On the Integrability and Twistor Theory of Real Almost-Grassmannian Manifolds

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Abstract of the Dissertation

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We present a twistor correspondence for half-flat almost-Grassmannian structures on real manifolds. An almost-Grassmannian structure is (essentially) a factorization of the tangent bundle, which determines two preferred families of tangent subspaces, and this structure is said to be half-flat if one of these families is integrable. We provide global results when the underlying manifold is a Grassmannian of 2-planes, and show there exist nontrivial deformations of the standard almost-Grassmannian structure. Whereas twistor constructions typically involve moduli of closed curves in a complex manifold, we utilize and expand upon the more flexible approach pioneered by LeBrun and Mason using moduli of curves-with-boundary.

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Chapter 1

Introduction

"Integrable systems, what are they? ... 'If you gotta ask, you'll never know!' "

– Nigel Hitchin

The notion of integrability is difficult to formulate. Although the field is well-developed, there is no single theory able to characterize all known cases. Mason and Woodhouse [MW96] suggest this is a consequence of the fact that research has been largely driven by individual cases, such that older definitions fail to capture the ever growing catalogue of examples. There is another, more insidious explanation – just like a manifold may require numerous patchwise descriptions to be stitched together, so too may the meaning of integrability resist any single global definition.

In any case, there exist loosely interpretable features to be expected of any integrable system. Hitchin [NH99] describes these as the existence of conserved quantities, the presence of algebraic geometry, and the ability to give explicit solutions. The property of integrability is highly nongeneric, but nonetheless important for two main reasons. First, almost by definition one can prove very elegant and powerful results in the presence of integrability. Secondly, nice properties of solutions often extend to those of non-integrable generalizations. As Ward [NH99] illustrates, shallow water waves are merely *approximated* by the Korgeweg-De Vries equation, and yet the full fluid-dynamical equations also enjoy the nice properties of having stable, solitonic solutions.

Mason and Woodhouse propose a different characterization. Indeed, the central thesis of [MW96] is that integrability is characterized by the existence of a twistor theory. The prototypical construction in twistor theory is the double fibration of flag manifolds

$$\begin{array}{ccc} & \mathbf{F}_{1,2}(\mathbb{C}^4) & \\ {}^{\mu}\swarrow & & \searrow^{\nu} \\ \mathbf{F}_1(\mathbb{C}^4) & & \mathbf{F}_2(\mathbb{C}^4), \end{array} \tag{1.0.1}$$

where

$$\begin{aligned} \mathbf{F}_{d_1,\ldots,d_k}(\mathbb{C}^4) = \{(S_1, \ldots, S_k) : S_j \text{ are subspaces of } \mathbb{C}^4 \\ \text{of dimension } d_j, S_1 \subset \ldots \subset S_k\}, \end{aligned} \tag{1.0.2}$$

and the maps μ, ν are first and second factor projections, respectively. This allows one to transform information from $\mathbf{F}_1(\mathbb{C}^4)$, called the "twistor space", to $\mathbf{F}_2(\mathbb{C}^4)$, called the "compactified complexified Minkowski space" or simply "spacetime". The correspondence between these two spaces has a simple geometric interpretation obtained by traversing up and down the diagram 1.0.1: an element $\Pi \in \mathbf{F}_2(\mathbb{C}^4)$ corresponds to the set of lines contained by Π, which is $\mu(\nu^{-1}(\Pi)) \cong \mathbb{CP}^1$, and an element $L \in \mathbf{F}_1(\mathbb{C}^4)$ corresponds to the set of planes containing L, which is $\nu(\mu^{-1}(L)) \cong \mathbb{CP}^2$. Using a more involved type of transform, one can also interpret data on the twistor space as solutions to certain differential equations on the Minkowski space. This was first accomplished by Roger Penrose for self-dual metrics [Pen76] and the zero rest-mass equation [Pen681], culminating in the seminal paper of Atiyah, Hitchin, and Singer [AHS78] which establishes the definitive twistor theory for 4-dimensional Riemannian geometry.

Theorem 1.0.1 (Atiyah, Hitchin, Singer)**.** *Let X be an oriented 4-manifold. Then a conformal structure on X defines a natural almost-complex structure on $\mathbb{P}(V_-)$, the projectivized bundle of (local) anti-self-dual spinors, and this almost-complex structure is integrable iff the conformal structure is self-dual.*

This construction is invertible, and therefore describes (at least locally) all self-dual metrics.

One of the underlying principles in twistor theory is to regard the "spacetime" as secondary to twistor space, which is in some sense more fundamental. In the 4-dimensional Riemannian version of the story, self-dual conformal manifolds arise as moduli spaces of compact complex curves in the twistor space. Such moduli spaces are themselves complex manifolds, and real geometries are obtained by imposing an anti-holomorphic involution on the twistor space; the curves invariant under this involution form the desired real manifold which is automatically endowed with a self-dual conformal metric. In the split-signature $(++--)$ case, the anti-holomorphic involution has fixed points and divides these invariant curves into two hemispheres. It was then observed by LeBrun and Mason that, rather than utilize an anti-holomorphic involution, one might instead focus on its fixed-point set. This subtle shift in perspective leads one to consider holomorphic curves *with boundary along a totally-real submanifold* inside the twistor space. Their approach led to a successful characterization of Zoll surfaces [LM0207] and self-dual Zollfrei 4-manifolds [LM0702].

In this document, the geometric structure of interest is what we call an *almost-Grassmannian structure*. Elsewhere in the literature this is sometimes referred to as a paraconformal structure, or a particular type of parabolic geometry. For the purposes of this introduction, a (p, q) almost-Grassmannian structure on a manifold M is a factorization of its tangent bundle,

$$TM \cong E \otimes H, \tag{1.0.3}$$

where the two factors have ranks $p, q \geq 2$ respectively. The terminology is motivated by the fact that a Grassmannian manifold $Gr(p, \mathbb{R}^{p+q})$ carries a canonical such structure. This definition can be seen as an attempt to capture the source of twistor theory in 4-dimensional conformal geometry. In fact, an almost-Grassmannian structure with $p = q = 2$ is equivalent to a conformal metric with a spin structure, and the factors in the tensor product are essentially spin bundles. The foundations of almost-Grassmannian geometry in the complex setting were laid out by Bailey and Eastwood in [BE91], which largely serves as a template upon which we overlay the holomorphic disk techniques developed by LeBrun and Mason.

1.1 Summary of Results

An almost-Grassmannian structure determines preferred subspaces, which are obtained by fixing an element in one of the factors and sweeping out the other.

$$\{e \otimes h \ : \ e \in E|_p \text{ free and } h \in H|_p \text{ fixed}\} \subset T_pM \qquad (1.1.1)$$

A natural question is then, are these preferred subspaces integrable? When this occurs the almost-Grassmannian structure is said to be right-flat. The first important result we prove is the almost-Grassmannian version of theorem 1.0.1.

Theorem 1.1.1. *A $(p, 2)$ almost-Grassmannian structure defines a natural distribution of complex $p+1$ planes on the projectivized complex spin bundle $\mathbb{P}(H \otimes \mathbb{C})$. This distribution is Frobenius integrable iff the almost-Grassmannian structure is right-flat.*

We also examine the perturbative theory, and characterize the linearized right-flat deformations (of a given right-flat almost-Grassmannian structure) by solutions $\phi \in \Gamma(End(TM))$ of the equation

$$\text{Trace-free part of } \{\nabla_{[a}\phi_{b]}{}^c\} = 0. \qquad (1.1.2)$$

Here the connection ∇ is determined by the almost-Grassmannian structure, but in the general case (at least one of $p, q > 2$) it is not a metric connection. Indeed, there is no metric involved whatsoever, which causes many of the proofs here to proceed along somewhat different lines than their 4-dimensional inspirations. In the special case $M = \widetilde{Gr}(2, \mathbb{R}^{p+2})$, we obtain a global twistor correspondence for right-flat perturbations of the standard almost-Grassmannian structure.

Theorem 1.1.2. *There is a one-to-one correspondence between right-flat almost-Grassmannian structures on the oriented Grassmannian $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ and smooth embeddings $\mathbb{RP}^{p+1} \hookrightarrow \mathbb{CP}^{p+1}$, at least near the standard ones.*

Construction of the twistor space $\mathbb{CP}^{p+1}$ largely follows from theorem 1.1.1 along with some topological considerations. To invert the construction, we rely on a regularity theory for holomorphic disks-with-boundary developed by LeBrun [LeB05]. It is seen that right-flat almost-Grassmannian structures on $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ naturally arise from moduli of holomorphic disks in $\mathbb{CP}^{p+1}$ with boundary along a totally real submanifold. In the above correspondence, two almost-Grassmannian structures are considered to be equivalent if one is a pullback of the other by some diffeomorphism, and two embeddings of $\mathbb{RP}^{p+1}$ are equivalent if they differ by some reparameterization and/or biholomorphism of $\mathbb{CP}^{p+1}$. By simply counting parameters, it follows that there is an infinite dimensional family of *non*-trivial right-flat deformations. This is later revealed more directly by computing a certain curvature invariant.

The situation with $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ is in stark contrast with that of the *un*-oriented Grassmannian $Gr(2, \mathbb{R}^{p+2})$, which does not admit any nontrivial right-flat deformations.

Theorem 1.1.3. *Every right-flat almost-Grassmannian structure on the unoriented Grassmannian $Gr(2, \mathbb{R}^{p+2})$ sufficiently close to the standard one is equivalent to the standard one.*

Local aspects of the geometry are then investigated in greater detail. The main tool here is the *local twistor bundle*, a particular vector bundle with connection, which is traditionally defined in a rather ad hoc manner. Using the Ward correspondence, which relates certain holomorphic vector bundles over a twistor space with vector bundles on the spacetime, we provide a more direct and geometric description of the local twistor bundle.

Theorem 1.1.4. *The Ward correspondence takes the dualized jet bundle $[J^1\mathcal{O}(1)]^*$ to the local twistor bundle.*

In analogy with the Weyl tensor of 4-dimensional Riemannian manifolds, almost-Grassmannian manifolds also have "scale invariant" curvature components, and these are what essentially govern the geometry. The local twistor bundle is useful because it precisely encodes these curvature components. As a first application, we prove a characterization of "locally flat" almost-Grassmannian structures in terms of torsion and curvature tensors.

Theorem 1.1.5. *A (p, q) almost-Grassmannian manifold is locally equivalent to the flat model if and only if*

$$T_{ab}{}^{c} = 0 \quad \textit{for} \quad p, q > 2$$

$$\widetilde{F}_{AA'BB'}{}^{CC'} = 0 \ \textit{and} \ \Psi_{ABC}{}^{D} = 0 \quad \textit{for} \quad p > 2, q = 2.$$

The cuvature $\Psi_{ABC}{}^{D}$, initially defined only in reference to the almost-Grassmannian manifold, is also given a twistor-theoretic interpretation. Lastly, we construct a special class of almost-Grasmannian manifolds with special holonomy. This is accomplished by representing solutions of the valence-2 twistor equation

$$\text{Trace-free part of } \{\nabla_{AA'}\omega^{BC}\} = 0 \tag{1.1.3}$$

by a certain nonvanishing cohomology group on the twistor space. Under a kind of special Lagrangian condition on the embedding $\mathbb{RP}^{2m+1} \hookrightarrow \mathbb{CP}^{2m+1}$, there are real solutions to eq. (1.1.3) which, by appropriate rescaling, can be made parallel.

Theorem 1.1.6. *Let v be a divergence-free vector field on $\mathbb{RP}^{2m+1}$ with respect to the standard metric, and denote by J the standard complex structure on $\mathbb{CP}^{2n+1}$. Then Jv determines a family of embeddings $\mathbb{RP}^{2n+1} \hookrightarrow \mathbb{CP}^{m+1}$ whose corresponding almost-Grassmannian structures are torsion-free and have holonomy group contained in $SO(2, \mathbb{R}) \cdot SL(2m, \mathbb{R})$.*

Chapter 2

The Geometry of Almost-Grassmannian Manifolds

This section is largely a summary of the relevant background, first developed by Bailey and Eastood in [BE91]. Although we are interested primarily in *real* geometry, it is natural to begin with the complex viewpoint. We now define the central objects of study in this dissertation, discuss the standard examples, and establish a framework for understanding deformations.

2.1 Complex Almost-Grassmannian Manifolds

Let M be a complex manifold, not necessarily compact, of dimension $n = pq$ with $p, q \geq 2$.

Definition 2.1.1. A (p, q) almost-Grassmannian structure on M is an isomorphism between its holomorphic tangent bundle and a tensor product of bundles,

$$\sigma : TM \xrightarrow{\cong} \mathcal{O}^A \otimes \mathcal{O}^{A'}, \tag{2.1.1}$$

where the bundles $\mathcal{O}^A, \mathcal{O}^{A'}$ have ranks p, q respectively. Additionally, there is an isomorphism between their top exterior powers,

$$\alpha : \wedge^p \mathcal{O}_A \xrightarrow{\cong} \wedge^q \mathcal{O}_{A'}. \tag{2.1.2}$$

When $p = q = 2$, an almost-Grassmannian structure is equivalent to a conformal class of metric with a spin structure. In analogy with this 4-dimensional case, we will refer to the bundles $\mathcal{O}^A, \mathcal{O}^{A'}$ as the (un)primed spin bundles. We will furthermore make use of abstract index notation, with lowercase indices representing tensorial quantities and uppercase indices representing spinorial quantities. These indices are related by eq. (2.1.1); explicitly, the isomorphism is given by a section $\sigma_a^{AA'} \in \Gamma(T^*M \otimes \mathcal{O}^A \otimes \mathcal{O}^{A'})$ which can be used to interchange tensorial indices with spinor indices, and vice versa. For instance, a vector field X^a has spinor representation $X^{AA'} = \sigma_a^{AA'} X^a$. However, we will often supress this notation and regard the indices a and AA' as completely identical. As usual, round and square brackets indicate symmetric and anti-symmetric parts, respectively.

