On The Irreducibility of the Spaces of Genus-0 Stable Maps and Quasimaps to Complete Intersections in Projective Space

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Abstract of the Dissertation

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The work of Harris, Roth and Starr shows the irreducibility of the Kontsevich space for smooth low degree hypersurfaces in projective space. In this dissertation, we generalize their result to smooth complete intersections in projective space in instances where the dimension of the projective space is large compared to the multidegree of the complete intersection. Moreover, we use our results and methods of Starr and Tian to show the irreducibility of the space of Quasimaps to every complete intersection within the same multidegree range.

Contents

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1 Introduction and Outline of Results

In [11], the authors prove that, over an algebraically closed field k of characteristic 0, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,r}(X, e)$ of genus-0 stable maps of degree e to a general hypersurface $X \subset \mathbb{P}^n$ of degree d is irreducible of the expected dimension $(n+1-d)e + (n-4)$, if $\frac{n+1}{2} > d$ and $n > 2$. The proof is an induction argument for which the base case uses the flatness of the evaluation map for pointed lines. Suppose that $X \subset \mathbb{P}^n$ is a projective variety. We denote by $F_{0,1}(X)$, the subscheme of the two step flag variety $F((1,2), n+1)$ which parameterizes pairs (p, L) where p is a point and L is a line such that $p \in L \subset X$. The evaluation map is the morphism:

$$\rho_X : F_{0,1}(X) \longrightarrow X$$

defined by $\rho_X(p, L) = p$. When X is a general hypersurface with $\frac{n+1}{2} > d$, the map ρ_X is flat with fiber dimension $n-d-1$, which equals the expected dimension. This flatness result and a version of Mori's Bend-and-Break lemma prove that every irreducible component of $\overline{\mathcal{M}}_{0,r}(X, e)$ has the expected dimension. The final step is a combinatorial argument that all the irreducible components are equal. There is also related work for the parameter space of lines on hypersurfaces by Riedl and Yang in [19].

In this thesis we extend the proof of the flatness of the evaluation map to pointed lines on complete intersections satisfying a system of inequalities (A), described in section 4.2. By $\mathbb{P}V_{\underline{d}}$ we denote a projective space parameterizing c-tuples of homogeneous polynomials of degree $d_1, d_2, ..., d_c$ up to scalar multiplication. Thus, an open subset of $\mathbb{P}V_{\underline{d}}$ parameterizes complete intersections of multidegree $(d_1, d_2, ..., d_c)$. If X is a complete intersection defined by the tuple $(F_1, F_2, \cdots F_c)$, we abuse notation and say $X \in \mathbb{P}V_{\underline{d}}$. We prove the following theorem:

Theorem 4.1. *Fix a non-increasing sequence of positive integers $(d_1, d_2, ..., d_c)$. If these integers satisfy the system (A) of inequalities, for a general complete intersection $X \in \mathbb{P}V_{\underline{d}}$ the morphism*

$$\rho_X : F_{0,1}(X) \longrightarrow X$$

is flat with fiber dimension $n - (d_1 + d_2 + ... + d_c) - c$

Using this new flatness result and the technique outlined above, we give a new proof of the irreducibility of the space of stable maps to a general complete intersection X of multidegree $(d_1, ..., d_c)$. This is different from the original proof in [1] by Beheshti and Kumar. Our methods further generalize to conclude flatness of the evaluation map for k-planes containing a given $k-1$ plane.

Formally, suppose $X \subset \mathbb{P}^n$ is a projective variety. Let $F((1, 2, \cdots, r+1), n+1)$ be the partial flag variety parameterizing flags of projective linear subspaces $(\Lambda_0, \Lambda_1, ..., \Lambda_r)$. Let

$F_{0,1,2,...,r}(X)$ be the subscheme of $F((1,2,\cdots r+1),n+1)$ which parameterizes flags of projective linear subspaces $(\Lambda_0,\Lambda_1,...,\Lambda_r)$ contained in X. Consider the map

$$\rho_X^r : F_{0,1,2,...,r}(X) \longrightarrow F_{0,1,2,...,r-1}(X)$$

defined by $\rho_X^r(\Lambda_0,\Lambda_1,...,\Lambda_r) = (\Lambda_0,\Lambda_1,...,\Lambda_{r-1})$. We prove the following generalization of Theorem 4.1:

Theorem 11.1. *Let X be a general complete intersection of type $(d_1,d_2,...,d_c)$. Set $1 \leq r \leq n$. For sufficiently large n and for r chosen small relative to n,the following is true:*

For a general flag $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-2}$ and for every $r-1$-plane Λ_{r-1} such that $\Lambda_0 \subset ... \subset \Lambda_{r-2} \subset \Lambda_{r-1}$, the set of r planes Λ_r containing Λ_{r-1} has the expected dimension. Equivalently, the map

$$\rho_X^r : F_{0,1,2,...,r}(X) \longrightarrow F_{0,1,2,...,r-1}(X)$$

is flat over an open dense subset of $F_{0,1,2,...,r-1}(X)$ where the first $r-2$ components in the flag are general.

Finally, In section 3, we extend the result of [21], to complete intersections. We use this along with results of [5] and the results from previous sections to prove the following for the space of quasimaps:

Theorem 12.3. *For every smooth complete intersection $X \subset \mathbb{P}^n$ of multidegree $\underline{d} = (d_1,d_2,...,d_c)$, for n sufficiently large, the space of quasimaps to X, $Y_{n,e}(X)$ is irreducible of the expected dimension.*

Throughout this thesis, we will assume that our base field k is algebraically closed and has characteristic 0. Our results show that for a general complete intersection of multidegree $(d_1,d_2,...,d_c)$ satisfying our system of inequalities, the Gromov-Witten invariants are enumerative.

2 Some Background on Stacks

2.1 Introduction

In this section, we provide a very brief introduction to the notion of algebraic stacks. Our main goal is to extend certain geometric notions of schemes to stacks, e.g irreducibility, properness, smoothness, etc. Both [6] and [16] develop the foundational theory of stacks used in this thesis.

2.2 Stacks

Definition 2.1. *Let C be a category with fiber products. A* ***Grothendieck topology*** *on C is the datum of a set $Cov(X)$ of coverings of X for each $X \in Obj(C)$, comprising a collection of morphisms $\{X_i \longrightarrow X\}_{i \in I}$ such that:*

1. *If $Y \longrightarrow X$ is an isomorphism,then $\{Y \longrightarrow X\} \in Cov(X)$*

2. *If $\{X_i \longrightarrow X\}_{i \in I} \in Cov(X)$ and $Y \longrightarrow X$ is a morphism, then $\{X_i \times_X Y \longrightarrow Y\}_{i \in I} \in Cov(Y)$*

3. *If $\{X_i \longrightarrow X\}_{i \in I} \in Cov(X)$ and for each $i \in I$, $\{V_{ij} \longrightarrow X_i\}_{j \in J_i} \in Cov(X_i)$, then for every possible $(i,j) \in \Pi_{i \in I}\{i\} \times J_i$, $\{V_{ij} \longrightarrow X_i \longrightarrow X\} \in Cov(X)$.*

A category C with a Grothendieck topology is called a site. This generalizes the notion of a topology on a set.

Example Consider a topological space X and the category, $Op(X)$ of open subsets of X, with the morphisms in $Hom(U,V)$ being inclusion if $U \subset V$ and empty otherwise. For $U \in Obj(Op(X))$, we may define $Cov(U)$ to be the collections $\{U_i \longrightarrow U\}$ such that $U = \bigcup_i U_i$. In particular, if X is a scheme, then the usual Zariski topology on X defines a site on $Op(X)$, called the small Zariski site.

Definition 2.2. *Let $p : C \longrightarrow S$ be a functor. We say C is* ***fibered in groupoids*** *over S if the following conditions hold:*

1. *For every morphism $f : V \longrightarrow U$ in S and every $x \in Obj(C)$ such that $p(x) = U$, there exists a morphism $\phi : y \longrightarrow x$ in C such that $p(y) = V$ and $p(\phi) = f$. In other words, for every lift of U, there is a lift of V.*

2. *For every pair of morphisms $\phi : Y \longrightarrow X$ and $\psi : Z \longrightarrow X$ in C and any morphism $f : p(Z) \longrightarrow p(Y)$ such that $p(\phi) \circ f = p(\psi)$ there exists a unique lift $\chi : Z \longrightarrow Y$ of f such that $\phi \circ \chi = \psi$.*

If $p : C \longrightarrow S$ is a category fibered in groupoids, we assume that for every morphism $f : A \longrightarrow B$ in S, and every object E over B, we have chosen a lifting $f_E : f^*E \longrightarrow E$ of f with target E.

Definition 2.3. *If $X \in S$ is an object, the category, S/X of* ***morphisms to*** *X has morphisms $\{Y \longrightarrow X\}$ as its objects. The morphisms in S/X are given by commutative triangles.*

Sending $\{Y \longrightarrow X\}$ to Y turns this into a fibered category over S.

Definition 2.4. *A* ***presheaf*** *on a category C is a functor*

$$F : C^{op} \longrightarrow Sets$$

Suppose now that C is a site. In the case that C is the category of open subsets of a topological space as above, this notion is exactly equivalent to that of a presheaf on the aforementioned topological space.

Suppose $p : C \longrightarrow S$ is a category fibered in groupoids as above and that $U \in Ob(S)$. We will denote by $C(U)$ the category whose objects are $E \in Ob(C)$ such that $p(E) = U$ and whose morphisms are $f : E \to E'$ in C such that $p(f) = id_U$.

Definition 2.5. *Suppose $X \in Ob(S)$ and let $x, x' \in Ob(C)$ such that $p(x) = p(x') = X$. We define the presheaf*

$$Isom(x, x') : (S/X)^{op} \longrightarrow Set$$

as follows. For any morphism $f : Y \longrightarrow X$, we choose pullbacks f^*x, f^*x' and set $Isom(x, x')(f : Y \longrightarrow X) = Isom_{C(Y)}(f^*x, f^*x')$.

To define stacks, we will need two further important notions, namely those of a sheaf on a site and a descent datum, which we now explain.

Definition 2.6. *A presheaf is a* ***sheaf*** *if for every object $U \in C$ and covering $\{U_i \longrightarrow U\}_{i \in I}$, the sequence:*

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \times_U U_j)$$

is exact.

Note that to say this sequence is exact, means that $F(U)$ can be identified as the equalizer of the two maps on the right. Once again, in the case when C is the category of open sets of a topological space, this notion coincides with that of a sheaf on the topological space.

Consider a scheme X with its Zariski topology. Given a vector bundle E on X, For an open covering $X_i \longrightarrow X$, if E_i is the restriction of E to X_i, we can reconstruct E from the E_i's and the induced isomorphisms between the E_i and E_j on overlaps. Notice that the isomorphisms satisfy the cocycle condition. Using this as a motivating example, we define a descent datum.

We will denote by $pr_1 : X \times X \to X$ and $pr_2 : X \times X \to X$, the canonical projections to X. Suppose we are given a category fibered in groupoids $p : C \to B$, where B has a Grothendieck topology.

Definition 2.7. *Given a category fibered in groupoids $p : C \longrightarrow B$ as above, a* ***descent datum*** *(E_i, f_{ij}) for $S \in Ob(B)$ consists of an open covering in B, $\{S_i \xrightarrow{g_i} S\}_{i \in I}$ and for each i, an object $E_i \in C(S_i)$ such that for each i, j there exists an isomorphism $f_{i,j} : pr_1^* E_i \to pr_2^* E_j$ in $C(S_i \times_S S_j)$ such that the for any three indices i, j, k, the following diagram is commutative in $C(S_i \times_S S_j \times_S S_k)$:*

$$\begin{array}{ccc} & & pr_3^* E_k \\ & \nearrow & \downarrow \\ pr_1^* E_i & & \\ & \searrow & \\ & & pr_2^* E_j \end{array}$$

The maps in the above diagram are the pullbacks of the f_{pq}-s along the appropriate projections.

Definition 2.8. *With the same setup as above, suppose that we have an object E of $C(S)$. The **trivial descent datum** is descent datum (E, id_E) with respect to the cover $\{S \xrightarrow{id_S} S\}$.*

Definition 2.9. *With the same setup as above, suppose that we have an object E of $C(S)$ and an open cover $\{S_i \xrightarrow{g_i} S\}_{i \in I}$. We have a descent datum on the family of objects $g_i^* E$, by pulling back the trivial descent datum (E, Id_E) on $\{S \xrightarrow{id_S} S\}$ via the map natural map $\{S_i \xrightarrow{g_i} S\} \to \{S \xrightarrow{id_S} S\}$. We call this the **canonical descent datum** on the family $g_i^* E$ and denote it by $(g_i^* E, can)$.*

Definition 2.10. *Consider a descent datum (E_i, f_{ij}) with respect to the covering $\{S_i \xrightarrow{g_i} S\}_{i \in I}$ of S. We say that (E_i, f_{ij}) is an **effective** descent datum if there exists an object $E \in C(S)$ such that (E_i, f_{ij}) is isomorphic to the canonical descent datum $(g_i^* E, can)$.*

For a more general definition, see [16, Chapter 4].

We can now finally define a stack:

Definition 2.11. *A category fibered in groupoids $p : F \longrightarrow C$ is a **stack** if the following hold:*

1. *For any $X \in C$ and objects $x, y \in F(X)$, the presheaf $Isom(x, y)$ on C/X is a sheaf.*
2. *Every descent datum is effective.*

2.3 Deligne Mumford Stacks and Geometric Properties:

We now come to the main kind of stack that we will use for the rest of our discussion. The additional notion that we will need to define Deligne Mumford stacks is representability. For Deligne-Mumford Stacks, we will be able to talk about its geometric properties such as irreducibility, dimension, etc. The first step in this direction is to understand the stack associated with a given scheme. By S, we will denote the category of schemes.

Definition 2.12. *Let S be a scheme. To S, we associate **the category of S-schemes**, Sch/S. Its objects are morphisms of schemes with target S, i.e., morphism of the form $T \longrightarrow S$. Given two objects $T \longrightarrow S$ and $T' \longrightarrow S$, a morphism between them in Sch/S is a morphism $h : T \longrightarrow T'$ of S-schemes that commutes with the two maps to S.*

Definition 2.13. *Let $p : X \longrightarrow S$ be a stack over S. We say that X is **representable** by a scheme S, if it is equivalent to Sch/S.*

Definition 2.14. *A* ***morphism of categories*** *over S is a covariant functor commuting with the projection to S.*

To every 1-morphism $f : Sch/S \longrightarrow X$ of categories over S, we associate the object $f(id : S \longrightarrow S)$. Suppose that in particular S and T are schemes and $f : S/T \longrightarrow Sch/S$ is a morphism of categories. Then $f(id : S \longrightarrow S)$ is itself a morphism $g : T \longrightarrow S$ of schemes, and f is completely described by g and vice versa. In other words, we can understand all morphism of schemes with target S by looking at morphisms of categories with target Sch/S, and hence the category Sch/S encodes all the information about the scheme S itself. This justifies the abuse of notation. After this, we will interchangeably use S to denote both the scheme and the category Sch/S.

Definition 2.15. *Suppose $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ are morphisms of stacks over a base category S. We define the* ***fiber product****, $X \times_Y Z$ as the following category:*

1. *The objects are triples (A, B, α) where $A \in X$, $B \in Z$. and $\alpha : f(A) \longrightarrow g(B)$ is a morphism in the same fiber of Y over S.*
2. *Given two triples (A, B, α) and (A', B', α'), a morphism between them is a pair $(\beta : A \longrightarrow A', \gamma : B \longrightarrow B')$, in the fibers of X and Z respectively such that $g(\gamma) \circ \alpha = \alpha' \circ f(\beta)$.*

A fiber product of stacks satisfies a universal property similar to that as the fiber product of schemes. See [6] for a formal statement.

We define the corresponding notion of representability for a morphism of stacks over S.

Definition 2.16. *Let $f : X \longrightarrow Y$ be a morphism of stacks over a category S. We say that the morphism f is* ***representable*** *if for every morphism $S \longrightarrow Y$, the fiber product $S \times_Y X$ is representable by a scheme.*

The composition of representable morphisms is representable, as is the base change of a representable morphism (See [20]). In the case of morphisms of schemes, we can talk about their geometric properties such as smoothness, properness, open and closed immersions, etc. If said property is stable under base change in the category of schemes, we can extend this notion to a representable morphism of stacks.

Definition 2.17. *Suppose a property P of morphisms of schemes is stable under base change and local on the target. Let $f : X \longrightarrow Y$ be a representable map of stacks. We say that f* ***has the property*** *P if for every morphism $S \longrightarrow Y$ with S being the stack associated to a scheme, the morphism $S \times_Y X \longrightarrow S$ has the property P.*

The meaning of local in the previous definition will depend on context. We will work only with Deligne Mumford stacks in this thesis and for us, by local we will mean local in the etale topology. Notice that because of representability, $S \times_Y X \longrightarrow S$ is an actual map of schemes which allows us to check the property P. Some examples of P that come up frequently are flatness, finiteness, being etale, etc. This will also come up in our discussion in the following context- we will often say $\mathcal{X}$ is a closed(resp. open) substack of the stack $\mathcal{Y}$. By this, we will mean that there is a morphism of stacks $i : \mathcal{X} \to \mathcal{Y}$ that is representable by a closed(resp. open) immersion of schemes.

