of a closed 4-manifold, we connect the Crane-Yetter theory to the signature of a 4-manifold with boundary and even corners.

Finally, we show that the Reshetikhin-Turaev TQFT is the boundary theory of the Crane-Yetter theory (up to a normalization).

To mom

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1. Introduction

The study of quantum invariants of knots and low-dimensional manifolds is a rich and active field of research. Central to the field is the Topological Quantum Field Theory (TQFT), which were introduced in [Ati1988] and [Wit1988]:

Definition 1.1. An n-dimensional TQFT τ is the following collection of data:

- to each $(n-1)$-dimensional manifold M, an assignment of a vector space $\tau(M)$, known as the "state space",
- to each n-dimensional manifold W, an assignment of a vector $\tau(W) \in \tau(\partial W)$, known as the "partition function",
- to a homeomorphism $\varphi : M \to M'$, a natural isomorphism $\tau(\varphi) : \tau(M) \simeq \tau(M')$,
- functorial isomorphisms $\tau(\overline{M}) \simeq \tau(M)^*$, $\tau(\varnothing) \simeq \mathbf{k}$, $\tau(M \sqcup M') \simeq \tau(M) \otimes \tau(M')$.

These data are required to satisfy a list of axioms (see e.g. [BakK2001, Chapter 4]).

In 1993, Crane and Yetter [CY1993] defined a 4d-TQFT based on "15-j"-symbols arising out of a category of representations of a quantum group, inspired by Ooguri [Oog1992], who proposed a formal expression for an invariant of 4-manifolds with an eye towards a theory of quantum gravity. It is constructed as a "state sum" which assigns and combines local invariants to each 4-simplex in a triangulation, similar to the Turaev-Viro state sum which defines a 3d-TQFT [TV1992]. The Crane-Yetter TQFT is known to compute the signature and Euler characteristic of a 4-manifold [CKY1993], [Rob1995].

The Turaev-Viro TQFT is known to be an extended 3-2-1-TQFT [BK], that is, the TQFT also makes an assignment of a category to each 1-manifold. The Crane-Yetter TQFT is widely expected to be an extended TQFT.

Another important object of study is the theory of skeins. These first arose in Kauffman's work [Kau1987] on the Jones polynomial [Jon1985]. The Kauffman skein relation imposes a relation on the space of link diagrams that also reflect the properties of a category of representations of a quantum group.

In [Kir], Kirillov shows that the state space of the Turaev-Viro TQFT can be defined as the space of skeins on the surface. It is widely expected that the state spaces of the Crane-Yetter TQFT can also be defined as the space of skeins in the 3-manifold.

1.1. **Main results.**

The main results of this paper are as follows:

- extend the Crane-Yetter TQFT to an extended TQFT based on a "PLCW decomposition" of the manifold (a generalization of triangulations),
- construct a parallel theory based on skeins, and prove equivalence with the previous constructions,
- develop the properties of skein categories associated to surfaces, in particular the annulus,
- discuss the signature formula
- show that Reshetikhin-Turaev TQFT is the boundary theory of the extended Crane-Yetter TQFT

2. Background

2.1. Categories.

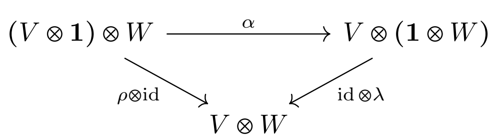
In this section, we review some definitions in category theory, leading up to the definition of modular categories.

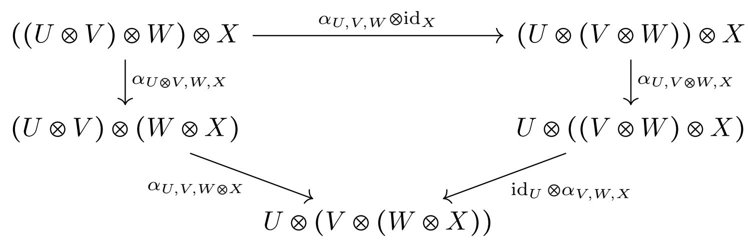
Most of the categories that we deal with will be abelian over some algebraically closed field $\mathbf{k}$ and semisimple; then the structure maps for the various definitions and constructions in this section will implicitly be assumed to be appropriately multilinear (see Remark 2.2).

Definition 2.1. Let $\mathcal{C}$ be a category. A *monoidal structure* on $\mathcal{C}$ is a collection of data as follows:

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *tensor product*;
- a natural isomorphism $\alpha : (-\otimes-)\otimes- \simeq -\otimes(-\otimes-)$, or more verbosely, for any three objects $U, V, W \in \mathcal{C}$, a natural isomorphism
$$\alpha_{U,V,W} : (U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$$
α is called the *associativity constraint*;
- a *unit object* $\mathbf{1} \in \mathcal{C}$ and *unit constraint* λ, ρ which are natural isomorphisms
$$\lambda_V : \mathbf{1} \otimes V \simeq V$$
$$\rho_V : V \otimes \mathbf{1} \simeq V$$

which are subject to the *triangle* and *pentagon* axioms:





We say that $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is a monoidal category.

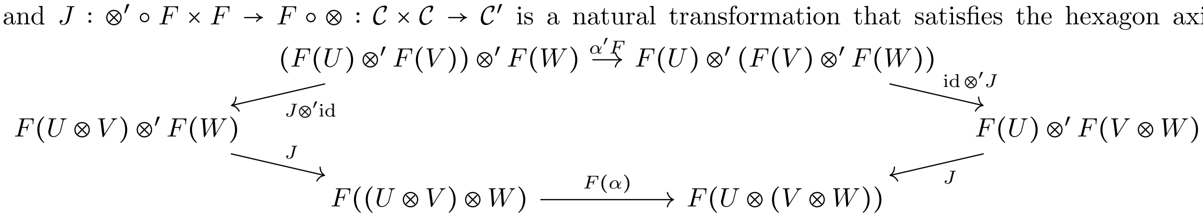
Sometimes we simply refer to $(\mathcal{C}, \otimes)$, or even just $\mathcal{C}$, as a monoidal category when the other structures are understood. We often hide the associativity and unit constraints for readability; this is justified by Theorem 2.6.

Remark 2.2. When $\mathcal{C}$ is abelian, the tensor product of morphisms should be a bilinear map: for $V, V', W, W' \in \mathcal{C}$, the map $\otimes : \mathrm{Hom}(V, V') \times \mathrm{Hom}(W, W') \to \mathrm{Hom}(V \otimes V', W \otimes W')$ is bilinear.

Example 2.3. Here are some common examples of monoidal categories:

(1) The category of vector spaces $\mathcal{V}ec_{\mathbf{k}}$ over some field $\mathbf{k}$, with $\otimes$ being the usual tensor product of vector spaces.
(2) The category $\mathrm{Rep}_{\mathbf{k}}(G)$ of representations of a finite group G over some field $\mathbf{k}$, with $\otimes$ being the usual tensor product of representations.

Definition 2.4. Let $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ and $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$ be monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor such that $F(\mathbf{1})$ is isomorphic to $\mathbf{1}'$, and $J : \otimes' \circ F \times F \to F \circ \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}'$ is a natural transformation that satisfies the hexagon axiom:



We call J a *monoidal structure* on F. If F is an equivalence of categories, we say that (F, J) is an *equivalence of monoidal categories.*

Let (F', J') be another monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$. A natural transformation of monoidal functors from (F, J) to (F', J') is a natural transformation $F \to F'$ that respects the monoidal structures,

Definition 2.5. A monoidal category $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ is *strict* if $(U \otimes V) \otimes W = U \otimes (V \otimes W)$ and $\mathbf{1} \otimes V = V = V \otimes \mathbf{1}$, and furthermore α, λ, ρ are the identity maps.

Theorem 2.6 (Maclane strictness)**.** *Any monoidal category is monoidally equivalent to a strict monoidal category.*

Definition 2.7. Let $\mathcal{C}$ be a monoidal category, and $V \in \mathcal{C}$ an object. A *left dual* to V is an object V^* together with morphisms $\mathrm{ev}_V : V^* \otimes V \to \mathbf{1}$ and $\mathrm{coev}_V : \mathbf{1} \to V \otimes V^*$, such that the compositions

$$V \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} V \otimes V^* \otimes V \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} V$$

$$V^* \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} V^* \otimes V \otimes V^* \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} V^*$$

are the identity morphisms. Note that we have suppressed the unit and associativity constraints.

Similarly, a *right dual*[1]to V is an object ${}^*V$ together with morphisms $\tilde{\mathrm{ev}}_V : V \otimes {}^*V \to \mathbf{1}$ and $\tilde{\mathrm{coev}}_V : \mathbf{1} \to {}^*V \otimes V$, such that the compositions

$$V \xrightarrow{\tilde{\mathrm{coev}} \otimes \mathrm{id}} V \otimes {}^*V \otimes V \xrightarrow{\mathrm{id} \otimes \tilde{\mathrm{ev}}} V$$

$${}^*V \xrightarrow{\mathrm{id} \otimes \tilde{\mathrm{coev}}} {}^*V \otimes V \otimes {}^*V \xrightarrow{\tilde{\mathrm{ev}} \otimes \mathrm{id}} {}^*V$$

are the identity morphisms.

If V, W have left duals, and $f \in \mathrm{Hom}(V, W)$, the *left dual* of f is the morphism $f^* \in \mathrm{Hom}(W^*, V^*)$ given by the composition

$$f^* : W^* \xrightarrow{\mathrm{coev}_V} W^* \otimes V \otimes V^* \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} W^* \otimes W \otimes V^* \xrightarrow{\mathrm{ev}_W} V^*$$

Similarly, if V, W have right duals, the right dual to $f \in \mathrm{Hom}(V, W)$ is the morphism ${}^*f \in \mathrm{Hom}({}^*W, {}^*V)$ given by the composition

$${}^*f : {}^*W \xrightarrow{\tilde{\mathrm{coev}}_V} {}^*V \otimes V \otimes {}^*W \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} {}^*V \otimes W \otimes {}^*W \xrightarrow{\tilde{\mathrm{ev}}_W} {}^*V$$

Proposition 2.8. *If $V \in \mathcal{C}$ has a left (respectively right) dual object, then it is unique up to unique isomorphism.*

Proof. See [EGNO2015, Prop 2.10.5]. □

Observe that V^* is a right dual to V, and ${}^*V$ is a left dual to V. Thus by the proposition above, we can naturally identify ${}^*(V^*) \simeq V \simeq ({}^*V)^*$.

Proposition 2.9. *Let $\mathcal{C}$ be a monoidal category and V an object in $\mathcal{C}$. If V has a left dual, then there are natural adjunction isomorphisms*

$$\mathrm{Hom}(U \otimes V, W) \simeq \mathrm{Hom}(U, W \otimes V^*)$$

$$\mathrm{Hom}({}^*V \otimes U, W) \simeq \mathrm{Hom}(U, V \otimes W)$$

Similarly, if V has a right dual, then there are natural adjunction isomorphisms

$$\mathrm{Hom}(U \otimes {}^*V, W) \simeq \mathrm{Hom}(U, W \otimes V)$$

$$\mathrm{Hom}({}^*V \otimes U, W) \simeq \mathrm{Hom}(U, {}^*V \otimes W)$$

Proof. The first natural isomorphism is given by $f \mapsto (f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_U \otimes \mathrm{coev}_V)$, with inverse $g \mapsto (\mathrm{id}_W \otimes \mathrm{ev}_V) \circ (g \otimes \mathrm{id}_V)$; other isomorphisms are similar. See [EGNO2015, Prop 2.10.8]. Abstractly, this proposition says that when the left dual exists, the multiplication functor $V^* \otimes -$ (respectively $- \otimes V^*$) are left (respectively right) adjoints to the multiplication functor $V \otimes -$ (respectively $- \otimes V$). (Similarly for right duals). □

[1]It is always confusing which side of V to put the asterisk on. I remember by thinking, left dual object evaluates from the left, and the asterisk should point towards the original object, thus $V^* \otimes V \to \mathbf{1}$ is the evaluation map for the left dual.

Definition 2.10. We say that a monoidal category is *rigid* if every object has a left and right dual.

In general, the left and right duals may not be isomorphic, let alone naturally isomorphic.

Definition 2.11. Let $\mathcal{C}$ be a rigid monoidal category. A *pivotal structure* on $\mathcal{C}$ is a natural isomorphism of monoidal functors $\delta : \mathrm{id} \simeq (-)^{**}$.

By the observation after Proposition 2.8, this is equivalent to having a natural isomorphism ${}^*V \simeq V^*$. Note that by virtue of being an isomorphism of *monoidal* functors, one has $\delta_{V\otimes Y} = \delta_V \otimes \delta_Y$.

Definition 2.12. Let V be an object in a pivotal category $\mathcal{C}$, and let $f \in \mathrm{End}_{\mathcal{C}}(V)$. The *left trace* of f is the composition

$$\mathrm{tr}^L(f) : \mathbf{1} \xrightarrow{\mathrm{coev}_V} V \otimes V^* \xrightarrow{\delta_V \otimes f^*} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}_{V^*}} \mathbf{1}$$

and similarly, the *right trace* is the composition

$$\mathrm{tr}^R(f) : \mathbf{1} \xrightarrow{\tilde{\mathrm{coev}}_V} {}^*V \otimes V \xrightarrow{\delta_{{}^*V} \otimes f} V^* \otimes V \xrightarrow{\mathrm{ev}_V} \mathbf{1}$$

In particular, we define the *left (respectively right) dimension*, denoted $\dim^L(V)$ (respectively $\dim^R(V)$ of an object V to be the left (respectively right) trace of id_V.

Definition 2.13. We say that a pivotal category $\mathcal{C}$ is *spherical* if $\dim^L(V) = \dim^R(V) \in \mathrm{End}(\mathbf{1})$ for every object $V \in \mathcal{C}$.

It is easy to check that $\dim^R(V^*) = \dim^L(V) = \dim^L(V^{**})$, so an equivalent definition of sphericality is $\dim^L(V) = \dim^L(V^*)$ for all $V \in \mathcal{C}$.

Definition 2.14. Let $\mathcal{C}$ be a monoidal category. A *braiding* on $\mathcal{C}$ is a natural isomorphism $c_{X,Y} : X \otimes Y \simeq Y \otimes X$ such that

$$c_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z}) \otimes (c_{X,Y} \otimes \mathrm{id}_Z) c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{id}_Y) \otimes (\mathrm{id}_X \otimes c_{Y,Z})$$

Definition 2.15. Let $\mathcal{C}$ be a braided monoidal category with braiding c. A *twist* (or *balancing transformation*) is a natural isomorphism $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$ such that

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$$

Definition 2.16. Let $\mathcal{C}$ be a monoidal category with a spherical and braided structure. A *ribbon structure* on $\mathcal{C}$ is a twist θ such that $(\theta_X)^* = \theta_{X^*}$.

Definition 2.17. Let $\mathcal{C}$ be a $\mathbf{k}$-linear abelian rigid monoidal category. We say that $\mathcal{C}$ is *multifusion* if $\mathcal{C}$ is finite semisimple and $\otimes$ is bilinear on morphisms. We say that $\mathcal{C}$ is *fusion* if $\mathrm{End}(\mathbf{1}) \cong \mathbf{k}$.

We denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of a multifusion category $\mathcal{C}$, and pick representatives X_i for each class $i \in \mathrm{Irr}(\mathcal{C})$. and denote by $\mathrm{Irr}_0(\mathcal{C}) \subseteq \mathrm{Irr}(\mathcal{C})$ the subset of simple objects appearing in the direct sum decomposition of the unit object $\mathbf{1}$; it is known (see [EGNO2015, Corollary 4.3.2]) that $\mathbf{1}$ decomposes into a direct sum of pairwise-distinct simples. We also assume that the dual functor sends our choice of representatives to themselves, resulting in an involution on $\mathrm{Irr}(\mathcal{C})$, which we denote i^* for $i \in \mathrm{Irr}(\mathcal{C})$. We will delve into more detail in Section 2.2.

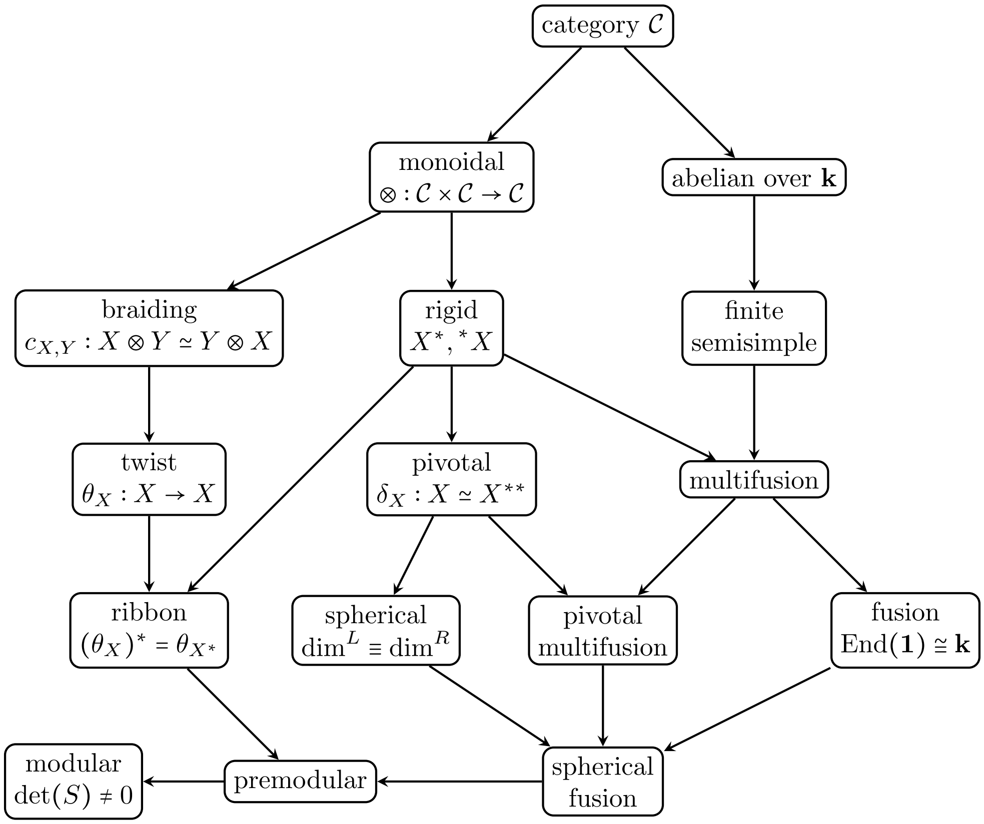
Definition 2.18. A *premodular* category is a ribbon fusion category.

Definition 2.19. The *S-matrix* of a premodular category $\mathcal{C}$ is the $\mathrm{Irr}(\mathcal{C}) \times \mathrm{Irr}(\mathcal{C})$ matrix

$$S := (s_{ij})_{i,j\in\mathrm{Irr}(\mathcal{C})} \quad \text{where} \quad s_{ij} = \mathrm{tr}(c_{X_j,X_i} \circ c_{X_i,X_j})$$

Definition 2.20. A premodular category $\mathcal{C}$ is *modular* if its S-matrix is non-degenerate.

We summarize the slew of definitions given above in the flow chart below:



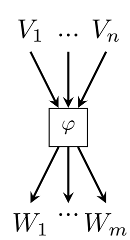
2.2. **Graphical Calculus and Conventions: Pivotal Multifusion Categories.**

Most of the categories that we encounter in this thesis will be pivotal multifusion categories. We therefore dedicate this section to notation and basic results about the "internal" structures of pivotal multifusion categories, i.e. objects and morphisms; the next section will be concerned with the "external structures", i.e. how pivotal multifusion categories relate to other categories via functors, module structures etc. It is mostly taken from [KT] which in turn was adapted from [Kir], modified to accommodate for the non-spherical non-fusion case. We also point the reader to [EGNO2015, Chapter 4] and [BakK2001] for further reference.

We express morphisms in the language of the graphical calculus [Pen1971], a convenient language to express morphisms and equations about them. As there are many conventions surrounding the use of the graphical calculus (see for example Remark 9.2), we explicitly state our conventions as follows:

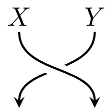
Convention 2.21. All diagrams are implicitly blackboard-framed.

Morphism run top to bottom:

(2.1) $\varphi \in \mathrm{Hom}(V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots \otimes W_m) \longleftrightarrow$ 

Convention 2.22. We draw diagrams in an ambient space that is oriented by the standard basis $((1,0,0),(0,1,0),(0,0,1))$ following "Fleming's right-hand rule" ($(1,0,0)$ points to the right, $(0,1,0)$ points into the page, $(0,0,1)$ points up).

Convention 2.23. The braiding c of a category is drawn as

(2.2) $c_{X,Y} =$ 

If a diagram is said to be drawn in $\mathbb{R}\times[0,1]\times[-\varepsilon,\varepsilon]$ with the opposite orientation, then applying the flipping map on the last coordinate (to change the orientation back to the standard one) amounts to changing all such crossings to its opposite.

Let $\mathcal{C}$ be a $\mathbf{k}$-linear pivotal multifusion category, where $\mathbf{k}$ is an algebraically closed field of characteristic 0. In all our formulas and computations, we will be suppressing the associativity and unit morphisms; we also suppress the pivotal morphism $V \simeq V^{**}$ when there is little cause for confusion.

We denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects in $\mathcal{C}$, and denote by $\mathrm{Irr}_0(\mathcal{C}) \subseteq \mathrm{Irr}(\mathcal{C})$ the subset of simple objects appearing in the direct sum decomposition of the unit object $\mathbf{1}$; it is known [EGNO2015, Corollary 4.3.2] that $\mathbf{1}$ decomposes into a direct sum of distinct simples, so $\mathrm{End}(\mathbf{1}) \cong \bigoplus_{l\in\mathrm{Irr}_0(\mathcal{C})}\mathrm{End}(\mathbf{1}_l)$. We fix a representative X_i for each isomorphism class $i \in \mathrm{Irr}(\mathcal{C})$; abusing language, we will frequently use the same letter i for denoting both a simple object and its isomorphism class. Rigidity gives us an involution $-^*$ on $\mathrm{Irr}(\mathcal{C})$; it is known that $l^* = l$ for $l \in \mathrm{Irr}_0(\mathcal{C})$. For $l \in \mathrm{Irr}_0(\mathcal{C})$, we may use the notation $\mathbf{1}_l := X_l$ to emphasize that it is part of the unit.

For $k,l \in \mathrm{Irr}_0(\mathcal{C})$, let $\mathcal{C}_{kl} := \mathbf{1}_k \otimes \mathcal{C} \otimes \mathbf{1}_l$, so that $\mathcal{C} = \bigoplus_{k,l\in\mathrm{Irr}_0(\mathcal{C})}\mathcal{C}_{kl}$. Any simple X_i is contained in exactly one of these $\mathcal{C}_{kl}$'s, or in other words, there are unique $k_i, l_i \in \mathrm{Irr}_0(\mathcal{C})$ such that $\mathbf{1}_{k_i} \otimes X_i \otimes \mathbf{1}_{l_i} \neq 0$. Since $\mathcal{C}_{kl}^* = \mathcal{C}_{lk}$, we have that $k_{i^*} = l_i$.

When $\mathcal{C}$ is spherical fusion, the categorical dimension is a scalar, defined as a trace, but here the non-simplicity of $\mathbf{1}$ and non-sphericality complicates things. To avoid confusion, denote by $\delta : V \to V^{**}$ the pivotal morphism. The *left dimension* of an object $V \in \mathrm{Obj}\,\mathcal{C}$ is the morphism

$$d_V^L := (\mathbf{1} \xrightarrow{\mathrm{coev}} V \otimes V^* \xrightarrow{\delta\otimes\mathrm{id}} V^{**}\otimes V^* \xrightarrow{\mathrm{ev}} \mathbf{1}) \in \mathrm{End}(\mathbf{1})$$

Similarly, the *right dimension* of V is the morphism

$$d_V^R := (\mathbf{1} \xrightarrow{\mathrm{coev}} {}^*V \otimes V \xrightarrow{\mathrm{id}\otimes\delta^{-1}} {}^*V\otimes {}^{**}V \xrightarrow{\mathrm{ev}} \mathbf{1}) \in \mathrm{End}(\mathbf{1})$$

Note that these are *vectors* and not scalars, since $\mathbf{1}$ may not be simple. It is easy to see that $d_V^R = d_{V^*}^L = d_{{}^*V}^L$. When $\mathcal{C}$ is spherical, we will drop the superscripts.

When $V = X_i$ is simple, we can interpret its left and right dimensions as scalars as follows. We have $X_i \in \mathcal{C}_{k_i l_i}$, so $\mathrm{Hom}(\mathbf{1}, X_i\otimes X_i^*) = \mathrm{Hom}(\mathbf{1}, \mathbf{1}_{k_i}\otimes X_i\otimes X_i^*) \simeq \mathrm{Hom}(\mathbf{1}_{k_i}, X_i\otimes X_i^*)$, and likewise $\mathrm{Hom}(X_i\otimes X_i^*, \mathbf{1}) \simeq \mathrm{Hom}(X_i\otimes X_i^*, \mathbf{1}_{k_i})$, so $d_{X_i}^L$ factors through $\mathbf{1}_{k_i}$, and hence we may interpret $d_{X_i}^L$ as an element of $\mathrm{End}(\mathbf{1}_{k_i}) \cong \mathbf{k}$. Similarly, $d_{X_i}^R$ may be interpreted as an element of $\mathrm{End}(\mathbf{1}_{l_i}) \cong \mathbf{k}$. We denote these scalar dimensions by d_i^L, d_i^R, and fix square roots such that $\sqrt{d_i^L} = \sqrt{d_{i^*}^R}$. The dimensions of simple objects are nonzero.

The *dimension* of $\mathcal{C}_{kl}$ is the sum

$$\mathcal{D} := \sum_{i\in\mathrm{Irr}(\mathcal{C}_{kl})} d_i^R d_i^L \tag{2.3}$$

or simply

$$\mathcal{D} = \sum d_i^2$$

if $\mathcal{C}$ is tensor. By [ENO2005, Proposition 2.17], this is the same for all pairs $k,l \in \mathrm{Irr}_0(\mathcal{C})$, and by [ENO2005, Theorem 2.3], they are nonzero.

We will fix fourth roots of d_i^R and $\mathcal{D}$ such that $(d_i^L)^{1/4} = (d_{i^*}^R)^{1/4}$.

We define functors $\mathcal{C}^{\boxtimes n} \to \mathcal{V}ec$ by

$$\langle V_1,\dots,V_n\rangle = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, V_1\otimes\cdots\otimes V_n) \tag{2.4}$$

$$\begin{aligned}\langle V_1,\dots,V_n\rangle_l &= \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}_l, V_1\otimes\cdots\otimes V_n) \text{ for } l\in\mathrm{Irr}_0(\mathcal{C}) \\ &\simeq \langle \mathbf{1}_l, V_1,\dots,V_n\rangle\end{aligned} \tag{2.5}$$

for any collection $V_1,\dots,V_n$ of objects of $\mathcal{C}$. Clearly $\langle V_1,\dots,V_n\rangle = \bigoplus_l \langle V_1,\dots,V_n\rangle_l$.

Note that the pivotal structure gives functorial isomorphisms

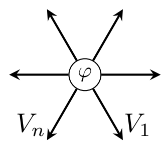
$$z\colon \langle V_1,\dots,V_n\rangle \simeq \langle V_n, V_1,\dots,V_{n-1}\rangle \tag{2.6}$$

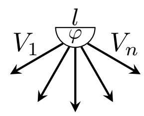
such that $z^n = \mathrm{id}$ (see [BakK2001, Section 5.3]); thus, up to a canonical isomorphism, the space $\langle V_1,\dots,V_n\rangle$ only depends on the cyclic order of $V_1,\dots,V_n$. In general, z does not preserve the direct sum decomposition of $\langle V_1,\dots,V_n\rangle$ above. For example, for a simple $X_i \in \mathcal{C}_{k_i l_i}$, we have $z : \langle X_i, X_i^*\rangle_{k_i} \simeq \langle X_i^*, X_i\rangle_{l_i}$.

We will commonly use graphic presentation of morphisms in a category, representing a morphism $W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \cdots \otimes V_n$ by a diagram with m strands at the top, labeled by $W_1,\dots,W_m$ and n strands at the bottom, labeled $V_1,\dots,V_n$ (Note: this differs from the convention in many other papers!). We will allow diagrams with with oriented strands, using the convention that a strand labeled by V is the same as the strands labeled by V^* with opposite orientation (suppressing isomorphisms $V \simeq V^{**}$).

We will show a morphism $\varphi \in \langle V_1,\dots,V_n\rangle$ by a round circle labeled by φ with outgoing edges labeled $V_1,\dots,V_n$ in counter-clockwise order, as shown in (2.7) By (2.6) and the fact that $z^n = \mathrm{id}$, this is unambiguous. We will show a morphism $\varphi \in \langle V_1,\dots,V_n\rangle_l$ by a semicircle labeled by φ and l as shown in (2.7); in contrast with a circular node, a semicircle imposes a strict ordering on the outgoing legs, not just a cyclic ordering.

(2.7)





We have a natural composition map

$$\begin{aligned}\langle V_1,\dots,V_n,X\rangle \otimes \langle X^*, W_1,\dots,W_m\rangle &\to \langle V_1,\dots,V_n,W_1,\dots,W_m\rangle\\ \varphi\otimes\psi &\mapsto \varphi \underset{X}{\circ} \psi = \mathrm{ev}_{X^*}\circ(\varphi\otimes\psi)\end{aligned} \tag{2.8}$$

where $\mathrm{ev}_{X^*}\colon X\otimes X^* \to \mathbf{1}$ is the evaluation morphism (the pivotal structure is suppressed).

Repeated applications of the composition map above gives us a non-degenerate pairing

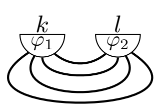
$$\mathrm{ev} : \langle V_1,\dots,V_n\rangle \otimes \langle V_n^*,\dots,V_1^*\rangle \to \mathrm{End}(\mathbf{1}) \tag{2.9}$$

More precisely, when restricted to the subspaces,

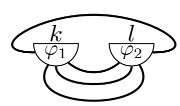
$$\langle V_1,\dots,V_n\rangle_k \otimes \langle V_n^*,\dots,V_1^*\rangle_l \to \mathrm{End}(\mathbf{1}) \tag{2.10}$$

the pairing is 0 if $k \neq l$, and is non-degenerate, factoring through $\mathrm{End}(\mathbf{1}_k) \simeq \mathbf{k}$, if $k = l$. The pairing is illustrated below for $\varphi_1 \in \langle V_1,\dots,V_n\rangle_k, \varphi_2 \in \langle V_n^*,\dots,V_1^*\rangle_l$:

$(\varphi_1,\varphi_2) =$



if $\mathcal{C}$ not spherical $\not\equiv$



$= (z^{-1}\cdot\varphi_1, z\cdot\varphi_2)$

Thus, we have functorial isomorphisms

$$\langle V_1,\dots,V_n\rangle^* \simeq \langle V_n^*,\dots,V_1^*\rangle \tag{2.11}$$

When $\mathcal{C}$ is spherical, this pairing is compatible with the cyclic permutations (2.6), in the sense that $(\varphi_1,\varphi_2) = (z\cdot\varphi_1, z^{-1}\cdot\varphi_2)$. Compatibility fails when $\mathcal{C}$ is not spherical; for example, it is easy to see that for $\varphi_1 = \varphi_2 = \mathrm{coev}_{X_i} \in \langle X_i, X_i^*\rangle$, one has $(\varphi_1,\varphi_2) = d_i^L$, while for $z\cdot\varphi_1 = z^{-1}\cdot\varphi_2 = \mathrm{coev}_{X_i^*} \in \langle X_i^*, X_i\rangle$, one has instead $(z\cdot\varphi_1, z^{-1}\varphi_2) = d_i^R$.

Lemma 2.24. *For $\varphi \in \langle V_1,\dots,V_n\rangle_l, \varphi' \in \langle V_n^*,\dots,V_1^*\rangle_l, \psi \in \langle W_n^*,\dots,W_1^*\rangle_l$, and $f \in \mathrm{Hom}(V_1\otimes\cdots\otimes V_n, W_1\otimes\cdots\otimes W_n)$, we have*

$$(\varphi,\varphi') = (\varphi',\varphi) \tag{2.12}$$

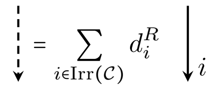
$$(f\circ\varphi_1,\varphi_2) = (\varphi_1, f^*\circ\varphi_2) \tag{2.13}$$

Proof. Straightforward from definitions. □

We will make two additional conventions related to the graphic presentation of morphisms.

Notation 2.25. A dashed line in the picture stands for the sum of all colorings of an edge by simple objects i, each taken with coefficient d_i^R:

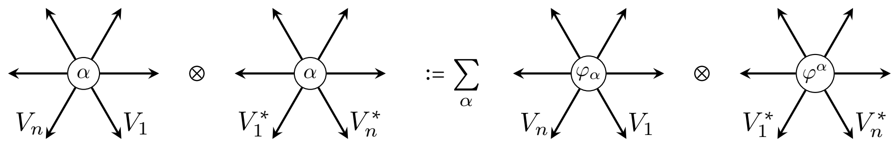
(2.14)



When $\mathcal{C}$ is spherical, the orientation of such a dashed line is irrelevant.

Notation 2.26. Let $\mathcal{C}$ be spherical. If a figure contains a pair of circles, one with outgoing edges labeled $V_1, \dots, V_n$ and the other with edges labeled $V_n^*, \dots, V_1^*$, and the vertices are labeled by the same letter α (or β, or …) it will stand for summation over the dual bases:

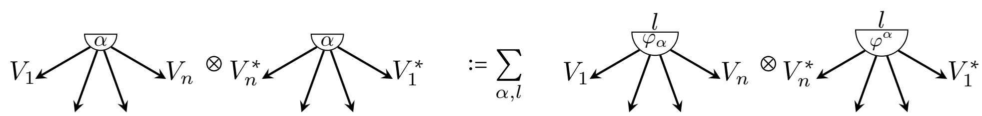
(2.15)



where $\varphi_\alpha \in \langle V_1, \dots, V_n \rangle$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^* \rangle$ are dual bases with respect to pairing (2.9). Note that the morphisms can be located far apart.

When $\mathcal{C}$ is not spherical, the pairing is no longer compatible with z from (2.6), so such notation can only make sense with semicircles:

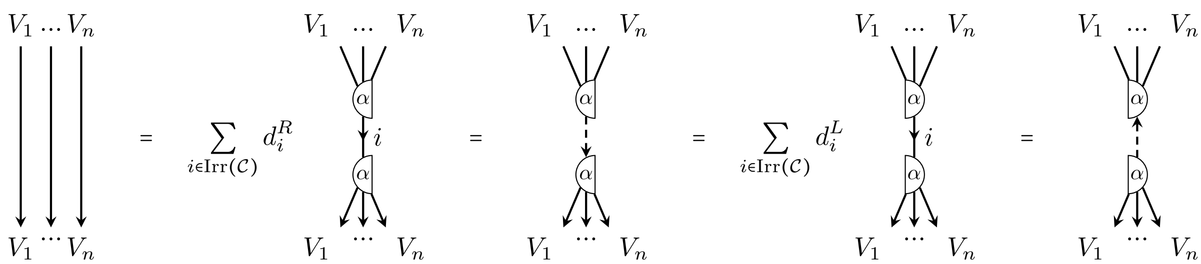
(2.16)



where $\varphi_\alpha \in \langle V_1, \dots, V_n \rangle_l$, $\varphi^\alpha \in \langle V_n^*, \dots, V_1^* \rangle_l$ are dual bases with respect to the pairing (2.9).

The following lemma illustrates the use of the notation above.

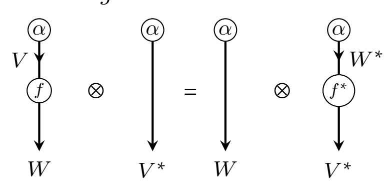
Lemma 2.27. *For any $V_1, \dots, V_n \in \mathcal{C}$, we have*



Proof of this lemma is straightforward: first show it for $V_1 \otimes \cdots \otimes V_n =$ simple, then follows for direct sums; interested reader can find a proof for spherical $\mathcal{C}$ in [Kir].

Lemma 2.28. *Morphisms can "tunnel through α":*

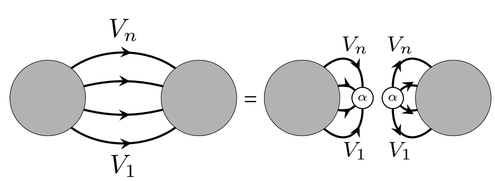
(2.17)



Proof. By identify V, W with a direct sum of simples, f becomes a collection of matrices (one for each simple), then the lemma follows easily from linear algebra. (See [Kir, Lemma 3.6] for details.) See Lemma 2.33, (5.16) for typical use cases. □

Lemma 2.29.

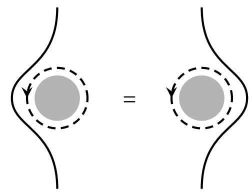
(2.18)



Proof. Apply Lemma 2.27, then notice that the "connecting edge" must be $i = \mathbf{1}$. (See [BK, Lemma 2.2.3]). ☐

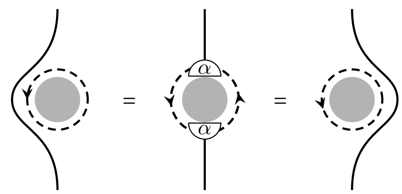
Lemma 2.30. *The following is a generalization of the "sliding lemma":*

(2.19)



These relations hold regardless of the contents of the shaded region.

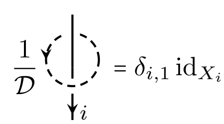
Proof.



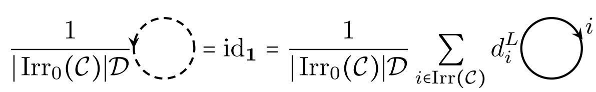
where we use Lemma 2.27 in the equalities. See also [Kir, Corollary 3.5]. Note this trick doesn't work when the circle is oriented the other way (unless of course if $\mathcal{C}$ is spherical). ☐

Lemma 2.31 (Killing Lemma, Charge Conservation)**.** *When $\mathcal{C}$ is modular,*

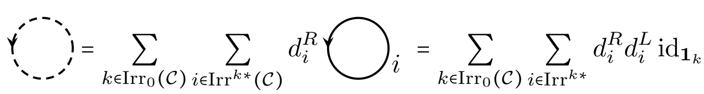
(2.20)



Lemma 2.32.



Proof. Let $\mathrm{Irr}^{kl} = \mathrm{Irr}(\mathcal{C}_{kl})$, and let $\mathrm{Irr}^{k*} := \bigcup_l \mathrm{Irr}(\mathcal{C}_{kl})$, i.e. the set of simples X_i such that $\mathbf{1}_k \otimes X_i = X_i$. Then



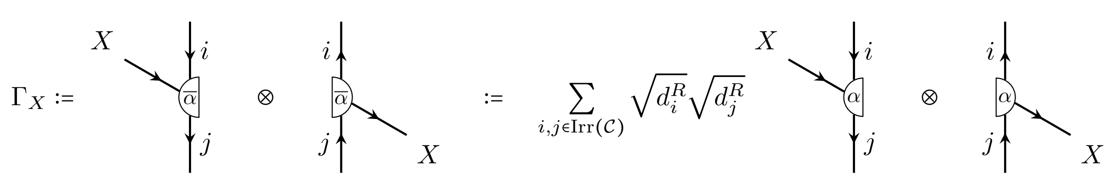
$$= \sum_{k \in \mathrm{Irr}_0(\mathcal{C})} \sum_{l \in \mathrm{Irr}_0(\mathcal{C})} \sum_{i \in \mathrm{Irr}^{kl}} d_i^R d_i^L \,\mathrm{id}_{\mathbf{1}_k} = \sum_{k \in \mathrm{Irr}_0(\mathcal{C})} \sum_{l \in \mathrm{Irr}_0(\mathcal{C})} \mathcal{D}\,\mathrm{id}_{\mathbf{1}_k} = |\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}\,\mathrm{id}_{\mathbf{1}}$$

The second equality is proved in a similar manner. ☐

The following lemma is used to prove that (2.27) is a half-braiding and the functor G in the proof of Theorem 2.46 respects composition:

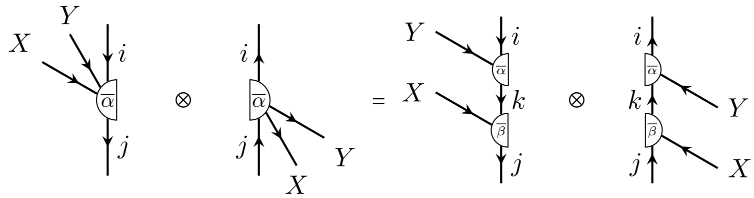
Lemma 2.33. *For $X \in \mathrm{Obj}\,\mathcal{C}$, define the following element Γ_X of $\mathrm{Hom}(X \otimes X_i, X_j) \otimes \mathrm{Hom}(X_i^*, X_j^* \otimes X)$:*

(2.21)

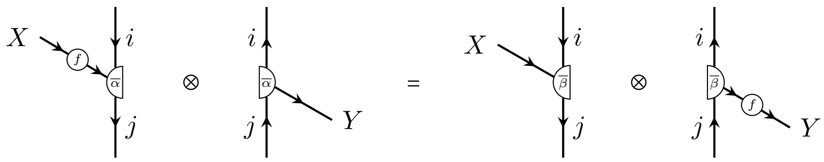


Note that the strands labeled i, j, for which we sum over simples, must be the two strands at the extremes of the semicircle; when $\mathcal{C}$ is spherical, this is not canonical, but usually it is clear which two strands are chosen. Γ_X satisfies the following properties:

(1) Γ_- *respects tensor products:*

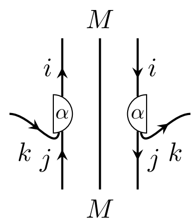
(2.22) 

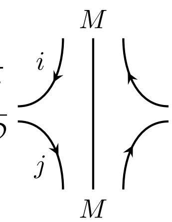
(2) Γ_X *is natural in* X*: for* $f : X \to Y$*,*

(2.23) 

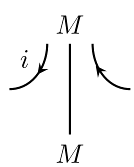
Proof. The second property follows from Lemma 2.28. The first property follows from using Lemma 2.28 to "pull" $\overline{\beta}$ through $\overline{\alpha}$, then use Lemma 2.27 to contract the k strand (note the $\sqrt{d_k^R}$ coefficients from $\overline{\alpha}$ and $\overline{\beta}$ combine to give d_k^R, allowing us to use Lemma 2.27). ☐

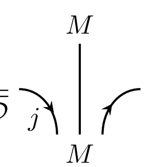
Lemma 2.34. *Let* $M \in \mathcal{M}$*. The morphism*

(2.24) $$P'_M := \sum_{i,j,k\in\mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L}d_k^R}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

$$= \sum_{i,j\in\mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L}}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

is a projection in $\mathrm{End}_{\mathrm{hTr}(\mathcal{M})}(\bigoplus X_i \triangleright M \triangleleft X_i^*)$*. Furthermore, it can be written as a composition* $P'_{(M,\gamma)} = \hat{P}'_{(M,\gamma)} \circ \check{P}'_{(M,\gamma)}$*, where*

(2.25) $$\check{P}'_{(M,\gamma)} := \sum_{i\in\mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$ 

$$, \quad \hat{P}'_{(M,\gamma)} := \sum_{j\in\mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_j^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$ 

such that $\check{P}'_{(M,\gamma)} \circ \hat{P}'_{(M,\gamma)} = \mathrm{id}_M$*, thus as objects in* $\mathrm{Kar}(\mathrm{hTr}(\mathcal{M}))$*, we have* $M \simeq (\bigoplus X_i \triangleright M \triangleleft X_i^*, P'_{(M,\gamma)})$*.*

Proof. Essentially the same as Lemma 2.39. (Note the use of left dimensions d_i^L instead of right dimensions d_i^R.) ☐

2.3. Module categories, balanced tensor product, and center.

In this section, we review the results about balanced tensor product of module categories. Our main goal is to give two constructions of the center of an $\mathcal{C}$-bimodule category $\mathcal{M}$ - $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ (Definition 2.35) and $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ (Definition 7.14), and show that when $\mathcal{C}$ is pivotal multifusion, they are equivalent (Theorem 2.46).

Recall our convention that all categories considered in this paper are locally finite $\mathbf{k}$-linear. Most of the time, they will be abelian; however, in some cases we will need to use $\mathbf{k}$-linear additive (but not necessarily abelian) categories. For such a category $\mathcal{A}$, we will denote by $\mathrm{Kar}(\mathcal{A})$ the Karoubi envelope (also known as idempotent completion) of $\mathcal{A}$. By definition, an object of $\mathrm{Kar}(\mathcal{A})$ is a pair (A, p), where A is an object of $\mathcal{A}$ and $p \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ is an idempotent: $p^2 = p$. Morphisms in $\mathrm{Kar}(\mathcal{A})$ are defined by

$$\mathrm{Hom}_{\mathrm{Kar}(\mathcal{A})}((A_1, p_1), (A_2, p_2)) = \{f \in \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \mid p_2 f p_1 = f\}$$

Throughout this section $\mathcal{C}$ is a pivotal category, though in the definitions $\mathcal{C}$ is only required to be monoidal. When $\mathcal{C}$ is multifusion, we use the conventions and notation laid out in the Appendix. In particular, $\mathrm{Irr}(\mathcal{C})$

is the set of isomorphism classes, $\mathrm{Irr}_0(\mathcal{C})$ are those simples appearing as direct summands of the unit $\mathbf{1}$, $\{X_i\}$ will be a fixed set of representatives of $\mathrm{Irr}(\mathcal{C})$, d_i^R is the (right) dimension of X_i, and we will be using graphical presentation of morphisms.

We assume that the reader is familiar with the notions of module categories and module structures on functors between module categories (refer to [EGNO2015, Chapter 7]); for a left module category $\mathcal{M}$ over $\mathcal{C}$, we will denote the action of $A \in \mathcal{C}$ on $M \in \mathcal{M}$ by $A \triangleright M$. Similarly, we use $M \triangleleft A$ for right action. In this paper, all module categories are assumed to be semisimple (as abelian categories).

This section is organized as follows: Subsection 2.3.1 provides the definition and some properties of $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, Subsection 2.3.2 does so for $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$, and Subsection 2.3.3 shows that when $\mathcal{C}$ is pivotal multifusion, these definitions are essentially the same.

2.3.1. $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$.

The following definition is essentially given in [GNN2009, Definition 2.1] (there $\mathcal{C}$ is assumed to be fusion).

Definition 2.35. Let $\mathcal{C}$ be a finite multitensor category, and let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category. The center of $\mathcal{M}$, denoted $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, is the category with the following objects and morphisms:

Objects: pairs (M, γ), where $M \in \mathcal{M}$ and γ is an isomorphism of functors $\gamma_A\colon A \triangleright M \to M \triangleleft A$, $A \in \mathcal{C}$ (half-braiding) satisfying natural compatibility conditions.

Morphisms: $\mathrm{Hom}((M,\gamma),(M',\gamma')) = \{f \in \mathrm{Hom}_{\mathcal{M}}(M, M') \mid f\gamma = \gamma' f\}$.

In particular, in the special case $\mathcal{M} = \mathcal{C}$, this construction gives the Drinfeld center $\mathcal{Z}(\mathcal{C})$.

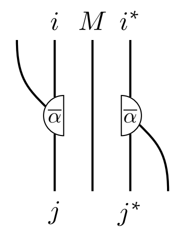
Remark 2.36. Equivalently, the center $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ can be described as the category of $\mathcal{C}$-bimodule functors $\mathcal{C} \to \mathcal{M}$; see [GNN2009] for details.

Theorem 2.37. *Let $\mathcal{C}$ be pivotal multifusion, and $\mathcal{M}$ a $\mathcal{C}$-bimodule category. Let $F\colon \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{M}$ be the natural forgetful functor $F\colon (M,\gamma) \mapsto M$. Then it has a two-sided adjoint functor $I\colon \mathcal{M} \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$, given by*

$$I(M) = \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} X_i \triangleright M \triangleleft X_i^* \tag{2.26}$$

with the half-braiding (recall Lemma 2.33)

(2.27)

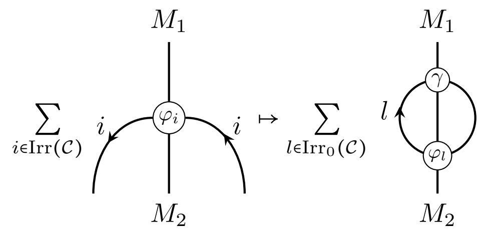


The adjunction isomorphism for $F\colon \mathcal{Z}(\mathcal{M}) \rightleftharpoons \mathcal{M}\colon I$,

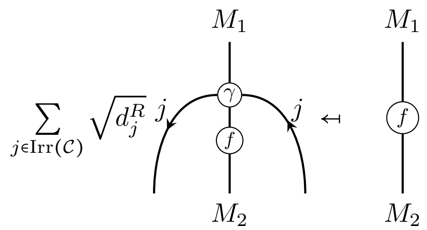
$$\mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}((M_1,\gamma), I(M_2)) \simeq \mathrm{Hom}_{\mathcal{M}}(M_1, M_2)$$

is given by:

(2.28)



(2.29)



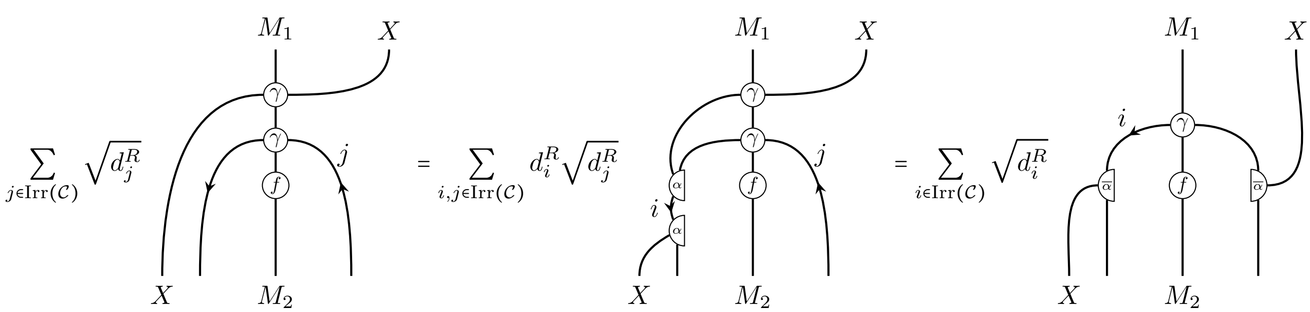
(Note the sum on the right in (2.28) is over $\mathrm{Irr}_0(\mathcal{C})$ *and not* $\mathrm{Irr}(\mathcal{C})$*.) The other adjunction isomorphism for* $I \colon \mathcal{M} \rightleftharpoons \mathcal{Z}(\mathcal{M}) : F$*,*

$$\mathrm{Hom}_{\mathcal{M}}(M_1, M_2) \simeq \mathrm{Hom}_{\mathcal{Z}(\mathcal{M})}(I(M_1), (M_2, \gamma))$$

is given by a similar formula, essentially obtained by rotating all the diagrams above.

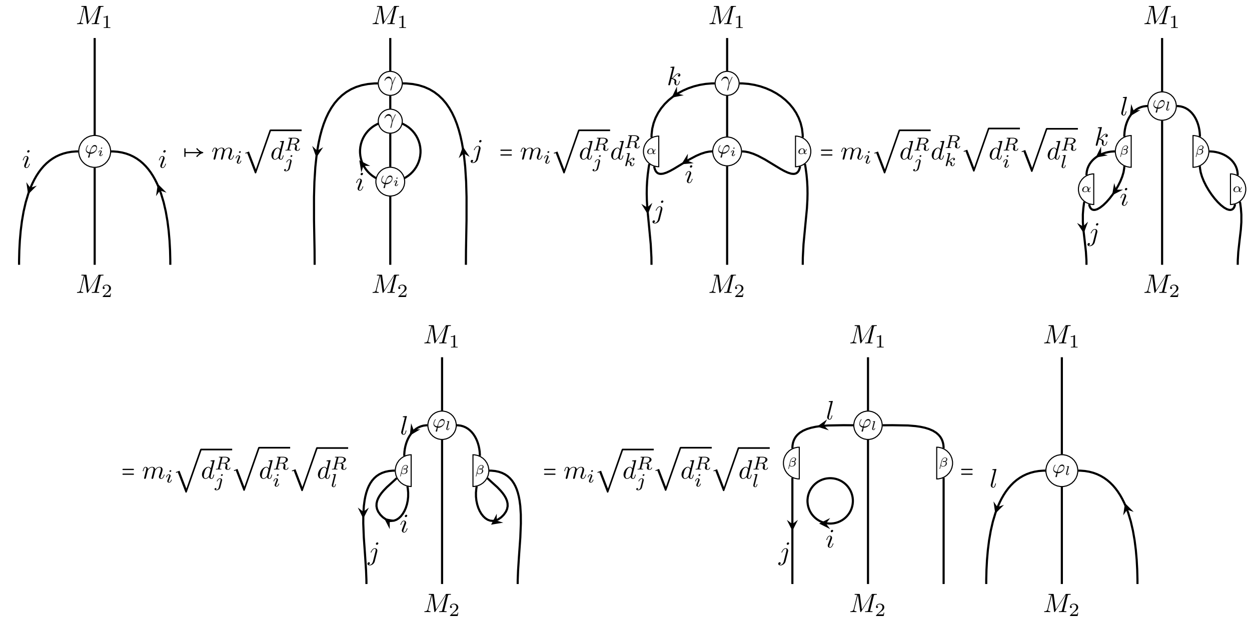
Proof. This is essentially the same as when $\mathcal{C}$ is spherical as in [Kir], but we provide it to assuage any doubts that the non-sphericality, manifested in requiring semicircular morphisms α, does not lead to problems. Note that the isomorphisms here differ slightly from that of [Kir].

Let us check that the morphism on the left side of (2.29) intertwines half-braidings:



In the first equality, we use Lemma 2.27; in the second equality, we use the naturality of γ to pull the top α to the right, and absorb the $\sqrt{d_i^R}\sqrt{d_j^R}$ factor into α to get $\overline{\alpha}$.

Next we check that applying (2.28) then (2.29) is the identity map. Let $m_i = 1$ if $i \in \mathrm{Irr}_0(\mathcal{C})$, 0 otherwise. In the following diagrams, we implicitly sum lowercase latin alphabets over $\mathrm{Irr}(\mathcal{C})$. Then the composition is the map



The first equality is the same as the previous computation. The second equality uses the fact that $\sum \varphi_i$ intertwines half-braidings, so that we "pull" the k strand through φ_i The third equality comes from "pulling" α through β. The fourth equality comes from "pulling" the i loop through β. Finally, for the last equality, we observe that (1) only $j = k$ terms in the sum contribute, and so we have a d_j^R coefficient, and we may apply Lemma 2.27; (2) since $d_i^R = 1$ for $i \in \mathrm{Irr}_0(\mathcal{C})$,

$$\sum_i m_i \sqrt{d_i^R} \bigcirc_i = \sum_{l \in \mathrm{Irr}_0(\mathcal{C})} \mathrm{id}_{\mathbf{1}_l} = \mathrm{id}_{\mathbf{1}}$$

□

An important special case is when $\mathcal{M} = \mathcal{M}_1 \boxtimes \mathcal{M}_2$, where $\mathcal{M}_1$ is a right module category over a pivotal multifusion category $\mathcal{C}$, and $\mathcal{M}_2$ is a left module category over $\mathcal{C}$. In this case, by [ENO2010, Proposition 3.8], one has that $\mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ is naturally equivalent to the balanced tensor product of categories:

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \simeq \mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \tag{2.30}$$

where the balanced tensor product is defined by the universal property: for any abelian category $\mathcal{A}$, we have a natural equivalence

$$Fun_{bal}(\mathcal{M}_1 \times \mathcal{M}_2, \mathcal{A}) = Fun(\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2, \mathcal{A})$$

where Fun, Fun_{bal} stand for category of $\mathbf{k}$-linear additive functors (respectively, category of $\mathbf{k}$-linear additive $\mathcal{C}$- balanced functors); see details in [ENO2010, Definition 3.3].

Under the equivalence (2.30), the natural functor $\mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2$ is identified with the functor $I: \mathcal{M}_1 \boxtimes \mathcal{M}_2 \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ constructed in Theorem 2.37.

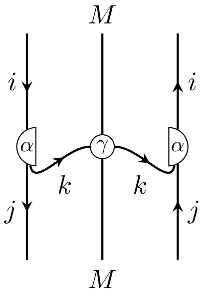
Recall that a functor $F: \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is abelian and $\mathcal{A}$ additive (not necessarily abelian) is called *dominant* if any object of $\mathcal{B}$ appears as a subquotient of $F(X)$ for some $X \in \mathrm{Obj}\,\mathcal{A}$. Similarly, we say that a full subcategory $\mathcal{A} \subset \mathcal{B}$ is dominant if any object of $\mathcal{B}$ appears as a subquotient of some $X \in \mathrm{Obj}\,\mathcal{A}$. In the case when $\mathcal{A}$ is a full additive subcategory in a semisimple abelian category $\mathcal{B}$, this immediately implies that the Karoubi envelope of $\mathcal{A}$ is equivalent to $\mathcal{B}$ (in particular, this implies that $\mathrm{Kar}(\mathcal{A})$ is abelian).

Proposition 2.38. *Under the hypotheses of Theorem 2.37, the functor $I: \mathcal{M} \to \mathcal{Z}_{\mathcal{M}}(\mathcal{M})$ is dominant.*

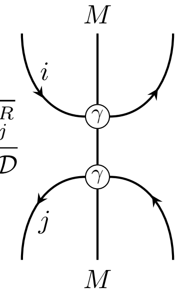
Proof. This follows immediately from Lemma 2.39 below. □

Lemma 2.39. *Let $(M,\gamma) \in \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$. The morphism*
(2.31)

$$P_{(M,\gamma)} := \frac{1}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}} G\left(\sum d_i^R \gamma_{X_i}\right) = \sum_{i,j,k \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}\sqrt{d_j^R}\, d_k^R}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$

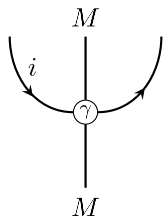


$$= \sum_{i,j \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}\sqrt{d_j^R}}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$

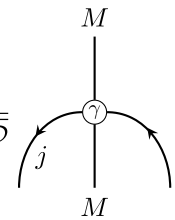


is a projection in $\operatorname{End}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}(I(M))$. Furthermore, it can be written as a composition $P_M = \hat{P}_M \circ \check{P}_M$, where

(2.32) $$\check{P}_{(M,\gamma)} := \sum_{i \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_i^R}}{\sqrt{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}}$$



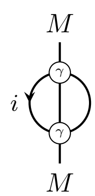
$$, \quad \hat{P}_{(M,\gamma)} := \sum_{j \in \operatorname{Irr}(\mathcal{C})} \frac{\sqrt{d_j^R}}{\sqrt{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}}$$



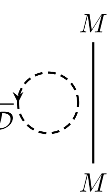
such that $\check{P}_{(M,\gamma)} \circ \hat{P}_{(M,\gamma)} = \operatorname{id}_{(M,\gamma)}$, thus exhibiting (M,γ) as a direct summand of $I(M)$. Note that $\check{P}_{(M,\gamma)}, \hat{P}_{(M,\gamma)}$ are multiples of the morphisms corresponding to id_M under the adjunctions of Theorem 2.37.

Proof. The second equality in (2.31) follows from pulling α through γ and using Lemma 2.27. $\hat{P}_{(M,\gamma)}$ was shown to be a morphism in $\operatorname{Hom}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}((M,\gamma), I(M))$ in the proof of Theorem 2.37, and one shows $\check{P}_{(M,\gamma)} \in \operatorname{Hom}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}(I(M), (M,\gamma))$ in a similar fashion. The following computation shows that $\check{P}_{(M,\gamma)} \circ \hat{P}_{(M,\gamma)} = \operatorname{id}_{(M,\gamma)}$:

$$\check{P}_{(M,\gamma)} \circ \hat{P}_{(M,\gamma)} = \sum_{i \in \operatorname{Irr}(\mathcal{C})} \frac{d_i^R}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$



$$= \frac{1}{|\operatorname{Irr}_0(\mathcal{C})|\mathcal{D}}$$



$$= \operatorname{id}_{(M,\gamma)}$$

The second equality comes from "pulling" the j loop out to the left, and the last equality follows from Lemma 2.32. □

Proposition 2.40. *Under the hypotheses of Theorem 2.37, if $\mathcal{M}$ is finite semisimple, then so is $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$.*

Proof. Using exactness in $\mathcal{M}$ of the left and right actions, abelianness of $\mathcal{M}$ transfers to $\mathcal{Z}(\mathcal{M})$. For example, the kernel K of a morphism $f : M_1 \to M_2$ such that $f \in \operatorname{Hom}_{\mathcal{Z}(\mathcal{M})}((M_1,\gamma^1),(M_2,\gamma^2))$ would inherit a half-braiding $\gamma^1|_K$. Semisimplicity follows from the semisimplicity of $\mathcal{M}$ and Lemma 2.39. Finiteness follows from Proposition 2.38; I ensures there can't be too many simples in $\mathcal{Z}(\mathcal{M})$. □

For applications, we will need to consider centers over a full, dominant, monoidal subcategory $\mathcal{C}' \subseteq \mathcal{C}$. Equivalently, $\mathcal{C}'$ is a pivotal category whose Karoubi envelope is multifusion.

Lemma 2.41. *Let $\mathcal{C}'$ be a pivotal locally finite **k**-linear additive category whose Karoubi envelope $\mathcal{C} = \operatorname{Kar}(\mathcal{C}')$ is multifusion. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category (as before, we assume that $\mathcal{M}$ is a semisimple abelian category). Then there is a natural equivalence*

$$\mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{Z}_{\mathcal{C}'}(\mathcal{M})$$

In particular, for right, left $\mathcal{C}$-module categories $\mathcal{M}_1, \mathcal{M}_2$, there is a natural equivalence

$$\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \simeq \mathcal{M}_1 \boxtimes_{\mathcal{C}'} \mathcal{M}_2.$$

Proof. The equivalence is given as follows: objects (M,γ) in $\mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ are naturally objects in $\mathcal{Z}_{\mathcal{C}'}(\mathcal{M})$ by forgetting some of the half-braiding, i.e. $(M,\gamma|_{\mathcal{C}'})$; morphisms $f:(M,\gamma)\to(M',\gamma')$ are naturally morphisms $f:(M,\gamma|_{\mathcal{C}'})\to(M',\gamma|_{\mathcal{C}'})$. We need to check that this is an equivalence.

The functor is essentially surjective: any half-braiding over $\mathcal{C}'$ can be completed to a half-braiding over $\mathcal{C}$. To see this, let γ be a half-braiding over $\mathcal{C}'$. Let $X\in\operatorname{Obj}\mathcal{C}\setminus\operatorname{Obj}\mathcal{C}'$, and let it be a direct summand of some $Y\in\operatorname{Obj}\mathcal{C}'$, $X\underset{p}{\overset{\iota}{\rightleftharpoons}}Y$. Then we define the extension of γ to X by $\gamma_X=(\mathrm{id}_{M_2}\triangleleft p)\circ\gamma_Y\circ(\iota\triangleright\mathrm{id}_{M_1})$. It is easy to check, using the semisimplicity of $\mathcal{C}$, that γ_X is independent on the choice of Y and p,ι. It is also easy to check that the resulting extension is indeed natural in X.