Any choice of connections on $\mathcal{O}^A, \mathcal{O}^{A'}$ determine a connection on TM, and thus a torsion $T_{ab}{}^c$. This is necessarily skew in a, b, and due to the decomposition

$$\wedge^2 (V \otimes W) = (\wedge^2 V \oplus \odot^2 W) \oplus (\odot^2 V \oplus \wedge^2 W), \tag{2.1.3}$$

we get the spinorial decomposition

$$T_{ab}{}^c = F_{AA'BB'}{}^{CC'} + \widetilde{F}_{AA'BB'}{}^{CC'} \tag{2.1.4}$$

where $F_{AA'BB'}{}^{CC'} = F_{(AB)[A'B']}{}^{CC'}, \widetilde{F}_{AA'BB'}{}^{CC'} = \widetilde{F}_{[AB](A'B')}{}^{CC'}$.

Theorem 2.1.2. *The totally trace-free parts of F and $\widetilde{F}$ are independent of the original choice of connections.*

A (spinorial) tensor is totally trace-free if all possible traces vanish. The significance of this theorem is that the totally trace-free parts are invariants of the almost-Grassmannian structure.

Definition 2.1.3. A *scale* is a non-vanishing volume form on $\mathcal{O}^A$, i.e. a section $\epsilon \in \Gamma(\wedge^p \mathcal{O}_A)$.

The unprimed spin bundle is in no way preferred over the primed spin bundle; due to the isomorphism in eq. (2.1.2), we could equally well have chosen a volume form on

$\mathcal{O}^{A'}$. Recall that in dimension 4, an almost-Grassmannian structure determines a conformal class of metrics. In this setting, a scale coincides with a particular choice of metric in the conformal class. For general almost-Grassmannian manifolds, there is the following analog of the Levi-Civita connection.

Theorem 2.1.4. *For any scale on an almost-Grassmannian structure, there are unique connections on $\mathcal{O}^A, \mathcal{O}^{A'}$ such that the torsion quantities F and $\widetilde{F}$ are totally trace-free, and*

$$\nabla_a \epsilon = 0, \quad \nabla_a \alpha(\epsilon) = 0. \tag{2.1.5}$$

A tanget vector V^a is said to be *null* if its spin representation is simple, i.e. if there exist $\mu^A, \nu^{A'}$ such that $V^{AA'} = \mu^A \nu^{A'}$. By fixing a primed spinor at a point and sweeping out all unprimed spinors, we obtain a null p-plane. Such a plane is called an α-plane. Similarly, fixing an unprimed spinor and sweeping through all the primed spinors yields a β-plane.

Definition 2.1.5. An α-*surface* in a (p, q) almost-Grassmannian manifold is a p-dimensional submanifold whose tangent space at every point is an α-plane.

Definition 2.1.6. An almost-Grassmannian manifold is *right-flat* if every α-plane is tangent to some α-surface. In this case, the space of α-surfaces is called the *twistor space*, which need not be a manifold.

Theorem 2.1.7. *A (p, q) almost-Grassmannian manifold is right-flat if and only if*

$$\widetilde{F}_{AABB'}{}^{CC'} = 0 \quad \textit{for} \quad p > 2;$$
$$\widetilde{\Psi}_{A'B'C'}{}^{D'} = 0 \quad \textit{for} \quad p = 2.$$

The quantity $\widetilde{\Psi}_{A'B'C'}{}^{D'}$ is a component of the curvature, which serves as the almost-Grassmannian analog of the anti-self-dual Weyl tensor in 4 dimensions. For a precise definition, see the appendix of [BE91].

2.2 Real Almost-Grassmannian Manifolds

In order to link the above machinery with real geometry we introduce the notion of *complexification* [Eas84]. A real analytic manifold M can always be embedded in some complex manifold $\mathbb{C}M$, which is unique up to germ equivalence. This complex manifold $\mathbb{C}M$ is called the complexification of M, and may be explicitly realized by allowing complex numbers in the power series transition data that describes M. A real manifold can be recovered from its complexification as the fixed-point set of an anti-holomorphic involution.

One can also work directly with a factorization of the real tangent bundle TM. In particular, if the real α-surfaces are real analytic, they will be propagated along with M to get complex α-surfaces, and this process is reversible via the anti-holomorphic involution. Thus at first glance there is no apparent advantange to complexification. The utility of the complex viewpoint will be made clear in later sections.

The namesake and primary example of this geometric structure is the Grassmannian manifold. The Grassmannian $Gr(q, \mathbb{C}^{p+q})$ carries a canonical (p, q) almost-Grassmannian structure, which is defined as

$$TGr(q, \mathbb{C}^{p+q}) = Hom(\gamma, \mathbb{C}^{p+q}/\gamma) = \gamma^* \otimes \mathbb{C}^{p+q}/\gamma$$

where γ is the tautological q-plane bundle. The isomorphism between top exterior powers is induced by the standard volume form on $\mathbb{C}^{p+q}$.

Proposition 2.2.1. *The Grassmannian $Gr(q, \mathbb{C}^{p+q})$ with its canonical almost-Grassmannian structure is right-flat, and its α-surfaces are all diffeomorphic to $\mathbb{CP}^p$.*

Proof. Fix an inner product on $\mathbb{C}^{p+q}$ so that $TGr(q, \mathbb{C}^{p+q}) = Hom(\gamma, \gamma^\perp)$. For each line $\ell \subset \gamma_x \subset \mathbb{C}^{p+q}$, there is a corresponding projective space $P_\ell \subset Gr(q, \mathbb{C}^{p+q})$ consisting of the q-planes spanned by $\ell^\perp \cap \gamma_x$ and a line in $\ell \oplus \gamma_x^\perp$. In particular, there is a natural identification $P_\ell = \mathbb{P}(\ell \oplus \gamma_x^\perp)$. We therefore obtain at each point x a $\mathbb{P}(\gamma_x) \cong \mathbb{CP}^{q-1}$ worth of $\mathbb{CP}^p$'s passing through x.

Now for some P_ℓ, consider the point y corresponding to a line $\tilde{\ell} \in \mathbb{P}(\ell \oplus \gamma_x^\perp)$. We have

$$T_y P_\ell = Hom(\tilde{\ell}, \tilde{\ell}^\perp \cap (\ell \oplus \gamma_x^\perp)), \tag{2.2.1}$$

which is an α-plane because y corresponds to the subspace $\tilde{\ell} \oplus (\ell^\perp \cap \gamma_x)$, and $(\tilde{\ell} \oplus (\ell^\perp \cap \gamma_x))^\perp = \tilde{\ell}^\perp \cap (\ell \oplus \gamma_x^\perp)$. This shows that each projective space P_ℓ is an α-surface, and by dimensionality considerations, every α-plane is exhausted in this manner. □

Corollary 2.2.2. *The Grassmannian $Gr(q, \mathbb{R}^{p+q})$ with its canonical almost-Grassmannian structure is right-flat, and its real α-surfaces are diffeomorphic to $\mathbb{RP}^p$.*

Corollary 2.2.3. *The oriented Grassmannian $\widetilde{Gr}(q, \mathbb{R}^{p+q})$ with its canonical almost-Grassmannian structure is right-flat, and its real α-surfaces are diffeomorphic to S^p.*

Proof. The first corollary follows proposition 2.2.1 by applying the anti-holomorphic involution given by ordinary complex-conjugation. In other words, $Gr(p, \mathbb{C}^{p+q})$ is the complexification of $Gr(q, \mathbb{R}^{p+q})$. Since right-flatness is a local condition, it must then be the case that the oriented Grassmannian is also right-flat. The fact that lifted real α-surfaces are connected may be seen by tracing through the argument in proposition 2.2.1 for $\widetilde{Gr}(q, \mathbb{R}^{p+q})$. □

2.3 Deformations of almost-Grassmannian structures

In light of corollary 2.2.2, we are led to ask whether the (un-)oriented Grassmannian carries other right-flat almost-Grassmannian structures. In this section we establish a general framework for discussing this question.

Let M be a manifold with an almost-Grassmannian structure. We shall view the induced connection on M as consisting of spin connections together with an Infeld-Van der Waerden symbol $\sigma_a^{AA'}$. By abuse of notation, we will denote all connections by the same symbol ∇_a. Explicitly,

$$\nabla_a X^b = \sigma_b^{BB'}(1 \otimes \nabla_a + \nabla_a \otimes 1)(\sigma_c^{BB'} V^c). \tag{2.3.1}$$

Recall that the spin connections are determined by σ, up to choice of scale, by the condition that the torsion be totally trace-free in spinor indices.

Any other almost-Grassmannian structure is given by some $\hat{\sigma}_a^{AA'}$, or equivalently an isomorphism of the tangent bundle $\Phi_a^b = \sigma_a^{AA'}\hat{\sigma}_{AA'}^{\ \ b}$. A deformation is therefore given by a 1-parameter family of isomorphisms of the tangent bundle through the identity. We will write this as a power series expansion

$$\Phi_a^b = \delta_a^b + t\phi_a^b + O(t^2). \tag{2.3.2}$$

Similarly, a general change in spin connections is given by

$$\hat{\nabla}_a\mu^C = \nabla_a\mu^C + tK_{aB}{}^C\mu^B + O(t^2), \tag{2.3.3}$$

$$\hat{\nabla}_a\mu^{C'} = \nabla_a\mu^{C'} + t\tilde{K}_{aB'}{}^{C'}\mu^{B'} + O(t^2). \tag{2.3.4}$$

Setting $Q_{ab}{}^c = K_{aB}{}^C\delta_{B'}^{C'} + \tilde{K}_{aB'}{}^{C'}\delta_B^C$, the torsion of the induced connection on TM is

$$\hat{T}_{ab}{}^c = T_{ab}{}^c - t(\nabla_{[a}\phi_{b]}{}^c + Q_{[ab]}{}^c) + O(t^2). \tag{2.3.5}$$

The contorsion tensors $K, \tilde{K}$ are thus determined up to scale by the condition that $\nabla_{[a}\phi_{b]}{}^c + Q_{[ab]}{}^c$ is totally trace-free, i.e. Q is exactly the trace part of $\nabla_{[a}\phi_{b]}{}^c$. One may choose, for example,

$$K_{aB}{}^C = -\nabla_{BD'}\phi_a^{CD'} \tag{2.3.6}$$

$$\tilde{K}_{aB'}{}^{C'} = -\nabla_{DB'}\phi_a^{DC'} + \nabla_{DD'}\phi_a^{DD'}\delta_{B'}^{C'}. \tag{2.3.7}$$

Equation 2.3.5 implies that the torsion is constant, up to first order, precisely when $\nabla_{[a}\phi_{b]}{}^c$ is pure trace. We can therefore characterize the right-flat deformations of the Grassmannian by solutions of the equation

$$\text{Trace-free part of } \{\nabla_{[a}\phi_{b]}{}^c\} = 0. \tag{2.3.8}$$

We are especially interested in the particular case that $q = 2$ and p is even.

Definition 2.3.1. A *quaternionic* almost-Grassmannian structure is a $(2m, 2)$ almost-Grassmannian structure.

One important property of quaternionic almost-Grassmannian structures is that the F component of the torsion automatically vanishes. This is because $F_{AA'BB'}{}^{CC'}$ is skew in $A'B'$, and if the primed spin bundle has rank 2, we must have a factorization $F_{ABA'B'}{}^{CC'} = G_{AB}{}^{CC'}\epsilon_{A'B'}$ where $\epsilon_{A'B'}$ is a scale form. But F is also trace-free, and tracing against ϵ is invertible, so if must be the case that F is automatically zero. If in addition $m \geq 2$, theorem 2.1.7 states that the right-flat condition is equivalent to the vanishing of $\widetilde{F}$. Thus, a right-flat quaternionic almost-Grassmannian manifold is *torsion-free.*

The decomposition in eq. (2.1.3) also applies to eq. (2.3.8), and so by the same reasoning above we only need to consider the component that is symmetric in $A'B'$ (and skew in AB). In this special case, the right-flat deformations are given by solutions to the equation

$$\text{Trace-free part of } \{\nabla_{[A|(A'}\phi_{B')|B]}{}^{CC'}\} = 0. \tag{2.3.9}$$

Chapter 3

The Twistor Correspondence

In this chapter we discuss the twistor construction and its inverse for right-flat $(p,2)$ almost-Grassmannian manifolds.

We have aready encountered the twistor space for complex almost-Grassmannian manifolds in definition 2.1.6. Recall in the $(p,2)$ case that there is a $\mathbb{CP}^1$ bundle of α-planes over M, given by $\mathbb{P}(\mathcal{O}^{A'})$. This total space is called the *correspondence space*, and there is a natural lifting of α-surfaces defined by sending the points in a given α-surface to their tangent α-planes. The space of leaves in this foliation is the twistor space, denoted by $\mathscr{Z}$. This space $\mathscr{Z}$ is not generally Hausdorff, but if we restrict ourselves to a neighborhood in M, then $\mathscr{Z}$ will be a complex manifold. The inverse construction is supplied by the following theorem.