Definition 2.18. *Let S be a scheme and let X/S be a stack over S. X is an* ***algebraic stack*** *if the following hold:*

1. *The diagonal $\Delta : X \longrightarrow X \times_S X$ is representable.*
2. *There exists a smooth surjective representable morphism $\pi : U \longrightarrow X$, where U is a scheme.*

Definition 2.19. *Suppose X/S is an algebraic stack as above with quasi-finite and unramified diagonal map Δ. If the map $\pi : U \to X$ can be chosen to be etale, we say that X is a* ***Deligne-Mumford*** *stack.*

The Deligne Mumford stacks that we will discuss will have an associated scheme called the coarse moduli space. See [9] a detailed description.

Lemma 2.1. *For an algebraic stack $\mathcal{X}$, if we have a morphism from a scheme $\phi : X \to \mathcal{X}$, then ϕ is representable.*

Proof. Suppose that $Y \longrightarrow \mathcal{X}$ is a morphism with Y a scheme. We have the Cartesian diagram:

$$\begin{array}{ccc} X \times_{\mathcal{X}} Y & \longrightarrow & X \times Y \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

The bottom arrow is representable, since $\mathcal{X}$ is algebraic, and hence, the top arrow is representable. But since $X \times Y$ is itself a scheme, we see that $X \times_{\mathcal{X}} Y$ is representable by a scheme. Thus, by the definition of representability of morphisms, $\phi : X \to \mathcal{X}$ is representable. □

We have already seen how to extend geometric ideas for morphisms of schemes to morphisms of stacks that are representable. In the case of Deligne-Mumford stacks, it is possible to do so even when a given morphism is not representable by a map of schemes.

Definition 2.20. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Deligne-Mumford stacks. Suppose that*

$$f : \mathcal{X} \to \mathcal{Y}$$

is a morphism of Deligne Mumford stacks that is not necessarily representable. Let P be a property that is etale local on the source and target. For every etale morphism $\phi : X \to \mathcal{X}$, if the composition $f \circ \phi : X \to \mathcal{Y}$ has the property P, then we say that f ***has the property*** *P.*

In the case where f is itself representable, the definitions coincide). Using this, we will frequently talk about the fiber dimension of a morphism of Deligne Mumford Stacks(Since fiber dimension is preserved by etale base change).

Some other important considerations for us are the geometric properties of the stacks themselves. Of particular interest are the properties of irreducibility, smoothness and dimension. We briefly outline the ideas in the context of the Deligne Mumford stacks we will encounter in the remainder of this thesis. More general definitions can be found at [20].

In our discussion, for every Deligne-Mumford Stack, there is an associated scheme known as the coarse moduli space, which we define in the next section. By the dimension of a stack, we will mean the dimension of its coarse moduli space in the usual sense of the dimension of schemes.

The geometric property of properness readily generalizes to stacks as follows:

Definition 2.21. *Suppose that $\mathcal{X}$ is a Deligne Mumford stack over an algebraically closed field $Spec(k)$ of characteristic 0. $\mathcal{X}$ is **proper** if it is locally finitely generated over $Spec(k)$ and for every discrete valuation ring R with quotient field K, there exists a field extension K' of K and a valuation ring $R' \subset K'$ which is a finite extension of R and dominates R, such that every morphism $Spec(K') \to \mathcal{X}$ extends uniquely to a morphism $Spec(R') \to \mathcal{X}$.*

Given a Deligne Mumford stack $\mathcal{X}$, let us consider an etale chart of our stack $U \to \mathcal{X}$, and a map $Spec(k) \to \mathcal{X}$, where k is an algebraically closed field of characteristic 0. There exists a lift of this map from $Spec(k)$ to the etale chart U, which gives us a point in U. Now, suppose we are given another etale chart V, and we get a further lift of this lift, and a point in V. Moreover we have a map of local rings from the point in U to the local ring at the point in V. Continuing this process, we can take a direct limit of this system of local rings. The ring we obtain is called the Henselized local ring of the stack $\mathcal{X}$ at the geometric point $Spec(K)$ and it is independent of the choice of etale charts.

Definition 2.22. *If this Henselized local ring is an integral domain at every geometric point of the stack, we say that the stack is **locally unibranch**.*

This is a generalization of the notion of a unibranch scheme.

There are two notions of irreducibility associated to a Deligne Mumford stack, one of which involves its coarse moduli space, while the second can be phrased in terms of the property of being unibranch.

Definition 2.23. *A Deligne Mumford stack $\mathcal{X}$ is **irreducible** if its coarse moduli space is irreducible as a scheme.*

The second notion is as follows:

Definition 2.24. *A conneced Deligne Mumford stack $\mathcal{X}$ is **unibranch** if the stack is everywhere locally unibranch.*

A Deligne Mumford stack which is unibranch is necessarily irreducible as well. The property of being unibranch implying irreducibility extends to schemes as well. Concretely, we can look at the Henselized local rings of a connected scheme. If these Henselized local rings are integral domains, then the scheme is irreducible.

We know that a locally finitely generated, separated k scheme X is smooth over $Spec(k)$ iff all of its local rings are regular. A Noetherian local ring R is regular if and only if its Henselization R^h is regular. Thus, a separated and connected scheme is smooth iff the Henselized local ring at every point is regular. This motivates our definition for the smoothness of a Deligne Mumford Stack:

Definition 2.25. *A Deligne Mumford Stack $\mathcal{X}$ over $Spec(k)$ which is locally finitely generated is **smooth** if its Henselized local ring at every point is regular.*

Definition 2.26. *A Deligne Mumford Stack $\mathcal{X}$ over $Spec(k)$ is **connected** if its coarse moduli space is connected.*

As we know, a smooth scheme X over $Spec(k)$ which is connected is necessarily irreducible since its Henselized local rings (and consequently its local rings) are regular. We note that the same is true for stacks: a smooth, connected stack is necessarily irreducible.

Definition 2.27. *A scheme X is a **local complete intersection** if its local ring at every point is a local complete intersection ring (That is,the quotient of a regular local ring by a regular sequence).*

It is true that a Noetherian local ring is a complete intersection ring iff its Henselization is. This leads us to the following definition of a complete intersection stack:

Definition 2.28. *A Deligne Mumford stack $\mathcal{X}$ is a **complete intersection stack** if its Henselized local ring at every point is a complete intersection ring.*

Just as in the case of schemes, this definition implies that a smooth Deligne Mumford Stack is a local complete intersection stack. These notions are relevant for us since in Section 2, we will prove in that a certain Deligne Mumford stack is an irreducible, local complete intersection stack by showing that it is smooth.

2.4 The Kontsevich Moduli Stack

The stacks that are relevant to our discussion are the Kontsevich Moduli spaces $\overline{\mathcal{M}}_{0,m}(X,\beta)$, which we now define.

Definition 2.29. *A **stable map** with r marked points is a triple $(C, f, x_1, ..., x_r)$ where C is a proper, reduced,connected, at worst nodal curve, $x_1, x_2, ..., x_r$ distinct, nonsingular points of C and a morphism $f : C \to X$ which satisfies the following stability condition- if f is constant on any component of C then that component must have at least 3 distinguished points. (By distinguished points, we mean either the marked points x_i or nodal points).*

Definition 2.30. *A **family of n-pointed stable maps to** X is a tuple $(\pi : C \to B, h : C \to X, p_1, p_2, ..., p_n)$ where π is a flat morphism, h is a morphism of* *schemes and each p_i is a section of π such that for every closed point $b \in B$, $(C_b, h|_{C_b}, p_1(b), p_2(b), ..., p_n(b))$ is a stable map with n marked points.*

Definition 2.31. *Given two families $(\pi : C \to B, h : C \to X, p_1, p_2, ..., p_n)$ and $(\pi' : C' \to B, h' : C' \to X, p'_1, p'_2, ..., p'_n)$ of stable maps over B, a **morphism** of families of stable maps is a fiber preserving isomorphism $F : C \to C'$, such that*

- $h' \circ F = h$
- $F \circ p_i = p'_i$

This defines a category of families of stable maps.

We define $\overline{\mathcal{M}}_{g,n}(X, \beta)$ to be the *Lax* 2-functor which associates to a scheme Y the groupoid of families of stable maps to $\pi : C \to Y$ over Y- i.e. the fiber over each closed point $y \in Y$ is a curve C_y of genus g with a stable map $h : C_y \to X$ - such that $h_*([C_y]) = \beta$ in $H_2(X, \mathbb{Q})$. To this *Lax 2*- functor, we can associate a stack which we also denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$. $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a Deligne Mumford stack called the Kontsevich Moduli space. We will be interested in the case of genus 0 curves only.

If X is projective over $Spec(k)$, and in our case it always will be, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is proper and of finite type over $Spec(k)$. If the Picard number of X is 1, as it will be when $X \subset \mathbb{P}^n$ is a complete intersection (which are our objects of interest), every integral curve will be the multiple of the class of a line. So if $[l]$ is the class of a line in $H_2(X, \mathbb{Q})$, any β can be expressed as $e.[l]$, and we will denote $\overline{\mathcal{M}}_{0,n}(X, e.[l])$ simply as $\overline{\mathcal{M}}_{0,n}(X, e)$.

$\overline{\mathcal{M}}_{0,n}(X, e)$ is equipped with a universal family $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{0,n}(X, e)$ as well as a morphism $h : \mathcal{U} \longrightarrow X$. For $i = 1, 2, ..., m$, we also have sections $s_1, s_2, ..., s_m$ of $\mathcal{U}$ which correspond to the marked points on a given stable curve. Moreover, for each j, we have the j-th evaluation map $ev_j : \overline{\mathcal{M}}_{0,n}(X, e) \to X$, which is the composition $h \circ s_j$.

Suppose now that X is a projective scheme over $\mathbb{C}$. Then there exists a projective scheme $\overline{M}_{0,m}(X, \beta)$ and a natural transformation of functors:

$$\phi : \overline{\mathcal{M}}_{0,m}(X, \beta) \to Hom(-, \overline{M}_{0,m}(X, \beta))$$

such that

- The induced map

$$\phi(Spec(\mathbb{C})) : \overline{\mathcal{M}}_{0,m}(X, \beta)(Spec(\mathbb{C})) \to Hom(Spec(\mathbb{C}), \overline{M}_{0,m}(X, \beta))$$

  is a set bijection.
- If Z is a scheme and $f : \overline{\mathcal{M}}_{0,m}(X, \beta) \to Hom(-, Z)$ is a natural transformation of functors, then there exists a unique morphism of schemes $\gamma : \overline{M}_{0,m}(X, \beta) \to Z$ such

that $f = \tilde{\gamma} \circ \phi$, where $\tilde{\gamma}$ is the natural transformation $\tilde{\gamma} : Hom(-, \overline{M}_{0,m}(X, \beta)) \to Hom(-, Z)$ induced by γ.

This is the definition in [9]. $\overline{M}_{0,m}(X, \beta)$ is the aforementioned coarse moduli space associated to the Deligne Mumford stack $\overline{\mathcal{M}}_{0,m}(X, \beta)$.

Definition 2.32. *In the situation above, we say that* $\overline{M}_{0,m}(X, \beta)$ ***coarsely represents*** *the functor* $\overline{\mathcal{M}}_{0,m}(X, \beta)$.

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3 Overview of Gromov-Witten Theory

3.1 Introduction

As mentioned in the introduction, the results of this thesis are motivated by questions of enumerativity in Gromov Witten Theory. In this subsection, we provide a broad overview of Gromov-Witten Theory and explain the implications of our results in this field. For more details, see [17, Section $1\frac{1}{2}$]. Gromov-Witten Theory is a method of counting the number of curves on a smooth projective variety X over $\mathbb{C}$ satisfying some set of incidence conditions. The subject has roots in symplectic geometry and topological strings.

We consider the moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(X, e)$ over $\mathbb{C}$, where X is a smooth projective variety. As mentioned in the previous section, this is a proper Deligne Mumford stack. It is important to have a proper space to obtain counts that are deformation invariant. Recall that $\overline{\mathcal{M}}_{0,n}(X, \beta)$ parameterizes families of stable maps to X, and as such, allows us to view curves on X as being parameterized by maps $f : C \to X$.

3.2 Gromov Witten Invariants and Curve Counting

Given a Deligne-Mumford stack $\mathcal{X}$ over $Spec(k)$ with a perfect obstruction theory, we have an associated integer known as the virtual (or expected) dimension. The virtual dimension provides a lower bound for the dimension of $\mathcal{X}$. Given a projective variety X as above, the Deligne-Mumford stack $\overline{\mathcal{M}}_{0,n}(X, \beta)$ has a naturally defined perfect obstruction theory and, as such, virtual dimension equal to $\int_\beta c_1(T_X) + n + dimX - 3$. For more details, see [2].

The main aspect of the perfect obstruction theory is the construction of the *virtual* fundamental class, carried out in [2]. In this paper, given a separated Deligne Mumford Stack $\mathcal{X}$ over k, Behrend and Fantechi construct a class in the rational Chow Group $A_d(\mathcal{X})$, where d is the virtual dimension of $\mathcal{X}$, which they define to be the virtual fundamental class. In particular, this class exists for $\overline{\mathcal{M}}_{0,n}(X, \beta)$. We can also think of it as an element

$$[\overline{\mathcal{M}}_{0,n}(X, \beta)]^{vir} \in H_*(\overline{\mathcal{M}}_{0,n}(X, \beta))$$

via the cycle class map.

With this in mind, our goal is to count curves on X. Recall that we have n evaluation maps $ev_i : \overline{\mathcal{M}}_{0,n}(X,\beta) \to X$. For a stable map $(C, f, x_1, x_2, ..., x_n)$, the evaluation map ev_i maps $(C, f, x_1, x_2, ..., x_n)$ to $f(x_i)$. To get curve counts, we could hope to integrate certain cohomology classes over the fundamental class of $\overline{\mathcal{M}}_{0,n}(X,\beta)$. The problem though is that $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is often reducible with components of various dimensions and is not smooth. In such cases, it does not make sense to talk about a fundamental class. To circumvent this issue, we instead work with the virtual fundamental class.

Suppose we have subvarieties $V_1, V_2, ..., V_n$ in X. Let α_i be the Poincare dual of the homology class of V_i. Pulling back α_i via the i-th evaluation map, we get a class $\gamma_i = ev_i^* \alpha_i$. We think of the Poincare dual of γ_i as the homology class of stable maps for which $f(x_i) \in V_i$.

Consider the cup product $\gamma_1 \smile \gamma_2 \smile ... \smile \gamma_n$. Extending our intuition from the previous paragraph, this expression parameterizes stable maps f such that $f(x_i) \in V_i$ where $(x_1, x_2, ..., x_n)$ is the tuple of points associated to the stable map. Once we let the points vary, we can think of this term as parameterizing stable maps $f : C \to X$ such that $f(C)$ intersects V_i.

We now pair with the virtual fundamental class to compute:

$$N^X_{0,\beta}(\alpha_1, \alpha_2, ..., \alpha_n) = \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]^{vir}} \gamma_1 \smile \gamma_2 \smile ... \smile \gamma_n \in \mathbb{Q}$$

These quantities are called the genus-0 Gromov-Witten Invariants. We interpret them as counts of rational curves intersecting the cycles V_i. Mathematicians such as Schubert studied classical invariants such as - How many rational curves in $\mathbb{P}^n$ intersect each of a specified yet general collection of linear subspaces? Gromov Witten Theory is often applied to classical questions like these. In such situations, the γ_i-s are powers of the hyperplane class. It is, however, often the case that the Gromov Witten invariants are not enumerative. That is, this number doesn't really count what we hope it will. But under certain assumptions, we do actually recover the naive count using this computation.

Let X be a smooth complete intersection in $\mathbb{P}^n$ of type $\underline{d} = (d_1, d_2, ..., d_c)$. Suppose that the Moduli space $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is irreducible of the expected dimension and has an open dense subset which parameterizes smooth embedded curves in X. In this situation, the virtual fundamental class equals the fundamental class of $\overline{\mathcal{M}}_{0,n}(X,\beta)$. Moreover for such an X, each γ_i is a power of the hyperplane class and The Gromov Witten invariants are indeed enumerative. That is to say, they agree with the classical invariants studied by Schubert et-al. The main result of this thesis proves that for a general smooth complete intersection X of degree $(d_1, d_2, ..., d_c)$ in $\mathbb{P}^n$ satisfying the system of polynomial inequalities (A), $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is irreducible of the expected dimension. Hence, in the context of Gromov Witten Theory, it shows that the Gromov Witten invariants are enumerative.

4 Lines on Complete Intersections

4.1 Notation

We begin by fixing notation. Let U be a vector space of dimension $n+1$ over a field k. By $\mathbb{P}^n$, we will mean the projective space $\mathbb{P}U = Proj(\oplus_{j=0}^{\infty} Sym^j(U^*))$, whose closed points correspond to 1 dimensional vector subspaces of U. Let us consider a fixed c-tuple $\underline{d} = (d_1, d_2, ..., d_c)$ of positive integers. Consider the vector space $V_{\underline{d}} = \prod_{i=1}^{c} H^0(\mathbb{P}^n, O(d_i))$. Thus, the points of $V_{\underline{d}}$ are c-tuples of homogeneous polynomials of the form $(F_1, F_2, ..., F_c)$ with $F_i \in H^0(\mathbb{P}^n, O(d_i))$. Let $\mathbb{P}V_{\underline{d}}$ be the projective space associated to this vector space. We note that an open subset of $\mathbb{P}V_{\underline{d}}$ parametrizes all complete intersections of type $(d_1, ..., d_c)$ in $\mathbb{P}^n$.