For morphisms, it is clear that this functor is faithful. To show fullness, consider $f\in\operatorname{Hom}_{\mathcal{Z}_{\mathcal{C}'}(\mathcal{M})}((M_1,\gamma^1),(M_2,\gamma^2))$. We need to check that it also intertwines half-braiding with $X\in\mathcal{C}$, but this follows easily from the definition of the extension of half-braiding given above.

Note since γ has a unique extension to all of $\mathcal{C}$, this proof actually shows that the equivalence is an isomorphism.

□

Note that in the proof above, we do not use the rigidity of $\mathcal{C}$, but we need it to conclude the second statement concerning balanced tensor products.

2.3.2. $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$.

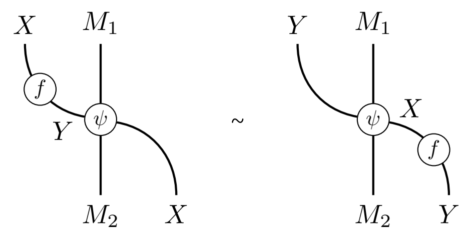
Next we define the other notion of center.

Definition 2.42. Let $\mathcal{C}$ be monoidal, and $\mathcal{M}$ a $\mathcal{C}$-bimodule category. Define the *horizontal trace* $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ as the category with the following objects and morphisms:

Objects: same as in $\mathcal{M}$

Morphisms: $\operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1,M_2)=\bigoplus_X\operatorname{Hom}^X_{\mathcal{M}}(M_1,M_2)/\sim$, where $\operatorname{Hom}^X_{\mathcal{M}}(M_1,M_2):=\operatorname{Hom}_{\mathcal{M}}(X\triangleright M_1,M_2\triangleleft X)$, the sum is over all (not necessarily simple) objects $X\in\mathcal{C}$, and $\sim$ is the equivalence relation generated by the following:

For any $\psi\in\operatorname{Hom}^{Y,X}_{\mathcal{M}}(M_1,M_2):=\operatorname{Hom}_{\mathcal{M}}(Y\triangleright M_1,M_2\triangleleft X)$ and $f\in\operatorname{Hom}_{\mathcal{C}}(X,Y)$, we have



In other words, $\operatorname{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1,M_2)=\int^X\operatorname{Hom}^{X,X}_{\mathcal{M}}(M_1,M_2)$ is the coend of the functor $\operatorname{Hom}^{-,-}_{\mathcal{M}}(M_1,M_2):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{V}ec$ (see e.g. [ML1998]).

Composition is given by

$$\operatorname{Hom}^Y_{\mathcal{M}}(M_2,M_3))\otimes\operatorname{Hom}^X_{\mathcal{M}}(M_1,M_2)\to\operatorname{Hom}^{Y\otimes X}_{\mathcal{M}}(M_1,M_3)$$

which sends $\psi\otimes\varphi$ to

$$Y\triangleright(X\triangleright M_1)\xrightarrow{\mathrm{id}_Y\triangleright\psi}Y\triangleright(M_2\triangleleft X)\simeq(Y\triangleright M_2)\triangleleft X\xrightarrow{\varphi\triangleleft\mathrm{id}_X}(M_3\triangleleft Y)\triangleleft X$$

For right, left $\mathcal{C}$-module categories $\mathcal{M}_1,\mathcal{M}_2$, we denote $\mathcal{M}_1\hat{\boxtimes}_{\mathcal{C}}\mathcal{M}_2=\mathrm{hTr}_{\mathcal{C}}(\mathcal{M}_1\boxtimes\mathcal{M}_2)$.

When the context is clear, we will drop the subscript $\mathrm{hTr}=\mathrm{hTr}_{\mathcal{C}}$. We will write $[\varphi]\in\operatorname{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1,M_2)$ for the morphism represented by $\varphi\in\operatorname{Hom}^X_{\mathcal{M}}(M_1,M_2)$ for some X.

It can be shown that in a certain sense this definition is dual to the definition of center given above and is closely related to the notion of co-center as described in [DSPS2013, Section 3.2.2]. However, we will not be discussing the exact relation here.

It is easy to see that the category $\mathrm{hTr}(\mathcal{M})$ is additive but not necessarily abelian.

There is a natural inclusion functor $\mathrm{hTr}\colon\mathcal{M}\to\mathrm{hTr}(\mathcal{M})$ which is identity on objects, and on morphisms it is the natural map $\operatorname{Hom}_{\mathcal{M}}(M_1,M_2)=\operatorname{Hom}^{\mathbf{1}}_{\mathcal{M}}(M_1,M_2)\to\operatorname{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1,M_2)$.

The horizontal trace construction is functorial with respect to bimodule functors:

Lemma 2.43. *Given a functor of $\mathcal{C}$-bimodule categories $(F,J):\mathcal{M}\to\mathcal{M}'$, there is a natural functor $\mathrm{hTr}(F,J):\mathrm{hTr}(\mathcal{M})\to\mathrm{hTr}(\mathcal{M}')$ that is the same as F on objects.*

Proof. We define the functor $\mathrm{hTr}(F,J)$ to act the same as F on objects, and sends a morphism $\varphi\in\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)$ to the composition

$$X\triangleright F(M_1)\simeq F(X\triangleright M_1)\xrightarrow{F(\varphi)}F(M_2\triangleleft X)\simeq F(M_2)\triangleleft X$$

where the equivalences are from the bimodule structure J. It is easy to check that composition is respected. □

We also consider $\mathcal{C}'\subseteq\mathcal{C}$ as in Lemma 2.41, but here we do not need rigidity nor semisimplicity on $\mathcal{C}$:

Lemma 2.44. *Let $\mathcal{C}'$ be monoidal, and let $\mathcal{C}=\mathrm{Kar}(\mathcal{C}')$ be its Karoubi envelope. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category. Then there is a natural equivalence*

$$\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})\simeq\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$$

In particular, for right, left $\mathcal{C}$-module categories $\mathcal{M}_1,\mathcal{M}_2$, there is a natural equivalence

$$\mathcal{M}_1\hat{\boxtimes}_{\mathcal{C}}\mathcal{M}_2\simeq\mathcal{M}_1\hat{\boxtimes}_{\mathcal{C}'}\mathcal{M}_2$$

Proof. The equivalence is given by the identity map on objects, and for two objects $M_1,M_2\in\mathrm{Obj}\,\mathcal{M}$, the map on morphisms is given by completing the bottom arrow:

$$\begin{array}{ccc}\bigoplus_{X\in\mathcal{C}'}\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2) & \longrightarrow & \bigoplus_{X\in\mathcal{C}}\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)\\ \downarrow & & \downarrow\\ \mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})}(M_1,M_2) & \longrightarrow & \mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1,M_2)\end{array}$$

It remains to prove that the bottom arrow is an isomorphism.

Let us first observe the following. Let $X,Y\in\mathrm{Obj}\,\mathcal{C}'$, and suppose X is a direct summand of Y, with $X\underset{p}{\overset{\iota}{\rightleftharpoons}}Y$. Let $\varphi\in\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)$. Then $\varphi=\varphi\circ p\circ\iota\sim\iota\circ\varphi\circ p\in\mathrm{Hom}^Y_{\mathcal{M}}(M_1,M_2)$, where we write p,ι instead of $p\triangleright\mathrm{id}_{M_1},\mathrm{id}_{M_2}\triangleleft\iota$ for simplicity. This works for $\mathcal{C}$ too. Thus one can identify $\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)$ with a subspace of $\mathrm{Hom}^Y_{\mathcal{M}}(M_1,M_2)$.

Surjectivity: Essentially, we need to show that any morphism in $\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})$ can be "absorbed" into $\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})$. Let $[\varphi]\in\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1,M_2)$ be represented by some $\varphi\in\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)$. By the above observation, we can choose $Y\in\mathrm{Obj}\,\mathcal{C}'$ with X a direct summand of Y, then $\varphi\in\mathrm{Hom}^X_{\mathcal{M}}(M_1,M_2)$ is identified with some morphism in $\mathrm{Hom}^Y_{\mathcal{M}}(M_1,M_2)$, so $[\varphi]$ is in the image.

Injectivity: Essentially, we need to show that relations can also be "absorbed" into $\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})$. Let $[\varphi]\in\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})}(M_1,M_2)$ that is sent to 0. By the observation above, we may represent it by some $\varphi\in\mathrm{Hom}^Y_{\mathcal{M}}(M_1,M_2)$ for some $Y\in\mathrm{Obj}\,\mathcal{C}'$. Since it is 0 in $\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})}(M_1,M_2)$, there exists

- a finite collection of objects $J=\{A_j\}\subset\mathrm{Obj}\,\mathcal{C}$ so that $A_0=Y$.
- $\Phi_i\in\mathrm{Hom}^{A_{m_i},A_{n_i}}_{\mathcal{M}}(M_1,M_2)$,
- $f_i:A_{n_i}\to A_{m_i}$,

such that $\varphi=\sum_i f_i\circ\Phi_i-\Phi_i\circ f_i\in\bigoplus_{A_j\in J}\mathrm{Hom}^{A_j}_{\mathcal{M}}(M_1,M_2)$.

We want to be able to replace the A_j's with objects in $\mathcal{C}'$. For each $j\neq 0$, choose some $B_j\in\mathrm{Obj}\,\mathcal{C}'$ such that A_j is a direct summand of B_j: $A_j\underset{p_j}{\overset{\iota_j}{\rightleftharpoons}}B_j$. For $j=0$, we take $B_0=A_0=Y$ and $\iota_0=p_0=\mathrm{id}_Y$. This gives us maps $\Theta_j:\psi\mapsto\iota_j\circ\psi\circ p_j:\mathrm{Hom}^{A_j}_{\mathcal{M}}(M_1,M_2)\to\mathrm{Hom}^{B_j}_{\mathcal{M}}(M_1,M_2)$. Denote $\Theta=\sum\Theta_j$.

Now consider

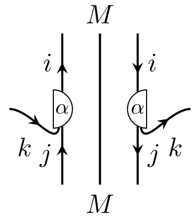
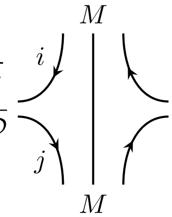
- $L=\{B_j\}\subset\mathrm{Obj}\,\mathcal{C}'$,
- $\Psi_i=\iota_{n_i}\circ\Phi_i\circ p_{m_i}\in\mathrm{Hom}^{B_{m_i},B_{n_i}}_{\mathcal{M}}(M_1,M_2)$,
- $g_i=\iota_{m_i}\circ f_i\circ p_{n_i}:B_{n_i}\to B_{m_i}$.

It is a simple matter to verify that $g_i \circ \Psi_i - \Psi \circ g_i = \Theta(f_i \circ \Phi_i - \Phi_i \circ f_i)$. Hence $\varphi = \Theta(\varphi) = \Theta(\sum f_i \circ \Phi_i - \Phi_i \circ f_i) = \sum g_i \circ \Psi_i - \Psi \circ g_i$ is 0 in $\mathrm{Hom}_{\mathrm{hTr}_{\mathcal{C}'}}(M_1, M_2)$.

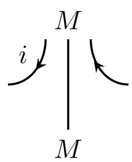
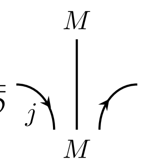
□

The following lemma is similar in spirit to Lemma 2.39.

Lemma 2.45. *Let $M \in \mathcal{M}$. The morphism*

(2.33) $$P'_M := \sum_{i,j,k \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L} d_k^R}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$  $$= \sum_{i,j \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}\sqrt{d_j^L}}{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}$$ 

is a projection in $\mathrm{End}_{\mathrm{hTr}(\mathcal{M})}(\bigoplus X_i \triangleright M \triangleleft X_i^*)$. Furthermore, it can be written as a composition $P'_M = \hat{P}'_M \circ \check{P}'_M$, where

(2.34) $$\check{P}'_M := \sum_{i \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_i^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$  , $$\hat{P}'_M := \sum_{j \in \mathrm{Irr}(\mathcal{C})} \frac{\sqrt{d_j^L}}{\sqrt{|\mathrm{Irr}_0(\mathcal{C})|\mathcal{D}}}$$ 

such that $\check{P}'_M \circ \hat{P}'_M = \mathrm{id}_M$, thus as objects in $\mathrm{Kar}(\mathrm{hTr}(\mathcal{M}))$, we have $M \simeq (\bigoplus X_i \triangleright M \triangleleft X_i^*, P'_M)$.

Proof. Essentially the same as Lemma 2.39. (Note the use of both left and right dimensions.) □

2.3.3. *Equivalence.*

Theorem 2.46. *Let $\mathcal{C}$ be pivotal multifusion, and $\mathcal{M}$ a $\mathcal{C}$-bimodule category. One has a natural equivalence*

$$\mathrm{Kar}(\mathrm{hTr}(\mathcal{M})) \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$$

Under this equivalence, the inclusion functor $\mathrm{hTr} : \mathcal{M} \to \mathrm{hTr}(\mathcal{M})$ is identified with the functor $I : \mathcal{M} \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$.

In particular, for right, left $\mathcal{C}$-modules $\mathcal{M}_1, \mathcal{M}_2$, we have

$$\mathcal{M}_1 \boxtimes_{\mathcal{C}} \mathcal{M}_2 \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \simeq \mathrm{Kar}(\mathcal{M}_1 \hat{\boxtimes}_{\mathcal{C}} \mathcal{M}_2)$$

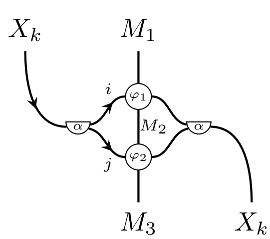
Before proving the theorem, we will need the following lemma.

Lemma 2.47. *The natural linear map*

(2.35) $$\bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{M}}^{X_i}(M_1, M_2) \to \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2)$$

is an isomorphism. Moreover, composition of morphisms $\mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_2, M_3) \otimes \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2) \to \mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_3)$ carries over to the composition rule

(2.36) $$\mathrm{Hom}_{\mathcal{M}}^{X_i}(M_2, M_3) \otimes \mathrm{Hom}_{\mathcal{M}}^{X_j}(M_1, M_2) \to \bigoplus_{k \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{M}}^{X_k}(M_1, M_3)$$

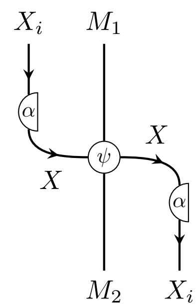
(2.37) $$\varphi_2 \otimes \varphi_1 \mapsto \sum_{k \in \mathrm{Irr}(\mathcal{C})} d_k^R$$ 

Proof. Define a linear map

$$\mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M_1, M_2) \to \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}^{X_i}_{\mathcal{M}}(M_1, M_2)$$

by

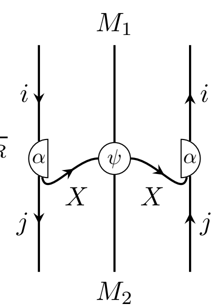
(2.38) $$\psi \mapsto \sum_{i \in \mathrm{Irr}(\mathcal{C})} d_i^R$$



for $\psi \in \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2)$; α is a sum over dual bases - see Notation 2.26. (2.38) is well-defined by Lemma 2.24. Using Lemma 2.27, it is easy to see that (2.35) and (2.38) are mutually inverse, and it is clear that they respect composition. □

Proof of Theorem 2.46. Define the functor $G : \mathrm{hTr}(\mathcal{M}) \to \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ on objects by $G(M) = I(M)$, and on morphisms by

(2.39) $$G(\psi) = \sum_{i,j\ \mathrm{Irr}(\mathcal{C})} \sqrt{d_i^R}\sqrt{d_j^R}$$



for $\psi \in \mathrm{Hom}^X_{\mathcal{M}}(M_1, M_2)$; once again see Notation 2.26 for definition of α.

It is easy to check the following properties:

(1) G is well-defined on morphisms (i.e. it preserves the equivalence relation): this follows from Lemma 2.33.
(2) G is dominant: any $Y \in \mathcal{Z}_{\mathcal{C}}(\mathcal{M})$ appears as a direct summand of $G(M)$ for some $M \in \mathcal{M}$. Namely, if $Y = (M, \gamma)$, then it appears as a direct summand of $G(M)$; the projection to Y is, up to a factor, $G(\sum d_i^R \gamma_{X_i})$ (see Lemma 2.39 in Appendix for proof; compare Proposition 2.38).
(3) G is bijective on morphisms: by adjointness property (Theorem 2.37), we have
$$\mathrm{Hom}_{\mathcal{Z}_{\mathcal{C}}(\mathcal{M})}(I(M), I(M')) \cong \mathrm{Hom}_{\mathcal{M}}(I(M), M') = \bigoplus_i \mathrm{Hom}_{\mathcal{M}}(X_i \triangleright M \triangleleft X_i^*, M')$$
and by Lemma 2.47, the right hand side coincides with $\mathrm{Hom}_{\mathrm{hTr}(\mathcal{M})}(M, M')$.

This immediately implies the statement of the theorem by the universal properties of Karoubi envelopes. □

By Lemma 2.41 and Lemma 2.44, we extend the above theorem to $\mathcal{C}' \subseteq \mathcal{C}$:

Corollary 2.48. *Let $\mathcal{C}'$ be a pivotal category whose Karoubi envelope $\mathcal{C} = \mathrm{Kar}(\mathcal{C}')$ is multifusion. Let $\mathcal{M}$ be a $\mathcal{C}$-bimodule category, and hence naturally a $\mathcal{C}'$-bimodule category. Then we have*

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}}(\mathcal{M})) \simeq \mathcal{Z}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{Z}_{\mathcal{C}'}(\mathcal{M}).$$

Note $\mathrm{Kar}(\mathcal{M})$ inherits a $\mathcal{C}'$-bimodule structure from $\mathcal{M}$. For example, $A \triangleright (M, p) = (A \triangleright M, \mathrm{id}_A \triangleright p)$. We compare these constructions for $\mathcal{M}$ and its Karoubi envelope:

Lemma 2.49. *Under the same hypotheses as Corollary 2.48,*

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M})) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathrm{Kar}(\mathcal{M})))$$

In particular, if $\mathcal{M}'$ is a dominant submodule category of $\mathcal{M}$, then

$$\mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M}')) \simeq \mathrm{Kar}(\mathrm{hTr}_{\mathcal{C}'}(\mathcal{M}))$$

Proof. The natural inclusion $\mathcal{M} \to \mathrm{Kar}(\mathcal{M})$ is a full, dominant functor of $\mathcal{C}'$-bimodules, and it is easy to see that the corresponding functor $\mathrm{hTr}(\mathcal{M}) \to \mathrm{hTr}(\mathrm{Kar}(\mathcal{M}))$ is also full and dominant. It follows that the induced functor on their Karoubi envelopes is an equivalence.

The second statement follows because $\mathrm{Kar}(\mathcal{M}') \simeq \mathrm{Kar}(\mathcal{M})$. □

2.4. **Piecewise-Linear Topology.**

In this section, we provide some background on piecewise-linear (henceforth abbreviated as PL) topology. Most of the background material in this section is obtained from [Hud1969]; we provide a lightning recap of basic definitions and quote results without proof.

As most of the PL topological arguments are in dimensions 4 and below, the reader who is unfamiliar with PL topology may quite safely replace all things PL with their smooth counterparts, as it is known that every PL manifold of dimension less than or equal to 6 admits a unique smooth structure up to diffeomorphism (see [HM1974], [KM1963]).

Most objects and maps discussed in this section are PL, and are implicitly assumed to be such when no qualifiers are added; when we want to emphasize that a map is merely continuous, we say it is C^0. Technically the term *affine linear* should be used instead of linear, but we simply say linear for brevity.

A *convex linear cell*, or simply a *cell*, in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$; the *dimension* of the cell is the dimension of the affine subspace spanned by the cell. A *face* of a k-dimensional cell is the intersection of the cell with an affine plane of dimension $k-1$ which does not meet the interior of the cell.

A *polyhedron* in $\mathbb{R}^n$ is the union of a finite collection of cells in $\mathbb{R}^n$.

A continuous map $f : P \to Q$ from one polyhedron $P \subset \mathbb{R}^p$ to another polyhedron $Q \subset \mathbb{R}^q$ is *PL* if the graph of f, $\Gamma_f = \{(x, f(x)|x \in P\} \subset \mathbb{R}^{p+q}$, is a polyhedron. The cells that make up Γ_f project to cells in $\mathbb{R}^p$ to make up P, such that the restriction of f to these cells is linear, so an equivalent definition of f being PL is if P can be presented as a union of cells such that f is linear on each cell. The composition of PL maps is PL.

The *cone* over a polyhedron $P \subset \mathbb{R}^p$, denoted cone(P), is the polyhedron $P * e_{p+1} \subset \mathbb{R}^{p+1}$, where $e_{p+1} = (0, \ldots, 0, 1)$. For a PL map $f : P \to Q$, the cone(f) is naturally defined and is also PL.

A *coordinate map* of a topological space X is a C^0-embedding $f : P \to X$ of a polyhedron P in X; we write (f, P) to denote such a map. Two maps are *compatible* if their overlap is a coordinate map; more precisely, (f, P) and (g, Q) are compatible if either their images don't intersect or there exists a third coordinate map (h, R) such that $h(R) = f(P) \cap g(Q)$ and $f^{-1}h$, $g^{-1}h$ are PL.

A *PL structure* on X is a family $\mathcal{F}$ of coordinate maps satisfying:

- Any two elements are compatible
- For every $x \in X$ has a coordinate neighborhood, i.e. there exists $(f, P) \in \mathcal{F}$ such that $f(P)$ is a neighborhood of x.
- $\mathcal{F}$ is maximal.

If $\mathcal{F}$ is not necessarily maximal, it is a *basis for a PL structure* on X.

A *convex linear cell complex* K in $\mathbb{R}^n$ is a finite collection of cells that is closed under taking faces of cells and taking intersections of pairs of cells that meet. We write $\sigma \leq \tau$ for $\sigma, \tau \in K$ when there is a sequence $\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_k = \tau$, $k \geq 0$, such that σ_i is a face of σ_{i+1}; note by the conditions of being a complex, if $\sigma \subset \tau$ as sets, $\sigma = \sigma \cap \tau$ is a face of τ, so the relations '$\leq$' and '$\subseteq$' are equivalent for complexes. We denote by $|K| \subset \mathbb{R}^n$ the union of cells.

An n-simplex in $\mathbb{R}^N$ is the convex hull of $(n+1)$ linearly independent points; a *simplicial complex* is simply a cell complex whose cells are simplices. A *triangulation* of a topological space X is a simplicial complex K together with an identification of X with $|K|$ as C^0-spaces. A triangulated space is naturally a PL space.

By [Hud1969, Chapter 3.2], any PL space X (i.e. a space with a PL structure) admits a triangulation $|K| \to X$ that is a PL map, i.e. the triangulation gives the same PL structure on X.

A cell complex K is a subdivision of another L if $|K| = |L|$ and every cell of K is a subset of some cell of L.

Here $I = [0,1]$.

An *isotopy* F between embeddings $f, g : M \to Q$ is a map $F : M \times I \to Q$ such that, writing $F_t = F(-,t)$, $F_0 = f, F_1 = g$ and F_t is an embedding for each t. (Warning: the linear maps $\mathrm{id}_{\mathbb{R}}$ and $x \mapsto 2x$ are indeed PL-isotopic, but $F_t(x) = tx$ is *not* a PL-isotopy, as F is not linear map as a map $\mathbb{R} \times I \to \mathbb{R}$.) An equivalent formulation of F is as a *level-preserving* embedding $\overline{F} : M \times I \to Q \times I$; we will write F instead of $\overline{F}$ when it does not lead to confusion. We say f and g are *ambient isotopic* if there exists an *ambient isotopy* $H : Q \times I \to Q$ such that $g = H_1 \circ f$; we say H *throws* $f(A)$ onto $g(A)$, where $A \subseteq M$ is some subset of M. If F is an isotopy from f to g, and $H|_M = F$, then we say that H *carries* F.

The *support* of a self-homeomorphism $h : Q \to Q$ is $\mathrm{supp}(h) := \{x \in Q | h(x) \neq x\}$; we say h is *supported by* $X \subseteq Q$ if $\mathrm{supp}(h) \subseteq X$. Similarly, an ambient isotopy is *supported by* $X \subseteq Q$ if the isotopy is fixed away from X.

An *m-ball* is a polyhedron which is PL-homeomorphic to an m-simplex. A *PL manifold* of dimension m is a polyhedron in which every point has a (closed) neighborhood which is a m-ball. We denote an m-ball by B^m or $\mathbb{D}^m$.

A *move* on a q-manifold Q is a homeomorphism h that is supported by a q-ball B in Q; we call B a *supporting ball.* A move h is a *proper move* if either h is fixed on ∂Q or B meets ∂Q in at most one face of B. By (the proof of) [Hud1969, Lemma 6.1], a proper move supported by a ball B is isotopic to the identity via an isotopy supported by B.

Two embeddings $f, g : M \to Q$ of manifolds are *isotopic by moves* if they are related by a finite sequence of moves, i.e. $g = h_k \circ \cdots \circ h_1 \circ f$. By [Hud1969, Theorem 6.2], if H is an ambient isotopy of a compact manifold Q, then H_1 is isotopic by moves to the identity. This result can be refined in several ways:

Theorem 2.50 ([Hud1969, Theorem 6.2.3])**.** *Given an open cover $\{U_\alpha\}$, if $f, g : M \to Q$ are ambient isotopic embeddings, where M is compact, then f, g are isotopic by moves such that each move is supported on a ball contained in some U_α. Moreover, if the ambient isotopy carrying f to g is fixed on the boundary, then each move can be chosen to be fixed on the boundary.*

A *manifold pair* (Q, M) is a pair of manifolds with M embedded in Q. We say (Q, M) is *proper* if $M \cap \partial Q = \partial M$, and is *locally unknotted* if for any $x \in M$, there exists a Q-neighborhood V of x such that $(V, V \cap M)$ is an unknotted ball pair (i.e. homeomorphic to the standard pair $(I^q, I^m \times 0)$; note locally unknotted implies proper). Furthermore, an isotopy $F : M \times I \to Q \times I$ is *locally unknotted* if, for all $0 \leq s \leq t \leq 1$, the proper manifold pair $(Q \times [s,t], F(M \times [s,t]))$ is locally unknotted.

Theorem 2.51 ([Hud1969, Theorem 6.12], Isotopy Extension Theorem)**.** *Let $F : M \times I \to Q \times I$, M compact, be a proper locally unknotted isotopy. Then there exists an ambient isotopy H of Q that carries F, i.e.*

$$F_t = H_t \circ F_0 \times \mathrm{id}_I$$

Furthermore, if F is fixed on ∂M, then we may choose H to be fixed on ∂Q.

Remark 2.52. We will be using this result particularly for extending an isotopy of a ribbon graph to the ambient manifold (see Section 3.2, Section 6.1). Strictly speaking, Theorem 2.51 does not apply there, as the graphs, as surfaces, have boundary in the interior (thus violating local-unknottedness). However, one can make a double of the ribbon graph, and glue it to the original graph, making it a closed surface except at the boundary. Then Theorem 2.51 applies to this double, and thus to the original graph.

Theorem 2.53 ([Hud1969, Theorem 6.11])**.** *A manifold with boundary admits a collar neighborhood, unique up to ambient isotopy fixing the boundary.*

Corollary 2.54 ([Hud1969, Corollary 6.12])**.** *Let (Q, M) be a locally unknotted compact proper manifold pair. Any boundary collar on M extends to a compatible boundary collar on Q.*

The following is adapted from [AZ1967, Definition 2]:

Definition 2.55. Let P, Q, M be manifolds with $P \subset Q$. Let $f : M \to Q$ be a map, and $x \in \mathrm{Int}(M)$ such that $f(x) \in P$. We say f is *transversal to P at x* if there is a commutative diagram

$$\begin{array}{ccccc} D^{q-p} \times D^{m+p-q}, 0 \times 0 & \xrightarrow{\mathrm{id} \times k} & D^{q-p} \times D^{p}, 0 \times 0 & \xleftarrow[0 \times \mathrm{id}]{} & 0 \times D^{p}, 0 \times 0 \\ \downarrow \phi & & \downarrow \psi & & \downarrow \psi \\ M, x & \xrightarrow{f} & Q, f(x) & \longleftarrow & P, f(x) \end{array}$$

where ϕ, ψ are embeddings onto neighborhoods of $x, f(x)$, respectively, and k is some map $D^{m+p-q} \to D^p$. We say f is *transversal to P* if it is transversal to P at all such points x.

Note that this definition was originally meant for closed manifolds, but we will be using it for M =surface with boundary (more specifically, ribbon graphs in Section 3.2). An obvious deficiency in this definition is that M may intersect P only at its boundary, but the definition says nothing about this intersection. We will not go into any detail on this, as our use case is very simple, and will be more interested in a stronger notion of transversality for ribbon graphs (see Definition 3.24).

Proposition 2.56. *Let $f : M \to \mathbb{R}$ be a proper map. We say $s \in \mathbb{R}$ is a* regular value *if f is transversal to s (as a 0-dim submanifold of $\mathbb{R}$). We say s is a* critical value *if it is not a regular value.*

The set of regular values is dense in $\mathbb{R}$.

Proof. By definition of PL maps, M can be presented as a union of cells such that the restriction of f to each cell is linear. Let P be the collection of cells on which f is constant (in particular, P contains all 0-cells). Then it is easy to see that any $s \notin f(P)$ is a regular value.

By propreness of f, the preimage $f^{-1}([a, b])$ of any compact interval only meets finitely many cells in M, thus $f^{-1}(P) \cap [a, b]$ is a finite set, and we are done. ☐

Remark 2.57. **Orientations and vectors.** It is somewhat tricky to define the tangent space of a point in a PL manifold, as the linear structure is broken, thus it is not immediately clear how to adapt the usual definition of orientation using an ordered tuple of vectors. Strictly speaking, one should use local homology groups to define orientations, but this is too cumbersome for our setup. Instead, we first pick a triangulation of the manifold, and since each simplex has a well-defined linear structure, the orientation at a point can be defined in the usual way; if a point belongs to multiple simplices, an orientation in one simplex naturally defines an orientation in another simplex. If we present the PL structure of a manifold M as a collection of charts $f : U \to M$, such that the transition maps are PL, then an orientation at a point can again be defined the usual way in $\mathbb{R}^N$. If the point belongs to another chart, it should define a corresponding orientation in the other chart as follows: propogate the orientation to the rest of the first chart, then, since the transition map must be linear on some open subset, the transition map transfers the propogated orientation to the other chart.

In short, we will define orientations at points by describing an ordered tuple of vectors at that point, implicitly assuming that some linear structure is chosen as reference.

We will also be discussing co-orientations of surfaces in a solid. As we only deal with locally unknotted manifold pairs, a co-orientation at an interior point of the surface can be defined as a choice of one component of $B^3 \backslash B^2$.

Finally, the convention on outward orientation of the boundary of an oriented manifold is: in some coordinate chart, at a point on the boundary, $o = (v_1, \ldots, v_{k-1})$ defines the outward orientation if $(\vec{n}, v_1, \ldots, v_{k-1})$ defines the orientation of the manifold, where $\vec{n}$ is an outward-pointing vector.

Next we discuss cobordisms and handle decompositions. We will only apply these results to 4-dimensional cobordisms, but we present them in general for clarity; the reader may assume $n = 4$, W is a 4-manifold, M is a 3-manifold, and N is a 2-manifold.

Definition 2.58. A *cobordism from M to M'* is an oriented manifold W with a partition of its boundary into disjoint manifolds $\partial W = \overline{M} \sqcup M'$ (so M' has the outward orientation, and M the inward orientation, with respect to W). We will denote a cobordism by $W : M \to M'$.

We will be considering 4-manifolds with corners as part of our extended TQFT. In particular, we will have to consider cobordisms between 3-manifolds with boundary. The following is a variation of a "cobordism with boundary" (as in [RS1972, Chapter 6]); Yetter calls them "cobordism with corners" [Yet1997].

Definition 2.59. A *cornered cobordism over N from M to M'* is an oriented manifold W with $\partial W = \overline{M} \cup_N M'$, $\overline{M} \cap M' = N = \partial M = \partial M'$.

We may regard a cobordism as a cornered cobordism over the empty manifold.

Definition 2.60. Let M be a manifold (with boundary N). The *identity cobordisms on M*, denoted id_M, is the cornered cobordism $W = M \times I : M \to_{N \times \{0\}} M$.

Definition 2.61. Given cornered cobordisms $W : M \to_N M'$, $W : M' \to_N M''$, their *composition*, denoted $W' \circ W$, is the cornered cobordism

$$W' \circ W = W' \cup_{M'} W : M \to_N M''$$

We also consider a slightly variation of composition, but first:

Definition 2.62. Let $W : M_0 \to_N M_1$ be a cornered cobordism, and let $M : N \to N'$ be a cobordism. The cornered cobordism *obtained from W by extending along M*, denoted $W_{M:N\to N'}$, is a cornered cobordism $M_0 \cup_N M \to_{N'} M_1 \cup_N M$ whose underlying manifold is obtained by gluing W to the identity cobordism on $M_1 \cup_N M$ (or equivalently, gluing W to the identity cobordism on $M_0 \cup_N M$).

Definition 2.63. Let W, W' be cornered cobordisms

$$W : M_0 \to_N M_1$$
$$W' : M_0' \to_{N'} M_1'$$

Suppose $M_1 \subset M_0'$ is a submanifold that does not meet N'. Then $M_0' \backslash M_1$ is a cobordism $N \to N'$. The *extended composition* of W and W' is

$$W' \tilde{\circ} W := W' \circ W_{M_0' \backslash M_1 : N \to N'} : M_0 \to_{N'} M_1' \cup (M_0' \backslash M_1)$$

Similarly, if $M_0' \subset M_1$ is a submanifold that does not meet N, then $M_1 \backslash M_0'$ is a cobordism $N' \to N$, and we define

$$W' \tilde{\circ} W := W'_{M_1 \backslash M_0' : N' \to N} \circ W : M_0 \cup (M_1 \backslash M_0') \to_N M_1'$$

The simplest cobordisms, besides the *identity cobordisms*, arise from attaching balls to M in a particular manner, known as *handles*. Here we present handles as cornered cobordisms:

Definition 2.64. Fix a dimension n, and let $0 \le k \le n$. The *n-dimensional k-handle*, denoted $\mathcal{H}_k^n$, or simply $\mathcal{H}_k$, is the cornered cobordism

$$\mathcal{H}_k = B^k \times B^{n-k} : \partial B^k \times B^{n-k} \to_{\partial B^k \times \partial B^{n-k}} B^k \times \partial B^{n-k}$$

Note that $\mathcal{H}_k$ and $\mathcal{H}_{n-k}$ are the same as manifolds with corners.

Some useful terminology associated to handles are as follows:

- *core*: $B^k \times \{0\}$; the handle $\mathcal{H}_k$ should be regarded as a thickening of the core,
- *co-core*: $\{0\} \times B^{n-k}$,
- *attaching sphere*: $\partial B^k \times \{0\}$; this will make more sense in the context of attaching handles,
- *attaching region*: $\partial B^k \times B^{n-k}$; again, this will make more sense in the context of attaching handles,
- *belt sphere*: $\{0\} \times \partial B^{n-k}$.

Definition 2.65. Let $W : M_0 \to_N M_1$ be a cornered cobordism. We say a cornered cobordisms $W' : M_0 \to_N M_1'$ is *obtained from W by attaching a k-handle* if $W' = W \cup_f \mathcal{H}_k$, where $f : \overline{\partial B^k \times B^{n-k}} \hookrightarrow M_1'$ is an embedding, called the *attaching map*, such that its image is disjoint from N. We call the image of the attaching sphere/region the *attaching sphere/region of f*.

Definition 2.66. An *elementary cornered cobordism of index k* is a cobordism obtained from an identity cobordism by attaching a k-handle. In other words, it is an extension of a k-handle by some cobordism.

Lemma 2.67. *Given an elementary cornered cobordism $W : M \to_N M'$ of index k, the* dual elementary cornered cobordism *is obtained by reversing the direction of W by reversing the orientations of M, M', so that we have*

$$W : \overline{M'} \to_{\overline{N}} \overline{M}$$

This is an elementary cornered cobordism of index $n - k$.

Proof. It is easy to check that if f is the attaching map of the k-handle to M, and f' is the inclusion map of the belt sphere into M', then

$$W = \mathcal{H}_k \cup_f \mathrm{id}_M = \mathrm{id}_{M'} \cup_{M'} \mathcal{H}_k \cup_f \mathrm{id}_M = \mathrm{id}_{M'} \cup_{f'} \mathcal{H}_k$$

∎

Definition 2.68. A *handle decomposition* of a cornered cobordism $W : M \to_N M'$ is an identification of W with a composition of elementary cornered cobordisms.

Definition 2.69. Given a handle decomposition $W = W_l \circ \cdots \circ W_1 : M \to_N M'$, the *dual handle decomposition* is the handle decomposition $W = W_1^* \circ \cdots \circ W_l^* : \overline{M'} \to_{\overline{N}} \overline{M}$, where W_i^* is the dual elementary cobordism from Lemma 2.67.

Proposition 2.70. *Handle decompositions exist for any cornered cobordism.*

Proof. In broad strokes, for a cornered cobordism $W : M \to_N M'$, take a triangulation of W, then each k-simplex not in M defines a k-handle (with the k-simplex as its core). Details can be found in [Bry2002, Section 6]. ☐

Definition 2.71. Let W_i, W_{i+1} be two successive elementary cornered cobordisms in a handle decomposition. Let $W_i = \mathcal{H}_k \widetilde{\circ} \mathrm{id}_{M_i} : M_i \to_{N_i} M_{i+1}$, and $W_{i+1} = \mathrm{id}_{M_{i+1}} \widetilde{\circ} \mathcal{H}_{k'} : M_{i+1} \to_{N_{i+1}} M_{i+2}$. Let f be the attaching map for $\mathcal{H}_{k'}$. There are several ways to modify a handle decomposition:

- reordering: if f misses $\mathcal{H}_k$, we may swap W_i and W_{i+1}. This can be arranged if $k' \leq k$ by general position and transversality arguments (isotope f to be transversal to the co-core of $\mathcal{H}_k$, so by dimensional considerations, the attaching sphere for f is disjoint from it; then apply an isotopy to "push away" from the co-core).
- handle slide, or handle addition/subtraction: when $k = k'$, arrange so f misses $\mathcal{H}_k$ as above, then isotope f by dragging a part of the attaching sphere through the core of $\mathcal{H}_k$, so that, if S, S' are the attaching spheres of $\mathcal{H}_k$ and $\mathcal{H}_{k'}$, then the new attaching sphere for $\mathcal{H}_{k'}$ is $S \# S'$.
- handle pair cancellation/creation: if $k' = k + 1$ and the attaching sphere of f intersects the co-core of $\mathcal{H}_k$ transversally, then $W_{i+1} \circ W_i$ is simply an identity cobordism, and we may eliminate this pair from the handle decomposition. (Pair creation is just the inverse operation of adding such a canceling pair).

Once again, we refer the reader to the reference texts for details.

Proposition 2.72. *Any two handle decompositions of a given cornered cobordism of dimension ≤ 6 are related by a sequence of modifications as in Definition 2.71.*

Proof. While we do rely on this proposition for a proof of Proposition 5.13, that result is also proved by a different method in Theorem 5.26. Thus, we merely outline a sketch of the proof of this proposition here, leaving the reader to details.

This is a well-known result in the smooth category, and is one of the main results of Cerf theory. The general outline of the proof goes as follows. A handle decomposition determines a Morse function on the cobordism $W \to \mathbb{R}$, i.e. a smooth function whose critical points are all non-degenerate. The two handle decompositions yield Morse functions f_0, f_1. For a generic smooth family $\{f_t\}$ interpolating them, f_t is either a Morse function or has "birth-death" degenerate critical points. Between f_t of the birth-death type, the f_t's lead to the same handle decomposition, and each passing of a birth-death type, the handle decomposition undergoes a modification of the type discussed in Definition 2.71. See for example [Mil1965] for details.

Now we transfer the smooth result to the PL category. Apply the unique smoothing of W, and embed it smoothly in some Euclidean space $\iota : W \to E^k$. Add an extra coordinate z to the Euclidean space, and modify the embedding to an isotopy $\iota_t = (\iota, f_t) : W \times I \to E^{k+1}$ such that the z coordinate is f_t. Then apply a PL approximation by choosing a dense (in the colloquial sense) set of points on $\iota_t(W \times I)$ as vertices and triangulate accordingly (see [Cai1961]). ☐

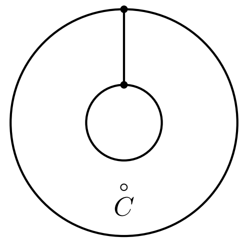


Figure 1. Example of generalized 2-cell [Kir2012, Figure 1 (a)]

3. Crane-Yetter Invariant for PLCW Decompositions

Recall that the Crane-Yetter invariant for 4-manifolds, as defined in [CKY1997], is an invariant constructed using a triangulation on the 4-manifold. We give a definition using a more general cellular decomposition, here called a "PLCW decomposition", and prove that they agree. The methods and exposition presented here closely mirror [BK], where they define the Turaev-Viro invariant using a polytope decomposition of a 3-manifold. We also draw heavily from [Kir2012], where PLCW decomposition is defined.

In what follows, the word "manifold" denotes a compact, oriented, piecewise-linear (PL) manifold; unless otherwise noted, we assume that it has no boundary. Similarly, all maps are assumed to be piecewise-linear unless otherwise specified.

3.1. **PLCW Decompositions.**

Most of the definitions and results here are taken from [Kir2012]; the main new additions are on "restricted elementary moves", "collapsible pairs", and Theorem 3.18 on relating relative PLCW decompositions via elementary moves.

Recall that a cellular decomposition of a topological manifold M is a collection of inclusion maps $\mathbb{D}^d \to M$, where $\mathbb{D}^d$ is the (open) d-dimensional ball, satisfying certain conditions. Equivalently, we can replace d-dimensional balls with d-dimensional cubes $(0,1)^d$.

We denote the n-dimensional (PL) ball by $B^n = [-1,1]^n$ for $n > 0$, and $B^0 = \{*\}$.

Definition 3.1 ([Kir2012, Definition 3.3])**.** Let M be a PL manifold possibly with boundary. A *generalized n-cell* in M is a subset $C \subseteq M$ together with a map $\varphi : B^n \to C \subseteq M$, such that the restriction of φ to the interior is a homeomorphism onto its image; φ is called the *characteristic map*. The *interior* $\mathring{C}$ of the generalized n-cell (C, φ) is the image $\varphi((-1,1)^n) \subseteq M$.

By abuse of notation, we sometimes refer to C as the generalized n-cell, which can be justified by the following lemma:

Proposition 3.2 ([Kir2012, Theorem 3.4])**.** *The characteristic map of a generalized n-cell is unique up to a homeomorphism of B^n; more precisely, given two generalized n-cells (C,φ), (C,φ') with the same image, there exists a PL homeomorphism $\psi : B^n \to B^n$ such that $\varphi' = \varphi \circ \psi$.*

Thus, we will consider generalized cells up to such equivalences, although we will often pick a characteristic map arbitrarily for concreteness.

Definition 3.3 ([Kir2012, Definition 4.1])**.** A *generalized cell complex structure* on a PL manifold M is a finite collection $\{(C, \varphi_C)\}$ of generalized cells in M such that

- $\mathring{C} \cap \mathring{C}' = \varnothing$ for distinct cells,
- $\bigcup C = M$.

The k-skeleton $\mathrm{sk}^k(M)$ is the union of all k-cells.

Definition 3.4 ([Kir2012, Definition 4.3])**.** Let M, M' be manifolds with generalized cell complex structures $\mathcal{M}, \mathcal{M}'$. A map $f : M \to M'$ is a *cellular map* if for each cell $(C, \varphi_C) \in \mathcal{M}$, the composition $f \circ \varphi_C$ is a characteristic map of some cell C' of $\mathcal{M}'$.

Definition 3.5 ([Kir2012, Definition 5.1])**.** A generalized cell complex structure on an n-manifold M is a *PLCW decomposition* or a *PLCW structure* if for each k-cell (C, φ) of M with $k > 0$, there exists a generalized cell complex structure $\mathcal{B}_\varphi$ on ∂B^k such that the restriction $\varphi|_{\partial B^k}$ is a cellular map.

We call a manifold M endowed with a PLCW decomposition $\mathcal{M}$ a *PLCW combinatorial manifold*, or simply *PLCW manifold*; we often refer to the PLCW manifold simply by $\mathcal{M}$.

We say that two PLCW decompositions on M are equivalent if there is a homeomorphism from M to itself that identifies them.

Note that by induction on k, the generalized cell complex structure $\mathcal{B}_\varphi$ on ∂B^k is itself a PLCW decomposition, hence this definition coincides with [Kir2012, Definition 5.1]; we call it the *induced PLCW decomposition*. We also call $\mathcal{B}_\varphi \cup B^k$ the induced PLCW decomposition on the k-cell $\varphi(B^k)$.

Definition 3.6. A PLCW decomposition $\mathcal{B}$ of B^k with exactly one k-cell is called *cell-like*.

In particular, the induced PLCW decomposition by a characteristic map is cell-like.

One convenient feature of PLCW decompositions is that products are easy to define:

Definition 3.7. Let $\mathcal{M}, \mathcal{M}'$ be two PLCW manifolds. For cells $(C, \varphi : C \to M), (C', \varphi' : C' \to M')$, we define their product cell as $(C \times C', \varphi \times \varphi' : C \times C' \to M \times M')$. It is easy to check that the product cells define a PLCW structure on $M \times M'$; we denote this structure by $\mathcal{M} \times \mathcal{M}'$.

A PLCW decomposition determines the PL structure of the underlying manifold. A natural question to ask is whether there is a simple set of modifications that can turn any PLCW decomposition into another (like the Pachner (a.k.a. bistellar) moves for triangulations). The answer is in the affirmative, given by the "elementary subdivisions" described later.

Definition 3.8 ([Kir2012, Definition 6.1])**.** Let $\mathcal{M}, \mathcal{M}'$ be PLCW decompositions of a manifold M. We say that $\mathcal{M}$ is *finer* than $\mathcal{M}'$, or is a *subdivision* of $\mathcal{M}'$, if every cell $C \in \mathcal{M}$ is contained in a cell $C' \in \mathcal{M}'$; we say $\mathcal{M}'$ is *coarser* than $\mathcal{M}$.

A subdivision of the induced PLCW decomposition on a cell yields a subdivision of the PLCW decomposition of the entire manifold. In particular, we consider the simplest possible subdivision:

Definition 3.9. Let $\mathcal{B}$ be a cell-like PLCW decomposition on B^k. An *elementary subdivision* of $\mathcal{B}$ is a PLCW decomposition on B^k that is the same as $\mathcal{B}$ on the boundary, but has exactly two k-cells, separated by exactly one $(k-1)$-cell; if F is the new $(k-1)$-cell, we say the elementary subdivision is *along* F.

Note that our definition differs slightly from [Kir2012, Section 7].

Definition 3.10. An elementary subdivision of a cell-like ball along F is *restricted* if F is embedded.

Clearly, such a subdivision on a cell in some PLCW decomposition induces a subdivision of the PLCW decomposition itself:

Definition 3.11 ([Kir2012, Lemma 7.1, Definition 7.2])**.** Let $\mathcal{M}$ be a PLCW decomposition of an n-manifold M. Let $C \in \mathcal{M}$ be an k-cell, $k > 0$. Consider a elementary subdivision of C along some $(k-1)$-cell F, say, $C \to \partial C \cup \{C_1, C_2, F\}$. Then this defines an *induced elementary subdivision of* $\mathcal{M}$ *along* F,

$$\mathcal{M} \to_e \mathcal{M}' := (\mathcal{M} \backslash C) \cup \{C_1, C_2, F\}$$

We say that $\mathcal{M}'$ is obtained from $\mathcal{M}$ by performing a *elementary move* if one is a elementary subdivision of the other, and denote it by

$$\mathcal{M} \leftrightarrow_e \mathcal{M}'$$

Lemma 3.12. *Bistellar (Pachner) moves can be obtained by sequence of elementary moves. (see* [Kir2012, Section 6, 8]*).*

It is not so easy to "undo" an elementary subdivision, e.g. after performing an elementary subdivisions along some k-cell, one might perform another subdivision that attaches a $(k+1)$-cell along the new k-cell, then one can no longer "undo" the first elementary subdivision. The following operation is in some sense a way to "undo" an elementary subdivision:

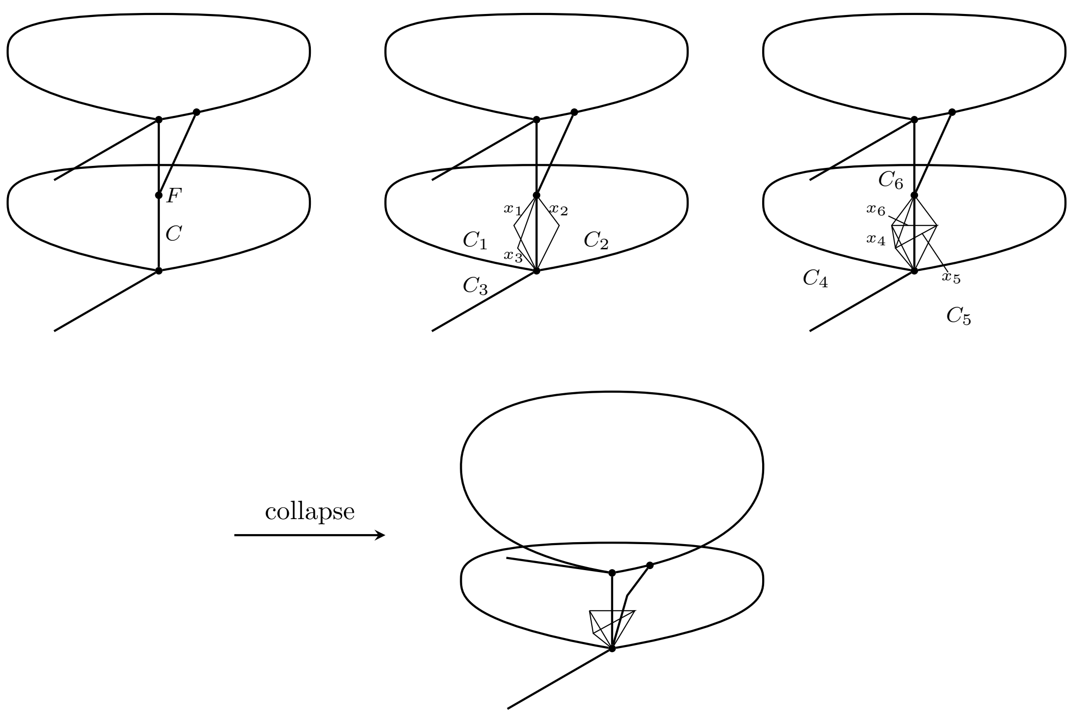
Definition 3.13. Let C be a k-cell in some PLCW manifold $\mathcal{M}$, and consider a $(k-1)$-cell F in its boundary, $k \geq 1$. Suppose F is not identified with any other boundary $(k-1)$-cell of C; that is, C only meets the interior of F once. Moreover, suppose the characteristic map of F is an embedding (not just on its interior). Then ∂F separates ∂C into two disks, $\partial C = F \cup_{\partial F} D$. We say the pair (C, F) is *collapsible.*

Arrange the cells that have C in its boundary in increasing dimension, $C_1, C_2, \ldots, C_m$ (beginning with dimension $k+1$); cells that have C in its boundary multiple times should appear once for each time. Let $X = \{C\}$; we iteratively add elements to X, each element will be a subspace of some C_i, with one such element for each C_i. From $i = 1$ to m, consider the union Y of elements $y \in X$ that are in the boundary of C_i, then consider a point $p \in C_i$ that is "close" to Y (say in some collar neighborhood of ∂C_i), and add the cone of p over Y, denoted x_i, to X. Thus we have $|X| = m+1$ elements.

Let $f : F \to D$ be some homeomorphism that fixes ∂F. Remove the interior of F from F, and remove the interior of x_i from C_i. We patch them up by apply f to glue F to D, then the cone over f in C_1 will glue the cone over F to the cone over D, and so on, iteratively performing cones over maps in the boundary. Now we have a new PLCW decomposition with all the same cells but F and C.

We call this operation *collapsing* (C, F). (Diagram below depicts $k = 1$.)

(3.1)



In particular, if F, C_1 are such that one could have obtained them by performing a *restricted* elementary subdivision of a cell C along F, splitting C into C_1, C_2, then we may collapse C_1 (or C_2) by F.

Lemma 3.14. *Let F restricted-elementarily subdivide C of $\mathcal{M}$ into C_1, C_2 of $\mathcal{M}'$. Then (C_1, F), (C_2, F) are collapsible pairs, and performing the collapsing operation on either one will return $\mathcal{M}'$ to $\mathcal{M}$.*

Proof. Obvious. □

Lemma 3.15. *Let (C, F) be a collapsible pair in $\mathcal{M}$. Consider an elementary move that does not affect C nor F (that is, neither C nor F is removed from $\mathcal{M}$ when performing this elementary move). Then the operation of collapsing (C, F) and the elementary move commute.*

Proof. Left as an exercise to the reader. □

Theorem 3.16 ([Kir2012, Theorem 8.1])**.** *Let $\mathcal{M}, \mathcal{M}'$ be two PLCW decompositions of an n-manifold M. Then there exists a finite sequence of elementary moves that take $\mathcal{M}$ to $\mathcal{M}'$.*

Proof. We simply sketch a proof, referring the reader to [Kir2012] for details. It suffices to show that there is a sequence of elementary moves that take any PLCW decomposition to a triangulation; the result then follows from Pachner's theorem [Pac1987], [Pac1991], which says that any two triangulations are related by bistellar moves, and Lemma 3.12.

One shows that one can perform, on any cell, a "radial subdivision" by a sequence of elementary moves; a radial subdivision of a cell-like ball is the cone over the boundary. Apply radial subdivisions on cells in order of increasing dimension, and after enough steps, we arrive at a triangulation. □

In order to prove a relative version of the result above, we need to strengthen the result about elementary moves to triangulations:

Lemma 3.17. *For $n \leq 3$, $\mathcal{M}$ a PLCW n-manifold, there exists a sequence of restricted elementary subdivisions that takes $\mathcal{M}$ to a triangulation; more precisely, there exists PLCW decompositions $\mathcal{M}_0 = \mathcal{M}, \mathcal{M}_1, \ldots, \mathcal{M}_k$ such that $\mathcal{M}_{i+1}$ is an elementary subdivision of $\mathcal{M}_i$, and $\mathcal{M}_k$ is a triangulation.*

Proof. For $n = 0, 1$ there is nothing to prove, and for $n = 2$ it is easy, simply subdivide each 1-cell many times, then perform a radial subdivision on every 2-cell.

Let $n = 3$. We can easily perform elementary subdivisions to make the 2-skeleton of $\mathcal{M}$ a simplicial complex. In particular, each 3-cell is now a cell-like ball whose boundary is a triangulated 2-sphere.

Claim: a cell-like 3-ball whose boundary is a triangulated 2-sphere can be converted to a triangulation by elementary subdivisions along interior cells.

We proceed by induction on the number k of triangles in the boundary of such a 3-cell C. The base case is $k = 4$, where C must be a tetrahedron and there is nothing to be done. So assume $k > 4$, and select a vertex v_0 arbitrarily; v_0 meets l triangles $T_1, \ldots, T_l$ (note that $3 < l < k$).

Apply an elementary subdivision to split C, separating all the T_i from the triangles not meeting v_0, then apply $l - 3$ elementary subdivisions (i.e. adding diagonals) on that new 2-cell F so it is made up of $l - 2$ triangles Now we have two 3-cells like C, one with $2l - 2$ triangles (the one meeting v_0), the other with $k - 2$ triangles, in their boundaries.

The former is easy to triangulate - simply add a 2-cell for each diagonal. To apply the induction hypothesis to the latter, we need to ensure that the resulting boundary is still a triangulation. It is a simple exercise to show that there exists a choice of triangulation of F that does this. Thus we are done.

(Note that all the elementary subdivisions performed here are restricted.) □

Theorem 3.18. *Let the boundary $N = \partial M$ of an n-manifold M have a PLCW decomposition $\mathcal{N}$, with $n \leq 4$. Then $\mathcal{N}$ can be extended into a PLCW decomposition for M, and this extension is unique up to elementary subdivisions of interior cells; more precisely, if $\mathcal{M}, \mathcal{M}'$ are two PLCW decompositions of M that agree on ∂M, then $\mathcal{M}, \mathcal{M}'$ are related by a sequence of elementary subdivisions as in Definition 3.11, where only interior cells are used for subdivision.*

Proof. The existence result is known for when $\mathcal{N}$ is a triangulation [Cas1995]. To prove the existence result for general PLCW $\mathcal{N}$, by Lemma 3.17, we may "induct" on PLCW decompositions, with "base case" $\mathcal{N}$ being triangulations, and "inductive steps" being applying inverse elementary subdivisions.

Suppose $\mathcal{N}'$ is a PLCW decomposition that can be extended to a PLCW decomposition on M. Consider $\mathcal{N}$ such that $\mathcal{N}'$ is an elementary subdivision of $\mathcal{N}$ along a cell C. By [Bry2002, Corollary 2.5], N has a collar neighborhood, i.e. there is a PL-embedding $N \times [0,1] \hookrightarrow M$ such that $N \times 0$ is sent to ∂M. Thus we may extend $\mathcal{N}$ to the collar neighborhood by using the product PLCW decomposition $\mathcal{N} \times [0,1]$. Apply the elementary subdivision to the copy of C in $\mathcal{N} \times 1$ so that $N \times 1$ now has the PLCW decomposition $\mathcal{N}'$. Now $M \backslash N \times [0,1] \simeq M$ as PL-manifolds, so by our "inductive hypothesis", we may extend the PLCW decomposition to the rest of M.

Now we show uniqueness. Once again we employ a similar "inductive" strategy, beginning with the "base case" of triangulations. For $\mathcal{N}$ a triangulation, uniqueness follows from [Cas1995], where it is shown that two triangulations that agree on the boundary are related by Pachner moves (which can be converted into a sequence of elementary moves involving interior cells only).

Now suppose, as before, a PLCW decomposition $\mathcal{N}'$ that has unique extensions to M up to interior elementary moves, and consider $\mathcal{N}$ such that $\mathcal{N}'$ is a *restricted* elementary subdivision of $\mathcal{N}$ along a cell F, splitting the cell C into C_1, C_2. Consider two extensions $\mathcal{M}, \mathcal{M}'$ of $\mathcal{N}$ to the interior of M. By the "inductive hypothesis", if we perform the restricted elementary subdivision to $\mathcal{N}$ and turn it into $\mathcal{N}'$, then there is a sequence of elementary moves that takes $\widetilde{\mathcal{M}}$ to $\widetilde{\mathcal{M}'}$ (obtained from $\mathcal{M}, \mathcal{M}'$ by the same subdivision), say $\widetilde{\mathcal{M}} = \mathcal{M}_0 \leftrightarrow_e \mathcal{M}_1 \leftrightarrow_e \dots \leftrightarrow_e \mathcal{M}_l = \widetilde{\mathcal{M}'}$. Now collapse (C_1, F) in each of $\mathcal{M}_i$; by Lemma 3.15, we have a sequence of elementary moves from $\mathcal{M}$ to $\mathcal{M}'$ that does not affect the boundary. □

Proposition 3.19. *Let $\mathcal{M}$ be a PLCW manifold, possibly with boundary, with underlying manifold M of dimension $n \le 4$, and let C be a top dimensional cell. A* single-cell subdivision *at C is a subdivison $\mathcal{M}'$ of $\mathcal{M}$ that keeps all cells except C fixed; as with Definition 3.11, we say that $\mathcal{M}$ and $\mathcal{M}'$ are related by a* single-cell move*. (In particular, it leaves the boundary fixed.)*

Then any two PLCW decompositions of M are related by a sequence of single-cell moves.

Proof. It suffices to check that an elementary subdivision of a k-cell can be achieved as a sequence of single-cell moves. It is clear that for $k = n$, an elementary subdivision is a special type of single-cell subdivision. Now, we induct on k from n down to 1. Let C be a k-cell that we wish to subdivide. For every l-cell containing C in its boundary, apply elementary subdivision to "cordon" off C. Let $\mathcal{M}''$ be this new PLCW decomposition. Then the union of the new cells that meet C is an n-ball, and applying an inverse single-cell subdivision, and then applying a single-cell subdivision that basically returns to $\mathcal{M}''$, except that C is subdivision. (This is essentially the same trick as in Figure 4. □

3.2. **Colored Ribbon Graphs.**

Next, we discuss (co-)ribbon graphs embedded in 3-manifolds, which is adapted from [RT1990], [RT1991]. Note that they work in the smooth category while we are in PL, but since these are equivalent in low dimensions, this should not be a problem.

Definition 3.20 ([RT1990, 4.1])**.** Let M be a PL 3-manifold. A *band* is the image of B^2 under a locally unknotted embedding into M; the (images of the) segments $[-1,1] \times \{-1\}$ and $[-1,1] \times \{1\}$ are its *bases*, and $\{0\} \times [-1,1]$ is its *core*.

An *annulus* is the image of $B^1 \times S^1$ under a locally unknotted embedding into M; the (image of) $\{0\} \times S^1$ is its *core*.

A band or annulus is *directed* if its core is oriented; we name the bases *incoming* and *outgoing* so that the orientation of the core points from the former to the latter. ([RT1990] calls them initial and final.)

The following is essentially [RT1990, 4.2], where they have $M = \mathbb{R}^2 \times [0,1]$, and ribbon graphs must meet the top and bottom boundary in particular positions.

Definition 3.21. A *ribbon graph (resp. co-ribbon graph)* in a PL 3-manifold M is an embedded oriented (resp. co-oriented) surface S that is presented as a union of a finite collection of bands and annuli, and each band is assigned a type of either *ribbon* or *coupon* (annuli are considered ribbons), such that

- bands of the same type are pairwise disjoint;
- a ribbon and coupon may only meet along their bases, in which case the base of a ribbon is entirely contained in the base of the coupon;
- coupons are disjoint from the boundary ∂M;
- a ribbon may only meet ∂M along its bases, in which case the base is entirely contained in ∂M.

A *directed* (co-)ribbon graph is one in which the core of each band is oriented.

The *boundary* of a ribbon graph Γ, denoted $\partial\Gamma$, is the intersection of Γ with the boundary ∂M, $\partial\Gamma = \Gamma \cap \partial M$ (not the boundary of the surface S); it is a collection of (co-)oriented arcs in ∂M.

Remark 3.22. Note also that we also consider *co-oriented* graphs in addition to the usual oriented ribbon graph (as is done in [RT1990]); it is merely a convenient way of assigning an orientation to the surface which changes when the orientation of the ambient 3-manifold changes. Of course, when the ambient 3-manifold is oriented, orientation and co-orientation of a surface are equivalent notions; we use the convention where, if $\hat{n}$ defines a co-orientation, and $(\hat{x}, \hat{y})$ defines an orientation, then $(\hat{n}, \hat{x}, \hat{y})$ defines the ambient orientation.

We will *never* color a co-oriented ribbon graph, unless the ambient manifold is oriented (so that, by the discussion above, the graph is automatically oriented).

Later, we will see that essentially only a circular ordering, rather than a linear ordering, on the ribbons attached to a coupon is needed (see Remark 3.43), so coupons may be drawn as a disk, without notion of a core or bases.

