The Nirenberg Problem on a Conical Sphere

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Abstract of the Dissertation

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We propose a new approach to the question of prescribing Gaussian curvature on the 2-sphere with at least three conical singularities and angles less than 2π, the main result being sufficient conditions for a positive function of class at least C^2 to be the Gaussian curvature of such a conformal conical metric on the round sphere. Our methods particularly differ from the variational approach in that they don't rely on the Moser-Trudinger inequality. Along the way, we also prove a general precompactness theorem for compact Riemann surfaces with at least three conical singularities and angles less than 2π.

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Chapter 1

Introduction

Conical surfaces have been extensively studied by many in different contexts, ranging from compactness theorems to Teichmuller dynamics. The simplest example of a conical surface is that obtained by taking the quotient of a Riemann surface by a discrete group of isometries. The general geometry of such quotients is that of a smooth surface with metric singularities that arise at the points $p_1, \dots p_n$ where the isometry group has nontrivial stabilizer. From a local perspective, these singularities can be characterized by the existence of neighborhoods around each p_i where the metric takes the form $g = e^{2u}|z|^{2\beta_i}|dz|^2$. Conical surfaces are the more general class of objects– not necessarily only those that arise as quotients– that enjoy this local conical geometry near a finite set of isolated points. The points $p_1, \dots p_n$ are referred to as *cone points* with corresponding cone angles $2\pi(\beta_i + 1)$.

A classical problem for conical surfaces is characterizing those smooth functions which arise as Gaussian curvatures of a pointwise conformal metric. Equivalently, one asks for necessary and sufficient conditions for existence of solutions to the Gaussian curvature equation of a pointwise conformal metric $\tilde{g} = e^{2u}g$, with g being some fixed background conical metric. Such equation is given by

$$K = e^{-2u}(K_g - \Delta_g u) \tag{1.0.1}$$

where K_g is the curvature of the metric g. In the next section, we will discuss some of the existence and non-existence results for Equation 1.0.1, due to M. Troyanov, which are very similar in spirit to the analogous results due to Kazdan-Warner in the case of smooth Riemann surfaces. A particularly delicate case is when the conical surface is a sphere, referred to in the literature as the *singular or conical* Nirenberg problem. The classical Nirenberg problem asks to characterize those smooth functions on the sphere which arise as the Gaussian curvature of a metric that is pointwise conformal to the round metric of curvature 1. Even though much progress has been done over the years, the problem still remains open in full generality. In passing to the singular case, one now allows metrics in a given conformal class to have conical singularities. While there are some results on this topic in the literature, many questions remain unanswered. In this work, we propose a different approach that recovers the known sufficient condition established by Troyanov, namely, if K is assumed to be strictly positive, then one can solve the singular Nirenberg problem on a conical 2-sphere with cone angles less than 2π and at least three cone points. In contrast to the standard variational approaches, we follow the ideas in recent work of Anderson [And17] where our methods now hinge on the degree theory of proper Fredholm operators and compactness theorems for quasi-conformal mappings. In what follows, we briefly highlight some of the results in the long history of the Nirenberg problem that either directly pertain to our work or serve as a point of contrast. We conclude this chapter with a summary of our own results on the subject and some suggestions for how to proceed in the future.

1.1 Prescribing Curvature on Riemann Surfaces

The problem of describing the set of possible curvatures on a given manifold has been studied extensively over the past 40 years or so. While there are numerous results for general dimension n (see for instance [KW75]), in this work we restrict our attention to compact (closed), connected surfaces. On such spaces there is essentially one notion of curvature and the question then reduces to characterizing the set of possible Gaussian curvatures. Formally, one asks for necessary and sufficient conditions for a function K on a Riemann surface Σ to arise as the Gaussian curvature of *some* metric g on Σ.