Let $F((1,2), n+1) \subset \mathbb{P}^n \times G(2, n+1)$ be the partial flag variety, parametrizing flags of the form (p, L) with $p \in L$. For a given projective variety X, let $F_{0,1}(X)$ parameterize pairs $(p, L) \in \mathbb{P}^n \times G(2, n+1)$ such that $p \in L \subset X$. Let $\mathcal{X} \subset \mathbb{P}V_{\underline{d}} \times \mathbb{P}^n$ be the subscheme parametrizing points pairs (X, p) with $p \in X$. For tuples When we restrict to an open subset of $\mathbb{P}V_{\underline{d}}$, X is a complete intersection of type $(d_1, d_2, ...d_c)$.

Let $F_{0,1}(\mathcal{X}) \subset \mathbb{P}V_{\underline{d}} \times F((1,2), n+1)$ be the subvariety parametrizing triples (X, p, L) with $p \in L \subset X$. Consider the projection maps:

$$\pi_0 : \mathbb{P}V_{\underline{d}} \times F((1,2), n+1) \longrightarrow \mathbb{P}V_{\underline{d}} \tag{1}$$

$$\pi_1 : \mathbb{P}V_{\underline{d}} \times F((1,2), n+1) \longrightarrow \mathbb{P}^n \tag{2}$$

$$\pi_2 : \mathbb{P}V_{\underline{d}} \times F((1,2), n+1) \longrightarrow G(2, n+1). \tag{3}$$

At the level of closed points, the map $(\pi_0, \pi_1)|_{F_{0,1}(\mathcal{X})} : F_{0,1}(\mathcal{X}) \longrightarrow \mathbb{P}V_{\underline{d}} \times \mathbb{P}^n$ sends the pair (X, p, L) with $p \in L \subset X$ to the pair (X, p) and as such, it factors through $\mathcal{X}$. We denote by ρ the induced morphism:

$$\rho : F_{0,1}(\mathcal{X}) \longrightarrow \mathcal{X}.$$

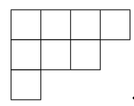
Given a complete intersection $X \in \mathbb{P}V_{\underline{d}}$ we have the fiber map,

$$\rho_X : F_{0,1}(X) \longrightarrow X$$

4.2 A system of polynomial inequalities

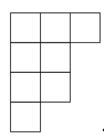
In this brief subsection, we state a system of polynomial inequalities with rational coefficients which, when satisfied, allow us to conclude the flatness of the map ρ_X. To write down our system in a compact and readable way, it is convenient to use Young Diagrams.

Suppose we have a non-increasing finite sequence of integers, $(d_1, d_2, ..., d_c)$. A Young Diagram of type $(d_1, ..., d_c)$ consists of a finite collection of cells arranged in rows. The i-th row has d_i cells. So for example, for the sequence of integers $(4, 3, 1)$ we have the following Young Diagram:



.

Given a Young Diagram as above, one can construct the conjugate Young Diagram by reflecting it along the main diagonal. The length of the i-th row in the transposed Young Diagram is the number of d_i which are greater than or equal to i. That is, for Young diagram λ of type $(a_1, a_2, ..., a_k)$, its conjugate is a Young diagram $\lambda*$ of type $(b_1, b_2, ..., b_{a_1})$ where $b_j = |\{i | a_i \geq j\}|$. In the example listed above, the conjugate Young Diagram is as follows:



.

In our case, the non-increasing sequence of integers will be the multidegrees(arranged in the necessary order) of the polynomials defining a complete intersection in $\mathbb{P}^n$. Given a complete intersection of type $(d_1, ...d_c)$, with $d_1 \geq d_2 \geq ... \geq d_c$, we draw the corresponding Young Diagram and look at its conjugate. We will denote by $(b_1, ..., b_{d_1})$, the sequence of integers corresponding to this conjugate. Now that we have fixed notation, we can state the aforementioned inequalities.

We denote by (A), the following system of inequalities:

$$I_1 : n - (d_1 + d_2 + ... + d_c) - c \geq 0$$
$$I_e : \binom{n - b_1 - b_2 - \cdots - b_e + e}{e} \geq n - c + 1, \text{for } e = 2, 3, ..., d_1.$$

Example We consider the case when each $d_i = 2$. The system then reduces to the following-

$$n - 2c - c \geq 0$$
$$\binom{n - 2c + 2}{1} \geq n - c + 1$$

The second inequality can be written as:

$$\frac{(n - 2c + 2)(n - 2c + 1)}{2} \geq n - c + 1$$

A straightforward computation shows that this inequality is satisfied whenever $\frac{n+1}{2} - \frac{\sqrt{n+1}}{2} \geq c$.

It is often the case that some of the inequalities in this system are redundant. We believe that in specific cases, for example when we have runs of the form $d_i = d_{i+1} = ... = d_j = d$, this system can be greatly simplified using approximations for the binomial coefficients.

4.3 Generic Flatness of the map ρ_X

For any closed subscheme X of a projective space $\mathbb{P}^r$, we will denote by $S(X)$ the homogeneous coordinate ring of X. We will need the following lemmas.

Lemma 4.1. *Let $X \subset \mathbb{P}^m$ be a subscheme of dimension l. The Hilbert Function of X, h_X is greater than or equal to that of a linear subvariety of the same dimension.*

Proof. We use the method in [10, Remark 13.10]. Pick a plane Λ disjoint from X and choose homogeneous co-ordinates $[Z_0, Z_1, ..., Z_m]$ on $\mathbb{P}^m$ such that Λ is given by $Z_0 = Z_1 = ... = Z_l = 0$. Then consider the projection

$$\pi_\Lambda : X \longrightarrow \mathbb{P}^l$$

This map must be surjective, and as such we have an inclusion of the homogenous co-ordinate ring $k[Z_0, Z_1, ..., Z_l]$ into the homogenous co-ordinate ring $S(X)$ of X in $\mathbb{P}^m$. This inclusion respects degrees. Thus, we get

$$h_X(r) \geq \binom{l+r}{r}$$

□

Lemma 4.2. *The incidence scheme $F_{0,1}(\mathcal{X})$ is smooth.*

Proof. This is a standard argument using homogeneity. Recall that $\mathbb{P}^n$ is the projective space parameterizing 1 dimensional linear subspaces of the vector space U. By definition, $F_{0,1}(\mathcal{X})$ is a closed subscheme of $F((1,2), n+1) \times \mathbb{P}V_{\underline{d}}$. Denote by pr_1 and pr_2 the projections from $F_{0,1}(\mathcal{X})$ to the first and second factor respectively. Inside the product $F((1,2), n+1) \times \mathbb{P}V_{\underline{d}}$, $F_{0,1}(\mathcal{X})$ is the zero scheme of the section σ of $\mathrm{pr}_2^* O(1) \otimes \mathrm{pr}_1^* \bigoplus_{i=1}^c \mathrm{Sym}^{d_i}(S^\vee)$ determined by the universal c-tuple $(F_1, \cdots, F_c)$ of global sections F_i of $\mathrm{Sym}^{d_i}(U^\vee \otimes O_{G(2,n+1)})$. Here $S^\vee$ is the universal rank 2 locally free quotient of the trivial free sheaf $U^\vee \otimes O_{G(2,n+1)}$ on the Grassmanian $G(2, n+1)$. We are abusing notation here slightly - $\mathrm{Sym}^{d_i}(S^\vee)$ is a sheaf on $G(2, n+1)$. But there is a natural projection from $F((1,2), n+1)$ to $G(2, n+1)$, and via this projection, we pull back $\mathrm{Sym}^{d_i}(S^\vee)$ to $F((1,2), n+1)$. We can then pull back further by pr_1 to make sense of $\mathrm{pr}_1^* \bigoplus_{i=1}^c \mathrm{Sym}^{d_i}(S^\vee)$. The section σ is linear on every fiber of pr_1 over a point of $F((1,2), n+1)$, and thus the fibers of pr_1 are projective linear spaces of the factor $\mathbb{P}V_{\underline{d}}$. Moreover, the pushforward via pr_1 of σ is adjoint to a morphism w of locally free sheaves from $\bigoplus_{i=1}^c \mathrm{Sym}^{d_i}(S)$ to the trivial locally free sheaf $V_{\underline{d}}^\vee \otimes O_{F((1,2),n+1)}$, and the zero scheme of the σ equals the relative Proj over $F((1,2), n+1)$ of the symmetric algebra of the cokernel of w. Denote the cokernel of w by $\mathcal{F}$. Note that $\mathcal{F}$ is a coherent sheaf.

This coherent sheaf $\mathcal{F}$ is equivariant under the natural action of the automorphism group $PGl(n+1)$ of $\mathbb{P}^n$ on $F((1,2),n+1)$. There is a maximal dense open subset, say $V \subset F((1,2),n+1)$ over which $\mathcal{F}$ is locally free. Since $\mathcal{F}$ is equivariant under the action of $PGl(n+1)$, V is invariant under the action of $PGl(n+1)$ on $F((1,2),n+1)$. But we know that the action of $PGl(n+1)$ on $F((1,2),n+1)$ is transitive, and so the only non-empty open subset of $F((1,2),n+1)$ which is invariant under this action is the whole space. Thus, $\mathcal{F}$ is locally free on $F((1,2),n+1)$, which implies that $F_{0,1}(\mathcal{X})$ is a projective space bundle over the $F((1,2),n+1)$. Being a projective bundle over a smooth base, $F_{0,1}(\mathcal{X})$ is smooth. □

Let $\mathcal{U} \subset \mathcal{X}$ be the set:

$$\mathcal{U} = \{(X,p) \in \mathcal{X} | \dim(\rho^{-1}(X,p)) \leq n - (d_1 + d_2 + ...d_c) - c\}.$$

$\mathcal{U}$ is Zariski open, by [12, Ch 2,Ex 3.22].

Lemma 4.3. *The fiber $\rho^{-1}(X,p)$ is non empty for every $(X,p) \in \mathcal{U}$.*

Proof. We note that $\rho^{-1}(X,p)$ can be identified as the subscheme $F(X,p)$ of $G(2,n+1)$ parametrizing lines on X passing through p.

Suppose the point p parametrizes the 1 dimensional vector subspace W of our original vector space U. Then the set of lines passing through p in $\mathbb{P}^n$ is parametrized by a certain Schubert cycle σ in $G(2,n+1)$. One observes that this Schubert cycle is isomorphic to the projective space:

$$P = \mathbb{P}(U/W)$$

Thus, $F(X,p)$ is the subscheme of P consisting of lines lying on X. Recall that $X = V(F_1, \cdots, F_c)$. Thus, X is the intersection of the hypersurfaces $V(F_i)$, for $i = 1,2,...,c$. Let us choose homogenous co-ordinates $[x_0, x_1, ..., x_n]$ on $\mathbb{P}^n$ such that $p = [0,0,0,...,1]$. Let us now do a power series expansion of the polynomials $F_1, F_2, ..., F_c$ around the point p in this co-ordinate system. We thus have:

$$F_1 = x_n^{d_1} F_{1,0} + x_n^{d_1-1} F_{1,1} + ... + F_{1,d_1}$$
$$F_2 = x_n^{d_2} F_{2,0} + x_n^{d_2-1} F_{2,1} + ... + F_{2,d_2}$$
$$.$$
$$.$$
$$.$$
$$F_c = x_n^{d_c} F_{c,0} + x_n^{d_c-1} F_{c,1} + ... + F_{c,d_c}$$

where each $F_{i,j}$ is a homogeneous polynomial of degree j in the co-ordinates $x_0, ..., x_{n-1}$. Note,the fact that p lies on X imposes the restriction $F_{i,0} = 0$ for $i = 1, 2, ..., c$.

A line L through p can be written parametrically in our co-ordinate system as the set $\{[a_0 s, a_1 s, ..., a_{n-1} s, t] | [s, t] \in \mathbb{P}^1\}$. Plugging these into the defining equation F_1 of the hypersurface $V(F_1)$, we see

$$F_1(a_0 s, a_1 s, ..., a_{n-1} s, t) = s^{d_1 - 1} t F_{1,1}(a_0, a_1, ..., a_{n-1}) + ... + F_{1,d_1}(a_0, ..., a_{n-1})$$

So, the line L lies on $V(F_1)$ iff $F_{1,j}(a_0, a_1, ..., a_{n-1}) = 0$ for $j = 1, 2, ..., d_1$. So, $F(V(F_1), p)$ is defined by d_1 homogenous equations in the $n-1$ dimensional projective space P. Hence, $F(V(F_1), p)$ is non empty and dim $F(V(F_1), p) \geq n - d_1 - 1$. Arguing similarly for each $i = 2, 3, ..., c$, we see that dim $F(V(F_i), p) \geq n - d_i - 1$. Moreover,

$$F(X, p) = F(V(F_1, p)) \cap F(V(F_2, p)) \cap ...F(V(F_c, p))$$

By our hypothesis $n - (d_1 + d_2 + ... + d_c) - c \geq 0$. Hence, using induction and [12] [Chapter 1, Theorem 7.2], $F(X, p)$ is non empty and has dimension greater than or equal to $n - (d_1 + d_2 + ... + d_c) - c$. □

We now state and prove the main theorem of this section.

Theorem 4.1. *Fix a non-increasing sequence of positive integers* $(d_1, d_2, ..., d_c)$. *If these integers satisfy the system* (A) *of inequalities, for a general complete intersection* $X \in \mathbb{P}V_{\underline{d}}$ *the morphism*

$$\rho_X : F_{0,1}(X) \longrightarrow X$$

is flat with fiber dimension $n - (d_1 + d_2 + ...d_c) - c$.

Proof. Let $\pi : \mathcal{X} \longrightarrow \mathbb{P}^n$ be the projection map. We note that $\mathcal{X}$ is projective bundle over $\mathbb{P}^n$. Indeed, consider the evaluation map:

$$ev : V_{\underline{d}} \otimes O_{\mathbb{P}^n} \longrightarrow \oplus_{i=1}^c O(d_i).$$

Let F be the kernel of the map ev. Then F is a locally free sheaf. We form the projective bundle $\mathbb{P}F$. The closed points of $\mathbb{P}F$ parameterize tuples $(F_1, F_2, ..., F_c, p)$ where $(F_1, F_2, ..., F_c) \in \mathbb{P}V_{\underline{d}}$ such $F_i(p) = 0$ for $i = 1, 2, ..., c$. Thus, $\mathbb{P}F = \mathcal{X}$. The fiber over $p \in \mathbb{P}^n$ can be viewed as a linear subvariety inside $\mathbb{P}V_{\underline{d}}$ corresponding to complete intersections containing p, once we restrict to the appropriate open subset of $\mathbb{P}V_{\underline{d}}$. So $\mathcal{X}$ is a smooth scheme and $\mathcal{U}$, being an open dense subset of a smooth scheme, is also smooth.

Let $\mathcal{U}$ be as in lemma 4.3. By definition of $\mathcal{U}$, the fiber dimension of

$$\rho|_{\rho^{-1}(\mathcal{U})} : \rho^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$$

is constant and equal to $n-(d_1+d_2+...+d_c)-c$. Since $F_{0,1}(\mathcal{X})$ is smooth, by Lemma 4.2, so is the open subset $\rho^{-1}(\mathcal{U}) \subset F_{0,1}(\mathcal{X})$. In particular, the source of the above morphism is Cohen Macaulay. Moreover the target $\mathcal{U}$ is smooth by our previous arguments and as such, is a regular scheme. As we mentioned above, the dimension of the fiber of ρ over U is constant. So by [15, Theorem 23.1] , ρ is flat over $\mathcal{U}$.

Let $Y \subset \mathcal{X}$ be the complement of $\mathcal{U}$. Y precisely consists of pairs (X,p) such that the map

$$\rho_X : F_{0,1}(X) \longrightarrow X$$

does not have the expected fiber dimension. Note, Y is a closed subscheme of a projective scheme, and as such, if we show that $\pi_0|_Y : Y \longrightarrow \mathbb{P}V_{\underline{d}}$ is not surjective then its image must be a proper closed subset of the the target. In other words, there is a dense open subset of $\mathbb{P}V_{\underline{d}}$ consisting of complete intersections for which ρ_X does have the expected fiber dimension and is flat.