Definition 3.23. An *isotopy* of (co-)ribbon graphs is a family $\{\Gamma_t\}_{t\in[0,1]}$ of (co-)ribbon graphs that can be presented as a union of isotopies of bands. More precisely, there is a collection of isotopies of bands $\{\psi_t^j : B^2 \to M\}_{t\in[0,1]}$, $j = 1, 2, \ldots, n$, such that $\bigcup_j \psi_t^j(B^2) = \Gamma_t$ for each t. An isotopy does not necessarily fix the boundary.

For a (co-)ribbon graph in a PLCW 3-manifold $\mathcal{M}$ with boundary, we typically assume that the graph meets the faces of $\mathcal{M}$ in "general position"; more precisely:

Definition 3.24. We say that a (co-)ribbon graph Γ intersects a (PLCW) surface N *transversally* if

- the coupons of Γ do not meet N,
- Γ only meets N in the interior (of 2-cells),
- the ribbons of Γ transversally intersect N in arcs, and each arc crosses every para-core of the ribbon exactly once, where a para-core is the image of $\{*\} \times [-1, 1]$ parallel to the core.

If the third condition is relaxed to the core being transverse to N, along with a small neighborhood of the core being transverse to N also, then we say Γ is *weakly transverse* to N.

Note that being transversal as a graph is stronger than having the surface of the graph S be transversal to the other surface N (in the sense of Definition 2.55). For example, S may intersect N in "unnecessary" circle components. However, the lemma below shows that one simply needs to "trim the fat".

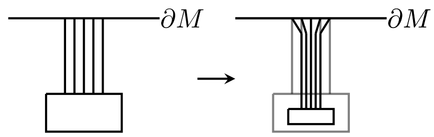
Definition 3.25. A *narrowing* of a (co-)ribbon graph Γ^0 in M is an isotopy $\Gamma^t = \Phi^t(\Gamma^0)$ such that

- $\Gamma^t \subset \Gamma^s$ for $t > s$,
- coupons are sent into themselves,
- Φ^t "respects the para-core foliation" of ribbons away from the boundary $\partial\Gamma^0$, i.e., for a ribbon $e : [-1, 1] \times [-1, 1] \to M$ of Γ^0, the only ribbon of Γ^0 that $e^t = \Phi^t(e)$ meets is e itself, and the para-cores match up except near ∂M (see (3.2)).

We also say that Γ^1 is a narrowing of Γ^0.

If the narrowing preserves the core of ribbons, then we say it is a *strict narrowing.*

(3.2)



Lemma 3.26. *Suppose the surface S of a (co-)ribbon graph Γ intersects another surface N in M transversally (see Definition 2.55), and N does not meet $\partial\Gamma$. Then there exists a narrowing Γ' of Γ that intersects N transversally (as a ribbon graph).*

Moreover, if Γ is weakly *transverse to N, then there exists a* strict *narrowing Γ' of Γ that intersects N transversally (as a ribbon graph).*

Note that we can always apply a small isotopy to make the surfaces transversal, so a simple corollary is that we can isotope a ribbon graph to be transverse to a surface.

Proof. By isotoping a coupon into itself, making it very small (with respect to some C^0 metric), and dragging attached ribbons along with it, we can make coupons avoid N.

Now consider a ribbon $e : [-1, 1] \times [-1, 1] \to M$ of the graph. The projection to the first factor (whose fibers are parallel to the core) defines a function $e \cap N \subset e \to [-1, 1]$. Take a regular value $x \neq -1, 1$, and perform a narrowing so that e is squeezed into a sufficiently small neighborhood of $\{x\} \times [-1, 1]$ (except at the bases of e, which may need to be fixed say if the base is on the boundary of M). Then it is clear that the graph is now transversal to N.

The weakly transverse condition essentially allows us to take the regular value to be $x = 0$, so the argument above applies, where we may use strict narrowings instead. □

It is also clear that narrowings preserve transversality (i.e. a narrowing of a graph that is already transverse to N is also transverse to N) if N is sufficiently far from the boundary. Furthermore, strict narrowings are essentially closed under intersection (with perhaps some inconsequential differences near the boundary).

Remark 3.27. From the discussions and lemmas above, we may assume that coupons are small enough and ribbons are always narrow enough. Indeed, it is possible to rewrite the definition of ribbon graphs in terms of "infinitesimal ribbon graphs", where we have a typical graph Γ of vertices and edges, but it comes with the germ of a surface $\mathcal{S}(\Gamma)$, i.e. a maximal collection of surfaces $\{S_\alpha\}$ such that each S_α includes Γ in its interior, the collection is closed under finite intersections. Similarly, markings (as discussed in the next section) can be taken to be a point with a germ of an arc.

Thus, in effect, we can just focus on the core of a ribbon when dealing with transversality.

3.2.1. *Co-Ribbon Graphs in PLCW Decompositions.* To define the Crane-Yetter invariant for PLCW decompositions, we will need to consider co-ribbon graphs in 3-cells of a particularly simple kind, which we call an *anchor*.

Definition 3.28. A *marking* of a 2-cell F in a PLCW decomposition is a co-oriented embedded line segment in the interior of F, $[0,1] \hookrightarrow \mathring{F}$. A *marked PLCW decomposition* is a PLCW decomposition with a choice of marking on each 2-cell.

Definition 3.29. An *anchor* of a cell-like PLCW 3-ball $(B^3, \mathcal{B})$ is a co-ribbon graph ψ that meets each 2-face of $\mathcal{B}$ at a marking, and is isotopic (not fixing the boundary) to the following co-ribbon graph, which we refer to as the *standard anchor*:

- exactly one coupon $[0, 0.5] \times [0, 0.5] \times 0$,
- a ribbon $[\frac{2i}{4k}, \frac{2i+1}{4k}] \times [-1, 0] \times 0$ for each $i = 0, 1, \ldots, k-1$, where k is the number of 2-cells of $\mathcal{B}$.

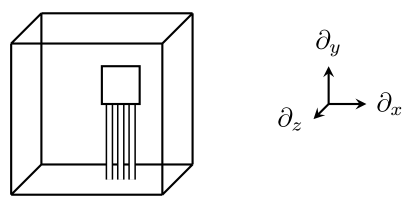


FIGURE 2. Standard anchor

An anchor endows the ribbons with a natural "monotonic" structure, in that there is a well-defined notion of one ribbon being in between two ribbon, but there is no canonical choice of ordering.

When B^3 is *oriented*, an anchor also defines an *ordering* on the ribbons, and hence the faces, as follows: let p be the point of intersection of the core of the coupon and the base meeting the ribbons; if u, v, w are vectors at p with

- u tangent to the base of the coupon, pointing in increasing ordering of the ribbons,
- v is tangent to the core of the coupon, pointing inwards,
- w is a vector giving the co-orientation,

then (u, v, w) gives the ambient orientation of B^3. (See Remark 2.57 on orientations in PL manifolds). In other words, if $B^3 \subset \mathbb{R}^3$ is given the standard orientation $(\partial_x, \partial_y, \partial_z)$, and we have isotoped the co-ribbon graph into the standard anchor, we have $u = \partial_x$, $v = \partial_y$, $w = \partial_z$, so the ordering of ribbons is from low to high x-value.

Two anchors are *equivalent* if they are isotopic fixing the boundary. △

The main point of an anchor is to make the identification of the boundary sphere ∂B^3 with some standard S^2 explicit and concrete. Such a graph is essentially "radial", having no knotting or linking. More precisely, if we thicken the coupon into a ball, the ribbons of an anchor should form a k-strand braid in $S^2 \times [0,1]$; this connection is pursued later.

Note that changing the orientation of B^3 flips the ordering on the faces.

It is clear that an anchor Γ is determined up to equivalence by the movement of its boundary; that is, if Φ_t is an isotopy from the standard anchor to the anchor Γ, and $s_i = [\frac{2i}{4k}, \frac{2i+1}{4k}] \times -1 \times 0$ is the i-th boundary segment of the standard anchor, then $\Phi_t|_{\{s_i\}}$, the isotopy restricted to s_i's, determines Γ up to equivalence. Moreover, if Φ'_t is another isotopy such that $\Phi'_t|_{\{s_i\}}$ is isotopic to $\Phi_t|_{\{s_i\}}$, then Φ'_t also determines an equivalent anchor.

Intuitively, if we imagine each face has a special marked point, then an anchor is essentially equivalent to a choice of how to move the special marked points into a line; this choice is not just an ordering on the faces, but also involves braid groups, as we discuss below.

Let $\mathfrak{F}_n$ be the *framed braid group* as defined in [KS1992], namely, $\mathfrak{F}_n$ is a semi-direct product $\mathbb{Z}^n \rtimes \mathfrak{B}_n$, where $\mathfrak{B}_n$ is the braid group on n points. The braid group naturally projects onto the symmetric group on n elements, $S_n = \mathrm{Aut}(\{1,\dots,n\})$, which gives a right action of $\mathfrak{B}_n$ on $\mathbb{Z}^n = \mathbb{Z}^{\{1,\dots,n\}}$, given by precomposition: $(r_1,\dots,r_n)\cdot\sigma = (r_{\sigma(1)},\dots,r_{\sigma(n)})$. $\mathfrak{F}_n$ is generated by σ_i, $i = \sigma_i, i = 1,2,\dots,n-1$, which generate $\mathfrak{B}_n$, and $t_j, j = 1,2,\dots,n$, which generate $\mathbb{Z}^n$, with the following relations:

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \ , \ \ |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ t_jt_k &= t_kt_j \\ \sigma_it_j &= t_{\sigma_i(j)}\sigma_i \end{aligned} \tag{3.3}$$

For $\mathbf{r} = (r_1,\dots,r_n)$, we write $t^{\mathbf{r}} = t_1^{r_1}\cdots t_n^{r_n}$, so we have

$$\sigma t^{\mathbf{r}} = t_{\sigma(1)}^{r_1}\cdots t_{\sigma(n)}^{r_n}\sigma = t_1^{r_{\sigma^{-1}(1)}}\cdots t_n^{r_{\sigma^{-1}(n)}}\sigma = t^{\mathbf{r}\cdot\sigma^{-1}}\sigma. \tag{3.4}$$

Geometrically, an element $t^{\mathbf{r}}\sigma$ is represented by the geometric braid σ with the label r_i on the i-th strand from the left according to the top endpoint. Then the product $t^{\mathbf{s}}\tau\cdot t^{\mathbf{r}}\sigma = t^{\mathbf{s}+\mathbf{r}\cdot\tau^{-1}}\tau\sigma$ amounts to stacking τ on top of σ, and labeling each strand with the sum of the two labelings (see Figure 3).

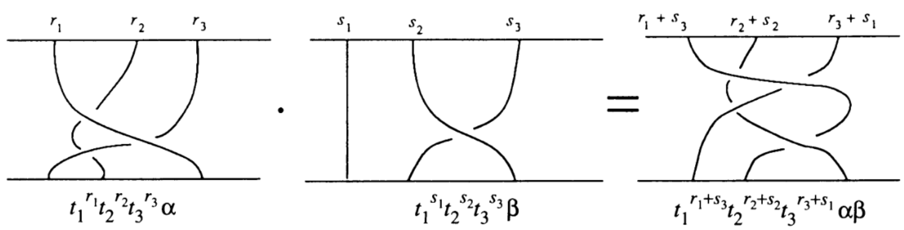
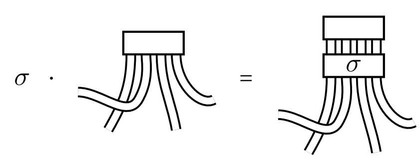


FIGURE 3. Product of framed braids; this was taken from [KS1992]

As the name of $\mathfrak{F}_n$ suggests, these labels can be interpreted as framings for the strands of the braid, or equivalently and more fittingly for this discussion, a strand labeled with r can be interpreted as a co-oriented ribbon, making r positive twists about its core relative to the blackboard framing.

Given an oriented cell-like PLCW 3-ball $(B^3,\mathcal{B})$, $\mathfrak{F}_n$ naturally acts (from the left) on the set of anchors up to equivalence, by inserting the framed braid near the coupon:

(3.5)



Lemma 3.30. *Let X be the boundary of some anchor in an oriented cell-like PLCW 3-ball $(B^3,\mathcal{B})$ (thus X is a collection of co-oriented arcs in ∂B^3, one for each 2-face of $\mathcal{B}$). Let $\mathfrak{A}_X$ be the set of anchors Γ, considered up to equivalence, such that $\partial\Gamma = X$ (as co-oriented 1-manifolds). Then $\mathfrak{F}_n$ acts transitively on $\mathfrak{A}_X$, and the stabilizer group for any anchor is the normalizer of $x = t_1^2\sigma_1\sigma_2\cdots\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1$.*

Proof. For simplicity, we may assume, without loss of generality, that X is the boundary of the standard anchor. It is clear that any isotopy of anchors can be isotoped to one that fixes the coupon pointwise, thus we may treat $\mathfrak{A}_X$ as the set of anchors with the same fixed coupon, up to isotopy of anchors that fixes the coupon and boundary pointwise.

As pointed out after Definition 3.29, upon thickening the coupon, we may identify an anchor with a framed braid in $S^2 \times [0,1]$, thus identifying $\mathfrak{A}_n$ with $\mathfrak{F}_n(S^2)$, the framed braid group on the sphere (see [BG2012]); clearly the actions of $\mathfrak{F}_n$ on $\mathfrak{A}_n$ and $\mathfrak{F}_n(S^2)$ agree. It suffices to show that $\mathfrak{F}_n(S^2) \cong \mathfrak{F}_n/\langle x\rangle$.

Let $\mathfrak{B}_n(S^2)$ be the braid group on the sphere. From [FVB1961], $\mathfrak{B}_n(S^2)$ has generators σ_i, $i = 1,\dots,n-1$, with the relations from $\mathfrak{B}_n$ and the additional $\bar{x} := \sigma_1 \cdots \sigma_{n-1}\sigma_{n-1}\cdots\sigma_1 = 1$. Thus we have the exact sequence

$$1 \to N(\bar{x}) \to \mathfrak{B}_n \to \mathfrak{B}_n(S^2) \to 1$$

where $N(\bar{x})$ is the normalizer of $\bar{x}$ in $\mathfrak{B}_n$. It is not hard to see that the natural forgetful map $\mathfrak{F}_n \to \mathfrak{B}_n$ that kills all t_i sends $N(x)$ onto $N(\bar{x})$. The exact sequence above fits in the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N(x) & \hookrightarrow & \mathfrak{F}_n & \longrightarrow & \mathfrak{F}_n(S^2) & & \\ & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow & & \\ 1 & \longrightarrow & N(\bar{x}) & \longrightarrow & \mathfrak{B}_n & \longrightarrow & \mathfrak{B}_n(S^2) & \longrightarrow & 1 \end{array}$$

Exactness of the bottom row and injectivity in top left implies the exactness of top row, thus $\mathfrak{F}_n(S^2) \cong \mathfrak{F}_n/\langle x\rangle$. □

3.2.2. *Colorings of Ribbon Graphs and Their Evaluations.* Now we consider colorings of ribbon graphs in an oriented 3-manifold (we will not need it for co-ribbon graphs).

Definition 3.31 ([RT1990, 4.4])**.** Let $\mathcal{A}$ be a premodular category. An *$\mathcal{A}$-coloring* Ψ of a ribbon graph Γ in an oriented 3-manifold M is the following data:

- Choice of an object $\Psi(\vec{e}) \in \operatorname{Obj}\mathcal{A}$ for each oriented directed ribbon $\vec{e}$, so that $\Psi(\overleftarrow{e}) = \Psi(\vec{e})^*$, where $\overleftarrow{e}$ is the oppositely directed ribbon.
- Choice of a vector $\Psi(\mathbf{c}) \in \operatorname{Hom}(\Psi(\vec{e}) \otimes \cdots \otimes \Psi(\vec{e}_k), \Psi(\vec{e}'_1) \otimes \cdots \otimes \Psi(\vec{e}'_l))$, for each directed coupon $\mathbf{c}$, where $\vec{e}_i, \vec{e}'_j$ are the incoming and outgoing edges, such that $\Psi(\overline{\mathbf{c}}) = \Psi(\mathbf{c})^*$.

We say (Γ,Ψ) is a $\mathcal{A}$-colored ribbon graph.

When the premodular category $\mathcal{A}$ is clear, we simply say coloring. We often denote (Γ,Ψ) by Γ for simplicity.

A coloring Ψ of a ribbon graph Γ defines a coloring of its boundary $\partial\Gamma$ in the following manner: for a base $b \subset \vec{e} \cap \partial M$, we assign it the object $\Psi(b) = \Psi(\vec{e})$, where $\vec{e}$ is taken with outward direction at b. (If $\vec{e}$ meets the boundary again at b', then b' is assigned $\Psi(b') = \Psi(b)^*$.) Such $\partial\Gamma$ with coloring is called a *boundary value* on ∂M; we will discuss this in more detail in Section 5.1.

Definition 3.32. An isomorphism $\Psi \simeq \Psi'$ between colorings is an isomorphism $\Psi(\vec{e}) \simeq \Psi'(\vec{e})$ for each directed ribbon which respects the dualities and intertwines the assignments to coupons.

We give two closely related notions of evaluating a colored ribbon graph, one is a "global" evaluation while the other is a "local" evaluation:

Proposition 3.33. *Given a colored ribbon graph (Γ,Ψ) in an oriented S^3, there is a well-defined* Reshetikhin-Turaev evaluation $Z_{RT}(\Gamma,\Psi) \in \mathbf{k}$ *that is invariant with respect to isotopy and isomorphism of colorings.*

Proposition 3.34. *Given a colored ribbon graph (Γ,Ψ) in an oriented ball D, and an identification of D with B^3 so that $\Gamma \cap \partial D$ is sent to $\Gamma' \cap \partial B^3$, where Γ' is the standard anchor Figure 2, there is a well-defined* Reshetikhin-Turaev evaluation $Z_{RT}((\Gamma,\Psi)) \in \operatorname{Hom}(\mathbf{1}, V_1 \otimes \cdots \otimes V_k)$*, that is invariant with respect to isotopy rel boundary and isomorphism of colorings (strictly speaking, covariant with respect to isomorphism of colorings of the ribbons meeting the boundary).*

Furthermore, for $\Gamma = \Gamma', D = B^3$, with the coupon labeled by $\varphi \in \operatorname{Hom}(\mathbf{1}, V_1 \otimes \cdots \otimes V_k)$, the Reshetikhin-Turaev evaluation is simply the labeling of the coupon, i.e. $Z_{RT}(\Gamma,\Psi) = \varphi$.

If Γ does not meet the boundary, then $Z_{RT}((\Gamma,\Psi)) \in \mathbf{k}$, with the empty graph identified with 1, and the evaluation is independent of the identification of D with B^3. (In general, the evaluation does depend on this identification.)

Thus, given a colored ribbon graph Γ in some oriented 3-manifold M, and some embedded ball D, we may speak of a "local evaluation" of Γ, and we can use this to cut down the space of colored ribbon graphs, saying two graphs are equivalent if they agree everywhere outside D, and have the same evaluation in D; this is the idea behind skeins, and we will discuss this in detail in Section 5.1.

Remark 3.35. Proposition 3.34 is often stated in terms of graphs in a thickened plane $\mathbb{R}^2 \times [0,1]$, and the graphs should have endpoints in particular positions. More precisely, treat $\mathbb{R}^2 \times [0,1]$ as a ball (missing an S^1 at its equator, which does not affect the discussion), and suppose the graph meets the boundary planes $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$ along their x-axes $\mathbb{R} \times 0 \times 0$, $\mathbb{R} \times 0 \times 1$, and the graph is always co-oriented towards the y-axis. Then under the graphical calculus, the graph defines a morphism $V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_m$, where the V's are the colors of the legs meeting $\mathbb{R}^2 \times 0$, directed inwardly, and the W's are the colors of the legs meeting $\mathbb{R}^2 \times 1$, directed outwardly.

Then the results of [RT1990] state that this morphism is invariant under isotopy of the graph. (See [BakK2001, Theorem 2.3.8].)

3.3. **The Crane-Yetter Invariant.**

The original Crane-Yetter invariant, as given in [CKY1997] (first announced in [CY1993]) is for a triangulated 4-manifold. We give a definition of the Crane-Yetter invariant for a 4-manifold with a PLCW decomposition; we prove it is equivalent to the original definition in the next section.

The constructions presented here closely mirror those of [BK], which recasts the definition of the Turaev-Viro invariant in terms of a 3-manifold with PLCW decomposition (there they call it a "polytope decomposition"). The following additional structure is meant to impose an ordering (and more) on the boundary 2-faces of a 3-cell; in one dimension lower, such a structure is not needed in the Turaev-Viro case as the orientation of a 2-cell furnishes a (cyclic) ordering on the boundary edges.

Definition 3.36. An *anchoring* of a combinatorial 3- or 4-manifold is the choice of an anchor for each 3-cell that agree on every 2-cell, i.e. there is a marking of each 2-cell such that every anchor meets a 2-cell in exactly that marking. We say that a combinatorial 4-manifold equipped with an anchoring is *anchored*.

Let $\mathcal{A}$ be a premodular category as in Section 2.1, and $\mathcal{W}$ a combinatorial 4-manifold, possibly with boundary. We denote by F^{or} the set of oriented 2-cells of $\mathcal{W}$. For an oriented 2-cell f, denote by $\overline{f}$ the same 2-cell with the opposite orientation.

Definition 3.37. A *labeling* of a combinatorial 2-, 3-, or 4-manifold $\mathcal{M}$ (possibly with boundary) is an assignment $l : F^{\mathrm{or}} \to \mathrm{Obj}\,\mathcal{C}$ such that $l(\overline{f}) \simeq l(f)^*$. A labeling is called *simple* if for every 2-cell, $l(f)$ is simple.

Two labelings l_1, l_2 are *equivalent* if $l_1(f) \simeq l_2(f)$ for every 2-cell f.

The *dual labeling* l^* assigns the dual objects: $l^*(f) = l(f)^*$.

If a labeling l is simple, we will assume that $l(f) = X_i$ (the chosen representative for isomorphism class i) for some i, so in particular we may write $l(\overline{\overline{f}}) = l(\overline{f})^* = l(f)^{**} = l(f)$.

We will often denote $d_l = \prod_f d_{l(f)}$ where the product is over all unoriented faces in the labeling l.

Definition 3.38. Let $\mathcal{W}$ be an anchored combinatorial 4-manifold with a labeling l, and $C \in F^{\mathrm{or}}$ be an oriented 3-cell. The *local state space* is the vector space

$$H(C, l, \psi_C) := \langle l(f_1), \dots, l(f_k) \rangle$$

where f_i are the faces of ∂C, with the outward orientation, taken in the order imposed by the anchor, and ψ_C is the anchor for C. We refer its elements as *local states*.

The definition of local state space depends on a choice of anchor, but different choices of anchors give local state spaces that are clearly isomorphic, and in fact canonically so:

Definition 3.39. Let C be as in Definition 3.38, and let ψ, ψ' be two anchors for C; by Lemma 3.30, $\psi' = \sigma \cdot \psi$ for some framed braid $\sigma \in \mathfrak{F}_n$. Define

$$f_\sigma : H(C, l, \psi) \simeq H(C, l, \psi')$$
$$\varphi \mapsto \sigma^{-1} \circ \varphi$$

to be the map obtained from interpreting the braid σ^{-1} in the graphical calculus. (Note it is "contravariant" in the sense that $f_{\tau\sigma} = f_\sigma f_\tau$.)

Observe that $f_{\tau\sigma} = f_\tau \circ f_\sigma$, and if $\sigma \in \mathfrak{F}_n$ fixes an anchor ψ, i.e. $\sigma \cdot \psi = \psi$, then clearly $f_\sigma = \mathrm{id}$. Thus, the isomorphisms $H(C,l,\psi) \simeq H(C,l,\psi')$ are canonical in the sense that it does not depend on the choice of σ that relates ψ and ψ'.

Definition 3.40. Let $\mathcal{M}$ be an anchored oriented combinatorial 3-manifold with a labeling l, with anchor ψ_C for 3-cell C. We define the *pre-state space of* $(\mathcal{M}, l, \psi_C)$ as the vector space

$$H(\mathcal{M}, l, \{\psi_C\}) := \bigotimes_C H(C, l, \psi_C)$$

where the tensor product is over all 3-cells C of $\mathcal{M}$; $H(\mathcal{M}, l, \{\psi_C\})$ transforms functorially with change of anchors, as in Definition 3.39, so we write $H(\mathcal{M}, l)$ for short.

The *pre-state space of* $\mathcal{M}$ is

$$H(\mathcal{M}) := \bigoplus_{\text{simple } l} H(\mathcal{M}, l)$$

where the sum is over all simple labelings up to equivalence. We refer to elements of $H(\mathcal{M}, l)$ and $H(\mathcal{M})$ as *pre-states.*

The corresponding state space will be defined in Proposition 3.50.

Definition 3.41. Let $\mathcal{M}, l, \psi_C$ be as in Definition 3.40, and suppose the underlying 3-manifold of $\mathcal{M}$ is S^3. Suppose we are given $\varphi_C \in H(C, l, \psi_C)$ for each 3-cell C (oriented the same as $\mathcal{M}$), defining an element $\otimes_C \varphi_C \in H(\mathcal{M}, l, \{\psi_C\})$. We define its *Reshetikhin-Turaev evaluation* as follows:

Consider the graph Γ obtained as the union of the anchors, $\Gamma = \cup \psi_C$. We make Γ a directed co-ribbon graph such that (1) the ribbons are attached to the outgoing base of the coupon and (2) ribbons are directed arbitrarily. Furthermore, the ambient orientation of C picks out an orientation on the graph, making it a ribbon graph. Label the coupon in C by φ_C, and label the ribbon intersecting the 2-face f by $l(f)$, where, if the core of the ribbon is directed outwardly from C, then f is taken with the outward orientation with respect to C. Then we define the Reshetikhin-Turaev evaluation of $\otimes_C \varphi_C$ to be $Z_{\mathrm{RT}}(\Gamma, l, \otimes_C \varphi_C)$ (see Proposition 3.33).

This is multilinear in φ_C's, so defines a linear map

$$Z_{\mathrm{RT}}(\Gamma, l, -) : H(\mathcal{M}, l, \{\psi_C\}) \to \mathbf{k} \tag{3.6}$$

Z_{RT} is also functorial with respect to anchor change: if $\{\psi'_C\}$ is another collection of anchors, related to the old by $\psi'_C = \sigma_C \cdot \psi_C$, with union $\Gamma' = \cup \psi'_C$, then clearly

$$Z_{\mathrm{RT}}(\Gamma', l, \bigotimes_C f_{\sigma_C}(\varphi_C)) = Z_{\mathrm{RT}}(\Gamma, l, \bigotimes_C \varphi_C)$$

Since $H(\overline{C}, l) = \langle l(f_k)^*, \dots, f(f_1)^* \rangle$ (recall that flipping orientation of C also flips the ordering on the faces), $H(\overline{C}, l)$ and $H(C, l)$ are naturally dual via (2.9). The following lemma states that the pairing is natural with respect to the choice of anchor:

Lemma 3.42. *Let C be an oriented 3-cell with an anchor ψ, as in Definition 3.38. Consider the orientation-preserving PL homeomorphism $C \cup_\partial \overline{C} \simeq S^3$, where ∂C and $\partial \overline{C}$ are glued via the identity map; from Definition 3.41, we have the following map:*

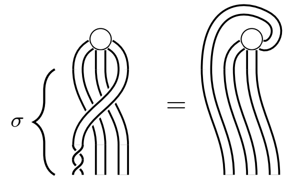
$$Z_{RT} : H(C, l, \psi) \otimes H(\overline{C}, l, \psi) \to \mathbf{k}$$

We call this the local state space pairing. *Then this pairing agrees with* ev *(see (2.9)). Furthurmore, this pairing is natural with respect to choice of anchors: if $\psi' = \sigma \cdot \psi$,*

$$\begin{array}{ccc} H(C,l,\psi) \otimes H(\overline{C},l,\psi) & \longrightarrow & \mathbf{k} \\ \big\downarrow{\scriptstyle f_\sigma \otimes f_\sigma} & & \big\| \\ H(C,l,\psi') \otimes H(\overline{C},l,\psi') & \longrightarrow & \mathbf{k} \end{array}$$

It is useful to keep in mind that when the ambient orientation is flipped, we can draw/visualize things by flipping them left/right, up/down, or front/back. For example, a braid that is mirrored up/down is taken to its inverse.

Remark 3.43. Note that for the braid σ defined below, $f_\sigma = z$ from (2.6), thus, we may use circular coupons instead of rectangular ones in (co-)ribbon graphs.



We may simply write $H(C,l)$ for the local state space when it does not lead to confusion; thus $H(C,l)$ is naturally dual to $H(\overline{C},l)$. When we construct elements of $H(C,l)$, typically a choice of anchor is needed, and the resulting element should vary functorially with change of anchor; when we say two elements of $H(C,l)$ are equal, we implicitly assumed that a choice of anchor ψ is made, and they are equal in $H(C,l,\psi)$.

Definition 3.44. Let $\mathcal{M}$ be an oriented combinatorial 3-manifold, and l a labeling of $\mathcal{M}$. Applying the local state space pairing of Lemma 3.42 to each 3-cell, we have a natural pairing

$$ev : H(\mathcal{M},l) \otimes H(\overline{\mathcal{M}},l) \to \mathbf{k} \tag{3.7}$$

Summing up over simple labeling, we thus get the *state space pairing*:

$$ev : H(\mathcal{M}) \otimes H(\overline{\mathcal{M}}) \to \mathbf{k} \tag{3.8}$$

Definition 3.45. Let T be a 4-cell in a combinatorial 4-manifold $\mathcal{W}$ with labeling l. We define the *local invariant*

$$Z(T,l) \in H(\partial T,l) \simeq H(\overline{\partial T},l)^* \tag{3.9}$$

as the functional that defines the Reshetikhin-Turaev evaluation from Definition 3.41; that is, upon choosing anchors $\{\psi_C\}_{C\in\partial T}$,

$$Z(T,l) : \bigotimes_C \varphi_{\overline{C}} \mapsto Z_{\mathrm{RT}}(\Gamma, l, \{\varphi_{\overline{C}}\})$$

where $\varphi_{\overline{C}} \in H(\overline{C},l,\psi_C)$ for each inwardly-oriented 3-cell $\overline{C} \in \overline{\partial T}$, and $\Gamma = \cup\psi_C$ is the graph obtained from some choice of anchoring of ∂T.

Equivalently, upon choosing a basis $\{\varphi_{C,\alpha}\}_\alpha$ of $H(C,l)$ for each 3-cell C,

$$Z(T,l) = \sum_\alpha \big(Z_{\mathrm{RT}}(\Gamma, l, \{\overline{\varphi}_{C,\alpha}\}) \bigotimes_C \varphi_{C,\alpha} \big)$$

where the sum is over all assignments of a basis vector $\varphi_{C,\alpha}$ to each C.

Definition 3.46. Let $\mathcal{W}$ be an oriented combinatorial 4-manifold, possibly with boundary. For a labeling l, we define

$$Z(\mathcal{W},l) = \mathrm{ev}(\bigotimes_T Z(T,l)) \in H(\partial\mathcal{W},l) \tag{3.10}$$

where ev applies the dual pairing of Lemma 3.42 to every interior 3-cell, and T runs over every 4-cell.

The *Crane-Yetter invariant* $Z_{\mathrm{CY}}(\mathcal{W})$ is

$$Z_{\mathrm{CY}}(\mathcal{W}) = \mathcal{D}^{x(\mathcal{W}\setminus\partial\mathcal{W}) + \frac{1}{2}x(\partial\mathcal{W})} \sum_l \prod_f d_{l(f)}^{n_f} Z(\mathcal{W},l) \in H(\partial\mathcal{W}) \tag{3.11}$$

where

- the sum is taken over all equivalence classes of simple labelings
- f runs over the set of unoriented 2-cells of $\mathcal{W}$
- $n_f = \begin{cases} 1 & \text{if } f \text{ is an internal 2-face} \\ \frac{1}{2} & f \in \partial\mathcal{W} \end{cases}$
- $\mathcal{D}$ is the dimension of the category (see (2.3))
- $x(X) = v(X) - e(X) =$ number of 0-cells − number of 1-cells
- $d_{l(f)}$ is the categorical dimension of $l(f)$

Note that the evaluation giving $Z(\mathcal{W},l)$ is well-defined by Lemma 3.42, in that it varies functorially with a change of anchors in boundary 3-cells, and is invariant under change of anchors in interior 3-cells.

The factor of 1/2 are meant to account for the fact that, when a boundary is glued to another, that boundary becomes part of the interior, so the coefficients combine to a whole.

It is helpful to write $Z_{\mathrm{CY}}(\mathcal{W})$ in terms of bases. Choose some anchoring of $\mathcal{W}$. For each labeling l, choose a basis $\{\varphi_{C,\alpha}\}$ of $H(C,l)$ for each oriented 3-cell C, so that C and $\overline{C}$ have dual bases. Then

$$Z_{\mathrm{CY}}(\mathcal{W},l) = \sum_{\alpha} \prod_{T\in\mathcal{W}} Z_{\mathrm{RT}}(\Gamma_T, l, \{\overline{\varphi}_{C,\alpha}\}) \cdot \mathrm{ev}\Big(\bigotimes_{\substack{C\in\partial T\\ T\in\mathcal{W}}} \varphi_{C,\alpha}\Big) \tag{3.12}$$

$$= \sum\nolimits'_{\alpha} \prod_{T\in\mathcal{W}} Z_{\mathrm{RT}}(\Gamma_T, l, \{\overline{\varphi}_{C,\alpha}\}) \cdot \bigotimes_{C\in\partial\mathcal{W}} \varphi_{C,\alpha} \tag{3.13}$$

$$= \sum\nolimits'_{\alpha} Z_{\mathrm{RT}}(\coprod \Gamma_T, l, \{\overline{\varphi}_{C,\alpha}\}) \cdot \bigotimes_{C\in\partial\mathcal{W}} \varphi_{C,\alpha} \tag{3.14}$$

$$Z_{\mathrm{CY}}(\mathcal{W}) = \mathcal{D}^{x(\mathcal{W}\setminus\partial\mathcal{W})+\frac{1}{2}x(\partial\mathcal{W})} \sum_{l}\prod_{f} d_{l(f)}^{n_f} \sum\nolimits'_{\alpha} Z_{\mathrm{RT}}(\coprod \Gamma_T, l, \{\overline{\varphi}_{C,\alpha}\}) \cdot \bigotimes_{C\in\partial\mathcal{W}} \varphi_{C,\alpha} \tag{3.15}$$

where the sum $\sum_\alpha$ is over all assignments of a basis vector $\varphi_{C,\alpha}$ to each oriented 3-cell C, while $\sum'_\alpha$ is restricted to those assignments where $\varphi_{C,\alpha}$ and $\varphi_{\overline{C},\alpha'}$ are dual (others are eliminated by the evaluation); Γ_T is graph obtained from the union of anchors in ∂T, and $\coprod \Gamma_T$ is the union of those graphs placed in a single S^3, each in a ball disjoint from the others. Each coupon of $\coprod \Gamma_T$ corresponds to an oriented 3-cell; internal 3-cells appear twice (once for each orientation), and boundary 3-cells only once (with the outward orientation).

Theorem 3.47. *For an oriented combinatorial 4-manifold $\mathcal{W}$, possibly with boundary, $Z_{CY}(\mathcal{W})$ is independent of the choice of PLCW decomposition in the interior; that is, if $\mathcal{W}'$ is another PLCW decomposition that agrees with $\mathcal{W}$ on the boundary, then $Z_{CY}(\mathcal{W}') = Z_{CY}(\mathcal{W}) \in H(\partial\mathcal{W})$. In particular, Z_{CY} is a well-defined invariant of closed PL 4-manifolds.*

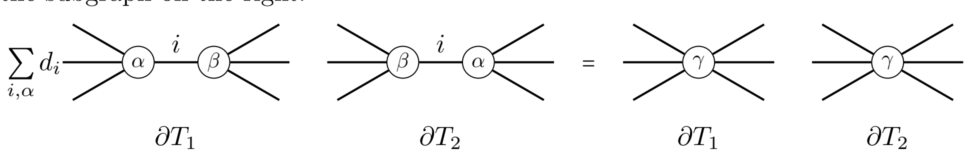
Proof. By Corollary 3.18, we just need to show that elementary subdivision of an interior k-cell C of $\mathcal{W}$ preserves $Z_{\mathrm{CY}}(\mathcal{W})$. Let $\mathcal{W}'$ be the new PLCW decomposition after subdivision. We consider different k in separate cases:

Case k = 1: The only thing that changes in $Z_{\mathrm{CY}}(\mathcal{W})$ are the number of internal 0- and 1-cells, but $x(\mathcal{W}\setminus\partial\mathcal{W})$ remains constant.

The rest of the cases follow from arguments very similar to the proof of invariance of the TV state sum in [BK, Section 5].

Case k = 3: This follows from the fact that if $\{\varphi^i_\alpha\}, \{\phi^i_\beta\}$ are bases for $\langle A_1,\dots,A_k,X_i\rangle, \langle X_i^*,B_1,\dots,B_l\rangle$, and $\{\overline{\varphi}^i_\alpha\}, \{\overline{\phi}^i_\beta\}$ are their dual bases for $\langle X_i^*,A_k^*,\dots,A_1^*\rangle, \langle B_l^*,\dots,B_1^*,X_i\rangle$, then $\{\sqrt{d_i}\,\mathrm{ev}_{X_i}(\varphi^i_\alpha\otimes\phi^i_\beta)\}, \{\sqrt{d_i}\,\mathrm{ev}_{X_i}(\overline{\phi}^i_\beta\otimes\overline{\varphi}^i_\alpha)\}$ are dual bases for $\langle A_1,\dots,A_k,B_1,\dots,B_l\rangle, \langle B_l^*,\dots,B_1^*,A_k^*,\dots,A_1^*\rangle$.

More precisely, let the new 2-face added be F, splitting an old 3-cell C of $\mathcal{W}$ into C_1 and C_2. There are exactly two 4-cells T_1,T_2 that sandwich C in $\mathcal{W}$, thus also sandwich C_1 and C_2 in $\mathcal{W}'$. To fix notation, we take C_1,C_2 to have the outward orientation with respect to T_1, and take F to have the outward orientation with respect to C_1. For a labeling with $l(F) = X_i$, $H(C_1,l) = \langle\dots,X_i\rangle$ and $H(C_2,l) = \langle X_i^*,\dots\rangle$; choose bases $\{\varphi^i_\alpha\}, \{\phi^i_\beta\}$ for them. Then in the $Z_{\mathrm{RT}}(\cdots)$ coefficient in (3.15), for $Z_{\mathrm{CY}}(\mathcal{W}')$, the 3-cells C_1,C_2 contribute the subgraph on the left (with the extra d_i coming from the $d_{l(f)}^{n_f}$ term, while for $Z_{\mathrm{CY}}(\mathcal{W})$, the 3-cell C contributes the subgraph on the right:



Case k = 4: This follows easily from Lemma 2.29.

Case k = 2: Let F be the 2-cell to be elementarily subdivided. We first use elementary subdivisions of 3-cells and 4-cells to modify $\mathcal{W}$ so that F meets exactly one 3-cell. Since F is of codimension 2, the (germ of neighborhood of F in the) 3-cells adjacent to F are naturally cyclically ordered (up to a choice of reversing the cyclic order). Let $C_1, \dots, C_m$ be the 3-cells in such an ordering; note that C_i are not necessarily distinct. Let T_i be 4-cell between C_i and C_{i+1} (indices taken modulo m); as with C_i, the T_i's are not necessarily distinct.

For each i, perform an elementary subdivision on C_i that separates F from the rest of the boundary; let C_i' be the new 3-cell that is adjacent to F. Now for each i, perform an elementary subdivision on T_i that separates C_i' and C_{i+1}' from the rest of the boundary; let T_i' be the new 4-cell that meets C_i' and C_{i+1}'. Note that C_i', T_i' are pairwise distinct. Then we sequentially remove $C_2', C_3', \dots, C_m'$ using inverse elementary subdivisions (of 4-cells), leaving C_1' as the only 3-cell meeting F.

We essentially performed the following operations, but in one dimension higher:

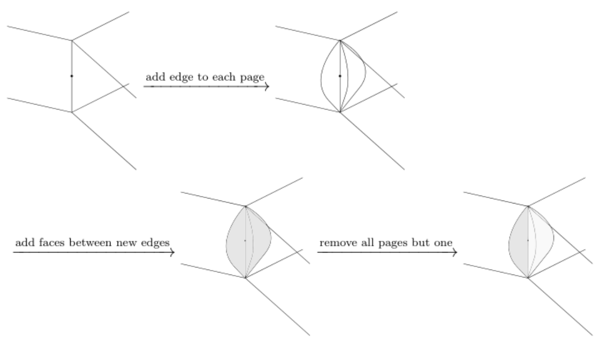
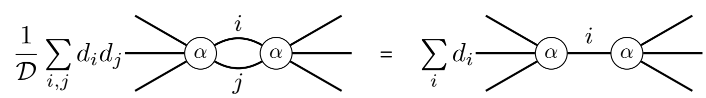


FIGURE 4. Preparing for elementary subdivision along 2-cell, here showing the same move but in one dimension lower. (Figure taken from [BK])

Thus we may assume F appears in the boundary of exactly one 3-cell C, and C appears in the boundary of exactly one 4-cell T (appearing twice, once for each orientation). Now the invariance follows from the following identity (the two nodes represent the oriented 3-cells $C, \overline{C}$, and edges represent 2-cells; on the right, the single edge between nodes is F, and on the left, the two edges are from the two 2-cells into which F is subdivided):



□

Corollary 3.48. *Z_{CY} satisfies the gluing axiom: if $\mathcal{W}$ is a combinatorial 4-manifold with boundary $\partial\mathcal{W} = \mathcal{M}_0 \sqcup \mathcal{M} \sqcup \overline{\mathcal{M}}$, and $\mathcal{W}'$ is the manifold obtained by identifying boundary components $\mathcal{M}, \overline{\mathcal{M}}$, then*

$$Z_{CY}(\mathcal{W}') = \mathrm{ev}(Z_{CY}(\mathcal{W})) \tag{3.16}$$

where the ev *applies the pairing of (3.8).*

Given a combinatorial 4-manifold $\mathcal{W}$, we may arbitrarily partition its boundary components into two sets, and call them *incoming* and *outgoing* boundary components, respectively. Denote by $\mathcal{M}_{in}$ the disjoint union of incoming boundary components, taken with inward orientation, and $\mathcal{M}_{out}$ the disjoint union of outgoing boundary components, taken with outgoing orientation. In other words, we treat $\mathcal{W}$ as a cobordism $\mathcal{W} : \mathcal{M}_{in} \to \mathcal{M}_{out}$. Then we may interpret

$$Z_{\mathrm{CY}}(W) \in H(\overline{\mathcal{M}_{in}}) \otimes H(\mathcal{M}_{out})$$

as a linear operator

$$Z_{\mathrm{CY}}(W) : H(\mathcal{M}_{in}) \to H(\mathcal{M}_{out}) \tag{3.17}$$

$$\varphi \mapsto \mathrm{ev}(Z_{\mathrm{CY}}(W), \varphi) \tag{3.18}$$

obtained by applying the pairing of (3.8) to $\mathcal{M}_{in}$. Here we may use W, the underlying PL manifold of $\mathcal{W}$, because of Theorem 3.47.

Lemma 3.49. *Let $\mathcal{W}$ be a combinatorial 4-manifold with $\partial\mathcal{W} = \overline{\mathcal{M}_{in}} \sqcup \mathcal{M}_{out}$, interpreted as a linear operator as in (3.17) above. Then its adjoint map is also given by $Z_{CY}(W)$:*

$$\begin{array}{ccc} H(\mathcal{M}_{out})^* & \xrightarrow{Z_{CY}(W)^*} & H(\mathcal{M}_{in})^* \\ \downarrow\simeq & & \downarrow\simeq \\ H(\overline{\mathcal{M}_{out}}) & \xrightarrow{Z_{CY}(W)} & H(\overline{\mathcal{M}_{in}}) \end{array}$$

Proposition 3.50. *For a combinatorial closed 3-manifold $\mathcal{M}$ with underlying PL manifold M, the linear map*

$$A_{\mathcal{M}} \coloneqq Z_{CY}(M \times I) : H(\mathcal{M}) \to H(\mathcal{M}) \tag{3.19}$$

is a projector. Moreover, the spaces

$$Z_{CY}(\mathcal{M}) \coloneqq \mathrm{im}(A_{\mathcal{M}}) \tag{3.20}$$

are canonically identified by the system of isomorphisms

$$A_{\mathcal{M}',\mathcal{M}''} \coloneqq Z_{CY}(M \times I) : H(\mathcal{M}') \to H(\mathcal{M}'')$$

where $\mathcal{M}', \mathcal{M}''$ are PLCW structures on M. Furthermore, $A_{\mathcal{M}}$ is self-adjoint, and the pairing of (3.8) restricts to a non-degenerate pairing

$$Z_{CY}(\mathcal{M}) \otimes Z_{CY}(\overline{\mathcal{M}}) \to \mathbf{k}$$

which transforms functorially with $A_{\cdot,\cdot}$. Thus, we may define $Z_{CY}(M) \coloneqq Z_{CY}(\mathcal{M})$ for some PLCW structure $\mathcal{M}$ on M, which we call the state space *of M,*

We will often simple denote $A_{\mathcal{M},\mathcal{M}'}$ by A when the context is clear.

Proof. Simple exercise left to the reader. □

By the existence of collar neighborhoods of boundaries and invariance under choice of PLCW decomposition, $Z_{\mathrm{CY}}(W) \in Z_{\mathrm{CY}}(\partial W)$ and is well-defined. We also have a canonical pairing

$$Z_{\mathrm{CY}}(M) \otimes Z_{\mathrm{CY}}(\overline{M}) \to \mathbf{k} \tag{3.21}$$

inherited from (3.8).

Theorem 3.51. *The invariants Z_{CY} above define a TQFT.*

It is often more convenient to consider elements of $H(\mathcal{M})$ instead of the subspace $Z_{\mathrm{CY}}(\mathcal{M}) \subset H(\mathcal{M})$:

Definition 3.52. A *lift* of a state $\varphi \in Z_{\mathrm{CY}}(\mathcal{M})$ is a pre-state $\widetilde{\varphi} \in H(\mathcal{M})$ that projects to φ under $A_{\mathcal{M}}$.

Definition 3.53. Let $\mathcal{M}, \mathcal{M}'$ be PLCW structures on a closed 3-manifold M. We say two pre-states $\varphi \in H(\mathcal{M}), \varphi' \in H(\mathcal{M}')$ are *equivalent as states*, or simple *equivalent*, denoted $\varphi \simeq \varphi'$, if for some (and hence any) PLCW decomposition $\mathcal{M}''$ of M, $A_{\mathcal{M},\mathcal{M}''}(\varphi) = A_{\mathcal{M}',\mathcal{M}''}(\varphi')$.

Lemma 3.54. *For a combinatorial 4-manifold $\mathcal{W}$, let $\widetilde{\varphi}$ be a lift of $\varphi \in H(\overline{\partial\mathcal{W}})$. Then*

$$\mathrm{ev}(Z_{CY}(\mathcal{W}), \varphi) = \mathrm{ev}(Z_{CY}(\mathcal{W}), \widetilde{\varphi})$$

Proof. By the invariance of Z_{CY} with respect to PLCW decomposition, $A_{\partial\mathcal{W}}(Z_{\mathrm{CY}}(\mathcal{W})) = Z_{\mathrm{CY}}(\mathcal{W})$, thus

$$\mathrm{ev}(Z_{\mathrm{CY}}(\mathcal{W}), \widetilde{\varphi}) = \mathrm{ev}(A(Z_{\mathrm{CY}}(\mathcal{W})), \widetilde{\varphi}) = \mathrm{ev}(Z_{\mathrm{CY}}(\mathcal{W}), A(\widetilde{\varphi})) = \mathrm{ev}(Z_{\mathrm{CY}}(\mathcal{W}), \varphi)$$

□

There is a canonical element of $H(\mathcal{M})$ in the subspace $H(\mathcal{M}, l \equiv \mathbf{1})$ where l labels all 2-cells with $\mathbf{1}$:

$$\widetilde{\varnothing}_{\mathcal{M}} \coloneqq \bigotimes \mathrm{id}_{\mathbf{1}} \in \bigotimes \mathrm{Hom}(\mathbf{1}, \mathbf{1} \otimes \cdots \otimes \mathbf{1}) = H(\mathcal{M}, l \equiv \mathbf{1}) \subset H(\mathcal{M}) \tag{3.22}$$

which we call the *empty pre-state*. We claim that these are, up to factors of $\mathcal{D}$, equivalent as states; these will be discussed further later (see (4.8) and Section 5.2).

However, we warn the reader that for lifts, $\widetilde{\varphi} \in H(\mathcal{M}), \widetilde{\varphi}' \in H(\overline{\mathcal{M}})$ of $\varphi \in Z_{\mathrm{CY}}(\mathcal{M}), \varphi' \in Z_{\mathrm{CY}}(\overline{\mathcal{M}})$, the pairing (3.8) and (3.21) disagree; they are related by

$$\mathrm{ev}(\varphi, \varphi') = \mathrm{ev}(\widetilde{\varphi}, A(\widetilde{\varphi})) \neq (\widetilde{\varphi}, \widetilde{\varphi}') \tag{3.23}$$

Thus, we make the following definition:

Definition 3.55. For $\varphi \in H(\mathcal{M}), \varphi' \in H(\overline{\mathcal{M}})$, their *reduced pre-state space pairing* is defined as

$$\overline{\mathrm{ev}}(\varphi, \varphi') := \mathrm{ev}(\varphi, A(\varphi')) \tag{3.24}$$

Equivalently, treating $M \times I$ as a cobordism $M \sqcup \overline{M} \to$, $Z_{\mathrm{CY}}(M \times I)$ defines the same pairing $H(\mathcal{M}) \otimes H(\overline{\mathcal{M}}) \to \mathbf{k}$.

Note that if either of φ or φ' is in the image of A, i.e. in Z_{CY}, then $\overline{\mathrm{ev}}(\varphi, \varphi') = \mathrm{ev}(\varphi, \varphi')$.

Remark 3.56. Warning. We point out another potentially confusing thing. A 4-ball ball $\mathcal{W} \simeq B^4$ defines a Crane-Yetter invariant $Z_{\mathrm{CY}}(\mathcal{W}) \in Z_{\mathrm{CY}}(\partial\mathcal{W})$. On the other hand, if we ignore the PLCW decomposition in the interior of $\mathcal{W}$ and treat it as a cell-like 4-ball, then we have $Z(\mathcal{W}, l) \in H(\partial\mathcal{W}, l)$. The $H(\partial\mathcal{W}, l)$ component of $Z_{\mathrm{CY}}(\mathcal{W})$ does not equal $Z(\mathcal{W}, l)$ (however the difference is simply by a factor coming from the $d_{l(f)}^{n_f}$ term).

Personally, I make the mistake of conflating the two most often when I am using them as functionals $H(\overline{\partial\mathcal{W}}, l) \to \mathbf{k}$, and particularly when using the skein definition of state spaces. This will be more apparent in computations as we will see in Section 9. (see (3.9)).

To summarize the section, we first have local state spaces $H(C, l)$ for each 3-cell C (Definition 3.38). The pre-state space of $\mathcal{M}$ is the tensor product of local state spaces, summed over all labelings (Definition 3.40). A 4-cell in a combinatorial 4-manifold $\mathcal{W}$ is given its local invariant $Z(T, l) \in H(\partial T, l)$, defined as the (dual to) evaluating colored ribbon graphs (Definition 3.45). The Crane-Yetter invariant $Z_{\mathrm{CY}}(\mathcal{W}) \in H(\partial\mathcal{W})$ is obtained by taking the tensor product of local invariants, applying the pairing (2.9) to matching 3-cells, and summing over all labelings with certain coefficients (Definition 3.46). Finally, the state space $Z_{\mathrm{CY}}(M)$ is the "common" subspace of $H(\mathcal{M})$ that is the image of the cylinder map $Z_{\mathrm{CY}}(\mathcal{M} \times I)$.

3.4. Equivalence of $Z_{\mathbf{CY}}$ with Crane-Yetter.

Let us recall the original construction of the Crane-Yetter state sum, as laid out in [CY1993] and generalized in [CKY1997].

As before, we fix a premodular category $\mathcal{A}$ ("semisimple tortile categories" in [CKY1997]). We will present their construction using our notation and conventions, but let us point out one potentially confusing difference. [2]

Let S be a simple object, and $\mathcal{B} = \{b_\alpha\}$ a basis of $\mathrm{Hom}(S, X)$. Then $\overline{\mathcal{B}} = \{\overline{b_\beta}\}$ is the basis of $\mathrm{Hom}(X, S)$ such that $\overline{b_\beta} \circ b_\alpha = \delta_{\alpha,\beta}\, \mathrm{id}_S$. In other words, $\overline{\mathcal{B}}$ is the dual basis to $\mathcal{B}$ with respect to the pairing

$$\mathrm{Hom}(X, S) \otimes \mathrm{Hom}(S, X) \to \mathbf{k} \tag{3.25}$$

$$\psi \otimes \varphi \mapsto a, \ \text{ where } \psi \circ \varphi = a \cdot \mathrm{id}_S \tag{3.26}$$

which is $1/d_S$ times the pairing defined by (2.9). Thus, $\overline{b_\alpha} = d_S \cdot b^\alpha$, where $\{b^\alpha\}$ is the dual basis to $\{b_\alpha\}$ as seen in (2.15).

For each triple of simple objects $i, j, k \in \mathrm{Irr}(\mathcal{A})$, choose a basis $\mathcal{B}_k^{ij}$ of $\mathrm{Hom}(X_i X_j, X_k)$, and write $\mathcal{B} = \bigsqcup \mathcal{B}_k^{ij}$.

Let $\mathbf{T}$ be an ordered triangulation on a 4-manifold M (that is, the set of vertices are ordered). Denote by $\mathbf{T}_{(i)}$ the set of i-simplices of $\mathbf{T}$. The ordering defines, on each simplex, an orientation as follows: if $\{v_0, v_1, \dots, v_i\}$ are the vertices of an i-simplex ($i > 0$) in increasing order, then the ordered set of directed line segments $(\overrightarrow{v_0 v_1}, \dots, \overrightarrow{v_0 v_i})$ defines an orientation at v_0, which is extended to the rest of the simplex. We call this the *ordering-orientation* on ξ.

---

[2] In [CKY1997],

- $\mathcal{A}[X, Y] = \mathrm{Hom}_{\mathcal{A}}(X, Y)$.
- Composition is written left to right: $\circ : \mathcal{A}[X, Y] \otimes \mathcal{A}[Y, Z] \to \mathcal{A}[X, Z]$.
- Morphisms in diagrams go from top to bottom

Definition 3.57 ([CKY1997], Definition 3.1)**.** A CSB-coloring of $\mathbf{T}$ is a triple of maps $\lambda = (\lambda_0, \lambda^+, \lambda^-)$, where

$$\lambda_0 : \mathbf{T}_{(2)} \sqcup \mathbf{T}_{(3)} \to \mathrm{Irr}(\mathcal{A})$$
$$\lambda^+ : \mathbf{T}_{(3)} \to \mathcal{B}$$
$$\lambda^- : \mathbf{T}_{(3)} \to \mathcal{B}$$

such that for 3-simplex τ, letting σ_k denote the boundary 2-simplex of τ opposite the k-th vertex (w.r.t. the ordering),

$$\lambda^+(\tau) \in \mathcal{B}^{\lambda_0(\sigma_0),\lambda_0(\sigma_2)}_{\lambda_0(\tau)}$$
$$\lambda^-(\tau) \in \mathcal{B}^{\lambda_0(\sigma_1),\lambda_0(\sigma_3)}_{\lambda_0(\tau)}$$

The set of CSB-colorings of $\mathbf{T}$ is denoted by $\Lambda_{CBS}(\mathbf{T})$.

Observe that the ordering-orientation on σ_0, σ_2 agrees with the outward orientation on $\partial\tau$ (when τ is given the ordering-orientation), while the orientations disagree for σ_1, σ_3.

We sometimes denote λ_0 simply by λ.

Definition 3.58. Let λ be a CSB-coloring of an ordered triangulation $\mathbf{T}$ of an oriented 4-manifold M, and let ξ be a 4-simplex in $\mathbf{T}$. We define the quantity $\|\lambda, \xi\| \in \mathbf{k}$ as follows. Let vertices of ξ be $v_0, \dots, v_4$ in increasing order. Suppose the ordering-orientation on ξ agrees with the orientation of M. Then $\|\lambda, \xi\|$ is defined as the evaluation of the graph in Figure 5. [3]

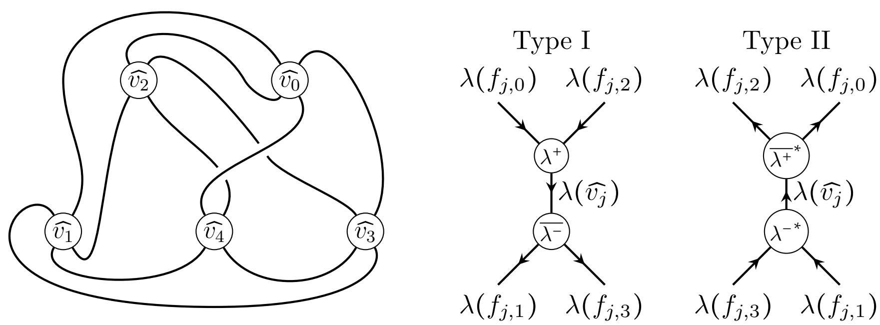


FIGURE 5. Evaluation of the left graph yields $\|\lambda, \xi\|$; the circles labeled with $\widehat{v_j}$ contain subgraphs, defined on the right; for $j = 0, 2, 4$, it contains the Type I subgraph, and for $j = 1, 3$, it contains the Type II subgraph. Here $f_{j,k}$ is the 2-simplex obtained from $\widehat{v_j}$ by omitting the k-th vertex (for example, $\widehat{v_2} = \{v_0, v_1, v_3, v_4\}$, so $f_{2,2} = \{v_0, v_1, v_4\}$).

If the ordering-orientation on ξ differs from the orientation of M, then we use the mirror image of the graph in Figure 5, and the choice of subgraphs for each $\widehat{v_j}$ is flipped (i.e. for $j = 0, 2, 4$, Type II subgraph is used, while for $j = 1, 3$, Type I subgraph is used).

Definition 3.59. For a CSB-coloring λ of an ordered triangulation $\mathbf{T}$, we define

$$\langle\langle\lambda\rangle\rangle = \mathcal{D}^{n_0 - n_1} \prod_{\sigma:2-splx.} d_{\lambda(\sigma)} \prod_{\tau:3-splx.} d^{-1}_{\lambda(\tau)} \prod_{\xi:4-splx.} \|\lambda, \xi\|$$

where n_0, n_1 are the number of vertices and edges, respectively.

Theorem 3.60 ([CKY1997], Theorem 3.2)**.** *The state-sum*

$$CY(M) = \sum_{\lambda \in \Lambda_{CSB}(\mathbf{T})} \langle\langle\lambda\rangle\rangle$$

is an invariant of the PL manifold M.

[3]When $\mathcal{A} = \mathrm{Rep}\, U_q sl_2$ (with q a root of unity), with appropriate choice of bases, $\|\lambda, \xi\|$ is the quantum 15-j symbol as described in [CY1993].

The main result of this section is to show that this definition is equivalent to ours:

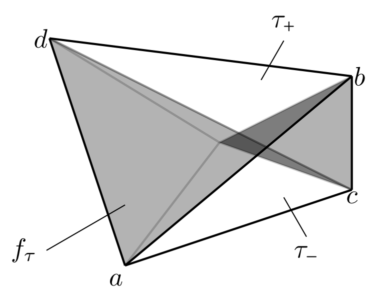
Theorem 3.61. *For a closed PL manifold M,*

$$CY(M) = Z_{CY}(M)$$

Proof. Fix an ordered triangulation $\mathbf{T}$ on M, so that we have

$$CY(M) = \mathcal{D}^{n_0-n_1} \sum_{\lambda\in\Lambda_{CSB}(\mathbf{T})} \prod_{\sigma:2-splx.} d_{\lambda(\sigma)} \prod_{\tau:3-splx.} d^{-1}_{\lambda(\tau)} \prod_{\xi:4-splx.} \|\lambda,\xi\|$$

Consider the PLCW structure $\mathcal{M} = \mathcal{M}_{\mathbf{T}}$ on M derived from $\mathbf{T}$, where we apply, to each 3-simplex $\tau = \{a,b,c,d\}$ (with $a < b < c < d$), the elementary subdivision move that adds the 2-cell f_τ with boundary $\overline{abcda}$, i.e. it separates τ into a top half τ^+ (bounded by 2-cells $\{a,b,d\},\{b,c,d\},f_\tau$) and bottom half τ^- (bounded by 2-cells $\{a,b,c\},\{a,c,d\},f_\tau$). They are called top and bottom by virtue of their ordering-orientations: the ordering-orientations of $\{a,b,d\}$ and $\{b,c,d\}$ coincide with the outward-orientation with respect to the ordering orientation of τ, while it is the opposite for $\{a,b,c\}$ and $\{a,c,d\}$. Thus, the 2-cells of $\mathcal{M}$ come in two varieties, the *triangular* type (coming from $\mathbf{T}$), and *quadrilateral* type (the added 2-cells f_τ).



Since M is closed, $v(\mathcal{M}) = n_0, e(\mathcal{M}) = n_1$, so the exponents of $\mathcal{D}$ agree. Every 2-cell in $\mathcal{M}$ corresponds to either an original 2-simplex of $\mathbf{T}$ or to a 3-simplex of $\mathbf{T}$ (as its separating 2-cell $f_\cdot$). Thus, we have

$$\begin{aligned} Z_{\mathrm{CY}}(\mathcal{M}) &= \mathcal{D}^{x(\mathcal{W})} \sum_l \prod_f d^{n_f}_{l(f)} Z(\mathcal{M},l) \\ &= \mathcal{D}^{n_0-n_1} \sum_l \prod_{\sigma:2-splx.} d_{l(\sigma)} \prod_{\tau:3-splx.} d_{l(f_\tau)} Z(\mathcal{M},l) \end{aligned}$$

which is intentionally written to resemble $CY(M)$. There are two main differences to note: the summing index (labelings vs. CSB-colorings) and the $d^{\pm}_{-}$ term in the product over 3-simplices.