In order to make the problem more tangible, Kazdan and Warner [KW75] propose to realize K in a very specific way, by first prescribing a metric g_0 on Σ and asking whether there exists a metric g which is *conformally equivalent* to g_0 (or even, *pointwise conformal* to g_0) with Gaussian curvature K. In this context, we say two metrics g, g_0 are *conformally equivalent* on Σ if there exists a diffeomorphism ϕ of Σ and a smooth function u such that $\phi^* g = e^{2u} g_0$. On the other hand, g, g_0 are *pointwise conformal* if there exists a smooth function u such that $g = e^{2u} g_0$. To put it another way, the pointwise conformal case is the special case of conformal equivalence in which we take the diffeomorphism ϕ to be the identity.

One of the main advantages in realizing K as the curvature of a pointwise metric $g = e^{2u} g_0$ is that now our question can be phrased in terms of finding solutions u to the differential equation

$$\Delta u = K_0 - K e^{2u} \tag{1.1.1}$$

where Δ and K_0 are the Gaussian curvature and Laplacian of the background metric g_0. On the other hand, the conformally equivalent case asks for a diffeomorphism ϕ such that the equation

$$\Delta u = K_0 - (K \circ \phi) e^{2u} \tag{1.1.2}$$

has a solution (since then the pullback of the metric $e^{2u} g_0$ by ϕ will have curvature K).

To address the question of necessity first, observe that the Gauss-Bonnet theorem imposes a restriction on the possible signs of K: if the Euler characteristic, $\chi(\Sigma)$, is positive, then K is positive somewhere, while if $\chi(\Sigma) < 0$, K is negative somewhere. Moreover, if $\chi(\Sigma) = 0$ then K must change sign (unless $K \equiv 0$). In light of this, it is natural to ask whether these sign conditions are also sufficient. In the conformal equivalent case, it turns out that this is indeed true when the Euler characteristic $\chi(\Sigma) \leq 0$. Explicitly, we have

Theorem 1.1.1. *(Kazdan and Warner, 1974). Let Σ be a compact Riemann surface and g_0 a given metric on Σ. Denote by K_0 the Gaussian curvature of g_0. Then,*

1. *If $\chi(\Sigma) = 0$, then a smooth function K is the curvature of a metric g conformally equivalent to g_0 if and only if either K changes sign or $K \equiv 0$*

2. *If $\chi(\Sigma) < 0$, then a smooth function K is the curvature of a metric g conformally equivalent to g_0 if and only if K is negative somewhere.*

The proof, as many of the existing results on the topic, relies on variational methods to find a solution. In this context, the variational formulation involves studying the critical points of the functional J, given by

$$J(u) = \int_\Sigma \frac{1}{2}|\nabla u|^2 dA - K_0 A \log \int_\Sigma K e^v dA + K_0 \cdot A \log(K_0 A) \tag{1.1.3}$$

It is not difficult to show that critical points of J are weak solutions of Equation 1.1.1. Using the Sobolev embedding theorem and standard elliptic regularity, one can further show that any critical point of J is in fact smooth and therefore a classical solution of Equation 1.1.1. The underlying details of variational methods will lead us too far astray from our main goal, but we highlight here the main two ingredients of such an approach in this context. First, one shows the functional J is bounded from below. This step has been classically dealt with using the Trudinger inequality:

Theorem 1.1.2. *The Trudinger Inequality. If M has dim $= 2$, then there exist positive constants β, γ such that for any $u \in W^{1,2}(M)$ with $\bar{u} = 0$ and $|\nabla u|_{L^2} \leq 1$ one has*

$$\int_M e^{\beta u^2} dA \leq \gamma \tag{1.1.4}$$

In fact, with some work, it follows from the Trudinger inequality that J is bounded below if $K_0 \leq \frac{2\beta}{A}$, where A denotes the area of Σ. The second main step is to use some form of compactness criteria in order to guarantee a minimum. It turns out that in the situation where $K_0 < \frac{2\beta}{A}$, one can show that minimizing sequences remain in a fixed ball in

$W^{1,2}$, which is weakly compact. At this point, one can simply select a weakly converging subsequence and use standard arguments to show the existence of a solution $u \in C^\infty(\Sigma)$.

In the case where $K_0 = \frac{2\beta}{A}$, the question of compactness is much more complicated, since the functional J may have no minimum (in fact, no critical points whatsoever). The optimal value of the constant β was later found by J. Moser in his very different proof of Trudinger's inequality, where he further shows that if $M = S^2$ or $\mathbb{RP}^2$ then the best constant in both cases is $\beta = 4\pi$. These two cases will be discussed in more detail in the next section, where we introduce the Nirenberg problem.