Theorem 3.0.1. *Let $\mathscr{Z}$ be a $p+1$ dimensional complex manifold, containing a line $L \cong \mathbb{CP}^1$ with normal bundle $\oplus^p \mathcal{O}_L(1)$. Suppose furthermore there is some line bundle $\mathcal{O}(1)$ on $\mathscr{Z}$ with $\mathcal{O}(1)|_L = \mathcal{O}_L(1)$ and $\mathcal{O}(-p-2) = K_{\mathscr{Z}}$. Then a neighborhood of L in $\mathscr{Z}$ is the twistor space of a $(p,2)$ almost-Grassmannian manifold.*

Typically, real geometries are extracted from this picture by way of an anti-holomorphic involution on $\mathscr{Z}$: the $\mathbb{CP}^1$'s fixed by an involution will form a real $(p,2)$ almost-Grassmannian manifold. However, instead of focusing on the involution, we may instead consider its fixed-point set P. In the case that this involution acts by conjugation on each fixed $\mathbb{CP}^1$ (as opposed to the antipodal map), P will divide each such $\mathbb{CP}^1$ into two hemispheres. From this perspective, the moduli space of interest is (up to double cover) the space of holomorphically embedded disks with boundary along P. A deformation of the associated almost-Grassmannian structure corresponds to a deformation of the fixed-point set $P \subset \mathscr{Z}$.

This approach was first carried out by LeBrun and Mason for Zoll surfaces and split-signature ASD 4-manifolds [LM0207, LM0702]. Here we will generalize their construction for $p > 2$.

Let M now be a real-analytic manifold with an almost-Grassmannian structure. We would like to consider the twistor space of its complexification $\mathbb{C}M$, which is a complex manifold that contains M as a totally real subspace. Concretely, one can form $\mathbb{C}M$ by allowing complex numbers in the power series transition data for M. The problem is that $\mathbb{C}M$ is only defined as the germ of the embedding of M, and therefore different complexifications might disagree about α-surfaces "far" from M. For this reason, we can only be concerned with complex α-surfaces that intersect M. Then, observe that any such α-surface either intersects M at a single point, or is the complexification of a real α-surface. This is a consequence of the following proposition.

Proposition 3.0.2. *Suppose that a complex α-plane contains a real null line. Then this complex α-plane is a complexified real α-plane.*

Proof. It will suffice to work within a single fixed vector space. Let $V \otimes W$ be a real vector space, and $\mathbb{C} \otimes (V \otimes W) = \mathbb{C}V \otimes \mathbb{C}W$ be its complexification. A complex α-plane is then of the form $\mathbb{C}V \otimes \mathfrak{w}$ for some $\mathfrak{w} \in \mathbb{C}W$. Now if this contains a real null line, then we have $v \otimes w \in \mathbb{C}V \otimes \mathfrak{w}$ for some $v \otimes w \in V \otimes W$, and it follows that $\mathfrak{w} = \mathbb{C} \otimes w$. Then $\mathbb{C}V \otimes \mathfrak{w} = \mathbb{C} \otimes (V \otimes w)$ as required. $\square$

Our twistor space will therefore be, in essence, the union of the real α-surfaces and the complex α-planes which are not complexified real α-planes. We cannot in general expect this space to even be Hausdorff, but we will soon show that for the special case $M = \widetilde{Gr}(2, \mathbb{R}^{p+2})$, the twistor space is a compact complex manifold.

3.1 The Real Twistor Space

We will now focus on $\widetilde{Gr}(2, \mathbb{R}^{p+2})$. The canonical almost-Grassmannian structure yields an isomorphism $T\widetilde{Gr}(2, \mathbb{R}^{p+2}) = E \otimes H$, with $\text{rank}(E) = p$, $\text{rank}(H) = 2$. The space $F := \mathbb{P}(H)$, called the (real) correspondence space, is foliated by lifted α-surfaces. Denote the leaf space, also called the *real twistor space*, by Z.

Proposition 3.1.1. *The real twistor space Z is a smooth manifold.*

Proof. This is essentially a corollary of the Reeb stability theorem. In [Thu74], Thurston proved the following generalization.

Theorem 3.1.2 (Thurston)**.** *Let $\mathcal{F}$ be a codimension k foliation, and L be a compact leaf of $\mathcal{F}$. If $H^1(L, \mathbb{R}) = 0$ and $H^1(L, Gl(k, \mathbb{R})) = 0$, then L has a neighborhood diffeomorphic to $L \times \mathbb{R}^k$ in such a way that each $L \times \{x\}$ is a leaf of $\mathcal{F}$.*

By corollary 2.2.3, each leaf in the foliation of $\mathbb{P}(H)$ is a sphere S^p, which satisfies the required cohomological conditions. The trivializations then yield a smooth atlas for Z. $\square$

We saw in the proof of proposition 2.2.1 and its corollaries that α-surfaces of $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ with its standard almost-Grassmannian structure correspond to lines in $\mathbb{R}^{p+2}$. In particular, its twistor space is identified with $\mathbb{RP}^{p+1}$. We therefore have the real double fibration of smooth manifolds

$$\begin{array}{ccc} & \mathbb{P}(H) & \\ {\scriptstyle p} \swarrow & & \searrow {\scriptstyle q} \\ \widetilde{Gr}(2, \mathbb{R}^{p+2}) & & \mathbb{RP}^{p+1} \end{array}$$

In fact, this picture holds for arbitrary right-flat deformations of the standard almost-Grassmannian structure. The key result is that α-surfaces being diffeomorphic to spheres is an open condition in the space of right-flat almost-Grassmannian structures. Then proposition 3.1.1 will again apply, and we will get a smooth family of compact manifolds. Since one member of this family (the standard one) is diffeomorphic to $\mathbb{RP}^{p+1}$, they all are.

Proposition 3.1.3. *Let M be a right-flat almost-Grassmannian manifold whose α-surfaces are all spheres. Then any other right-flat almost-Grassmannian structure sufficiently close in the C^1 topology also has α-surfaces that are spheres.*

Proof. The key result we need to apply is a theorem due to Langevin and Rosenberg [LR77].

Theorem 3.1.4 (Langevin, Rosenberg)**.** *Let $X \to Y$ be a C^1 fiber bundle with compact, simply connected fibers and a compact base. Let $\mathcal{F}$ be the foliation of X by these fibers. Then $\mathcal{F}$ has a neigborhood in the C^1 Epstein topology on the space of foliations whose elements are all of the form $\phi^*\mathcal{F}$ for some C^1 diffeomorphism $\phi : X \to X$.*

Any two almost-Grassmannian structures are related by some endomorphism $A : TM \to TM$. This endomorphism allows us to identify spin bundles, so that we can compare lifted α-planes by pulling back. Explicitly, we have two bundles of α-planes $F, F' \to M$ carrying distributions D, D' respectively. Using A we can regard D' as a distribution on F.

By assumption, both distributions are integrable, and therefore are tangent to foliations $\mathcal{F}, \mathcal{F}'$ of lifted α-surfaces. If the endomorphism A is C^1 close to the identity, then the foliations $\mathcal{F}, \mathcal{F}'$ will also be C^1 close. We can therefore apply theorem 3.1.4 to conclude that there is a C^1 diffeomorphism of F sending $\mathcal{F}$ to $\mathcal{F}'$, so the leaves of $\mathcal{F}'$ must also be spheres.

□

All together, we have proven the following.

Proposition 3.1.5. *The real twistor space of a sufficiently small right-flat deformation of $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ is diffeomorphic to $\mathbb{RP}^{p+1}$.*

3.2 The Complex Twistor Space

Let $M = \widetilde{Gr}(2, \mathbb{R}^{p+2})$ be endowed with an almost-Grassmannian structure near the standard one. We begin our journey into the complex setting by complexifying the bundle H. This is motivated by the aforementioned notion of complexification, in which the complex spin bundles $\mathcal{O}^A, \mathcal{O}^{A'}$ restrict along M to $E \otimes \mathbb{C}, H \otimes \mathbb{C}$. Note, however, that we do not actually require M to be real-analytic. For notational convenience, we will drop the restriction and identify $\mathbb{C} \otimes H = \mathcal{O}^{A'}$ as bundles over $\widetilde{Gr}(2, \mathbb{R}^{p+2})$. The (complex) correspondence space $\mathscr{F} := \mathbb{P}(\mathcal{O}^{A'})$ is then the $\mathbb{CP}^1$ bundle of complex α-planes over $\widetilde{Gr}(2, \mathbb{R}^{p+2})$.

To understand the complex geometry of this bundle a little better, consider a real local basis h_1, h_2 for H. We can then have a fiber coordinate ζ given by $\zeta \mapsto [h_1 + \zeta h_2]$. If we now set $h = h_1 + \zeta h_2$ and choose a real local basis $e_1, \ldots, e_p$ for E, we can define a map $\mathscr{F} \to \mathbb{P}(\wedge^p TM)$ fiber-wise by

$$\zeta \mapsto [(h \otimes e_1) \wedge \cdots \wedge (h \otimes e_p)]. \tag{3.2.1}$$

The almost-Grassmannian structure induces a decomposition of $\wedge^p T\widetilde{Gr}(2, \mathbb{R}^{p+2})$ into irreducible $SL(2, \mathbb{R}) \times SL(p, \mathbb{R})$ representations. In particular, there is a projection map $\wedge^p T\widetilde{Gr}(2, \mathbb{R}^{p+2}) \to \odot^p H \otimes \wedge^p E$. Since the image of eq. (3.2.1) is symmetric in H and skew in E, we can compose it with this projection to obtain a map $\mathscr{F} \to \mathbb{P}(\odot^p H \otimes \wedge^p E)$. The fibers of $\mathscr{F}$ are then seen to be rational normal curves within $\mathbb{P}(\odot^p H \otimes \wedge^p E)$.

The correspondence space in fact carries a natural distribution of complex $p+1$ planes. Fix a scale ϵ and let ∇ be the associated connection (on H). This determines a parallel transport of points in $\mathscr{F}$ along curves in $\widetilde{Gr}(2, \mathbb{R}^{p+2})$, and therefore a horizontal distribution in $T\mathscr{F}$. By complexifying these bundles, we may then construct a distribution $\mathcal{D}_h \subset T_{\mathbb{C}}\mathscr{F}$ of horizontally lifted complex α-planes. Let $\mathcal{V}^{0,1} \subset T_{\mathbb{C}}\mathscr{F}$ be the $(0,1)$-tangent bundle of the fibers, and set $\mathcal{D} = \mathcal{D}_h + \mathcal{V}^{0,1}$.

Proposition 3.2.1. *The distribution $\mathcal{D}$ is involutive if and only if the almost-Grassmannian* *structure on* $\widetilde{Gr}(2,\mathbb{R}^{p+2})$ *is right-flat.*

Proof. Let's first dispose of the easier direction. Suppose that $\mathcal{D}$ is involutive. Then since $T_{\mathbb{C}}F$ is trivially involutive, their intersection $\mathcal{D}\cap T_{\mathbb{C}}F = \mathcal{D}_h|_F$ is as well. This is just the complexified bundle of lifted α-planes, and so the un-complexified bundle of α-planes must also be involutive. By the Frobenius theorem it must be integrable, and thus the almost-Grassmannian structure is right-flat.

For the other direction, we will first set up a coordinate representation of $\mathcal{D}$. Let x denote a local coordinate on some open set $U\subset\widetilde{Gr}(2,\mathbb{R}^{p+2})$, and ζ be the fiber coordinate described above. Define

$$\phi_j = \binom{p}{j} h_1^j h_2^{p-j}\otimes\epsilon^*\in\Gamma(\odot^p H\otimes\wedge^p E), \tag{3.2.2}$$

so from eq. (3.2.1) we have the coordinates

$$(x,\zeta)\mapsto[\phi_0+\zeta\phi_1+\cdots+\zeta^p\phi_p]_x \tag{3.2.3}$$

viewed as a submanifold of $\mathbb{P}(\odot^p H\otimes\wedge^p E)$. The complex α-plane corresponding to each (x,ζ) is simply the complex span

$$span\{(h_1+\zeta h_2)\otimes e_1,\ldots,(h_1+\zeta h_2)\otimes e_p\}, \tag{3.2.4}$$

which we want to horizontally lift. With a dual basis $u^{k\ell}=(h_k\otimes e_\ell)^*$, we can write the induced connection as

$$\nabla\phi_j = \theta_j{}^i\phi_i, \tag{3.2.5}$$

where $\theta_j{}^i$ is a connection 1-form that can be expanded in terms of the dual basis as $\theta_j{}^i = \theta_{k\ell j}{}^i u^{k\ell}$. Here we are not using abstract index notation, and the $\theta_{k\ell j}{}^i$ are simply functions. The distribution is then given by

$$\mathcal{D} = span\left\{\mathfrak{w}_1,\ldots,\mathfrak{w}_p,\frac{\partial}{\partial\bar{\zeta}}\right\} \tag{3.2.6}$$

where the $\mathfrak{w}_\ell$ are horizontal lifts of $(h_1+\zeta h_2)\otimes e_j$. Explicitly,

$$\mathfrak{w}_\ell = (h_1+\zeta h_2)\otimes e_\ell + Q_\ell(x,\zeta)\frac{\partial}{\partial\zeta} \tag{3.2.7}$$

with

$$Q_\ell(x,\zeta) = \sum_{i,j=0}^{p} \zeta^j ({\theta_{1\ell j}}^i + \zeta {\theta_{2\ell j}}^i). \tag{3.2.8}$$

Here these are not quite horizontal lifts of $(h_1 + \zeta h_2) \otimes e_j$, but instead differ by a $\partial/\partial\overline{\zeta}$ component. They are still horizontal (and real) when ζ is real, but the removal of the $\partial/\partial\overline{\zeta}$ component additionally makes the vector fields holomorphic in ζ (in terms of the basis $h_j \otimes e_\ell, \partial/\partial\zeta$).