The second difference arise simply because of the different convention of dual pairing used. Namely, recall that $\overline{b_\alpha} = d_S b^\alpha$ for a basis $\{b_\alpha\}$ of $\mathrm{Hom}(S,X)$. Thus, if we define $|\lambda,\xi|$ to be the evaluation of the same graph as in $\|\lambda,\xi\|$, except that we use b^α instead of $\overline{\lambda^\pm} = \overline{b_\alpha}$ in the Type I,II subgraphs. Then

$$\|\lambda,\xi\| = |\lambda,\xi| \prod_{\tau\subset\xi} d_{\lambda(\tau)}$$

where the product is over 3-simplices of ξ. Since each 3-simplex appears in exactly two 4-simplices (and hence exactly two terms $\|\lambda,\xi\|$), we see that

$$CY(M) = \mathcal{D}^{n_0-n_1} \sum_{\lambda\in\Lambda_{CSB}(\mathbf{T})} \prod_{\sigma:2-splx.} d_{\lambda(\sigma)} \prod_{\tau:3-splx.} d_{\lambda(\tau)} \prod_{\xi:4-splx.} |\lambda,\xi|$$

Thus, we move on to addressing the first difference, that of summing over labelings vs. CSB-colorings, for which it suffices to show that

$$Z(\mathcal{M},l) = \sum_{\lambda|\lambda_0=l} \prod_\xi |\lambda,\xi|$$

From (3.13),

$$Z(\mathcal{M},l) = \sum\nolimits'_\alpha \prod_\xi Z_{\mathrm{RT}}(\Gamma_\xi, l, \{\varphi_{\overline{C},\alpha}\})$$

where recall that Γ_ξ is the union of anchors in $\partial\xi$, a basis $\{\varphi_{C,\alpha}\}$ of $H(C,l)$ is chosen for each oriented 3-cell C such that C and $\bar{C}$ have dual bases (w.r.t. (2.9)), and the sum $\sum'_\alpha$ is summed over all choices of assignments $C \mapsto \varphi_{C,\alpha}$ such that the assignments of C and $\bar{C}$ are dual. We may choose the basis of $H(C,l)$ to be the appropriate $\mathcal{B}^{ij}_k$; then it is apparent that the sum $\sum'_\alpha$ is the same as $\sum_{\lambda|\lambda_0=l}$. With this choice of basis, it remains to show that there is a choice of anchoring of $\mathcal{M}$ such that for any 4-simplex ξ,

$$|\lambda,\xi| = Z_{\mathrm{RT}}(\Gamma_\xi, \lambda_0, \{\overline{\lambda^\pm}\}) \tag{3.27}$$

where $\overline{\lambda^\pm}$ is the dual to $\lambda^\pm$ with respect to the pairing of (2.9) (not from (3.26)).

We claim that the anchors as in Figure 6 fits the bill. Figure 7 show how the anchors fit together when the ordering orientation of ξ agrees with the orientation of M, and Figure 8 show a more simplified version, which can be easily seen to match Figure 5, both in terms of the graph and the labels.

Let us point out a sticky orientation matter that may be confusing. Recall that our convention for drawing figures/graphical calculus is that we follow the right-hand rule (see Convention 2.22). When $\xi = \{v_0,\dots,v_4\}$ has the ordering orientation, the face $\hat{v}_4 = \{v_0,\dots,v_3\}$ has the outward-orientation given by $(\overrightarrow{v_0v_1}, \overrightarrow{v_0v_2}, \overrightarrow{v_0v_3})$. This means that Figure 7 is drawing $\overline{\partial\xi}$. This is exactly what we want, since the Reshetikhin-Turaev evaluation $Z_{\mathrm{RT}}(\Gamma_\xi, \lambda_0, \{\overline{\lambda^\pm}\})$ takes place in $\overline{\partial\xi}$, not $\partial\xi$.

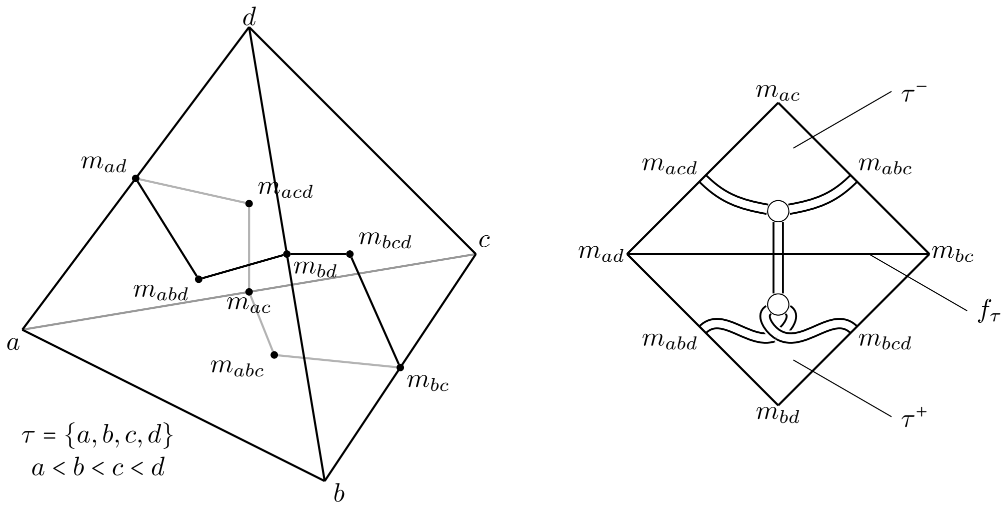


FIGURE 6. Choice of anchor for $\tau^\pm$. m_{ab}, m_{abc} stand for the midpoint and centroids of ab, abc respectively. The anchors lie in a disk (except at the half braiding), a "normal quadrilateral" in the sense of normal surface theory, as seen on the right; this disk is not part of the PLCW structure, it is only there to guide the definition of the anchors. The line connecting m_{ad} to m_{bc} is the intersection of f_τ with this disk. Note that the intersection of the anchor with the 2-simplex efg, with $e < f < g$, is along a segment lying on the line segment $\overline{m_{eg}m_{egh}}$ (is irrespective of which 3-simplex the anchor belongs to), so the anchoring condition of agreeing at 2-cells is satisfied.

It remains to check that, for a 4-simplex ξ, the anchors in $\partial\xi$ glue up to be equivalent to the graph of Figure 5, and that the labelings by objects and morphisms agree.

□

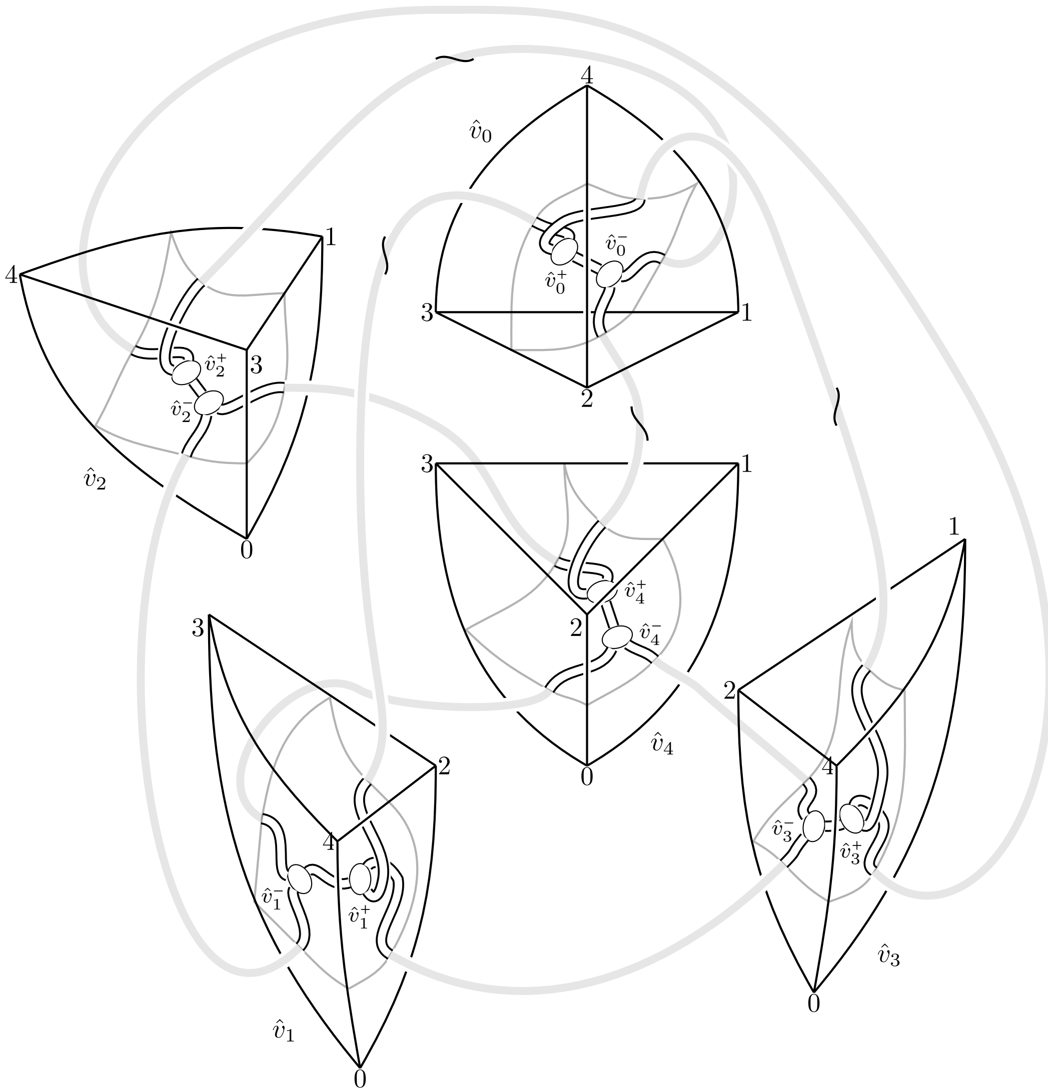


FIGURE 7. Anchors in $\overline{\partial\xi}$ fitting together. The co-orientation of the co-ribbon graph is pointing towards us (except near $\hat{v}_0$). The squiggles on the gray ribbons going into $\hat{v}_0$ indicate that there should be a half-twist.

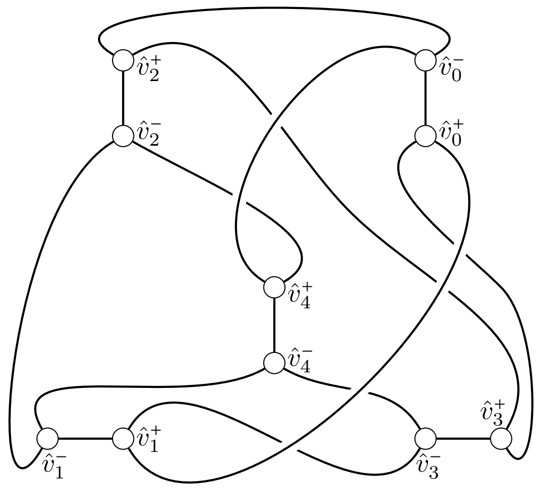


FIGURE 8. Simplified version of Figure 7.

4.1. $Z_{\mathbf{CY}}$ for 4-Manifolds with Corners.

We discuss an extension of the definition of the Crane-Yetter invariant to 4-manifolds with corners. The modifications we make to the original state sum seem very similar to [BFMGI2007] (they also work with the category of representations of quantum $\mathfrak{sl}_2$), but we were not aware of their work until recently.

Definition 4.1. Let N be a closed surface. A *colored marked PLCW decomposition* $(\mathcal{N}, l)$ of N is a marked PLCW decomposition $\mathcal{N}$ with a simple labeling l.

It is best to think of N as an unoriented (but possibly orientable) surface unless the orientation is explicitly needed; for example, N may be a closed surface in the boundary of an oriented 4-manifold, say $\partial W = M \cup_N M'$, wherein the outward orientations on N with respect to M and M', themselves given the outward orientation with respect to W, are different.

Definition 4.2. For an oriented PLCW 3-manifold $\mathcal{M}$ with boundary $\mathcal{N}$, let $l_{\mathcal{N}}$ be a simple labeling of $\mathcal{N}$. We define the *relative pre-state space* to be

$$H(\mathcal{M}; (\mathcal{N}, l_{\mathcal{N}})) \coloneqq \bigoplus_{l|_{\mathcal{N}} = l_{\mathcal{N}}} H(\mathcal{M}; l)$$

i.e. the sum of pre-state spaces over all labelings on $\mathcal{M}$ agreeing with $l_{\mathcal{N}}$ on $\mathcal{N}$.

The corresponding relative state space will be defined in Definition 4.11.

Definition 4.3. The *relative pre-state space pairing* is the perfect pairing

$$\mathrm{ev}^{\partial} : H(\mathcal{M}; (\mathcal{N}, l_{\mathcal{N}})) \otimes H(\overline{\mathcal{M}}; (\mathcal{N}, l_{\mathcal{N}})) \to \mathbf{k} \tag{4.1}$$

$$\bigotimes \varphi_C \otimes \bigotimes \varphi'_C \mapsto d_{l_{\mathcal{N}}}^{1/2} \prod_C (\varphi_C, \varphi'_C) \tag{4.2}$$

that applies the local state space pairing of Lemma 3.42 to each corresponding pair of 3-cells. (Note we do not have to deal with the orientation on $\mathcal{N}$, using the same colored marked PLCW decomposition $(\mathcal{N}, l_{\mathcal{N}})$ for both M and $\overline{M}$.)

Definition 4.4. Let $\mathcal{W}$ be an oriented PLCW 4-manifold, and let $\mathcal{N} \subset \partial\mathcal{W}$ be a PLCW closed surface in its boundary. Recall from (3.10):

$$Z(\mathcal{W}, l) = \mathrm{ev}(\bigotimes_T Z(T, l)) \in H(\partial\mathcal{W}, l) \tag{4.3}$$

Given a simple labeling $l_{\mathcal{N}}$ of $\mathcal{N}$, the *restricted Crane-Yetter invariant* $Z_{\mathrm{CY}}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}}))$ is

$$Z_{\mathrm{CY}}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})) \coloneqq \mathcal{D}^{x(\mathcal{W} \setminus \partial\mathcal{W}) + \frac{1}{2} x(\partial\mathcal{W} \setminus \mathcal{N})} \sum_l \prod_f d_{l(f)}^{n_f} Z(\mathcal{W}, l) \in H(\partial\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})) \tag{4.4}$$

where

- the sum is taken over all equivalence classes of simple labelings l that agree with $l_{\mathcal{N}}$ on $\mathcal{N}$,
- f runs over the set of unoriented 2-cells of $\mathcal{W}$ not in $\mathcal{N}$,
- $n_f = \begin{cases} 1 & \text{if } f \text{ is an internal 2-face} \\ \frac{1}{2} & f \in \partial\mathcal{W} \setminus \mathcal{N} \\ 0 & f \in \mathcal{N} \text{ (redundant by previous point)} \end{cases}$
- $\mathcal{D}$ is the dimension of the category (see (2.3))
- $x(X) = v(X) - e(X) =$ number of 0-cells − number of 1-cells
- $d_{l(f)}$ is the categorical dimension of $l(f)$

The *extended Crane-Yetter invariant* $Z_{\mathrm{CY}}(\mathcal{W}; \mathcal{N})$ is

$$Z_{\mathrm{CY}}(\mathcal{W}; \mathcal{N}) \coloneqq \sum_{l_{\mathcal{N}}} Z_{\mathrm{CY}}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})) \in H(\partial\mathcal{W}) \tag{4.5}$$

Essentially, the definition is the same as Definition 3.46, but the coefficient for each labeling is weighted as if $\mathcal{N}$ was removed from $\mathcal{W}$; this was cooked up to make the gluing along manifolds with boundary to hold (similar to the 1/2's that appear in $Z_{\mathrm{CY}}(\mathcal{W})$ (Definition 3.46) to account for gluing along boundaries).

Theorem 4.5. *For an oriented PLCW 4-manifold $\mathcal{W}$, with corner $\mathcal{N} \subset \partial\mathcal{W}$, $Z_{CY}(\mathcal{W};\mathcal{N})$ is independent of the choice of PLCW decomposition in the interior; that is, if $\mathcal{W}'$ is another PLCW decomposition that agrees with $\mathcal{W}$ on the boundary, then $Z_{CY}(\mathcal{W}';\mathcal{N}) = Z_{CY}(\mathcal{W};\mathcal{N}) \in H(\partial\mathcal{W})$.*

Proof. Follows from Theorem 3.47; we just have to note that in the proof of invariance under elementary moves, only coefficients $d_{l(f)}$ for 2-cells f in the interior are involved. □

The gluing result holds:

Proposition 4.6. *Let $\mathcal{W}$ be an oriented PLCW 4-manifold, and suppose the 3-cells of $\partial\mathcal{W}$ are partitioned into three 3-manifolds, $\partial\mathcal{W} = \mathcal{M}_0 \cup \mathcal{M} \cup \mathcal{M}'$, such that $\mathcal{M}$ and $\mathcal{M}'$ are disjoint, and $\mathcal{M}' \simeq \overline{\mathcal{M}}$. In particular, $\partial\mathcal{M}_0 = \partial\mathcal{M} \sqcup \partial\mathcal{M}'$ as unoriented manifolds. Let $\mathcal{W}'$ be the PLCW manifold obtained from $\mathcal{W}$ by identifying $\mathcal{M}, \mathcal{M}'$. Then*

$$Z_{CY}(\mathcal{W}';\partial\mathcal{M}_0) = \mathrm{ev}(Z_{CY}(\mathcal{W};\partial\mathcal{M}_0))$$
$$Z_{CY}(\mathcal{W}') = \mathrm{ev}^{\partial}(Z_{CY}(\mathcal{W}))$$

where ev *applies the local state space pairing (Lemma 3.42) to each 3-cell of $\mathcal{M}$, and* ev^{∂} *is the relative pre-state space pairing Definition 4.3.*

Proof. Essentially by definition. □

It is often convenient to state the gluing result as the composition of maps:

Proposition 4.7. *Let $\mathcal{W}, \mathcal{W}'$ be an oriented PLCW 4-manifolds, and suppose $\partial\mathcal{W} = \overline{\mathcal{M}} \cup_{\mathcal{N}} \mathcal{M}'$, $\partial\mathcal{W}' = \overline{\mathcal{M}'} \cup_{\mathcal{N}} \mathcal{M}''$. We say $\mathcal{W}$ is a* cornered cobordism over $\mathcal{N}$ from $\mathcal{M}$ to $\mathcal{M}'$, *denoted $\mathcal{W} : \mathcal{M} \to_{\mathcal{N}} \mathcal{M}'$ (likewise $\mathcal{W}' : \mathcal{M}' \to_{\mathcal{N}} \mathcal{M}''$; see also Definition 2.59).*

Then for a simple labeling $l_{\mathcal{N}}$ of $\mathcal{N}$,

$$Z_{CY}(\mathcal{W} \cup_{\mathcal{M}'} \mathcal{W}'; (\mathcal{N}, l_{\mathcal{N}})) = \mathrm{ev}(Z_{CY}(\mathcal{W}'; (\mathcal{N}, l_{\mathcal{N}})) \otimes Z_{CY}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})))$$

where the evaluation applies the local state space pairing (Lemma 3.42) to each 3-cell of $\mathcal{M}'$. Thus

$$Z_{CY}(\mathcal{W} \cup_{\mathcal{M}'} \mathcal{W}'; \mathcal{N}) = \mathrm{ev}(Z_{CY}(\mathcal{W}';\mathcal{N}) \otimes Z_{CY}(\mathcal{W};\mathcal{N}))$$

We may interpret $Z_{CY}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}}))$ as a map

$$Z_{CY}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})) : H(\mathcal{M}; (\mathcal{N}, l_{\mathcal{N}})) \to H(\mathcal{M}'; (\mathcal{N}, l_{\mathcal{N}}))$$

and similarly, $Z_{CY}(\mathcal{W};\mathcal{N})$ as a map

$$Z_{CY}(\mathcal{W};\mathcal{N}) : H(\mathcal{M}) \to H(\mathcal{M}')$$

Then

$$Z_{CY}(\mathcal{W} \cup_{\mathcal{M}'} \mathcal{W}'; (\mathcal{N}, l_{\mathcal{N}})) = Z_{CY}(\mathcal{W}'; (\mathcal{N}, l_{\mathcal{N}})) \circ Z_{CY}(\mathcal{W}; (\mathcal{N}, l_{\mathcal{N}})) : H(\mathcal{M}; (\mathcal{N}, l_{\mathcal{N}})) \to H(\mathcal{M}'; (\mathcal{N}, l_{\mathcal{N}}))$$
$$Z_{CY}(\mathcal{W} \cup_{\mathcal{M}'} \mathcal{W}'; \mathcal{N}) = Z_{CY}(\mathcal{W}';\mathcal{N}) \circ Z_{CY}(\mathcal{W};\mathcal{N}) : H(\mathcal{M}) \to H(\mathcal{M}')$$

The extended Z_{CY} also respects the extended composition (see Definition 2.63) wherein the corners can be different, which in fact is a generalization of the above result:

Proposition 4.8. *Let $\mathcal{W}_1, \mathcal{W}_2$ be cornered cobordisms*

$$\mathcal{W}_1 : \mathcal{M}_1 \to_{\mathcal{N}_1} \mathcal{M}_1'$$
$$\mathcal{W}_2 : \mathcal{M}_2 \to_{\mathcal{N}_2} \mathcal{M}_2'$$

Suppose $\mathcal{M}_1' \subseteq \mathcal{M}_2$ is a PLCW submanifold. Let $\mathcal{W} = \mathcal{W}_1 \cup_{\mathcal{M}_1'} \mathcal{W}_2$, and $\mathcal{M} = (\mathcal{M}_2 \backslash \mathcal{M}_1') \cup \mathcal{M}_1$, so that $\mathcal{W}$ is naturally a cornered cobordism

$$\mathcal{W} : \mathcal{M} \to_{\mathcal{N}_2} \mathcal{M}_2'$$

Then $Z_{CY}(\mathcal{W}_1;\mathcal{N}_1)$ *and* $Z_{CY}(\mathcal{W}_2;\mathcal{N}_2)$ *compose to give* $Z_{CY}(\mathcal{W};\mathcal{N}_2)$*; more precisely,*

$$Z_{CY}(\mathcal{W};\mathcal{N}_2) = Z_{CY}(\mathcal{W}_2;\mathcal{N}_2) \circ (\mathrm{id} \otimes Z_{CY}(\mathcal{W}_1;\mathcal{N}_1))$$
$$Z_{CY}(\mathcal{W};\mathcal{N}_2)(\varphi \otimes \psi) = Z_{CY}(\mathcal{W}_2;\mathcal{N}_2)(\varphi \otimes Z_{CY}(\mathcal{W}_1;\mathcal{N}_1)(\psi))$$

for $\psi \in H(\mathcal{M}_1;(\mathcal{N}_1,l)), \psi \in \bigotimes_{C\in\mathcal{M}_2\backslash\mathcal{M}_1'} H(C,l)$*,* l *a simple labeling of* $\mathcal{M}$*.*

Similarly, suppose that we have the reverse inclusion $\mathcal{M}_2 \subseteq \mathcal{M}_1'$*. Let* $\mathcal{W} = \mathcal{W}_1 \cup_{\mathcal{M}_2} \mathcal{W}_2$*, and* $\mathcal{M}' = (\mathcal{M}_1'\backslash\mathcal{M}_2) \cup \mathcal{M}_2'$*, so that* $\mathcal{W}$ *is naturally a cornered cobordism*

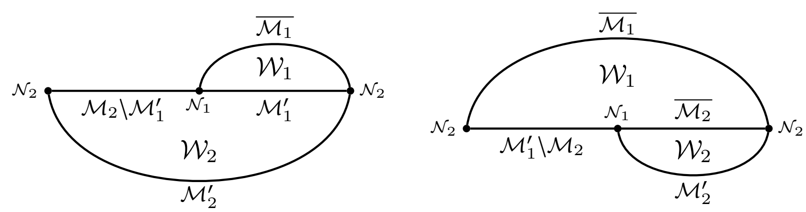
$$\mathcal{W} : \mathcal{M}_1 \to_{\mathcal{N}_1} \mathcal{M}'$$

Then $Z_{CY}(\mathcal{W}_1;\mathcal{N}_1)$ *and* $Z_{CY}(\mathcal{W}_2;\mathcal{N}_2)$ *compose to give* $Z_{CY}(\mathcal{W};\mathcal{N}_1)$*; more precisely,*

$$Z_{CY}(\mathcal{W};\mathcal{N}_1) = (\mathrm{id} \otimes Z_{CY}(\mathcal{W}_2;\mathcal{N}_2)) \circ (Z_{CY}(\mathcal{W}_1;\mathcal{N}_1))$$
$$Z_{CY}(\mathcal{W};\mathcal{N}_1)(\Phi) = \sum_a \varphi^{(a)} \otimes Z_{CY}(\mathcal{W}_2;\mathcal{N}_2)(\psi^{(a)})$$

where $Z_{CY}(\mathcal{W}_1;\mathcal{N}_1)(\Phi) = \sum_a \varphi^{(a)} \otimes \psi^{(a)}$*, with* $\psi^{(a)} \in H(\mathcal{M}_2)$*.*

(4.6)



(We label the 3-manifolds with the orientation that makes the side on which the label appears the "outside".)[4]

Proof. Note that we have a slightly more general situation than Definition 2.63, as $\mathcal{N},\mathcal{N}'$ can meet here. Once again, this is a straightforward consequence of the definitions. As mentioned before, $Z_{\mathrm{CY}}(\mathcal{W};\mathcal{N})$ is essentially the same as $Z_{\mathrm{CY}}(\mathcal{W})$, but the weights are designed to "omit" $\mathcal{N}$. Thus, intuitively, in the composition $Z_{\mathrm{CY}}(\mathcal{W}_2;\mathcal{N}_2) \circ (\mathrm{id} \otimes Z_{\mathrm{CY}}(\mathcal{W}_1;\mathcal{N}_1))$, $\mathcal{N}_1$ appears in both terms, but contributes to the $\mathcal{D}$ and $d_{l(f)}$ weights in the $Z_{\mathrm{CY}}(\mathcal{W}_2;\mathcal{N}_2)$ and not in $(\mathrm{id} \otimes Z_{\mathrm{CY}}(\mathcal{W}_1;\mathcal{N}_1))$.

□

Note that $\mathcal{N}_1$ and $\mathcal{N}_2$ may not be disjoint, so $\mathcal{M}_2\backslash\mathcal{M}_1'$ may not be a submanifold.

The following lemma is useful for computations:

Lemma 4.9. *Let* $\mathcal{W}$ *be a cell-like 4-ball, and suppose* $\partial\mathcal{W} = \overline{\mathcal{M}}_{in} \cup_{\mathcal{N}} \mathcal{M}_{out}$*, where* $\mathcal{M}_-$ *are 3-balls,so* $\mathcal{W} : \mathcal{M}_{in} \to_{\mathcal{N}} \mathcal{M}_{out}$*. Furthermore, suppose the bottom hemisphere* $\mathcal{M}_{out}$ *consist of exactly one 3-cell* C_{out}*.*

Choose some anchoring of $\mathcal{W}$ *and simple labeling* l*. Consider some* $\Phi = \bigotimes \varphi_C \in H(\mathcal{M}_{in};(\mathcal{N},l))$*. Assemble the anchors in* $\mathcal{M}_{in}$ *to a ribbon graph* Γ*, and color the coupons by* φ_C*, giving us a colored ribbon graph* (Γ,Φ)*. Under an identification* $\mathcal{M}_{in} \simeq \mathcal{M}_{out}$ *as PL manifolds,* (Γ,Φ) *may be viewed in* C_{out}*, and has some Reshetikhin-Turaev evaluation (Definition 3.41)* $\varphi_{ev} \in H(C_{out},l) = H(\mathcal{M}_{out};(\mathcal{N},l))$*. Then*

$$Z_{CY}(\mathcal{W};\mathcal{N})(\Phi) = \mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in}\backslash\mathcal{N})} d_{l^\circ}^{1/2} \cdot \varphi_{ev}$$

where $d_{l^\circ}^{1/2} = \prod_{f\in\mathcal{M}_{in}\backslash\mathcal{N}} d_{l(f)}$*.*

Thus, if C_{out} *is a 3-cell in some PLCW 3-manifold* $\mathcal{M}$*, and* $\mathcal{M}' = (\mathcal{M}\backslash C_{out}) \cup \mathcal{M}_{in}$*, then for simple labeling* l *of* $\mathcal{M}'$*, and* $\Psi \in \prod_{C\in\mathcal{M}\backslash C_{out}} H(C,l)$*, by Proposition 4.8,* $\Psi \otimes \Phi$ *and* $\Psi \otimes (\mathcal{D}^{x(\mathcal{M}_{in}\backslash\partial\mathcal{M}_{in})/2} d_{l^\circ}^{1/2} \cdot \varphi_{ev})$ *are equivalent as states (Definition 3.53), i.e.*

$$A_{\mathcal{M}',\mathcal{M}''}(\Psi \otimes \Phi) = A_{\mathcal{M},\mathcal{M}''}(\Psi \otimes (\mathcal{D}^{x(\mathcal{M}_{in}\backslash\partial\mathcal{M}_{in})/2} d_{l^\circ}^{1/2} \cdot \varphi_{ev})) \tag{4.7}$$

for any $\mathcal{M}''$*.*

In words, if $\mathcal{M}$ *is obtained from* $\mathcal{M}'$ *by "merging" several 3-cells into one 3-cell, then a pre-state of* $\mathcal{M}'$ *is equivalent (up to a factor) to the pre-state obtained by "evaluating" the subgraph in those 3-cells.*

[4]"Are those ... invariants ... on a cob?"

Proof. Note that $x(\mathcal{W}\backslash\partial\mathcal{W}) = 0$ and $x(\partial\mathcal{W}\backslash\mathcal{N}) = x(\mathcal{M}_{in}\backslash\mathcal{N})$. We have, for any $\varphi' \in H(\overline{C_{out}}, l)$,

$$\begin{aligned}
\mathrm{ev}(Z_{\mathrm{CY}}(\mathcal{W};\mathcal{N})(\Phi),\varphi') &= \mathcal{D}^{x(\mathcal{W}\backslash\partial\mathcal{W})+\frac{1}{2}x(\partial\mathcal{W}\backslash\mathcal{N})} d_{l(f)}^{n_f}\,\mathrm{ev}(Z(\mathcal{W},l),\Phi\otimes\varphi') \\
&= \mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in}\backslash\mathcal{N})} d_{l^\circ}^{1/2}\,\mathrm{ev}(Z(\mathcal{W},l),\Phi\otimes\varphi') \\
&= \mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in}\backslash\mathcal{N})} d_{l^\circ}^{1/2} Z_{\mathrm{RT}}(\Gamma\cup\Gamma', l, \Phi\otimes\varphi') \text{ where } \Gamma' \text{ is anchor for } C_{out} \\
&= \mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in}\backslash\mathcal{N})} d_{l^\circ}^{1/2} Z_{\mathrm{RT}}(\Gamma'\cup\Gamma', l, \varphi_{ev}\otimes\varphi') \\
&= \mathrm{ev}(\mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in}\backslash\mathcal{N})} d_{l^\circ}^{1/2}\cdot\varphi_{ev},\varphi').
\end{aligned}$$

(recall Definition 3.45). ☐

This fact is useful for dealing with lifts of states in $H(\mathcal{M})$, as it relates them to states for different PLCW decomposition.

Recall the "empty states" $\widetilde{\varnothing}_{\mathcal{M}}$ from (3.22). It follows from Lemma 4.9 that, for PLCW structures $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ on a closed 3-manifold,

$$A_{\mathcal{M},\mathcal{M}''}(\mathcal{D}^{-\frac{1}{2}x(\mathcal{M})}\widetilde{\varnothing}_{\mathcal{M}}) = A_{\mathcal{M}',\mathcal{M}''}(\mathcal{D}^{-\frac{1}{2}x(\mathcal{M}')}\widetilde{\varnothing}_{\mathcal{M}'}) \tag{4.8}$$

Indeed, by Theorem 3.18, PLCW decompositions $\mathcal{M}, \mathcal{M}'$ are related by elementary moves fixing the boundary. The operation of changing C_{out} to $\mathcal{M}_{in}$ is a single-cell subdivision, and by Proposition 3.19, can relate any two PLCW decompositions. It is clear that the relation above holds for $\mathcal{M}, \mathcal{M}'$ related by a single-cell move, so we are done. This will be made more clear in Section 5.2.

Remark 4.10. Lemma 4.9 is the analog of "null graphs with respect to D" in the skeins context (see Section 5), in the sense that Lemma 4.9 provides an explicit description of how states transform when the PLCW decomposition changes locally, while a null graph defines an equivalence between graphs that are the same outside a local region.

We may define the relative state space similar to Proposition 3.50:

Definition 4.11. Let M be a 3-manifold with boundary $\partial M = N$. Consider $W = M \times I$ with corner $N \times 0$, so that it is a cornered cobordism from M to M. Let $\mathcal{M}, \mathcal{M}'$ be PLCW decompositions of M with the same boundary $\mathcal{N}$. Let $\mathcal{W}$ be a PLCW structure on W that extends $\mathcal{M}$ on the incoming M and $\mathcal{M}'$ on the outgoing M.

We define, for simple labeling $l_{\mathcal{N}}$ on $\mathcal{N}$,

$$\begin{aligned}
A_{\mathcal{M},\mathcal{M}';l_{\mathcal{N}}} &:= Z_{\mathrm{CY}}(\mathcal{W};(\mathcal{N},l_{\mathcal{N}})) : H(\mathcal{M};(\mathcal{N},l_{\mathcal{N}})) \to H(\mathcal{M}';(\mathcal{N},l_{\mathcal{N}})) \\
A_{\mathcal{M},\mathcal{M}'} &:= Z_{\mathrm{CY}}(\mathcal{W};\mathcal{N}) : H(\mathcal{M}) \to H(\mathcal{M}')
\end{aligned}$$

Note $A_{\mathcal{M},\mathcal{M}'} = \sum_l A_{\mathcal{M},\mathcal{M}';l}$, summed over all simple labelings of $\mathcal{N}$.

The maps $A_{\mathcal{M},\mathcal{M}'}$ compose as in Proposition 3.50, i.e. $A_{\mathcal{M},\mathcal{M}''} = A_{\mathcal{M}',\mathcal{M}''} \circ A_{\mathcal{M},\mathcal{M}'}$, so in particular $A_{\mathcal{M}} := A_{\mathcal{M},\mathcal{M}}$ is a projection, and the spaces

$$Z_{\mathrm{CY}}(\mathcal{M};(\mathcal{N},l)) := \mathrm{im}(A_{\mathcal{M},\mathcal{M};l})$$

are canonically identified, thus we may define $Z_{\mathrm{CY}}(M;(\mathcal{N},l)) = Z_{\mathrm{CY}}(\mathcal{M};(\mathcal{N},l))$ without ambiguity. We call $Z_{\mathrm{CY}}(M;(\mathcal{N},l))$ the *relative state space of* M.

We will often simple denote $A_{\mathcal{M},\mathcal{M}'}$ by A when the context is clear.

Similar to before, we may talk about a *lift* of a state in $Z_{\mathrm{CY}}(\mathcal{M};(\mathcal{N},l))$ to a pre-state in $H(\mathcal{M};(\mathcal{N},l))$.

We also consider the extension of Definition 3.55 to 3-manifolds with boundary:

Definition 4.12. For $\varphi \in H(\mathcal{M};(\mathcal{N},l)), \varphi' \in H(\overline{\mathcal{M}};(\mathcal{N},l))$, the *reduced relative pre-state space pairing* is defined as

$$\overline{ev}^{\partial}(\varphi,\varphi') := \mathrm{ev}^{\partial}(\varphi, A(\varphi')) = \mathrm{ev}^{\partial}(A(\varphi),\varphi'). \tag{4.9}$$

Just as in Definition 3.55, the reduced pairing can also be given a more TQFT-esque flavor:

Lemma 4.13. *Observe that $M \times I$ has boundary $M \cup_N \overline{M}$ (where $N = \partial M$); we treat it as a cobordism $M \times I : M \cup_N \overline{M} \to \varnothing$. Give $M \cup_N \overline{M}$ the PLCW structure $\mathcal{M} \cup_{\mathcal{N}} \overline{\mathcal{M}}$. Then for $\varphi \in H(\mathcal{M}; (\mathcal{N}, l)), \varphi' \in H(\overline{\mathcal{M}}; (\mathcal{N}, l))$, $\varphi \otimes \varphi'$ defines a pre-state in $H(\mathcal{M} \cup_{\mathcal{N}} \overline{\mathcal{M}})$, and*

$$Z_{CY}(M \times I)(\varphi \otimes \varphi') = d_l^{1/2} \operatorname{ev}(Z_{CY}(M \times I; \mathcal{N})(\varphi), \varphi') = \overline{ev}^{\partial}(\varphi, \varphi')$$

(here the ev *is just* ev^{∂} *but without the* $d_l^{1/2}$ *coefficient).*

Proof. Obvious. □

4.2. **Category of Boundary Values: PLCW version.**
As an extended theory, Z_{CY} should associate to each surface N a category $Z_{\mathrm{CY}}(N)$, known as the category of boundary values. Here we define such a category; however, we only develop many of its properties later, in the more convenient language of skein categories.

Definition 4.14. Let N be an *oriented* closed surface. Define the category $\hat{Z}_{\mathrm{CY}}(N)$ whose objects are "direct sums" of colored marked PLCW compositions $\bigoplus(\mathcal{N}, l)$. We describe morphisms for objects $(\mathcal{N}, l)$ and extend to direct sums.

Morphisms are given by

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(N)}((\mathcal{N}, l), (\mathcal{N}', l')) = Z_{\mathrm{CY}}(N \times I; (\mathcal{N}, l), (\mathcal{N}', l')),$$

where $(\mathcal{N}, l)$ is imposed on $N \times 0$ and $(\mathcal{N}', l')$ on $N \times 1$, and $N \times I$ is given the orientation such that the outward orientation at $N \times 1$ is the orientation of N (under the obvious identification $N \simeq N \times 1$).

The identity morphism for $(\mathcal{N}, l)$ given by

$$\mathrm{id}_{(\mathcal{N}, l)} = \mathcal{D}^{\frac{1}{2} e(\mathcal{N})} d_l^{-1/2} A(\widetilde{\mathrm{id}}_{(\mathcal{N}, l)})$$

where

$$\widetilde{\mathrm{id}}_{(\mathcal{N}, l)} = \bigotimes_{f \in \mathcal{N}} \mathrm{coev}_{l(f)} \in H(\mathcal{N} \times I, l \times I)$$
$$d_l^{-1/2} = \prod_{f \in \mathcal{N}} d_{l(f)}^{-1/2}$$

where $l \times I$ is the labeling with l on both copies $\mathcal{N} \times 0$ and $\mathcal{N} \times 1$, and $\mathbf{1}$ on all other 2-faces $e \times I$ of $\mathcal{N} \times I$; more intuitively, it is the "vertical graph" $m \times I$, where m is the marking. (See Lemma 4.15.)

The composition of morphisms is given by:

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(N)}((\mathcal{N}', l'), (\mathcal{N}'', l'')) \otimes \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(N)}((\mathcal{N}, l), (\mathcal{N}', l')) \to \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(N)}((\mathcal{N}, l), (\mathcal{N}'', l''))$$
$$\varphi' \otimes \varphi \mapsto \varphi' \circ \varphi := A(\varphi' \otimes \varphi)$$

We define $Z_{\mathrm{CY}}(N)$ to be the Karoubi envelope of $\hat{Z}_{\mathrm{CY}}(N)$:

$$Z_{\mathrm{CY}}(N) = \mathrm{Kar}(\hat{Z}_{\mathrm{CY}}(N))$$

Lemma 4.15. *The morphism* $\mathrm{id}_{(\mathcal{N}, l)}$ *defined in Definition 4.14 indeed satisfy the properties of an identity morphism.*

Proof. Let $\widetilde{\varphi} \in H(\mathcal{M})$ represent some morphism in $\mathrm{Hom}_{Z_{\mathrm{CY}}(N)}((\mathcal{N}, l), (\mathcal{N}', l'))$, where $\mathcal{M}$ is some PLCW structure on $N \times I$. Consider $\widetilde{\varphi} \otimes \widetilde{\mathrm{id}}_{(\mathcal{N}, l)} \in H(\mathcal{M} \cup_{\mathcal{N}} \mathcal{N} \times I)$. Using Lemma 4.9, we can merge each 3-cell of $\mathcal{N} \times I$ with the adjacent 3-cell in $\mathcal{M}$. We then have essentially $\widetilde{\varphi}$ again, with a factor of $d_l^{1/2}$ from the merging operations (number of 0- and 1-cells remains the same, so no $\mathcal{D}$ factor). We then perform more single-cell moves (Proposition 3.19) to modify the PLCW structure back to $\mathcal{M}$; this accumulates another factor of $\mathcal{D}^{-\frac{1}{2} e(\mathcal{N})}$ (since $\frac{1}{2} x(\mathcal{M} \cup \mathcal{N} \times I) - \frac{1}{2} x(\mathcal{M}) = \frac{1}{2}(x(\mathcal{N} \times I) - x(\mathcal{N})) = \frac{1}{2}((2x(\mathcal{N}) - v(\mathcal{N})) - x(\mathcal{N})) = -\frac{1}{2} e(\mathcal{N})$). Thus it follows that $A(\widetilde{\varphi} \otimes A(\mathcal{D}^{\frac{1}{2} e(\mathcal{N})} d_l^{-1/2} \widetilde{\mathrm{id}})) = A(\widetilde{\varphi} \otimes \mathcal{D}^{\frac{1}{2} e(\mathcal{N})} d_l^{-1/2} \widetilde{\mathrm{id}}) = A(\widetilde{\varphi})$. □

Proposition 4.16. *Let N be an oriented closed surface. A 3-manifold M bounded by $\partial M = N$ defines a functor*

$$\mathcal{F}_M : Z_{CY}(N) \to \mathcal{V}ec$$
$$(\mathcal{N}, l) \mapsto Z_{CY}(M; (\mathcal{N}, l))$$

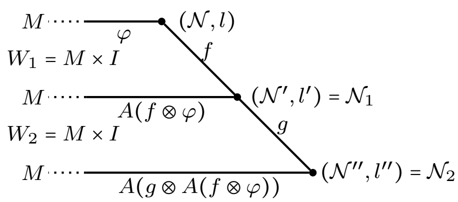
and for a morphism $f : (\mathcal{N}, l) \to (\mathcal{N}', l')$,

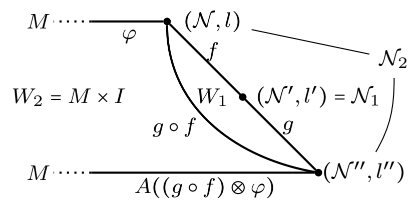
$$\mathcal{F}_M(f) = f \circ - := A(f \otimes -)$$

Furthermore, a cornered cobordism $W : M \to_{\mathcal{N}} M'$ defines a natural transformation

$$\mathcal{F}_W : \mathcal{F}_M \to \mathcal{F}_{M'}$$
$$(\mathcal{F}_W)_{(\mathcal{N},l)} = Z_{CY}(W; l) : Z_{CY}(M; (\mathcal{N}, l)) \to Z_{CY}(M'; (\mathcal{N}, l))$$

Proof. We first prove this for $\hat{Z}_{\mathrm{CY}}(N)$. The functor $\mathcal{F}_M$ is shown to respect composition as follows. For $\varphi \in Z_{\mathrm{CY}}(M; (\mathcal{N}, l))$, $f \in \mathrm{Hom}_{Z_{\mathrm{CY}}(N)}((\mathcal{N}, l), (\mathcal{N}', l'))$, $g \in \mathrm{Hom}_{Z_{\mathrm{CY}}(N)}((\mathcal{N}', l'), (\mathcal{N}'', l''))$, consider the two diagrams

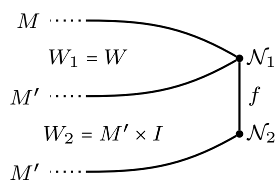


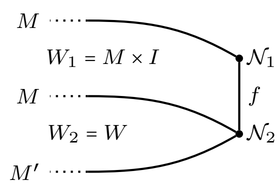


To apply Proposition 4.8, on the left, we give W_1 corner $\mathcal{N}_1 = \mathcal{N}'$, and W_2 corner $\mathcal{N}_2 = \mathcal{N}''$; on the right, we give W_1 corner $\mathcal{N}_1 = \mathcal{N} \sqcup \mathcal{N}''$, and W_2 corner $\mathcal{N}_2 = \mathcal{N}''$. Then,

$$\mathcal{F}_M(g)(\mathcal{F}_M(f)(\varphi)) = A(g \otimes A(f \otimes \varphi)) = A(g \otimes f \otimes \varphi) = A((g \circ f) \otimes \varphi) = \mathcal{F}_M(g \circ f)(\varphi)$$

The naturality of $\mathcal{F}_W$ follows from a similar argument: from





we have

$$\mathcal{F}_{M'}(f)(\mathcal{F}_{W_1}(\varphi)) = A(f \otimes Z_{\mathrm{CY}}(W_1; \mathcal{N}_1)(\varphi)) = Z_{\mathrm{CY}}(W_2; \mathcal{N}_2)(A(f \otimes \varphi)) = \mathcal{F}_{W_2}(\mathcal{F}_M(f)(\varphi))$$

The naturality of $\mathcal{F}_W$ allows us to extend $\mathcal{F}_M$ and $\mathcal{F}_W$ to the Karoubi envelope: for an object $((\mathcal{N}, l), P) \in Z_{\mathrm{CY}}(N)$,

$$\mathcal{F}_M(((\mathcal{N}, l), P)) = P \cdot \mathcal{F}_M((\mathcal{N}, l))$$

(here P is acting by $(P \circ -)$) and for a state $\varphi = P \circ \varphi \in P \cdot \mathcal{F}_M((\mathcal{N}, l))$,

$$\mathcal{F}_W(\varphi) = \mathcal{F}_W(A(P \otimes \varphi)) = A(P \otimes \mathcal{F}_W(\varphi)) \in P \cdot \mathcal{F}_{M'}((\mathcal{N}, l))$$

□

5.1. **Skeins.**

In this section, we give a definition of colored graphs/skeins. This definition essentially coincides with those given in [KT], [JF2015], [Coo2019], however, here everything is performed in the PL category.

Throughout this section, all 3-manifolds are assumed to be oriented, and may be non-compact and/or with boundary.

In [KT], we define skeins using framed graphs, that is, an embedded graph with a normal vector field. We can convert a (smooth) ribbon graph into a framed graph by shrinking the coupons to points, and taking the cores of ribbons to be the edges of the graph, with normal vector field given by the normal to the surface of the ribbon graph (these are defined up to a contractible choice).

The constructions in this section mirror those of Section 4; boundary values (Definition 5.1) correspond to colored marked PLCW decompositions on oriented surfaces, and skeins (Definition 5.6) correspond to states. This will be made explicit in Section 5.2.

Definition 5.1. Let N be an (unoriented) surface, possibly noncompact or with boundary. A *boundary value* $\mathbf{V} = (B, \{V_{\vec{b}}\}_{\vec{b}\in\vec{B}})$ is a finite collection B of embedded unoriented arcs $b : [-1,1] \to N$, together with an assignment of an object $V_{\vec{b}} \in \mathcal{A}$ to each oriented arc $\vec{b}$, such that $v_{\overleftarrow{b}} = v_{\vec{b}}^*$.

The assignment $V_{\vec{b}}$, in particular the duality under changing orientations, is meant to mirror labelings (see Section 5.2).

Definition 5.2. The *empty configuration* or *empty boundary value* on the surface N, denoted $\mathbf{E}_N$ or simply $\mathbf{E}$, is the boundary value with no marked points, $\mathbf{E} = (\varnothing, \{\}) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(N)$.

Definition 5.3. Given an embedding of surfaces $f : N \hookrightarrow N'$ and a boundary value $\mathbf{V}$ on N, we may define the boundary value $f_*(\mathbf{V}) = (f(B), \{V_{\vec{b}}\})$ on N'.

Definition 5.4. Let M be an oriented 3-manifold with boundary $N = \partial M$. A colored ribbon graph (Γ, Ψ) in M defines a boundary value $\mathbf{V} = (B, \{V_{\vec{b}}\})$ on N, with $B = \Gamma \cap N$, and for $\vec{b} = \vec{e} \cap N$, where $\vec{e}$ is a leg of Γ taken with direction pointing outwards at b, and $\vec{b}$ is given the outward orientation with respect to Γ as an oriented surface, we take $V_{\vec{b}} = \Psi(\vec{e})$, and $V_{\overleftarrow{b}} = \Psi(\overleftarrow{e})$. We define

$$\mathrm{Graph}(M;\mathbf{V}) = \text{set of all colored ribbon graphs in } M \text{ with boundary value } \mathbf{V}$$

and similarly consider formal linear combinations:

$$\mathrm{VGraph}(M;\mathbf{V}) = \{\text{formal linear combinations of graphs } \Gamma \in \mathrm{Graph}(M;\mathbf{V})\}$$

We denote the empty graph in M by $\varnothing_M^\Gamma$; it is a basis element of $\mathrm{VGraph}(M;\mathbf{E})$, and should not be confused with the 0 element in $\mathrm{VGraph}(M;\mathbf{E})$.

Definition 5.5. Let $\Gamma = \sum c_i\Gamma_i \in \mathrm{VGraph}(M;\mathbf{V})$ be a formal linear combination of colored graphs, and D an embedded closed ball in M. (Note D is allowed to touch the boundary ∂M.) We say that Γ is *null with respect to D* if either

(1) $\Gamma = \zeta^t(\Gamma^0) - \Gamma^0$, where ζ^t is an ambient isotopy supported on D, or
(2) each Γ_i meets ∂D transversally, all Γ_i coincide outside of D as colored graphs, and

$$\langle\Gamma\rangle_D = \sum c_i \langle\Gamma_i\rangle_D = 0$$

where $\langle\Gamma\rangle_D$ is the Reshetikhin-Turaev evaluation of Γ in a ball D (see Proposition 3.34; some choice of identification $D \simeq B^3$ is needed, but it is clear that the choice is irrelevant in this context).

Note that if we are in the first situation, and Γ is transversal to ∂D, then the second condition subsumes the first, since isotopic graphs have the same Reshetikhin-Turaev evaluation. Thus, if we imagine all ribbon graphs to be narrow enough (or even infinitesimally narrow as in Remark 3.27), then we can essentially disregard the first condition.

Definition 5.6. For an oriented 3-manifold M and boundary condition $\mathbf{V} = (B, \{V_b\})$ on ∂M, we define the *skein module of M with boundary value* $\mathbf{V}$ as

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(M; \mathbf{V}) = \mathrm{VGraph}(M; \mathbf{V})/Q$$

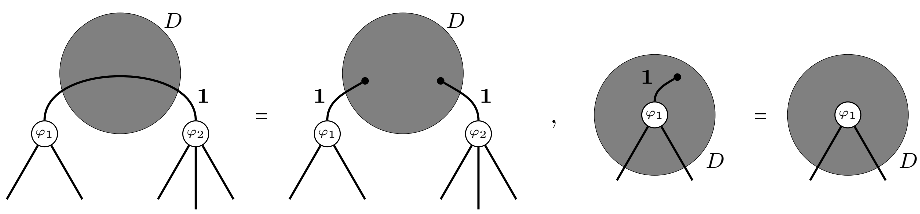
where Q is the subspace spanned by all null graphs (for all possible embedded balls). An element of $Z_{\mathrm{CY}}^{\mathrm{sk}}(M; \mathbf{V})$ is called a *skein with boundary value* $\mathbf{V}$, or simple a skein.

Definition 5.7. The *empty skein*, denoted $\varnothing_M^{\mathrm{sk}}$, is the equivalence class of the empty graph $[\varnothing_M^{\Gamma}] \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M, \mathbf{E})$.

For example, $Z_{\mathrm{CY}}^{\mathrm{sk}}(S^3) \simeq \mathbf{k}$, as a ribbon graph $\Gamma \subset S^3$ is always contained in the interior of some ball D, which has some Reshetikhin-Turaev evaluation $Z_{\mathrm{RT}}(\Gamma) \in \mathbf{k}$, and so Γ is equivalent, as a skein, to $Z_{\mathrm{RT}}(\Gamma) \cdot \varnothing_M^{\mathrm{sk}}$.

We also have $Z_{\mathrm{CY}}^{\mathrm{sk}}(\varnothing) \simeq \mathbf{k}$, as expected from a 4-dimensional TQFT, since the empty graph is the only graph in the empty 3-manifold, and there are no relations.

Ribbons labeled with $\mathbf{1}$ can essentially be ignored:



Definition 5.8. Given a homeomorphism $f : M \simeq M'$ and a boundary value $\mathbf{V}$ on M, we define

$$f_* : Z_{\mathrm{CY}}^{\mathrm{sk}}(M; \mathbf{V}) \simeq Z_{\mathrm{CY}}^{\mathrm{sk}}(M'; f_*(\mathbf{V}))$$

by simply applying f to graphs.

Definition 5.9. Let M' be a 3-manifold obtained from M by gluing two boundary components N, N' together via a diffeomorphism $f : \overline{N} \simeq N'$. Let $\mathbf{V}$ be a boundary value on N, and $\mathbf{V}' = f_*(\mathbf{V})$.

Then for $\varphi \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M; \mathbf{V}, \mathbf{V}', \mathbf{W})$, where $\mathbf{W}$ is some boundary value on the other boundary components of M', we define $\varphi/_N$ to be the skein obtained as the image of φ under the gluing $M \to M/f = M'$. If N separates $M = M_1 \cup_N M_2$, and $\varphi_1 \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M_1; \mathbf{V}, ..), \varphi_2 \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M_2; , \mathbf{V}', ..)$, we also use the notation

$$\varphi_1 \cup_N \varphi_2 = (\varphi_1 \otimes \varphi_2)/_N$$

We can now define the category of boundary conditions.

Definition 5.10. Let N be an oriented surface, possibly non-compact. Suppose first N has no boundary. Define $\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}(N)$ as the category whose objects are boundary values on N, and the morphisms are given by:

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}(N)}(\mathbf{V}, \mathbf{V}') = Z_{\mathrm{CY}}^{\mathrm{sk}}(N \times [0,1]; \iota_*^0(\mathbf{V}), \iota_*^1(\mathbf{V}')),$$

where $\iota^j : N \simeq N \times \{j\}$, and $N \times [0,1]$ is oriented so that $N \times 1$ (with the orientation from N) has the outward orientation. The identity morphism $\mathrm{id}_{\mathbf{V}}$, for $\mathbf{V} = (B, \{V_b\})$, is given by the "vertical" graph $B \times [0,1]$ with the obvious coloring.

Composition of morphisms $\varphi \in \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}}(\mathbf{V}, \mathbf{V}')$, $\varphi' \in \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}}(\mathbf{V}', \mathbf{V}'')$ is given by

$$\varphi' \circ \varphi = \varphi' \cup_N \varphi$$

i.e. joining the underlying graphs along $\mathbf{V}'$.

$\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}$ is additive and $\mathbf{k}$-linear. We define

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(N) = \mathrm{Kar}(\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}(N)), \tag{5.1}$$

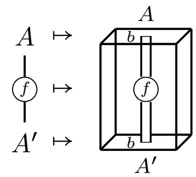
its Karoubi envelope.

For N with boundary, we define $\hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}(N) = \hat{Z}_{\mathrm{CY}}^{\mathrm{sk}}(N \backslash \partial N)$, $Z_{\mathrm{CY}}^{\mathrm{sk}}(N) = Z_{\mathrm{CY}}^{\mathrm{sk}}(N \backslash \partial N)$.

We call $Z_{\mathrm{CY}}^{\mathrm{sk}}(N)$ the *skein category of N*, or the *category of boundary values on N*.

It is immediate from the definition that for a 2-disk $\mathbb{D}^2$, $Z^{\mathrm{sk}}_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A}$; by choosing some arc b in $\mathbb{D}^2$, for $A, A' \in \mathrm{Hom}_{\mathcal{A}}(A, A')$,

(5.2)



Next let us address the 4-dimensional component of $Z^{\mathrm{sk}}_{\mathrm{CY}}$.

Definition 5.11. Let $W = \mathcal{H}_k \widetilde{\circ} \mathrm{id}_M : M \to_N M'$ be an elementary cornered cobordism of index k. (see Definition 2.66). For a boundary value $\mathbf{V} \in Z^{\mathrm{sk}}_{\mathrm{CY}}(N)$, we define

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N) : Z^{\mathrm{sk}}_{\mathrm{CY}}(M; \mathbf{V}) \to Z^{\mathrm{sk}}_{\mathrm{CY}}(M'; \mathbf{V})$$

case-by-case: given a colored ribbon graph $\Gamma \in Z^{\mathrm{sk}}_{\mathrm{CY}}(M; \mathbf{V})$, $Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma)$ is constructed as follows:

- $k = 0$: $\mathcal{H}_0$ adds a new S^3 component to M; we define
$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma) \coloneqq \mathcal{D} \cdot \Gamma \cup \varnothing^{\mathrm{sk}}_{S^3}$$
- $k = 4$: $\mathcal{H}_4$ kills off an S^3 component of M; writing $\Gamma = \Gamma' \sqcup \Gamma''$ with Γ' in the S^3 component and Γ'' in the other component of M,
$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma) \coloneqq Z_{\mathrm{RT}}(\Gamma') \cdot \Gamma''$$
- $k = 1$: the attaching region of $\mathcal{H}_1$ is a pair of balls; by an isotopy, we may arrange that Γ is disjoint from the attaching region, and regard Γ as a graph in M', and define
$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma) \coloneqq \mathcal{D}^{-1} \cdot \Gamma$$
- $k = 2$: similar to the $k = 1$ case, arrange Γ to be disjoint from the attaching region. The belt sphere of $\mathcal{H}_2$ (and its neighborhood) defines an embedding of the solid torus $B^2 \times \partial B^2 \to M'$. Let $\gamma = [-\varepsilon, \varepsilon] \times \partial B^2 \subset B^2 \times \partial B^2$ be the core of the solid torus, trivially framed, and let $\Gamma' = (\gamma, \text{regular})$ be the colored ribbon graph obtained by applying the regular coloring to γ. Then, sending Γ' to M' under the embedding,
$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma) = \Gamma \cup \Gamma'$$
- $k = 3$: first suppose that Γ intersects the co-core of $\mathcal{H}_3$ transversally and in exactly one ribbon, and suppose the label of this ribbon is a simple object i. If $i = \mathbf{1}$, we may ignore this ribbon and isotope Γ to be disjoint from $\mathcal{H}_3$; then we may define
$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma) = \delta_{i,\mathbf{1}} \cdot \Gamma$$
In general, by isotopy, we may arrange Γ to be transverse to the co-core of $\mathcal{H}_3$, and then apply Lemma 2.27 to get $\Gamma = \sum_j \Gamma_j$, where each Γ_j satisfies the previous assumption (see (5.16)). Then we extend to this case by linearity.

It is not hard to see that this definition is well-posed:

- for $k = 0, 3, 4$, it is clear that $Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N)(\Gamma)$ is well-defined,
- for $k = 2$, well-defined-ness follows from the sliding lemma Lemma 2.30, and
- for $k = 3$, well-defined-ness follows from Lemma 2.28 (see also (5.16)).

Definition 5.12. Let $W : M \to_N M'$ be a cornered cobordism, and let $W = W_l \circ \cdots \circ W_1$ be a handle decomposition (Definition 2.68). We define

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W; N) = Z^{\mathrm{sk}}_{\mathrm{CY}}(W_l; N) \circ \cdots \circ Z^{\mathrm{sk}}_{\mathrm{CY}}(W_1; N)$$

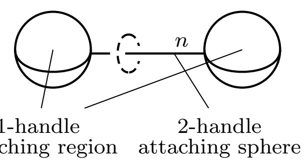
Proposition 5.13. *The map $Z^{sk}_{CY}(W; N)$ defined in Definition 5.12 is independent of handle decomposition of W.*

Proof. By Proposition 2.72, it suffices to check that any modification as in Definition 2.71 leaves the overall $Z^{\mathrm{sk}}_{\mathrm{CY}}(W;N)$ unchanged.

It is clear that reordering handles does not affect $Z^{\mathrm{sk}}_{\mathrm{CY}}(W;N)$.

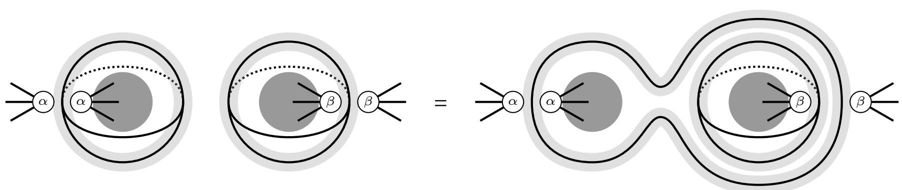
Let us consider handle pair cancellation. We consider a graph Γ in M, then consider a pair of canceling elementary cobordisms, and observe what happens to Γ. Canceling a 0-1 pair is clear, as the $\mathcal{D}^{\pm}$ factors cancel. A 1-2 canceling pair looks like

$\frac{1}{\mathcal{D}}$



The dashed loop, coming from the belt sphere of $\mathcal{H}_2$, can be pull out, and evaluated to $\mathcal{D}$, thus canceling the $1/\mathcal{D}$ factor. For a 2-3 pair, we first have $\Gamma \cup \Gamma'$ from $\mathcal{H}_2$. Then the graph γ from $\mathcal{H}_2$ must meet the co-core of $\mathcal{H}_3$ exactly once, and is cut, leaving only the $i = \mathbf{1}$ strand, which leaves us with Γ again. For a 3-4 pair, the attaching region of $\mathcal{H}_3$ is $\partial B^3 \times B^1$, and the fact that $\mathcal{H}_3$ cancels with $\mathcal{H}_4$ implies this attaching region separates M into $M' \sqcup B^3$ (the B^3, after attaching $\mathcal{H}_3$, becomes an S^3 component that gets killed by $\mathcal{H}_4$). Thus, Γ can be isotoped away from B^3, and then away from the attaching region, thus it is not affected by attaching both $\mathcal{H}_3$ and $\mathcal{H}_4$.

Finally let us consider handle slides. There is nothing to prove for 0- and 4-handles, and for 1-handles it is clear. For 2-handles, independence follows from the sliding lemma (Lemma 2.30), which is the same way that the Reshetikhin-Turaev invariant for 3-manifolds is shown to be invariant with respect to handle slides in Kirby calculus. For 1-handles, we have



□

Note that Theorem 5.26 provides an alternative proof to Proposition 5.13.

It is easy to get a formula relating connected sums to their components:

Proposition 5.14. *For non-empty closed 4-manifolds W_1, W_2,*

$$Z^{sk}_{CY}(W_1 \# W_2) = \mathcal{D}^{-1} \cdot Z^{sk}_{CY}(W_1) \cdot Z^{sk}_{CY}(W_2)$$

More generally, for non-empty 4-manifolds W_1, W_2, possibly with boundary,

$$Z^{sk}_{CY}(W_1 \# W_2) = \mathcal{D}^{-1} \cdot Z^{sk}_{CY}(W_1) \otimes Z^{sk}_{CY}(W_2) \in Z^{sk}_{CY}(\partial W_1) \otimes Z^{sk}_{CY}(\partial W_2)$$

Proof. Let $m = 1, 2$. Choose some handle decomposition for W_m as a cobordism $W_m : \varnothing \to \partial W_m$ that contains at least one handle (we can create one using handle pair creation; see Definition 2.71 or proof of Proposition 5.13); let $\mathcal{H}_4^{(m)}$ be such a 4-handle.