1.1.1 The Nirenberg Problem

In 1974, L. Nirenberg asked the following question, "Is any given strictly positive function K on S^2 the Gaussian curvature of some metric that is pointwise conformal to the standard metric?" In other words, can we solve equation (1.0.1) on S^2 where g_0 is the round metric of curvature 1 under the assumption that $K > 0$?

Observe that requiring that K be strictly positive is stronger than the necessary condition imposed by the Gauss-Bonnet theorem. If we asked Nirenberg's question in the conformally equivalent case, then this condition is in fact sufficient: under the assumption that $K > 0$, H.Gluck [Glu72] shows that given a smooth function f one can find a diffeomorphism ϕ of $S^2 \subset \mathbb{R}^3$ such that

$$\int_{S^2} (f \circ \phi) \cdot n dA = 0 \tag{1.1.5}$$

where n is the unit normal vector field. With $f = \frac{1}{K}$, one only needs Equation 1.1.5 to hold in order to prove the existence of a convex surface in $\mathbb{R}^3$ whose curvature is $K \circ \phi$. Pulling back the round metric by ϕ then gives the desired solution.

The pointwise case is however not as straightforward. In fact, the answer to Nirenberg's original question is "no": Kazdan and Warner have shown that one can construct strictly positive functions K, which are known to be curvatures by Gluck's work, and cannot be realized as curvatures of a metric pointwise conformal to the round metric (see Theorem

8.8 in [KW74]). Their work led to a now well-known obstruction to existence: if K is the Gaussian curvature of a metric pointwise conformal to the round metric g_0, then

$$\int_{S^2} X(K) dA_0 = 0 \tag{1.1.6}$$

for any conformal Killing field X on (S^2, g_0). As an explicit example of this obstruction, consider the vector field $X = \nabla l$, where l is any linear function on $\mathbb{R}^3$ restricted to (S^2, g_0). Then any function K of the form $K = 1 + l$ cannot be the Gaussian curvature of a pointwise conformal metric on (S^2, g_0).

Interestingly, under the further restriction that $K(x) = K(-x)$ (that is, that K is also an even function), D. Koutroufiotis proved the result for K sufficiently close to 1 while the general case was established by J. Moser using the methods discussed in the previous section.

What are then sufficient and necessary conditions which characterize Gaussian curvatures of pointwise conformal metrics on the 2-sphere? This more general question has been the subject of much study and is referred to as the *Nirenberg problem.* The literature on the topic is vast, so we refer the interested reader to [CGY93], [CL93], [CY87], [Han90], [Ji04], [And17] for a more comprehensive story.

Moser's aforementioned work serves as a stepping stone for many of the subsequent results on the Nirenberg problem. As we hinted at previously, the difficulty here is in obtaining some sort of compactness criteria that guarantees existence of a minimum of J. To elaborate on this, suppose ϕ is a conformal transformation of (S^2, g_0), where g_0 is again the round metric of curvature 1. This means that $\phi^* g_0 = e^{2\Psi_\phi} g_0$ for some function $\Psi_\phi \in C^\infty(S^2)$. Therefore, if $g = e^{2u} g_0$, then $\phi^* g = e^{2u_\phi} g_0$, where $u_\phi = u \circ \phi + \Psi_\phi$. One can show that the functional J of Equation 1.1.3 satisfies

$$J(u) = J(u_\phi)$$

In other words, J is invariant under the conformal group of (S^2, g_0). Moreover, since the conformal group in this case is noncompact, J is in fact invariant under the action of a noncompact group. As a consequence, J fails to satisfy the Palais-Smale condition, which is a compactness-kind of condition that is usually employed in proving the existence of stationary points in the variational approach (for more details refer to [Nir01]).