Now suppose the almost-Grassmannian structure is right-flat. As explained above, this implies the distribution $\mathcal{D}_h|_F \subset T_{\mathbb{C}}F$ is involutive. By construction, this distribution is spanned by $\mathfrak{m}_j$, and therefore

$$[\mathfrak{w}_j, \mathfrak{w}_k] \wedge \mathfrak{w}_1 \wedge \cdots \wedge \mathfrak{w}_p = 0 \tag{3.2.9}$$

for any j, k. But the components of each $\mathfrak{w}_j$ are holomorphic in ζ, and the same must be true of the quantity in eq. (3.2.9). Since it is zero for ζ real, it must be zero for all ζ. This shows that the distribution spanned by $\mathfrak{w}_j$ is in fact involutive everywhere. Furthermore, again by holomorphicity we must have that $[\partial/\partial\zeta, \mathfrak{w}_j] = 0$. This proves that $\mathcal{D}$ is involutive on the region parameterized by our coordinates (x, ζ). In other words, the O'Neill tensor

$$\begin{aligned} A_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} &\to T_{\mathbb{C}}\mathscr{F} \\ (u, v) &\mapsto [u, v] \bmod \mathcal{D}, \end{aligned} \tag{3.2.10}$$

vanishes along this region. The O'Neill tensor is continuous, and therefore must vanish on the entire set $p^{-1}(U)$. We can build an open cover of $\mathscr{F}$ of such sets, and therefore $\mathcal{D}$ is involutive everywhere. □

Proposition 3.2.2. *The distribution $\mathcal{D}$ is independent of the initial choice of scale.*

Proof. Observe that the distribution $\mathcal{D}_h|_F$ is independent of any choice of scale, since it is defined as the distribution of tangent spaces to lifted α-surfaces. If we choose some other scale and construct a new distribution $\hat{\mathcal{D}}_h$, they must then coincide along F. But the coefficents in terms of the basis $h_i \otimes e_j, \partial/\partial\zeta$ are holomorphic, so the two distribution must in fact coincide everywhere in the region parameterized by (x, ζ) defined above. Again by continuity and taking an open cover, the two distributions coincide on all of $\mathscr{F}$. $\square$

We can very nearly interpret $\mathcal{D}$ as the $T^{0,1}$ component of an almost-complex structure. The problem is that $\mathcal{D}$ is real along F, i.e. $\mathcal{D}_F = \overline{\mathcal{D}}_F$. But away from this equatorial section, we indeed have $\mathcal{D} \cap \overline{\mathcal{D}} = 0$, and therefore an almost-complex structure which is integrable when its associated almost-Grassmannian structure is right-flat.

Since the bundle H is orientable, the inclusion $\mathbb{P}(H) \subset \mathbb{P}(\mathcal{O}^{A'})$ divides the complex correspondence space into two connected components, each of which is a disk bundle over M. Our chosen scale determines a particular orientation on H, and and thus an orientation on $\mathbb{P}(H)$ in the sense that each fiber is an oriented circle. Similarly, each disk bundle induces an orientation on its boundary $\mathbb{P}(H)$ according to the fiber-wise complex structure. We may then define $\mathscr{F}_+$ as the component whose induced orientation on $\mathbb{P}(H)$ agrees with the scale.

Now consider the quotient map Ψ on $\mathscr{F}_+$, defined as the identity on the interior and $q : \mathbb{P}(H) \to \mathbb{RP}^{p+1}$ on the boundary. The image, denoted by $\mathscr{Z}$, is a smooth $2p+2$-dimensional manifold. This may be seen via coordinate representation of the map near the boundary $\partial\mathscr{F}_+ \approx \mathbb{RP}^{p+1} \times S^p$

$$\mathbb{RP}^{p+1} \times S^p \times [0, 1) \to \mathbb{RP}^{p+1} \times \mathbb{R}^{p+1} \tag{3.2.11}$$

$$(p, \vec{x}, t) \mapsto (p, t\vec{x}). \tag{3.2.12}$$

The 4-dimensional case of the following theorem is proven in [LM0702], and the proof carries over without modification.

Theorem 3.2.3. *Let $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ be endowed with a right-flat almost-Grassmannian structure near the standard one. Then $\mathscr{Z} = \Psi(\mathscr{F}_+)$ obtained as above carries a unique complex structure such that $\Psi_*\mathcal{D} \subset T^{0,1}\mathscr{Z}$.*

The complex manifold $\mathscr{Z}$ is called the (complex) twistor space.

A right-flat deformation corresponds to a complex deformation of $\mathscr{F}$, and thus a deformation of $\mathscr{Z}$. But $\mathscr{Z} \cong \mathbb{CP}^{p+1}$ is holomorphically rigid, yielding the following corollary.

Corollary 3.2.4. *The twistor space of $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ with a right-flat almost-Grassmannian structure near the standard one is biholomorphic to $\mathbb{CP}^{p+1}$.*

Although the twistor spaces themselves are indistinguisahable, they come equipped with additional data. Namely, there is a real slice $\Psi(\partial\mathscr{F}_+) \approx \mathbb{RP}^{p+1} \subset \mathbb{CP}^{p+1}$, which is deformed along with the almost-Grassmannian structure. As one would hope, the canonical almost-Grassmannian structure corresponds to the standard embedding $\mathbb{RP}^{p+1} \subset \mathbb{CP}^{p+1}$. In general, the real slice allows us to completely reconstruct the original almost-Grassmannian strucure, as we show in the next section.

3.3 The Inverse Construction

Our goal is to prove the following theorem.

Theorem 3.3.1. *Let $P \subset \mathbb{CP}^{p+1}$ be the image of a smooth embedding $\mathbb{RP}^{p+1} \hookrightarrow \mathbb{CP}^{p+1}$ near the standard one. Then the family of embedded holomorphic disks with boundary along P is a smooth manifold diffeomorphic to $\widetilde{Gr}(2, \mathbb{R}^{p+2})$, which carries a natural right-flat almost-Grassmannian structure.*

First we need that the space of disks is a smooth manifold. In the usual twistor correspondence for 4-dimensional Riemannian manifold, this is essentially a consequence of the deformation theory for compact curves in a complex manifold.

Theorem 3.3.2 (Kodaira)**.** *Let $X \subset Z$ be a compact complex submanifold whose normal bundle satisfies $H^1(X, \mathcal{O}(N)) = 0$. Then any small deformation of Z contains an $h^0(X, \mathcal{O}(N))$-complex dimensional family of compact complex submanifolds, obtained by deforming X.*

In the Riemannian setting, this theorem guarantees that we can recover a moduli space of curves after deforming the twistor space. But we are not concerned with moduli of compact curves, and instead have a space of curves with boundary along a real submanifold. The required machinery was developed by LeBrun [LeB05], which we now summarize.

Given a Riemann surface X with boundary, we can build the *abstract double* $\mathbb{X}$ by gluing a mirrored copy $\bar{X}$ via the identity map along their boundaries.

$$\mathbb{X} = X \cup_{\partial X} \bar{X} \tag{3.3.1}$$

The abstract double carries an anti-holomorphic involution ρ given by interchanging $X, \bar{X}$ and fixing ∂X. Moreover, holomorphic vector bundles on X can be extended to $\mathbb{X}$ by assigning the conjugate bundle over $\bar{X}$, which does in fact yield a locally trivial structure near ∂X. The involution ρ then extends to these bundles in the obvious manner. Now suppose that $X \subset Z$ is an embedded curve with boundary. Then X has a normal bundle N, which extends to a bundle $\mathcal{N} \to \mathbb{X}$. Let $H^0_\rho(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ denote the sections fixed by ρ, which is a real vector space since ρ acts as an anti-holomorphic involution on $H^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))$.

Theorem 3.3.3 (LeBrun)**.** *Let (Z, P) denote a complex manifold Z with a totally real submanifold P. Suppose that $(X, \partial X) \subset (Z, P)$ is an embedded curve with boundary along P. If $H^1(\mathbb{X}, \mathcal{O}(\mathcal{N})) = 0$, then any small deformation (Z', P') contains a $h^0(\mathbb{X}, \mathcal{O}(\mathcal{N}))$ real dimensional family of curves-with-boundary obtained by deforming X.*

Corollary 3.3.4. *Suppose that $P \subset \mathbb{CP}^{p+1}$ is the image of a smooth embedding $\mathbb{RP}^{m+1} \hookrightarrow \mathbb{CP}^{p+1}$ near the standard one. Then there is a smooth $2p$-dimensional family of embedded holomorphic disks with boundary along P.*

Now that we know this family of disks actually exists, it is a simple matter to establish a diffeomorphism with $\widetilde{Gr}(2, \mathbb{R}^{p+1})$. We have previously seen that in the standard case, the disks come in conjugate pairs forming closed $\mathbb{CP}^1$'s. Each such curve must intersect the standard complex p-quadric $Q = \{z_0^2 + \cdots + z_{p+1}^2 = 0\} \subset \mathbb{CP}^{p+1}$ at a conjugate pair of points; one per hemisphere. A deformed disk will continue to intersect Q at a single point, and conversely, since the disks must foliate $\mathbb{CP}^{p+1} - P$, every point along Q corresponds to a disk. We therefore have a smooth family of compact manifolds, so they are all diffeomorphic to one another.

It will be convenient to continue identifying the parameter space of disks with Q (as a real manifold). This space carries a tautological closed disk bundle $\mathscr{F}_+ \to Q$, with a map $\Psi : \mathscr{F}_+ \to \mathbb{CP}^{p+1}$. By construction, Ψ is a diffeomorphism between the interior of $\mathscr{F}_+$ and $\mathbb{CP}^{p+1} - P$. After complexifying the tangent bundles, we have a map $\Psi_* : T_{\mathbb{C}}\mathscr{F}_+ \to T_{\mathbb{C}}\mathbb{CP}^{p+1}$. Let $\Psi_*^{1,0}$ denote the composition of Ψ_* followed by projection onto the holomorphic tangent space $T^{1,0}\mathbb{CP}^{m+1}$. Set $\mathcal{D} = \ker \Psi_*^{1,0}$. If we assume that Ψ is C^1 close to that of the flat model, then Ψ_* will also have maximal rank, so that $\mathcal{D}$ is a rank $p+1$ complex distribution on all of $\mathscr{F}_+$. The boundary $\partial\mathscr{F}_+$, which is $2p+1$-real dimensional, is mapped to $\mathbb{RP}^{p+1}$ and therefore

$$E := \ker \Psi_*|_{\partial\mathscr{F}_+} \tag{3.3.2}$$

has rank at least p. On the other hand, Ψ is fiberwise holomorphic, and so $\mathcal{D}$ contains the vertical tangent space $V^{0,1}$ of the fibers. Thus, $(E \otimes \mathbb{C}) \oplus V^{0,1} \subset \mathcal{D}$. It follows that E has rank exactly p.

Now form the abstract double $\mathscr{F}$ of $\mathscr{F}_+$, defined fiberwise as above. Explicitly, one may take a duplicate copy $\mathscr{F}_-$ of $\mathscr{F}_+$ and glue them together along their boundaries. The distribution $\mathcal{D}$ extends over the double by defining it to be the conjugate $\overline{\mathcal{D}}$ on $\mathscr{F}_-$. We thus have a $\mathbb{CP}^1$ bundle $\wp : \mathscr{F} \to Q$ with a distribution $\mathcal{D}$ of complex $p+1$ planes. This distribution is furthermore involutive away from the fiberwise equator.

Proposition 3.3.5. *The evaluation of $c_1(\mathcal{D})$ on a fiber of $\wp$ is $-p-2$.*

Proof. Consider a fiber of $\mathscr{F}_+$. By extending the normal bundle of this disk in $\mathbb{CP}^{p+1}$ across the abstract double, we get a splitting $N = \oplus_{j=1}^{p}\mathcal{O}(\kappa_j)$. The sum $\kappa = \kappa_1 + \cdots + \kappa_p$ is called the Maslov index of the disk, and for the standard $\mathbb{RP}^{p+1} \subset \mathbb{CP}^{p+1}$ we have $\kappa_j = 1$, so $\kappa = p$.