Let us denote by e the codimension of $\mathcal{Y}$ in $\mathcal{X}$. Note that the fiber dimension of

$$\pi_0 : \mathcal{X} \longrightarrow \mathbb{P}V_{\underline{d}}$$

is $n-c$. To show that $\pi_0|_Y : Y \longrightarrow \mathbb{P}V_{\underline{d}}$ is not surjective, it is sufficient to show that $e \geq n-c+1$. Note that $\mathcal{X}$ also has a natural projection map to $\mathbb{P}^n$,

$$\pi_1 : \mathcal{X} \longrightarrow \mathbb{P}^n$$

given by $\pi_1(X,p) = p$. We note that $e \geq \operatorname{codim}(Y \cap \pi_1^{-1}(p), \pi^{-1}(p))$. To prove that $e \geq n-c+1$, it is thus sufficient to show that

$$\operatorname{codim}(Y \cap \pi_1^{-1}(p), \pi_1^{-1}(p)) \geq n-c+1.$$

We now return to the problem at hand. Let us choose homogenous co-ordinates $[x_0, x_1, ..., x_n]$ on $\mathbb{P}^n$ such that $p = [0,0,0,...,1]$ and do a power series expansion of the polynomials $F_1, F_2, ..., F_c$ around the point p as in lemma 4.3. As before, we have:

$$F_1 = x_n^{d_1} F_{1,0} + x_n^{d_1-1} F_{1,1} + ... + F_{1,d_1}$$
$$F_2 = x_n^{d_2} F_{2,0} + x_n^{d_2-1} F_{2,1} + ... + F_{2,d_2}$$
$$.$$
$$.$$
$$.$$
$$F_c = x_n^{d_c} F_{c,0} + x_n^{d_c-1} F_{c,1} + ... + F_{c,d_c}$$

where each $F_{i,j}$ is a homogeneous polynomial of degree j in the co-ordinates $x_0, ..., x_{n-1}$.

The point p lying on our complete intersection X in this co-ordinate system forces $F_{i,0} = 0$ for $i = 1, 2, ..., c$. Moreover, in the projective space P parametrizing lines through p mentioned above, the Fano Scheme $F(X, p)$ is defined precisely by the equations $F_{i,j}$ for $1 \leq i \leq c$ and $1 \leq j \leq d_i$.

We note that the fiber $\pi_1^{-1}(p)$ can be identified with subscheme of $\mathbb{P}V_{\underline{d}}$ parametrizing complete intersections containing p. This is in fact a linear subspace of $\mathbb{P}V_{\underline{d}}$ of codimension c. As such, it is a projective space of dimension c lower than the dimension of P. For brevity, we denote this space by H_p. The subscheme $Y \cap \pi_1^{-1}(p)$ is precisely the locus of complete intersections containing p, for which the equations $F_{i,j}$ cut out something of higher than expected dimension (or equivalently, lower than expected codimension) in P.

We now follow the induction argument outlined in [11] to show that the locus $Y \cap \pi_1^{-1}(p)$ indeed has codimension greater than or equal to $n-c+1$ in $\pi_1^{-1}(p)$ as long as our inequalities are satisfied.

As mentioned above, $F(X, p) = V(F_{1,1}, ..., F_{1,d_1}, F_{2,1}, ..., F_{2,d_2}, ..., F_{c,d_c})$. The order of the defining equations is irrelevant, and as such, we may rearrange them to group them together by degree. In other words, we may write

$$F(X, p) = V(F_{1,1}, F_{2,1} ..., F_{b_1,1}, F_{1,2}, F_{2,2}, ..., F_{b_2,2}, ..., F_{b_{d_1},d_1})$$

Here, all the linear terms are written first and by our comments in section 1.2, there are b_1 of them. Immediately after the linear terms, we have the quadratic terms and as before, there are b_2 of them. We continue this process until we end with the b_{d_1} terms of degree d_1. [A small note here: We might as well assume all $b_i \geq 2$, because if not, then the zero locus of the linear F_i-s is just a linear subspace in $\mathbb{P}^n$ and as such, is a projective space of lower dimension. In this case, we would run the same argument on this lower dimensional projective space, where $b_i \geq 2$ is satisfied for all i].

It is useful to view the construction of $F(X, p) \subset P$ as an iterated process, which we now describe.

Let us look at the zero scheme $V(F_{1,1}) \subset H_p$. Then, let us restrict $F_{2,1}$ to $V(F_{1,1})$ and look at its zero set inside $V(F_{1,1})$. This is precisely equal to $V(F_{1,1}, F_{2,1})$. We then restrict $F_{3,1}$ to this subscheme and look at its zero set, which will of course be $V(F_{1,1}, F_{2,1}, F_{3,1})$. We repeat this procedure until we have cycled through all of the linear polynomials.

Once we have cycled through all of the linear parts, we have the subscheme $V(F_{1,1}, ..., F_{b_1,1})$. We then look at the quadratic parts. We begin by restricting the first quadratic polynomial $F_{1,2}$ to $V(F_{1,1}, ..., F_{b_1,1})$ and looking at its zero set, whereby we have constructed $V(F_{1,1}, ..., F_{b_1,1}, F_{1,2})$. We then repeat this with each of the $F_{j,2}$, for $j = 1, 2, ..., b_2$. We continue this process, cycling through all the homogeneous parts of a given degree, and subsequently moving to the next degree until we have exhausted all the degree $F_{i,j}$. At the end of this process,we have constructed $F(X, p)$.

For the Fano scheme is to have higher than expected dimension, it must happen at some stage of this iterative process that one of the $F_{i,j}$ when restricted to the previous subscheme fails to cut out something of the correct codimension. We now explicitly describe the locus of such 'bad' tuples $(F_1, F_2, ..., F_c)$.

Let $B_{i,j}$ be the subscheme which parametrizes c-tuples $(F_1, ..., F_c)$ inside H_p where $F_{i,j}$ fails to cut out a subscheme of the correct codimension when restricted to the previous subscheme in the iterative process. Thus,

$$Y \cap H_p = \cup_{i,j} B_{i,j}$$

Let us begin by looking at the polynomial $F_{1,1}$. This is, by definition, a linear polynomial in n variables. This fails to cut out a codimension 1 subvariety inside P if it is identically 0. Thus, $B_{1,1}$ parametrizes those tuples $(F_1, F_2, ..., F_c)$ such that $F_{1,1} \equiv 0$. This happens precisely when all the n coefficients of $F_{1,1}$ are 0. But the coefficients of $F_{1,1}$ are co-ordinates in the projective space H_p, and thus $B_{1,1}$ is a linear subvariety of H_p defined by n linear equations, and as such, satisfies

$$\mathrm{codim}(B_{1,1}, H_p) \geq n$$

Let us now look at the open complement of $B_{1,1}$, i.e., $H_p \setminus B_{1,1}$.

For $(F_1, F_2, ..., F_c) \in H_p \setminus B_{1,1}$, $F_{1,1}$ does not vanish identically, and therefore, $V(F_{1,1})$ is a subscheme of dimension $(n-1)-1$ in P. We now restrict $F_{2,1}$ to each irreducible component of $V(F_{1,1})$ [Note, $V(F_{1,1})$ is itself irreducible, being a linear subvariety, But once we cycle through all the linear parts and move to higher degree, this will no longer be the case. So for the sake of consistency, we mention irreducible components at this stage.] The only way $F_{2,1}$ does not cut out a codimension 1 subscheme of $V(F_{1,1})$ is if it vanished identically on some irreducible component, say C_1 of dimension $(n-1)-1$ of $V(F_{1,1})$. This is equivalent to saying that the image of $F_{2,1}$ in the homogeneous co-ordinate ring $S(C_1)$ of C_1 is 0 under the projection morphism from $S(P)$.

The fact that $F_{2,1}$ has degree 1 implies that its image must lie in the degree 1 part of $S(C_1)$. If we expand the image of $F_{2,1}$ in $(S(C_1))_1$ in terms of a basis, then the coefficients will be linear polynomials in the coefficients of $F_{2,1}$. The image of $F_{2,1}$ being 0 entails that these linear polynomials must all vanish. That is to say, $B_{2,1} \setminus B_{1,1}$ is cut out in $H_p \setminus B_{1,1}$ by the vanishing of these linear polynomials. The number of such linear polynomials equals the dimension of the vector space $(S(C_1))_2$. By Lemma 4.1,

$$h_{C_1}(1) \geq \binom{n-2+1}{1}$$

The above restriction on the Hilbert Function tells us that there are at least $\binom{n-2+1}{1}$ such linear conditions. Thus,

$$\operatorname{codim}(B_{2,1} \setminus B_{1,1}, H_p \setminus B_{1,1}) \geq \binom{n-2+1}{1}$$

We now look at the quasiprojective variety $H_p \setminus (B_{1,1} \cup B_{2,1})$. For c-tuples $(F_1, ..., F_c) \in H_p \setminus (B_{1,1} \cup B_{2,1})$, $V(F_{1,1}, F_{2,1})$ has dimension $n-3$. We now restrict $F_{3,1}$ to $V(F_{1,1}, F_{2,1})$. By the exact same argument as before $F_{3,1}$ being in the bad locus $B_{3,1}$ entails that $F_{3,1}$ vanishes identically on some irreducible component of dimension $(n-3)$ of $V(F_{1,1}, F_{2,1})$. Just as before, this is equivalent to the image of $F_{3,1}$ being 0 in the degree 1 part of the coordinate ring of the irreducible component. As before, we translate this into the vanishing of some linear polynomials in the coefficients of $F_{3,1}$. Hence, $B_{1,3}$ is defined by the vanishing of linear polynomials. By Lemma 4.1 there are at least $\binom{n-3+1}{1}$ such conditions. Hence,

$$\operatorname{codim}(B_{3,1} \setminus (B_{1,1} \cup B_{2,1}), H_p \setminus (B_{1,1} \cup B_{2,1}) \geq \binom{n-3+1}{1}.$$

Continuing this procedure until we exhaust all b_1 linear parts and repeatedly applying Lemma 4.1, We will get that

$$\operatorname{codim}((B_{i,1} \setminus (\bigcup_{j=1}^{j=i-1} B_{j,1})), (H_p \setminus (\bigcup_{j=1}^{j=i-1} B_{j,1})) \geq \binom{n-i+1}{1} \text{ for } i = 1, 2, ..., b_1$$

Note that the smallest lower bound for codimension occurs at the last step in the process, since the binomial coefficient $\binom{m}{s}$ decreases with m if s is held fixed.
The smallest lower bound for the codimension is $\binom{n-c+1}{1} = n-c+1$. Thus, at every step, the codimension of the bad locus inside H_p is larger than or equal to $n-c+1$, which is what we required.

Let $\Delta_1 = \bigcup_{j=1}^{j=b_1} B_{j,1}$. For tuples $(F_1, ..., F_c) \in H_p \setminus \Delta_1$, $V(F_{1,1}, F_{2,1}, ..., F_{b_1,1})$ has dimension precisely $n - b_1 - 1$. We now look at the quadratic parts. Restrict $F_{1,2}$ to $V(F_{1,1}, F_{2,1}, ..., F_{b_1,1})$ and look at each irreducible component. $F_{1,2}$ fails to cut out a subscheme of codimension 1 if and only if the restriction of $F_{1,2}$ to some irreducible component,say C_2, of $V(F_{1,1}, F_{2,1}, ..., F_{b_1,1})$ vanishes identically. Again, this can be expressed in terms of the image of $F_{1,2}$ being 0 in the homogeneous co-ordinate ring $S(C_2)$ of C_2. The only difference from the case of the linear parts, is that the image of $F_{1,2}$ lies in the vector space $(S(C_2))_2$, i.e., the degree 2 part of the grading. As before, this translates to the vanishing of linear equations in the coefficients of $F_{1,2}$ and hence, $B_{1,2} \setminus \Delta_1$ is defined by the vanishing of linear equations in $H_p \setminus \Delta_1$. The number of linear equations equals the dimension of $(S(C_2))_2$ as a vector space. Again, by Lemma 4.1, $h_{C_2}(2) \geq \binom{n-c-1+2}{2}$. Hence, the vector space dimension of $(S(C_2))_2$ is at least $\binom{n-c-1+2}{2}$ Thus,

$$\operatorname{codim}(B_{1,2} \setminus \Delta_1, H_p \setminus \Delta_1) \geq \binom{n-c-1+2}{2}$$

Now we remove $(B_{1,2} \cup \Delta_1)$ and look at the open complement $H_p \setminus (B_{1,2} \cup \Delta_1)$. For c-tuples of polynomials in this locus, $V(F_{1,1}, ..., F_{b_1,1}, F_{1,2})$ has dimension $(n - b_i - 1) - 1$. We now iterate this process for all of the quadratic polynomials and repeatedly use lemma 4.1. As mentioned earlier, because the binomial coefficients $\binom{m}{s}$ are decreasing in m if s is held fixed, we will get the least codimension at the last step, namely,

$$\operatorname{codim}(B_{b_2,2} \setminus (\bigcup_{j=1}^{j=b_2-1} B_{j,2} \cup \Delta_1), H_p \setminus (\bigcup_{j=1}^{j=b_2-1} B_{j,2} \cup \Delta_1)) \geq \binom{n-b_1-b_2+2}{2}$$

By our system (A) of polynomial inequalities, $\binom{n-b_1-b_2+2}{2} \geq n-c+1$, which is what we required. We now remove the locus $\Delta_2 = \bigcup_{j=1}^{j=b_2} B_{j,2} \cup \Delta_1$ and for c-tuples of polynomials in the complement, we cycle through the cubic terms as before. We then repeat the process as necessary, until we have exhausted all the homogeneous parts.

We keep in mind that for each degree e between 1 and d_1, the smallest codimension of the bad locus will occur when we reach the last homogenous polynomial of degree e. Lemma 4.1 ensures that the codimension is at least as large as $\binom{n-b_1-b_2-...-b_e+e}{e}$. At every stage, our system (A) of polynomial inequalities ensures that the codimension of the bad locus is at least $n-c+1$.

Thus, for a general $X \in \mathbb{P}V_{\underline{d}}$, we know that the fiber dimension of

$$\rho_X : F_{0,1}(X) \to X$$

equals $n - (d_1 + d_2 + ... + d_c) - c$. Also, we know that ρ is flat over $\mathcal{U}$. Using the argument in the proof of [7], Theorem 2.1, we conclude that $\rho_X : F_{0,1}(X) \to X$ is flat for a general X with fiber dimension $n - (d_1 + d_2 + ... + d_c) - c$. Thus, the proof is complete.

□

5 Graphs and Trees

In the remainder of the paper, we will show that for a general complete intersection of multidegree $(d_1, d_2, ..., d_c)$ satisfying the polynomial inequalities (A), the Kontsevich space of stable maps $\overline{\mathcal{M}}_{0,r}(X, e)$ is an irreducible complete intersection stack of the expected dimension. To do so, we will use the technique discussed in [11]. To this end, it is necessary to introduce the notions of stable A-graphs and trees, which is done in the following section.

5.1 Notation

Definition 5.1. *A **graph** τ is the datum of a 4-tuple $(F_\tau, W_\tau, j_\tau, \delta_\tau)$, where*
(1) F_τ is a finite set called the set of flags,
(2) W_τ is a finite set of called the set of vertices
(3) $j_\tau : F_\tau \longrightarrow F_\tau$ is an involution
(4) $\delta_\tau : F_\tau \longrightarrow W_\tau$ is a map called the evaluation map.

In addition, we have the set $S_\tau \subset F_\tau$ of tails, which are the fixed points of the involution j_τ. The set E_τ of edges is the quotient of $F_\tau \setminus S_\tau$ by the action of j_τ. For a vertex $v \in W_\tau$ the valence of v is defined to be the cardinality of the set $\delta_\tau^{-1}(v)$. For simplicity, we will write $Flag(\tau), Vertex(\tau), Tail(\tau)$ and $Edge(\tau)$ for F_τ, W_τ, S_τ and E_τ respectively.

A graph τ can be realized geometrically as a 1 dimensional CW-complex $|\tau|$ in the following way:

To each element of $Vertex(\tau) \sqcup Tail(\tau)$ we associate a 0-simplex. The 1-simplices correspond to elements of $Edge(\tau) \sqcup Tail(\tau)$. Consider a 1-simplex $[0, 1]$ associated to an edge $\{f, \bar{f}\}$. The point 0 is glued to the 0 simplex $\delta_\tau f$ and the point 1 is glued to the 0-simplex $\delta_\tau \bar{f}$. Conversely, for a 1-simplex $[0, 1]$ associated to a tail f, 0 is glued to the 0 simplex $\delta_\tau f$ and 1 is glued to the 0-simplex f.

Definition 5.2. *A **tree** is a graph for which the associated simplicial complex satisfies $H_1(|\tau|, \mathbb{Z}) = 0$. This is equivalent to saying that $|\tau|$ has no closed loops.*

Definition 5.3. *An A-**graph** (τ, β_τ) is a graph τ, together with a map*

$$\beta_\tau : Vertex(\tau) \longrightarrow \mathbb{N} \cup \{0\}$$

which we call the A-structure map. We shall write τ instead of (τ, β_τ) to simplify notation. An A-graph τ is said to be stable if for each vertex $v \in Vertex(\tau)$ such that $\beta_\tau = 0$, the valence of v is at least 3.

Definition 5.4. *For an A-graph τ, we associate the following integers:*

$$\beta(\tau) = \sum_{v \in \tau} \beta(v)$$

$$E(\tau) = \max_{v \in \tau} \beta(v)$$

5.2 Some important examples of A-graphs

1. One important example of a graph is the empty tree, $\tau_{\emptyset}$. This is the tree with underlying vertex set, $Vertex(\tau) = \emptyset$.

2. For each pair of non-negative integers r and e, consider the tree τ_r with one vertex, i.e., $Vertex(\tau_r) = \{v\}$, and $Tail(\tau_r) = \{f_1, f_2, ..., f_r\}$. In this case, we define the A-structure by setting $\beta(v) = e$. We refer to this A-graph as $\tau_r(e)$. Thus, $\tau_r(e)$ is a stable graph iff either $e \geq 0$ or $r \geq 3$.