One can concretely see this through the following example. Let $p \in S^2 \subset \mathbb{R}^3$ and $\sigma_p : S^2 - \{p\} \to \mathbb{R}^2$ be the stereographic projection from p. Let $\delta_\lambda(x) = \lambda x$ for $\lambda > 0$ be a dilation of $\mathbb{R}^3$ and define

$$\phi_\lambda := \sigma_p^{-1} \circ \delta_\lambda \circ \sigma : S^2 \to S^2 \tag{1.1.7}$$

where the composition is extended to the whole sphere by sending $p \to p$. One can compute that

$$\phi_\lambda^* g_0 = \Psi_\lambda^2 g_0$$

where

$$\Psi_\lambda(x, y, z) = \frac{2\lambda}{(1+z) + \lambda^2(1-z)}$$

It follows that ϕ_λ is a conformal transformation for each λ. Moreover, observe that as $\lambda \to \infty$ the conformal factors Ψ_λ will concentrate (or "bubble") at p while converging to zero at all other points. In regards to J, we see that since these metrics arise from pulling back the round metric g_0, $J(\log \Psi_\lambda) = 0$ for all λ.

Many of the existing results in the literature address the conformal invariance of the problem by directly studying this type of bubbling phenomenon. The results are rather technical and outside of the scope of our work, and we refer the reader to [CGY93], [CL93], [CL93], to mention a few.

Recently, M. Anderson has proposed a new non-variational approach to the Nirenberg problem. Since the ideas in his work are at the core of our generalization to conical singularities, we will be discussing this approach in a more technical setting in the subsequent chapters, rather than in this introductory framework.

1.1.2 Extension to Orbifolds and Conical Metrics

It follows from the result of Moser mentioned above on antipodally symmetric functions on S^2, that the answer to Nirenberg's original question for $\mathbb{RP}^2$ is "yes". In fact, $\mathbb{RP}^2$ is the only 2-manifold for which the conditions imposed by Gauss-Bonnet are necessary and sufficient in the pointwise conformal case. The fundamental difference between the two spaces is that by taking the quotient of S^2 and passing to $\mathbb{RP}^2$, one "kills" the noncompactness of the conformal group discussed above. A natural question is then how far one can extend this line of reasoning: given a subgroup of isometries of S^2, such that the quotient has compact conformal group, can we answer Nirenberg's problem?

The question can be posed for an even more general class objects that enjoy a similar local geometry and are known as conical surfaces. Many of the usual topological invariants defined for smooth surfaces extend to conical ones. For instance, one defines the *generalized Euler characteristic* for the conical surface (Σ, g, β) by

$$\chi(\Sigma, \beta) := \chi(\Sigma) + \sum_{i=1}^{n} \beta_i \tag{1.1.8}$$

As in the smooth case, one can ask for necessary and sufficient conditions for existence of solutions to the Gaussian curvature equation of a pointwise conformal metric $\tilde{g} = e^{2u} g$, with g a conical metric representing a given divisor β. As mentioned in the introduction, this equation is given by

$$K = e^{-2u}(K_g - \Delta_g u) \tag{1.1.9}$$

where K_g is the curvature of the metric g. Using a variational approach, M. Troyanov proves several existence and uniqueness results for Equation 1.1.9. Once again, a particularly delicate case is when the conical surface is a sphere. We summarize in the next theorem the known results obtained by Troyanov in the case $\Sigma = S^2$.

Theorem 1.1.3. *Suppose β is a divisor on S^2 and K is a function on S^2. Then*

1. *Negative Curvature: If* $\sup K < 0$*, then there exists a unique conformal metric representing* β *with Gaussian curvature* K *if and only if* $\chi(S^2, \beta) < 0$*.*

2. *Zero Curvature: If* $K = 0$*, then there exists a conformal flat metric representing the divisor* β *if and only if* $\chi(S^2, \beta) = 0$*. The metric is unique up to homothety.*

Remark. Conditions (1) and (2) in Theorem 1.1.3 imply that if $K \leq 0$ then the cone angles $\theta_i = 2\pi(\beta_i + 1)$ satisfy

$$0 < \sum_{i=1}^{n} \theta_i \leq (2n - 4)\pi \tag{1.1.10}$$

so in particular the number n of prescribed cone points must be at least 3.