The Maslov index is invariant under complex deformations [MS04], and so $\kappa = p$ for all $P \approx \mathbb{RP}^{p+1}$ near the standard one. The argument in [LM0702] then implies $c_1(\mathcal{D})$ on a fiber of $\wp$ is $-p-2$. $\square$

We have previously observed that $V^{0,1} \subset \mathcal{D}$, so it makes sense to consider the rank m distribution $\mho := \mathcal{D}/V^{1,0}$. By construction, $\mho$ is mapped injectively by $\wp_*$ into $T_{\mathbb{C}}Q$, and we therefore obtain a map

$$\begin{aligned} \Phi : \mathscr{F} &\to Gr(p, T_{\mathbb{C}}Q) \\ x &\mapsto \wp_*(\mho|_x) = \wp_*(\mathcal{D}|_x). \end{aligned} \tag{3.3.3}$$

This map is holomorphic on the interior of $\mathscr{F}_+$. To see this, let ζ be a holomorphic fiber coordinate of $\mathscr{F}_+$ and let $\mathfrak{w}_j$ be m sections of $\mathcal{D}$ which, with $\partial/\partial\bar{\zeta}$, span $\mathcal{D}$. Then since $\mathcal{D}$ is involutive,

$$\frac{\partial}{\partial\bar{\zeta}}(\wp_*\mathfrak{w}_j) = \wp_*(\mathcal{L}_{\partial/\partial\bar{\zeta}}\,\mathfrak{w}_j) = \wp_*([\partial/\partial\bar{\zeta}, \mathfrak{w}_j]) \equiv 0 \bmod \operatorname{span}\{\mathfrak{w}_1, \ldots, \mathfrak{w}_p\}, \tag{3.3.4}$$

which implies that Φ is fiberwise holomorphic on the interior of $\mathscr{F}_+$. Then Φ must be fiberwise holomorphic on the interior of $\mathscr{F}_-$ as well, and by continuity must in fact be fiberwise holomorphic on the entirety of $\mathscr{F}$.

Let $\iota : Gr(p, T_{\mathbb{C}}Q) \hookrightarrow \mathbb{P}(\wedge^p T_{\mathbb{C}}Q)$ denote the Plucker embedding, and consider the composition $\iota \circ \Phi$. Since $c_1(V^{0,1}) = -2$ on a fiber of $\wp$, proposition 3.3.5 tells us that $c_1(\mho) = -p$. Note also that the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}(\wedge^p T_{\mathbb{C}}Q)$ pulls back via ι to the tautological p-plane bundle over $Gr(p, T_{\mathbb{C}}Q)$, and therefore $(\iota \circ \Phi)^*\mathcal{O}(-1) = \wedge^p \mho$. This implies that the restriction of $\iota \circ \Phi$ to a fiber of $\wp$ must be a degree p map.

Proposition 3.3.6. *Isomorphisms $\mathbb{C}^{2p} \cong \mathbb{C}^2 \otimes \mathbb{C}^p$ correspond exactly to degree p rational curves in $Gr(p, \mathbb{C}^{2p}) \subset \mathbb{P}(\wedge^p \mathbb{C}^{2p})$.*

Proof. Let $\{u_1, \ldots, u_{2p}\}$, $\{v_1, v_2\}$, and $\{w_1, \ldots, w_p\}$ be bases such that the isomorphism is given by $u_j = v_1 \otimes w_j, u_{j+p} = v_2 \otimes w_j$ for $1 \le j \le p$. Then there is a natural map

$$\begin{aligned} \mathbb{CP}^1 &\to Gr(p, \mathbb{C}^{2p}) \\ [z_0, z_1] &\mapsto span\{(z_0 v_1 + z_1 v_2) \otimes w_1, \dots, (z_0 v_1 + z_1 v_2) \otimes w_p\} \\ &= span\{z_0 u_1 + z_2 u_{p+1}, \dots, z_0 u_p + z_1 u_{2p}\}. \end{aligned} \tag{3.3.5}$$

The image under the Plucker embedding is then

$$(z_0^p v_1^p + z_0^{p-1} z_1 v_1^{p-1} v_2 + \cdots + z_1^p v_2^p) \otimes (\wedge_{j=1}^p w_j), \tag{3.3.6}$$

which is manifestly a degree m rational curve; it is in fact a rational normal curve in the subspace $\mathbb{P}(\odot^p \mathbb{C}^2 \otimes \wedge^p \mathbb{C}^p)$. Conversely, all rational normal curves are projectively equivalent, and all projective subspaces of a fixed dimension are also projectively equivalent. Thus, there exists some projective linear transformation of $\mathbb{P}(\wedge^p \mathbb{C}^{2p})$ taking a degree p rational curve to any other. This is a change of basis for $\mathbb{C}^{2p}$, providing the required isomorphism. $\square$

With proposition 3.3.6, we just need to show that our degree p maps are embeddings rather than ramified covers of lower degree curves. Suppose that a fiber maps onto a curve of lower degree $p' < p$. But a degree p' curve lives in a p' dimensional projective subspace, and therefore the p-planes correponding to this curve must share a common $p - p'$ dimensional subspace. But $\mathcal{D} \cap \overline{\mathcal{D}} = 0$ away from the equator, so there cannot be a nontrivial common subspace. It follows that $p - p' = 0$, which concludes the proof of theorem 3.3.1.

Chapter 4

Local Aspects of Almost-Grassmannian Manifolds

In the twistor picture thus far, points in $\widetilde{Gr}(2, \mathbb{R}^{p+2})$ correspond to holomorphically embedded disks in $\mathbb{CP}^{p+1}$. This correspondence with disks, rather than closed curves, is essential for capturing global phenomena, owing to the fact that a perturbation of the standard almost-Grassmannian structure causes the hemispheres of each curve to split apart. One peculiar consequence is that the basic twistor correspondence of some open set $U \in M$, given by $U'' = \mu(\nu^{-1}(U))$, is not an open set but in fact has a boundary.

However, in consideration of local phenomena, we may always form the abstract double to work with moduli of closed curves. For a single Riemann surface X with boundary, the abstract double is obtained by attaching a conjugated copy $\overline{X}$ by the identity along their boundaries. Holomorphic data, e.g. vector bundles, uniquely extends over the abstract double so that we can effecively work with closed twistor lines. This procedure can also be done on U'', either by constructing its double all at once or line by line. By abuse of notation, the double will also be denoted by U''. We then have a correspondence between points $x \in U$ and closed twistor lines $L_x \subset U''$.

4.1 The Local Twistor Bundle

For any (p, q) almost-Grassmannian manifold, the local twistor bundle τ is defined as an extension

$$0 \to \mathcal{O}_{A'} \to \tau \to \mathcal{O}^{A} \to 0, \tag{4.1.1}$$

which is trivialized $\tau = \mathcal{O}^A \oplus \mathcal{O}_{A'}$ for any (local) choice of scale. In a choice of scale, a section of the local twistor bundle is thus represented by a pair of spinors $(\omega^A, \pi_{A'})$ which transforms under rescaling $\epsilon \mapsto f\epsilon$ by

$$\begin{pmatrix} \hat{\omega}^A \\ \hat{\pi}_{A'} \end{pmatrix} = \begin{pmatrix} \omega^A \\ \pi_{A'} - \Upsilon_{AA'}\omega^A \end{pmatrix} \tag{4.1.2}$$

where $\Upsilon_{AA'} = f^{-1}\nabla_{AA'}f$. This bundle carries a natural $SL(p+q)$-invariant connection given by

$$D_{AA'} \begin{pmatrix} \omega^B \\ \pi_{B'} \end{pmatrix} = \begin{pmatrix} \nabla_{AA'}\omega^B + \delta_A^B \pi_{A'} \\ \nabla_{AA'}\pi_{B'} - P_{AA'BB'}\omega^B \end{pmatrix} \tag{4.1.3}$$

where P is a curvature tensor depending on ∇. For a precise definition see [BE91]; the important feature is that P transforms under change of scale by

$$P_{AA'BB'} \to P_{AA'BB'} - \nabla_{AA'}\Upsilon_{BB'} + \Upsilon_{AB'}\Upsilon_{BA'}.$$

This connection is derived from the twistor equation so that parallel sections correpond exactly to solutions of the twistor equation. This construction is rather ad hoc, and we will first provide a twistor characterization of this bundle in the quaternionic case.

To set this up, we must introduce the Ward correspondence [War77, PR86]. Let M be a real right-flat $(2k, 2)$ almost-Grassmannian manifold. The Ward correspondence assigns to every holomorphic vector bundle on the twistor space, which is additionally trivial along each twistor line, a vector bundle with connection on M that is flat along α-surfaces. The basic idea is that, given such a holomorphic vector bundle $\mathscr{V} \to \mathscr{Z}$, one can define a bundle on M whose fiber at a point x is the space of sections $\Gamma(L_x, \mathscr{V})$. This provides a natural identification of the fibers along a single α-surface with the fiber over the corresponding point in $\mathscr{Z}$, which is, in effect, parallel transport over α-surfaces. Under mild hypothesis, this correspondence is one-to-one, and we can give an alternate characterization in the reverse direction. Starting with a vector bundle on M that is flat along α-surfaces, there is a holomorphic vector bundle over $\mathscr{Z}$ whose fiber is the space of covariantly constant sections along the corresponding α-surface.

Next, define an *auto-parallel spinor* on an α-surface Σ to be a parallel section $\pi_{A'} \in \Gamma(\Sigma, \mathcal{O}_{A'})$ such that $v^{AA'}\pi_{A'} = 0$ for any v tangent to Σ. Then denote by $\mathcal{O}(-1)$ the bundle on $\mathscr{Z}$ whose fiber at a point is the space of autoparallel spinors along the corresponding α-surface. This is in fact a line bundle, because the condition $v^{AA'}\pi_{A'} = 0$ algebraically determines $\pi_{A'}$ at any point, and this can be propagated across α-surfaces in a consistent manner because they are simply connected. We can then consider the 1-jet bundle $J^1\mathcal{O}(-1)$, whose fiber at z is defined to be equivalence classes of germs of sections of $\mathcal{O}(-1)$, where two germs are equivalent if their first derivatives at z coincide.

In general, a 1-jet bundle contains information about the first derivatives of local sections, and one should therefore expect a close relation with the cotangent bundle of the original vector bundle. Let L denote the total space of $\mathcal{O}(-1) - \mathbf{0}$, where $\mathbf{0}$ denotes the zero section. Then L is a smooth manifold in its own right, and has a tangent bundle $TL \to L$. The space L also comes with a $\mathbb{C}^*$ action given by rescaling the fibers, such that $L/\mathbb{C}^* \cong \mathscr{Z}$. Each choice of $\zeta \in \mathbb{C}^*$ has a Jacobian so that we have an induced action on TL. The quotient is a rank $m+2$ bundle $\mathscr{L} = TL/\mathbb{C}^* \to L/\mathbb{C}^* = \mathscr{Z}$.

Lemma 4.1.1. *There is a natural isomorphism* $J^1\mathcal{O}(1) \cong \mathscr{L}^* \otimes \mathcal{O}(1)$.

Proof. A section of $\mathcal{O}(1)$ can be interpreted as a function on L which is homogeneous of degree 1 along the fibers. Similarly, a section of $\mathscr{L}$ is a vector field on L which is invariant under the $\mathbb{C}^*$ action, i.e. also homogenous of degree 1 along the fibers. We therefore have a differentiation map $\mathcal{O}(\mathscr{L}) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$, which is a perfect pairing of $\mathcal{O}$-modules when restricted to $\mathcal{O}(\mathscr{L}) \otimes J^1\mathcal{O}(1)$. □

Theorem 4.1.2. *The Ward correspondence takes the dualized jet bundle $[J^1\mathcal{O}(1)]^*$ to the local twistor bundle.*

Proof. According to lemma 4.1.1, we can instead consider $\mathscr{L} \otimes \mathcal{O}(-1)$. A point in L is given by a pair $(\Sigma, \pi_{A'})$ consisting of an α-surface and an autoparallel spinor along it. Now consider a 1-parameter family of such points. Clearly the derivative of this family can be represented by another pair, one component to measure how Σ varies and the other for $\pi_{A'}$. There are, however, two distinct cases: either the α-surface Σ varies, or it doesn't. In the first case, there is a nonzero vector field $J^{AA'}$ along Σ connecting it to an infinitesimally separated one. Note that $J^{AA'}$ is only well-defined up to a tangential component, which we can eliminate by contracting with the autoparallel spinor $\pi_{A'}$. The contraction $\omega^A = J^{AA'}\pi_{A'}$ exactly encodes how Σ varies, and then $\eta_{A'} = J^{BB'}\nabla_{BB'}\pi_{A'}$ is of course how $\pi_{A'}$ varies. We therefore represent an element of TL by the pair $(\omega^A, \eta_{A'})$. In the case that Σ does not vary, the 1-parameter family is (to first order) a simple rescaling of $\pi_{A'}$. Then we can again use a pair $(\omega^A, \eta_{A'})$, but here $\omega^A = 0$ and $\eta_{A'}$ is proportional to $\pi_{A'}$.

Observe that in either case the $\mathbb{C}^*$ action $\pi_{A'} \mapsto \zeta\pi_{A'}$ induces $(\omega^A, \pi_{A'}) \mapsto (\zeta\omega^A, \zeta\pi_{A'})$. An element of $\mathscr{L}$ is then an equivalence class $[(\omega^A, \pi_{A'})]$ under this $\mathbb{C}^*$ action, and the twist by $\mathcal{O}(-1)$ then allows us to select a particular representative. The twist, in effect, cancels the $\mathbb{C}^*$ action. We therefore can represent an element of $\mathscr{L} \otimes \mathcal{O}(-1)$ by a pair $(\omega^A, \pi_{A'})$.

All that remains is to interpret $(\omega^A, \eta_{A'})$ in terms of the geometry on M. Recall that ω^A is the contraction of a connecting vector field with an autoparallel spinor $\pi_{A'}$. But connecting vector fields are just tangent vectors on $\mathscr{Z}$, which have the following characterization [BE91].

Lemma 4.1.3 (Bailey, Eastwood)**.** *The fiber of Θ at a point in $\mathscr{Z}$ is naturally isomorphic to the space of solutions ξ to the equation*

$$\textit{Trace-free part of } (\pi^{A'}\pi^{B'}\nabla_{BB'}\xi^A_{A'}) = 0 \tag{4.1.4}$$

along the corresponding α-surface, where $\pi^{A'}$ generates that α-surface.