Let $W'_m = W_m \backslash \mathcal{H}_4^{(m)}$. Note that $\partial W'_m = \partial W_m \sqcup S^3$. It is clear from Definition 5.11 that

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W'_m) = Z^{\mathrm{sk}}_{\mathrm{CY}}(W_m) \otimes \varnothing^{\mathrm{sk}}_{S^3} \in Z^{\mathrm{sk}}_{\mathrm{CY}}(\partial W_m) \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(S^3) \simeq Z^{\mathrm{sk}}_{\mathrm{CY}}(\partial W'_m)$$

We can obtain $W_1 \# W_2$ by attaching a 1-handle that connects the S^3 components from $\partial W'_1$ and $\partial W'_2$, then capping the resulting S^3 component off with a 4-handle. Together these operations give

$$\begin{aligned} Z^{\mathrm{sk}}_{\mathrm{CY}}(W'_1) \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(W'_2) &= (Z^{\mathrm{sk}}_{\mathrm{CY}}(W_1) \otimes \varnothing^{\mathrm{sk}}_{S^3}) \otimes (Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2) \otimes \varnothing^{\mathrm{sk}}_{S^3}) \\ &\overset{\text{1-handle}}{\mapsto} \mathcal{D}^{-1} \cdot Z^{\mathrm{sk}}_{\mathrm{CY}}(W_1) \otimes \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2) \otimes \varnothing^{\mathrm{sk}}_{S^3} \\ &\overset{\text{4-handle}}{\mapsto} \mathcal{D}^{-1} \cdot Z^{\mathrm{sk}}_{\mathrm{CY}}(W_1) \otimes \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2) \end{aligned}$$

(Note that the arguments above work for a self-connect sum, i.e. if $W_1 = W_2$.) □

Lemma 5.15. *Let $W : M_0 \to_N M_1$ be a cornered cobordism, and let $M : N \to N'$ be a cobordism. Recall the cornered cobordism obtained from W by extending along M from Definition 2.62 is the cornered cobordism $W_{M:N\to N'} = W \tilde{\circ} \mathrm{id}_{M_0 \cup_N M} : M_0 \cup_N M \to_{N'} M_1 \cup_N M$. Then for $\varphi \in Z^{sk}_{CY}(M_0; \mathbf{V}), \varphi' \in Z^{sk}_{CY}(M; \mathbf{V}, \mathbf{V}')$,*

$$Z^{sk}_{CY}(W_{M:N\to N'}; N')(\varphi' \cup_N \varphi) = \varphi' \cup_N Z^{sk}_{CY}(W)(\varphi)$$

Proof. By construction. ☐

We consider a skein-theoretic analog of Proposition 4.8:

Proposition 5.16. *Let W_1, W_2 be cornered cobordisms*

$$W_1 : M_1 \to_{N_1} M_1'$$
$$W_2 : M_2 \to_{N_2} M_2'$$

Suppose $M_1' \subseteq M_2$ is a submanifold. Let $W = W_1 \cup_{M_1'} W_2$, and $M = (M_2 \backslash M_1') \cup M_1$, so that W is the extended cobordism (Definition 2.63)

$$W : M \to_{N_2} M_2'$$

Then $Z^{sk}_{CY}(W_1; N_1)$ and $Z^{sk}_{CY}(W_2; N_2)$ compose to give $Z^{sk}_{CY}(W; N_2)$; more precisely,

$$Z^{sk}_{CY}(W; N_2) = Z^{sk}_{CY}(W_2; N_2) \circ (\mathrm{id} \cup Z^{sk}_{CY}(W_1; N_1))$$
$$Z^{sk}_{CY}(W; N_2)(\Phi) = Z^{sk}_{CY}(W_2; N_2)(\psi \cup Z^{sk}_{CY}(W_1; N_1)(\varphi))$$

where $\Phi \in Z^{sk}_{CY}(M; \mathbf{V})$ can be obtained by gluing colored graphs $\varphi \subset M_1, \psi \subset M \backslash M_1$, and $\mathbf{V}$ is some boundary value on N_2,

Similarly, suppose that we have the reverse inclusion $M_2 \subseteq M_1'$. Let $W = W_1 \cup_{M_2} W_2$, and $M' = (M_1' \backslash M_2) \cup M_2'$, so that W is the extended cobordism (Definition 2.63)

$$W : M_1 \to_{N_1} M_2'$$

Then $Z^{sk}_{CY}(W_1; N_1)$ and $Z^{sk}_{CY}(W_2; N_2)$ compose to give $Z^{sk}_{CY}(W; N_1)$; more precisely,

$$Z^{sk}_{CY}(W; N_1) = (\mathrm{id} \otimes Z^{sk}_{CY}(W_2; N_2)) \circ (Z^{sk}_{CY}(W_1; N_1))$$
$$Z^{sk}_{CY}(W; N_1)(\Phi) = \sum_a \varphi^{(a)} \cup Z^{sk}_{CY}(W_2; N_2)(\psi^{(a)})$$

where $Z^{sk}_{CY}(W_1; N_1)(\Phi) = \sum_a \varphi^{(a)} \cup \psi^{(a)}$, with $\psi^{(a)} \in Z^{sk}_{CY}(M_2; \mathbf{V})$ for some boundary value $\mathbf{V}$ on N_2.

Proof. Also by construction. ☐

We have an analog of Proposition 4.16:

Proposition 5.17. *Let N be an oriented closed surface. A 3-manifold M bounded by $\partial M = N$ defines a functor*

$$\mathcal{F}^{sk}_M : Z_{CY}(N) \to \mathcal{V}ec$$
$$\mathbf{V} \mapsto Z^{sk}_{CY}(M; \mathbf{V})$$

and for a morphism $f : \mathbf{V} \to \mathbf{V}'$,

$$\mathcal{F}^{sk}_M(f) = f \cup_N -$$

Furthermore, a cornered cobordism $W : M \to_N M'$ defines a natural transformation

$$\mathcal{F}^{sk}_W : \mathcal{F}^{sk}_M \to \mathcal{F}^{sk}_{M'}$$
$$(\mathcal{F}^{sk}_W)_{\mathbf{V}} = Z^{sk}_{CY}(W; N) : Z^{sk}_{CY}(M; \mathbf{V}) \to Z^{sk}_{CY}(M'; \mathbf{V})$$

Proof. Similar to Proposition 4.16. ☐

We also consider the analog of the reduced relative state space pairing (Definition 4.12), based on the more TQFT-esque interpretation (see Lemma 4.13):

Definition 5.18. Observe that $W = M \times I$ has boundary $M \cup_N \overline{M}$ (where $N = \partial M$); we treat it as a cobordism $W : M \cup_N \overline{M} \to \varnothing$ (see Lemma 4.13). For a boundary value $\mathbf{V}$ on N and skeins $\varphi \in Z^{\mathrm{sk}}_{\mathrm{CY}}(M;\mathbf{V}), \varphi' \in Z^{\mathrm{sk}}_{\mathrm{CY}}(\overline{M};\mathbf{V})$, we define the *skein pairing*

$$\mathrm{ev}^{sk}(\varphi,\varphi') = Z^{\mathrm{sk}}_{\mathrm{CY}}(W)(\varphi \cup_N \varphi') \tag{5.3}$$

We will see in Section 5.2 that this pairing is equivalent to the pairing in Definition 4.12, which is non-degenerate.

Lemma 5.19. *A cornered cobordism $W : M \to_N M'$ defines a map*

$$Z^{sk}_{CY}(W;N) : Z^{sk}_{CY}(M;\mathbf{V}) \to Z^{sk}_{CY}(M';\mathbf{V})$$

but as a cobordism $W : M \cup_N \overline{M'} \to \varnothing$, we have

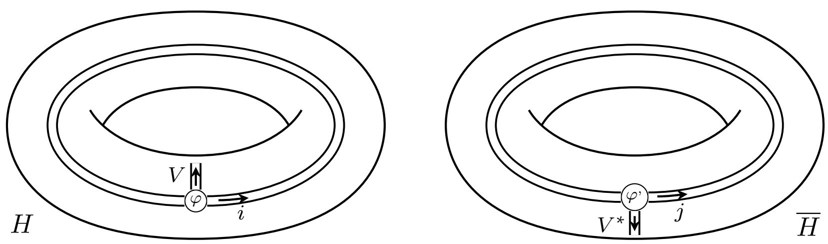
$$Z^{sk}_{CY}(W) : Z^{sk}_{CY}(M;\mathbf{V}) \otimes Z^{sk}_{CY}(\overline{M'};\mathbf{V}) \to Z^{sk}_{CY}(M \cup_N \overline{M'}) \to \mathbf{k}$$

Then for $\varphi \in Z^{sk}_{CY}(M;\mathbf{V})$, $\varphi' \in Z^{sk}_{CY}(\overline{M'};\mathbf{V})$,

$$\mathrm{ev}^{sk}(Z^{sk}_{CY}(W;N)(\varphi),\varphi') = Z^{sk}_{CY}(W)(\varphi \cup \varphi')$$

Proof. Follows immediately from Proposition 5.16. ☐

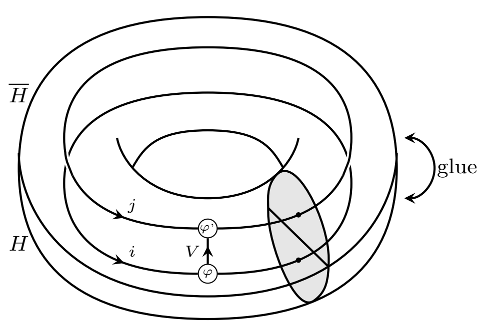
Example 5.20. Let us give a more explicit description of the skein pairing for handlebodies, and show that it agrees (up to a factor of $\mathcal{D}^{g/2}$) with [BakK2001, (4.4.5)]. Consider the following skeins in the genus 1 handlebodies (i.e. solid tori) H and $\overline{H}$:

(5.4)


Note that $\overline{H}$ is drawn after reflecting H across a horizontal plane, so that its interior appears with the same orientation as H.

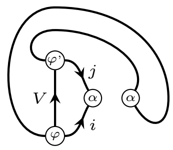
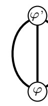
Let $N = \partial H$. Observe that H is built from N by first attaching a 2-handle (its core is a vertical disc), then a 3-handle (a ball that fills the rest of the interior of H).

The 4-manifold defining the skein pairing (i.e. $H \times I$) is built similarly, but with each handle bumped up one index. We start with $H \cup_N \overline{H}$, which we can visualize as follows:

(5.5)


The gray disks are the cores of the (3-dimensional) 2-handles making up H and $\overline{H}$; they glue up in $H \cup_N \overline{H}$ to form a 2-sphere, to which we attach a (4-dimensional) 3-handle. Similarly, the (3-dimensional) 3-handles making up the rest of H and $\overline{H}$ glue up in $H \cup_N \overline{H}$ to form a 3-sphere, to which we attach a (4-dimensional)

4-handle. Thus, based on Definition 5.11, we have that (writing φ for both the coloring of the coupon and the skein that results from the coloring)

(5.6) $\quad \mathrm{ev}^{sk}(\varphi,\varphi') = Z^{\mathrm{sk}}_{\mathrm{CY}}(H\times I)(\varphi\cup\varphi') =$  $= \delta_{i*,j}\cdot d_i^{-1}\cdot$  $= \delta_{i*,j}\cdot d_j^{-1}\cdot \mathrm{ev}(\varphi,\varphi')$

The computation above easily generalizes to handlebodies of arbitrary genus: in a handlebody H of genus g, with a boundary value $\mathbf{V} = (B, \{V_{\vec{b}}\})$, choose a co-ribbon graph with a single coupon like in (5.4) (more precisely, choose PLCW structure on H with exactly one 3-cell, and appropriate number of 2-cells on the boundary, and take the anchor as our co-ribbon graph). Then for labeling $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, (\otimes_b V_{\vec{b}}) \otimes (X_{i_1} \otimes X^*_{i_1}) \otimes \cdots \otimes (X_{i_g} \otimes X^*_{i_g}))$ of the coupon in H, where $i_1,\ldots,i_g$ are simple labels of ribbons in the interior, and labeling $\varphi' \in \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, (X^*_{j_g}\otimes X_{j_g})\otimes\cdots(X^*_{j_1}\otimes X_{j_1})\otimes(\otimes_b V^*_{\vec{b}}))$ of the coupon in $\overline{H}$, we have

$$\mathrm{ev}^{sk}(\varphi,\varphi') = \prod_{k=1}^{g} \delta_{i_k^*,j_k}\cdot d_{j_g}^{-1}\cdot \mathrm{ev}(\varphi,\varphi') \tag{5.7}$$

It is easy to see that this pairing agrees (up to a factor of $\mathcal{D}^{g/2}$) with [BakK2001, (4.4.5)].

5.2. Equivalence with $Z_{\mathbf{CY}}$.

We show an equivalence between the theory of skeins and the theory established with PLCW decompositions. The proof of Lemma 5.22 and Lemma 5.24 mirror the methods used in [Kir], except that all dimensions increase by 1, and we deal with manifolds with boundaries directly.

Definition 5.21. Let N be a closed surface. For a colored marked PLCW decomposition $(\mathcal{N}, l)$, we assign a boundary value $\Upsilon((\mathcal{N},l)) = (B,\{V_{\vec{b}}\})$ where B is the marking of $(\mathcal{N},l)$ (forgetting the co-orientation), and, for the marking b, with orientation $\vec{b}$, of some 2-cell f with the orientation $(\vec{b},\hat{n})$, we set $V_{\vec{b}} = l(f)$. (Note this is opposite from the convention discussed in Remark 3.22, but the reason for violating the convention will be clear soon.)

Lemma 5.22. *Let $\mathcal{M}$ be an oriented PLCW 3-manifold with boundary $\mathcal{N}$. Arbitrarily choose an anchoring of $\mathcal{M}$, giving us a marking of $\mathcal{N}$. Let $l_{\mathcal{N}}$ be a simple labeling of $\mathcal{N}$, and let $\mathbf{V} = \Upsilon((\mathcal{N}, l_{\mathcal{N}}))$.*

Consider some vector $\Phi = \bigotimes \varphi_C \in \bigotimes H(C,l) \subset H(\mathcal{M};(\mathcal{N},l_{\mathcal{N}}))$, where l is a labeling agreeing with $l_{\mathcal{N}}$ on the boundary. The anchors assemble into a ribbon graph Γ, and Φ naturally defines a coloring of Γ which we also denote Φ; we consider a differently-normalized coloring $\mathcal{D}^{\frac{1}{2}x(\mathcal{M}\backslash\mathcal{N})+\frac{1}{4}x(\mathcal{N})} d_{l^\circ}^{1/2} d_{l_{\mathcal{N}}}^{1/4}\Phi$, where $d_{l^\circ}^{1/2} = \prod d_{l(f)}^{1/2}$, the product taken over unoriented internal 2-cells of $\mathcal{M}$, $d_{l_{\mathcal{N}}}^{1/4} = \prod_{f\in\mathcal{N}} d_{l(f)}^{1/4}$, and x is from Lemma 4.9, This defines a linear map

$$\upsilon : H(\mathcal{M};(\mathcal{N},l_{\mathcal{N}})) \to Z^{sk}_{CY}(M;\mathbf{V}) \tag{5.8}$$

$$\Phi \mapsto (\Gamma, \mathcal{D}^{\frac{1}{2}x(\mathcal{M}\backslash\mathcal{N})+\frac{1}{4}x(\mathcal{N})} d_{l^\circ}^{1/2} d_{l_{\mathcal{N}}}^{1/4}\Phi \tag{5.9}$$

This map factors through another map. Let $M^\circ = M\backslash \mathrm{sk}^1(\mathcal{M})$, i.e. remove the 1-skeleton of $\mathcal{M}$ from M. Then (5.9) factors through

$$\upsilon : H(\mathcal{M};(\mathcal{N},l_{\mathcal{N}})) \to Z^{sk}_{CY}(M^\circ;\mathbf{V}) \tag{5.10}$$

$$\Phi \mapsto (\Gamma, \mathcal{D}^{\frac{1}{2}x(\mathcal{M}\backslash\mathcal{N})+\frac{1}{4}x(\mathcal{N})} d_{l^\circ}^{1/2} d_{l_{\mathcal{N}}}^{1/4}\Phi \tag{5.11}$$

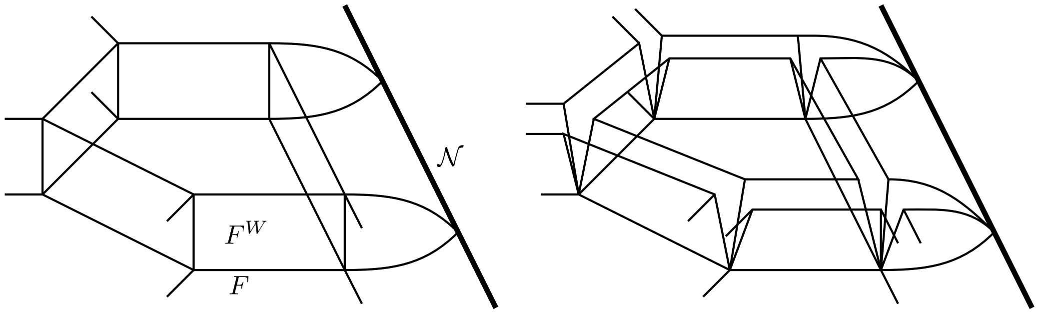
Now for an interior 1-cell e of $\mathcal{M}$, define the map $B_e : Z^{sk}_{CY}(M^\circ;\mathbf{V}) \to Z^{sk}_{CY}(M^\circ;\mathbf{V})$, where we add a ribbon loop, label with $\frac{1}{\mathcal{D}}$ times the regular coloring. By Lemma 2.30, B_e is a projection. It is also clear that $B_e, B_{e'}$ commute for any two interior 1-cells e,e'. Thus, we may consider the projection

$$B_{\mathcal{M}} = \prod B_e : Z^{sk}_{CY}(M^\circ;\mathbf{V}) \to Z^{sk}_{CY}(M^\circ;\mathbf{V}) \tag{5.12}$$

Then υ intertwines $A_{\mathcal{M}}$ and $B_{\mathcal{M}}$, i.e. $\upsilon\circ A_{\mathcal{M}} = B_{\mathcal{M}}\circ\upsilon$.

Proof. Recall that $A_{\mathcal{M}} = Z_{\mathrm{CY}}(W;\mathcal{N})$, where $W = M \times I$ with corner $N \times 0$ (see Definition 4.11). Consider the PLCW decomposition $\mathcal{W}$ that has exactly one interior $(k+1)$-cell C^W for each k-cell C of $\mathcal{M}$ (essentially, $\mathcal{W} = \mathcal{M} \times I$, but the boundary $N \times I$ is squashed; see (5.13)).

(5.13)



Recall

$$Z(\mathcal{W}, l) = \mathrm{ev}(\bigotimes_C Z(C^W, l))$$

where ev applies the pairing (2.9) to each interior 3-cell of $\mathcal{W}$; by construction, each interior 3-cell is of the form $F^W = F \times I$ for some interior 2-cell F of $\mathcal{M}$. Choose dual bases $\{\varphi_{F,\alpha}\}, \{\varphi_{\overline{F},\alpha}\}$ for $H(F^W, l), H(\overline{F^W}, l)$ for each such F. Then, for $\Phi = \otimes\varphi_C$ as in the proposition statement,

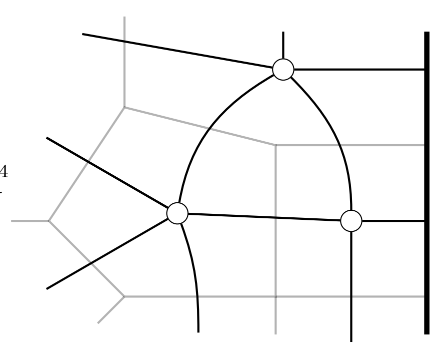
$$\mathrm{ev}(Z(\mathcal{W}, l), \Phi) = \mathrm{ev}(\bigotimes_C Z(C^W, l), \bigotimes_C \varphi_C \otimes \bigotimes_F \varphi_{F,\alpha} \otimes \varphi_{\overline{F},\alpha}) = \sum_\alpha \bigotimes_C \mathrm{ev}(Z(C^W, l), \varphi_C \otimes \bigotimes_F \varphi_{\overline{F},\alpha})$$

where, in the middle expression, the sum over α is implicit, and in the last expression, the F runs over interior 2-cells in ∂C. (If F has outward orientation with respect to C, then $F^W = F \times I$ has the outward orientation with respect to C^W, so we should evaluate $Z(C^W, l)$ against $\varphi_{\overline{F},\alpha} \in H(\overline{F^W}, l)$, not $\varphi_{F,\alpha} \in H(F^W, l)$.)

Now each term $\mathrm{ev}(Z(C^W, l), \varphi_C \otimes \bigotimes_F \varphi_{\overline{F},\alpha})$ in the last expression is a local state in $H(C, l)$, and is φ_{ev} as discussed in Lemma 4.9.

Thus, if $\upsilon(\Phi)$ is of the form

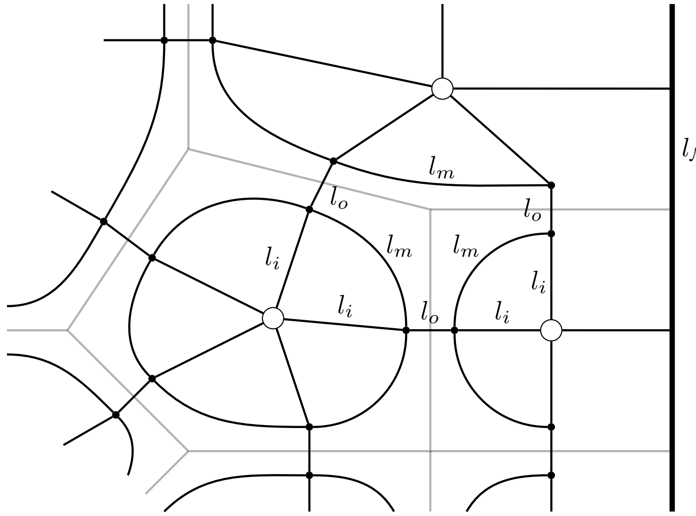
(5.14) $\upsilon(\Phi) = \mathcal{D}^y d_{l^\circ}^{1/2} d_{l_{\mathcal{N}}}^{1/4}$



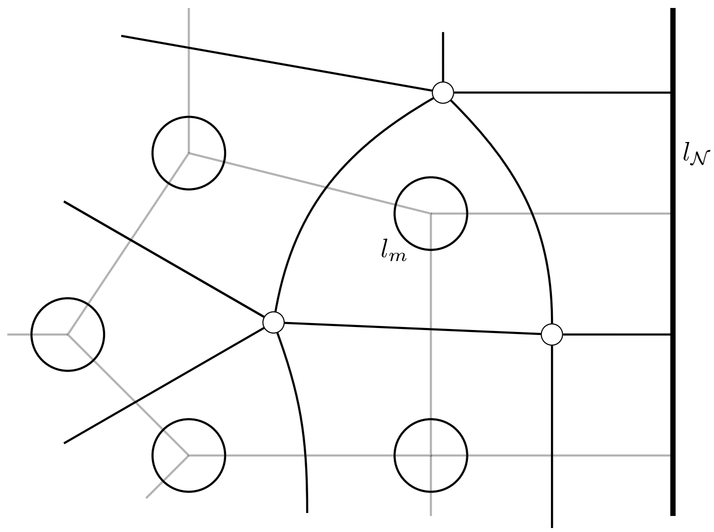
where the circles should be labeled by φ_C's, and $y = \frac{1}{2}x(\mathcal{M}\backslash\mathcal{N}) + \frac{1}{4}x(\mathcal{M})$, then we have, writing $l_i = l^\circ$, $l_o = l^\circ_{out}$, $l_m = l_{mid}$ (for the labeling on the 2-cells in the incoming boundary, outgoing boundary, and interior of $\mathcal{W}$, respectively), and $y' = x(\mathcal{W}\backslash\partial\mathcal{W}) + \frac{1}{2}x(\partial\mathcal{W}\backslash\mathcal{N}) = -v(\mathcal{M}\backslash\mathcal{N}) + x(\mathcal{M}\backslash\mathcal{M}) = -e(\mathcal{M}\backslash\mathcal{N})$,

(5.15)

$$\upsilon(A(\Phi)) = \sum_{l_m, l_o} \underbrace{\mathcal{D}^{y} d_{l_o}^{1/2} d_{l_{\mathcal{N}}}^{1/4}}_{\text{from } \upsilon} \underbrace{\mathcal{D}^{y'} d_{l_i}^{1/2} d_{l_m} d_{l_o}^{1/2}}_{\text{from } Z_{\mathrm{CY}}(\mathcal{W};\mathcal{N})}$$



$$= \sum_{l_m} \mathcal{D}^{-e(\mathcal{M}\setminus\mathcal{N})} d_{l_m} \cdot \mathcal{D}^{y} d_{l_i}^{1/2} d_{l_{\mathcal{N}}}^{1/4}$$



$$= B(\upsilon(\Phi))$$

□

Lemma 5.23. *The maps υ is "invariant under A", that is, if $\upsilon_{\mathcal{M}} = \upsilon : H(\mathcal{M};(\mathcal{N},l)) \to Z^{sk}_{CY}(M;\mathbf{V})$, then $\upsilon_{\mathcal{M}} = \upsilon_{\mathcal{M}'} \circ A_{\mathcal{M},\mathcal{M}'}$. In other words, υ gives a well-defined map $\upsilon : Z_{CY}(\mathcal{M};(\mathcal{N},l)) \to Z^{sk}_{CY}(M;\mathbf{V})$.*

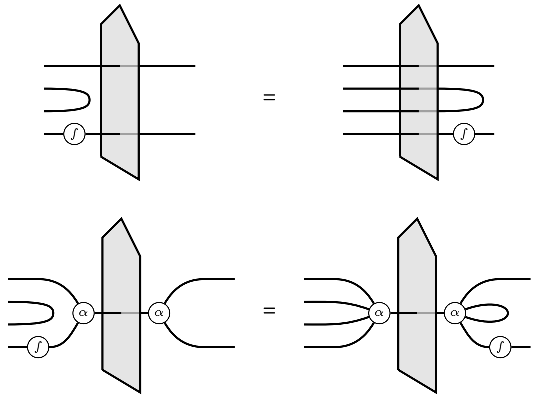
Proof. It is clear from Lemma 4.9 that if $\mathcal{M},\mathcal{M}'$ are related by a single-cell move, then the lemma holds; then by Proposition 3.19, a sequence of single-cell moves relate any two PLCW decompositions, so we are done. □

Lemma 5.24. *The map $\upsilon : H(\mathcal{M};(\mathcal{N},l_{\mathcal{N}})) \to Z^{sk}_{CY}(M^\circ;\mathbf{V})$ from Lemma 5.22 is an isomorphism.*

Proof. We define an inverse map υ^{-1}. Let Γ be the graph built from arbitrarily chosen anchors, as in (5.14).

Let Γ' be some arbitrary colored ribbon graph in $Z^{\mathrm{sk}}_{\mathrm{CY}}(M^\circ;\mathbf{V})$. We define $\upsilon^{-1}(\Gamma')$ as follows. Apply an isotopy to make Γ' weakly transverse to every 2-cell, then by Lemma 3.26, there is a strict narrowing that makes Γ' transverse to every 2-cell (we discuss how this choice of isotopy is irrelevant). Using Lemma 2.27, we can make the graph meet each 2-cell in a single ribbon, or more precisely, Γ' is equivalent as a skein to $\sum c_j \Gamma''_j$, where each Γ'_j agrees exactly with Γ in a neighborhood of each 2-cell (see (5.16)); this is well-defined by Lemma 2.28. Then, for each 3-cell C, take a ball D that is a just a slightly shrunk C, and evaluate the graphs Γ''_j, and replace it with a subgraph that looks like Γ but with different colorings.

(5.16)



Now let us discuss why the isotopies of Γ' to make it transverse to 2-cells is irrelevant. It is easy to see that the choice of strict narrowing is irrelevant, thus we will assume that all ribbon graphs are already "narrow enough" (see also Remark 3.27). Suppose we have isotoped Γ' to Γ'_1 and Γ'_2, each of which is weakly transverse to every 2-cell. Let Φ^t be an ambient isotopy that throws Γ'_1 onto Γ'_2.

For a 2-cell F of $\mathcal{M}$, let U_F be a small open neighborhood of $\mathrm{Int}(F) \subset M^\circ$ (say, take a point on either side of F close to it, and take U_F to be the cone over F, then remove its boundary). Then $\{\mathrm{Int}(C), U_F\}_{C,F}$ is an open cover of M°, where C, F run over all 3-,2-cells of $\mathcal{M}$, respectively. By Theorem 2.50, the isotopy Φ^t can be replace with a sequence of moves, each of which is supported in a ball in some $\mathrm{Int}(C)$ or U_F.

Let $\Gamma_0 = \Gamma'_1, \Gamma_1, \dots, \Gamma_l = \Gamma'_2$ be the graphs between the moves, and let h^t_k be the move that takes Γ_k to Γ_{k+1}.

We show that we can assume that all Γ_k's are weakly transverse to all 2-cells, so that it suffices to show that $\upsilon^{-1}(\Gamma_k) = \upsilon^{-1}(\Gamma_{k+1})$. Consider a move supported in a ball $D \subset U_F$ for some 2-cell F; say this move h^t_k takes Γ_k to Γ_{k+1}. Suppose Γ_k is weakly transverse to F, If Γ_{k+1} is not weakly transverse to F, we apply a small isotopy ϕ^t (in particularly, supported in D and away from all other supporting balls of moves supported in $\mathrm{Int}(C)$), so that $\phi^1(\Gamma_{k+1})$ is weakly transverse to F. Somewhere "down the line", there is some minimal $k' > k$ such that the k'-th move $h^t_{k'}$ is supported in U_F (because Γ_{k+1} is not weakly transversal while the target Γ_l is, and moves supported in $\mathrm{Int}(C)$ cannot help). We modify all the graphs and moves up to k' so that we have

$$\dots, \Gamma_k, \phi^1(\Gamma_{k+1}), \phi^1(\Gamma_{k+2}), \dots, \phi^1(\Gamma_{k'}), \Gamma_{k'+1}, \dots$$

and for isotopies, the k-th isotopy is post-concatenated with ϕ^t, while the k'-th isotopy is pre-concatenated with ϕ^{1-t}. Thus, we may assume that all Γ_k's are weakly transverse to all 2-cells.

Clearly the moves supported in $\mathrm{Int}(C)$ do not affect $\upsilon^{-1}(\Gamma_k)$. Now consider some h^t_k supported in a ball $D \subset U_F$ for some 2-cell F as above. Let h'^t be an isotopy supported in U_F that moves D into the interior of an adjacent cell C of F. We choose h'^t to be composed of simple "displacements", say, if F is given a collar neighborhood $F \times [-1, 1]$, then such "displacements" should be essentially of the form $(x, s) \mapsto (x, s + \alpha)$ for some fixed α (except perhaps near ∂F); furthermore, we ensure that each such "displacement" ends with the graph being weakly transverse to F. These "displacements" look like (5.16), and thus h' do not affect $\upsilon^{-1}(\Gamma_k)$. Finally, the move that was supported in D is now the move $h'^1 \circ h^t_k$, supported in $\mathrm{Int}(C)$. Thus, we can replace h^t_k by moves $h'^t, h'^1 \circ h^t_k, h'^{1-t}$, each of which does not affect the outcome of υ^{-1}, so we are done.

(The arguments above on the irrelevance of isotopies are very similar to the proof of Lemma 6.13.) □

Theorem 5.25. *For an oriented closed surface N, the assignments Υ, υ from Definition 5.21, Lemma 5.22 define an equivalence*

$$\Upsilon : Z_{CY}(N) \simeq Z^{sk}_{CY}(N)$$

Proof. It suffices to prove equivalence for the hat versions:

$$\Upsilon : \hat{Z}_{\mathrm{CY}}(N) \simeq \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)$$

Υ is clearly essentially surjective. By the previous lemmas applied to $M = N \times I$, we see that υ defines isomorphisms $\upsilon : \mathrm{Hom}_{Z_{\mathrm{CY}}(N)}((\mathcal{N},l),(\mathcal{N}',l')) \simeq \mathrm{Hom}_{Z^{\mathrm{sk}}_{\mathrm{CY}}(N)}(\Upsilon((\mathcal{N},l)),\Upsilon((\mathcal{N}',l')))$. It is straightforward to check that Υ respects composition (this is the reason for the $\mathcal{D}^{\frac{1}{4}(x(\mathcal{N}))} d_{l_{\mathcal{N}}}^{1/4}$ factors in υ), and that the identity morphism $\mathrm{id}_{(\mathcal{N},l)}$ is sent to the identity morphism. □

Theorem 5.26. *We have*

$$\upsilon : \mathcal{F}_M \simeq \mathcal{F}^{sk}_M \circ \Upsilon$$

where υ, Υ are from Theorem 5.25 above, and $\mathcal{F}_M, \mathcal{F}^{sk}_M$ are from Propositions 4.16, 5.17.

Furthermore, υ intertwines $\mathcal{F}^{sk}_W \circ \Upsilon$ and $\mathcal{F}_W$: for $\mathbf{V} = \Upsilon((\mathcal{N},l))$, we have the commutative diagram

$$\begin{array}{ccc} Z_{CY}(M;(\mathcal{N},l)) & \xrightarrow{Z_{CY}(W;N)} & Z_{CY}(M';(\mathcal{N},l)) \\ \downarrow \upsilon & & \downarrow \upsilon \\ Z^{sk}_{CY}(M;\mathbf{V}) & \xrightarrow{Z^{sk}_{CY}(W;N)} & Z^{sk}_{CY}(M';\mathbf{V}) \end{array}$$

Proof. The first part follows directly from Theorem 5.25 and the preceding lemmas.

To prove the second part, it suffices to consider elementary cornered cobordisms W. More specifically, we just have to consider handles $\mathcal{H}_k$ with (mostly) "empty" input. For a cornered cobordism $W : M \to_N M'$, let $\widetilde{\mathcal{F}}^{sk}_W : \mathcal{F}^{sk}_M \circ \Upsilon \to \mathcal{F}^{sk}_{M'} \circ \Upsilon$ be the natural transformation that is obtained from "transferring" $\mathcal{F}_W$ via Υ; more precisely, for $(\mathcal{N},l) \in Z_{\mathrm{CY}}(N)$, we define

$$(\widetilde{\mathcal{F}}^{sk}_W)_{(\mathcal{N},l)} = \upsilon_{(\mathcal{N},l)} \circ (\mathcal{F}_W)_{(\mathcal{N},l)} \circ \upsilon^{-1}_{(\mathcal{N},l)}$$

(In other words, the second part claims that $\widetilde{\mathcal{F}}^{sk}_W = \mathcal{F}^{sk}_W \circ \Upsilon$.)

We compute $\widetilde{\mathcal{F}}^{sk}_W(\mathcal{H}_k)$ case-by-case. We choose different PLCW decompositions on $B^4 = B^k \times B^{4-k}$ for different cases, but in each case, the interior of $\mathcal{H}_k$ will consist of just one 4-cell, i.e. we will choose a cell-like PLCW decomposition, and we denote by $\mathcal{M}_{in}, \mathcal{M}_{out}$ the PLCW decomposition on $B^k \times \partial B^{4-k}, \partial B^k \times B^{4-k}$.

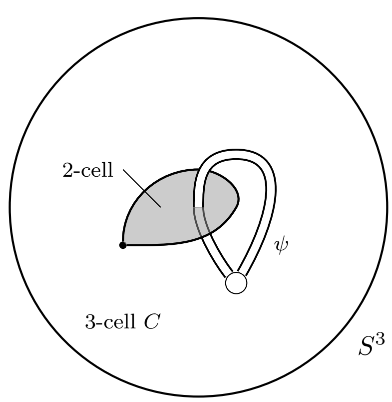
Case $k = 4$**:** $\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_4}(\varnothing^{\mathrm{sk}}_{S^3}) = 1$. Take any PLCW structure $\mathcal{M}_{in}$ on S^3. The empty skein corresponds to $\upsilon^{-1}(\varnothing^{\mathrm{sk}}_{S^3}) = \mathcal{D}^{-\frac{1}{2}x(\mathcal{M}_{in})} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}$. Then

$$\begin{aligned} \widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_4}(\varnothing^{\mathrm{sk}}_{S^3}) &= Z_{\mathrm{CY}}(\mathcal{H}_4)(\mathcal{D}^{-\frac{1}{2}x(\mathcal{M}_{in})} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\ &= \mathrm{ev}(Z_{\mathrm{CY}}(\mathcal{H}_4), \mathcal{D}^{-\frac{1}{2}x(\mathcal{M}_{in})} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\ &= \mathrm{ev}(\mathcal{D}^{\frac{1}{2}x(\mathcal{M}_{in})} Z(B^4, l \equiv \mathbf{1}), \mathcal{D}^{-\frac{1}{2}x(\mathcal{M}_{in})} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\ &= Z_{\mathrm{RT}}(\widetilde{\varnothing}_{\mathcal{M}_{in}}) = 1 \end{aligned}$$

where in the last line, $\widetilde{\varnothing}_{\mathcal{M}_{in}}$ is interpreted as a colored graph as in Definition 3.45.

Case $k = 0$**:** $\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_0} = \mathcal{D} \cdot \varnothing^{\mathrm{sk}}_{S^3}$. Consider the PLCW structure $\mathcal{M}_{out}$ on $S^3 = \partial\mathcal{H}_0$ with exactly one j-cell for each dimension $j = 0,1,2,3$ as follows:

(5.17)



For the labeling $l(f) = i$, the state $\varphi = \mathrm{coev}_i \in H(C, l)$ should have coefficient $\frac{1}{d_i} \cdot d_i = 1$ in $Z_{\mathrm{CY}}(\mathcal{H}_0, l)$, because the dual to coev_i is $\frac{1}{d_i} \cdot \mathrm{coev}_i \in H(\overline{C}, l)$, and that graph evaluates to d_i. Thus

$$Z_{\mathrm{CY}}(\mathcal{H}_0) = \sum_i d_i^{1/2} \mathrm{coev}_i \overset{\upsilon}{\mapsto} \sum_i d_i^{1/2} (\Gamma, d_i^{1/2} \mathrm{coev}_i) = \sum_i d_i^2 \cdot \varnothing^{\mathrm{sk}}_{S^3} = \mathcal{D} \cdot \varnothing^{\mathrm{sk}}_{S^3}$$

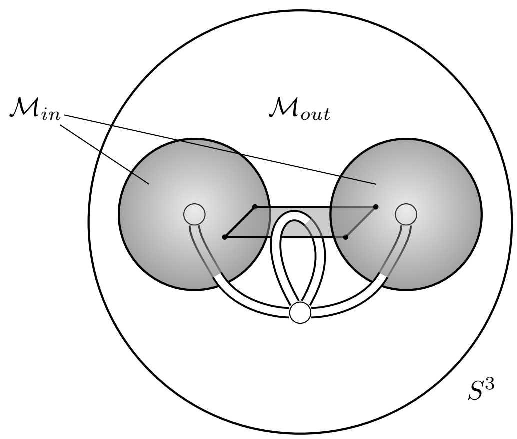
where Γ is the ribbon graph obtained from the anchor. (Note $(\Gamma, \mathrm{coev}_i) = d_i$.)

Case $k = 1$**:** $\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_1}(\varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}) = \mathcal{D}^{-1} \cdot \varnothing^{\mathrm{sk}}_{S^2 \times B^1}$

(Strictly speaking, $\varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}$ has empty boundary value $\mathbf{E}$, which is not in the image of Υ; however, an object $(B, \{\mathbf{1}\}) = \Upsilon((\mathcal{N}, l \cong \mathbf{1}))$ is canonically isomorphic to $\mathbf{E}$ by the simplest possible graph (one ribbon and one coupon for each marking, labeled with $\mathbf{1}$ and $\mathrm{coev}_{\mathbf{1}}$ respectively.)

A 1-handle $\mathcal{H}_1$ is a cornered cobordism $\mathcal{H}_1 : \mathcal{D}^3 \sqcup \mathcal{D}^3 \to_{S^2 \sqcup S^2} S^2 \times B^1$. We use the following PLCW decomposition:

(5.18)



We have $\upsilon(\widetilde{\varnothing}_{B^3}) = \mathcal{D}^{\frac{1}{4}} \cdot \varnothing^{\mathrm{sk}}_{B^3}$, so $\upsilon(\widetilde{\varnothing}_{\mathcal{M}_{in}}) = \mathcal{D}^{\frac{1}{2}} \cdot \varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}$, and $\upsilon(\widetilde{\varnothing}_{\mathcal{M}_{out}}) = \mathcal{D}^{-\frac{1}{2}} \cdot \varnothing^{\mathrm{sk}}_{S^2 \times B^1}$. Then

$$\begin{aligned}
\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_1}(\varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}) &= Z_{\mathrm{CY}}(\mathcal{H}_1; S^2 \sqcup S^2)(\mathcal{D}^{-1/2} \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\
&= \mathcal{D}^{-1/2} \sum_i \mathrm{ev}(\mathcal{D}^{-1} d_i^{1/2} Z(\mathcal{H}_1, l(f) = i), \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\
&= \mathcal{D}^{-3/2} \sum_i d_i^{1/2} \mathrm{coev}_i \\
&\overset{\upsilon}{\mapsto} \mathcal{D}^{-3/2} \sum_i d_i^2 \mathcal{D}^{-1/2} \cdot \varnothing^{\mathrm{sk}}_{S^2 \times B^1} \\
&= \mathcal{D}^{-1} \cdot \varnothing^{\mathrm{sk}}_{S^2 \times B^1}
\end{aligned}$$

Case $k = 3$**:** $\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_3}(\zeta_i) = \delta_{i,\mathbf{1}} \cdot \varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}$, where ζ_i is the identity morphism for an object $(\{b\}, \{X_i\}) \in Z^{\mathrm{sk}}_{\mathrm{CY}}(S^2)$.

We will use the same PLCW decomposition as for $\mathcal{H}_1$ (but $\mathcal{M}_{in}, \mathcal{M}_{out}$ are reversed). For the pre-state $\upsilon^{-1}(\zeta_i)$ corresponding to the skein ζ_i, the 2-cell in the center of the diagram (5.18) should be labeled $\mathbf{1}$, and the 2-cells bounding the 3-balls should be labeled i and i^* respectively. Then for $i \neq \mathbf{1}$, the local state space for the two 3-balls is 0, thus we must have the coefficient $\delta_{i,\mathbf{1}}$.

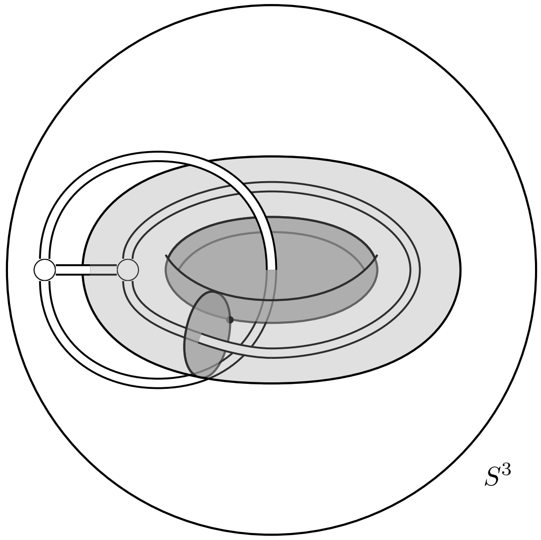
For $i = \mathbf{1}$, $\zeta_{\mathbf{1}}$ is essentially the empty skein $\varnothing^{\mathrm{sk}}_{B^1 \times S^2}$, which corresponds to $\upsilon^{-1}(\varnothing^{\mathrm{sk}}_{B^1 \times S^2}) = \mathcal{D}^{1/2} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}$ as before. Thus

$$\begin{aligned}
\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_3}(\varnothing^{\mathrm{sk}}_{B^1 \times S^2}) &= Z_{\mathrm{CY}}(\mathcal{H}_3; S^2 \sqcup S^2)(\mathcal{D}^{1/2} \cdot \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\
&= \mathcal{D}^{1/2} \,\mathrm{ev}(\mathcal{D}^{-1} Z(\mathcal{H}_3, l \equiv \mathbf{1}), \widetilde{\varnothing}_{\mathcal{M}_{in}}) \\
&= \mathcal{D}^{-1/2} \cdot \widetilde{\varnothing}_{\mathcal{M}_{out}} \\
&\overset{\upsilon}{\mapsto} \varnothing^{\mathrm{sk}}_{B^3 \sqcup B^3}
\end{aligned}$$

Case $k = 2$**:** $\widetilde{\mathcal{F}}^{sk}_{\mathcal{H}_2}(\varnothing^{\mathrm{sk}}_{B^2 \times S^1}) = \sum_i d_i \phi_i$, where ϕ_i is the belt sphere labeled with object i. (Technically, ϕ_i should be $S^1 \times [-\varepsilon, \varepsilon] \times \{0\} \subset S^1 \times B^2$, i.e. framed with 0 self-intersection when included into $S^3 = \partial \mathcal{H}_2$.)

We use the following PLCW decomposition, with corresponding anchor:

(5.19)



We have $\upsilon(\widetilde{\varnothing}_{B^2\times S^1}) = \mathcal{D}^{-1/4}\cdot\varnothing^{\text{sk}}_{B^2\times S^1}$ for both solid tori; then

$$\begin{aligned} Z^{\text{sk}}_{\text{CY}}(\mathcal{H}_2; S^1\times S^1)(\varnothing^{\text{sk}}_{B^2\times S^1}) &= Z_{\text{CY}}(\mathcal{H}_2; S^1\times S^1)(\mathcal{D}^{1/4}\cdot\widetilde{\varnothing}_{B^2\times S^1}) \\ &= \mathcal{D}^{1/4}\sum_i d_i^{1/2}\,\text{coev}_i \\ &\overset{\upsilon}{\mapsto} \sum_i d_i\phi_i \end{aligned}$$

It follows easily from the above computations that $\widetilde{\mathcal{F}}^{sk}_W = \mathcal{F}^{sk}_W \circ \Upsilon$ for elementary cornered cobordisms W, thus we are done. □

Corollary 5.27. *The skein pairing* ev^{sk} *(Definition 5.18) is equivalent to the reduced relative pre-state space pairing* $\overline{ev}^{\partial}$ *(Definition 4.12), i.e. for* $\widetilde{\varphi}\in H(\mathcal{M};(\mathcal{N},l)), \widetilde{\varphi}'\in H(\overline{\mathcal{M}};(\mathcal{N},l))$*,*

$$\overline{ev}^{\partial}(\widetilde{\varphi},\widetilde{\varphi}') = \text{ev}^{sk}(\upsilon(\varphi),\upsilon(\varphi'))$$

In particular, since $\overline{ev}^{\partial}$*, when restricted to* $Z_{CY}(M;(\mathcal{N},l))$*, is non-degenerate, it follows that* ev^{sk} *is non-degenerate as well.*

Proof. Follows immediately from Theorem 5.26 and Lemma 4.13. □

Remark 5.28. To draw very loose parallels, the PLCW definition of Crane-Yetter is to the skein definition as simplicial homology is to singular homology. $H(\mathcal{M})$ would play the role of the simplicial chain complex $C^{\Delta}_*(M)$, both of which are already finite-dimensional, but are dependent on the (PLCW) combinatorial structure; VGraph(M) would play the role of the singular chain complex $C_*(M)$, both of which are manifestly independent of additional structures on M, but are infinite-dimensional and unwieldy.

In this section, we consider properties of skein modules and categories. Subsection 6.1 is focused on the space of relations (i.e. the null graphs $Q \subset \mathrm{VGraph}(Y, \mathbf{V})$), in particular how they are generated. In Subsection 6.2, we exhibit a "stacking" monoidal structure on the category of boundary values of manifolds of the form $P \times (0,1)$, and show it to be pivotal. Finally in Subsection 6.3, we show that skein categories satisfy excision.

Most of this section is taken from [KT], but adapted to the PL category. All PL topology results used in this section is quoted in Section 2.4 for the reader's convenience.

All surfaces in this section will be of finite type, i.e. closed surfaces with finitely many (open or closed) disks removed.

6.1. **Skein Modules.**

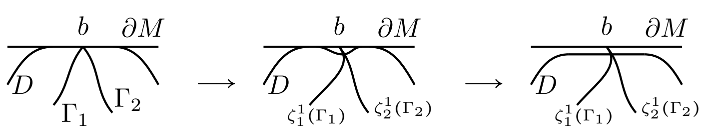
Recall that a null graph in M is null with respect to some 3-ball D, and D is allowed to touch the boundary ∂M. In future applications, it will be convenient to only consider balls D that do not meet ∂M, since such balls can be displaced by ambient isotopy but balls meeting ∂M may not. If we exclude balls D that meet ∂M, the resulting space of null graphs Q' will be strictly smaller than Q, but not by much; the following lemma says we just need to include equivalence of graphs under ambient isotopy rel boundary:

Lemma 6.1. *Let M be a 3-manifold, possibly with boundary or non-compact, and let $\mathbf{V} =\in \mathrm{Obj}\, \hat{Z}^{sk}_{CY}(\partial M)$ be a fixed boundary value. Define $Q' \subset Q \subset VGraph(M, \mathbf{V})$ to be the subspace generated by graphs that are null with respect to a ball that does not meet the boundary ∂M. Let U be an open neighborhood of $\mathbf{V}$ (more precisely, of the set of arcs of $\mathbf{V}$), and let $Q'' \subset VGraph(M, \mathbf{V})$ be relations obtained by ambient isotopy supported on U fixing the boundary, i.e. generated by graphs $\Gamma^1 - \Gamma^0$, where $\Gamma^t = \zeta^t(\Gamma)$, ζ^t is a compactly-supported ambient isotopy fixing ∂M and is supported on U.*

Then $Q = Q' + Q''$.

Proof. It is clear that $Q', Q'' \subset Q$, so it suffices to show that $Q \subset Q' + Q''$. Let $\Gamma = \sum c_i \Gamma_i$ be a null graph with boundary value $\mathbf{V}$, null with respect to a ball $D \subset M$, and suppose D meets the boundary ∂M. We would like to shrink D to not meet ∂M while maintaining that Γ be null with respect to it. Clearly if D does not meet any point in $\mathbf{V}$ then we can do this, and then $\Gamma \in Q'$.

Suppose D does contain some arc $b \in \mathbf{V}$. For each i, apply a small ambient isotopy ζ_i^t supported in a small neighborhood of b so that the resulting graphs $\zeta_i^1(\Gamma_i)$ agree in a (possibly smaller) neighborhood of b (this follows from uniqueness of collar neighborhoods, see Theorem 2.53).



Then we can push D slightly inwards away from the boundary at b, and note that this new graph $\Gamma' = \sum c_i \zeta_i^1(\Gamma_i)$ will be null with respect to the deformed D. This reduces the number of arcs of $\mathbf{V}$ that D meets, so after performing this finitely many times, we are back to the case considered above where D does not meet $\mathbf{V}$. Thus we see that repeated applications of isotopies (i.e. relations in Q'') takes Γ to another graph $\Gamma' \in Q'$; in other words, $\Gamma \in Q'' + Q'$. □

Proposition 6.2. *Let M be an 3-manifold, possibly with boundary or non-compact. Let $\mathbf{V} \in \mathrm{Obj}\, \hat{Z}^{sk}_{CY}(\partial M)$ be a fixed boundary value. Let $\{U_\alpha\}$ be a finite open cover of M such that each arc of $\mathbf{V}$ is contained in some U_α. Define $Q_\alpha \subset Q \subset VGraph(M, \mathbf{V})$ to be the subspace of null graphs in M with boundary value $\mathbf{V}$ that are null with respect to some closed ball D contained in U_α. Then the space of null graphs is generated by Q_α's, i.e.*

$$Q = \sum Q_\alpha$$

Proof. Let $\Gamma \in Q$ be a null graph. Let V_b be an open neighborhood of $b \in \mathbf{V}$ that is contained in some U_α, and let $V = \bigcup V_b$. By Lemma 6.1, Γ can be written as a sum of null graphs $\Gamma' + \Gamma''$, where $\Gamma' = \sum c'_j \Gamma'_j$ is a sum of graphs, each Γ'_j is null with respect to some ball not meeting ∂M, and $\Gamma'' = \sum((\Gamma''_j)^1 - (\Gamma''_j)^0)$ for some smooth isotopies $(\Gamma''_j)^t$ supported on V. Clearly $\Gamma'' \in \sum Q_\alpha$, so it suffices to consider Γ'.

Consider a term Γ'_j in Γ', and suppose it is null with respect to some ball D not meeting ∂M. There exists an ambient isotopy $\zeta^t : M \to M$ that moves D into some open set U_a. Then $\zeta^1(\Gamma'_j) \in Q_a$. But by Theorem 2.50 the isotopy ζ can be chosen to be a sequence of moves, each supported in some U_α, thus $\zeta^1(\Gamma'_j) - \Gamma'_j \in \sum Q_\alpha$. □

6.2. Properties of Categories of Boundary Values.

Let $I' = (0,1)$, the open interval.

Lemma 6.3. *Let N_1, N_2 be surfaces without boundary, possibly non-compact. Let $\varphi : N_1 \to N_2$ be an orientation-preserving embedding. Then φ induces an obvious inclusion functor*

$$\varphi_* : \hat{Z}^{sk}_{CY}(N_1) \to \hat{Z}^{sk}_{CY}(N_2)$$

that sends objects to their image under φ, and sends morphisms to their image under $\varphi \times \mathrm{id}_I$. This extends to the Karoubi envelopes

$$\varphi_* : Z^{sk}_{CY}(N_1) \to Z^{sk}_{CY}(N_2)$$

Furthermore, an isotopy $\varphi^t : N_1 \to N_2$ induces a natural isomorphism from φ^0_* to φ^1_*, and isotopic isotopies induce the same natural isomorphisms.

Proof. Clear. □

Lemma 6.4. *Under the same hypothesis above,*

$$\hat{Z}^{sk}_{CY}(N_1 \sqcup N_2) \simeq \hat{Z}^{sk}_{CY}(N_1) \boxtimes \hat{Z}^{sk}_{CY}(N_2)$$
$$Z^{sk}_{CY}(N_1 \sqcup N_2) \simeq Z^{sk}_{CY}(N_1) \boxtimes Z^{sk}_{CY}(N_2)$$

Proof. The proof for $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}$ is clear: the inclusions of N_1 and N_2 into $N_1 \sqcup N_2$ together induce $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_1) \boxtimes \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_2) \to \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_1 \sqcup N_2)$, and this is easily seen to be an isomorphism of categories. The equivalence for $Z^{\mathrm{sk}}_{\mathrm{CY}}$ then follows by universal property, and the fact that the Deligne-Kelly tensor product of two finite semisimple abelian categories is also a finite semisimple abelian category. □

Next we discuss the "stacking" monoidal structure of some special surfaces. Let $I' = (0,1)$, and let $m : I' \sqcup I' \to I'$ be $x/2$ on the first I' and $(x+1)/2$ on the second I'. This can be made part of an A_∞-space structure, as defined in [Sta1963]: m is not associative, but there is an isotopy $m^t_3 : I' \sqcup I' \sqcup I' \to I'$ from $m^0_3 = m \circ (m \sqcup \mathrm{id}_{I'})$ to $m^1_3 = m \circ (\mathrm{id}_{I'} \sqcup m)$, relating the two ways of including three intervals into one. Note that the "straight line isotopy" that is often used to define m^t_3 is not a PL isotopy; we may explicitly define $m^t_3 = \varphi^t \circ m^0_3$, where

$$\varphi^t(x) = \begin{cases} 2x & \text{if } x < t/4 \\ (1+x)/2 & \text{if } x > 1 - t/2 \\ x + t/4 & \text{else} \end{cases}$$

Let P be a finite collection of open intervals and circles. By abuse of notation, we denote $\mathrm{id}_P \times m$, $\mathrm{id}_P \times m^t_3$ by m, m^t_3, respectively.

Proposition 6.5. *There is a monoidal structure on $\hat{Z}^{sk}_{CY}(P \times I')$ given as follows:*

- *The tensor product is*

$$\otimes := m_* : \hat{Z}^{sk}_{CY}(P \times I') \boxtimes \hat{Z}^{sk}_{CY}(P \times I') \to \hat{Z}^{sk}_{CY}(P \times I')$$

- *The unit $\mathbf{1}$ is the empty configuration $\mathbf{E}$. (Left, right unit constraints are given in proof.)*
- *The associativity constraint α is the natural isomorphism that is induced by m^t_3.*

This extends to a monoidal structure on $Z^{sk}_{CY}(P \times I')$.

Proof. Left unit constraint $l_A : A \otimes \mathbf{1} \to A$ is given by a "straight line" graph, likewise for right unit constraint. That α satisfies the pentagon relations follows from the fact that any two inclusions $I'^{\sqcup 4} \hookrightarrow I'$ are isotopic, and any two isotopies are themselves isotopic. The result for $Z^{\mathrm{sk}}_{\mathrm{CY}}(P \times I')$ follows from universal property. □

Proposition 6.6. *The monoidal structure on $\hat{Z}^{sk}_{CY}(P \times I')$ and $Z^{sk}_{CY}(P \times I')$ given in Proposition 6.5 is pivotal.*

Remark 6.7. The input category $\mathcal{A}$ has to be spherical, but the resulting categories $Z^{\text{sk}}_{\text{CY}}(P \times I')$ may not be; we will see in Section 7 that $Z^{\text{sk}}_{\text{CY}}(S^1 \times I')$ is pivotal but not spherical.

Proof. It suffices to prove this for $\hat{Z}^{\text{sk}}_{\text{CY}}(P \times I')$, since its Karoubi envelope will inherit the pivotal structure.

The rigid and pivotal structures come from topological constructions. Denote by $\theta : P \times I' \to P \times I'$ be the orientation-reversing diffeomorphism which flips I', i.e. $(p, x) \mapsto (p, 1 - x)$. Denote by $\Theta : P \times I' \times [0, 1] \to P \times I' \times [0, 1]$ the orientation-preserving diffeomorphism that rotates the $I' \times [0, 1]$ rectangle by 180°, i.e. $(p, x, t) \mapsto (p, 1 - x, 1 - t)$.

Denote by ν the map that takes $P \times I' \times [0, 1]$, squeezes it in half along the I' direction, bends it like an accordion so that the left side collapses, and puts it back in $P \times I' \times [0, 1]$ so that the top and bottom are now attached to the top (see Figure 9); ν', η, η' are defined similarly.

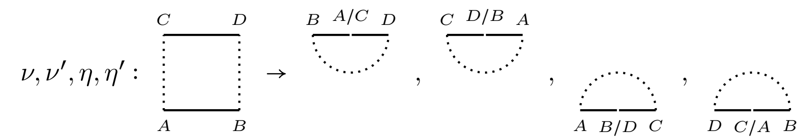


FIGURE 9. The maps ν, ν', η, η' for $P = \{*\}$

Let $\mathbf{V} = (B, \{V_b\}) \in \text{Obj}\, \hat{Z}^{\text{sk}}_{\text{CY}}(P \times I')$. Its left dual $\mathbf{V}^*$ is given by $(\theta(B), \{V_b^*\})$, that is, apply the flipping diffeomorphism θ defined above to the marked points, and label them by the left duals of the original labeling. Similarly, the right dual is ${}^*\mathbf{V} = (\theta(B), \{{}^*V_b\})$. (It is not too important to distinguish V_b^* from ${}^*V_b$ since $\mathcal{A}$ itself is pivotal.)

The left evaluation and coevaluation morphisms for $\mathbf{V}$ are obtained by applying ν and η to $\text{id}_{\mathbf{V}}$, respectively. Similarly, the right evaluation and coevaluation morphisms for $\mathbf{V}$ are obtained by applying ν' and η' to $\text{id}_{\mathbf{V}}$, respectively. It is easy to see that these morphisms have the required properties.

Given a morphism $f \in \text{Hom}_{\hat{Z}^{\text{sk}}_{\text{CY}}(P \times I')}(\mathbf{V}, \mathbf{V}')$ represented by a graph Γ, it is easy to check that its left and right duals are given by applying the rotation Θ to Γ, and keeping all orientations and labels of the edges of Γ.

The pivotal structure is essentially the identity morphism, but with one vertex on each vertical line labeled by δ, the pivotal structure of $\mathcal{A}$. □

Example 6.8. Recall that $Z_{\text{CY}}(I' \times I') \simeq Z_{\text{CY}}(D^2) \simeq \mathcal{A}$. $I' \times I'$ can stack in two ways, along the first copy of I' (horizontal stacking) or the second (vertical stacking). They both give monoidal structures equivalent to $\mathcal{A}$'s. This is explored further in Section 8.2

Next we consider module categories over $Z^{\text{sk}}_{\text{CY}}(P \times I')$.

Definition 6.9. Let N be a surface, and suppose there is a surface N' with a boundary component $P \subseteq \partial N'$, such that $N = N' \backslash P$.

A *0-collaring* of N is a proper embedding $\iota : P \times I' \hookrightarrow N$ that extends to $\bar{\iota} : P \times [0, 1) \hookrightarrow N'$; a *1-collaring* extends to $P \times (0, 1] \hookrightarrow N'$.

Let N be a 0-collared surface. By crossing with P, we can carry, via the collaring, the left module structure n on I' to N, obtaining a left multiplication $n : P \times I' \sqcup N \to N$ and an isotopy n_3^t from $n \circ (\text{id}_{P \times I'} \sqcup n)$ to $n \circ (m \sqcup \text{id}_N)$.

Proposition 6.10. *Given a 0-collared surface N, the multiplication m on $P \times I'$ defines a left module map*

$$n : P \times I' \sqcup N \to N$$

and an isotopy

$$\left(n_3^t : n \circ (\text{id}_{P \times I'} \sqcup n) \to n \circ (m \sqcup \text{id}_N)\right) : P \times I' \sqcup P \times I' \sqcup N \to N$$

Then there is a left $\hat{Z}^{sk}_{CY}(P \times I')$-module category structure on $\hat{Z}^{sk}_{CY}(N)$ given by

$$\triangleright := n_* : \hat{Z}^{sk}_{CY}(P \times I') \boxtimes \hat{Z}^{sk}_{CY}(N) \to \hat{Z}^{sk}_{CY}(N)$$

and the associativity constraint is given by the natural isomorphism induced by the isotopy n_3^t. This extends to a left $Z^{sk}_{CY}(P \times I')$-module category structure on $Z^{sk}_{CY}(N)$.

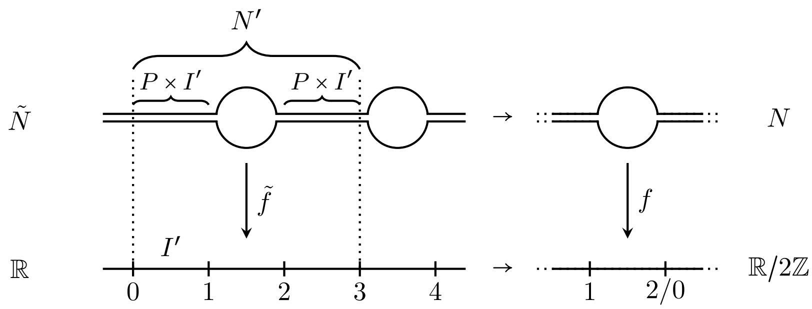
Proof. Similar to Proposition 6.5. □

There is a similar story for right module structure, where N is a collared n-manifold so that 1 escapes to infinity. Of course, the results above extend to surfaces with collared boundaries, as Z_{CY} is defined by removing the boundaries.

6.3. Excision for $Z^{\mathbf{sk}}_{\mathbf{CY}}$.

In this subsection, we prove the main result of this section, that $Z^{\mathrm{sk}}_{\mathrm{CY}}$ satisfies excision. Let $I' = (0,1)$ and $I = [0,1]$ as before.

Let N be a finite type surface without boundary. To present N as the quotient of some surface N' by some gluing, consider a map $f : N \to S^1 = \mathbb{R}/2\mathbb{Z}$, together with a trivialization of P-bundles $P \times I' \simeq f^{-1}(I')$, where P is some finite collection of open intervals and circles, as in Section 6.2. Take N' to be the "preimage of $(0,3)$ under f"; more precisely, pullback f along the universal covering map $\mathbb{R} \to \mathbb{R}/2\mathbb{Z}$ to get $\widetilde{f} : \widetilde{N} \to \mathbb{R}$, and take $N' = \widetilde{f}^{-1}((0,3))$ (see figure below). So N is obtained from N' by gluing the parts over $(0,1)$ and $(2,3)$.