As before, the case of positive curvature is not as simple. The following result of Troyanov generalizes Moser's result for $\mathbb{RP}^2$ to conical spheres satisfying a special inequality.

Theorem 1.1.4. *Suppose* $\beta = \sum_{i=1}^{n} \beta_i p_i$ *is a divisor on* S^2*. If*

$$0 < \chi(S^2, \beta) < \min(2, 2\beta_1 + 2) \tag{1.1.11}$$

then any function K *on* S^2 *which is positive somewhere is the curvature of a conformal conical metric* g *representing the divisor* β*.*

The upper bound on inequality (1.1.11) is a consequence of Trundinger's inequality in the conical case, which, as in the smooth case, plays a central role in prescribing curvature on conical surfaces via a variational approach. Since the pioneering work of Troyanov in [Tro91] several other methods have been employed. These include complex analytic ideas [Ere04], minmax theory [CM12] and recently, synthetic geometric methods when the surface is a sphere [MP19]. In the case of constant curvature, there is a complete existence theory developed over the years [Ere04; McO88; LT92; Tro91] for conical surfaces with at least three conical singularities and angles less than 2π. In particular, it has been observed by many that a necessary condition for the existence and uniqueness of such conformal conical metric of constant curvature 1 on S^2 is

$$\sum_{i \neq j} \beta_i < \beta_j, \text{ for all } j \tag{1.1.12}$$

Condition (1.1.12) has come to be known as the *Troyanov condition* and we refer the reader to [MW15] for a geometric interpretation.

We conclude this chapter with some examples of conical metrics on S^2 that will hopefully illuminate some of the previous discussion on this topic.

1.1.3 Examples of Conical Metrics on S^2

Example 1: The football. Let $\Sigma = S^2(1)/\mathbb{Z}_k$ where $\mathbb{Z}_k$ acts by rotations. The quotient is known as the American football and it is a topological sphere with two conical singularities, each of angle $\frac{2\pi}{k}$. If $\bar{g}$ is the induced metric on the quotient, i.e. $\pi^* \bar{g} = g_{+1}$, where $\pi : S^2 \to S^2/\mathbb{Z}_k$ is the quotient map, then $Conf(S^2/\mathbb{Z}_k, \bar{g})$ is noncompact. Indeed, let $\phi_\lambda : S^2 \to S^2$ be as in 1.1.7. The action of $\mathbb{Z}_k$ on S^2 can be viewed as an action of $\mathbb{Z}_k$ on $\mathbb{C}$ after identifying $S^2 - p$ with the complex plane via the stereographic projection. From this point of view, for each element $[m] \in \mathbb{Z}_k$ we get a map $\psi_m(z) = \zeta^m \cdot z$, where ζ is a kth root of unity. Then one can check that

$$\phi_\lambda \circ \psi_m = \psi_m \circ \phi_\lambda$$

for every $[m] \in \mathbb{Z}_k$. In particular, the map ϕ_λ descends to the quotient and it will be a conformal map of $(S^2/\mathbb{Z}_k, \bar{g})$.

Example 2. Variation on the Football. Another way to obtain the American football of Example 1 is by cutting out two neighborhoods of say, the north and south pole and gluing back two different cones, e.g we can replace a neighborhood U_1 of the north pole by a quotient of the disk $D/\mathbb{Z}_n$ and a neighborhood U_2 of the south pole by the quotient $D/\mathbb{Z}_m$. When $n = m$, upon choosing appropriate gluing maps and metrics on the cone pieces, this space is just the quotient S^2/Z_n of Example 1. In the case when $n \neq m$, we cannot represent it as a global quotient by a subgroup of isometries anymore, although we can still define an orbifold structure. More importantly, the conformal group with respect to the induced orbifold metric is now compact.

Example 3. Double of a Spherical Triangle. Let T be a spherical triangle in $S^2(1)$ with angles $\alpha = \frac{2\pi}{n}, \beta = \frac{2\pi}{m}, \gamma = \frac{2\pi}{p}$. Construct the double of T by identifying T with itself via the identity. The resulting space M is a topological sphere with 3 conical singularities of angles $2\alpha, 2\beta, 2\gamma$. The conformal group is the dihedral group D_6.