3. For a pair of non-negative integers r_1, r_2, let τ_{r_1,r_2} be the graph obtained by joining the vertices v_1 and v_2 of the two trees τ_{r_1} and τ_{r_2} by a single edge. If in addition we have A-structure maps of the individual trees $\lambda_{r_1}(e_1)$ and $\tau_{r_2}(e_2)$ as in example (2), we can define an A-structure on τ_{r_1,r_2} by setting $\beta(v_1) = e_1$ and $\beta(v_2) = e_2$.

5.3 The Category of A-graphs

As in the paper by [3], we can define a category whose objects are stable A-graphs, in which morphisms are compositions of the following two basic types.We refer the reader to [3] for the precise definitions.

A *contraction* $\alpha : \sigma \to \tau$ is a pair of maps:

$$\alpha_V : Vertex(\sigma) \to Vertex(\tau)$$

$$\alpha_F : Flag(\tau) \to Flag(\sigma)$$

such that α_V is surjective and α_F is injective, such that for $w \in Vertex(\tau)$, $\beta(w) = \sum_{v \in \alpha_v^{-1}(w)} \beta(v)$. A contraction α collapses edges of σ to obtain τ, while preserving adjacency, i.e., the image of two vertices joined by an edge in the source are also joined by an edge in the target.

A *combinatorial* morphism $\alpha : \tau \hookleftarrow \sigma$ is the inclusion of the subgraph σ into τ. We adopt the convention used in [11] and write the arrow backwards. (We will justify this notation shortly.)

6 Prestable Curves, A-graphs and strict τ-maps

Definition 6.1. *A* ***prestable*** *curve with r marked points is a tuple of the form* $(C, \{x_1, , ..., x_r\})$ *where* C *is a proper, reduced,connected, at worst nodal curve and* $x_1, x_2, ..., x_r$ *distinct, non-singular points of* C*.*

Suppose we are given a pre-stable curve $(C, \{x_1, x_2, ..., x_r\})$ of arithmetic genus 0. To it, we associate an dual graph $\Delta(C, x)$ as follows:

To each irreducible component of C_i of C, we associate a vertex v_i. Two vertices are joined by an edge if the irreducible components corresponding to them meet at a node. The tails correspond to the marked points of the curve. One observes that this graph is in fact a tree.

Let (X, L) be a polarized variety.

Definition 6.2. *A* ***pre-stable map*** *is a pair*

$$((C, \{x_1, x_2, ..., x_r\}), h : C \longrightarrow X)$$

where $(C, \{x_1, x_2, ..., x_r\})$ *is a pre-stable curve and* $h : C \longrightarrow X$ *is a morphism of schemes over* $\mathbb{C}$*.*

To a pre-stable map h as above, we can associate an A-graph $\Delta(C, x, h)$. The underlying graph is simply $\Delta(C, x)$. For each vertex v of $\Delta(C, x)$, we look at the corresponding irreducible component C_v. Define the A-structure by setting $\beta(v) = deg(h^*(L)|_{C_v})$. We say the pre-stable map $((C, \{x_1, x_2, ..., x_r\}), h : C \longrightarrow X)$ is stable if the A-graph $\Delta(C, x, h)$ is stable.

Definition 6.3. *Let* X *be a variety,* L *a line bundle on* X *and* τ *a stable* A*-graph. A* ***strict*** τ***-map*** *is the datum:*

$$((C_v), (h_v : C_v \to X), (q_f))$$

defined as follows:

- (C_v) *is a set of rational curves parameterized by* $v \in Vertex(\tau)$
- $(h_v : C_v \to X)$ *is a set of morphisms of* $\mathbb{C}$*-schemes parameterized by* $v \in Vertex(\tau)$
- (q_f) *is a set of closed points* $q_f \in C_{\delta f}$ *parameterized by* $f \in Flag(\tau)$

and satisfying the following conditions:

- *For* $v \in Vertex(\tau)$ *the degree of* $h_v^*(L)$ *as a line bundle on* C_v *is* $\beta_\tau(v)$*,*
- *for* $f_1, f_2 \in Flag(\tau)$ *distinct flags with* $\delta f_1 = \delta f_2$*,* $q_{f_1} \neq q_{f_2}$
- *for* $f \in Flag(\tau)$*, we have* $h_{\delta f}(q_f) = h_{\delta \overline{f}}(q_{\overline{f}})$*.*

Definition 6.4. *A **family** of strict τ-maps over a base B is a tuple (π, h, q) where,*

- $\pi_v : C_v \longrightarrow B$ *is a set of smoooth, proper morphisms, one for each vertex v of τ, whose fibres are rational curves.*
- h *is a collection of maps $h_v : C_v \longrightarrow X$ such on each geometric fiber of π_v, $h_v^*(L)$ has degree $\beta(v)$.*
- q *is a collection of maps $q_f : B \longrightarrow C_{\delta f}$ such that:*

  1. *$\pi_{\delta f} \circ q_f = id_B$, i.e. q_f is a section of $pi_{\delta f}$.*
  2. *If f and f' are distinct flags, q_f is disjoint from q'_f.*
  3. *if $f_2 = j_\tau(f_1)$, then $h_{\delta f_1} \circ q_{f_1} = h_{\delta f_2} \circ q_{f_2}$*

Definition 6.5. *If $\xi = (\pi, h, q)$ and $\eta = (\pi', h', q')$ are two families of strict τ maps over the same base, a **morphism of families of strict τ-maps** $\phi : \xi \longrightarrow \eta$ is a collection of isomorphisms $\phi_v : C_v \longrightarrow C'_v$ indexed by $Vertex(\tau)$ such that:*

- $h'_v \circ \phi_v = h_v$
- $\phi_{\delta_f} \circ q_f = q'_f$

We can define the composition of morphisms in the obvious way. Given a scheme S, the category of families of strict τ maps over S is thus a groupoid. Given a morphism of schemes $u : S' \longrightarrow S$, we can pull back families of strict τ maps over S' to S by considering the pullback along u. In this way, we have the functor $\mathcal{M}(X, \tau)$ from the category of $\mathbb{C}$-schemes to the category of groupoids which associates to each scheme the families of strict τ maps over that scheme.

One can observe that $\mathcal{M}(X, \tau)$ is a stack in groupoids over $\mathbb{C}$. We refer the reader to [3] for a more detailed construction. In what follows, we will outline what is necessary for our discussion. The first property to note is that when X is projective and L is ample, $\mathcal{M}(X, \tau)$ is a Deligne-Mumford stack [11, Theorem 3.10], and has a modular compactification $\overline{\mathcal{M}}(X, \tau)$ corresponding to the irreducible components degenerating to a reducible curves.

Definition 6.6. *Suppose X is a projective variety and τ is a stable A-graph. Define the **evaluation map***

$$ev_f : \overline{\mathcal{M}}(X, \tau) \longrightarrow X$$

by sending a family (π, h, q) of strict τ-maps to $h_{\delta_f} \circ q_f$.

We note that $\overline{\mathcal{M}}(X, \tau_0(1))$ is just the classical Fano variety $F_{0,1}(X)$, and the evaluation map is the morphism ρ_X in Theorem 4.1.

Given a projective variety X, and a contraction $\alpha : \sigma \longrightarrow \tau$ of A-graphs, the functor $\overline{\mathcal{M}}(X, \alpha) : \overline{\mathcal{M}}(X, \sigma) \longrightarrow \overline{\mathcal{M}}(X, \tau)$ forgets the labelling of some components of the domain curve. By $\mathcal{M}(X, \alpha)$, we will mean its restriction to the open substack $\mathcal{M}(X, \tau)$.

Given a combinatorial morphism $\phi : \tau \hookleftarrow \sigma$, the contravariant functor

$$\mathcal{M}(X, \phi) : \overline{\mathcal{M}}(X, \tau) :\longrightarrow \overline{\mathcal{M}}(X, \sigma)$$

is the forgetful morphism that remembers those components of τ-maps contained in σ. The fact that this functor is contravariant explains our convention of drawing the arrow backwards.

7 Flatness and Dimension results for $\mathcal{M}(X, \tau)$

Let (X, L) be a polarized variety in $\mathbb{P}^n$ with $K_X \overset{num}{=} mL$ we define the expected dimension of $\mathcal{M}(X, \tau)$ to be the positive integer

$$dim(X, \tau) = -m\beta(\tau) + \dim X - 3 + \#Tail(\tau) - \#Edge(\tau) \tag{4}$$

We say that $\mathcal{LCI}(X, \tau)$ holds if $\mathcal{M}(X, \tau)$ is a local complete intersection stack of dimension exactly $dim(X, \tau)$. As we prove later, $\mathcal{M}(X, \tau)$ is automatically a local complete intersection stack if every irreducible component has the expected dimension.

Theorem 7.1. *Every irreducible component of $\mathcal{M}(X, \tau)$ has dimension at least $dim(X, \tau)$.*

Proof. The proof is exactly identical to that of [11]. The key idea is to show that $\overline{\mathcal{M}}(X, \tau)$ is an open substack of the relative morphism-stack. The result of [13][Thoerem 2.17.1], then gives a lower bound on the dimension of the scheme of morphisms which establishes the aforementioned result. □

Given a stable A-graph τ and complete intersection X in $\mathbb{P}^n$, we say that $\mathcal{FE}(X, \tau)$ holds if the evaluation map has the fiber dimension $dim(X, \tau) - dim(X)$. We will see momentarily, that in this case, the evaluation map is also flat.

Lemma 7.1. *For a complete intersection $X \subset \mathbb{P}^n$,*

1. *$\mathcal{M}(X, \tau)$ is a local complete intersection stack iff every irreducible component has the expected dimension.*

2. *The evaluation map is flat iff $\mathcal{FE}(X, \tau)$ holds.*

Proof. For both the above claims, one of the directions is obvious. For the harder direction, we outline the main idea of the proof given in [11]. Suppose that X is defined $\mathbb{P}^n$ by polynomials of $F_1, F_2, ..., F_c$ degree $d_1, ..., d_c$. The defining equations of X give a section of the vector bundle $O(d_1) \oplus O(d_2) ... \oplus O(d_c)$. $\mathcal{M}(\mathbb{P}^n, \tau)$ is smooth by [3]. Consider the universal curve $\pi : \mathcal{C} \longrightarrow \mathcal{M}(\mathbb{P}^n, \tau)$. If $h : \mathcal{C} \longrightarrow \mathbb{P}^n$ is the universal map, the sections F_i pull back to a give section σ of $\pi_*(h^*(O(d_1) \oplus O(d_2) ... \oplus O(d_c)))$ whose vanishing locus is exactly $\mathcal{M}(X, \tau)$.

If every irreducible component of $\mathcal{M}(X,\tau)$ has the expected dimension, then its codimension equals the rank of the locally free sheaf $\pi_*(h^*(O(d_1)\oplus O(d_2)...\oplus O(d_c)))$, which proves that it is a local complete intersection. If $\mathcal{FE}(X,\tau)$ holds, then in particular $\mathcal{LCI}(X,\tau)$ holds. In this case, the evaluation map is a dominant morphism from a Cohen Macaulay scheme to a smooth scheme of constant fiber dimension. Thus by using the Local Flatness Theorem [15, Theorem 23.1], we conclude that the evaluation map is flat. □

We end this section with the following result, which appears as [11] [Proposition 4.8].

Lemma 7.2. *Suppose that τ is a stable A-graph with $E(\tau)=E$. If for e=0,...,E we have $\mathcal{FE}(X,\tau_1(e),f)$, then for every flag $f\in Flag(\tau)$ we have that $\mathcal{FE}(X,\tau,f)$ holds.*

The proof uses induction on the number of vertices. We refer the reader to [11] for details.

8 The irreducible components of $\mathcal{M}(X,\tau)$

In this section, we summarize the results in [11, Section 5]. The last result of the previous section gave us a criterion for checking the flatness of the evaluation map for a general stable A- graph τ. Here, we will reduce this to checking a finite number of cases. We will also use specializations to better understand the irreducible components of $\mathcal{M}(X,\tau)$.

We have the following version of Bend-and-Break:

Lemma 8.1. *[11, Lemma 5.1] Let X be a projective variety and let τ be a stable A-graph. If $f_1,f_2\in Flag(\tau)$ then there is no complete curve C in any fiber of the map*

$$ev_{f_1,f_2}:\mathcal{M}(X,\tau_2(e))\longrightarrow X\times X$$

This allows us to conclude the following:

Lemma 8.2. *If $X\subset\mathbb{P}^n$ is a complete intersection. Suppose that $\mathcal{FE}(X,\tau_1(e),f_1)$ holds for $e<E$ and that every irreducible component of $\mathcal{M}(X,\tau_1(E))$ has dimension at least $2dim(X)$. Then $\mathcal{FE}(X,\tau_1(e),f_1)$ holds.*

Definition 8.1. *Let $X\subset\mathbb{P}^n$ be a complete intersection of multidegree $(d_1,d_2,...,d_c)$. We define the **threshold degree** of X to be*

$$E(X)=\left\lfloor\frac{n+2-c}{n+1-d_1-d_2-...-d_c}\right\rfloor \tag{5}$$

As promised, we will now reduce checking flatness of the evaluation map for complete intersections to a finite number of cases.

Lemma 8.3. *Let $X\subset\mathbb{P}^n$ be a complete intersection of multidegree $(d_1,d_2,...,d_c)$ and let τ be a stable A-graph. If $\mathcal{FE}(X,\tau_1(e),f_1)$ holds for every $1\le e\le E(X)$, then for every $f\in Flag(\tau)$, $\mathcal{FE}(X,\tau,f)$ holds.*

Proof. By Lemma 7.2, to establish $\mathcal{FE}(X,\tau,f)$, it is sufficient to show $\mathcal{FE}(X,\tau_1(e),f_1)$ for $e=1,...,E(\tau)$. Suppose $\mathcal{FE}(X,\tau_1(e),f_1)$ holds for $1\le e\le E(X)$. By Theorem 4.1, for $e>E(X)$, we have that every component of $\mathcal{M}(X,\tau_1(e))$ has dimension at least

$$n+1-(d_1+d_2+...+d_c)e-(n-c-3)+1\ge 2n-2c$$

which is the dimension of X, since X is a complete intersection. So by repeated applications of lemma 8.2, we have $\mathcal{FE}(X,\tau_1(e),f_1)$ for $e\le E(\tau)$. Thus, we are done. □

Definition 8.2. *A stable A-graph τ is said to be* ***basic*** *if for every $v\in Vertex(\tau)$, $\beta(v)\le E(X)$ or equivalently, if $E(\tau)\le E(X)$.*

Definition 8.3. *For a stable A graph τ define the* ***degree zero subgraph*** *τ^0 to be the maximal subgraph τ^0 of τ such that every vertex of τ^0 has degree 0, i.e.,*

$$Vertex(\tau^0)=\{v\in Vertex(\tau)|\beta(v)=0\}$$
$$Flag(\tau^0)=\{f\in Flag(\tau)|\delta_\tau f\in Vertex(\tau^0)\}$$

Definition 8.4. *A contraction $\alpha:\sigma\longrightarrow\tau$ is said to be a* ***nice contraction*** *if it induces an isomorphism between the degree 0 subgraphs of σ and τ, i.e. $\sigma^0\cong\tau^0$*

We can now state the most important result of this section:

Theorem 8.1. *[11, Theorem 5.10] Let $X\subset\mathbb{P}^n$ be a complete intersection and τ be a stable A-graph. Let $M\subset\mathcal{M}(X,\tau)$ be an irreducible component. Suppose that $\mathcal{FE}(X,\tau_1(e),f_1)$ holds for $1\le e\le E(X)$. Then there exists a nice contraction $\phi:\sigma\longrightarrow\tau$ and an irreducible component $N\subset\mathcal{M}(X,\sigma)$ such that σ is basic and that $N\subset\overline{M}$.*

9 Properties of the evaluation Morphism

In this section, we will deduce some important properties of the evaluation map. In particular, we will show that the general fiber is irreducible and we will use it to show a correspondence between nice contractions and the irreducible components of the stack $\mathcal{M}(X,\tau)$

To this end, we will need the following results:

Lemma 9.1. *For a general complete intersection $X\in\mathbb{P}^n$ of type $(d_1,...,d_c)$, with $d_1+d_2+...+d_c\le n-1$ there is a line l such that the normal bundle $N_{l/X}$ is of the form*

$$O_l(1)^{n-1-(d_1+d_2+...+d_c)}\oplus O_l^{(d_1+d_2+...+d_c)-c}$$

Proof. Consider the incidence correspondence

$$\{(l,X)|l\subset X\}\subset G(2,n+1)\times\mathbb{P}V_{\underline{d}}$$

parametrizing pairs (l, X) of a line and a complete intersection containing the line. Suppose that $X = V(F_1, F_2, ..., F_c)$, where F_i has degree d_i in keeping with our choice of notation. We then have an exact sequence of locally free sheaves:

$$0 \longrightarrow N_{l/X} \longrightarrow O_l(1)^{n-1} \longrightarrow (N_{X/\mathbb{P}^n})|_l \longrightarrow 0 \tag{6}$$