Remark 6.11. Excision is usually phrased in terms of gluing two collared manifolds. In the above language, that will correspond to the case when $N' = N_1 \sqcup N_2$, where $N_1 = \widetilde{f}^{-1}((0,1.5))$, $N_2 = \widetilde{f}^{-1}((1.5,3))$, so that the pullback map $N' \to N$ is the gluing/overlapping of N_1 and N_2 over $(0,1)$, the collared neighborhoods.

Since $\widetilde{f}^{-1}((0,1)) \simeq \widetilde{f}^{-1}((2,3))$ naturally, the trivialization $P \times I' \simeq f^{-1}(I')$ gives a left and right $P \times I'$-module structure on N', and makes $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N')$ a $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P \times I')$-bimodule category (likewise for $Z^{\mathrm{sk}}_{\mathrm{CY}}$).

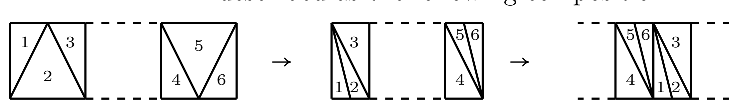
The natural gluing map $N' \to N$ is the composition $N' \subset \widetilde{N} \to N$. We can also embed N' in N as follows: consider a following sequence of maps $N' \to P \times I' \sqcup N' \sqcup P \times I' \to N' \to N$; the first map is just the obvious inclusion, the second is the left and right module maps "squeezing" N' into itself, and the third map is the natural quotient map. It is easy to see that the composition is an embedding, in fact a diffeomorphism onto $N \backslash f^{-1}(0.5)$.

We denote the composition above by i, and denote $X = f^{-1}(0.5)$, which we call the *seam*.

Since $i : N' \to N$ is an embedding, it induces a functor $i_* : \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N') \to \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)$. Recall that there is a natural functor $\mathrm{hTr} : \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N') \to \mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))$ that is identity on objects.

Lemma 6.12. *The inclusion functor $i_* : \hat{Z}^{sk}_{CY}(N') \to \hat{Z}^{sk}_{CY}(N)$ extends along* hTr *to a functor $i_* : \mathrm{hTr}(\hat{Z}^{sk}_{CY}(N')) \to \hat{Z}^{sk}_{CY}(N)$.*

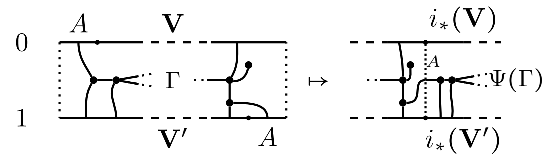
Proof. Consider a map $\Psi : N' \times I \to N \times I$ described as the following composition:

(6.1) $\Psi :$ 

The vertical direction corresponds to the I factor in $N' \times I$, while a horizontal slice is N or N' (the diagram depicts cross-sections of the form $\{*\} \times I' \times I \subset N \times I$). The numbered regions are in the $(P \times I') \times I$ portions of $N' \times I$ (i.e. the collared part $\times I$); Ψ is linear on these regions. The first map sends $N' \times I$ into itself, and the second map is simply the gluing map $N' \to N$ (crossed with I); the left and right vertical sides on the middle figure get identified as $f^{-1}(0.5) \subset N \times I$.

Below we depict a graph Γ in $N' \times I$ with incoming boundary value $A \triangleright \mathbf{V}$ and outgoing boundary value $\mathbf{V}' \triangleleft A$, representing an element of $\mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))}(\mathbf{V}, \mathbf{V}')$. It is sent to a graph $\Psi(\Gamma)$ in $N \times I$ with incoming boundary value $\mathbf{V}$ and outgoing boundary value $\mathbf{V}'$, representing an element of $\mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)}(\mathbf{V}, \mathbf{V}')$.

(6.2)



The only points in $N \times I$ that are hit more than once are in $X \times I$; we sometimes also call this the seam. We denote $L_0 = X \times I$.

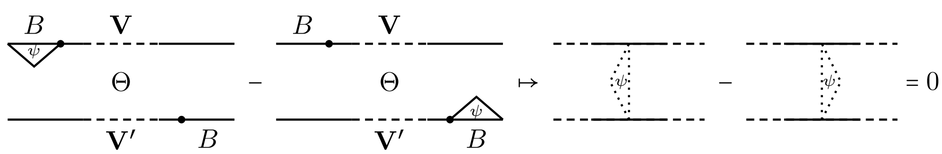
In the figure above, the seam L_0 is depicted as the vertical dotted line labeled with A in the right figure. The seam is also the image of the top left and bottom right boundary pieces (the parts labeled A in the left figure). The image of $\Psi|_{N' \times I'}$ is exactly $N \times I' \backslash L_0$.

We claim that the following map is well-defined:

$$\mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))}(\mathbf{V}, \mathbf{V}') \to \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)}(i_*(\mathbf{V}), i_*(\mathbf{V}'))$$
$$\Gamma \mapsto \Psi(\Gamma)$$

It is not hard to see that the assignment $\Gamma \mapsto \Psi(\Gamma)$ yields a well-defined map $\mathrm{Hom}^{A}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N')}(\mathbf{V}, \mathbf{V}') \to \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)}(i_*(\mathbf{V}), i_*(\mathbf{V}'))$; a graph $\Gamma = \sum c_i \Gamma_i$ that is null with respect to some ball D would have image $\Psi(\Gamma)$ null with respect to $\Psi(D)$. We need to check that the relations $\sim$ in $\mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))}(\mathbf{V}, \mathbf{V}') = \bigoplus \mathrm{Hom}^{A}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N')}(\mathbf{V}, \mathbf{V}')/\sim$ are satisfied. Recall that relations are generated by $\Theta \circ (\psi \triangleright \mathrm{id}_{\mathbf{V}}) - (\mathrm{id}_{\mathbf{V}'} \triangleleft \psi) \circ \Theta$, where $\Theta \in \mathrm{Hom}^{A,B}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N')}(\mathbf{V}, \mathbf{V}')$ and $\psi \in \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P \times I')}(B, A)$. We see that

(6.3)



We leave checking that composition is respected as a simple exercise. □

We want to show that i_* is an equivalence, and will be considering Ψ^{-1} applied to graphs. It is not clear that this is well-defined, e.g. moving parts of a graph in $N \times I$ across the seam L_0. could result in different graphs with different boundary conditions in $N' \times I$. However, the relation $\Theta \circ (\psi \triangleright \mathrm{id}_{\mathbf{V}}) - (\mathrm{id}_{\mathbf{V}} \triangleleft \psi) \circ \Theta$ essentially takes care of this ambiguity.

Let us make this precise. Let $\beta \in (0, 0.5)$ be such that $i_*(\mathbf{V})$ and $i_*(\mathbf{V}')$ do not have any arcs in the neighborhood $f^{-1}((0.5 - \beta, 0.5 + \beta)) \subset N$ of X. Let ζ_α, $\alpha \in (-\beta', \beta')$ for some smaller $0 < \beta' < \beta$, be an ambient isotopy (indexed by α) of $N \times I$ defined, in a smaller neighborhood

$$U_1 = f^{-1}((0.5 - \beta', 0.5 + \beta')) \subset N \times I \tag{6.4}$$

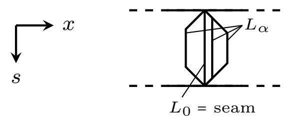
of the seam, by

$$\zeta_\alpha : (p, x, s) \mapsto \begin{cases} (p, x \pm s, s) & \text{if } x < |\alpha| \\ (p, x \pm (1 - s), s) & \text{if } s > 1 - |\alpha| \\ (p, x + \alpha, s) & \text{else} \end{cases} \tag{6.5}$$

where the $\pm$ is the sign of α, $(p, x) \in P \times I'$ (so that $X = \{x = 0\}$), s parameterizes the up-down axis (see (6.6)). Let $L_\alpha = \zeta_\alpha(L_0)$.

Define $\Psi_\alpha = \zeta_\alpha \circ \Psi : N' \times I \to N \times I$, so that the new "seam" is L_α.

(6.6)



Note that L_α's overlap near the top and bottom $X \times \{0,1\}$, but away from there, they form parallel copies of the seam $X \times I$. Thus, for a compact submanifold that does not meet $X \times \{0,1\}$, intersections with L_α coincide with level sets of x.

Lemma 6.13. *Let Γ be a graph in $N \times I$ that represents a morphism in $\mathrm{Hom}_{\hat{Z}^{sk}_{CY}(N)}(i_*(\mathbf{V}), i_*(\mathbf{V}'))$ for some $\mathbf{V}, \mathbf{V}' \in \mathrm{hTr}(\hat{Z}^{sk}_{CY}(N'))$, and let $S(\Gamma)$ denote the surface of the graph.*

Since $i_*(\mathbf{V}), i_*(\mathbf{V}')$ does not meet the seam, α defines a function

$$\alpha : S(\Gamma) \cap U_1 \to (0.5 - \beta', 0.5 + \beta')$$

By Proposition 2.56 there exists an α such that Γ is weakly transverse to L_α, and by Lemma 3.26, there is a strict narrowing Γ' of Γ that is transverse to L_α, thus defining a boundary value $A_\alpha = \Gamma' \cup L_\alpha$ on L_α. We see that $\Psi_\alpha^{-1}(\Gamma')$ is a graph in $N' \times I$ representing a morphism in $\mathrm{Hom}_{\hat{Z}^{sk}_{CY}(N')}(A_\alpha \triangleright \mathbf{V}, \mathbf{V}' \triangleleft A_\alpha)$. Then as a morphism in $\mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{sk}_{CY}(N'))}(\mathbf{V}, \mathbf{V}')$, $\Psi_\alpha^{-1}(\Gamma')$ is independent of such a choice of α and strict narrowing of Γ.

Proof. First note that if Γ is transverse (as a graph) to $L_\alpha, L_{\alpha'}$, then it is clear from the picture (6.3) that $\Psi_\alpha^{-1}(\Gamma) = \Psi_{\alpha'}^{-1}(\Gamma)$.

Next, it is easy to see that if Γ is transverse to L_α, then a further strict narrowing of Γ will also not affect $\Psi_\alpha^{-1}(\Gamma)$ (the arcs of the object A_α just get shorter). Thus we may assume that the graphs discussed below are "sufficiently narrow" (see Remark 3.27 and proof of Lemma 5.24).

Finally, we show that given α, the choice of strict narrowing is irrelevant; then to prove the lemma, take α, α' with $L_\alpha, L_{\alpha'}$ transverse to $S(\Gamma)$, and find a narrowing of Γ that is transverse to both $L_\alpha, L_{\alpha'}$, then by the first observation, we are done.

To show irrelevance of choice of strict narrowing, consider the more general situation where Γ, Γ' are isotopic graphs that are transverse to $L_\alpha, L_{\alpha'}$ respectively. Let $U_\pm = N \times \backslash L_{\pm\beta'}$; $\{U_+, U_-\}$ forms an open cover of $N \times I \backslash X \times \{0,1\}$.

By Theorem 2.50, the isotopy from Γ to Γ' can be broken into moves supported in $U_\pm$. In particular, each move is supported in some closed ball that does not meet some L_α; more precisely, if $\{h_k^t\}$ is the sequence of moves, with h_k^t supported on a closed ball D_k, then there exists $\alpha_k \in (0.5 - \beta', 0.5 + \beta')$ such that D_k does not meet L_{α_k}, and we may take α_k such that L_{α_k} is weakly transverse to the surface of the graph during the isotopy h_k^t.

Let $\Gamma_k = h_k^1 \circ \cdots \circ h_1^1(\Gamma)$ for $0 \le k \le l$ (with $\Gamma_0 = \Gamma, \Gamma_l = \Gamma'$), and let $\alpha_0 = \alpha, \alpha_{l+1} = \alpha'$. By the second observation above, we assume Γ is narrow enough to begin with, or more precisely, we retroactively apply a strict narrowing to Γ so that each Γ_k is transverse to L_{α_k}; this does not affect $\Psi_\alpha^{-1}(\Gamma)$ nor $\Psi_{\alpha'}^{-1}(\Gamma')$. We consider the sequence of graphs in $N' \times I$:

$$\Psi_\alpha^{-1}(\Gamma) = \Psi_{\alpha_0}^{-1}(\Gamma_0), \Psi_{\alpha_1}^{-1}(\Gamma_0), \Psi_{\alpha_1}^{-1}(\Gamma_1), \Psi_{\alpha_2}^{-1}(\Gamma_1), \ldots, \Psi_{\alpha_l}^{-1}(\Gamma_{l-1}), \Psi_{\alpha_l}^{-1}(\Gamma_l), \Psi_{\alpha_{l+1}}^{-1}(\Gamma_l) = \Psi_{\alpha'}^{-1}(\Gamma')$$

The adjacent graphs $\Psi_{\alpha_l}^{-1}(\Gamma_l), \Psi_{\alpha_{l+1}}^{-1}(\Gamma_l)$ are equivalent as skeins by the first observation, while $\Psi_{\alpha_l}^{-1}(\Gamma_{l-1}), \Psi_{\alpha_l}^{-1}(\Gamma_l)$ are isotopic. □

Theorem 6.14. *The extension $i_* : \mathrm{hTr}(\hat{Z}^{sk}_{CY}(N')) \to \hat{Z}^{sk}_{CY}(N)$ is an equivalence.*

Proof. It was already evident from the object map that $\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N')) \to \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)$ is essentially surjective - it only misses objects that have marked points on $f^{-1}(0.5)$, but such an object is isomorphic to an object with points moved slightly off of $f^{-1}(0.5)$.

To show that i_* is fully faithful, fix objects $\mathbf{V}, \mathbf{V}' \in \mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))$. Let Γ be a colored ribbon graph in $N \times I$ with boundary values $i_*(\mathbf{V}), i_*(\mathbf{V}')$.

By Lemma 6.13, we have a well-defined map $\Psi_-^{-1} : \mathrm{VGraph}(N \times I; i_*(\mathbf{V}), i_*(\mathbf{V}')) \to \mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))}(\mathbf{V}, \mathbf{V}')$. This map factors through the quotient map $\mathrm{VGraph}(N \times I; i_*(\mathbf{V}), i_*(\mathbf{V}')) \to \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)}(i_*(\mathbf{V}), i_*(\mathbf{V}'))$:

using the same open cover $\{U_\pm\}$ as in the proof of Lemma 6.13, a null graph in $N \times I$, null with respect to some $D \subset U_\pm$, is clearly sent to a null graph in $N' \times I$, null with respect to $\Psi_\alpha^{-1}(D)$.

It is clear that for a graph Γ' in $N' \times I$ representing an element of $\mathrm{Hom}_{\mathrm{hTr}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))}(\mathbf{V}, \mathbf{V}')$, $\Psi_0^{-1}(\Psi(\Gamma')) = \Gamma'$. It is also clear that for a graph Γ in $N \times I$, $\Psi(\Psi_\alpha^{-1}(\Gamma))$ is isotopic to Γ. Thus i_* is fully faithful. □

Combining the topological result above with the algebraic results of Section 6.1, we have the main result of this section:

Theorem 6.15. *There is an equivalence*

$$\mathcal{Z}_{Z^{sk}_{CY}(P\times I)}(Z^{sk}_{CY}(N')) \simeq Z^{sk}_{CY}(N)$$

In particular, when $N = N_1 \cup N_2$ as in Remark 6.11,

$$Z^{sk}_{CY}(N_1) \boxtimes_{Z^{sk}_{CY}(P\times I)} Z^{sk}_{CY}(N_2) \simeq \mathcal{Z}_{Z^{sk}_{CY}(P\times I)}(Z^{sk}_{CY}(N_1 \sqcup N_2)) \simeq Z^{sk}_{CY}(N)$$

Proof. We claim that $Z^{\mathrm{sk}}_{\mathrm{CY}}(P \times I)$ is multifusion; we justify this claim later. By Proposition 6.6, $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P \times I)$ is pivotal. In reference to the notation in Section 6.1, take $\mathcal{C}' = \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P\times I), \mathcal{C} = Z(P\times I), \mathcal{M}' = \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'), \mathcal{M} = Z(N')$. Then we have

$$\begin{aligned}
\mathcal{Z}_{Z^{\mathrm{sk}}_{\mathrm{CY}}(P\times I)}(Z^{\mathrm{sk}}_{\mathrm{CY}}(N')) &\simeq \mathrm{Kar}(\mathrm{hTr}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P\times I)}(Z^{\mathrm{sk}}_{\mathrm{CY}}(N'))) && \text{by Corollary 2.48,}\\
&\simeq \mathrm{Kar}(\mathrm{hTr}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(P\times I)}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N'))) && \text{by Lemma 2.49,}\\
&\simeq \mathrm{Kar}(\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N)) && \text{by extending } i_* \text{ from Theorem 6.14 to Kar}\\
&= Z^{\mathrm{sk}}_{\mathrm{CY}}(N)
\end{aligned}$$

The second statement follows from the first by applying Lemma 6.4 and (2.30).

Now we need to justify $Z^{\mathrm{sk}}_{\mathrm{CY}}(P \times I)$ being multifusion, in particular, that it is semisimple, as Proposition 6.6 guarantees it is rigid and pivotal. This is true for $P = I$. We apply this result with $N' = I \times I, N = S^1 \times I$, and by Proposition 2.40, $Z^{\mathrm{sk}}_{\mathrm{CY}}(S^1 \times I)$ is semisimple. In fact, $Z^{\mathrm{sk}}_{\mathrm{CY}}(S^1 \times I) \simeq \mathcal{Z}_{Z^{\mathrm{sk}}_{\mathrm{CY}}(I\times I)}(Z^{\mathrm{sk}}_{\mathrm{CY}}(I \times I)) \simeq \mathcal{Z}(\mathcal{A})$, the Drinfeld center of $\mathcal{A}$, which is known to be modular, in particular multifusion; we discuss this equivalence in more detail in Section 7.4.

So $Z^{\mathrm{sk}}_{\mathrm{CY}}(P \times I)$ is pivotal multifusion for any connected P; the claim easily follows for a disjoint union of finitely many such P's. □

Corollary 6.16. *The functor $i_* : Z^{sk}_{CY}(N') \to Z^{sk}_{CY}(N)$ is dominant.*

Proof. Follows immediately from Proposition 2.38 □

Corollary 6.17. *$Z^{sk}_{CY}(N)$ is a finite semisimple category.*

Proof. Any connected N can be built from I^n by a sequence of gluings of collared manifolds. For example, for $n = 2$, gluing opposite edges of a square gives an annulus, and gluing boundaries of the annulus together gives the torus.

By Theorem 6.15, the corresponding category $Z^{\mathrm{sk}}_{\mathrm{CY}}(N)$ thus can be constructed from the Deligne product of several copies of $Z^{\mathrm{sk}}_{\mathrm{CY}}(I^n) \simeq \mathcal{A}$ by repeatedly applying the center construction, replacing a category $\mathcal{M}$ by $\mathcal{Z}_{Z^{\mathrm{sk}}_{\mathrm{CY}}(P\times I)}(\mathcal{M})$. Since it was shown in the proof of Theorem 6.15 that $Z^{\mathrm{sk}}_{\mathrm{CY}}(P \times I)$ is pivotal multifusion, it now follows from Proposition 2.40 that applying the center construction always gives a finite semisimple category. Thus, $Z^{\mathrm{sk}}_{\mathrm{CY}}(N)$ is a finite semisimple category.

□

7. The Reduced Tensor Product on $Z^{\mathrm{SK}}_{\mathrm{CY}}(\mathrm{Ann})$

In this section, we focus on the category of boundary values on the annulus. The results here are reproduced from [Tha2020]. We will fix a premodular category $\mathcal{A}$; concatenating objects denotes tensor product (of $\mathcal{A}$).

7.1. **Background.**

Recall that $\mathcal{Z}(\mathcal{A})$, the Drinfeld center of $\mathcal{A}$, is the category with

Objects: pairs (X,γ), where $X \in \mathcal{A}$ and γ is a half-braiding, i.e. a natural isomorphism of functors $\gamma_A : A \otimes X \to X \otimes A$, $A \in \mathcal{C}$ satisfying natural compatibility conditions.

Morphisms: $\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((X,\gamma),(X',\gamma')) = \{f \in \mathrm{Hom}_{\mathcal{A}}(X,X') \mid f\gamma = \gamma' f\}$.

We recall some well-known properties of $\mathcal{Z}(\mathcal{A})$. These do not require the braiding on $\mathcal{A}$, only its spherical fusion structure.

The following is standard (see e.g. [EGNO2015, Corollary 8.20.14]):

Definition 7.1. $\mathcal{Z}(\mathcal{A})$ is modular, with tensor product

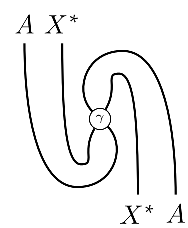
$$(X,\gamma) \otimes (Y,\mu) := (X \otimes Y, \gamma \otimes \mu) \tag{7.1}$$

where

$$(\gamma \otimes \mu)_A := (\mathrm{id}_X \otimes \mu_A) \circ (\gamma_A \otimes \mathrm{id}_Y) \tag{7.2}$$

and left dual given by

$$(X,\gamma)^* = (X^*,\gamma^*), \text{ where } (\gamma^*)_A = (\gamma_{{}^*A})^* = \tag{7.3}$$



,

and similarly the right dual is ${}^*(X,\gamma) = ({}^*X, {}^*\gamma)$, where $({}^*\gamma)_A = {}^*(\gamma_{A^*})$.

The pivotal structure is given by that of $\mathcal{A}$. △

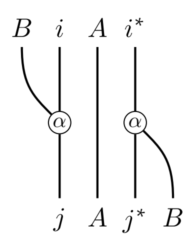
The following is taken from [Kir, Theorem 8.2]: (these are direct specializations of results in Section 2.2 and Section 6.1, with $\mathcal{M} = \mathcal{A}$ as a bimodule over itself.)

Proposition 7.2. *Let $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ be the natural forgetful functor $F : (X,\gamma) \mapsto X$. Then it has a two-sided adjoint functor $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$, given by*

$$I(A) = \Big(\bigoplus_{i \in \mathrm{Irr}(\mathcal{A})} X_i \otimes A \otimes X_i^*, \Gamma\Big) \tag{7.4}$$

where Γ is the half-braiding given by

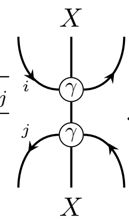
$$\Gamma_B = \sum_{i,j \in \mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j} \tag{7.5}$$



.

We refer the reader to [Kir] *for more details.*

Proposition 7.3. *The adjoint functor $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ above is dominant. More explicitly, the object (X,γ) is a direct summand of $I(X)$, given by the projection $P_{(X,\gamma)}$, described below:*

(7.6)
$$P_{(X,\gamma)} := \sum_{i,j \in \mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}}$$


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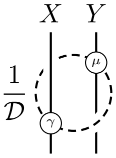
7.2. **Reduced Tensor Product on $\mathcal{Z}(\mathcal{A})$.**

From here on, we will assume that $\mathcal{A}$ is premodular. We define a different monoidal structure on $\mathcal{Z}(\mathcal{A})$, which we call the *reduced tensor product*, denoted by $\overline{\otimes}$; we emphasize that *braiding* is required to define this monoidal structure. We will see in the coming sections that the definitions and results proved here have topological origin, but we first present them purely algebraically.

We also note again that [Was] defines a similar tensor product that coincides with ours when $\mathcal{A}$ is symmetric.

The definition of $\overline{\otimes}$ will be given in several steps.

Definition 7.4. Let X, Y be objects in $\mathcal{A}$, and let γ, μ be half-braidings on X, Y respectively. The *reduced tensor product of X and Y with respect to* γ, μ is defined as the image of the projection $Q_{\gamma,\mu} : X \otimes Y \to X \otimes Y$ defined as follows:

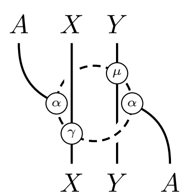
(7.7)
$$X_\gamma \overline{\otimes}_\mu Y := \mathrm{im}(Q_{\gamma,\mu}) \ , \ Q_{\gamma,\mu} := \frac{1}{\mathcal{D}}$$


It is easy to check that $Q^2_{\gamma,\mu} = Q_{\gamma,\mu}$. △

(Compare [Was, Equation (11)])

There is an accompanying definition of $\overline{\otimes}$ for half-braidings:

Definition 7.5. Let γ, μ be half-braidings on X, Y respectively. Define $\gamma \overline{\otimes} \mu$ to be natural transformation $- \otimes XY \to XY \otimes -$ given by

$$(\gamma \overline{\otimes} \mu)_A = \frac{1}{\mathcal{D}}$$


△

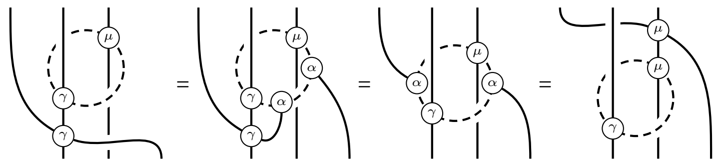
In general, $\gamma \overline{\otimes} \mu$ fails to be a half-braiding, but only insofar as it is not an isomorphism. Observe that $\gamma \overline{\otimes} \mu$ commutes with $Q_{\gamma,\mu}$, so it descends to a natural transformation $- \otimes (X_\gamma \overline{\otimes}_\mu Y) \to (X_\gamma \overline{\otimes}_\mu Y) \otimes -$. This is in fact a half-braiding on $X_\gamma \overline{\otimes}_\mu Y$:

Lemma 7.6. *Let γ, μ be half-braidings on X, Y respectively. Consider the half-braidings $\gamma \otimes c$ and $c^{-1} \otimes \mu$ on $X \otimes Y$, where recall c is the braiding on $\mathcal{A}$. Observe that the projection $Q_{\gamma,\mu}$ intertwines both $\gamma \otimes c$ and $c^{-1} \otimes \mu$, thus they restrict to half-braiding on $X_\gamma \overline{\otimes}_\mu Y$. Then as half-braidings on $X_\gamma \overline{\otimes}_\mu Y$, we have*

$$\gamma \overline{\otimes} \mu = \gamma \otimes c = c^{-1} \otimes \mu$$

Proof. This follows from the following computation:

(7.8)



∎

(Compare [Was, Lemma 10].)

Definition 7.7. Let $(X,\gamma),(Y,\mu)$ be objects in $\mathcal{Z}(\mathcal{A})$. Their *reduced tensor product* is defined as follows:

$$(X,\gamma)\,\overline{\otimes}\,(Y,\mu) := (X\,{}_{\gamma}\overline{\otimes}_{\mu}Y, \gamma\,\overline{\otimes}\,\mu)$$

For $f:(X,\gamma)\to(X,\gamma'), g:(Y,\mu)\to(Y',\mu')$, their reduced tensor product is

$$f\,\overline{\otimes}\,g := Q_{\gamma',\mu'}\circ(f\otimes f')\circ Q_{\gamma,\mu}$$

or more simply, it is $f\otimes f'$ restricted to $X\,{}_{\gamma}\overline{\otimes}_{\mu}Y$. △

Lemma 7.8. *The reduced tensor product of Definition 7.7 is associative. More precisely, if $a:(X_1\otimes X_2)\otimes X_3\simeq X_1\otimes(X_2\otimes X_3)$ is the associativity constraint of $\mathcal{A}$, and $\gamma_1,\gamma_2,\gamma_3$ are half-braidings on X_1,X_2,X_3 respectively, then a restricts to an isomorphism*

$$a:(X_1\,{}_{\gamma_1}\overline{\otimes}_{\gamma_2}X_2)\,{}_{\gamma_1\overline{\otimes}\gamma_2}\overline{\otimes}_{\gamma_3}X_3\simeq X_1\,{}_{\gamma_1}\overline{\otimes}_{\gamma_2\overline{\otimes}\gamma_3}(X_2\,{}_{\gamma_2}\overline{\otimes}_{\gamma_3}X_3)$$

and hence

$$a:\big((X_1,\gamma_1)\,\overline{\otimes}\,(X_2,\gamma_2)\big)\,\overline{\otimes}\,(X_3,\gamma_3)\simeq(X_1,\gamma_1)\,\overline{\otimes}\big((X_2,\gamma_2)\,\overline{\otimes}\,(X_3,\gamma_3)\big)$$

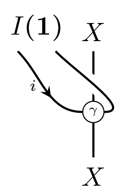
Furthermore, a is natural in (X_i,γ_i), and satisfies the pentagon equation.

Proof. Follows easily from Lemma 7.6. (Compare [Was, Lemma 17].) ∎

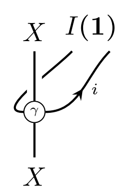
Proposition 7.9. $(\mathcal{Z}(\mathcal{A}),\overline{\otimes})$ *is a pivotal multifusion category. More precisely,*

- *the associativity constraint is given by the associativity constraint of $\mathcal{A}$ (see Lemma 7.8);*
- *the unit object, denoted $\overline{\mathbf{1}}$, is $I(\mathbf{1})$ (see Proposition 7.2), with left and right unit constraints given by*

(7.9) $\quad l_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$



$r_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$

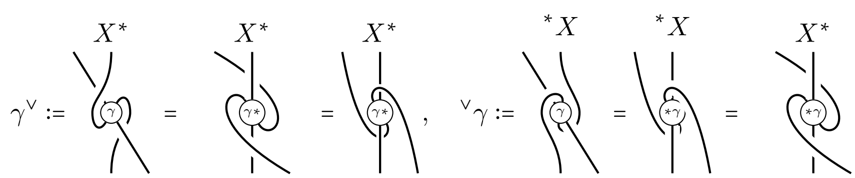


- *the duals are given by*

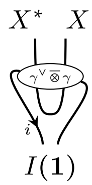
$$(X,\gamma)^\vee := (X^*,\gamma^\vee), \quad {}^\vee(X,\gamma) := ({}^*X, {}^\vee\gamma)$$

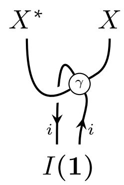
where

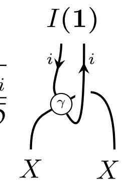
(7.10)



with evaluation and coevaluation maps given by (the projections $Q_{\gamma^\vee,\gamma}$ are implicit)

(7.11) $\mathrm{ev}_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$ 

$= \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$ 

, $\mathrm{coev}_{(X,\gamma)} := \sum_i \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$ 

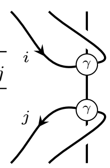
;

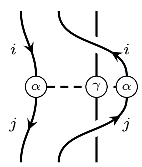
and similarly for right duals.

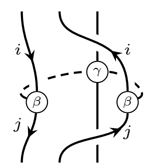
- *the pivotal structure is given by that of $\mathcal{A}$.*

(Compare [Was, Theorem 24])

Proof. The inverses of the unit constraints are given by reflecting the diagrams vertically. For example, we have

$\sum_{i,j} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}}$ 

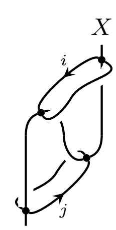
$= \sum_{i,j} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}}$ 

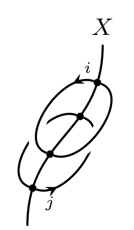
$= \sum_{i,j} \frac{\sqrt{d_i}\sqrt{d_j}}{\mathcal{D}}$ 

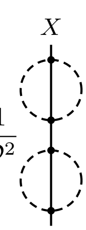
$= Q_{\Gamma,\gamma} = \mathrm{id}_{I(\mathbf{1})\overline{\otimes}(X,\gamma)}$

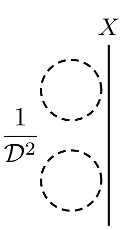
The last two forms of $\gamma^\vee$ are equivalent from the following consideration: pulling the diagonal strands across the γ^* morphism accumulates a (left) right-hand twist, i.e. (inverse) Drinfeld morphism, and these cancel out by the naturality of half-braidings.

It is also easy to check that the (co)evaluation maps described have the desired properties. The only potentially confusing thing is that the "empty strand" in graphical calculus is no longer the unit of $\mathcal{Z}(\mathcal{A})$, so the unit object and unit constraints have to be explicitly included. For example, (all dot nodes represent γ, and sum over $i, j \in \mathrm{Irr}(\mathcal{A})$ is implicit)

$\mathrm{ev}_{(X,\gamma)} \circ \mathrm{coev}_{(X,\gamma)} = \frac{d_i d_j}{\mathcal{D}^2}$ 

$= \frac{d_i d_j}{\mathcal{D}^2}$ 

$= \frac{1}{\mathcal{D}^2}$ 

$= \frac{1}{\mathcal{D}^2}$ 

$= \mathrm{id}_{(X,\gamma)}$

As for the pivotal structure, it is straightforward to check that $\gamma^{\vee\vee} = \gamma^{**}$. □

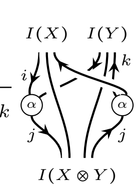
Remark 7.10. $\overline{\otimes}$ is not braided; indeed, we will see later (e.g. (7.22)) that there can be objects $(X,\gamma),(Y,\mu) \in \mathcal{Z}(\mathcal{A})$ such that $(X,\gamma)\overline{\otimes}(Y,\mu) \not\simeq 0$ and $(Y,\mu)\overline{\otimes}(X,\gamma) \simeq 0$. In particular, this necessitates the "multi" in multifusion. △

When considering the usual monoidal structure on $\mathcal{Z}(\mathcal{A})$, the forgetful functor $F : \mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ is naturally a tensor functor, but its adjoint, $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is not. With the reduced tensor product, however, F is clearly not tensor, but I is:

Proposition 7.11. *The functor I from Proposition 7.2 is a tensor functor. More precisely, for $X, Y \in \mathcal{A}$, define the morphism*

$$J_{X,Y} : I(X)\overline{\otimes} I(Y) \to I(X \otimes Y)$$

by

$J_{X,Y} := \sqrt{d_i}\sqrt{d_j}\sqrt{d_k}$ 

Then

$$(I, J) : (\mathcal{A}, \otimes) \to (\mathcal{Z}(\mathcal{A}), \overline{\otimes})$$

is a pivotal tensor functor.

Proof. Straightforward computations. □

Remark 7.12. In [Was2], Wasserman showed that the Drinfeld center of a symmetric $\mathcal{A}$ is a "bilax 2-fold tensor category", which loosely means a category with two monoidal structures that almost commute. More precisely, it consists of a pair of natural transformations

$$\eta : ((X,\gamma)\otimes(X',\gamma'))\,\overline{\otimes}\,((Y,\mu)\otimes(Y',\mu')) \rightleftharpoons ((X,\gamma)\,\overline{\otimes}\,(Y,\mu))\otimes((X',\gamma')\,\overline{\otimes}\,(Y',\mu')) : \zeta$$

such that $\eta\circ\zeta = \mathrm{id}$, together with several morphisms relating the units $\mathbf{1}$ and $\overline{\mathbf{1}}$, and they satisfy a cocktail of compatibility axioms. We claim that $\mathcal{Z}(\mathcal{A})$ also has such a structure when $\mathcal{A}$ is not symmetric. The only difference to the structure maps is where we have to distinguish the braiding c from its inverse: $\eta = \mathrm{id}_X \otimes c^{-1}_{X',Y}\otimes \mathrm{id}_{Y'}$ and $\zeta = \mathrm{id}_X \otimes c_{Y,X'}\otimes \mathrm{id}_{Y'}$ (with the various projections implicit). The proof that they satisfy the various compatibility axioms is a lengthy calculation that we do not share here. There is a more topological approach, see Remark 7.25. We do not prove this claim in this thesis, as it will take us too far afield. △

7.3. **Horizontal Trace of $\mathcal{A}$.**

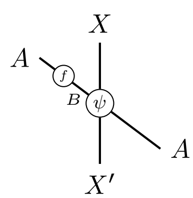
In this section, we will consider the *horizontal trace* of $\mathcal{A}$, as defined in [BHLŽ2017, Section 2.4], which is a generalization of Ocneanu's tube algebra [O]. We follow the exposition in Section 2.3 (see also [KT]), where we also considered a minor generalization to bimodule categories $\mathcal{M}$; here we only consider $\mathcal{M} = \mathcal{A}$.

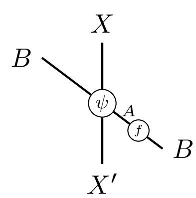
Definition 7.13. Consider $\mathcal{A}$ as a bimodule category over itself by left and right multiplication. Its *horizontal trace*, denoted $\mathrm{hTr}(\mathcal{A})$ or simply $\hat{\mathcal{A}}$, is the category with the following objects and morphisms:

Objects: same as in $\mathcal{A}$

Morphisms: $\mathrm{Hom}_{\hat{\mathcal{A}}}(X,X') := \bigoplus_A \mathrm{Hom}^A_{\mathcal{A}}(X,X')/\sim$, where $\mathrm{Hom}^A_{\mathcal{A}}(X,X') := \mathrm{Hom}_{\mathcal{A}}(A\otimes X, X'\otimes A)$, the sum is over all objects $A\in\mathcal{A}$, and $\sim$ is the equivalence relation generated by the following:

For any $\psi \in \mathrm{Hom}^{B,A}_{\mathcal{A}}(X,X') := \mathrm{Hom}_{\mathcal{A}}(B\otimes X, X'\otimes A)$ and $f\in\mathrm{Hom}_{\mathcal{C}}(A,B)$, we have

(7.12) $\mathrm{Hom}^A_{\mathcal{A}}(X,X') \ni$ 

$\sim$ 

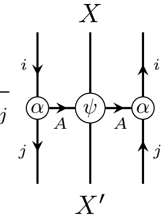
$\in \mathrm{Hom}^B_{\mathcal{A}}(X,X')$

In other words, $\mathrm{Hom}_{\hat{\mathcal{A}}}(X,X') = \int^A \mathrm{Hom}^A_{\mathcal{A}}(X,X')$. △

Definition 7.14. Let $\mathrm{hTr} : \mathcal{A}\to\hat{\mathcal{A}}$ be the natural inclusion functor that is identity on objects, and on morphisms is the natural map $\mathrm{Hom}_{\mathcal{A}}(X,X') = \mathrm{Hom}^{\mathbf{1}}_{\mathcal{A}}(X,X') \to \mathrm{Hom}_{\hat{\mathcal{A}}}(X,X')$. △

The adjective "inclusion" further justified by the following proposition (along with the fact that I is faithful), which implies hTr is faithful also.

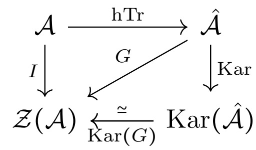
Proposition 7.15. [KT, Theorem 3.9] *Let $G : \hat{\mathcal{A}} \to \mathcal{Z}(\mathcal{A})$ be defined as follows: on objects, $G(X) = I(X)$, and on morphisms, for $\psi\in\mathrm{Hom}^A_{\mathcal{M}}(X,X')$,*

(7.13) $$G(\psi) = \sum_{i,j\ \mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j}$$ 

Then the extension to the Karoubi envelope is an equivalence of abelian categories:

$$\mathrm{Kar}(G) : \mathrm{Kar}(\hat{\mathcal{A}}) \simeq \mathcal{Z}(\mathcal{A})$$

Under this equivalence, the natural functor $\mathrm{hTr} : \mathcal{A} \to \hat{\mathcal{A}}$ *is identified with* $I : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$*, i.e. we have the commutative diagram*

(7.14)


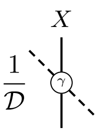
We note that in [KT], the proposition would establish an equivalence $\mathrm{Kar}(\mathrm{hTr}(\mathcal{M})) \simeq \mathcal{Z}(\mathcal{M})$, where $\mathrm{hTr}(\mathcal{M})$ is the horizontal trace of an $\mathcal{A}$-bimodule category, and $\mathcal{Z}(\mathcal{M})$ is the center of $\mathcal{M}$ [GNN2009, Definition 2.1] which is analogous to the Drinfeld center.

It is useful to construct an inverse to $\mathrm{Kar}(G)$:

Proposition 7.16. *An inverse to* $\mathrm{Kar}(G)$ *is given by:*

$$\mathrm{Kar}(G)^{-1} : \mathcal{Z}(\mathcal{A}) \simeq \mathrm{Kar}(\hat{\mathcal{A}})$$
$$(X, \gamma) \mapsto (X, \hat{P}_\gamma)$$

where

$$\hat{P}_\gamma := \sum_{i \in \mathrm{Irr}(\mathcal{A})} \frac{d_i}{\mathcal{D}} \gamma_{X_i} = \frac{1}{\mathcal{D}}$$


and on morphisms, for $f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}((X, \gamma), (Y, \mu))$,

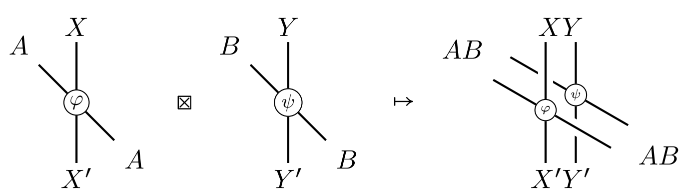
$$f \mapsto \hat{P}_\mu \circ f = f \circ \hat{P}_\gamma$$

Proof. Straightforward. □

When $\mathcal{A}$ is premodular, in particular when $\mathcal{A}$ is *braided*, $\hat{\mathcal{A}}$ has a natural monoidal structure. This was discussed in [KT, Example 8.2]; here we spell it out more explicitly.

Proposition 7.17. *There is a tensor product on* $\hat{\mathcal{A}}$*, denoted* $\hat{\otimes}$*: on objects, it is simply the same as* $\mathcal{A}$*, and on morphisms,*

$$\hat{\otimes} : \ \mathrm{Hom}^{A}_{\mathcal{A}}(X, X') \ \boxtimes \ \mathrm{Hom}^{B}_{\mathcal{A}}(Y, Y') \ \longrightarrow \ \mathrm{Hom}^{AB}_{\mathcal{A}}(XY, X'Y')$$



The unit object is the same as that of $\mathcal{A}$.

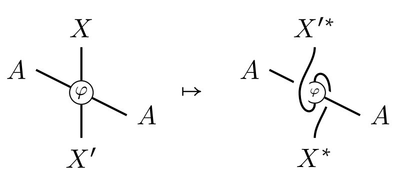
Furthermore, this tensor product structure is compatible with the rigid and pivotal structures of $\mathcal{A}$.

In other words, this is an extension the tensor product on $\mathcal{A}$*, so that the inclusion functor* $\mathrm{hTr} : \mathcal{A} \to \hat{\mathcal{A}}$ *is a pivotal tensor functor.*

Proof. Straightforward. □

It is useful to work out the left dual of a morphism (right dual being similar):

$$\mathrm{Hom}^{A}_{\mathcal{A}}(X, X') \ \to \ \mathrm{Hom}^{A}_{\mathcal{A}}(X'^{*}, X^{*})$$

(7.15)


In particular, when $X' = X$, and $\varphi = \gamma$ is a half-braiding on X, the right side is nothing but $\gamma^\vee$.

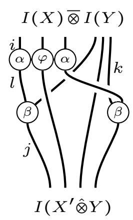
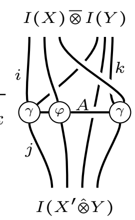
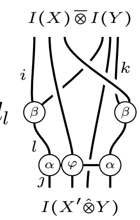
The following proposition is an upgrade of Proposition 7.15:

Proposition 7.18. *When $\mathcal{Z}(\mathcal{A})$ is endowed with the reduced tensor product, the equivalence in Proposition 7.15 is an equivalence of pivotal multifusion categories. More precisely, let J be the tensor structure on I from Proposition 7.11. Then*

$$(G, J) : (\hat{\mathcal{A}}, \hat{\otimes}) \to (\mathcal{Z}(\mathcal{A}), \overline{\otimes})$$

is a pivotal tensor functor, and hence its completion to $\mathrm{Kar}(\hat{\mathcal{A}})$ *is a pivotal tensor equivalence.*

Proof. As Proposition 7.11 takes care of objects, it remains to check naturality of J with respect to morphisms of $\hat{\mathcal{A}}$. This is easy to do: for example, for $\varphi \in \mathrm{Hom}^A_{\mathcal{A}}(X, X')$, we see that $J_{X',Y} \circ (I(\varphi) \overline{\otimes} I(\mathrm{id}_Y)) = I(\varphi \otimes \mathrm{id}_Y) \circ J_{X,Y}$ by

$\sqrt{d_i}\sqrt{d_j}\sqrt{d_k}d_l$  $= \sqrt{d_i}\sqrt{d_j}\sqrt{d_k}$  $= \sqrt{d_i}\sqrt{d_j}\sqrt{d_k}d_l$ 

using (2.15) and Lemma 2.27. ∎

Proposition 7.19. *The inverse functor* $\mathrm{Kar}(G)^{-1}$ *defined in Proposition 7.16 is naturally a tensor functor.*

Proof. For the tensor structure on $\mathrm{Kar}(G)^{-1}$, there is a natural isomorphism $(X, \hat{P}_\gamma) \otimes (Y, \hat{P}_\mu) = (XY, \hat{P}_\gamma \otimes \hat{P}_\mu) \simeq (X_\gamma \overline{\otimes}_\mu Y, \hat{P}_{\gamma \overline{\otimes} \mu})$, which is given by the natural projection $Q_{\gamma,\mu} : XY \to X_\gamma \overline{\otimes}_\mu Y$. This follows from two simple observations:

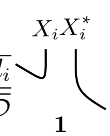
$$\hat{P}_\gamma \otimes \hat{P}_\mu = \hat{P}_{\gamma \overline{\otimes} \mu} \in \mathrm{End}_{\hat{\mathcal{A}}}(XY)$$

(using Lemma 2.27), which implies $(XY, \hat{P}_\gamma \otimes \hat{P}_\mu) = (XY, \hat{P}_{\gamma \overline{\otimes} \mu})$, and

$$\hat{P}_{\gamma \overline{\otimes} \mu} = \hat{P}_\gamma \circ Q_{\gamma,\mu} = \hat{P}_\gamma \circ Q_{\gamma,\mu} \circ Q_{\gamma,\mu} = \hat{P}_{\gamma \overline{\otimes} \mu} \circ Q_{\gamma,\mu} \in \mathrm{End}_{\hat{\mathcal{A}}}(XY)$$

(see (7.8)) which implies $Q_{\gamma,\mu} : (XY, \hat{P}_{\gamma \overline{\otimes} \mu}) \to (X_\gamma \overline{\otimes}_{,\mu} Y, \hat{P}_{\gamma \overline{\otimes} \mu})$ is an isomorphism. It is easy to check that this satisfies the hexagon axiom for tensor structure.

The isomorphism $\mathrm{Kar}(G)^{-1}(\overline{\mathbf{1}}) \simeq \mathbf{1}$ is given by

(7.16) $$\sum_{i \in \mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$ 

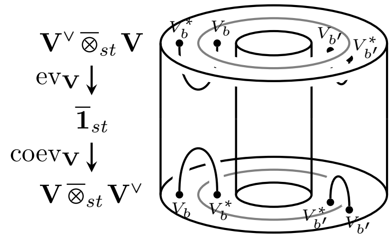
∎

7.4. **Equivalence to Stacking Product.**

As we have seen in the proof of Theorem 6.15, $Z^{\mathrm{sk}}_{\mathrm{CY}}(\mathrm{Ann}) \simeq \mathcal{Z}(\mathcal{A})$ as abelian categories. Here we show that the reduced tensor product on $\mathcal{Z}(\mathcal{A})$ is equivalent to the stacking product on $Z^{\mathrm{sk}}_{\mathrm{CY}}(\mathrm{Ann})$. The following is a visual summary of results in Section 6.2 specialized to $N = \mathrm{Ann}$:

Proposition 7.20 ([KT, Proposition 6.7], [Tha2020])**.** *The stacking tensor product on $Z^{sk}_{CY}(Ann)$, as defined in Section 6.2, denoted by $\overline{\otimes}_{st}$, makes $\hat{Z}_{CY}(Ann)$ a pivotal multifusion category as follows:*

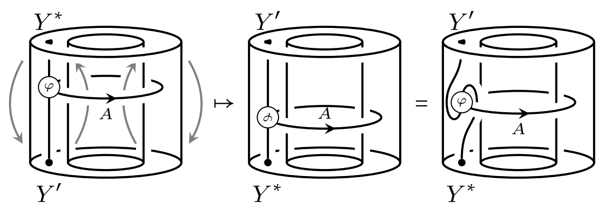
- *For an object* $\mathbf{V} = (B, \{V_b\})$*, its left dual is the object* $\mathbf{V}^\vee := (\theta(B), \{V_b^*\})$*, where* θ *is the operation on Ann that flips* $(0,1)$*, and has the following evaluation and coevaluation maps:*



(The gray lines indicate $S^1 \times \{1/2\}$ *which separates* $\mathbf{V}^\vee$ *from* $\mathbf{V}$ *in the tensor product, and play no role in defining these morphisms.) We think of the outside half of the annulus as the left side in the tensor product. The right dual is* ${}^\vee\mathbf{V} := (\theta(B), \{{}^*V_b\})$*, with essentially the same (co)evaluation maps.*
- *The pivotal map* $\delta_{\mathbf{V}} : \mathbf{V} \to \mathbf{V}^{\vee\vee}$ *is simply the graph with vertical strands running down, and one node on each strand labeled with* δ_{V_b}*.*

$Z_{CY}(Ann)$*, as the Karoubi envelope of* $\hat{Z}_{CY}(Ann)$*, naturally inherits these structures.*

It is not hard to see that the left dual of a morphism is obtained by turning the solid annulus "inside-out"; for example,

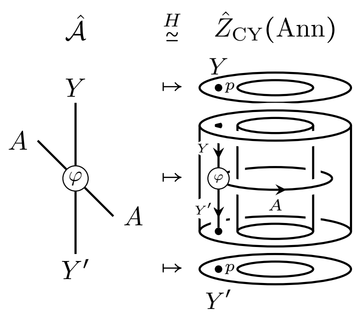


The gray arrows indicate the "inside-out" operation. [5] Note the upside-down 'φ' in the second diagram; the last diagram can be turned into the second by pulling on the upward and downward strands, forcing the φ node to turn upside-down.

The last diagram is reminiscent of duals in $\hat{\mathcal{A}}$ (see (7.15)). In particular, when $Y' = Y$, and φ is a half-braiding on Y, the extra bending in the last diagram can be incorporated into the node to become $\varphi^\vee$ (see Proposition 7.9).

In [KT], in Example 8.2, we provided an explicit equivalence $H = H_p : \hat{\mathcal{A}} \simeq \hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, where $p \in \mathrm{Ann}$, given as follows: [6]

(7.17)



This is an equivalence by Theorem 6.14, and it is not hard to see that, endowing $\hat{\mathcal{A}}$ with the tensor product $\hat{\otimes}$, this equivalence is in fact tensor and pivotal. For example, for $X, Y \in \mathrm{Obj}\,\hat{\mathcal{A}}$, the tensor product

[5] Imagine pulling your hand out of the sleeve of a tight sweater - the second diagram is inside-out the same way the sleeve is.

[6] H_p is dependent on the choice of a point $p \in \mathrm{Ann}$, but all H_p are naturally isomorphic (by non-unique natural isomorphism). One can consider the full subcategory with objects of the form $(\{p\}, \{X\})$. This is strictly speaking not a tensor subcategory, since the tensor product of such objects would have two arcs. However, one can put a different tensor product, $(\{p\}, \{X\}) \overline{\otimes}_{\mathrm{st}}{}'(\{p\}, \{Y\}) = (\{p\}, \{XY\})$, and the inclusion can be made a pivotal tensor equivalence.

in $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, $H(X)\,\overline{\otimes}_{\mathrm{st}}\,H(Y)$, is an object with two arcs, labeled with X and Y. The tensor structure on H would be a trivalent graph connecting $H(X)\,\overline{\otimes}_{\mathrm{st}}\,H(Y)$ to $H(XY)$, with the unique vertex labeled by id_{XY} (which is naturally identified with coev $\in \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, XY(XY)^*)$). The unit object $\overline{\mathbf{1}}_{\mathrm{st}}$ in $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, i.e. the empty configuration, is isomorphic to the object $H(\mathbf{1}) = (\{p\}, \{\mathbf{1}\})$. Hence we have the following:

Theorem 7.21. *We have the following commutative diagram, where all functors are pivotal tensor functors:*

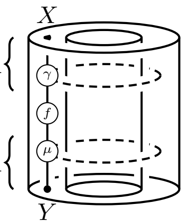
$$\begin{array}{ccccc}
(\mathcal{A}, \otimes) & \xrightarrow{\mathrm{hTr}} & (\hat{\mathcal{A}}, \hat{\otimes}) & \xrightarrow[\simeq]{H} & (\hat{Z}_{CY}(Ann), \overline{\otimes}_{st}) \\
\downarrow I & \swarrow G & \downarrow \mathrm{Kar} & & \downarrow \mathrm{Kar} \\
(\mathcal{Z}(\mathcal{A}), \overline{\otimes}) & \xleftarrow[\mathrm{Kar}(G)]{\simeq} & (\mathrm{Kar}(\hat{\mathcal{A}}), \hat{\otimes}) & \xrightarrow[\mathrm{Kar}(H)]{\simeq} & (Z_{CY}(Ann), \overline{\otimes}_{st})
\end{array}$$

In other words, the reduced tensor product $\overline{\otimes}$ on $\mathcal{Z}(\mathcal{A})$ encodes the stacking tensor product on $Z_{CY}(Ann)$.

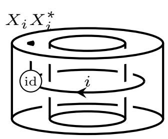
Together with Proposition 7.16, the pivotal tensor equivalence $(\mathcal{Z}(\mathcal{A}), \overline{\otimes}) \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$ is given by

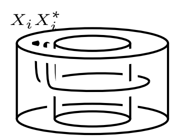
$$\mathrm{Kar}(H) \circ \mathrm{Kar}(G)^{-1} : (\mathcal{Z}(\mathcal{A}), \overline{\otimes}) \simeq Z_{\mathrm{CY}}(\mathrm{Ann}) \tag{7.18}$$

$$\begin{array}{ccc}
(X, \gamma) & \mapsto & \mathrm{im}\Big\{ \\
\downarrow f & \mapsto & \\
(Y, \mu) & \mapsto & \mathrm{im}\Big\{
\end{array} \tag{7.19}$$



Once again this functor doesn't send unit to unit; the isomorphism is given by (compare (7.16))

$$\sum_{i \in \mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$ 

, or more intuitively, $$\sum_{i \in \mathrm{Irr}(\mathcal{A})} \frac{\sqrt{d_i}}{\sqrt{\mathcal{D}}}$$ 

As an application of Theorem 7.21, we describe $Z_{\mathrm{CY}}(\mathbf{T}^2)$ purely algebraically in terms of $\mathcal{A}$. We can produce $\mathbf{T}^2$ from Ann by gluing (neighborhoods of) $S^1 \times \{0\}$ and $S^1 \times \{1\}$. The excision property of Z_{CY} as stated in the introduction doesn't work as is, but [KT, Theorem 7.5] actually proves an apparently slightly more general but ultimately equivalent form of excision, which allows Σ to be obtained by gluing two boundaries of a single surface Σ'; the balanced tensor product is then replaced by the center of $Z_{\mathrm{CY}}(\Sigma')$ as a $Z_{\mathrm{CY}}(\mathrm{Ann})$-bimodule (as defined in [GNN2009, Definition 2.1], repeated in [KT, Definition 3.1]; for applications here, it suffices to know that this notion of center for a monoidal category as a bimodule over itself coincides with the Drinfeld center.)

We can view $Z_{\mathrm{CY}}(\mathrm{Ann})$ as a bimodule category over itself, thinking of the left and right actions as "insertions" from the left ($S^1 \times \{0\}$) and right ($S^1 \times \{0\}$). Thus we have the following corollary of [KT, Theorem 7.5]:

Proposition 7.22.

$$Z_{CY}(\mathbf{T}^2) \simeq \mathcal{Z}((Z_{CY}(Ann), \overline{\otimes}_{st}))$$

Proof. Take $X' = S^1 \times (0, 3)$ in [KT, Theorem 7.5]. □

As an immediate corollary of this and Theorem 7.21, we have

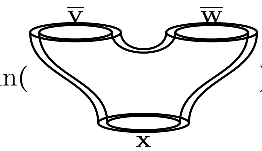
Corollary 7.23.

$$Z_{CY}(\mathbf{T}^2) \simeq \mathcal{Z}((\mathcal{Z}(\mathcal{A}), \overline{\otimes}))$$

as abelian categories.

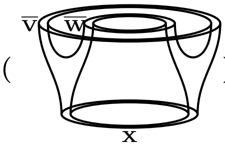
Remark 7.24. Since $\overline{\otimes}$ on $\mathcal{Z}(\mathcal{A})$ has a nice topological interpretation as $\overline{\otimes}_{\mathrm{st}}$ on $Z_{\mathrm{CY}}(\mathrm{Ann})$, it is also nice to have a topological interpretation for the standard tensor product on $\mathcal{Z}(\mathcal{A})$. This comes from the (thickened) pair of pants, denoted M_{POP}, in the following manner. For $(X, \gamma), (Y, \mu) \in \mathcal{Z}(\mathcal{A})$, with corresponding objects $\mathbf{V} = (H(X), H(\hat{P}_\gamma)), \mathbf{W} = (H(Y), H(\hat{P}_\mu)) \in Z_{\mathrm{CY}}(\mathrm{Ann})$, the object $\mathbf{V} \otimes \mathbf{W}$ which corresponds to $(X, \gamma) \otimes (Y, \mu)$ is characterized by a natural isomorphism

$$\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathrm{Ann})}(\mathbf{V} \otimes \mathbf{W}, \mathbf{X}) \simeq \mathrm{Skein}(M_{\mathrm{POP}}; \overline{\mathbf{V}}, \overline{\mathbf{W}}, \mathbf{X}) = \mathrm{Skein}(\quad)$$



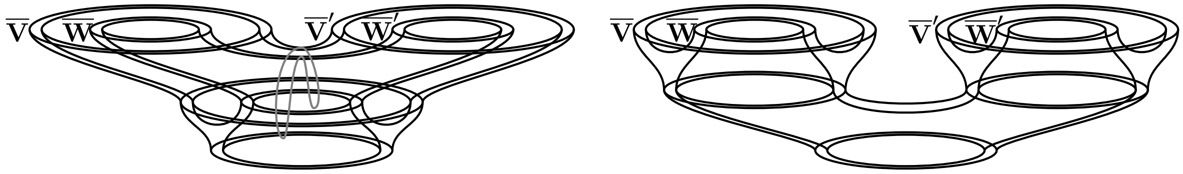
for all $\mathbf{X} \in Z_{\mathrm{CY}}(\mathrm{Ann})$. (This is well-known for the extended Turaev-Viro theory [TV1992], where given spherical fusion $\mathcal{A}$, one has $Z_{\mathrm{TV}}(S^1) = \mathcal{Z}(\mathcal{A})$ [Kir], and the standard tensor product on $\mathcal{Z}(\mathcal{A})$ is given by the pair of pants just as above.) The stacking product can also be described this way, but instead of the usual pair of pants M_{POP}, we use a different cobordism, M_{Y}:

$$\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathrm{Ann})}(\mathbf{V} \mathbin{\overline{\otimes}_{\mathrm{st}}} \mathbf{W}, \mathbf{X}) \simeq \mathrm{Skein}(M_{\mathrm{Y}}; \overline{\mathbf{V}}, \overline{\mathbf{W}}, \mathbf{X}) = \mathrm{Skein}(\quad)$$



M_{Y} is a thickened 'Y' crossed with S^1. We do not prove these claims here, which are not hard to prove after all the work in this section. △

Remark 7.25. The topological interpretations of $\otimes$ and $\overline{\otimes}_{\mathrm{st}}$ above can also elucidate the structure morphisms mentioned in Remark 7.12. Consider the two cobordisms below:



The cobordism on the left (ignoring the gray curve) corresponds to $(\mathbf{V} \otimes \mathbf{V}') \mathbin{\overline{\otimes}_{\mathrm{st}}} (\mathbf{W} \otimes \mathbf{W}')$, or $((X, \gamma) \otimes (X', \gamma')) \mathbin{\overline{\otimes}} ((Y, \mu) \otimes (Y', \mu'))$, while the cobordism on the right corresponds to $(\mathbf{V} \mathbin{\overline{\otimes}_{\mathrm{st}}} \mathbf{W}) \otimes (\mathbf{V}' \mathbin{\overline{\otimes}_{\mathrm{st}}} \mathbf{W}')$, or $((X, \gamma) \mathbin{\overline{\otimes}} (Y, \mu)) \otimes ((X', \gamma') \mathbin{\overline{\otimes}} (Y', \mu'))$. They are different, but not by much: the right can be obtained from the left by surgery, specifically, by attaching a 2-handle along the gray curve (the gray curve starts in the outside M_{POP} for $\mathbf{V}$'s, goes down into the bottom M_{Y}, back up into the M_{POP} for $\mathbf{W}$'s, then goes down again into M_{Y}, and closes up, all the while staying close to the "inner boundary", i.e. keeping as tight as possible like a rubber band). One way to visualize this is to consider the reverse process of removing a 2-handle: start with the right side, push the two "troughs" - between $\mathbf{V}$ and $\mathbf{W}$ and between $\mathbf{V}'$ and $\mathbf{W}'$ - downward and into the bottom pair of pants, and when they are about to meet in the middle, drill a hole through the wall.

On the level of skein modules, this surgery is the same as adding the gray curve colored by the regular coloring (up to a factor, see (2.19)); this is the topological interpretation of η. Conversely, graphs in the right cobordism can be lifted to a graph in the left cobordism plus the gray curve with regular coloring; this is the topological interpretation of ζ.

The other structure morphisms can also be described very easily. For example, one of them is a morphism $v_2 : \overline{\mathbf{1}} \otimes \overline{\mathbf{1}} \to \overline{\mathbf{1}}$, which is simply the empty graph in M_{POP} (up to a factor) interpreted as a morphism $\overline{\mathbf{1}}_{\mathrm{st}} \mathbin{\overline{\otimes}_{\mathrm{st}}} \overline{\mathbf{1}}_{\mathrm{st}} \to \overline{\mathbf{1}}_{\mathrm{st}}$.

Thus, checking the compatibility axioms becomes an exercise in topology. Once again, we do not show the work in this paper.

As mentioned in the introduction, such 2-fold monoidal structures are related to iterated loop spaces. It may be interesting to see if these two topological aspects of $(\mathcal{Z}(\mathcal{A}), \otimes, \overline{\otimes})$ are directly related. △

7.5. $\mathcal{A}$ Modular.

When $\mathcal{A}$ is modular, we have ([Müg2003], see also [EGNO2015, Proposition 8.20.12]):

$$\begin{aligned} \mathcal{A}^{\mathrm{bop}} \boxtimes \mathcal{A} &\simeq_{\otimes,\mathrm{br}} (\mathcal{Z}(\mathcal{A}), \otimes) \\ X \boxtimes Y &\mapsto (X \otimes Y, c^{-1} \otimes c) = (X, c^{-1}) \otimes (Y, c) \end{aligned} \tag{7.20}$$

where $\mathcal{A}^{\text{bop}}$ is $\mathcal{A}$ with the opposite braiding, and c is the braiding on $\mathcal{A}$. Here the monoidal structure on $\mathcal{A}^{\text{bop}} \boxtimes \mathcal{A}$ is defined component-wise. In particular, duals are given by $(X \boxtimes Y)^* = X^* \boxtimes Y^*$.