1.2 Outline of the dissertation and main results

In this work we propose a new approach to the singular Nirenberg problem when there are at least three cone points and the angles are less than 2π. Using our methods we find sufficient conditions for a function K to arise as the Gaussian curvature of a conformal conical metric in this setting. Specifically, we show

Theorem 1.2.1. *Suppose $n \geq 3$, and $\beta = \sum_{i=1}^{n} \beta_i p_i$ is a divisor on S^2 satisfying the Troyanov condition (1.1.12) and there exists i, j, k distinct for which $\beta_i, \beta_j, \beta_k$ are all distinct. Assume $\chi(S^2, \beta) > 0$ and let g_β be the unique conical metric on S^2 representing the divisor β of Gaussian curvature $K_\beta = 1$. Then a function K on S^2 is the Gaussian curvature of a metric g conformal to g_β if K is a positive function in $C^{m,\alpha}_{\gamma}$, $k \geq 2, \alpha \in (0,1)$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n, \gamma_i \neq \frac{m}{\beta_j}$, $\gamma_i > 0$ and m is an integer.*

The space $C^{m,\alpha}_{\gamma}$ consists of Hölder continuous functions which are $(k-2)$-times differentiable and satisfy growth conditions near the cone points that are determined by the weights γ (for a precise definition see Chapter 2). Following the ideas in recent work of Anderson [And17], our approach is to choose appropriate Banach spaces such that the differential operator defined by (1.1.9) is a proper Fredholm map of index zero. For such operators, there is a well-defined notion of degree, and one can perform a degree count in order to study the surjectivity of this map. In establishing properness, a preliminary step is a compactness theorem for conical surfaces. Although compactness theorems for conical surfaces have been shown in [Deb20] for the uniform topology and in [Ram18] for the $C^{m,\alpha}$ topology but angles less than π, to our knowledge no result of the form required for our proof of Thm 1.2.1 exists in the literature. Thus in Chapter 2, we show

Theorem 1.2.2. *: Let Σ be a compact Riemann surface without boundary and fix a divisor $\beta = \sum_{j=1}^{n} \beta_j p_j$ on Σ such that $-1 < \beta_j < 0$ for all j. If $M_i = (\Sigma, g_i, \beta)$ is a sequence of smooth conformal conical metrics on Σ representing the divisor β such that there exist constants $D_0, \Lambda, v_0 > 0$ for which*

1. *$diam(M_i) \leq D_0$*
2. *there exists $t_0 > 0$ such that $vol_{g_i}(B(r)) \geq v_0$ for every $r \leq t_0$*
3. *$||K_{g_i}(x)||_0 \leq \Lambda$ away from the cone points*

Then for any $\gamma \in \mathbb{R}^n$, there exists a subsequence of (g_i), $C^{2,\alpha}$ diffeomorphisms $F_i : \Sigma \to \Sigma$ and a $C^{1,\alpha}_{\gamma}$ conformal conical metric g representing a divisor β' such that

$$||(F_i^* g_i)_{st} - g_{st}||_{1,\alpha;\gamma} \to 0 \tag{1.2.1}$$

as $i \to \infty$, where $\beta' = \sum_{j=1}^{m} \beta'_j q_j$ with $-1 < \beta'_j \leq 0$, $m \leq n$.

This result, however, only guarantees control of the metrics $g = e^{2u_i} g_\beta$ in a weighted $C^{m,\alpha}$ topology modulo diffeomorphisms. In order to obtain properness of the differential operator defined by (1.3), one needs to ensure the conformal factors u_i themselves converge on a subsequence. As in [And17], one can further show that also in the conical case we actually have control of the metrics modulo the conformal group of g_β, where Astala's theorem ([Ast94]) plays a central role in this step. To obtain that the conformal factors u_i themselves converge, we rely on the fact that a conformal conical metric on a sphere with at least three cone points has compact conformal group (see Thm 3.1.4).