Moreover, note that since X is a complete intersection in $\mathbb{P}^n$, $N_{X/\mathbb{P}^n} = \oplus_1^c O_X(d_i)$ Recall, by Grothendieck's lemma, that $N_{l/X} = \oplus_1^{n-c-1} O_l(a_i)$ for some integers a_i.Let us now twist down the equation (1) by $O_l(-1)$ and take the long exact sequence of cohomology corresponding to it. We will then have the following map as a part of the exact sequence:

$$H^0(l, O_l)^{n-1} \xrightarrow{\alpha} H^0(l, (N_{X/\mathbb{P}^n}(-1))|_l) \tag{7}$$

i.e., we have the map

$$H^0(l, O_l)^{n-1} \xrightarrow{\alpha} \oplus_1^c H^0(l, O_l(d_i - 1))$$

We choose co-ordinates $[x_0, ..., x_n]$ on $\mathbb{P}^n$ such that $l = V(x_2, ..., x_n)$ and do a power series expansion to say:

$$F_i = \sum_2^n x_j F_{i,j}$$

In these co-ordinates the matrix of α is given by $((F_{i,j})|_l)_{n \times c}$. Taking X general, we can say that this matrix has full rank, and hence, that α is surjective. This means that the next term in the exact sequence, i.e., $H^1(l, N_{l/X}(-1))$ is zero. By our previous arguments, since $N_{l/X} = \oplus_1^{n-c-1} O_l(a_i)$, we must have each $a_i = 0$ or 1. □

Lemma 9.2. *In characteristic* 0*, for a general complete intersection X in $\mathbb{P}^n$, $F_{0,1}(X)$ is smooth and the general fiber of ev_f is irreducible.*

Proof. We note that we have the projection map as previously defined:

$$\rho = (\pi_0, \pi_1) : F_{0,1}(\mathcal{X}) \longrightarrow \mathcal{X}$$

which sends tuples (X, p, L) to tuples to (X, p). For a given pair (X, p), the fiber $\pi_0^{-1}(X, p) = F_{0,1}(X)$. We argued in the proof of Theorem 4.1 that $\mathcal{X}$ is smooth, being a projective bundle over a smooth base. Note that $F_{0,1}(\mathcal{X})$ is a projective bundle over the partial flag variety $F((1, 2), n+1)$. Therefore, $F_{0,1}(\mathcal{X})$ is smooth. By our arguments in the proof of theorem 4.1, the map ρ is dominant. Thus, applying [12, Chapter 3, Theorem 10.7] we conclude that the general fiber $\pi_0^{-1}(X, p)$ is smooth and therefore, $F_{0,1}(X)$ is smooth for general X. Finally, by previous arguments, the fiber of ev_f is a complete intersection of ample hypersurfaces in a projective space of dimension $n - 1$. Hence, the fiber of ev_f is connected by [12, Chapter 3, Exercise 5.5] □

We now invoke the methods used in [11, Section 6].

Definition 9.1. *For a smooth subvariety $X \subset \mathbb{P}^n$ we say that $\mathcal{B}(X, \tau, f)$ holds if:*

1. *$\mathcal{FE}(X, \tau, f)$ holds.*

2. *The general fiber of the evaluation map ev_f is irreducible.*

3. *There is a point $[h : C \longrightarrow X]$ on $\mathcal{M}(X, \tau)$ which is free, i.e., h^*T_X is generated by global sections.*

With this setup, we have the following:

Lemma 9.3. *For a general complete intersection X of multidegree $(d_1, d_2, ..., d_c)$ satisfying our system (A) of inequalities, $\mathcal{B}(X, \tau_1(1), f_1)$ holds.*

Proof. Indeed, 1) of $\mathcal{B}(X, \tau_1(1), f_1)$ holds by Theorem 4.1. 2) follows from the lemma 9.2 above. Finally, 3) follows from lemma 9.1, since our system of inequalities necessitates that $d_1 + d_2 + ... + d_c < n - 1$. □

Lemma 9.4. *Let $X \subset \mathbb{P}^n$ be a smooth subvariety which satisfies $\mathcal{B}(X, \tau_1(e), f_1)$ for $e = 1, 2, ..., E$. Let τ be an A-graph such that $E(\tau) \leq E$. Then we have:*

1. *For each flag $f \in Flag(\tau)$, we have $\mathcal{B}(X, \tau, f)$*

2. *$\mathcal{M}(X, \tau)$ is an irreducible stack.*

Proof. This is just a restatement of [11, Proposition 6.8] and the proof is identical. □

Lemma 9.5. *Let $X \subset \mathbb{P}^n$ be a smooth subvariety which satsisfies $\mathcal{B}(X, \tau_1(e), f_1)$ for $e = 1, 2, ..., E$. Let τ be an A-graph such that $E(\tau) \leq E$. Suppose in addition that*

$$\alpha : \tau \longrightarrow \sigma$$

is a contraction. Then the morphism

$$\mathcal{M}(X, \alpha) : \mathcal{M}(X, \tau) \longrightarrow \mathcal{M}(X, \sigma)$$

maps a general point of $\mathcal{M}(X, \tau)$ to a smooth point of $\mathcal{M}(X, \sigma)$.

Proof. This is [11, proposition 6.8]. □

Suppose that we have a smooth subvariety $X \subset \mathbb{P}^n$ satisfying $\mathcal{B}(X, \tau_1(e), f_1)$ for $e = 1, 2, ..., E$ and $\alpha : \tau \longrightarrow \sigma$ is a contraction. Lemma 9.4 tells us that $\mathcal{M}(X, \tau)$ is an irreducible stack and lemma 9.5 tells us that $\mathcal{M}(X, \alpha)$ maps a general point of $\mathcal{M}(X, \tau)$ to a smooth point of $\mathcal{M}(X, \sigma)$.

We know by Theorem 8.1 that given an irreducible component $M \subset \mathcal{M}(X, \sigma)$ there is a nice contraction $\beta : \gamma \longrightarrow \sigma$ and an irreducible component $N \subset \mathcal{M}(X, \gamma)$ such that $A \subset \overline{M}$. So in particular, for the contraction $\alpha : \tau \longrightarrow \sigma$ above, $\mathcal{M}(X, \tau)$ is an irreducible stack by lemma 9.4 and it only has 1 irreducible component i.e., $N = \mathcal{M}(X, \tau)$ itself. So, we can say that the image of *the* irreducible component N is a subset of $\overline{M}$. Lemma 9.5 tells us that a general point of the image of N is smooth. Hence, N is contained in a unique irreducible component of $\mathcal{M}(X,\sigma)$. Summing up, we have shown that:

- Every irreducible component M of $\overline{\mathcal{M}}(X,\sigma)$ contains a basic component N
- As long as $\mathcal{B}(X,\tau_1(e),f_1)$ holds for $e = 1,2,...,E$ the basic components of N are contained in unique irreducible components.

In the above situation, for each nice contraction $\alpha : \tau \longrightarrow \sigma$ we can look at the unique irreducible component $M(\alpha)$ containing the image of $\mathcal{M}(\alpha)$. Finally, we have proposition 6.9 of [11]

Lemma 9.6. *Suppose $X \subset \mathbb{P}^n$ is a smooth complete intersection of threshold degree $E(X)$ satisfying:*

1. *$\mathcal{B}(X,\tau_1(1),f_1)$ holds,*
2. *$\mathcal{FE}(X,\tau_1(e),f_1)$ holds for $e = 1,2,...,E(X)$*
3. *$\mathcal{M}(X,\tau)$ is irreducible for $e = 1,...,E(X)$*

Then we have:

1. *For every basic A-graph τ with $E(\tau) \leq E$, and each flag $f \in Flag(\tau)$, $\mathcal{B}(X,\tau,f)$ holds.*
2. *For each stable A-graph τ and every contraction $\alpha : \sigma \longrightarrow \tau$ of a basic A graph σ there is a unique irreducible component $M(\alpha)$ of*
   $overline M(X,\tau)$ which contains the image of $\mathcal{M}(X,\alpha)$. Moreover, $M(\alpha)$ is smooth of the expected dimension at a general point of the image.
3. *$\overline{M}(X,\tau_0(e))$ is the union of irreducible components $M(\alpha)$ as $\alpha : \sigma \longrightarrow \tau$ ranges over nice contractions such that $E(\sigma) \leq 1$.*

Hence, to understand the irreducible components of $\mathcal{M}(X,\tau)$ it is sufficient to determine the nice contractions $\alpha : \sigma \longrightarrow \tau$ with $E(\sigma) \leq 1$

10 Equating Irreducible Components

In this section, we will follow the technique outlined in [11] to conclude that all the irreducible components of $\overline{\mathcal{M}}(X,\tau)$ are equal. Hitherto, we have estabilished that for every stable A graph τ each irreducible component of $\overline{\mathcal{M}}(X,\tau)$ corresponds to a nice contraction $\alpha : \sigma \longrightarrow \tau$ with $E(\sigma) \leq 1$. We will now show that all such nice contractions are isomorphic to each other.

Let $X \subset \mathbb{P}^n$ be a complete intersection satisfying the conditions of lemma 9.6. Suppose $\mathcal{B}(X,\tau_1(e),f_1)$ holds for $e = 1,2,...,E$ with $E \geq E(X)$. For a stable A-graph τ, let $S_E(\tau)$ be the set of isomorphism classes of nice contractions $\alpha : \sigma \longrightarrow \tau$ with $E(\sigma) \leq E$.Define two nice contractions σ and σ' to be related if there exists a contraction $\epsilon : \sigma \longrightarrow \sigma'$ with

$\alpha = \alpha \circ \epsilon$. This defines an equivalence relation on the set $S_E(\tau)$. We notice that if α and α' are equivalent, then the irreducible components $M(\alpha)$ and $M(\alpha')$ corresponding to α and α' are equal. We will now show that there is only one such equivalence class, and hence, only one irreducible component of $\mathcal{M}(X, \tau)$.

Definition 10.1. *Given $X \subset \mathbb{P}^n$ a smooth complete intersection as above, we define the* ***modified threshold degree*** *to be $E'(X) = max(E(X), 2)$.*

Theorem 10.1. *Let X be a smooth complete intersection satisfying $\mathcal{B}(X, \tau_1(e), f)$ for $e = 1, 2, ..., E'(X)$. Then $S_{E'(X)}(\tau_0(e))$ is a singleton set - i.e., $\overline{\mathcal{M}}(X, \tau_0(e))$ is irreducible.*

Proof. We will proceed by showing that all the nice contractions in the set $S_{E'(X)}(\tau_0(e))$ are equivalent to the nice contraction $\alpha_e : \sigma_e \longrightarrow \tau_0(e)$ where σ_e is a path, i.e., there are no vertices of degree greater than 2. We note that if we have any nice contraction $\beta : \sigma \longrightarrow \tau_0(e)$ then the number of vertices of σ is at most e.

We define the length of a path to be the number of vertices in the path and the diameter of a graph to be the maximum length of the paths in that graph. We note that if a graph with k vertices has diameter k, then the graph has to be path. We will use the fact to prove the assertion made in the previous paragraph. We have already stated that any nice contraction $\beta : \sigma \longrightarrow \tau_0(e)$ has at most e vertices. If σ has diameter e, then we are done. If not, we will construct a nice contraction $\gamma : \sigma' \longrightarrow \tau_0(e)$ which is related to β which has diameter strictly larger than that of σ. We can thus iterate this process until we attain a nice contraction with diameter of the source equal to e, i.e. until we obtain a path.

Now suppose we have a nice contraction $\alpha : \sigma \longrightarrow \tau_0(e)$ with $E(\sigma) = 1$. Let us choose a maximal path s in σ. If $s \neq \sigma$, then there is at least vertex v in s with valence at least 3. Pick an edge x which passes through v and is not an edge in the path s. Consider the nice contraction $\epsilon : \sigma \longrightarrow \rho$ which contracts the edge x to a single vertex. The nice contraction α factors through ρ as $\alpha_\rho : \rho \longrightarrow \tau_0(e)$.

The image of s is a path s_ρ which contains v. Construct a nice contraction $\epsilon : \sigma' \longrightarrow \tau$ as follows. We consider a path s' of length $diam(s) + 1$ with a nice contraction that collapses two adjacent vertices w_1 and w_2 to v. To get σ', we attach the subgraph $\rho \setminus s_\rho$ of ρ to s' suitably. The attachment is done by adding edges such that the vertices of s_ρ that are adjacent to vertices in $\rho \setminus s_\rho$ are also adjacent in σ'. Then there is a unique nice contraction $\epsilon : \sigma' \longrightarrow \tau$ such that ϵ restricted to s' is the contraction of the two vertices w_1 and w_2 above and is an isomorphism from $\sigma' \setminus s' \longrightarrow \rho \setminus s_\rho$. Define α' to be $\alpha_\rho \circ \epsilon$. Then $\alpha' : \sigma' \longrightarrow \tau_0(e)$ is a nice contraction with $diam(\sigma') = diam(\sigma) + 1$. Thus, using the iterative argument mentioned earlier, we are done. □

Corollary 10.1. *With the same hypotheses as the previous Theorem, for each stable A-graph τ we have:*

- $\overline{\mathcal{M}}(X,\tau)$ *is an integral, local complete intersection stack of the expected dimension and* $\mathcal{M}(X,\tau)$ *is the unique dense stratum in the Behrend-Manin decomposition.*

- *For each flag* $f \in Flag(\tau)$, $\mathcal{B}(X,\tau,f)$ *holds.*

- *For each contraction* $\alpha : \sigma \longrightarrow \tau$, $\overline{\mathcal{M}}(X,\tau)$ *is smooth at the general point of the image* $\mathcal{M}(X,\alpha) : \mathcal{M}(X,\sigma) \longrightarrow \overline{\mathcal{M}}(X,\tau)$.

Proof. The proof is identical to that of [11]. We only emphasize the fact that since $\overline{\mathcal{M}}(X,\tau)$ has the expected dimension, it is a local complete intersection stack by previous statements. Since it is generically smooth, it is reduced. So $\overline{\mathcal{M}}(X,\tau)$ is an integral, local complete intersection stack of the expected dimension. □

Theorem 10.2. *Suppose* $(d_1, ..., d_c)$ *satisfy our system of polynomial inequalities* (A). *For every such complete intersection* X *of multidegree* $(d_1, d_2, ..., d_c)$*, the modified threshold degree is* $E'(X) = 2$. *For a general such complete intersection* $\overline{\mathcal{M}}(X,\tau)$ *is an irreducible, local complete intersection stack of the expected dimension and hence* $\mathcal{M}(X,\tau)$*, being the unique dense stratum in the Behrend-Manin decomposition, is also irreducible of the expected dimension.*

Proof. We note firstly that for a general such X, by lemma 9.3, $\mathcal{B}(X,\tau_1(1),f_1)$ holds. We note that by Lemma 8.3, $\mathcal{FE}(X,\tau_1(2),f_1)$ holds and hence, so does $\mathcal{LCI}(X,\tau_1(2),f_1)$. Since $\mathcal{M}(X,\tau_0(2))$ is the unique dense stratum in the Behrend-Manin decomposition, to show that it is irreducible, it is sufficient to show that $\overline{\mathcal{M}}(X,\tau_0(2))$ is irreducible. This is immediate from the fact there is a unique nice contraction from $\alpha_2 : \sigma_2 \longrightarrow \tau_0(2)$ with $E(\sigma_2) = e$ and our earlier remarks relating the irreducible components of such a stack to nice contractions.

Note that in the case mentioned in the statement of the theorem, the modified threshold degree is given by $E'(X) = 2$. Noting that a general complete intersection is smooth and using the fact that we have established $\mathcal{B}(X,\tau_1(1),f_1)$, we use Lemma 9.6 to conclude that $\mathcal{B}(X,\tau_1(2),f_1)$ holds. We may now apply Corollary 10.1 to conclude that for every stable A graph, $\overline{\mathcal{M}}(X,\tau)$ is irreducible.

Note that in particular, by Theorem 10.1, we have showed that $\overline{\mathcal{M}}(X,\tau_0(e))$ is irreducible for a general complete intersection X satisfying (A). □

11 Generalizing the inductive argument to k-planes

In the previous section, the main result generalizes the induction argument in [11] from the case of pointed lines on a hypersurface to pointed lines on a complete intersection. In this section, we bootstrap to make a similar conclusion about a the space of k-planes contained in a complete intersection X. The key idea is to inductively use Theorem 4.1.

11.1 Setup

Let us recall that for a fixed vector space U of dimension $n+1$ over our field k, by $\mathbb{P}^n$, we will mean the projective space $\mathbb{P}U = Proj(\oplus_{j=0}^{\infty} Sym^j(U^*))$. For each $1 \leq r \leq n+1$ by $F((1,2,...,r),U)$, we mean the partial flag variety parametrizing projective flags of subspaces:

$$\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-1} \subset \mathbb{P}^n$$

Moreover, we note that for each r, the flag variety $F((1,2,...,r),U)$ comes with a natural projection morphism down to the flag variety of $r-1$ flags. This natural projection simply forgets the r-plane in the flag $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-1}$ and projects down to $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-2}$.

Lemma 11.1. *Let*

$$P_r : F((1,2,...,r+1),U) \longrightarrow F((1,2,...,r),U)$$

be the aforementioned natural projection. Each fiber of P_r is isomorphic to a projective space of dimension $n-r$.