One can choose $\pi^{A'}$ to be dual to our autoparallel spinor $\pi_{A'}$. It then follows immediately that ω^A solves the equation

$$\text{Trace-free part of } (\pi^{B'}\nabla_{BB'}\omega^A) = 0. \tag{4.1.5}$$

From here it is a straightforward verification that $(\omega^A, \eta_{A'})$ is parallel under local twistor transport along the corresponding α-surface.

□

The local twistor bundle will be a crucial tool in our study of the local geometry of almost-Grassmannian manifolds. To motivate our first application, recall the following theorem in conformal geometry.

Theorem 4.1.4 (Weyl)**.** *Let $(M, [g])$ be a conformal manifold of dimension at least 3. Then the Weyl tensor $W = 0$ if and only if M is locally conformally flat.*

This theorem has such practical use that it is often taken as a definition. There is an almost-Grassmannian analog of this theorem, first proposed by Bailey and Eastwood in [BE91], but never proven. Let us first establish a small lemma before proving their theorem.

Lemma 4.1.5. *Let $(\omega^A, \pi_{A'})$ be a nonzero covariantly constant local twistor field. Then the zero set $\{\omega^A = 0\}$ is an α-surface. If the local twistor connection is flat, then every α-surface arises in this way.*

Proof. Let $\Sigma = \{\omega^A = 0\}$ with $D_{BB'}(\omega^A, \pi_{A'}) = 0$. According to the first component of the local twistor connection, $\nabla_{BB'}\omega^A = -\delta_B^A \pi_{B'}$, so ω^A is constant in directions $X^{BB'}$ with $X^{BB'}\pi_{B'} = 0$. If $(\omega^A, \pi_{A'}) = (0, 0)$ at any point, then it is zero everywhere since $D_{AA'}$ is an $SL(p+q)$ connection. Therefore at any point in Σ, we must have $\pi_{A'} \neq 0$, so the tanget space is a null p-plane. If the local twistor bundle is furthermore flat, we can arrange for any poinwise value of $(\omega^A, \pi_{A'})$, and therefore every α-plane may be realized as the tanget space of such a zero set. □

Theorem 4.1.6. *Let M be an almost-Grassmannian manifold. Then M is locally equivalent to the flat model if and only if*

$$T_{ab}{}^{c} = 0 \quad \textit{for} \quad p, q > 2$$
$$\widetilde{F}_{AA'BB'}{}^{CC'} = 0 \quad \textit{and} \quad \Psi_{ABC}{}^{D} = 0 \quad \textit{for} \quad p > 2, q = 2. \tag{4.1.6}$$

Proof. First note that the given tensors are zero for the flat model, and therefore any map preserving the standard almost-Grassmannian structure must also preserve the vanishing of these tensors. This establishes one direction of the proof.

For the other direction, we will first show that eq. (4.1.6) implies the local twistor connection is flat. The local twistor curvature is manifestly composed of curvature components of the connection on M, and since the local twistor connection is scale-invariant, its curvature can only involve scale-invariant curvature components. In the case that $p, q > 2$, all scale-invariant curvatures are determined by the torsion, and in the case $p > 2, q = 2$ all but $\Psi_{ABC}{}^{D}$ are determined by the torsion. Thus eq. (4.1.6) implies the local twistor connection is flat.

Now consider the Grassmannian bundle $Gr(q, \tau) \to M$. Roughly speaking, we are attaching a copy of the flat model to every point of M, analogous to the construction of a tangent bundle. Then the local twistors at a point correspond to the global twistors of the associated flat model. This bundle carries a canonical section Δ, given by

$$\Delta(x) = \{(\omega^{A}, \pi_{A'}) \in \tau_x \mid \omega^{A} = 0\}, \tag{4.1.7}$$

which is well-defined because the unprimed coordinate is scale-invariant. In other words, the image of Δ is just the subspace $H^* \subset \tau$.

Fix a point $O \in M$ and a neighborhood U of O. Then we can identify the space of covariantly constant local twistor fields with the vector space $\mathbb{T} = \tau_O$. This trivializes $\tau|_U$, and therefore the associated Grassmannian bundle

$$\iota : Gr(q, \tau)|_U \to Gr(q, \mathbb{T}) \times U. \tag{4.1.8}$$

Lastly, let $\rho_1 : Gr(q, \mathbb{T}) \times U \to Gr(q, \mathbb{T})$ be the first factor projection and consider the composition

$$\varphi := \rho_1 \circ \iota \circ \Delta : U \to Gr(q, \mathbb{T}). \tag{4.1.9}$$

Concretely, this composition takes the q-plane $H^* \subset \tau$ over a point in M and parallel transports it via the local twistor connection to O. We claim that this map preserves the almost-Grassmannian structure.

Let Σ be an α-surface in U. By lemma 4.1.5, this is equivalent to some covariantly constant local twistor field $(\omega^A, \pi_{A'})$ up to scale, and so may be regarded as a line $\ell \subset \mathbb{T}$. Similarly, any point $x \in \Sigma$ is a q-plane $\Pi_x \subset \mathbb{T}$ consisting of the twistors vanishing at x. But now recall from our examination of the flat model in proposition 2.2.1 that lines in a q-dimensional plane $\Pi \subset \mathbb{T}$ correspond exactly to the standard α-planes passing through $\Pi \in Gr(q, \mathbb{T})$. Since $\ell \subset \Pi_x$ for all $x \in \Sigma$, it follows that the standard α-surface corresponding to ℓ passes through every point of $\varphi(\Sigma)$.

This proves that $\varphi : U \to Gr(q, \mathbb{T})$ takes α-surfaces to standard ones. Since an almost-Grassmannian structure is determined by its null directions, and by extension its α-surfaces, this shows that M is locally equivalent to the standard model. □

Analysis of curvature in the $(p, 2)$ case, as mentioned in the proof above, shows that the tensor $\Psi_{ABC}{}^D$ essentially governs all of the geometry on M, to the extent that any other curvature component can be "scaled away" by chosing an appropriate scale. We investigate this curvature in more detail in the next section.

4.2 The Nonlinear Graviton

We have already established in chapter 3 that right-flat deformations of an almost-Grassmnannian structure correspond to complex deformations of the twistor space, and vice versa. In this section we will carry out this correspondence in greater detail by examining the Penrose transform of $H^1(U'', \Theta)$.

The Penrose transform [BE90, Eas85] is a machine for interpreting the cohomology of sheaves on a twistor space in terms of solutions of diffential equations on M. First, under suitable topological hypotheses there is an isomorphism

$$H^k(U'', \mathscr{F}) = H^k(U', \mu^{-1}\mathscr{F}) \tag{4.2.1}$$

for any sheaf $\mathscr{F}$ on U''. The latter group is then computed via some resolution

$$0 \to \mu^{-1}\mathscr{F} \to \mathscr{R}^0 \to \mathscr{R}^1 \to \cdots , \tag{4.2.2}$$

which is pushed down onto U. The differentials in eq. (4.2.2) induce differential operators on the direct image sheaves in M, completing the Penrose transform. As a small, but important example, let us compute the Penrose transform of $H^1(U'', \mathcal{O}(1))$.

Take a resolution

$$0 \to \mu^{-1}\mathcal{O}(1) \to \mu^*\mathcal{O}(1) \to \Omega^1_\mu(1) \to \Omega^2_\mu(1) \to \cdots \tag{4.2.3}$$

with maps given by diffentiation along the fibers of μ, i.e. the partial connection $\nabla_A = \pi^{A'}\nabla_{AA'}$. We can also identify $\Omega^1_\mu = \mathcal{O}_A(1)$, so

$$\begin{aligned} \mathscr{R}^0 &= \mu^*\mathcal{O}(1) \\ \mathscr{R}^1 &= \mathcal{O}_A(2) \\ \mathscr{R}^2 &= \mathcal{O}_{[AB]}(3), \end{aligned} \tag{4.2.4}$$

and

$$\begin{aligned} \nu^0_*\mathscr{R}^0 &= \mathcal{O}_{A'} \\ \nu^0_*\mathscr{R}^1 &= \mathcal{O}_{A(A'B')} \\ \nu^0_*\mathscr{R}^2 &= \mathcal{O}_{[AB](A'B'C')}, \end{aligned} \tag{4.2.5}$$

with the induced maps being given by the canonical connection and appropriate symmetrizing. These identifications with the deRham theorem say that

$$H^1(U', \mu^{-1}\mathcal{O}(1)) \cong \frac{ker\ \nabla_A : \Gamma(U', \mathcal{O}_A(2)) \to \Gamma(U', \mathcal{O}_{[AB]}(3))}{im\ \nabla_A : \Gamma(U', \mathcal{O}(1)) \to \Gamma(U', \mathcal{O}_A(2))}$$
$$\cong \frac{ker\ D : \Gamma(U, \mathcal{O}_{A(A'B')}) \to \Gamma(U, \mathcal{O}_{[AB](A'B'C')})}{im\ D : \Gamma(U, \mathcal{O}_{A'}) \to \Gamma(U, \mathcal{O}_{A(A'B')})}, \tag{4.2.6}$$

where D denotes the induced differential operators on the sheaves in eq. (4.2.5). In particular, an element of $H^1(U', \mu^{-1}\mathcal{O}(1))$ may be represented by a solution $f_{A(A'B')}$ of the equation $\nabla_{(C'|[C} f_{A]|A'B')} = 0$.

We now return to the original group of interest, $H^1(U'', \Theta)$. Our starting point is the Atiyah jet sequence twisted by $\mathcal{O}(1)$,

$$0 \to \mathcal{O} \to [J^1\mathcal{O}(1)]^* \otimes \mathcal{O}(1) \to \Theta \to 0. \tag{4.2.7}$$

The induced long exact sequence in cohomology looks like

$$\cdots \to H^1(U'', \mathcal{O}) \to H^1(U'', [J^1\mathcal{O}(1)]^* \otimes \mathcal{O}(1)) \to H^1(U'', \Theta) \to H^2(U'', \mathcal{O}) \to \cdots \tag{4.2.8}$$

There is a map $\mu^* : H^2(U'', \mathcal{O}) \to H^2(U', \mu^{-1}\mathcal{O})$, which is injective by an argument in [EPW81]. Morever, $H^2(U', \mu^{-1}\mathcal{O})$ is in fact zero due to the resolution of sheaves

$$0 \to \mu^{-1}\mathcal{O} \to \mathcal{O}_{U'} \to \Omega^1_\mu \to \Omega^2_\mu \to 0. \tag{4.2.9}$$

It follows that $H^2(U'', \mathcal{O}) = 0$, and so

$$H^1(U'', \Theta) = H^1(U'', [J^1\mathcal{O}(1)]^* \otimes \mathcal{O}(1))/im\ H^1(U'', \mathcal{O}). \tag{4.2.10}$$

Recall now that $J^1\mathcal{O}(1)$ is trivial along each twistor line, which makes computing the Penrose transform of $H^1(U'', [J^1\mathcal{O}(1)]^* \otimes \mathcal{O}(1))$ especially simple. In particular, we can express the Penrose transform of $H^1(U'', [J^1\mathcal{O}(1)]^* \otimes \mathcal{O}(1))$ in terms of the Ward correspondence of $[J^1\mathcal{O}(1)]^*$ and the Penrose transform of $H^1(U'', \mathcal{O}(1))$, both of which we have already computed.

According to theorem 4.1.2, the Ward correspondence of $[J^1\mathcal{O}(1)]^*$ is the local twistor bundle τ with connection ∇. We then have the operators

$$\nabla \times D : \tau \otimes \mathcal{O}_{A'} \to \tau \otimes \mathcal{O}_{A(A'B')} \tag{4.2.11}$$

$$\nabla \times D : \tau \otimes \mathcal{O}_{A(A'B')} \to \tau \otimes \mathcal{O}_{[AB](A'B'C')}, \tag{4.2.12}$$

where D is as above and we are taking the usual product connection. In particular,

$$\nabla \times D \begin{pmatrix} \omega^C \otimes f_{A(A'B')} \\ \pi_{C'} \otimes f_{A(A'B')} \end{pmatrix} = \begin{pmatrix} (\nabla_{DD'}\omega^C + \delta^C_D \pi_{D'}) \otimes f_{A(A'B')} + \omega^C \otimes \nabla_{(D'|[D} f_{A]A'B')} \\ (\nabla_{DD'}\pi_{C'} - P_{CC'DD'}\omega^C) \otimes f_{A(A'B')} + \pi_{C'} \otimes \nabla_{(D'|[D} f_{A]A'B')} \end{pmatrix}. \tag{4.2.13}$$

Suppose that an element lies in the kernel of this operator. Then, reading off the top line, we have a solution ${g_{A(A'B')}}^B$ of

$$\text{Trace-free part of } (\nabla_{[C|(C'} {g_{A')|A]B'}}^B) = 0. \tag{4.2.14}$$

This is equivalent to eq. (2.3.9) derived earlier, and therefore provides an alternate derivation of the equation characterizing right-flat deformations.

4.3 A Twistor Description of ${\Psi_{ABC}}^D$

We have previously seen that the obstruction to (locally) standardizing a $(p, 2)$ almost-Grassmannian structure is given by the curvature component ${\Psi_{ABC}}^D$. In the twistor space, one can also consider obstructions to "trivializing" the neighborhood of a twistor line. (In this context, trivial means that a neighborhood of the twistor line is isomorphic to a neighborhood of a linearly embedded curve in the standard projective space.) The aim of this section is to describe how these two ideas correspond.