Chapter 2

A Compactness Theorem for Conical Singularities

A fundamental step in proving an existence theorem is that of finding an appropriate compactness criteria, which usually comes in the form of a priori bounds on the solutions. In our setting, such control is initially obtained, modulo diffeomorphisms, by making use of the analogs of Cheeger-Gromov and Anderson-Cheeger compactness theorems for conical surfaces. As no such results existed in the literature for conical surfaces –at the level of generality required for our analysis of the conical Nirenberg problem– we present a proof here of a precompactness theorem for conical surfaces.

We further note that the existence of an associated compactness or precompactness theorem for the class of conical Riemann surfaces with curvature, diameter and volume bounds is interesting in its own right, and independent of the results in the subsequent chapter.

2.1 Conformal Metrics with Conical Singularities

We begin with a formal, more technical definition of a conical surface than the one given in the introduction. Let Σ be a compact Riemann surface without boundary and g_0 be a fixed smooth metric on Σ. Given points $p_1, \dots p_n \in \Sigma$, set $\Sigma^* = \Sigma - \{p_1, \dots p_n\}$. For $R > 0$ small enough, let $B_R(p)$ be a geodesic ball in Σ of radius R centered at p with respect to the metric g_0 and set $B^*_R(p) := B_R(p) - \{p\}$. Moreover, let $\Sigma_R := \Sigma - \cup_{i=1}^n \bar{B}_R(p_i)$, where $\bar{B}_R(p)$ is the closure of $B_R(p)$ in Σ.

Definition 2.1.1. Given points $p_1, \ldots p_n$ on the Riemann surface Σ, we say a smooth function $\rho : \Sigma^* \to (0, 1]$ is a *radius function* on (Σ^*, g_0), if $\rho(z) \equiv 1$ on Σ_R and $\rho(z) = O(|z|)$ in isothermal coordinates z on each $B_R(p_i)$ for $i = 1, \ldots, n$. We further define ρ^γ for $\gamma = (\gamma_1, \ldots, \gamma_n)$ in $\mathbb{R}^n$ as follows

$$\rho^\gamma = \rho^{\gamma_i} \text{ on } B_R(p_i)$$

$$\rho^\gamma \equiv 1 \text{ otherwise}$$

Moreover, for $a \in \mathbb{R}$, we define $\gamma + a := (\gamma_1 + a, \ldots \gamma_n + a)$.

Definition 2.1.2. Given a divisor $\beta = \sum_{i=1}^n \beta_i p_i$ on (Σ, g_0), we say g_β is a *conical metric* on (Σ, g_0) representing the divisor β if there exists a radius function $\rho : \Sigma^* \to \mathbb{R}$ such that

$$g_\beta = \rho^2 g_0$$

where ρ is smooth and positive outside of the set of *cone points* $\{p_1, \ldots, p_n\}$ and if z is a holomorphic coordinate in a neighborhood of p_i such that $z(p_i) = 0$, then $\rho(z) = O(|z|^{\beta_i})$ as $z \to 0$. We say g is a *conical conformal metric* on (Σ, ρ) if there exists a smooth positive function $u : \Sigma \to \mathbb{R}$ such that

$$g = e^{2u} g_\beta \tag{2.1.1}$$

where g_β is a conical metric on Σ representing the divisor β.

The pair (Σ, g_β) will be referred to as a *conical surface* for brevity, where g_β is always assumed to be a conical metric on (Σ, g_0) representing the divisor β unless otherwise specified.

Any conformal conical metric has an associated curvature function defined on the complement of the cone points. This curvature function K is just the Gaussian curvature of the smooth metric on $\Sigma - \{p_1, \ldots, p_n\}$. In fact, if we write $g_\beta = e^{2v} g_0$ for a conformal conical metric representing the divisor β, then K can be defined such that

$$K dA = dA_1 - d * dv \tag{2.1.2}$$

where dA_1 is the area element for the metric g_0, dA the area element for the conformal conical metric and $*$ is the Hodge star operator on forms: in the coordinate z

$$*dv = -i\left(\frac{\partial v}{\partial z}\right)dz + i\left(\frac{\partial v}{\partial \bar{z}}\right)d\bar{z}$$

Moreover, we can explicitly compute the Gaussian curvature in a neighborhood of any given point as follows: by the uniformization theorem, for any point $p \in \Sigma$ we can find a neighborhood of p and a holomorphic coordinate z such that $g_0 = e^{2\psi}|dz|^2$ for some $\psi > 0$ and smooth. Hence, locally, we can assume that conformal conical metrics g_β can be written in the form

$$g_\beta = e^{2u}|z|^{2\beta}|dz|^2 \tag{2.1.3}$$

for some $\beta \in (-1, 0]$ which depends on p.