Proof. Let $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-1}$ be an arbitrary but fixed point in $F((1,2,...,r),U)$. The fiber $P_r^{-1}(\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-1})$ is the set of all r-flags which have first r components equal to $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-1}$. This is equivalent to specifying all projective r planes in $\mathbb{P}^n$ containing the $r-1$ plane Λ_{r-1}. As in Lemma 1.1, these are parametrized by a Schubert cycle in $G(r+1,n+1)$. This Schubert cycle isomorphic to projective space $\mathbb{P}(U/\overline{\Lambda}_{r-1})$, where $\overline{\Lambda}_{r-1}$ is the r dimensional subspace of the vector space U corresponding to Λ_{r-1}. □

Let $\mathcal{X}$ denote the scheme parametrizing pairs (X,p) of a point and a complete intersection X containing the point p. Consistent with our earlier notation, we denote by $F_{0,1,2,...,r}(\mathcal{X})$ the subscheme of $F((1,2,...,r+1),U)$, parametrizing flags contained in X, i.e. flags such that $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_r \subset X$. In addition, for a given complete intersection X, we will use $F_{0,1,2,...,r}(X)$ to denote all r flags contained in X. The argument showing the smoothness of $F_{0,1}(\mathcal{X})$ extends almost verbatim to show that $F_{0,1,2,...,r}(\mathcal{X})$ is also smooth for each r. The natural projection $P_r : F((1,2,...,r+1),U) \longrightarrow F((1,2,...,r),U)$ induces a map

$$\rho^r : F_{0,1,2,...,r}(\mathcal{X}) \longrightarrow F_{0,1,2,...,r-1}(\mathcal{X})$$

Given a complete intersection X, we denote by

$$\rho^r_X : F_{0,1,2,...,r}(X) \longrightarrow F_{0,1,2,...,r-1}(X)$$

the fiber map of ρ.

11.2 Flatness and dimension result for ρ^r_X

Theorem 11.1. *Fix a sequence of positive integers $(d_1, d_2, .., d_c)$ as in Theorem 4.1. Let X be a general complete intersection of multidegree $(d_1, d_2, ..., d_c)$. Set $1 \leq r \leq n$. For sufficiently large n and r chosen small relative to n,the following is true:*

For a general flag $\Lambda_0 \subset \Lambda_1 \subset ... \subset \Lambda_{r-2}$ and for every $r-1$-plane Λ_{r-1} such that $\Lambda_0 \subset ... \subset \Lambda_{r-2} \subset \Lambda_{r-1}$, the set of r planes Λ_r, which are contained in X, and themselves contain Λ_{r-1} has the expected dimension. Equivalently, the map

$$\rho^r_X : F_{0,1,2,...,r}(X) \longrightarrow F_{0,1,2,...,r-1}(X)$$

is flat over an open dense subset of $F_{0,1,2,...,r-1}(X)$ where the first $r-2$ components in the flag are general.

Proof. As we mentioned before, for each s, the varieties $F_{0,1,2,...,s}(\mathcal{X})$ are smooth. Thus, by the local flatness theorem [15, Theorem 23.1], showing the flatness of ρ^s_X is equivalent to showing that the fiber dimension is constant and is equal to the expected dimension. This is precisely what we will show.

Let us fix an r, $2 \leq r \leq n$. For a point $(X, \Lambda_0, ..., \Lambda_{r-2}, \Lambda_{r-1}) \in F_{0,1,2,...,r-1}(\mathcal{X})$, the fiber $(\rho^r)^{-1}(X, \Lambda_0, ..., \Lambda_{r-2}, \Lambda_{r-1})$ is a subscheme of the fiber
$(P_r)^{-1}(\Lambda_0, ..., \Lambda_{r-2}, \Lambda_{r-1})$. Specifically, it is the subscheme parameterizing r-planes Λ_r contained in X, for which $\Lambda_{r-1} \subset \Lambda_r$. For simplicity, let us call this subscheme A_r. Similarly, inside the fiber $(P_{r-1})^{-1}(\Lambda_0, ..., \Lambda_{r-2})$,over the point $(\Lambda_0, ..., \Lambda_{r-2})$, we have the corresponding subscheme A_{r-1}, parameterizing $r-1$ planes Λ_{r-1} such that $\Lambda_{r-1} \subset X$ and $\Lambda_{r-2} \subset \Lambda_{r-1}$. A_{r-1} is of course equal to $(\rho^{r-1})^{-1}(X, \Lambda_0, ..., \Lambda_{r-2})$

By what we proved in Lemma 11.1 we know that $(P_r)^{-1}(\Lambda_0, ..., \Lambda_{r-1})$ is isomorphic to the projective space $\mathbb{P}(U/\overline{\Lambda}_{r-1})$ and also that $(P_{r-1})^{-1}(\Lambda_0, ..., \Lambda_{r-2})$ is isomorphic to $\mathbb{P}(U/\overline{\Lambda}_{r-2})$. By definition, a point $p \in \mathbb{P}(U/\overline{\Lambda}_{r-2})$ corresponds to an $r-1$-plane, say Λ_p in $\mathbb{P}^n$ which contains Λ_{r-2}. We now make the following obvious, but crucial observation - a line L in $\mathbb{P}(U/\overline{\Lambda}_{r-2})$ containing p corresponds to an r-plane in $\mathbb{P}^n$, containing the $r-1$-plane Λ_p. Moreover, every r-plane in $\mathbb{P}^n$, containing a fixed $r-1$-plane, corresponds to a line containing a point in $\mathbb{P}(U/\overline{\Lambda}_{r-2})$. So there is a one to one correspondence between r-planes containing a fixed $r-1$ plane in $\mathbb{P}^n$ and pointed lines in $\mathbb{P}(U/\overline{\Lambda}_{r-2})$. Finally, the pointed lines in $\mathbb{P}(U/\overline{\Lambda}_{r-2})$ that are contained in A_r, correspond to r planes Λ_r containing an $r-1$ plane such that $\Lambda_r \subset X$. In other words, A_r is isomorphic to the space of pointed lines contained in A_{r-1}. With this in mind, we now use induction.

The base case, $r = 1$, is precisely Theorem 4.1. To rephrase Theorem 4.1 in our new notation, the map $\rho_X : F_{0,1}(X) \longrightarrow X$ is the map ρ^1_X. We fix a point $\Lambda_0 \in X$. As in

Theorem 1.1, because of homogeneity, this same argument works irrespective of the choice of Λ_0. The fiber $(\rho_X^1)^{-1}(\Lambda_0)$ is A_1 and concretely, it parameterizes pointed lines on X, i.e. A_0. The proof of Theorem 1.1 then shows that if n is large enough to satisfy our system (A) of polynomial inequalities, then for general X, A_1 has the expected dimension, i.e., A_1 is itself a complete intersection inside $\mathbb{P}(U/\overline{\Lambda}_0)$. Moreover, as we saw earlier, if $(F_1, F_2, ..., F_c)$ are the homogeneous polynomials defining $X \subset \mathbb{P}^n$, the defining equations of A_1 are obtained by taking the Taylor expansion of the F_i-s around the point Λ_0.

Now, suppose we have proved this for all $r \leq k$. Let us now consider the map ρ_X^{k+1} : $F_{0,1,2,...,k+1}(X) \longrightarrow F_{0,1,2,...,k}(X)$. We have, by the induction hypothesis, that for a general flag $(\Lambda_0, \Lambda_1, ..., \Lambda_{k-2})$, A_k has the correct dimension, i.e., is a complete intersection inside the projective space $\mathbb{P}(U/\overline{\Lambda}_{k-1})$. (As before,its defining equations are obtained by doing a Taylor expansion of the defining equations of A_{k-1} around Λ_{k-1}). However, we note that for a subset of $k-1$-flags $(\Lambda_0, \Lambda_1, ..., \Lambda_{k-1})$, A_k is singular, despite having the correct dimension. For these A_k's, A_{k+1} will fail to have the correct dimension, since the linear part of the Taylor expansion will be identically 0. Of course, the set of such bad $k-1$-flags is parameterized by a closed subset of the partial flag variety $F((1, 2, ..., k), U)$.

Let us now look at $k-1$-flags contained in the open complement of this bad subset. As mentioned above, A_{k+1} is the scheme of pointed lines on A_k. Its defining equations are obtained by taking the Taylor expansion of the defining equations of A_k around the point Λ_k. We noted earlier that A_k is a complete intersection in a projective space of dimension $n-k$, (namely $\mathbb{P}(U/\overline{\Lambda}_{k-1})$) and we know its defining equations and multidegree. If n is chosen large enough that our system of polynomial inequalities is satisfied when n is replaced by $n-k$ and $(d_1, d_2, ..., d_c)$ is replaced by the multidegree of A_k, then we can mimic the proof of Theorem 4.1, after removing the subset of "bad" $k-1$-flags. Thus, on the open subset of good $k-1$-flags, A_{k+1} has the correct dimension, i.e. it is a complete intersection. It is always possible to do so when n is large and k is small compared to n.

We note that the genericity of the first part of the flag becomes relevant only for flags of length 3 or higher, which is why did not have to address this in Theorem 4.1. This completes the proof.

□

12 Irreducibility of the space of Quasimaps

12.1 Outline

Using the results of section 1, we prove an irreducibility result about the space of quasimaps into a complete intersection X of type $(d_1, d_2, ..., d_c)$. We refer the reader to [18] and [14] for more details on the space of quasimaps.

Unlike in [14], we consider quasi maps that are unparameterized- that is, we do not parameterize the domain of the quasimap (To obtain unparameterized quasimaps from parameterized ones, we quotient by the automorphism group of the domain curve). The key difference from section 1 is that we are able to conclude irreducibility for the space of quasimaps into every complete intersection X (provided our inequalities hold) - not just a general one. We follow the technique used by Jason Starr and Zhiyu Tian for hypersurfaces.

12.2 Setup

We outline a construction for the space of quasimaps:

By the Universal property of projective space, a morphism $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ is defined by $n+1$ global sections of $O_{\mathbb{P}^1}(e)$ which do not vanish simultaneously. It is helpful to think about these global sections as the co-ordinate functions of ϕ. Concretely, suppose $[x_0, x_1, ..., x_n]$ and $[s, t]$ are homogeneous co-ordinate systems on $\mathbb{P}^n$ and on $\mathbb{P}^1$ respectively. We will have

$$\phi([s,t]) = [\phi_0(s,t), \phi_1(s,t), ..., \phi_n(s,t)]$$

where the homogeneous polynomials ϕ_j are precisely the aforementioned global sections of $O_{\mathbb{P}^1}(e)$. So, given a morphism ϕ as above, we have a vector space homomorphism

$$\tilde{\phi} : H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(e))$$

which sends the co-ordinate x_j to $\phi_j([s,t])$. Conversely, given a vector space homomorphism $\tilde{\phi}$ as above, we can define a map $\phi : \mathbb{P}^1 \to \mathbb{P}^n$, provided we the polynomials $\phi_j([s,t])$ do not vanish simultaneously. So, there is a correspondence between morphisms $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ and elements of an open subset of $\mathrm{Hom}(H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(1)), H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(e)))$. A morphism ϕ obtained as above is defined to be a quasimap. Of course, it is unique up to scaling by an element of the underlying field k. Thus, we may instead consider the projective space $\mathbb{P}H = \mathbb{P}(\mathrm{Hom}(H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(1)), H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(e))))$. By what we have discussed, there is a correspondence between the closed points of an open subset (To be precise, the open subset where the $\phi_j([s,t])$ do not simultaneously vanish) of $\mathbb{P}H$ and morphisms $\phi : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$. The automorphism group $PGl(2)$ of $\mathbb{P}^1$ acts on the domain by reparameterizations. The action of this automorphism group on the domain induces an action on $\mathbb{P}H$.

Definition 12.1. *The* ***space of quasimaps*** *into* $\mathbb{P}^n$ *is defined to be the GIT quotient of* $\mathbb{P}H$ *by this action, i.e.,* $Y_{n,e} = \mathbb{P}H//PGl(2)$.

We thus have a quotient map

$$\pi : \mathbb{P}H \longrightarrow Y_{n,e}$$

Given a subvariety $X \subset \mathbb{P}^n$, we construct the space of quasimaps into X as follows. A quasimap $\phi : \mathbb{P}^1 \longrightarrow X$ is simply thought of as a quasimap into $\mathbb{P}^n$ whose image lies inside

X. The condition for the image of ϕ to lie in X is easy to see - the defining equations of X must vanish identically when the co-ordinate functions of ϕ are plugged into them. So in particular, suppose $X \subset \mathbb{P}^n$ is a complete intersection with defining equations $F_1, F_2, ..., F_c$. Given an arbitrary $(n+1)$-tuple of homogenous polynomials $\phi_0(s,t), \phi_1(s,t), ..., \phi_n(s,t)$, we plug them in to each F_i and gather coefficients. For each i and j, we will obtain $e.d_i + 1$ homogeneous polynomials in the coefficients of ϕ_j. The common zero locus of these $(e.\sum_{i=1}^{c} d_i + c)$ polynomials, which we call A_X, is precisely the locus parameterizing quasimaps whose image is contained in X. We will abuse notation and use A_X to also mean the intersection of A_X with the open set of semi-stable points in $\mathbb{P}H$.

Definition 12.2. *The* ***space of quasimaps to*** X *is the subscheme* $Y_{n,e}(X)$ *inside* $Y_{n,e}$*, parameterizing quasimaps to* X*.*

The inverse image of $Y_{n,e}(X)$ in the open subset of semi-stable points is precisely the subvariety A_X.

There is also a second description of the space of quasimaps to $\mathbb{P}^n$ as the image of a contraction from the space of stable maps.

$$cont_{n,e} : \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e) \longrightarrow Y_{n,e}$$

The construction of this contraction morphism is carried out in [18] as well as in [5]. By our discussions in Lemma 7.1, we know that for a complete intersection X, $\overline{\mathcal{M}}_{0,0}(X, e)$ is defined as a subvariety of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, e)$ by the equations obtained by plugging in the co-ordinates of the stable map into the defining equations of X. The image of $\overline{\mathcal{M}}_{0,0}(X, e)$ is precisely the subvariety $Y_{n,e}(X)$ inside $Y_{n,e}$. The expected dimension of $Y_{n,e}(X)$ is equal to:

$$n - c + e(n + 1 - \sum_{i=1}^{c} d_i) - 3$$

As in the case of stable maps, the expected dimension is a lower bound on the dimension of $Y_{n,e}(X)$.

12.3 Irreducibility results

Lemma 12.1. *For a general complete intersection* X *of multidegree* $(d_1, d_2, ..., d_c)$ *satisfying our system of polynomial inequalities* (A)*, the space of quasimaps* $Y_{n,e}(X)$ *is irreducible of the expected dimension.*

Proof. By Theorem 7.2, we know that for a general complete intersection X, $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible. We note that by [5] Theorem 1.8, that $Y_{n,e}(X)$ is the surjective image of $\overline{\mathcal{M}}_{0,0}(X, e)$ under the contraction $cont_{n,e}$. But the surjective image of an irreducible variety is also irreducible. Hence, for a general complete intersection X, $Y_{n,e}(X)$ is irreducible.

Moreover, by [5, Theorem 1.9], $cont_{n,e}$ is a birational map. Hence, $Y_{n,e}$ has the same dimension as $\overline{\mathcal{M}}_{0,0}(X,e)$, which for a general smooth complete intersection X as in the statement of the lemma, is
$n - c + e(n+1-\sum_{i=1}^{c} d_i)$. □

In order to prove our result, we will need to generalize the following result of [21]. Let $\mathbb{G}(k,n)$ be the Grassmanian parameterizing linear k planes in $\mathbb{P}^n$. Let $\mathbb{P}^{N_d}//PGl(k+1)$ be the moduli space of semistable degree d hypersurfaces in $\mathbb{P}^k$. Let $X \subset \mathbb{P}^n$ be a degree d hypersurface. Given a k-plane $\Lambda \subset \mathbb{P}^n$, the map

$$\phi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{N_d}//PGl(k+1)$$

sending Λ to $\Lambda \cap X$ defines a rational transformation. Theorem 1.1 in [21] states:

Theorem 12.1. *If X is a smooth hypersurface, the map ϕ is dominant if*

$$n \geq \binom{d+k-1}{k} + k - 1$$

We now extend this result to complete intersections $X \subset \mathbb{P}^n$. To this end, we proceed exactly as in [21]. For a complete intersection of multidegree $\underline{d} = (d_1, d_2, ..., d_c)$, we define the integer

$$N_k(n,\underline{d}) = n - k - \sum_{i=1}^{c} \binom{d_i+k-1}{k}$$

We then have the following:

Lemma 12.2. *Let X be a smooth complete intersection inside $\mathbb{P}^n$. If the integer $N_k(n,\underline{d})$ is nonnegative, then there exists an irreducible component I of $F((1,2,3,...,(k+1)),X)$ having the expected dimension equal to*
$dim(I) = \sum_0^k N_m(n,\underline{d})$. Here, $F((1,2,3,...,(k+1)),X)$ is the variety parameterizing partial k flags contained in X.