It is natural to ask: what is the reduced tensor product on $\mathcal{A}^{\text{bop}} \boxtimes \mathcal{A}$ under this equivalence? We claim (proven below in Theorem 7.28) that the following definition is the answer, which justifies the repeated use of the name "reduced" and notation like $\overline{\otimes}$ and $\overline{\mathbf{1}}$. (The reduced tensor product on $\mathcal{Z}(\mathcal{A})$ cannot in general be braided, as we shall see soon, so we will ignore the difference in braiding on $\mathcal{A}$.)

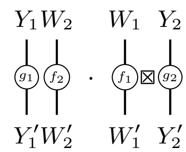
Definition 7.26. Let $W_1 \boxtimes Y_1, W_2 \boxtimes Y_2 \in \mathcal{A} \boxtimes \mathcal{A}$. Define their *reduced tensor product* to be

$$(W_1 \boxtimes Y_1) \overline{\otimes} (W_2 \boxtimes Y_2) := \langle Y_1, W_2 \rangle \cdot W_1 \boxtimes Y_2$$

where recall $\langle Y_1, W_2 \rangle := \text{Hom}_{\mathcal{A}}(\mathbf{1}, Y_1 W_2)$. $\overline{\otimes}$ naturally extends to direct sums, and is clearly associative.

For morphisms $f_1 \boxtimes g_1 : W_1 \boxtimes Y_1 \to W_1' \boxtimes Y_1'$, $f_2 \boxtimes g_2 : W_2 \boxtimes Y_2 \to W_2' \boxtimes Y_2'$, their *reduced tensor product* is given by

$$(f_1 \boxtimes g_1) \overline{\otimes} (f_2 \boxtimes g_2) := \langle g_1, f_2 \rangle \cdot f_1 \boxtimes g_2 =$$



where the left side, the "coefficient" $\langle g_1, f_2 \rangle$, is to be interpreted as a linear map $\langle Y_1, W_2 \rangle \to \langle Y_1', W_2' \rangle$ by composition. △

For example,

$$(X_i \boxtimes X_j^*) \overline{\otimes} (X_k \boxtimes X_l^*) \simeq \delta_{j,k} X_i \boxtimes X_l^* \tag{7.21}$$

In particular, when $i \neq j$,

$$\begin{aligned} &(X_i \boxtimes X_i^*) \overline{\otimes} (X_i \boxtimes X_j^*) \simeq X_i \boxtimes X_j^* \\ &(X_i \boxtimes X_j^*) \overline{\otimes} (X_i \boxtimes X_i^*) \simeq 0 \end{aligned} \tag{7.22}$$

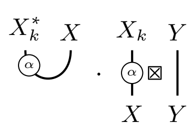
so $\overline{\otimes}$ cannot be braided.

Proposition 7.27. $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ *is a pivotal multifusion category. More precisely,*

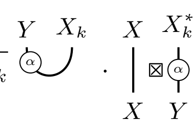
- *The unit object, denoted* $\overline{\mathbf{1}}$, *is* $\bigoplus_{i \in \text{Irr}(\mathcal{A})} X_i \boxtimes X_i^*$;
- $(X \boxtimes Y)^\vee = Y^* \boxtimes X^*$, ${}^\vee(X \boxtimes Y) = {}^\vee Y \boxtimes {}^* X$, *the (co)evaluation maps are described in the proof;*
- *The pivotal structure is defined component-wise:* $\delta_{X \boxtimes Y} = \delta_X \boxtimes \delta_Y$.

Proof. Fix $X \boxtimes Y \in \mathcal{A} \boxtimes \mathcal{A}$. The left and right unit constraint are given by

$$l_{X \boxtimes Y} := \sum_{k \in \text{Irr}(\mathcal{A})} \sqrt{d_k}$$



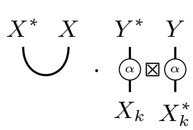
$$; \quad r_{X \boxtimes Y} := \sum_{k \in \text{Irr}(\mathcal{A})} \sqrt{d_k}$$



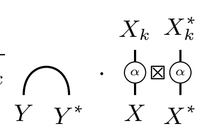
where we recall that α is a sum over a pair of dual bases (see (2.15)). Their inverses are given by flipping the diagram upside down.

The left (co)evaluation maps are given by

$$\text{ev}_{X \boxtimes Y} := \sum_{k \in \text{Irr}(\mathcal{A})} \sqrt{d_k}$$



$$; \quad \text{coev}_{X \boxtimes Y} := \sum_{k \in \text{Irr}(\mathcal{A})} \sqrt{d_k}$$



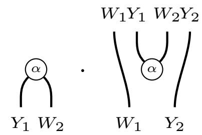
The right (co)evaluation maps are given by similar diagrams. It is straightforward to check that these have the right properties. □

Theorem 7.28. *There is an equivalence of pivotal multifusion categories*

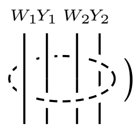
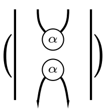
$$\begin{aligned} K : \mathcal{A} \boxtimes \mathcal{A} &\simeq_{\otimes} (\mathcal{Z}(\mathcal{A}), \overline{\otimes}) \\ X \boxtimes Y &\mapsto (XY, c^{-1} c) \end{aligned}$$

Proof. The tensor structure L on K is given as follows: for $W_1 \boxtimes Y_1, W_2 \boxtimes Y_2$, the isomorphism $L : K(W_1 \boxtimes Y_1) \overline{\otimes} K(W_2 \boxtimes Y_2) \simeq K((W_1 \boxtimes Y_1) \overline{\otimes} (W_2 \boxtimes Y_2))$ is given by

$$L : (W_1Y_1, c^{-1}c) \overline{\otimes} (W_2Y_2, c^{-1}c) \simeq \langle Y_1, W_2 \rangle \cdot (W_1Y_2, c^{-1}c)$$

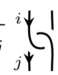


The inverse to L is given by flipping the diagram upside down. The following observation is helpful: for $(W_1Y_1, c^{-1}c), (W_2Y_2, c^{-1}c) \in \mathcal{Z}(\mathcal{A})$, we have

$$W_1Y_1 {}_{c^{-1}c}\overline{\otimes}_{c^{-1}c} W_2Y_2 = \mathrm{im}\Big(\frac{1}{\mathcal{D}}$$  $$\Big) = \mathrm{im}\Big($$  $$\Big)$$

It is easy to check that L satisfies the hexagon axiom.

Note that K does not send unit to unit - the half-braiding on $K(\overline{\mathbf{1}}) = (\bigoplus X_iX_i^*, c^{-1}c)$ is not the same as $\overline{\mathbf{1}} = (\bigoplus X_iX_i^*, \Gamma)$; the isomorphism $K(\overline{\mathbf{1}}) \simeq \overline{\mathbf{1}}$ is essentially given by the S-matrix:

$$S = \sum_{i,j \in \mathrm{Irr}(\mathcal{A})} \sqrt{d_i}\sqrt{d_j}$$ 

Clearly the pivotal structures agree. ☐

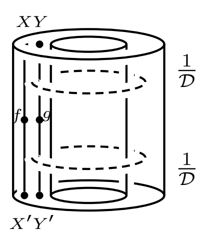
The equivalence given above has a nice interpretation in $Z_{\mathrm{CY}}(\mathrm{Ann})$. Namely, the composition $\mathrm{Kar}(H) \circ \mathrm{Kar}(G)^{-1} \circ K : \mathcal{A} \boxtimes \mathcal{A} \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$ is naturally isomorphic to the following functor:

$$\mathcal{A} \boxtimes \mathcal{A} \simeq Z_{\mathrm{CY}}(\mathrm{Ann})$$

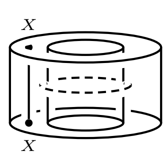
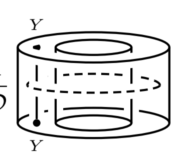
$X \boxtimes Y \mapsto \mathrm{im}\{$

$f \boxtimes g \mapsto$

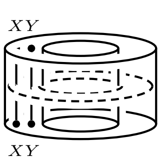
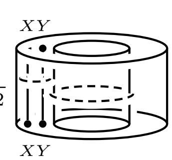
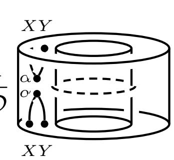
$X' \boxtimes Y' \mapsto \mathrm{im}\{$



The composition $\mathrm{Kar}(H) \circ \mathrm{Kar}(G)^{-1} \circ K$ itself only hits objects with one marked point p. The functor presented above is more intuitive from the following perspective. Restricting to the first factor, i.e. setting $Y = \mathbf{1}$, this is like including $\mathcal{A}$ into $Z_{\mathrm{CY}}(\mathrm{Ann})$ along the outer boundary; likewise, the second factor is including $\mathcal{A}$ into $Z_{\mathrm{CY}}(\mathrm{Ann})$ along the inner boundary:

$$X \mapsto (X, c^{-1}) \mapsto \mathrm{im}\Big(\frac{1}{\mathcal{D}}$$  $$\Big) \quad ; \quad Y \mapsto (Y, c) \mapsto \mathrm{im}\Big(\frac{1}{\mathcal{D}}$$  $$\Big)$$

The functor presented before is the $\overline{\otimes}_{\mathrm{st}}$-tensor product of these two. This picture also elucidates the definition of $\overline{\otimes}$ on $\mathcal{A} \boxtimes \mathcal{A}$: if we take the tensor product of the two functors above in opposite order, essentially looking at $(\mathbf{1} \boxtimes X) \overline{\otimes} (Y \boxtimes \mathbf{1})$, we get

$$K(\mathbf{1} \boxtimes X) \overline{\otimes}_{\mathrm{st}} K(Y \boxtimes \mathbf{1}) = \mathrm{im}\Big(\frac{1}{\mathcal{D}^2}$$  $$\Big) = \mathrm{im}\Big(\frac{1}{\mathcal{D}^2}$$  $$\Big) = \mathrm{im}\Big(\frac{1}{\mathcal{D}}$$  $$\Big) \simeq \langle X, Y \rangle \cdot \overline{\mathbf{1}}_{\mathrm{st}}$$

where we used (2.19) and Lemma 2.31.

Note that the equivalence $\mathcal{A}^{\mathrm{bop}} \boxtimes \mathcal{A} \simeq_{\otimes,\mathrm{br}} (\mathcal{Z}(\mathcal{A}), \otimes)$ mentioned in the beginning of this section is also built by tensoring the same two functors together; it just happens that [7]

$$(X, c^{-1}) \otimes (Y, c) = (X, c^{-1}) \mathbin{\overline{\otimes}} (Y, c)$$

Remark 7.29. One can consider a similar tensor product on $\mathcal{A} \boxtimes \mathcal{A}$ when $\mathcal{A}$ is not modular. Definition 7.26 of $\overline{\otimes}$ will look the same, but $\langle Y_1, W_2\rangle$ would be replaced by the symmetric part of $Y_1 W_2$, that is, the direct summands of $Y_1 W_2$ that belong to the symmetric center of $\mathcal{A}$. The functor $K : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ will still respect $\overline{\otimes}$, but K will not be an equivalence. Furthermore, it is not clear whether $\overline{\otimes}$ on $\mathcal{A} \boxtimes \mathcal{A}$ possesses a unit. As evidence suggestive of this, recall that in the modular case, even showing $K(\overline{\mathbf{1}}) \simeq \overline{\mathbf{1}}$ required the non-degeneracy of the S-matrix. △

Finally, we compare $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ to Mat(Vec), the category of Vec-valued matrices (see [EGNO2015, Example 4.1.3]). Let $\mathcal{S}$ be some finite set. The objects of $\mathrm{Mat}_{\mathcal{S}}(\mathrm{Vec})$ are bigraded vector spaces $V = \bigoplus_{i,j\in\mathcal{S}} V_i^j$, and the tensor product is given by

$$(V \otimes W)_i^j = \bigoplus_{k\in\mathcal{S}} V_i^k \otimes W_k^j$$

Let $\mathbf{k}_i^j$ be $\mathbf{k}$ with bigrading i, j. Then the unit is $\bigoplus_{i\in\mathcal{S}} \mathbf{k}_i^i$. Duals are given by transposing the matrix and then taking duals componentwise, i.e. $(V^*)_i^j = (V_j^i)^*$.

Proposition 7.30. *There is a tensor equivalence*

$$\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec}) \simeq_{\otimes} (\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$$
$$\mathbf{k}_i^j \mapsto X_i \boxtimes X_j^*$$

Proof. Clearly the functor above is an equivalence of abelian categories, and sends unit to unit.

Denote $X_i^j := X_i \boxtimes X_j^*$, so $(X_i^j)^\vee = X_j^i$. One has $X_i^j \mathbin{\overline{\otimes}} X_j^l = \langle X_j^*, X_k\rangle X_i^l$, which is 0 if $j \neq k$. When $j = k$, define J_k to be the map

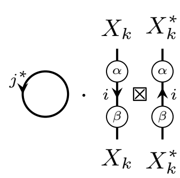
$$(\mathrm{ev}_{X_k} \circ -) \cdot \mathrm{id}_{X_i^l} : X_i^k \mathbin{\overline{\otimes}} X_k^l \simeq X_i^l$$

Then it is easy to check that

$$J_{\mathbf{k}_i^j, \mathbf{k}_k^l} = \delta_{j,k} J_k$$

is a tensor structure on the functor above. □

While $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ and $\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec})$ are tensor equivalent, they are not pivotal tensor equivalent (assuming the natural pivotal structure on $\mathrm{Mat}_{\mathrm{Irr}(\mathcal{A})}(\mathrm{Vec})$ coming from Vec). This can be seen from computing dimensions. For example, the left trace of $\mathrm{id}_{X_i^j}$ is

$$\sum_{k\in\mathrm{Irr}(\mathcal{A})} d_k$$


$$= \sum_k \delta_{i,k} \frac{d_j}{d_k} \mathrm{id}_{X_k^k}$$

so the left dimension of X_i^j is $d^L_{X_i^j} = \frac{d_j}{d_i}$, which cannot always be 1 for all pairs i, j. Its right dimension is $d^R_{X_i^j} = d^L_{X_j^i} = \frac{d_i}{d_j}$. Thus $(\mathcal{A} \boxtimes \mathcal{A}, \overline{\otimes})$ can be thought of as a non-pivotal deformation of Mat(Vec).

[7]Coincidence? I think NOT!

8. Computations for Other Surfaces

In this section, we consider the categories associated to other surfaces. We will dedicate one subsection to study the once-punctured torus.

We want to relate the skein category of a surface N with that of a punctured one N_0, that is, $N_0 = N\backslash\{p\}$. We will think of N as obtained from N_0 by gluing with an open disk, "sealing" the puncture: $N = N_0 \cup \mathbb{D}^2$, implicitly choosing some collared structure on N_0 and $\mathbb{D}^2$.

Recall $\hat{\mathcal{A}} \coloneqq \mathrm{hTr}(\mathcal{A})$ from Section 7.3. The unit object $\mathbf{1}$ in $\hat{\mathcal{A}}$ corresponds to the empty configuration in $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)$. There is a right action of $\mathrm{Hom}_{\hat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$ on the morphisms of $\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)$, by "pushing in" from the puncture, i.e. $\mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y,Y') \otimes \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathbf{1},\mathbf{1}) \to \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y \triangleleft \mathbf{1}, Y' \triangleleft \mathbf{1}) \cong \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y,Y')$. We have that $\Gamma \in \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y,Y')$ and $f,g \in \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$, $\Gamma \triangleleft (f \circ g) = (\Gamma \triangleleft f) \triangleleft g$. Moreover, for $\Gamma' \in \mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y',Y'')$, $(\Gamma' \circ \Gamma) \triangleleft (f \circ g) = (\Gamma' \triangleleft f) \circ (\Gamma \triangleleft g)$.

Let $\pi = \sum d_i/\mathcal{D} \cdot \mathrm{id}_{X_i} \in \bigoplus \mathrm{Hom}^{X_i}_{\mathcal{A}}(\mathbf{1},\mathbf{1}) = \mathrm{Hom}_{\hat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$. (Note: $\mathcal{D}$ and simples X_i are of $\mathcal{A}$, and not of $\mathcal{Z}(\mathcal{A})$.) π is an idempotent in $\mathrm{Hom}_{\hat{\mathcal{A}}}(\mathbf{1},\mathbf{1})$, and hence also acts as an idempotent on $\mathrm{Hom}_{\hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)}(Y,Y')$. As a morphism in $Z^{\mathrm{sk}}_{\mathrm{CY}}(\mathrm{Ann})$ under the equivalence discussed in Section 7 (see Theorem 7.21), π is a graph in $\mathrm{Ann} \times [0,1] \simeq S^1 \times D^2$ (a solid torus), consisting of a loop going along a core of the solid torus, labeled with the regular coloring/$\mathcal{D}$.

Proposition 8.1. *Let $N_0 = N\backslash\{p\}$ as above. Consider the category $\hat{\mathcal{B}}$ consisting of the same objects as $\hat{Z}^{sk}_{CY}(N_0)$, but morphisms given by*

$$\mathrm{Hom}_{\hat{\mathcal{B}}}(Y,Y') = \mathrm{im}(\mathrm{Hom}_{\hat{Z}^{sk}_{CY}(N_0)}(Y,Y') \circlearrowleft \pi)$$

Then the restriction to $\hat{\mathcal{B}}$ of the inclusion functor corresponding to $i : N_0 \hookrightarrow N$ is an equivalence:

$$i_*|_{\hat{\mathcal{B}}} : \hat{\mathcal{B}} \simeq \hat{Z}^{sk}_{CY}(N)$$

The Karoubi envelope $\mathcal{B} \coloneqq \mathrm{Kar}(\hat{\mathcal{B}})$ is obtained from $Z^{sk}_{CY}(N_0)$ the same way: same objects, and morphisms are

$$\mathrm{Hom}_{\mathcal{B}}(Y,Y') = \mathrm{im}(\mathrm{Hom}_{Z^{sk}_{CY}(N_0)}(Y,Y') \circlearrowleft \pi)$$

and the equivalence above extends to an equivalence

$$i_*|_{\mathcal{B}} : \mathcal{B} \simeq Z^{sk}_{CY}(N)$$

Proof. First note that $\hat{\mathcal{B}}$ is indeed closed under composition of morphisms because π is idempotent:

$$(\Gamma' \triangleleft \pi) \circ (\Gamma \triangleleft \pi) = (\Gamma' \circ \Gamma) \triangleleft (\pi \circ \pi) = (\Gamma' \circ \Gamma) \triangleleft \pi \tag{8.1}$$

This also immediately implies that $\mathrm{Kar}(\hat{\mathcal{B}})$ is indeed has the description in the proposition statement.

It is clear that $i_*|_{\hat{\mathcal{B}}}$ is essentially surjective. To prove fully faithfulness, consider two objects $Y, Y' \in \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)$. Abusing notation, we also denote $i_*(Y), i_*(Y') \in \mathrm{Obj}\, \hat{Z}^{\mathrm{sk}}_{\mathrm{CY}}(N_0)$ by Y, Y'. We call the vertical segment $p \times [0,1] \subset N \times [0,1]$ the *pole*, so that $N_0 \times [0,1] = N \times [0,1] \backslash \mathrm{pole}$.

We construct an inverse map to i_*. Let U be a small open neighborhood of p in N, and let $\mathcal{N} = U \times [0,1] \subset N \times [0,1]$ be a small open neighborhood of the pole. Choose U small enough so that it does not contain any marked points of Y, Y'. Consider a graph $\Gamma \in \mathrm{Graph}(N \times [0,1]; Y^*, Y')$. Define $j(\Gamma)$ as follows: if Γ intersects the pole, then use an isotopy supported in $\mathcal{N}$ to push Γ off of it, resulting in a new graph Γ'. Now Γ' can be considered a graph in $\mathrm{Graph}(N_0 \times [0,1]; Y^*, Y')$. Then we define $j(\Gamma) = \Gamma' \triangleleft \pi$.

We need to check that j is well-defined. Firstly, the (linear combination of) graphs $\Gamma' \triangleleft \pi$ is independent of the choice of isotopy - this follows from the sliding lemma (Lemma 2.30). More generally, it means that for any isotopy φ of $N \times [0,1]$ supported on $\mathcal{N}$, $j(\Gamma) = j(\varphi(\Gamma))$.

Now we check that j sends null graphs to 0. Take the two set open cover $\{\mathcal{N}, N_0 \times [0,1]\}$ of $N \times [0,1]$, and apply Proposition 6.2. Let $\Gamma = \sum c_i \Gamma_i$ be null with respect to some ball D. If $D \subset N_0 \times [0,1]$, clearly $j(\Gamma)$ is null with respect to D. If $D \subset \mathcal{N}$, we may assume D doesn't touch the boundary (by choice of U), so we can isotope it with some isotopy φ supported on $\mathcal{N}$ so that $\varphi(D)$ doesn't meet the pole. Then clearly $j(\varphi(\Gamma))$ is null with respect to $\varphi(D)$.

Finally, it is easy to see that j is inverse to i_*. For example, $i_* \circ j$ amounts to adding a trivial dashed circle, which is equivalent to 1 by Lemma 2.32. ☐

8.1. The Sphere, S^2.

Definition 8.2 ([Müg2003, Definition 2.9]). The *Müger center* of a braided tensor category $\mathcal{A}$, denoted $\mathcal{Z}^{\text{Mü}}(\mathcal{A})$, is the full subcategory consisting of *transparent objects*, i.e. objects X such that for all objects $Y \in \mathcal{A}$,

$$c_{Y,X} \circ c_{X,Y} = \text{id}_{XY}$$

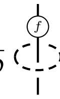
Theorem 8.3. *The category of boundary values associated to the sphere is the Muger center:*

$$Z_{CY}^{sk}(S^2) \simeq \mathcal{Z}^{M\ddot{u}}(\mathcal{A})$$

and in particular, when $\mathcal{A}$ is modular, $Z_{CY}^{sk}(S^2) \simeq \mathcal{V}ec$.

Proof. Think of the disk $\mathbb{D}^2$ as a punctured sphere, so by Proposition 8.1, we have that $\hat{Z}_{\text{CY}}^{\text{sk}}(S^2) \simeq \hat{\mathcal{B}}$, where, as in Proposition 8.1, $\hat{\mathcal{B}}$ is the category with the same objects as $\hat{Z}_{\text{CY}}^{\text{sk}}(\mathbb{D}^2) \simeq \mathcal{A}$, but morphisms are, for $A, A' \in \mathcal{A}$,

$$\text{Hom}_{\hat{\mathcal{B}}}(A, A') = \left\{ \frac{1}{\mathcal{D}} \quad \middle| \; f \in \text{Hom}_{\mathcal{A}}(A, A') \right\}$$



In particular, when $A = A' = X_i$ a simple object, it follows from [Müg2003, Corollary 2.14] that simple objects that are not transparent, i.e. not in the Müger center, are killed:

$$\text{End}_{\hat{\mathcal{B}}}(X_i) = \begin{cases} \text{End}_{\mathcal{A}}(X_i) & \text{if } X_i \in \mathcal{Z}^{\text{Mü}}(\mathcal{A}) \\ 0 & \text{otherwise} \end{cases}$$

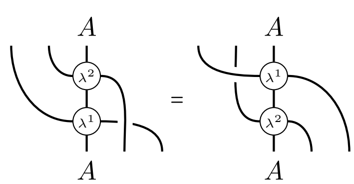
It follows that $\hat{\mathcal{B}}$ coincides with the Müger center, which is already abelian, and so $Z_{\text{CY}}^{\text{sk}}(S^2) \simeq \text{Kar}(\hat{\mathcal{B}}) = \mathcal{Z}^{\text{Mü}}(\mathcal{A})$. When $\mathcal{A}$ is modular, only the unit object is transparent, so $Z_{\text{CY}}^{\text{sk}}(S^2) \simeq \mathcal{Z}^{\text{Mü}}(\mathcal{A}) \simeq \mathcal{V}ec$. ☐

8.2. The Once-Punctured Torus, $\mathbf{T}_0^2$.

In [Tha2019], we defined a category, which we called the *elliptic Drinfeld center*, and claimed that it is equivalent to the category of boundary values associated to the once-punctured torus. We then proved this claim in [KT]. We present the results here.

Definition 8.4. Let $\mathcal{A}$ be a premodular category. The *elliptic Drinfeld center of* $\mathcal{A}$, denoted $\mathcal{Z}^{\text{el}}(\mathcal{A})$, is defined as follows: the objects are tuples $(A, \lambda^1, \lambda^2)$, where λ^1, λ^2 are half-braidings on A that satisfy

(8.2)



We call the relation (8.2) "COMM". The morphisms $\text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{A})}((A, \lambda^1, \lambda^2), (A', \mu^1, \mu^2)$ are morphisms of $\mathcal{A}$ that intertwine both half-braidings, i.e.

$$\text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{A})}((A, \lambda^1, \lambda^2), (A', \mu^1, \mu^2)) \coloneqq \text{Hom}_{\mathcal{Z}(\mathcal{A})}((A, \lambda^1), (A', \mu^1)) \cap \text{Hom}_{\mathcal{Z}(\mathcal{A})}((A, \lambda^2), (A', \mu^2))$$

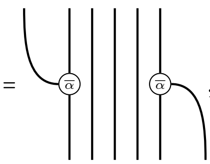
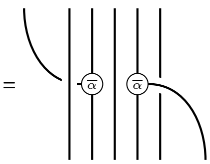
In [Tha2019], we proved the following:

Proposition 8.5 ([Tha2019], Prop 3.4). *$\mathcal{Z}^{el}(\mathcal{A})$ is a finite semisimple category.*

Proposition 8.6 ([Tha2019], Prop 3.5, Prop 3.8)**.** *The forgetful functor $\mathcal{F}^{el} : \mathcal{Z}^{el}(\mathcal{A}) \to \mathcal{A}$ has a two-sided adjoint $\mathcal{I}^{el} : \mathcal{A} \to \mathcal{Z}^{el}(\mathcal{A})$, where on objects, $\mathcal{I}^{el}$ sends*

$$A \mapsto (\bigoplus_{i,j} X_i X_j A X_j^* X_i^*, \Gamma^1, \Gamma^2)$$

where

$\Gamma^1 =$  $, \Gamma^2 =$ 

where $\overline{\alpha}$ is defined in Lemma 2.33.

On morphisms, $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$,

$$\mathcal{I}^{el}(f) = \bigoplus_{i,j} \mathrm{id}_{X_i X_j} \otimes f \otimes \mathrm{id}_{X_j^* X_i^*}$$

We refer to [Tha2019] *for the functorial isomorphisms giving the adjunction.*

Furthermore, $\mathcal{I}^{el}$ is dominant.

Theorem 8.7 ([Tha2019], Theorem 4.3)**.** *When $\mathcal{A}$ is modular, there is an equivalence*

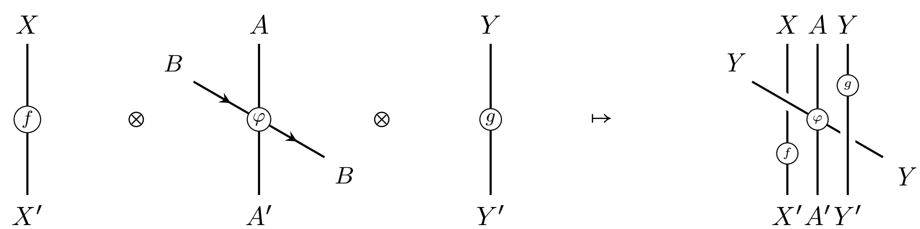
$$\mathcal{A} \simeq \mathcal{Z}^{el}(\mathcal{A})$$
$$A \mapsto (\bigoplus_i X_i A X_i^*, \Gamma, \Omega)$$

where Γ is the half-braiding on $I(X)$ in Theorem 2.37, and $\Omega = c^{-1} \otimes c^{-1} \otimes c$, where $c_{-,-}$ is the braiding on $\mathcal{A}$.

These results were proved purely algebraically in [Tha2019]. We will reprove them here, making use of the dictionary between algebra and topology developed in previous sections.

Proposition 8.8. *Let $\hat{\mathcal{A}} = \mathrm{hTr}(\mathcal{A})$ as in Section 7.3. Consider an $\mathcal{A}$-bimodule structure on $\hat{\mathcal{A}}$, where on objects it is the usual $X \triangleright A \triangleleft Y = X \otimes A \otimes Y$, and on morphisms, for $f \in \mathrm{Hom}_{\mathcal{A}}(X, X'), g \in \mathrm{Hom}_{\mathcal{A}}(Y, Y')$ and $\varphi \in \mathrm{Hom}^B_{\hat{\mathcal{A}}}(A, A')$, we have*

$$\mathrm{Hom}_{\mathcal{A}}(X, X') \otimes \mathrm{Hom}_{\hat{\mathcal{A}}}(A, A') \otimes \mathrm{Hom}_{\mathcal{A}}(X, X') \to \mathrm{Hom}_{\hat{\mathcal{A}}}(X \otimes A \otimes Y, X' \otimes A' \otimes Y')$$



$=: f \triangleright \varphi \triangleleft g$

The $\mathcal{A}$-bimodule structure extends to $\mathcal{Z}(\mathcal{A}) = \mathrm{Kar}(\hat{\mathcal{A}})$,

$$X \triangleright (A, \gamma) \triangleleft Y = (X \otimes A \otimes Y, c \otimes \gamma \otimes c^{-1})$$

(see (7.2) for $\otimes$ of half-braidings), and the action of morphisms is simply by tensor product.

Then we have

$$\mathrm{Kar}(\mathrm{hTr}(\hat{\mathcal{A}})) = \mathcal{Z}_{\mathcal{A}}(\hat{\mathcal{A}}) \simeq \mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A})) \simeq \mathcal{Z}^{el}(\mathcal{A})$$

In particular, by Proposition 2.40, $\mathcal{Z}^{el}(\mathcal{A})$ is finite semisimple.

Proof. Let us first establish the last equivalence, $\mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A})) \simeq \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$. An object in $\mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))$ is of the form $((A, \gamma), \gamma')$, where γ' is a half-braiding on (A, γ) as an $\mathcal{A}$-bimodule category: for $X \in \mathcal{A}$, γ' gives an isomorphism

$$\gamma'_X : X \triangleright (A, \gamma) = (X \otimes A, c \otimes \gamma) \simeq (A \otimes X, \gamma \otimes c^{-1}) = (A, \gamma) \triangleleft X$$

Moreover, γ'_X must be a $\mathcal{Z}(\mathcal{A})$-morphism, and it is easy to see that that is equivalent to γ, γ' satisfying the COMM relation (8.2). Thus, $((A, \gamma), \gamma')$ defines an object $(A, \gamma, \gamma') \in \mathcal{Z}^{\mathrm{el}}(\mathcal{A})$, and vice versa. Hence, there is a bijection between the objects of $\mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))$ and $\mathcal{Z}^{\mathrm{el}}(\mathcal{A})$. It is also clear that $\mathrm{Hom}_{\mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))}(((A,\gamma),\gamma'),((B,\mu),\mu')) = \mathrm{Hom}_{\mathcal{Z}^{\mathrm{el}}(\mathcal{A})}((A,\gamma,\gamma'),(B,\mu,\mu'))$ as subspaces of $\mathrm{Hom}_{\mathcal{A}}(A,B)$.

The functor $G : \hat{\mathcal{A}} \to \mathcal{Z}(\mathcal{A})$ from Proposition 7.15 admits a bimodule structure J', The structure J' should provide a natural isomorphism $J' : G(B \triangleright A \triangleleft C) \simeq B \triangleright G(A) \triangleleft C$. It is easy to check that $J' = c \otimes \mathrm{id} \otimes c$ works. Then we have a functor $\mathrm{hTr}(G,J') : \mathrm{hTr}(\hat{\mathcal{A}}) \to \mathrm{hTr}(\mathcal{Z}(\mathcal{A}))$ which is dominant (because G is dominant); thus, by Lemma 2.49, the Karoubi envelopes are equivalent: $\mathrm{Kar}(\mathrm{hTr}(G,J')) : \mathcal{Z}_{\mathcal{A}}(\hat{\mathcal{A}}) \simeq \mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))$.

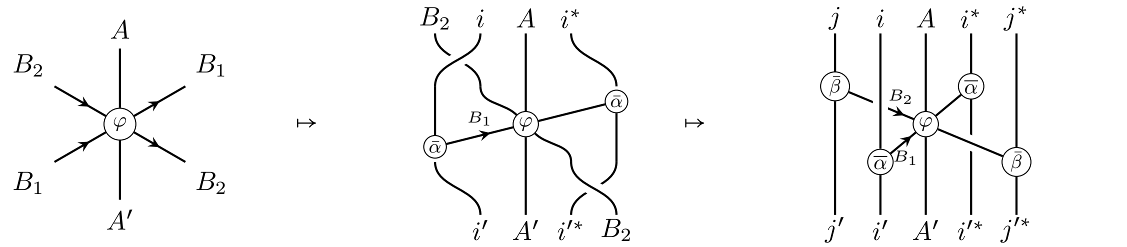
It is useful to work out the functors $\mathrm{hTr}(\hat{\mathcal{A}}) \to \mathrm{hTr}(\mathcal{Z}(\mathcal{A})) \to \mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))$ explicitly. For a morphism $\varphi \in \mathrm{Hom}^X_{\hat{\mathcal{A}}}(A,A')$, Recall that, for $A, A' \in \mathrm{Obj}\,\mathcal{A}$,

$$\mathrm{Hom}_{\mathrm{hTr}(\hat{\mathcal{A}})}(A,A') \cong \int^{B_2} \mathrm{Hom}_{\hat{\mathcal{A}}}(B_2 \triangleright A, A' \triangleleft B_2) \cong \int^{B_2}\int^{B_1} \mathrm{Hom}_{\mathcal{A}}(B_1 \triangleright B_2 \triangleright A, A' \triangleleft B_2 \triangleleft B_1)$$

For a morphism $\varphi \in \mathrm{Hom}^{B_2}_{\hat{\mathcal{A}}}(A,A') \to \mathrm{Hom}_{\mathrm{hTr}(\hat{\mathcal{A}})}(A,A')$,

$$\mathrm{Hom}^{B_2}_{\hat{\mathcal{A}}}(A,A') \xrightarrow{\mathrm{hTr}(G,J')} \mathrm{Hom}^{B_2}_{\mathcal{Z}(\mathcal{A})}(G(A),G(A')) \xrightarrow{I} \mathrm{Hom}_{\mathcal{Z}_{\mathcal{A}}(\mathcal{Z}(\mathcal{A}))}(I(G(A)),I(G(A')))$$

(8.3)



where I is the two-sided adjoint to the forgetful functor (see Theorem 2.37, Theorem 2.46). □

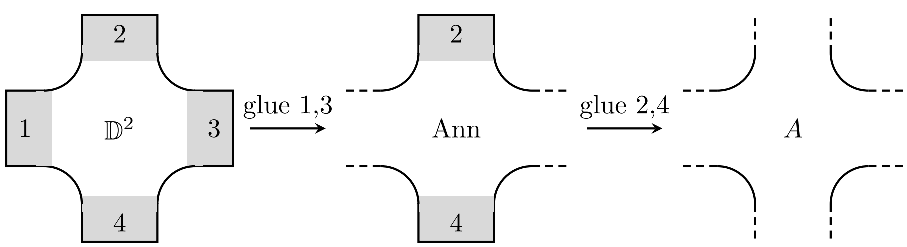
It is easy to see that $I(G(A))$ is naturally isomorphic to $\mathcal{I}^{\mathrm{el}}(A)$.

Proposition 8.9. *Let* $\mathbf{T}^2_0$ *be the once-punctured torus. There is an equivalence*

$$Z_{CY}(\mathbf{T}^2_0) \simeq \mathcal{Z}^{el}(\mathcal{A})$$

Under this equivalence, the inclusion functor $\mathcal{A} \simeq Z_{CY}(\mathbb{D}^2) \to Z_{CY}(\mathbf{T}^2_0)$ *is identified with* $\mathcal{I}^{el} : \mathcal{A} \to \mathcal{Z}^{el}(\mathcal{A})$.

Proof. Think of the once-punctured torus as an open disk, drawn like a '+' sign, with opposite sides identified ($\mathrm{Ann} = S^1 \times I$):



The left most figure shows how $Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A}$ is a module category over $Z_{\mathrm{CY}}(I \times I) \simeq \mathcal{A}$ in four ways; we think of the 1,2 edges as acting on the left, 3,4 edges as acting on the right. The actions are just usual left and right multiplication.

By Theorem 6.14, the first "glue 1,3" arrow induces an equivalence

$$i^{1,3}_* : \mathrm{hTr}_{\hat{Z}_{\mathrm{CY}}(I\times I)}(\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)) \simeq \hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$$

Again by Theorem 6.14, the second "glue 2,4" arrow induces an equivalence

$$i^{2,4}_* : \mathrm{hTr}_{\hat{Z}_{\mathrm{CY}}(I\times I)}(\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})) \simeq \hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)$$

It is easy to see that, under the equivalences $\mathcal{A} \simeq \hat{Z}_{\mathrm{CY}}(I \times I) \simeq \hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)$, $\hat{\mathcal{A}} \simeq \hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$, the $\mathcal{A}$-bimodule structure on $\hat{\mathcal{A}}$ and the $\hat{Z}_{\mathrm{CY}}(I \times I)$-bimodule structure on $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$ are equivalent. Thus we have, by

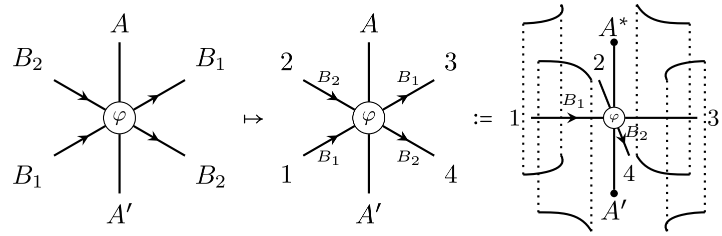
Proposition 8.8,

$$\begin{array}{ccccccccc}
\mathcal{A} & \xrightarrow{\text{hTr}} & \hat{\mathcal{A}} & \xrightarrow{\text{hTr}(G,J')} & \text{hTr}(\hat{\mathcal{A}}) & & \xrightarrow{\text{Kar}} & & \mathcal{Z}^{\text{el}}(\mathcal{A}) \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} & & & & \downarrow{\scriptstyle\simeq} \\
\hat{Z}_{\text{CY}}(\mathbb{D}^2) & \xrightarrow{i_*^{1,3}} & \hat{Z}_{\text{CY}}(\text{Ann}) & \xrightarrow{\text{hTr}} & \text{hTr}_{\hat{Z}_{\text{CY}}(I\times I)}(\hat{Z}_{\text{CY}}(\text{Ann})) & \xrightarrow[\simeq]{i_*^{2,4}} & \hat{Z}_{\text{CY}}(\mathbf{T}_0^2) & \xrightarrow{\text{Kar}} & Z_{\text{CY}}(\mathbf{T}_0^2)
\end{array}$$

As noted before, the top arrows compose to a functor that is naturally isomorphic to $\mathcal{I}^{\text{el}}$.

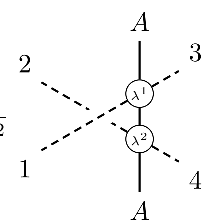
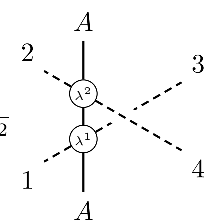
Let us give a more explicit description of the equivalence $Z_{\text{CY}}(\mathbf{T}_0^2) \simeq \mathcal{Z}^{\text{el}}(\mathcal{A})$. The equivalence $\text{hTr}(\hat{\mathcal{A}}) \simeq \hat{Z}_{\text{CY}}(\mathbf{T}_0^2)$ sends a morphism $\varphi \in \text{Hom}_{\mathcal{A}}(B_1 \triangleright B_2 \triangleright A, A' \triangleleft B_2 \triangleleft B_1) \to \text{Hom}_{\text{hTr}(\hat{\mathcal{A}})}(A, A')$ to the graph in $\mathbf{T}_0^2 \times [0,1]$ shown on the right (which we represent more conveniently by the diagram in the middle):

(8.4)



As we can see, the two diagrams on the left are essentially the same, except for the extra labels 1,2,3,4 in the middle diagram. We may sometimes abuse notation and conflate them.

Then $\mathcal{Z}^{\text{el}}(\mathcal{A}) \simeq Z_{\text{CY}}(\mathbf{T}_0^2)$ is given as follows: on objects,

(8.5) $\quad (A, \lambda^1, \lambda^2) \mapsto \text{im}(P_{(A,\lambda^1,\lambda^2)})$, where $P_{(A,\lambda^1,\lambda^2)} := \dfrac{1}{\mathcal{D}^2}$  $= \dfrac{1}{\mathcal{D}^2}$ 

where the equality of diagrams follows from the COMM requirement (8.2), and the dashed line represents a weighted sum over simples (see Notation 2.25). On morphisms,

$$\text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{A})}((A,\lambda^1,\lambda^2),(A',\mu^1,\mu^2)) \ni f \mapsto P_{(A',\mu^1,\mu^2)} \circ f \circ P_{(A,\lambda^1,\lambda^2)}$$

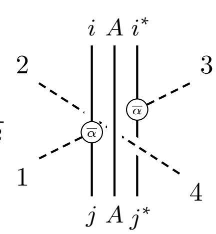
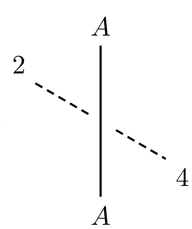
Thus we have a 2-commutative diagram

(8.6)
$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\mathcal{I}^{\text{el}}} & \mathcal{Z}^{\text{el}}(\mathcal{A}) \\
\downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\
Z_{\text{CY}}(\mathbb{D}^2) & \xrightarrow{\text{incl.}_*} & Z_{\text{CY}}(\mathbf{T}_0^2)
\end{array}$$

□

Next we will prove Theorem 8.7, that, when $\mathcal{A}$ is modular, $\mathcal{Z}^{\text{el}}(\mathcal{A}) \simeq \mathcal{A}$. We present a different proof, one that uses the equivalence $Z_{\text{CY}}(\mathbf{T}_0^2) \simeq \mathcal{Z}^{\text{el}}(\mathcal{A})$ from Proposition 8.9.

Proof of Theorem 8.7. Under the equivalence (8.5), the object $(\bigoplus X_i A X_i^*, \Gamma, \Omega)$ is sent to

(8.7) $\quad (\bigoplus X_i A X_i^*, \Gamma, \Omega) \mapsto \text{im}\left(\dfrac{1}{\mathcal{D}^2}\right.$  $\left.\right) \cong \text{im}\left(\dfrac{1}{\mathcal{D}}\right.$  $\left. =: P_\Omega\right)$

where the isomorphism essentially follows from Lemma 2.45.

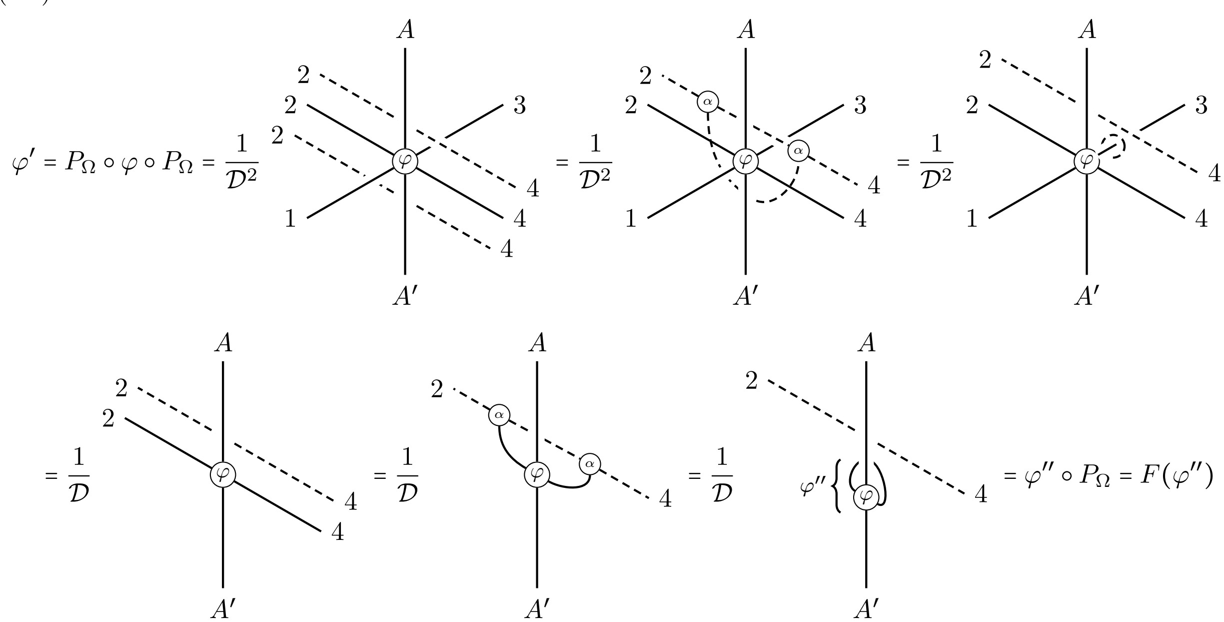
Let $F : Z_{\mathrm{CY}}(\mathbb{D}^2) \to Z_{\mathrm{CY}}(\mathbf{T}^2_0)$ be the functor that sends objects $A \mapsto \mathrm{im}(P_\Omega)$, and morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ to $P_\Omega \circ f \circ P_\Omega$ (here we've identified $\mathcal{A} \simeq Z_{\mathrm{CY}}(\mathbb{D}^2)$ by choosing a marked point, so $A \in Z_{\mathrm{CY}}(\mathbb{D}^2)$ is the object with one marked point labeled by A). Note that in fact $P_\Omega \circ f \circ P_\Omega = P_\Omega \circ f = f \circ P_\Omega$, but this is not true in general if f is replaced by some arbitrary morphism of $\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbf{T}^2_0)}(F(A), F(A'))$.

We first show that F is full. Denote by $i_*(A) \in Z_{\mathrm{CY}}(\mathbf{T}^2_0)$ the object with one marked point labeled by A, that is, the object pushed forward under an inclusion $i : \mathbb{D}^2 \hookrightarrow \mathbf{T}^2_0$. By definition, $F(A)$ is a summand of $i_*(A)$, as P_Ω is a projection on $i_*(A)$, $P_\Omega : i_*(A) \to i_*(A)$. Now

$$\mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbf{T}^2_0)}(F(A), F(A')) = P_\Omega \circ \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)}(i_*(A), i_*(A')) \circ P_\Omega$$

as subspaces of $\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)}(i_*(A), i_*(A'))$. In other words, any morphism $\varphi' \in \mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbf{T}^2_0)}(F(A), F(A'))$ is of the form $P_\Omega \circ \varphi \circ P_\Omega$. Then

(8.8)



so any $\varphi' \in \mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbf{T}^2_0)}(F(A), F(A'))$ is equal to $\varphi'' \circ P_\Omega = F(\varphi'')$ for some $\varphi'' \in \mathrm{Hom}_{\mathcal{A}}(A, A')$.

Next we show that F is faithful. Suppose some $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ is sent to $F(f) = f \circ P_\Omega = 0 \in \mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbf{T}^2_0)}(F(A), F(A'))$, viewed as a subspace of $\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)}(i_*(A), i_*(A'))$ as above. By Proposition 8.9 and Lemma 2.47,

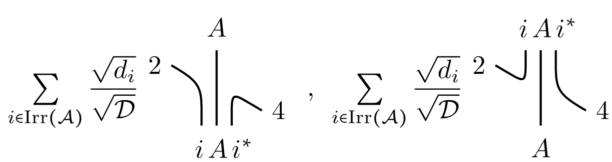
$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)}(i_*(A), i_*(A')) \cong \int^{B_2}\int^{B_1} \mathrm{Hom}_{Z_{\mathrm{CY}}(\mathbb{D}^2)}(B_1 \triangleright B_2 \triangleright A, A' \triangleleft B_2 \triangleleft B_1) \cong \bigoplus_{i,j \in \mathrm{Irr}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(X_i \triangleright X_j \triangleright A, A' \triangleleft X_j \triangleleft X_i$$

It is clear that $f \circ P_\Omega \in \bigoplus_{j \in \mathrm{Irr}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(\mathbf{1} \triangleright X_j \triangleright A, A' \triangleleft X_j \triangleleft \mathbf{1})$, and in fact the $j = 0$ (i.e. $X_j = \mathbf{1}$) component of $f \circ P_\Omega$ is simply $1/\mathcal{D} \cdot f$. Thus, since $f \circ P_\Omega = 0$, $f = 0$ as well.

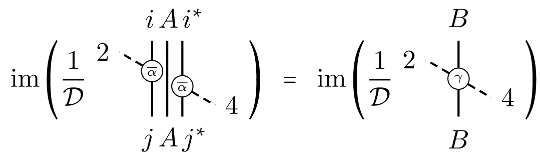
Finally, we show that F is essentially surjective. Since $i_* : Z_{\mathrm{CY}}(\mathbb{D}^2) \to Z_{\mathrm{CY}}(\mathbf{T}^2_0)$ is dominant by Corollary 6.16, and $Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq \mathcal{A}$ is semisimple, it suffices to show that any $i_*(A)$ is isomorphic to some $F(A')$.

Consider

(8.9)



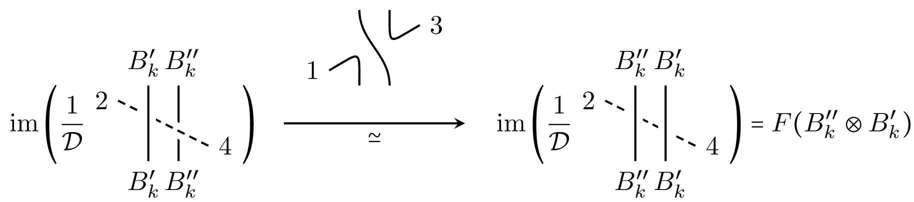
which are analogs of the $\hat{P}'_{(M,\gamma)}, \check{P}'_{(M,\gamma)}$, in Lemma 2.45. By similar reasoning, these morphisms are isomorphisms to and from the object



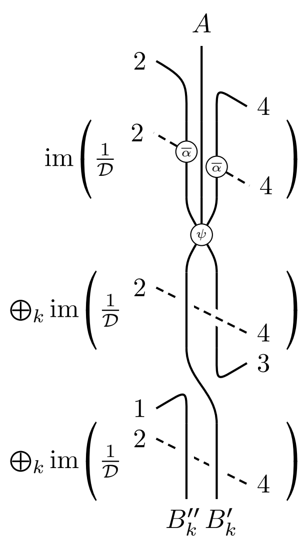
where of course $(B,\gamma) = (\bigoplus X_i A X_i^*, \Gamma)$. Now, from (7.20), there is an isomorphism

$$\psi : (B,\gamma) \simeq \bigoplus_k (B'_k \otimes B''_k, c^{-1} \otimes c)$$

for some finite collection of objects B'_k, B''_k. Thus, the object above is a direct sum of objects



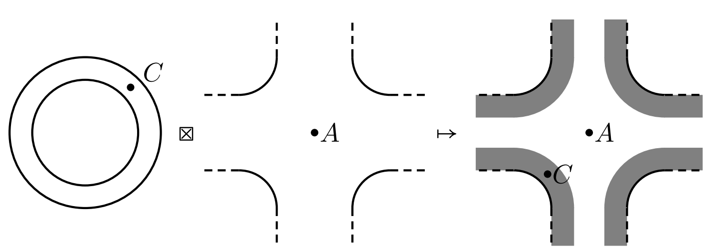
In summary, we have thing following chain of isomorphisms:



□

Our next task is to upgrade this equivalence to an equivalence of left $Z_{\mathrm{CY}}(\mathrm{Ann})$-modules. In the rightmost figure in (8.10), the gray area is a collar neighborhood of the puncture of $\mathbf{T}^2_0$. By Proposition 6.10, there is a (left) $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $\hat{Z}_{\mathrm{CY}}(\mathbf{T}^2_0)$: on objects,

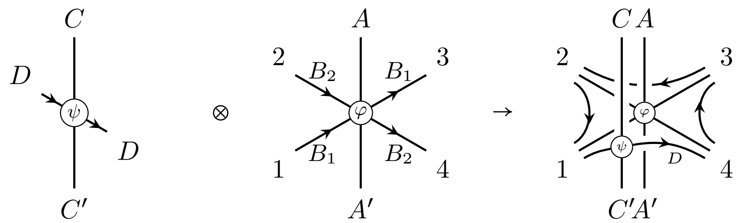
(8.10)



while on morphisms, the module structure is given as follows:

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}(C, C') \;\otimes\; \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}_0^2)}(A, A') \;\to\; \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbf{T}_0^2)}(A, A')$$

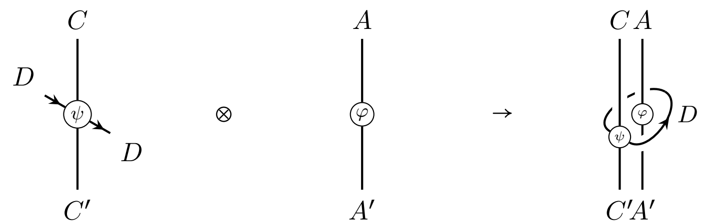
(8.11)



(The D-labeled strand originally goes around the annulus in $\mathrm{Ann} \times [0,1]$; after inserting into $\mathbf{T}_0^2 \times [0,1]$, it wraps around like the gray area in (8.10)). This extends to a left $Z_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $Z_{\mathrm{CY}}(\mathbf{T}_0^2)$.

Similarly, there is a left $\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})$-module structure on $\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)$ (which extends to Z_{CY}):

$$\mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}(C, C') \;\otimes\; \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)}(A, A') \;\to\; \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)}(A, A')$$



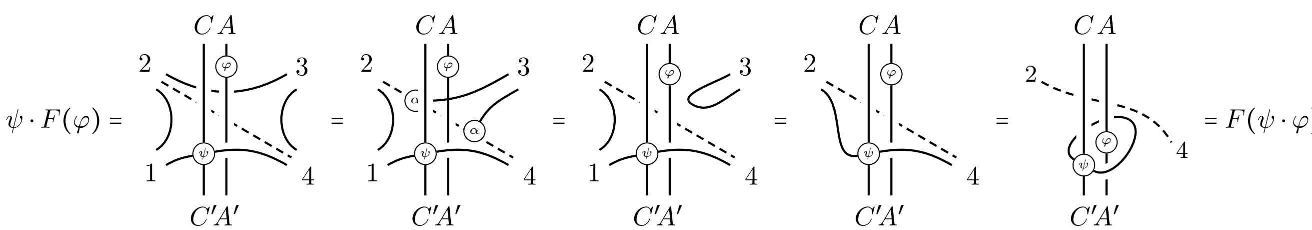
(8.12)

In light of (8.6), the following theorem is an upgrade of Theorem 8.7:

Theorem 8.10. *Let $\mathcal{A}$ be modular. There is an equivalence of left $Z_{CY}(Ann)$-modules*

$$Z_{CY}(\mathbb{D}^2) \simeq Z_{CY}(\mathbf{T}_0^2)$$

Proof. For $\psi \in \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathrm{Ann})}(C, C')$, $\varphi' = \mathcal{D} \cdot \varphi \in \mathrm{Hom}_{\hat{Z}_{\mathrm{CY}}(\mathbb{D}^2)}(A, A')$, the action of ψ on $F(\varphi')$ (where F is the functor from Theorem 8.7; the factor of $\mathcal{D}$ is to simplify the diagrams below) is given by:



The first and last equalities are by definition. The second and third equalities follow from Lemma 2.27 in order to perform the "sliding lemma" as in proof of Lemma 2.30. The fourth equality is by isotopy. The fifth equality applies the similar method of the second and third equalities. Hence, the equivalence does respect the module structure, and we are done. □

Finally, we state the main result of this section:

Theorem 8.11. *Let $\mathcal{A}$ be modular. Let N be a connected compact oriented surface with b boundary components and genus g, and let $S_{0,b} = S^2 \backslash (\mathbb{D}^2)^{\sqcup b}$ be a genus 0 surface with b boundary components. Then*

$$Z_{CY}(N) \simeq Z_{CY}(S_{0,b})$$

In particular, $Z_{CY}(closed\ surface) \simeq Z_{CY}(S^2) \simeq \mathcal{V}ec$ and $Z_{CY}(once\text{-}punctured\ surface) \simeq Z_{CY}(\mathbb{D}^2) \simeq \mathcal{A}$.

Proof. Suppose $g > 0$, so that we can present N as a connect sum $N' \# \mathbf{T}^2$, where N' is a connected compact oriented surface with b boundary components and genus $g - 1$. We think of the connect sum as $N = N'_0 \cup_{\mathrm{Ann}} (\mathbf{T}_0^2)$, where $N'_0 = N' \backslash \{pt\}$ is a punctured surface. Then by Theorem 6.15 and Theorem 8.10, $Z_{\mathrm{CY}}(N) \simeq Z_{\mathrm{CY}}(N'_0) \boxtimes_{Z_{\mathrm{CY}}(\mathrm{Ann})} Z_{\mathrm{CY}}(\mathbf{T}_0^2) \simeq Z_{\mathrm{CY}}(N'_0) \boxtimes_{Z_{\mathrm{CY}}(\mathrm{Ann})} Z_{\mathrm{CY}}(\mathbb{D}^2) \simeq Z_{\mathrm{CY}}(N'_0 \cup_{\mathrm{Ann}} \mathbb{D}^2) = Z_{\mathrm{CY}}(N')$. Thus, by induction on the genus, we have $Z_{\mathrm{CY}}(N) \simeq Z_{\mathrm{CY}}(S_{0,b})$.

The final statements follow from the $b = 0, 1$ cases. □

8.3. **Closed Surfaces.**

Let $\mathcal{A}$ be modular.

Let us consider $Z_{\mathrm{CY}}(N)$ for a closed surface in more detail. By Theorem 8.11, $Z_{\mathrm{CY}}(N) \simeq \mathcal{V}ec$, so a natural question to ask is: what do the simple objects, i.e. the objects corresponding to $\mathbf{k} \in \mathcal{V}ec$, look like in $Z_{\mathrm{CY}}(N)$?

Definition 8.12. Let N be a surface, and let ξ be an embedded circle in N. Let $E = i_*(\mathbf{1}) \in \hat{Z}_{\mathrm{CY}}(N)$ be the empty configuration. Define $P_\xi \in \mathrm{End}_{\hat{Z}_{\mathrm{CY}}(N)}(E)$ to be the morphism defined by the graph $\xi \times 1/2 \subset N \times [0,1]$, colored with $1/\mathcal{D}\cdot$ regular coloring.

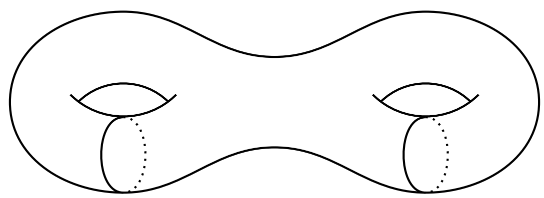
Furthermore, we may extend this definition to multicurves, i.e. a finite collection of pairwise disjoint embedded circles (see [FM2012, Section 1.2.4]). Let $\Xi = \{\xi_i\}$ be a multicurve. Then we define $P_\Xi = \prod P_{\xi_i}$.

Note that P_Ω (with $A = \mathbf{1}$) from (8.7) in the proof of Theorem 8.7 is P_ξ for some loop.

Clearly, if two multicurves Ξ, Ξ' are isotopic, then $P_\Xi = P_{\Xi'}$. In particular, if a multicurve Ξ has two components $\xi_1, \xi_2 \in \Xi$ that are isotopic, then $P_\Xi = P_{\Xi'}$, where $\Xi' = \Xi \backslash \{\xi_1\}$.

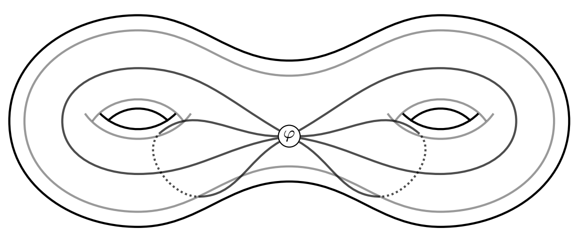
Definition 8.13. We say a multicurve in a closed surface N is *full* if performing surgery along it (removing a neighborhood and gluing in disks) results in a sphere.

For example, each set of attaching curves in a Heegaard diagram is full. It is clear by definition that a full multicurve always looks, under suitable diffeomorphism of N, as follows:



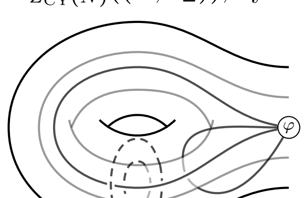
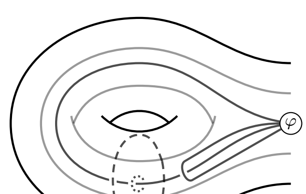
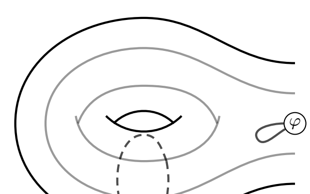
Proposition 8.14. *Let N be a closed surface, and let E be the empty configuration (as in Definition 8.12). For any full multicurve Ξ, the object $(E, P_\Xi) \in Z_{CY}(N)$ is simple. In particular, all such objects are isomorphic.*

Proof. It is easy to see that an element of $\mathrm{End}_{Z_{\mathrm{CY}}(N)}(E)$ is always equivalent to a sum of graphs as follows:

(8.13) 

(each loop starting and ending at the node represents a basis element of H_1.)

Then in $\mathrm{End}_{Z_{\mathrm{CY}}(N)}((E, P_\Xi))$, by killing and sliding lemma, the graph can be "shrunk" into a ball:

(8.14) $\dfrac{1}{\mathcal{D}^2}$  $= \dfrac{1}{\mathcal{D}^2}$  $= \dfrac{1}{\mathcal{D}}$ 

Thus, any skein in $\mathrm{End}_{Z_{\mathrm{CY}}(N)}((E, P_\Xi))$ is represented by a multiple of the empty graph in $N \times [0,1]$, hence $\mathrm{End}_{Z_{\mathrm{CY}}(N)}((E, P_\Xi)) \cong \mathbf{k}$. ☐

Lemma 8.15. *Let M be a 3-manifold with boundary component $N \subseteq \partial M$. Let $\Xi \subset N$ be a multicurve, and let M_ξ be the 3-manifold obtained from M by adding 2-handles along Ξ. Then the map $Z^{sk}_{CY}(M; E_M) \to Z^{sk}_{CY}(M_\xi; E_{M_\xi})$ induce by inclusion restricts to an isomorphism*

$$Z^{sk}_{CY}(M; (E_M, P_\xi)) = P_\xi \cdot Z^{sk}_{CY}(M; E_M) \cong Z^{sk}_{CY}(M_\xi; E_{M_\xi}) \tag{8.15}$$

where E_M, E_{M_ξ} are the empty configurations in $\partial M, \partial M_\xi$.

Proof. The inverse is given by the operation that takes a graph Γ in M_ξ and gives a graph in M by pushing (i.e. isotoping) the graph out of the 2-handle. In general, the resulting graph may be dependent on extra choices. We claim that this operation gives a well-defined isomorphism (8.15).