We must first introduce infinitesimal neighborhoods and deformations of complex manifolds. Let X be a complex submanifold of a complex manifold Y. The nth infinitesimal neighborhood $X^{(n)}$ of X in Y is the space X with the augmented ring of functions

$$\mathcal{O}_{X^{(n)}} := \mathcal{O}_Y / \mathcal{I}_X^{n+1}, \tag{4.3.1}$$

where $\mathcal{I}_X$ is the ideal sheaf of functions vanishing along X. If X has codimension k, then this augmented ring of functions is locally of the form

$$\mathcal{O}_{X^{(n)}} \cong \mathcal{O}_X[\zeta_1, \ldots, \zeta_k]/(\zeta_1 \ldots, \zeta_k)^{n+1}. \tag{4.3.2}$$

For example, the nth infinitesimal neighorhood of the origin in the complex plane carries the polynomial ring $\mathbb{C}[\zeta]/\zeta^{n+1}$.

More generally, a ringed space $(X, \mathcal{O}_{X^{(n)}})$ is called an nth order fattening of X if $\mathcal{O}_{X^{(n)}}$ locally satisfies eq. (4.3.2). Notice that a fattening of order n determines a collection of fattenings for all orders $k < n$ by the short exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^k \to \mathcal{O}_{X^{(n)}} \to \mathcal{O}_{X^{(k)}} \to 0. \tag{4.3.3}$$

A fattening need not arise from a genuine embedding of X into another complex manifold, but if it does, then $\mathcal{I}^2/\mathcal{I}$ is the co-normal bundle N^*. We therefore adopt this as the definition of the co-normal bundle in general, and then $\mathcal{I}^n/\mathcal{I}^{n-1} \cong \mathcal{O}(\odot^n N^*)$. We also need the extended tangent bundle $\hat{T} := Der(\mathcal{O}_{(1)}, \mathcal{O})$ which fits into the short exact sequence

$$0 \to T \to \hat{T} \to N \to 0. \tag{4.3.4}$$

In the case that the fattening arises from an embedding $X \hookrightarrow Y$, then $\hat{T} = T_Y|_X$ is the usual restricted tangent bundle.

Given a fattening $X^{(n)}$, one may ask whether it can be extended to a higher order fattening. The result of interest is due to Eastwood and LeBrun [EL92].

Theorem 4.3.1 (Eastwood, LeBrun)**.** *The obstruction to finding an extension $X^{(n+1)}$ of $X^{(n)}$, for $n \geq 1$, is in $H^2(X, \mathcal{O}(\hat{T} \otimes \odot^{n+1} N^*))$. If this obstruction vanishes, then $H^1(X, \mathcal{O}(\hat{T} \otimes \odot^{n+1} N^*))$ acts freely and transitively on the family of extensions.*

Now consider a holomorphic disk $(\mathbb{D}, \partial\mathbb{D}) \subset (\mathscr{Z}, P)$ corresponding to some point in M. We are ultimately interested in how the almost-Grassmnannian geomtry changes when the complex structure on a neighborhood of this disk is deformed. By restricting a deformation to infinitesimal neighborhoods, we obtain a series of fattenings as described above. This deformation can then be measured step-by-step against the initial complex structure by fixing the initial one as the zero element in $H^1(X, \mathcal{O}(\hat{T} \otimes \odot^{n+1} N^*))$. Although we are primarily concerned with deformations of the standard model, the technique here works for any right-flat almost-Grassmannian structure.

The only issue is that the above machinery is designed for closed manifolds. To circumvent this technicality, we may construct the abstract double [LeB05] of $\mathbb{D}$, which is obtained by attaching a conjugated copy $\overline{\mathbb{D}}$ to $\mathbb{D}$ by identifying their boundaries. This double $X = \mathbb{D} \cup_{\partial\mathbb{D}} \overline{\mathbb{D}}$ is a closed Riemann surface, and carries unique extensions of holomorphic vector bundles on $\mathbb{D}$. It is important to realize that the double is an abstract Riemann surface, and is not a submanifold of the twistor space.

Let's begin with the first order fattening. This is a special case not covered by theorem 4.3.1: a first order fattening is equivalent to the trivial one if the short exact sequence in eq. (4.3.4) splits. In our case of interest, $X \approx \mathbb{CP}^1$ with normal bundle $N = \mathcal{O}^A|x \otimes \mathcal{O}(1)$. For notational convenience we will write $N = \mathcal{O}^A(1)$, with the understanding that $\mathcal{O}^A$ is really a fixed vector space when we restrict to X. Then $H^1(X, T \otimes N^*) = H^1(\mathbb{CP}^1, \mathcal{O}_A(-1)) = 0$, so any extension $\hat{T}$ automatically splits. To first order, there is therefore no deviation from the trivial fattening, and we may assume a fixed isomorphism $\hat{T} \cong \mathcal{O}(2) \oplus \mathcal{O}^A(1)$.

For higher order fattenings we need to analyze $H^1(X, \mathcal{O}(\hat{T} \otimes \odot^k N^*))$ for $k \geq 2$. We have $\odot^k N^* = \mathcal{O}_{(A\ldots C)}(-k)$ where there are k many lower indices, so

$$\hat{T} \otimes \odot^k N^* = \mathcal{O}_{(A\ldots C)}(2-k) \oplus \mathcal{O}_{(A\ldots C)}{}^D(1-k). \tag{4.3.5}$$

From this we can immediately read off $H^1(X, \mathcal{O}(\hat{T} \otimes \odot^2 N^*)) = 0$ since $H^1(\mathbb{CP}^1, \mathcal{O}(m)) = 0$ for $m \geq -1$, so that second order fattenings also cannot deviate from the trivial one. However, with $k = 3$ we get that

$$H^1(X, \mathcal{O}(\hat{T} \otimes \odot^3 N^*)) \cong \mathcal{O}_{(ABC)}{}^D \otimes H^1(X, \mathcal{O}(-2)). \tag{4.3.6}$$

To see how 3rd order deformations $H^1(X, \mathcal{O}_{X^{(3)}}(\hat{T}))$ relate to the group $H^1(X, \mathcal{O}(\hat{T} \otimes$

$\odot^3 N^*)$), take the short exact sequence

$$0 \to \mathcal{O}(\hat{T} \otimes \odot^n N^*) \to \mathcal{O}_{X^{(n)}}(\hat{T}) \to \mathcal{O}_{X^{(n-1)}}(\hat{T}) \to 0. \tag{4.3.7}$$

Analysis of the long exact sequence in cohomology for increasing values of n shows that

$$H^1(X, \mathcal{O}_{X^{(3)}}(\hat{T})) \cong H^1(X, \mathcal{O}(\hat{T} \otimes \odot^3 N^*)). \tag{4.3.8}$$

The 3rd order fattening is therefore the first obstruction to trivializing a deformation. The group $H^1(X, \mathcal{O}(-2))$ is furthermore trivialized by a choice of scale. To see this, consider the Wronskian $\wedge^2 H \to H^0(X, \Omega^1(2))$ given by

$$u \wedge v \mapsto u \otimes dv - v \otimes du, \tag{4.3.9}$$

where elements in H are also interpreted as sections of $\mathcal{O}(1)$ over the corresponding twistor line. Both vector spaces are one dimensional and the map is nonzero, so it must be an isomorphism. The required trivialization then follows from Serre duality $H^0(X, \Omega^1(2)) \cong [H^1(X, \mathcal{O}(-2))]^*$.

Proposition 4.3.2. *There is a natural map $H^1(X, \mathcal{O}_{X^{(3)}}(\hat{T})) \to \mathcal{O}_{(ABC)}{}^D[-1]$ which is surjective.*

Proof. The isomorphism eq. (4.3.8) is given by taking the 3rd normal derivatives of a Cech representative, and can therefore be given as a Penrose-type contour integral. From eq. (4.3.5) it is seen that an element of $H^1(X, \mathcal{O}_{X^{(3)}}(\hat{T}))$ can be given by an expression of the form

$$F^A\left(\frac{\mu}{\pi_1}\right)\frac{\pi_1^2}{\pi_2}$$

where π_1, π_2 are homogeneous coordinates along the twistor line, and μ indicates all the remaining homogeneous coordinates. The vector-valued function F^A is holomorphic in the affine coordinates μ/π_1 and vanishes to 2nd order, and the expression π_1^2/π_2 is to be understood as a Cech representative of a nontrivial element in $H^1(X, \mathcal{O}(1))$. Now fix a scale, which by the previous discussion provides a specific scalar multiple of $\pi_1 d\pi_2 - \pi_2 d\pi_1 \in H^0(X, \Omega^1(2))$; denote the multiple by ξ. Then we have the contour integral formula

$$\begin{aligned}
&\frac{1}{2\pi i}\oint \frac{\partial}{\partial \mu^B}\frac{\partial}{\partial \mu^C}\frac{\partial}{\partial \mu^D} F^A\left(\frac{\mu}{\pi_1}\right)\frac{\pi_1^2}{\pi_2}(\pi_1 d\pi_2 - \pi_2 d\pi_1)\cdot \xi \\
&\qquad = \frac{1}{2\pi i}\oint F_{BCD}{}^A \frac{\pi_1^2}{\pi_1^3 \pi_2}(\pi_1 d\pi_2 - \pi_2 d\pi_1)\cdot \xi \\
&\qquad = \frac{1}{2\pi i}\oint F_{BCD}{}^A \left(\frac{d\pi_2}{\pi_2} - \frac{d\pi_1}{\pi_1}\right)\cdot \xi \\
&\qquad = F_{BCD}{}^A \cdot \xi
\end{aligned} \tag{4.3.10}$$

where $F_{BCD}{}^A$ is the 3rd order coefficient of F^A;

$$F^A\left(\frac{\mu}{\pi_1}\right) = F_{BCD}{}^A \frac{\mu^B \mu^B \mu^C}{\pi_1^3} + O(\mu^4). \tag{4.3.11}$$

From this equation it is clear that $F_{BCD}{}^A$ can be chosen arbitrarily, and since the result is proportional to the multiple ξ, it follows that the contour integral formula surjects onto $\mathcal{O}_{(BCD)}{}^A[-1]$. □

Remark. The above integral formula is manifestly scale-invariant. Meanwhile, $\Psi_{ABC}{}^D \epsilon_{A'B'}$ is the only scale-invariant curvature quantity on a $(p, 2)$ almost-Grassmannian manifold, and since both are nonzero they must coincide. A proof of this is left to future work.

4.4 Special Holonomy

The aim of this final section is to provide an almost-Grassmannian analog of Pontecorvo's characterization of Kahler metrics on compact surfaces [Pon92].

Theorem 4.4.1 (Pontecorvo)**.** *Let M be a complex surface with an ASD Hermitian metric. Its complex structure J and its conjugation $-J$ define two sections of the twistor projection $Z \to M$. Denote the images of these sections by $\Sigma, \overline{\Sigma}$. Then the divisor line bundle $[\Sigma + \overline{\Sigma}]$ satisfies*

$$[\Sigma + \overline{\Sigma}] \cong K_Z^{-1/2} \tag{4.4.1}$$

if and only if M is conformally Kahler.

We will not attept to prove such a general result, and instead focus on the case that M is diffeomorphic to the quaternionic flat model $\widetilde{Gr}(2, \mathbb{R}^{2m+2})$. Although this manifold does not come equipped with a preferred complex structure, there is still a notion of compatibility of an almost-complex structure with a given quaternionic almost-Grassmannian structure. In particular, J is compatible with $TM \cong E \otimes H$ if it arises from an almost-complex structure on the rank 2 bundle H. Equivalently there exist local bases $e_1, \dots, e_m$ and h_1, h_2 of E, H such that J acts by

$$\begin{aligned} J(e_i \otimes h_1) &= e_i \otimes h_2 \\ J(e_i \otimes h_2) &= -e_i \otimes h_1. \end{aligned} \tag{4.4.2}$$

A choice of J then determines an m-dimensional complex distribution $T^{1,0}M \subset T_{\mathbb{C}}M$, which consists of α-planes. We can therefore interpret J as a section of the disk bundle $\mathscr{F}_+ \to M$. Since these are genuine complex α-planes, as opposed to complexified real ones, this section survives the blowing down map $\Psi : \mathscr{F}_+ \to \mathbb{CP}^{2m+1}$, and its image here is the m-dimensional quadric Q previously seen in section 3.2. Note, in analogy with eq. (4.4.1), that

$$[Q] \cong K_{\mathscr{X}}^{-1/(m+1)}. \tag{4.4.3}$$

Now consider on $\mathbb{C}^{2m+2}$ the standard volume form paired with the Euler vector field,

$$\alpha = \left(z_0 \frac{\partial}{\partial z_0} + \cdots + z_{2m+2} \frac{\partial}{\partial z_{2m+2}} \right) \lrcorner \, (dz^0 \wedge \cdots \wedge dz^{2m+2}). \tag{4.4.4}$$

Since the Euler vector field has homogeneity 1, we may regard this as a weight $2m+2$ holomorphic volume form on $\mathbb{CP}^{2m+1}$, i.e. as a section of $K \otimes \mathcal{O}(2m+2)$. The quadric $Q = \{q = 0\} \subset \mathbb{CP}^{2m+1}$ given by J from above then determines a volume form

$$\Omega = \frac{\alpha}{q^{m+1}} \tag{4.4.5}$$

which is holomorphic away from Q. Whereas the setting of theorem 4.4.1 provides a real structure that interchanges Σ and $\overline{\Sigma}$, we will investigate the condition that Ω restricts to a real form along the real slice $P \subset \mathbb{CP}^{2m+1}$.