Proof. The proof in [21] carries over almost ad-verbum. We repeat it only for completeness. We define I to be the closure of any connected component of U_k, where U_k is the open subset described in [21, Proposition 2.3]. We need only check that U_k is non-empty. As in [21], with our notation, this is automatic if ρ_X^r is surjective for $r = 1,2,...,k$. Of course, as in the proof of Theorem 11.1, The fiber over a given point is defined by the intersection in $\mathbb{P}^{n-r}$ of divisors, defined by the equations obtained from a Taylor series expansion. As such, by [12, Chapter 1, Theorem 7.2] if the number of equations defining the fiber is less than the dimension of the projective space, i.e., $(n-r)$, then the fiber is non-empty. In other words, the fiber over every point is non-empty if precisely the condition $N_k(n,\underline{d}) \geq 0$ holds. □

We now have the analogue of [21, Proposition 1.3].

Lemma 12.3. *Let X be a smooth complete intersection of type $\underline{d}$ in $\mathbb{P}^n$ and let $F_k(X)$ be the Fano variety of k-planes contained in X. Then there exists an irreducible component C of $F_k(X)$ having the expected dimension if $N_k(n,\underline{d}) \geq 0$. Moreover, if $N_k(n,d) = -1$ then there is a non-empty open subset $U_{k-1} \subset F_{k-1}(X)$ such that for every $\Lambda_{k-1} \in U_{k-1}$ there exists no k-plane in X containing Λ_{k-1}.*

Proof. The proof also works very similarly to the one in [21]. We only outline the idea. The first part follows immediately from the previous Lemma. For the second part, since $N_{k-1}(n,\underline{d})$ is non-negative, U_{k-1} is non-empty. Moreover, by [21, Proposition 2.3] the map $\rho^k_X : (\rho^k_X)^{-1}(U_{k-1}) \longrightarrow U_{k-1}$ is smooth of the expected dimension. As we see, the expected dimension is negative and hence the $(\rho^k_X)^{-1}(U_{k-1})$ is empty. Thus, for every $\Lambda_{k-1} \in U_{k-1}$, there is no k-plane in X containing Λ_{k-1}. □

Finally, we generalize the theorem in [21]. As in the case of hypersurfaces, for a complete intersection X, we have a rational transformation

$$\phi : \mathbb{G}(k,n) \dashrightarrow \mathbb{P}^{N_{d_1,d_2,\ldots,d_c}}//PGl(k+1)$$

where $\mathbb{P}^{N_{d_1,d_2,\ldots,d_c}}//PGl(k+1)$ parameterizes complete intersections in $\mathbb{P}^k$ and given a complete intersection $X \subset \mathbb{P}^n$, the map ϕ sends a k-plane Λ to $\Lambda \cap X \subset \Lambda$.

Theorem 12.2. *Let X be a smooth complete intersection of type $\underline{d} = (d_1, d_2, ..., d_c)$. If $N_k(n,\underline{d}) \geq 0$, then the map ϕ is dominant.*

Proof. We build on the technique of the proof of [21, Theorem 1.1]. Let $H_{k,n}$ be the open subset of $\mathbb{P}(\mathbf{Hom}(\mathbb{C}^{k+1}, \mathbb{C}^{n+1}))$ parameterizing injective matrices. Thus, $H_{k,n}$ parameterizes linear embeddings of $\mathbb{P}^k$ into $\mathbb{P}^n$. There is a natural action of $PGl(k+1)$ on $H_{k,n}$ and the quotient by this action is the Grassmanian $G(k,n)$. If $F_k(X)$ is the Fano Scheme of k-planes on X, let its inverse image in $H_{k,n}$ be $\tilde{F}_k(X)$. That is, $\tilde{F}_k(X)$ parameterizes linear embeddings of $\mathbb{P}^k$ into X.

If $F_1, F_2, ..., F_c$ are the defining equations of X, then for each i restricting F_i gives a global section of $H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(d_i))$. We then have a regular morphism

$$\tilde{\phi} : H_{k,n} \longrightarrow \bigoplus_{i=1}^{c} H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(d_i))$$

Let V be the open subset of $H_{k,n}$ of points where the dimension of the fiber $\tilde{\phi}^{-1}(\tilde{\phi}(p))$ equals $\dim H_{k,n} - \sum_{i=1}^{c} \dim H^0(\mathbb{P}^k, O_{\mathbb{P}^k}(d_i))$. We note that $\tilde{\phi}$ is dominant if and only if V is nonempty.

We note that $\tilde{F}_k(X)$ is the fiber of $(\tilde{\phi})^{-1}(0)$. Initially, we deal with the case $N_k(n,d) \geq 0$. If this is true,then by Lemma 12.3 there exists an irreducible component C of $F_k(X)$ of the expected dimension. Thus, the inverse image $\tilde{I}$ is also an irreducible component of $\tilde{F}_k(X)$

of the expected dimension, or equivalently, the expected codimension. But the expected codimension is precisely

$$\sum_{i=1}^{c} \binom{d_i + k}{k}$$

Thus, the generic point of $\tilde{I}$ satisfies the fiber dimension condition defining V and in particular, belongs to V - i.e. V is non-empty.

□

We note the following:

- The space $\mathbb{P}H$ is a projective space of dimension $(n+1)(e+1)-1$ and the locus A_X parameterizing quasimaps to X is defined by $(e.\sum_{i=1}^{c} d_i + c)$ equations. Thus, the codimension of A_X is at most $(e.\sum_{i=1}^{c} d_i + c)$, by [12, Chapter 1, Theorem 7.2]. As such, A_X must intersect all subschemes of $\mathbb{P}H$ of dimension greater than or equal to $(e.\sum_{i=1}^{c} d_i + c)$.

- Suppose $L \subset \mathbb{P}^n$ is an r-plane in $\mathbb{P}^n$. The subscheme A_L corresponds to quasimaps with image contained in L. But L is itself isomorphic to a projective space of dimension r. Thus, $A_L \subset \mathbb{P}H$ is in fact the projective space

$$\mathbb{P}(\mathrm{Hom}(H^0(L, O_L(1)), H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(e))))$$

  of dimension $(r+1)(e+1)-1$, linearly embedded in $\mathbb{P}H$. This can also be observed directly from the defining equations of L by the same method we defined A_X. Indeed there are $(n-r)$ defining equations of L inside $\mathbb{P}^n$, each of degree 1. Thus, the locus A_L is defined by the vanishing of $e.(n-r)+(n-r)$ linear equations in $\mathbb{P}H$ and so, is a linearly embedded $(e+1)(r+1)-1$ plane in $\mathbb{P}H$.

- If U, V are subvarieties of $\mathbb{P}^n$, then $A_U \cap A_V$ and $A_{U \cap V}$ both exactly parameterize quasimaps with image in $U \cap V$. Thus, $A_U \cap A_V = A_{U \cap V}$. This too can be observed directly from the equations defining U and V.

We now make an arbitrary choice for the multidegree $(d_1, d_2, ..., d_c)$ of the complete intersection X and the degree of the quasimap e and hold them fixed. We say that a positive integer n satisfies (B) if the following hold:

1. $n \geq \frac{2(e.\sum_{i=1}^{c} d_i + c)}{e+1} - 2$

2. $N_{\lfloor \frac{n}{2} \rfloor}(n, \underline{d}) \geq 0$

3. $\lfloor \frac{n}{2} \rfloor$ satisfies our system (A) of polynomial inequalities.

Note, the smallest positive integer n_0 satisfying (B) is effectively computable and all $n \geq n_0$ satisfy (B).

We are now ready to prove our theorem:

Theorem 12.3. *For every smooth complete intersection $X \subset \mathbb{P}^n$ of multidegree $\underline{d} = (d_1, d_2, ..., d_c)$, for $n \geq n_0$ where n_0 is the smallest positive integer satisfying (B), the space of quasimaps to X, $Y_{n,e}(X)$ is irreducible of the expected dimension.*

Proof. Let $\mathbb{P}H$ and A_X be as before. Our goal will be to show that A_X is irreducible. If we show this, then because A_X maps surjectively onto $Y_{n,e}(X)$, we will conclude that $Y_{n,e}(X)$ is irreducible. To do this, we use proof by contradiction.

Suppose that A_X is not irreducible. Thus, it must have a finite number of irreducible components, say $V_1, V_2, ..., V_p$. Let us choose an r such that

$$r \geq \frac{(e.\sum_{i=1}^{c} d_i + c)}{e+1} - 1$$

For such an r, by our previous arguments, the dimension of A_L for an r-plane L, is $(e+1)(r+1) - 1$, which is greater or equal to the expected codimension of A_X, that is $(e.\sum_{i=1}^{c} d_i + c)$. So for all such r-planes L, A_L must intersect each irreducible component V_i nontrivially and moreover, and the intersection of $A_L \cap V_i$ has dimension at least

$$(r+1)(e+1) - 1 - e\sum_{i=1}^{c} d_i - c = (r-c) + e(r+1-\sum_{i=1}^{c} d_i)$$

Since n satisfies (B), we know that $n \geq \frac{2(e.\sum_{i=1}^{c} d_i + c)}{e+1} - 2$. We could thus choose $r = \lfloor \frac{n}{2} \rfloor$.

For a general r-plane L, $L \cap X$ is a complete intersection of the same multidegree $\underline{d}$ inside L. By our previous remarks, $A_L \cap A_X = A_{L\cap X}$ is subscheme of A_L parameterizing quasimaps to $L \cap X \subset L$. Moreover, by our assumption $A_X = \cup_{i=1}^{p} V_i$, which implies that

$$A_{L\cap X} = A_L \cap A_X = A_L \cap (\cup_{i=1}^{p} V_i) = \cup_{i=1}^{p} (A_L \cap V_i)$$

As noted earlier, each $A_L \cap V_i$ is nonempty. Thus, $A_{L\cap X}$ can be written as the union of the closed subsets $A_L \cap V_i$.

However, we know that if we choose r such that $N_r(n, d) \geq 0$, then by Theorem 12.2 the map ϕ is dominant. So, by taking L general, we have that the complete intersection $L \cap X \subset L$ is general. The condition of not being smooth is equivalent to the Jacobian of the defining equations of $L \cap X$ inside L not having full rank, which is a closed condition. Therefore, $L \cap X \subset L$ is smooth, since it is general. This can also be observed using an iterated application of Bertini's Theorem. Since n satisfies, among other things, the second condition of (B), choosing $r = \lfloor \frac{n}{2} \rfloor$ suffices.

Finally, if r is chosen to satisfy our system of polynomial inequalities (A), then by lemma 12.1, the space of quasimaps $Y_{r,e}(L \cap X)$ is irreducible for a general $L \cap X \subset L$. Again, from

(B), we may choose $r = \lfloor \frac{n}{2} \rfloor$ for this to hold true. Note that the fibers of the map

$$\pi : \mathbb{P}(Hom(H^0(L, O_L(1)), H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(e))) \longrightarrow Y_{r,e}$$

are irreducible. Thus, if r is chosen in this way, $A_L \cap A_X$ is irreducible of dimension $r - c + e(r + 1 - \sum_{i=1}^{c} d_i)$ (Since π maps $A_L \cap A_X$ surjectively onto $Y_{n,e}(L \cap X)$ with irreducible fibers).

We noted earlier that each $A_L \cap V_i$ has dimension at least equal to this same integer. So if L is general, we see that $A_L \cap A_X$ is irreducible of the expected dimension, and each $A_L \cap V_i$ is a closed subset of $A_L \cap A_X$ of the same dimension as the ambient space. So, by [12, Chapter 1, Exercise 1.10 d] we have that $A_L \cap V_i = A_L \cap A_X$ for all pairs i. In particular, we have that $A_L \cap V_i = A_L \cap V_j$ for all pairs of i, j and each $A_L \cap V_i$ is irreducible.

Let E be the expected dimension of A_X, that is, $E = n - c + e(n+1-\sum_{i=1}^{c} d_i)$. We know that $\dim V_i \geq E$ for each i. A_L is an $(r+1)(e+1)-1$- plane in $\mathbb{P}H$. Applying [12, Theorem 7.2], we have that

$$\dim A_L \cap V_i \geq \dim V_i - (e+1)(n-r) \geq E - (e+1)(n-r)$$

But we know that $A_L \cap V_i = A_{L \cap X}$ has dimension exactly $r - c + e(n+1-\sum_{i=1}^{c} d_i)$. Note that

$$E - (e+1)(n-r) = n - c + e(n+1-\sum_{i=1}^{c} d_i) - (e+1)(n-r) = r - c + e(n+1-\sum_{i=1}^{c} d_i)$$

and so we see that $\dim A_L \cap V_i = E - (e+1)(n-r)$. So we get that

$$E - (e+1)(n-r) = \dim A_L \cap V_i \geq \dim V_i - (e+1)(n-r) \geq E - (e+1)(n-r).$$

Thus, all the inequalities must be equalities and we must have

$$\dim V_i - (e+1)(n-r) = E - (e+1)(n-r)$$

Thus $\dim V_i = E$ for each i. So A_X is a pure dimensional subscheme of the expected dimension.

We have established that all irreducible components V_i of A_X have dimension equal to the expected dimension. We give V_i the induced reduced structure as an integral closed subscheme of $\mathbb{P}H$. Suppose V_i has multiplicity m_i, where each m_i is a positive integer. Let $[A_X]$ be the cycle associated with A_X. We then have $[A_X] = \sum_i m_i[V_i]$. The intersection of A_L with A_X equals $A_{L \cap X}$, which is a reduced, irreducible, complete intersection in A_L with dimension equal to the expected dimension. We know that the space of stable maps has a nonempty smooth locus - for example, the multiple covers of free lines are smooth points of the moduli space. Being non-empty, the smooth locus is open and dense in $\overline{\mathcal{M}}_{0,0}(X,e)$. Because of the birationality of $cont_{n,e}$, we conclude that the space of quasimaps has nonempty smooth locus, hence it is everywhere reduced. (Local complete intersection schemes are Cohen-Macaulay, hence they are S1, and S1 schemes that are generically reduced are also everywhere reduced).

Thus, the cycle of $A_{L\cap X}$ equals $[A_{L\cap X}]$ with multiplicity 1. Denote by

$$j_L : A_L \to \mathbb{P}H$$

the regular embedding of A_L inside the space of quasimaps. $[A_{L\cap X}]$ also equals the pullback along of $[A_X]$ along j_L and we have that

$$[A_{L\cap X}] = \sum_{i=1}^{p} m_i(j_L^*[V_i])$$

This equality follows by the results of [8, Section 7.1], since each dimension equals the expected dimension and hence the pullbacks of the cycles are proper intersections. By Krull's Hauptidealsatz, the pullback of each prime cycle $[V_i]$ to A_L is a nonempty, effective cycle.

If we did not know that the pullbacks of cycles were proper intersections, it could have happened that the intersection multiplicity of some $j_L^*[V_i]$ along a prime cycle was negative, and would cancel out the positive intersection multiplicity of $j_L^*[V_k]$, for some other V_k, along the same prime cycle. But this is not the case, since the pullbacks of the cycles are proper intersections and their intersection multiplicities can be computed in the usual way (See [8, Section 7.1]). In particular, all the intersection multiplicities are positive integers. Since the sum over all of these nonempty cycles with positive integer multiplicities equals the prime cycle $[A_{L\cap X}]$ with multiplicity 1, it follows that there is a unique V_i and the multiplicity m_i equals 1, i.e., A_X is irreducible and reduced.

As previously mentioned, all the arguments of the proof work by choosing $r = \lfloor \frac{n}{2} \rfloor$. Hence, we are done. □

13 Future Questions

We mentioned in the introduction that our techniques provided sufficient conditions for the Gromov-Witten invariants to be enumerative for complete intersections $X \in \mathbb{P}^n$. In this final section we consider further potential questions in this area.

1. **Can the system of inequalities mentioned in section 4.2 be simplified using approximations for the binomial coefficients or other numerical methods for large n?**

The system (A) is a set of inequalities involving binomial coefficients. We believe that by using Stirling's approximation and other such methods, it is possible to simplify this system.

2. **Can the system of inequalities obtained in Theorem 4.1 be improved by taking into account the geometry at each step of the iteration?**

   In our inductive argument Theorem 4.1, we obtained our system of inequalities by successively restricting the homogenous parts $F_{i,j}$ of the Taylor expansions to the irreducible components of $V = V(F_{1,1}, F_{2,1}, ..., F_{i,j-1})$. However, we did not make any considerations about the geometry of V. It is our belief that, at least when there are some restrictions on the d_i-s, it is possible to obtain a weaker system of inequalities by more closely examining the nature of each of the varieties obtained at the various stages our iterative process.

3. **Can the inductive argument in section 11 be generalized to k planes containing r planes where $r < k - 1$?**

   In our proof of Theorem 11.1, we use the fact that the fibers of the projection map ρ_X^r are complete intersections of some multidegree in a different projective space. Instead, if we took our forgetful map to be one which forgets not the last element in a flag of subspaces, but the last $k - r$ elements, the fiber would be some subvariety of a Grassmanian. We can ask whether it is possible to formulate a similar iterative argument to tackle this case as well.

4. **Can the methods of Behesti and Kumar be used in the context of Grassmanians?**

   In [7], Robert Findley generalized the result of [11] to hypersurfaces of low degree in $G(k, n)$. It is our hope that one can apply the methods of [1] to obtain new degree inequalities for hypersurfaces in $G(k, n)$.

5. **Can the techniques of Browning and Vishe be applied to the complete intersections?**

   In [4], Browning and Vishe used techniques in Analytic Number Theory to show that the space of quasimaps to X is irreducible for every hypersurface $X \subset \mathbb{P}^n$. We ask whether it is possible to extend their techniques to the case of complete intersections.

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