This claim is similar to Proposition 8.1. Indeed, we may consider the "pole" here to be a cocore of the 2-handle in M_ξ. Then P_ξ corresponds to the action π in Proposition 8.1. We leave the details to the reader. □

Lemma 8.16. *Let M, N, Ξ be as in Lemma 8.15. Let M' be a handlebody (i.e. a 3-ball with 1-handles) such that $\partial M' \simeq \overline{N}$. Furthermore, let the diffeomorphism $\partial M' \simeq \overline{N}$ send Ξ to null-homotopic curves in M'. Thus, $M_\Xi^* := M \cup_N M' = M_\Xi \cup 3-balls$. Then*

$$Z_{CY}^{sk}(M, (E_M, P_\Xi)) \cong Z_{CY}^{sk}(M_\Xi^*, E_{M_\Xi^*})$$

the map induced by inclusion $M \subset M_\Xi^*$.

Proof. Indeed, $M \cup M'$ is just M_Ξ with an extra 3-handle to "fill in the hole in M'", and the addition of a 3-handle does not affect the skein module of a 3-manifold, as graphs and isotopies of graphs can avoid the hole. (Here the bonudary values in question are the empty configurations, but this works in general, since $Z_{\mathrm{CY}}(S^2) \cong \mathcal{V}ec$.) □

Corollary 8.17. *Let M be a 3-manifold, with boundary components $\partial M = \cup N_k$. Let Ξ_k be a full multicurve for N_k, and let $P_\Xi = \prod \Xi_k$. Then*

$$Z_{CY}^{sk}(M; (E, P_\Xi)) \cong \mathbf{k}$$

In particular, $Z_{CY}^{sk}(M) \cong \mathbf{k}$ for a closed 3-manifold.

Proof. By Lemma 8.16, this reduces to the case when M is closed.

Present M by Heegaard splitting $M = M' \cup_N M''$ (see [Sav2012]) and let Ξ', Ξ'' be the multicurves that define the attaching of M', M'' respectively. Let Ξ be some multicurve such that the 3-manifold defined by Ξ and Ξ'' is the 3-sphere. Then again by Lemma 8.16,

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(M) \cong Z_{\mathrm{CY}}^{\mathrm{sk}}(M''; (E, P_{\Xi'})) \cong Z_{\mathrm{CY}}^{\mathrm{sk}}(M''; (E, P_\Xi)) \cong Z_{\mathrm{CY}}^{\mathrm{sk}}(S^3) \cong \mathbf{k}$$

where the middle equality holds by Proposition 8.14. □

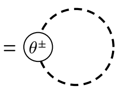
9. Relation of Z_{CY} to Other Invariants

9.1. **Relation of $Z_{\mathbf{CY}}$ to the Signature and Euler Characteristic of 4-Manifolds.** In [CKY1993], the authors prove that, for $\mathcal{A} = \operatorname{Rep} U_q sl_2$ (with q a root of unity),

$$Z_{\text{CY}}(W) = \mathcal{D}^{\chi(W)/2} \kappa^{\sigma(W)} \tag{9.1}$$

where χ, σ are the Euler characteristic and signature of W, respectively, and

$$\kappa = \sqrt{p_+/p_-} \ , \ \ p_\pm = \sum \theta_i^\pm d_i^2 =$$



Note $\mathcal{D} = p_+ p_-$ ($= \eta^{-2}$ from [CKY1993],[Rob1995]). In this section, we extended this result to 4-manifolds with boundary and arbitrary modular categories $\mathcal{A}$, and then to 4-manifolds with corners.

The Crane-Yetter invariant associated to a 4-manifold with boundary is not a scalar, but a vector $Z_{\text{CY}}(W) \in Z_{\text{CY}}(\partial W)$. By Corollary 8.17, $Z_{\text{CY}}(\partial W)$ is 1-dimensional, so we need to choose the correct identification $Z_{\text{CY}}(\partial W) \cong \mathbf{k}$ to reproduce (9.1). It is natural to guess that the identification $Z_{\text{CY}}(\partial W) \cong \mathbf{k}$ that we are after should send this element $\varnothing_M^{\text{sk}} \mapsto 1$, but this is not correct; the correct normalization is given in Definition 9.3 below. In order to motivate that normalization, we first consider the following example:

Example 9.1. Let $\mathcal{A}$ be modular. Let $W_L = \mathcal{H}_0 \cup (\mathcal{H}_2^{(1)} \sqcup \cdots \sqcup \mathcal{H}_2^{(n)})$, where the 2-handle $\mathcal{H}_2^{(m)}$ is attached along the m-th component L_m of a framed link $L = \bigsqcup L_m \subset \partial \mathcal{H}_0$; Then W_L has a natural handle decomposition as a cobordism $W_L : \varnothing \to M_L$. Consider its dual handle decomposition as a cobordism $W_L : \overline{M_L} \to \varnothing$ (Definition 2.69). Recall that the belt spheres of the dual handles are the attaching spheres of the original handles, so in particular the belt sphere of $\mathcal{H}_2^{*(m)}$, the dual 2-handle to $\mathcal{H}_2^{(m)}$, is L_m. Then, using Definition 5.11, these 2-handles define a cobordism $\bigsqcup \mathcal{H}_2^{*(m)} : \overline{M_L} \to \overline{\partial \mathcal{H}_0}$, such that

$$Z_{\text{CY}}^{\text{sk}}(\bigsqcup \mathcal{H}_2^{(m)})(\varnothing_{\overline{M_L}}^{\text{sk}}) = \Omega L \in Z_{\text{CY}}^{\text{sk}}(\overline{\partial \mathcal{H}_0}) \tag{9.2}$$

where ΩL means we color each component of L by the regular color $\sum_i d_i \operatorname{id}_i$. Thus, we have

$$Z_{\text{CY}}^{\text{sk}}(W_L)(\varnothing_{\overline{M_L}}^{\text{sk}}) = Z_{\text{CY}}^{\text{sk}}(\mathcal{H}_0^*)(\Omega L) = Z_{\text{CY}}^{\text{sk}}(\mathcal{H}_4)(\Omega L) = Z_{\text{RT}}(\Omega \overline{L}) \tag{9.3}$$

where $\overline{L}$ is the mirror image of L and $\mathcal{H}_0^*$ is the dual handle to $\mathcal{H}_0$. (The formula in Definition 5.11 for a 4-handle states that $Z_{\text{CY}}^{\text{sk}}(\mathcal{H}_4)(\Gamma) = Z_{\text{RT}}(\Gamma)$, but Γ should be a graph in $S^3 = \overline{\partial \mathcal{H}_4}$, hence since L was given as a link in $\partial \mathcal{H}_0$, we need to take the mirror image $\overline{L}$, which is isotopic to the image of L under an orientation-reversing homeomorphism of S^3.)

This is essentially the Reshetikhin-Turaev invariant of M_L [RT1991], up to a normalizing factor of $\mathcal{D}^{-1/2}$ (see [BakK2001]):

$$Z_{\text{RT}}(M_L) := \kappa^{-\sigma(L)} \cdot \mathcal{D}^{(-|L|-1)/2} \cdot Z_{\text{RT}}(\Omega \overline{L}) \tag{9.4}$$

where $\sigma(L)$ is the signature of the linking matrix of L; this invariant only depends on the 3-manifold M_L, and not the link L. (See Remark 9.2 for clarification.)

Thus, since $\sigma(L) = \sigma(W_L)$ and $|L| + 1 = \chi(W_L) = \chi(\overline{W_L})$, we see that

$$Z_{\text{CY}}^{\text{sk}}(W_L)\left(\frac{1}{Z_{\text{RT}}(M_L)} \cdot \varnothing_{\overline{M_L}}^{\text{sk}}\right) = \kappa^{\sigma(W)} \cdot \mathcal{D}^{\chi(W_L)/2}$$

Note that under this normalization of Z_{RT}, we have

$$Z_{\text{RT}}(S^3) = \mathcal{D}^{-1/2}$$

$$Z_{\text{RT}}(M_1 \# M_2) = \mathcal{D}^{1/2} Z_{\text{RT}}(M_1) Z_{\text{RT}}(M_2)$$

We will discuss Reshetikhin-Turaev invariants in greater detail in Section 9.3. (see also Remark 9.24).

Remark 9.2. In [BakK2001, Theorem 4.1.16], the formula for $Z_{\mathrm{RT}}(M_L)$ is the same as (9.4) except it is ΩL instead of $\Omega\overline{L}$. However, this apparent discrepancy is due to our conventions. More specifically, in our graphical calculus, we treat morphisms as going from top to bottom, and our braiding and twist is the left-hand twist, while [BakK2001] uses the opposite conventions. In particular, flipping a diagram through a horizontal plane switches between our and their conventions, hence evaluating $Z_{\mathrm{RT}}(\Omega\overline{L})$ with our conventions is the same as evaluating $Z_{\mathrm{RT}}(\Omega L)$ with the conventions of [BakK2001].

Motivated by the example above, we define the following:

Definition 9.3. Let $\mathcal{A}$ be modular. For a closed 3-manifold M, define the *normalized empty skein* ε_M by

$$\varepsilon_M := \frac{1}{Z_{\mathrm{RT}}(\overline{M})} \cdot \varnothing_M^{\mathrm{sk}} \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M)$$

Note the orientation reversal in $Z_{\mathrm{RT}}(\overline{M})$.

Note that ε_M is sensitive to the orientation of M, as $Z_{\mathrm{RT}}(M) \neq Z_{\mathrm{RT}}(\overline{M})$ in general.

Proposition 9.4. *Let $\mathcal{A}$ be modular. For a 4-manifold W,*

$$Z_{CY}^{sk}(W)(\varepsilon_{\overline{\partial W}}) = \kappa^{\sigma(W)}\mathcal{D}^{\chi(W)/2}$$

More generally, let W be a cobordism $W : M \to M'$ between closed 3-manifolds. Then

$$Z_{CY}^{sk}(W)(\varepsilon_M) = \kappa^{\sigma(W)}\mathcal{D}^{\chi(W)/2} \cdot \varepsilon_{M'}$$

Proof. The more general statement follows directly from the first statement. Indeed, let W' be a 4-manifold with boundary $\partial W' = \overline{M'}$. Then, assuming the first statement has been proven,

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(W \cup_{M'} W')(\varepsilon_{\overline{\partial W}}) = \kappa^{\sigma(W\cup_{M'}W')}\mathcal{D}^{\chi(W\cup_{M'}W')/2}$$

By Novikov additivity [AS1968, 7.1], $\sigma(W \cup_{M'} W') = \sigma(W) + \sigma(W')$, and by additivity of Euler characteristic (together with $\chi(M^3) = 0$ for any closed 3-manifold), $\chi(W \cup_{M'} W') = \chi(W) + \chi(W')$. Then, using the fact that $Z_{\mathrm{CY}}(M^3)$ is 1-dimensional for closed 3-manifolds (Corollary 8.17), $Z_{\mathrm{CY}}^{\mathrm{sk}}(W)(\varepsilon_M) = y \cdot \varepsilon_{M'}$ for some scalar $y \in \mathbf{k}$. The above observations imply that $y = \kappa^{\sigma(W)}\mathcal{D}^{\chi(W)/2}$. Thus, we will focus on proving the first statement.

Let W be presented by a handle decomposition, arranged in increasing index, and let W_k be the union of j-handles, where $j \leq k$ (so $W_4 = W, W_{-1} = \varnothing$). Denote $M_k = \overline{\partial W_k}$, and $W_k' = W_k \setminus \mathrm{Int}(W_{k-1})$, so W_k' is a cobordism

$$W_k' : M_k \to M_{k-1}$$

which is a composition of several elementary cornered cobordisms of index $4-k$ (dual to the original k-handles). Let n_k be the number of original k-handles (i.e. the number of $4-k$-handles that make up W_k').

Observe that $M_3 = M_4 \sqcup \coprod_{n_4} S^3$, so

$$Z_{\mathrm{RT}}(\overline{M_3}) = Z_{\mathrm{RT}}(\overline{M_4}) \cdot \mathcal{D}^{-n_4/2}$$

By Definition 5.11 (using $k = 0$), we have

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(W_4')(\varnothing_{M_4}^{\mathrm{sk}}) = \mathcal{D}^{n_4} \cdot \varnothing_{M_3}^{\mathrm{sk}}$$

So

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(W_4')(\varepsilon_{M_4}) = \mathcal{D}^{n_4/2}\varepsilon_{M_3}$$

Now consider $k = 3$. Again by Definition 5.11 (using $k = 1$),

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(W_3')(\varnothing_{M_3}^{\mathrm{sk}}) = \mathcal{D}^{-n_3} \cdot \varnothing_{M_2}^{\mathrm{sk}}$$

Suppose $n_3 = 1$, and the removal of that 3-handle (i.e. addition of dual 1-handle) connects two components of M_3, so that if $M_3 = M_3' \sqcup M_3''$, then $M_2 = M_3' \# M_3''$. Then

$$Z_{\mathrm{RT}}(\overline{M_2}) = \mathcal{D}^{1/2} \cdot Z_{\mathrm{RT}}(\overline{M_3})$$

Suppose the removal of the 3-handle does not connect two components, but connects one component $M_3' \subseteq M_3$ to itself, resulting in $M_2' \subseteq M_2$. If L is a framed link such that $M_3' = M_L$, then $M_2' = M_{L\cup\bigcirc}$, where $L \cup \bigcirc$ is

L with an extra unknotted 0-framed component that is unlinked with L. Then it is easy to see that we also have $Z_{\mathrm{RT}}(\overline{M_2}) = \mathcal{D}^{1/2} \cdot Z_{\mathrm{RT}}(\overline{M_3})$. Thus, for multiple 3-handles,

$$Z_{\mathrm{RT}}(\overline{M_2}) = \mathcal{D}^{n_3/2} \cdot Z_{\mathrm{RT}}(\overline{M_3})$$

and so

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W_3')(\varepsilon_{M_3}) = \mathcal{D}^{-n_3/2}\varepsilon_{M_2}$$

Finally we come to $k = 2$. We will consider W_2 all at once. By multiplicativity under disjoint unions, we may assume that W_2 is connected. Furthermore, we cancel 0- and 1-handles so that $n_0 = 1$.

The attaching maps of 1-handles may be visualized as a pair of balls in S^3. The attaching maps of 2-handles form a framed link L, and links that go through the 1-handle go into one ball and appear at the other one.

We use the "dotted circle" notation for 1-handles, as developed in [Akb1977] (see also [GS1999, Chapter 5.4]). We may bring the pairs of balls together, joining the arcs from L appropriately, and put a new unknotted circle around them; we put a dot on these new circles to distinguish them from L, and let L' be the unlink consisting of these dotted circles.

Thus, ∂W_1 is depicted as $S^3 = \partial W_0$ with n_1 dotted circles forming an unlink L', and the attaching maps for the 2-handles is simply the framed link L in the complement of L'. By Definition 5.11,

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2')(\varnothing^{\mathrm{sk}}_{M_2}) = \Omega L \in Z^{\mathrm{sk}}_{\mathrm{CY}}(M_1)$$

Once again by Definition 5.11, removing 1-handles (i.e. attaching dual 3-handles to M_1) amounts to removing all strands (after "bunching" them up with Lemma 2.27) with $i \neq \mathbf{1}$ that goes through the 1-handle. Thus, by Lemma 2.31, this is equivalent to coloring the dotted circles by $1/\mathcal{D}\cdot$ regular color:

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2)(\varnothing^{\mathrm{sk}}_{M_2}) = Z^{\mathrm{sk}}_{\mathrm{CY}}(W_0)(\mathcal{D}^{-n_1} \cdot \Omega(L \cup L')) = \mathcal{D}^{-n_1} \cdot Z_{\mathrm{RT}}(\Omega(\overline{L \cup L'}))$$

Now $\overline{M_2} = \partial W_2 \simeq M_{L\cup L'}$, so

$$Z_{\mathrm{RT}}(\overline{M_2}) = Z_{\mathrm{RT}}(M_{L\cup L'}) = \kappa^{-\sigma(L\cup L')} \cdot \mathcal{D}^{(-|L'|-|L|-1)/2} \cdot Z_{\mathrm{RT}}(\Omega(\overline{L \cup L'}))$$

Since $|L| = n_2, |L'| = n_1$, we have

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W_2)(\varepsilon_{M_2}) = \kappa^{\sigma(L\cup L')} \cdot \mathcal{D}^{(n_2-n_1+1)/2}$$

Thus, we have

$$Z^{\mathrm{sk}}_{\mathrm{CY}}(W)(\varepsilon_{\overline{\partial W}}) = \kappa^{\sigma(L\cup L')} \cdot \mathcal{D}^{(n_4-n_3+n_2-n_1+1)/2} = \kappa^{\sigma(L\cup L')} \cdot \mathcal{D}^{\chi(W)/2}$$

It remains to see that $\sigma(L \cup L') = \sigma(W)$. Observe that $W_4' \cup W_3'$ is a disjoint union of 4-dimensional handlebodies ($\natural mB^3 \times S^1$), so $\sigma(W_4' \cup W_3') = 0$. and hence by Novikov additivity [AS1968], $\sigma(W) = \sigma(W_2)$. Finally, replacing a dotted circle by a 0-framed unknot can be achieved by surgery on the 4-manifold (see [GS1999, Chapter 5.4]), and thus W_2 and $W_{L\cup L'}$ are cobordant, and hence, by [Tho1954], have the same signature $\sigma(W_2) = \sigma(W_{L\cup L'}) = \sigma(L \cup L')$. (It is also not hard to show directly that $\sigma(L \cup L') = \sigma(W_2)$, e.g. using handle cancellations on the homology level.) □

Remark 9.5. It seems that the key point where the proof requires $\mathcal{A}$ to be modular is when we traded 1-handles for 2-handles. It will be interesting to see what one might obtain for a non-modular $\mathcal{A}$.

Corollary 9.6. *Let $\mathcal{A}$ be modular, and let M be a closed 3-manifold, Then*

$$\mathrm{ev}^{sk}(\varnothing^{sk}_M, \varnothing^{sk}_{\overline{M}}) = Z_{TV}(M)$$

Proof. Since $\sigma(M \times I) = \chi(M \times I) = 0$, we have $Z^{\mathrm{sk}}_{\mathrm{CY}}(M \times I)(\varepsilon_M \otimes \varepsilon_{\overline{M}}) = 1$, so $\mathrm{ev}^{sk}(\varnothing^{\mathrm{sk}}_M, \varnothing^{\mathrm{sk}}_{\overline{M}}) = Z^{\mathrm{sk}}_{\mathrm{CY}}(M \times I)(\varnothing^{\mathrm{sk}}_M \otimes \varnothing^{\mathrm{sk}}_{\overline{M}}) = Z_{\mathrm{RT}}(M)Z_{\mathrm{RT}}(\overline{M}) = Z_{\mathrm{TV}}(M)$. (The last equality is proven in [Rob1995], where he works with $\mathcal{A} = \mathrm{Rep}\, U_q sl_2$ (with q a root of unity), but all the arguments work for general modular category). □

9.2. **4-Manifolds with Corners, Wall Non-Additivity.**

Observe that, for 4-manifolds W, W', thought of as cobordisms $W : M \to M', W' : M' \to M''$, composition aligns with additivity of signature (Novikov additivity [AS1968]) and Euler characteristic (since $\chi(M^3) = 0$ for any closed 3-manifold).

As discussed in Section 5.2, 4-manifolds with corners may be viewed as cornered cobordism, and we might hope that a similar additivity result would follow. However, Wall [Wal1969] proved that when two 4-manifolds are glued along a 3-manifold with boundary, the signatures do not add on the nose, but an error term is introduced.

Another issue that arises in this situation is that the skein module of a 3-manifold with boundary is not necessarily 1-dimensional. This will be remedied by considering an appropriate boundary value, in particular, as one may naturally guess, the simple objects from Proposition 8.14.

Let us recall the main definitions and results of [Wal1969], and refer the reader to the paper for details.

Definition 9.7. Let L_1, L_2, L_3 be Lagrangian subspaces of a symplectic vector space (V, ω) over $\mathbb{R}$. For $x_1, x_1' \in L_1 \cap (L_2 + L_3)$, define the form

$$\Psi(x_1, x_1') = \omega(x_1, x_2') = \omega(x_1, -x_3')$$

where $x_2' \in L_2, x_3' \in L_3$ such that $x_1' + x_2' + x_3' = 0$. It is easy to check

- Ψ is independent of the choice of x_2', x_3';
- Ψ is symmetric;
- Ψ vanishes on $L_1 \cap L_2 + L_1 \cap L_3$.

Thus, Ψ descends to a symmetric bilinear form on $\overline{V} = (L_1 \cap (L_2 + L_3))/(L_1 \cap L_2 + L_1 \cap L_3)$; We denote the signature of Ψ by

$$\sigma(V; L_1, L_2, L_3)$$

Note that the signature of Ψ as a symmetric bilinear form on $L_1 \cap (L_2 + L_3)$ and $\overline{V}$ are the same, so there is no ambiguity. It is clear that

$$\sigma(V; L_{\tau(1)}, L_{\tau(2)}, L_3\tau(3)) = \mathrm{sgn}(\tau) \cdot \sigma(V; L_1, L_2, L_3)$$

We also note that if any pair of L's are the same, say $L_1 = L_2$, then $\sigma(V; L_1, L_2, L_3) = 0$.

The main result of [Wal1969] is as follows, recast in our setting:

Theorem 9.8 ([Wal1969, Theorem.]). *Let N be a closed surface, and let $W : M_1 \to_N M_2$, $W' : M_2 \to_N M_3$ be cornered cobordism. Let N be given the outward orientation with respect to M_3 (equivalently M_1, M_2). Let $V = H_1(N; \mathbb{R})$, with the intersection pairing as the symplectic structure. Let $L_j = \ker(i_* : V \to H_1(M_j; \mathbb{R}))$ be the kernel of the map on homology induced by the inclusion $i : N \to M_j$; it is known that L_j is a Lagrangian subspace. Then*

$$\sigma(W \cup_{M_2} W') = \sigma(W) + \sigma(W') - \sigma(V; L_1, L_2, L_3)$$

Note that if we change the orientations of W and W', we need to either flip the orientations of M_j's, or reverse the direction of the cobordisms; for the former, the orientation on N is flipped, while in the latter, L_1 and L_3 are swapped. In both cases, the sign of $\sigma(V; L_1, L_2, L_3)$ is flipped, which is consistent with the rest of the equation.

We want to incorporate the $\sigma(V; L_1, L_2, L_3)$ term into our Crane-Yetter setting. First, a generalization of Definition 9.3 to 3-manifolds with boundary:

Definition 9.9. Let M be a 3-manifold with boundary $\partial M = N$. Let Ξ be a full multicurve on N (see Proposition 8.14. Let M_Ξ^* be the closed 3-manifold obtained by gluing 2-handles to M along Ξ, and then adding 3-handles (see Lemma 8.16). Define

$$\varepsilon_{M,\Xi} = \frac{1}{Z_{\mathrm{RT}}(\overline{M_\Xi^*})} \cdot P_\Xi \circ \varnothing_M^{\mathrm{sk}}$$

where P_Ξ is the projection in $\mathrm{End}_{Z_{\mathrm{CY}}}(N)(E)$ defined by Ξ (see Definition 8.12).

Theorem 9.10. *Let $\mathcal{A}$ be modular. Let $W : M_1 \to_N M_2$ be a cornered cobordism, N outwardly oriented with respect to M_1 (and M_2), and let $L_1, L_2 \subseteq V = H_1(N;\mathbb{R})$ be the kernels of the inclusions of N into M_1, M_2, as in Theorem 9.8. Let $[\Xi] \subseteq V$ be the subspace spanned by $\{[\xi] \mid \xi \in \Xi\}$. Then*

$$Z_{CY}^{sk}(W)(\varepsilon_{M_1,\Xi}) = \kappa^{\sigma(W)-\sigma(V;L_1,L_2,[\Xi])}\mathcal{D}^{(\chi(W)-(1-g))/2} \cdot \varepsilon_{M_2,\Xi}$$

Proof. Let $M_j' = (M_j)^*_\Xi = M_j \cup H$ (see Definition 9.9 above). Let $\mathrm{id}_{M_2'} : M_2' \to M_2'$ be the identity cobordism on M_2', and let $W' = W \cup_{M_2} \mathrm{id}_{M_2'}$. By Lemma 8.16, $P_\Xi \circ \varnothing^{\mathrm{sk}}_{M_j} \cup \varnothing^{\mathrm{sk}}_H = \varnothing^{\mathrm{sk}}_{M_j'}$.

Suppose $Z_{\mathrm{CY}}(W)(\varepsilon_{M_1,\Xi}) = z \cdot \varepsilon_{M_2,\Xi}$. Then

$$\begin{aligned} Z_{\mathrm{CY}}(W')(\varepsilon_{M_1'}) &= Z_{\mathrm{CY}}(W')(\varepsilon_{M_1,\Xi} \cup \varnothing^{\mathrm{sk}}_H) \\ &= Z_{\mathrm{CY}}(\mathrm{id}_{M_2'})(Z_{\mathrm{CY}}(W)(\varepsilon_{M_1,\Xi}) \cup \varnothing^{\mathrm{sk}}_H) \\ &= z \cdot \varepsilon_{M_2,\Xi} \cup \varnothing^{\mathrm{sk}}_H \\ &= z \cdot \varepsilon_{M_2'} \end{aligned}$$

But by Proposition 9.4, $Z_{\mathrm{CY}}(W')(\varepsilon_{M_1'}) = \kappa^{\sigma(W')}\mathcal{D}^{\chi(W')/2} \cdot \varepsilon_{M_2'}$. By Theorem 9.8, $\sigma(W') = \sigma(W) + 0 - \sigma(V;L_1,L_2,[\Xi])$ (since $\sigma(id_{M_2'}) = \sigma(M_2' \times [0,1]) = 0$. It is also easy to see that $\chi(M_2) = 1-g$, so $\chi(W') = \chi(W) + 0 - (1-g)$. (since $\chi(id_{M_2'}) = \chi(M_2' \times [0,1]) = 0$). Thus we are done. □

Note that one may verify directly that the formula given in Theorem 9.10 respects composition of cornered cobordisms. Namely, for $W' : M_2 \to_N M_3$ (not the W' from the proof), we should have

$$\sigma(W \cup W') - \sigma(V;L_1,L_3,[\Xi]) = (\sigma(W) - \sigma(V;L_1,L_2,[\Xi])) + (\sigma(W') - \sigma(V;L_2,L_3,[\Xi]))$$

which, by Theorem 9.8, can be rearranged to

$$\sigma(V;L_1,L_2,L_3) + \sigma(V;L_1,L_3,[\Xi]) = \sigma(V;L_1,L_2,[\Xi])) + \sigma(V;L_2,L_3,[\Xi]))$$

This also follows from Theorem 9.8: consider $W'' = M_3' \times [0,1]$, where $M_3' = M_3 \cup H$ as in the proof of Theorem 9.10 Lemma 8.16 then the left side of the equation is the error term from gluing W to W' first followed by W'', while the right side is from gluing W'' to W' first followed by W. In a sense, this equation expresses an associativity constraint.

9.3. **Relation to Reshetikhin-Turaev invariants.**

It is widely expected that the Reshetikhin-Turaev TQFT is the "boundary theory" of the Crane-Yetter TQFT; some version of this has appeared in [BFMGI2007]. Indeed, the heavy reliance of our constructions of the Crane-Yetter invariants on evaluations of ribbon graphs in terms of morphisms in $\mathcal{A}$ would heavily suggest it (although perhaps it is a self-fulfilling prophecy). In this section we construct such a "boundary theory". Throughout this section, we assume $\mathcal{A}$ is modular.

First we address the state space in the Reshetikhin-Turaev TQFT. As noted in Example 5.20, for a handlebody $H = \natural g S^1 \times B^2$ and some boundary value $\mathbf{V}$ on ∂H,

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(H;\mathbf{V}) \simeq \bigoplus \langle (\bigotimes_b V_{\bar{b}}) X_{i_1} X_{i_1}^* \cdots X_{i_g} X_{i_g}^* \rangle \simeq \tau(\partial H, \mathbf{V}) \tag{9.5}$$

where $\tau(\partial H, \mathbf{V})$ is the state space associated to $(\partial H, \mathbf{V})$ under the Reshetikhin-Turaev TQFT (see [BakK2001, (4.4.4)]). This holds for any 3-manifold, as is expected from the "boundary theory" hypothesis:

Theorem 9.11. *Let $\mathcal{A}$ be modular. For a 3-manifold with boundary M, the space of skeins with some boundary value $\mathbf{V}$ is isomorphic to the state space $\tau(\partial M, \mathbf{V})$ associated to its boundary under the Reshetikhin-Turaev TQFT:*

$$Z_{CY}^{sk}(M;\mathbf{V}) \simeq \tau(\partial M, \mathbf{V}) \tag{9.6}$$

In particular, $Z_{CY}^{sk}(M;\mathbf{E})$ only depends on $N = \partial M$.

Proof. Let $W : M \to_N \sqcup H$ be some cornered cobordism of W from M to a disjoint union of handlebodies. Let $\mathbf{V} \simeq \bigoplus(\mathbf{E}, P_\Xi)$ be a decomposition of $\mathbf{V}$ into simples (see Proposition 8.14). By Theorem 9.10, for each $(\mathbf{E}, P_\Xi)$, $Z_{\mathrm{CY}}^{\mathrm{sk}}(W;N)$ is an isomorphism

$$Z_{\mathrm{CY}}^{\mathrm{sk}}(W;N) : Z_{\mathrm{CY}}^{\mathrm{sk}}(M;(\mathbf{E},P_\Xi)) \simeq Z_{\mathrm{CY}}^{\mathrm{sk}}(\sqcup H;(\mathbf{E};P_\Xi))$$

and thus it is an isomorphism

$$Z^{\text{sk}}_{\text{CY}}(W;N) : Z^{\text{sk}}_{\text{CY}}(M;\mathbf{V}) \simeq Z^{\text{sk}}_{\text{CY}}(\sqcup H;\mathbf{V})$$

By the preceding discussion, we are done. ☐

Definition 9.12. Let $\mathcal{A}$ be premodular. We define the *category of extended cobordisms*, denoted $\mathbf{Cob^x}$, as follows. An object of $\mathbf{Cob^x}$ is a pair $(M,\mathbf{V})$, where M is a 3-manifold, and $\mathbf{V} \in Z^{\text{sk}}_{\text{CY}}(\partial M)$ is a boundary value.

A morphism from $(M,\mathbf{V})$ to $(M',\mathbf{V}')$ is a triple (W, M_0, φ_0), where W is a 4-manifold with $\partial W = \overline{M} \cup_{\partial M} M_0 \cup_{\partial M'} M'$, and $\varphi_0 \in Z^{\text{sk}}_{\text{CY}}(M_0;\mathbf{V} \cup \mathbf{V}')$.

Composition is given by simply gluing the manifolds appropriately, and composing the skeins; more precisely, for morphisms

$$(W, M_0, \varphi_0) : (M,\mathbf{V}) \to (M',\mathbf{V}')$$
$$(W', M_0', \varphi_0') : (M',\mathbf{V}') \to (M'',\mathbf{V}'')$$

their composition is given by

$$(W', M_0', \varphi_0') \circ (W, M_0, \varphi_0) \coloneqq (W' \cup_{M'} W, M_0' \cup_{\partial M'} M_0, \varphi_0' \cup_{\partial M'} \varphi_0)$$

The identity morphism of $(M,\mathbf{V})$ is given by $(M \times I, \partial M \times I, \text{id}_{\mathbf{V}})$.

Lemma 9.13. *With disjoint union as the monoidal structure on* $\mathbf{Cob^x}$*, it is rigid. The unit object is the empty manifold* $(\varnothing,\mathbf{E})$*. For an object* $(M,\mathbf{V})$*, the same triple* $(M \times I, \partial M \times I, \text{id}_{\mathbf{V}})$ *that defines the identify morphism also defines the (co)evaluation morphisms*

$$\text{ev}_{(M,\mathbf{V})} = (M \times I, \partial M \times I, \text{id}_{\mathbf{V}}) : (M,\mathbf{V}) \sqcup (\overline{M},\mathbf{V}) \to (\varnothing,\mathbf{E})$$
$$\text{coev}_{(M,\mathbf{V})} = (M \times I, \partial M \times I, \text{id}_{\mathbf{V}}) : (\varnothing,\mathbf{E}) \to (M,\mathbf{V}) \sqcup (\overline{M},\mathbf{V})$$

Proof. Straightforward. ☐

Proposition 9.14. *For an object* $(M,\mathbf{V})$ *of* $\mathbf{Cob^x}$*, define*

$$\mathcal{F}^{\mathbf{x}}((M,\mathbf{V})) \coloneqq Z^{sk}_{CY}(M;\mathbf{V})$$

and for a morphism $(W, M_0, \varphi_0) : (M,\mathbf{V}) \to (M',\mathbf{V}')$*, we define*

$$\mathcal{F}^{\mathbf{x}}((W, M_0, \varphi_0)) : \mathcal{F}^{\mathbf{x}}((M,\mathbf{V})) \to \mathcal{F}^{\mathbf{x}}((M',\mathbf{V}'))$$
$$\varphi \mapsto Z^{sk}_{CY}(W;\partial M')(\varphi_0 \cup_{\partial M} \varphi)$$

Then $\mathcal{F}^{\mathbf{x}}$ *is a monoidal functor.*

Proof. Straightforward.

We note that the image of the evaluation morphism under $\mathcal{F}^{\mathbf{x}}$ gives the skein pairing (Definition 5.18). ☐

Let us recall the Reshetikhin-Turaev invariants of 3-manifolds with links in more detail [RT1991]. (The constructions here are adapted from the exposition in [BakK2001, Chapter 4.4].) Consider a coupon c of a ribbon graph Γ, and let $r_1, \ldots, r_g$ be some ribbons that attach to c twice (not necessarily all such ribbons). Let $H(c, r_1, \ldots, r_g)$ be a small neighborhood of $c \cup r_1 \cup \cdots \cup r_g$; it is a genus g handlebody.

Definition 9.15. A *special graph* X is an (uncolored) ribbon graph in S^3 with a distinguished collection of annuli L, and a distinguished collection $\Psi = \{(c, r_1, \ldots, r_g), \ldots\}$ of coupons and ribbons as above. We define

$$M_X = M_L \backslash H_\Psi$$

where $M_L = \partial W_L$, W_L is the 4-manifold obtained by attaching 2-handles to B^4 along $L \subset S^3 = \partial B^4$, and $H_\Psi \coloneqq \bigcup H(c, r_1, \ldots, r_g)$ is the union of handlebodies associated to the distinguished coupons and ribbons.

In the following, $Z^{\text{sk}}_{\text{CY}}(H_\Psi;\mathbf{V})$ replaces $\tau(\partial H_\Psi, \mathbf{V})$ in the usual definition (say in [BakK2001]).

Definition 9.16. Let X be a special graph, and consider the graph $\Gamma = (X\backslash L)\backslash H_\Psi$, interpreted as a graph in M_X. Given a coloring Φ of Γ, and a coloring φ of the coupons of Ψ, we consider the evaluation $Z_{\mathrm{RT}}((X\backslash L, \Phi \cup \varphi) \cup \Omega L)$, that is, $X\backslash L$ (as a graph in S^3) is colored by Φ and φ, and each component of L is colored by the regular coloring (see Example 9.1). Thus, if we hold Φ fixed, so that it defines a boundary value $\mathbf{V}$ on ∂H_Ψ, then the evaluation defines a functional

$$Z_{\mathrm{RT}}((X\backslash L, \Phi \cup -) \cup \Omega L) : Z_{\mathrm{CY}}^{\mathrm{sk}}(H_\Psi; \mathbf{V}) \to \mathbf{k} \tag{9.7}$$

(it is clear that the possible colorings φ of the coupons of Ψ span $Z_{\mathrm{CY}}^{\mathrm{sk}}(H_\Psi; \mathbf{V})$). Then by using the non-degenerate skein pairing Definition 5.18, this defines a vector

$$\tau(M_X, (\Gamma, \Phi)) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H_\Psi}; \mathbf{V}) \tag{9.8}$$

(By Example 5.20, the skein pairing agrees (up to a factor) with the pairing defined in [BakK2001, (4.4.5)].)

The *Reshetikhin-Turaev of the colored ribbon graph* (Γ, Φ) *in* M_X is defined to be

$$Z_{\mathrm{RT}}((M_X, (\Gamma, \Phi))) = \kappa^{-\sigma(L)} \cdot \mathcal{D}^{(-|L|-1)/2} \cdot \tau(M_X, (\Gamma, \Phi)) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H_\Psi}; \mathbf{V}) \tag{9.9}$$

More generally, given a 3-manifold M with a colored ribbon graph Γ' with boundary value $\mathbf{V}$', if we have homeomorphisms $M \simeq M_X$ and $H \simeq H_\Psi$ that glue to a homeomorphism $M \cup_{\partial M \simeq \overline{\partial H}} H \simeq M_X \cup H_\Psi = M_L$, and it sends (Γ', Φ') to (Γ, Φ), $\mathbf{V}'$ to $\mathbf{V}$, then we define

$$Z_{\mathrm{RT}}((M, (\Gamma', \Phi'))) = Z_{\mathrm{RT}}((M_X, (\Gamma, \Phi)))$$

under the identification $Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H}; \mathbf{V}') \simeq Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H_\Psi}; \mathbf{V})$.

The definition above is a generalization of (9.4) to 3-manifolds with boundary; it is shown to be independent of the choice of special graph X (i.e. the choice of homeomorphism in the last paragraph) in Theorem 9.19. Now we give a definition of the Reshetikhin-Turaev invariant that is based on $\mathbf{Cob}^{\mathbf{x}}$:

Definition 9.17. Let X be a special graph, with $\Gamma = (X\backslash L)\backslash H_\Psi \subset M_X$ and Φ a coloring of Γ as in Definition 9.16 above. Let $N = \partial H_\Psi$, and let $\mathbf{V}$ be the boundary value on N defined by Φ. Consider the 4-manifold W_L obtained from attaching 2-handles along L, so that $\partial W_L = M_L$; treat $\overline{W_L}$ as a cornered cobordism $\overline{W_L} : M_X \to_N \overline{H_\Psi}$. Then we define

$$\tau_{CY}(M_X, (\Gamma, \Phi)) = Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{W_L}; N)((\Gamma, \Phi)) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H_\Psi}; \mathbf{V})$$

Equivalently, we can consider the morphism $(\overline{W_L}, M_X, (\Gamma, \Phi)) : (\varnothing, \mathbf{E}) \to (\overline{H_\Psi}; \mathbf{V})$, and define

$$\tau_{CY}(M_X, (\Gamma, \Phi)) = \mathcal{F}^{\mathbf{x}}((\overline{W_L}, M_X, (\Gamma, \Phi)))(\varnothing_\varnothing^{\mathrm{sk}}) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H_\Psi}; \mathbf{V})$$

We then define

$$Z_{\mathrm{RT}}^{\mathrm{CY}}(M_X, (\Gamma, \Phi)) = \kappa^{-\sigma(\overline{W_L})} \cdot \mathcal{D}^{-\chi(\overline{W_L})/2} \cdot \tau_{CY}(M_X, (\Gamma, \Phi))$$

More generally, for an arbitrary 3-manifold M with a colored graph $(\Gamma, \Phi) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(M; \mathbf{V})$, and a disjoint union of handlebodies H with $\partial M = N = \overline{\partial H}$, we define

$$Z_{\mathrm{RT}}^{\mathrm{CY}}(M, (\Gamma, \Phi)) = \kappa^{-\sigma(W)} \cdot \mathcal{D}^{-\chi(W)/2} \cdot Z_{\mathrm{CY}}^{\mathrm{sk}}(W; N)((\Gamma, \Phi)) \in Z_{\mathrm{CY}}^{\mathrm{sk}}(\overline{H}; \mathbf{V})$$

where W is some 4-manifold with boundary $\partial W = \overline{M \cup_N H}$, i.e. a cornered cobordism $W : M \to_N \overline{H}$.

Lemma 9.18. *The value of* $Z_{RT}^{CY}(M, (\Gamma, \Phi)) \in Z_{CY}^{sk}(\overline{H}; \mathbf{V})$ *depends only on* M *and* H *(and the identification of their boundaries), and is independent of* W.

Proof. Follows easily from Proposition 9.4; details are left to the reader. □

The following theorem shows that the invariants Z_{RT} and $Z_{\mathrm{RT}}^{\mathrm{CY}}$ are equivalent (we repeat some definitions for the reader's convenience):

Theorem 9.19. *Let* M *be a 3-manifold, and let* Γ *be a colored ribbon graph in* M *with boundary value* $\mathbf{V}$ *on* $N := \partial M$. *Let* H *be a disjoint union of handlebodies with an identification* $\partial H \simeq \overline{N}$, *so that we have an identification of the Reshetikhin-Turaev state space* $\tau(N, \mathbf{V})$ *with the skein space* $Z_{CY}^{sk}(\overline{H}; \mathbf{V})$. *Then for any cornered cobordism* $W : M \to \overline{H}$,

$$Z_{RT}(M, \Gamma) = \kappa^{-\sigma(W)} \cdot \mathcal{D}^{-\chi(W)/2} \cdot Z_{CY}(W; N)(\Gamma) \in Z_{CY}^{sk}(\overline{H}; \mathbf{V})$$

Proof. Choose some special graph X with identifications $M \simeq M_X$, $H \simeq H_\Psi$, as in Definition 9.17.

The right-hand side is $Z^{\mathrm{CY}}_{\mathrm{RT}}(M, \Gamma)$, which by Lemma 9.18 is independent of the choice of W. Take $W = \overline{W_L}$. Since $\chi(W) = |L| + 1$ and $\sigma(W) = \sigma(L)$, it suffices to show that from $\tau = \tau_{CY}$.

As we have seen in Example 9.1, the 2-handles of W_L give rise to ΩL. Thus, by Lemma 5.19,

$$\begin{aligned} \mathrm{ev}^{sk}(\varphi, \tau_{CY}(M_X, (\Gamma, \Phi))) &= \mathrm{ev}^{sk}(\varphi, Z^{\mathrm{sk}}_{\mathrm{CY}}(W_L; N)((\Gamma, \Phi))) \\ &= Z^{\mathrm{sk}}_{\mathrm{CY}}(W_L)((X\backslash L, \Phi \cup \varphi)) \\ &= Z_{\mathrm{RT}}((X\backslash L, \Phi \cup \varphi) \cup \Omega L) \\ &= \mathrm{ev}^{sk}(\varphi, \tau(M_X, (\Gamma, \Phi))) \end{aligned}$$

□

The gluing axiom follows easily from Definition 9.17:

Lemma 9.20. *Let M be a 3-manifold with boundary $\partial M = N_1 \sqcup N_2 \sqcup N'$, where N_1, N_2 are connected closed surfaces, and let $f : N_1 \simeq \overline{N_2}$ be a homeomorphism. Let H_1, H_2 be handlebodies with $\partial H_1 = N_1, \partial H_2 = N_2$, and suppose f extends to a homeomorphism $f : H_1 \simeq \overline{H_2}$. Let Γ be a colored ribbon graph (omitting the coloring) with boundary value $\mathbf{V} = \mathbf{V}_1 \sqcup \mathbf{V}_2 \sqcup \mathbf{V}'$, and suppose that $f_*(\mathbf{V}_1) = \mathbf{V}_2$. Then we have a graph $\Gamma' = \Gamma/f$ in $M' = M/f$, and*

$$\mathrm{ev}^{sk}(Z^{CY}_{RT}(M, \Gamma)) = Z^{CY}_{RT}(M', \Gamma') \in Z_{CY}(H'; \mathbf{V}')$$

for any handlebodies H' with $\partial H' = N'$, where ev^{sk} applies the skein pairing

$$Z^{sk}_{CY}(H_1; \mathbf{V}_1) \otimes Z^{sk}_{CY}(H_2; \mathbf{V}_2) \to \mathbf{k}$$

Proof. Applying the skein pairing amounts to composing with $\mathrm{ev}_{(H_1, \mathbf{V}_1)}$, so that, if W is a 4-manifold with $\partial W = M \cup (H_1 \sqcup H_2 \sqcup H')$, then $W \cup_{H_1, H_2} H_1 \times I$ has boundary M'. □

It is also helpful to connect the gluing result above to Definition 9.16. Let X be a special graph, and let $\psi_1, \psi_2 \in \Psi$ define handlebodies H_{ψ_1}, H_{ψ_2}. Let Φ be some coloring of $\Gamma = (X\backslash L)\backslash H_\Psi$, leaving a boundary value $\mathbf{V}$ on ∂H_Ψ. Let $\Psi' = \Psi \backslash \{\psi_1, \psi_2\}$, and let $\mathbf{V}', \mathbf{V}_1, \mathbf{V}_2$ be the boundary value $\mathbf{V}$ restricted to $\partial H_{\Psi'}, \partial H_{\psi_1}, H_{\psi_2}$, respectively. We have

$$Z_{\mathrm{RT}}(M_X, (\Gamma, \Phi)) \in Z^{\mathrm{sk}}_{\mathrm{CY}}(\overline{H_\Psi}; \mathbf{V}) = Z^{\mathrm{sk}}_{\mathrm{CY}}(\overline{H_{\psi_1}}; \mathbf{V}_1) \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(\overline{H_{\psi_1}}; \mathbf{V}_2) \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(\overline{H_{\Psi'}}; \mathbf{V}') \tag{9.10}$$

Suppose that there is an identification $f : \psi_1 \simeq \psi_2$, extending to an identification $f : H_{\psi_1} \simeq \overline{H_{\psi_2}}$ which also identifies the boundary values, i.e. $f_*(\mathbf{V}_1) = \mathbf{V}_2$. Then, as in the lemma above, we may perform the skein pairing $Z^{\mathrm{sk}}_{\mathrm{CY}}(H_{\psi_1}; \mathbf{V}_1) \otimes Z^{\mathrm{sk}}_{\mathrm{CY}}(H_{\psi_2}; \mathbf{V}_2) \to \mathbf{k}$ on $Z_{\mathrm{RT}}(M_X, (\Gamma, \Phi))$, or equivalently, compose $(W_L, M_X, (\Gamma, \Phi))$ with the evaluation morphism for $(H_{\psi_1}, \mathbf{V}_1)$, to obtain

$$\mathrm{ev}^{sk}(Z_{\mathrm{RT}}(M_X, (\Gamma, \Phi))) = \mathrm{ev}_{(H_{\psi_1}, \mathbf{V}_1)} \circ (W_L, M_X, (\Gamma, \Phi))(\varnothing^{\mathrm{sk}}_{\varnothing}) \in Z^{\mathrm{sk}}_{\mathrm{CY}}(H_{\Psi'}; \mathbf{V}') \tag{9.11}$$

Again, the 4-manifold associated to $\mathrm{ev}_{(H_{\psi_1}, \mathbf{V}_1)} \circ (W_L, M_X, (\Gamma, \Phi))$ i.e. $W' := W_L \cup_{H_{\psi_1}, H_{\psi_2}} H_{\psi_1} \times I$, has boundary $M' := \partial W' = M_X/f$. The graph $\Gamma \subset M_X$ glues up under f to some colored graph $\Gamma' = \Gamma/f$.

We want a surgery description for M'. The 4-manifold W' is built from W_L by attaching 1- and 2-handles; we can see this from the dual handle decomposition to the description of the cobordism $H \cup_N \overline{H} \to \varnothing$ in Example 5.20: attach a 1-handle to neighborhoods of c_1 and c_2 (the coupons in ψ_1 and ψ_2), then corresponding ribbons would glue up into g annuli, so we attach 2-handles to these annuli.

More concretely, start with the special graph X. Bring c_1 and c_2 close together via an ambient isotopy. Remove c_1, c_2 and join the ribbons together according to the identification f, then add a dotted circle around these ribbons. Add the g annuli formed from the ribbons of ψ_1, ψ_2 to L (note that these ribbons are not part of Γ). Finally, replace the dotted circle with a 0-framed annulus, and add it to L (see discussion in proof of Proposition 9.4). Now we have a new special link X', with distinguished link L' (having $|L| + g + 1$ components), and distinguished collection $\Psi' = \Psi \backslash \{\psi_1, \psi_2\}$ of coupons and ribbons, such that $M_{X'} = M'$. (See [BakK2001, Figure 4.11]).

Given a homeomorphism $f : N \simeq N'$ which sends some boundary value $\mathbf{V}$ on N to $\mathbf{V}' = f_*(\mathbf{V})$, the Reshetikhin-Turaev construction gives an isomorphism $\tau(f) : \tau(N, \mathbf{V}) \simeq \tau(N', \mathbf{V}')$ that is defined up to some factors of κ. The isomorphism $\tau(f)$ is unchanged under isotopy (maintaining $f_*(\mathbf{V}) = \mathbf{V}'$), so it gives projective representations of certain mapping class groups of surfaces.

Let us construct $\tau(f)$ in terms of Crane-Yetter theory. We assume for simplicity that N, N' are connected, the disconnected case follows naturally.

Definition 9.21. Let $f : N \simeq N'$ be a homeomorphism between connected surfaces, and let f sends some boundary value $\mathbf{V}$ on N to boundary value $\mathbf{V}' = f_*(\mathbf{V})$ on N'.

Let H, H' be handlebodies with $\partial H = N, \partial H' = N'$, so that $Z^{\mathrm{sk}}_{\mathrm{CY}}(H;\mathbf{V}), Z^{\mathrm{sk}}_{\mathrm{CY}}(H';\mathbf{V}')$ are stand-ins for the Reshetikhin-Turaev state spaces $\tau(N,\mathbf{V}), \tau(N',\mathbf{V}')$. Let H'' be some handlebody with $\partial H'' = N$, and such that $f : N \simeq N'$ extends to a homeomorphism $f : H'' \simeq H'$. [8] For $\varphi \in Z^{\mathrm{sk}}_{\mathrm{CY}}(H;\mathbf{V})$, we have $Z_{\mathrm{RT}}(H,\varphi) \in Z^{\mathrm{sk}}_{\mathrm{CY}}(H'';\mathbf{V})$. Then we define

$$\tau(f)(\varphi) = f_*(Z_{\mathrm{RT}}(H,\varphi)) \in Z^{\mathrm{sk}}_{\mathrm{CY}}(H';\mathbf{V}') \tag{9.12}$$

In particular, if $f : N \simeq N'$ extends to $f : H \simeq H'$, then $\tau(f) = f_*$ simply acts on the skeins directly.

Note that in general, $\tau(g \circ f) \neq \tau(g) \circ \tau(f)$; the discrepancy is a factor arising from the error terms in gluing cornered cobordisms:

Lemma 9.22. *Let $f : N_1 \simeq N_2, f' : N_2 \simeq N_3$ be homeomorphisms between closed genus g surfaces, sending boundary value $\mathbf{V}_1$ to $\mathbf{V}_2 = f_*(\mathbf{V}_1)$ and $\mathbf{V}_2$ to $\mathbf{V}_3 = f'_*(\mathbf{V}_2)$. Let H_j be a handlebody with boundary N_j, so that $\tau(f) : Z^{sk}_{CY}(H_1;\mathbf{V}_1) \simeq Z^{sk}_{CY}(H_2;\mathbf{V}_2)$ and $\tau(f') : Z^{sk}_{CY}(H_2;\mathbf{V}_2) \simeq Z^{sk}_{CY}(H_3;\mathbf{V}_3)$. Let $L_j \subset H_1(N_j;\mathbb{R})$ be the kernel of the inclusion $N_j \subset H_j$ as in Theorem 9.8; we identify them as subspaces of $V := H_1(N_3;\mathbb{R})$ under f_*, f'_*. Then*

$$\tau(f' \circ f) = \kappa^{\sigma(V;L,L_2,L_3)} \cdot \mathcal{D}^{(1-g)/2} \cdot \tau(f') \circ \tau(f) \tag{9.13}$$

Proof. Let H, H' be the auxiliary handlebodies used to define $\tau(f), \tau(f')$; i.e. $\partial H = N_1$, $\partial H' = N_2$, and f, f' extend to $f : H \simeq H_2, f' : H' \simeq H_3$. Let $W : H_1 \to_{N_1} H, W' : H_2 \to_{N_2} H'$ be cornered cobordisms. They compose to give $W'' := W' \cup_f W : H_1 \to_{N_2} H'$. Then

$$\tau(f) = f_* \circ \kappa^{-\sigma(W)} \cdot \mathcal{D}^{-\chi(W)/2} \cdot Z^{\mathrm{sk}}_{\mathrm{CY}}(W;N_1)$$
$$\tau(f') = f'_* \circ \kappa^{-\sigma(W')} \cdot \mathcal{D}^{-\chi(W')/2} \cdot Z^{\mathrm{sk}}_{\mathrm{CY}}(W';N_2)$$
$$\tau(f' \circ f) = f'_* \circ \kappa^{-\sigma(W'')} \cdot \mathcal{D}^{-\chi(W'')/2} \cdot Z^{\mathrm{sk}}_{\mathrm{CY}}(W'';N_2)$$

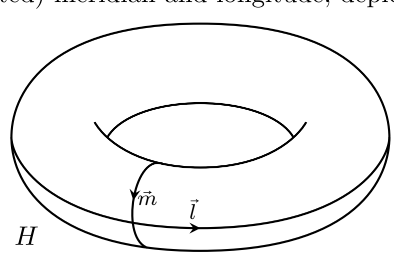
Note that $Z^{\mathrm{sk}}_{\mathrm{CY}}(W'';N_2) = Z^{\mathrm{sk}}_{\mathrm{CY}}(W';N_2) \circ f_* \circ Z^{\mathrm{sk}}_{\mathrm{CY}}(W;N_1)$, so

$$\tau(f' \circ f) = \kappa^{\sigma(W)+\sigma(W')-\sigma(W'')} \cdot \mathcal{D}^{(\chi(W)+\chi(W')-\chi(W''))/2} \cdot \tau(f') \circ \tau(f)$$

and the result follows from Theorem 9.8 and a simple Euler characteristic computation. $\square$

Let us work out the action of the mapping class group of a torus. Let $N = S^1 \times S^1$, and $H = B^2 \times S^1$. Let $\vec{m} = \partial B^2 \times \{*\}, \vec{l} = \{*\} \times S^1$ be (directed) meridian and longitude, depicted as follows:

(9.14)



The outward orientation on N is $(\vec{m},\vec{l})$. A matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ acts on N by sending $\vec{m} \mapsto a \cdot \vec{m} + b \cdot \vec{l}$ and $\vec{l} \mapsto c \cdot \vec{m} + d \cdot \vec{l}$. In particular, the S-matrix[9] $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ swaps the meridian and longitude, $\vec{m} \mapsto \overleftarrow{l}, \vec{l} \mapsto \vec{m}$. Note

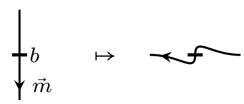
[8]One may take $H'' = H'$, but this extra step can be helpful for conceptual clarity and simplify computation. Note also that such an extension of f, if it exists, is unique up to isotopy.

that with boundary values, an integer matrix is not enough for specifying an element of the mapping class group.

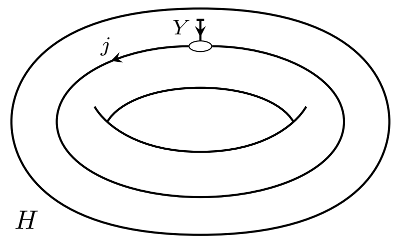
Let $\mathbf{V}$ be a boundary value on N with exactly one arc b, with no projectors (i.e. $\mathbf{V} \in \hat{Z}^{\text{sk}}_{\text{CY}}(N)$), and let $V_{\vec{b}} = Y$ so that an inwardly-directed boundary ribbon in H has label Y. Then we have

$$Z^{\text{sk}}_{\text{CY}}(H;\mathbf{V}) \simeq \bigoplus_j \text{Hom}_{\mathcal{A}}(Y, X_j X_j^*) \tag{9.15}$$

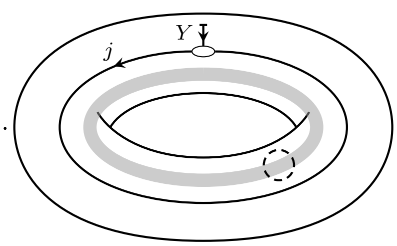
Let $\widetilde{S}$ be a homeomorphism of N preserving $\mathbf{V}$ which descends to S when we forget $\mathbf{V}$, and looks like this near b (drawn from an outside perspective):

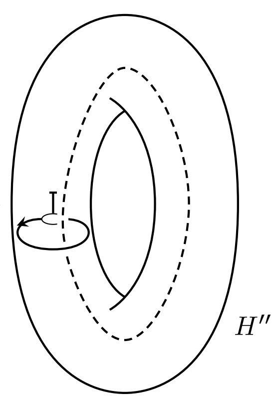
(9.16) 

Let us compute the map associated to $\widetilde{S}_*$. Consider $\varphi \in Z^{\text{sk}}_{\text{CY}}(H;\mathbf{V})$,

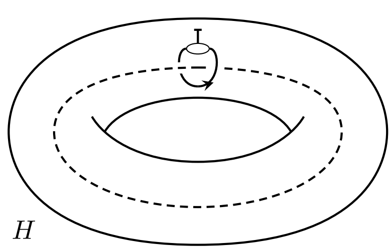
(9.17) $\varphi =$ 

By adding a 2-handle to H along its core circle (with blackboard framing, depicted in (9.18) as the gray loop), we get a cornered cobordism $\mathcal{H}_2 : H \to_N H''$, and we have

(9.18) $Z_{\text{RT}}(H,\varphi) = \mathcal{D}^{-1/2} \cdot$ 

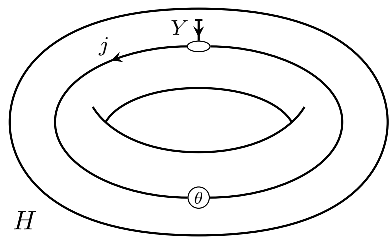
$= \mathcal{D}^{-1/2} \cdot$ 

Then applying $\widetilde{S}_*$,

(9.19) $\tau(\widetilde{S})(\varphi) = \mathcal{D}^{-1/2} \cdot$ 

The T-matrix[9] $T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ acts by left-hand Dehn twist. Choosing a good $\widetilde{T}$ that fixes $\mathbf{V}$, it is clear that

(9.20) $\tau(\widetilde{T})(\varphi) =$



It is clear that these agree with [BakK2001, Definition 3.1.15].

Proposition 9.23 ([BakK2001, Theorem 3.1.16])**.** *Using the identification* $Z^{sk}_{CY}(H;\mathbf{V}) \simeq \bigoplus_j \mathrm{Hom}_{\mathcal{A}}(Y, X_j X_j^*)$ *from (9.15),* $\tau(\widetilde{S})^4 = - \circ \theta_Y^{-1}$, $(\tau(\widetilde{S})\tau(\widetilde{T}))^3 = \kappa \cdot \tau(\widetilde{S})^2$.

Proof. We will use Lemma 9.22 repeatedly. Let us compare $\tau(\widetilde{S^2})$ against $\tau(\widetilde{S})^2$. Since $\chi(H) = 1 - g = 0$, the $\mathcal{D}$ factor does not contribute to the error factor (this is the case for all computations here). The kernel space $L \subset H_1(N;\mathbb{R}) = V$ for the inclusion $N \subset H$ is spanned by the meridian. The homeomorphism $\widetilde{S}$ swaps meridian m and longitude l, so if $L' \subset H_1(N;\mathbb{R})$ is spanned by l, then the exponent of κ in the error factor is $\sigma(V; L, L', L) = 0$. Thus, $\tau(\widetilde{S^2}) = \tau(\widetilde{S})^2$, and similarly, $\tau(\widetilde{S^4}) = \tau(\widetilde{S^2})^2 = \tau(\widetilde{S})^4$.

Since $\widetilde{S}^4$ is isotopic to the identity, it extends to H. More explicitly, it is easy to see from (9.16) that $\widetilde{S}^4$ is the identity everywhere expect near b, where it is rotated 360° counter-clockwise, dragging its surroundings with it. It extends into the interior, introducing a right-hand twist to the ribbon attached to b. Thus, on the level of the skein space, this amounts to precomposing with θ_Y^{-1} (which corresponds to a right-hand twist under our convention).

Now let us consider the second equation. Since $\widetilde{T}$ preserves the meridian, by similar considerations as above, we have $\tau(\widetilde{S}\widetilde{T}) = \tau(\widetilde{S})\tau(\widetilde{T})$.

Write $f = \widetilde{S}\widetilde{T}$ for simplicity, and let $L \subseteq H_1(N;\mathbb{R}) = V$ be spanned by the meridian as before. We have

$$\tau(f^2) = \kappa^{\sigma(V; f_*^2(L), f_*(L), L)} \cdot \tau(f)^2$$
$$\tau(f^3) = \kappa^{\sigma(V; f_*^3(L), f_*(L), L)} \cdot \tau(f^2)\tau(f)$$

It is easy to check that $f_*(\vec{m}) = -\vec{l}$, $f_*^2(\vec{m}) = -\vec{m} - \vec{l}$, $f_*^3(\vec{m}) = -\vec{m}$. So $\sigma(V; f_*^3(L), f_*(L), L) = \sigma(V; L, f_*(L), L) = 0$.

Recall that $\sigma(V; f_*^2(L), f_*(L), L)$ is the signature of a certain symmetric bilinear form Ψ on

$$\overline{V} = (f_*^2(L) \cap (f_*(L) + L))/(f_*^2(L) \cap f_*(L) + f_*^2(L) \cap L) = f_*^2(L)$$

Consider $\vec{m} + \vec{l} \in f_*^2(L)$, and $-\vec{l} \in f_*(L), \vec{m} \in L$; they sum to 0, so

$$\Psi(\vec{m} + \vec{l}, \vec{m} + \vec{l}) = \omega(\vec{m} + \vec{l}, -\vec{l}) = -1$$

where ω is the intersection form on V. Thus,

$$\tau(f^3) = \tau(f^2)\tau(f) = \kappa^{-1} \cdot \tau(f)^3$$

Since $(\widetilde{S}\widetilde{T})^3 = \widetilde{S}^2$,

$$(\tau(\widetilde{S})\tau(\widetilde{T}))^3 = \tau(f)^3 = \kappa \cdot \tau(f^3) = \kappa \cdot \tau(\widetilde{S^2}) = \kappa \cdot \tau(\widetilde{S})^2$$

□

Remark 9.24. Comparison with notation in [RT1991] (symbols on left are from [RT1991], which we recast in terms of our notation):

- $d_i = \dim X_i / p_+$ (see page 558, around (3.1.6))
- $v_i = \theta_i$
- F simply means evaluate colored ribbon graph as morphism
- $S_{i,j} = tr(c_{X_j,X_i} c_{X_i,X_j})$ but here c is right-hand braid

[9] Note that these are not the same as the S and T-matrices in [BakK2001], but are conjugate to them under $\mathrm{diag}(-1,1)$.

- $C = p_-/p_+$.
- $\sigma_-(L)$ is number of non-positive eigenvalues. Then $\sigma_-(L) = (|L| - \sigma(L) + null(L))/2$, where $null(L) = \dim H_1$ is the nullity of the intersection matrix.
- $\{L\}$ is label all strands by $d_i \,\mathrm{id}_i$, but here d_i is their definition, so its $\{L\} = p_+^{-|L|} \cdot \Omega L$, ΩL is label with regular coloring (our definition)
- Thus $F(M) = F(M;L) = C^{-\sigma_-(L)}\{L\} = (p_+/p_-)^{(|L|-\sigma(L)+null(L))/2} p_+^{-|L|} \cdot \Omega L = \kappa^{-\sigma(L)+null(L)} \mathcal{D}^{-|L|/2} \cdot \Omega L$

We also note that the Ω coloring in [Rob1995] is actually $\eta \cdot \Omega$ in our notation, where $\eta = \mathcal{D}^{-1/2}$.

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