With J and the bases of E, H as above, define a scale ϵ by its dual $\epsilon^* = h_1 \wedge h_2 = h_1 \wedge J(h_1)$. Let ζ be a fiber coordinate of $\mathscr{F}_+$ where $\zeta \leftrightarrow h_1 + \zeta h_2$. Then by construction, $\Psi^{-1}(Q)$ correponds to $\zeta = i$. Since $\Psi^*\Omega$ is a section of the canonical bundle, it must annihilate $\mathfrak{w}_j, \partial/\partial\zeta$. We there have the coordinate expression

$$\Psi^*\Omega = \frac{f}{(1+\zeta^2)^{m+1}}[(h_1^* + \zeta h_2^*)^{2m} \otimes \epsilon)] \wedge [d\zeta + F(\theta_j{}^i)] \qquad (4.4.6)$$

where f is some bounded holomorphic function on $\mathscr{F}_+$, and F is a function in the connection 1-forms $\theta_j{}^i$. Note that the totally real submanifold $P \subset \mathbb{CP}^{2m+1}$ pulls back to the boundary of $\mathscr{F}_+$, i.e. where ζ is real. If Ω is real along P, it follows that f must be real where ζ is real. Along each ζ fiber, the reflection principle allows us to extend f to all of $\mathbb{C}$, and then by boundedness f must be constant. We may therefore regard f as a function on M.

Define $\hat{\Omega}$ to be the residue of $\Psi^*\Omega$ along $\Psi^{-1}(Q)$. Since $\Psi^*\Omega$ is holomorphic away from $\zeta = i$, we can choose to compute $\hat{\Omega}$ by integrating along the boundary $\partial\mathscr{F}_+$. By the preceding paragraph, $\hat{\Omega}$ is in fact real, with the explicit formula

$$\hat{\Omega} = \frac{1}{2\pi i}\oint \Psi^*\Omega = f \cdot [(h_1^*)^2 + (h_2^*)^2]^m \otimes \epsilon. \qquad (4.4.7)$$

Here the contour integral is meant to be taken over each fiber. Since the entire expression is real, and the tensor product is over $\mathbb{C}$, we can assume without loss of generality that each of the terms in the right-hand side are real. We can then factor out the rescaled scale form $f\epsilon$ to obtain a symmetric rank m spinor field, which we will denote by $\omega_{A'\ldots D'}$. In other words, $\hat{\Omega} = \omega_{A'\ldots D'} \otimes f\epsilon$. The construction of ω is a very concrete form of the Penrose transform for $H^1(\mathbb{CP}^{m+1}, \mathcal{O}(-2m-2))$, and will automatically satisfy the helicity $2m$ zero rest-mass equation

$$\nabla_{E[E'}\omega_{A']\ldots D'} = 0; \qquad (4.4.8)$$

see [EPW81, HT85] for details. The field ω additionally solves the twistor equation.

Lemma 4.4.2. *The spinor field* $\omega_{A'\ldots D'}$ *satisfies* $\nabla_{E(E'}\omega_{A'\ldots D')} = 0$.

Proof. On $\mathscr{F}_+$ there is a tautological object $\pi^{A'}$. Concretely, the total space of the bundle $\nu : \mathcal{O}^{A'} \to M$ carries the pullback bundle $\nu^*\mathcal{O}^{A'}$ which has a tautological section. This has homogeneity 1, and so descends through projectivization of the fibers to yield a section of $\nu^*\mathcal{O}^{A'} \otimes \mathcal{O}(1)$. Write $h^{A'} = h_1$ where the left-hand side uses spinor notation, and the right-hand side is one of the generating sections from before. Then we can parameterize $\pi^{A'} = h^{A'} + \zeta J(h^{A'})$, and in the same notation we have

$$\omega_{A'\ldots D'} = \underbrace{[h_{A'}h_{B'} + J(h_{A'})J(h_{B'})] \cdots [h_{C'}h_{D'} + J(h_{C'})J(h_{D'})]}_{m}.$$

(Strictly speaking, J only acts on $\mathcal{O}^{A'}$. By abuse of notation, $J(h_{A'})$ indicates the dual of $J(h^{A'})$.) We then have

$$\pi^{A'} \cdots \pi^{D'} \omega_{A'\ldots D'} = (1 + \zeta^2)^m \tag{4.4.9}$$

as an equation on $\mathscr{F}_+$. The important feature is that the right-hand side depends only on ζ and not upon any of the coordinates on M. Thus, the operator $\pi^{D'}\nabla_{DD'}$ must annihilate the left-hand side above. But the tautological object commutes with $\pi^{D'}\nabla_{DD'}$, and so

$$\pi^{A'} \cdots \pi^{C'} \pi^{D'} \nabla_{DD'} \omega_{A'\ldots C'} = 0. \tag{4.4.10}$$

Since $\pi^{A'}$ sweeps out all possible spinors up to scale, we conclude that the primed symmetric part of $\nabla_{DD'}\omega_{A'\ldots C'}$, as an object on M, must vanish. □

Theorem 4.4.3. *Suppose that Ω restricts to a real n-form along P. Then M carries a parallel section of $\odot^2 H^*$.*

Proof. Recall that ω is symmetric in all of its indices, and therefore the twistor equation of lemma 4.4.2 is equivalent to the vanishing of $\nabla_{E(E'}\omega_{A')\ldots D'}$. Since ω solves eq. (4.4.8), we have

$$\nabla_{EE'}\omega_{A'\ldots D'} = \nabla_{E[E'}\omega_{A']\ldots D'} + \nabla_{E(E'}\omega_{A')\ldots D'} = 0,$$

and although ω is by construction a section of $\mathcal{O}_{A'\ldots D'}$, it is real and must therefore really be a section of $\odot^{2m}H^*$. In fact, ω has an mth-root given in local coordinates by $\tilde{\omega} =$

$h_{A'}h_{B'} + J(h_{A'})J(h_{B'})$ and the arguments showing that ω is parallel apply equally well to the root $\tilde{\omega}$. The only additional consideration is that $\tilde{\omega}$ may only be well-defined up to an m-th root of unity, but M is simply connected, so it is indeed well-defined and we may also take it to be real. □

Remark. It may seem odd that ω, though initially constructed to solve the zero rest-mass equation, also happens to solve the twistor equation. The picture is a little more clear seen in reverse: the twistor equation is overdetermined and therefore the existence of a solution imposes additional structure. In fact, starting with a solution of the twistor equation, it is very nearly an algebraic consequence that it also satisfies the zero rest-mass equation.

Proposition 4.4.4. *Suppose that $\omega_{A'B'} \in \Gamma(\mathcal{O}_{(A'B')})$ satisfies $\nabla_{C(C'}\omega_{A'B')} = 0$, is real, nondegenerate, and $|\omega| = const$. Then ω also satisfies $\nabla_{C[C'}\omega_{A']B'} = 0$.*

Proof. The inner product on valence 2 spinors is given by

$$\langle \phi_{A'B'}, \psi_{C'D'} \rangle = \phi_{A'B'}\overline{\psi}^{A'B'} = \phi_{A'B'}\overline{\psi}_{C'D'}\varepsilon^{A'C'}\varepsilon^{B'D'}.$$

Therefore the constant norm hypothesis is equivalent to $\nabla_{CC'}(\omega_{A'B'}\omega^{A'B'}) = 0$, which by the product rule and index raising/lowering yields $\omega^{A'B'}\nabla_{CC'}\omega_{A'B'} = 0$. Then

$$\begin{aligned}
0 &= \tfrac{1}{2}\omega^{A'B'}\nabla_{C(C'}\omega_{A'B')} \\
&= \omega^{A'B'}\nabla_{CC'}\omega_{A'B'} + \omega^{A'B'}\nabla_{CA'}\omega_{B'C'} + \omega^{A'B'}\nabla_{CB'}\omega_{C'A'} \\
&= \omega^{A'B'}\nabla_{CA'}\omega_{B'C'} + \omega^{A'B'}\nabla_{CB'}\omega_{C'A'} \\
&= \omega^{A'B'}\nabla_{CA'}\omega_{B'C'} + \omega^{B'A'}\nabla_{CB'}\omega_{A'C'},
\end{aligned}$$

which shows that $\omega^{A'B'}\nabla_{CA'}\omega_{B'C'} = 0$.

Now we consider the partial contraction $\omega_{A'B'}\omega^{A'}_{C'}$, which is automatically skew in $B'C'$ and therefore a multiple of $\epsilon_{B'C'}$. The hypothesis that the full contraction (i.e. norm) is constant shows furthermore that this multiple must be constant, so $\omega_{A'B'}\omega^{A'}_{C'} = \lambda\varepsilon_{B'C'}$. The covariant derivative yields

$$\omega_{A'B'}\nabla_{EE'}\omega^{A'}_{C'} + \omega^{A'}_{C'}\nabla_{EE'}\omega_{A'B'} = 0.$$

Now trace with $\varepsilon^{C'E'}$ to get

$$\omega_{A'B'}\nabla_{EE'}\omega^{E'A'} + \omega^{E'A'}\nabla_{EE'}\omega_{A'B'} = 0.$$

The second term vanishes by the first paragraph, and by nondegeneracy of ω one then concludes that $\nabla_{EE'}\omega^{E'A'} = 0$. This is equivalent to $\nabla_{C[C'}\omega_{A']B'} = 0$. $\square$

In order to make use of theorem 4.4.3, we need to know that there really exist nontrivial deformations of $\mathbb{RP}^{2m+1} \subset \mathbb{CP}^{2m+1}$ satisfying the necessary reality condition. The key is to recognize that Ω restricts to the standard real volume form on $\mathbb{RP}^{m+1}$, and so to preserve this condition we will require deformations essentially corresponding to divergence-free vector fields.

In fact, such deformations are relatively easy to come by in the real analytic case. Suppose v is a real analytic vector field on $\mathbb{RP}^{2m+1}$ which is divergence-free with respect to the standard metric. Then if J denotes the ambient complex structure, Jv is a normal vector which we want to flow $\mathbb{RP}^{2m+1}$ along. Note that $Jv + iv$ is a section of $T^{1,0}(\mathbb{CP}^{2m+1})|_{\mathbb{RP}^{2m+1}}$ and by analyticity must have an extension to some neighborhood of $\mathbb{RP}^{2m+1}$. We therefore get a family of biholomorphisms ψ_t of this neighborhood. Furthermore, since $\mathcal{L}_{Jv+iv}\Omega = \operatorname{div}(Jv + iv)\Omega$ and $\operatorname{div}(iv)|_{\mathbb{RP}^{2m+1}} = 0$, we see that $\mathcal{L}_{Jv+iv}\Omega$ is real along $\mathbb{RP}_{2m+1}$. Then $\operatorname{Im}\psi_t^*\Omega|_{\mathbb{RP}^{2m+1}} = 0$, showing that divergence-free vector fields yield deformations with the desired reality condition.

In fact, we can dispense with the analyticity assumption as shown by the following proposition.

Proposition 4.4.5. *For $k \geq 1, \alpha \in (0,1)$, the space of $C^{k,\alpha}$ deformations of the standard $\mathbb{RP}^{2m+1} \subset \mathbb{CP}^{2m+1}$ along which $Im\,\Omega = 0$ is a Banach manifold whose tangent space consists of divegence-free $C^{k,\alpha}$ vector fields on $\mathbb{RP}^{2m+1}$ with respect to the standard metric.*

Proof. We use J as above to identify $T\mathbb{RP}^{2m+1}$ with the normal bundle $N\mathbb{RP}^{2m+1}$. Then a vector field $v \in \Gamma(T\mathbb{RP}^{2m+1})$ yields a deformation by exponentiating $Jv + iv$, denoted by ψ_v.

The map ψ_v is a diffeomorphism onto its image, and all $C^{k,\alpha}$ deformations of $\mathbb{RP}^{2m+1}$ arise this way. For convenince, we will use the fixed metric to identify v as a 1-form, and now examine the map

$$F : C^{k,\alpha}(\wedge^1) \to C^{k-1,\alpha}(\wedge^{2m+1}) \oplus C^{k-1,\alpha}(\wedge^2) \tag{4.4.11}$$

$$v \mapsto (\operatorname{Im} \psi_v^* \Omega, d\alpha). \tag{4.4.12}$$

The linearization of the first term at 0 is seen to be $\operatorname{Im} \mathcal{L}_{v^*}\Omega = \operatorname{Im}\ \operatorname{div}(Jv + iv)\Omega$. But Ω is real along $\mathbb{RP}^{2m+1}$, so the linearization is seen to be $\operatorname{div}(v)\Omega$. The second term is included so that the full linearization of F at 0 is essentially given by $d^* \oplus d$. The kernel of $DF(0)$ is evidently the harmonic 1-forms $\mathcal{H}^1(\mathbb{RP}^{m+1}) = 0$. Moreover, F maps into

$$\{C^{k-1,\alpha} \text{ exact } 2m+1\text{-forms}\} \oplus \{C^{k-1,\alpha} \text{ exact 2-forms}\},$$

and under this restriction $DF(0)$ is surjetive. The Schauder esimates then show F is a Banach space isomorphism near the origin, and therefore the desired space $F^{-1}(0, B)$ (for some ball around 0 in the space of $C^{k-1,\alpha}$ exact 2-forms) has tangent space given by divergence-free vector fields as claimed. $\square$

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