Lemma 2.1.3. *If z is a holomorphic coordinate in a neighborhood of $p \in (S^2, g)$ such that $z(p) = 0$ and the conformal conical metric $g = e^{2u}|z|^{2\beta}|dz|^2$, then the Gaussian curvature K_g of g satisfies*

$$K_g = -\frac{e^{-2u}\Delta u}{|z|^{2\beta}} \tag{2.1.4}$$

for $z \neq 0$.

Proof. Let $f(z) = 2(u + \log|z|^\beta)$. By rewriting $g = e^{2f}|dz|^2$, we compute that the scalar curvature of g is

$$R_g = e^{-2f}(-2\Delta f) = e^{-2f}(-2\Delta(u(z) + \log|z|^\beta))$$

where Δ is the laplacian with respect to the flat metric $|dz|^2$. Now observe that

$$\Delta \log|z| = 0$$

Indeed, $\Delta \log|z| = 0$ if and only if $\Delta \log|z|^2 = 0$. Since $\Delta = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$, we have

$$\begin{aligned}
\Delta \log|z|^2 &= 4\frac{\partial^2}{\partial z \partial \bar{z}} \log z\bar{z} \\
&= 4\frac{\partial}{\partial z}\left(\frac{1}{|z|^2}z\right) \\
&= 4\frac{\partial}{\partial z}\left(\frac{1}{z\bar{z}}z\right) = 4\frac{\partial}{\partial z}\left(\frac{1}{\bar{z}}\right) = 0
\end{aligned}$$

Therefore,

$$R_g = e^{-2f}(-2\Delta u) = e^{-2u}|z|^{-2\beta}(-2\Delta u)$$

Since the Gaussian curvature $K_g = \dfrac{R_g}{2}$, this gives the result. □

One can further prove a corresponding Gauss-Bonnet theorem for conical surfaces, which we record below for completeness.

Theorem 2.1.4. *If* (Σ, g_β) *is a conical surface with Gaussian curvature* K *and area element* dA*, then*

$$\int_\Sigma K dA = 2\pi\chi(\Sigma) + 2\pi \sum \beta_i = 2\pi\chi(\Sigma, \beta) \tag{2.1.5}$$

2.2 Compactness Theorems for Conical Surfaces

To begin, a chart H will be referred to as an *isothermal chart* if there exists a holomorphic coordinate z such that

$$(H^{-1})^* g = e^{2\phi(z)} \prod_{i=1}^{m} |z - z^j|^{2\alpha_j} |dz|^2 \tag{2.2.1}$$

Observe that the existence of such charts is guaranteed by the uniformization theorem.

In order to directly apply our results in the next chapter, we will work with weighted Hölder spaces, which we define here as follows. For g_β a conical metric on (Σ, g_0) representing a divisor β and ρ a radius function on (Σ^*, g_0) such that $g_\beta = \rho^{2\beta} g_0$, if $u \in C^k_{loc}(\Sigma^*)$ and $\gamma \in \mathbb{R}^n$, set

$$||u||_{C^k_\gamma} := \sum_{j=0}^{k} \sup_{x \in \Sigma^*} |\rho(x)^{-\gamma+j} \nabla^j u(x)| \tag{2.2.2}$$

Define the space of $C^{k,\alpha}_\gamma(\Sigma, \beta)$ functions on (Σ, β) to be

$$C^{k,\alpha}_\gamma(\Sigma, \beta) = \left\{ u \in C^k_{loc}(\Sigma^*) : ||u||_{k,\alpha;\gamma} < \infty \right\} \tag{2.2.3}$$

where the norm $|| \cdot ||_{k,\alpha;\gamma}$ is given by

$$||u||_{k,\alpha;\gamma} := ||u||_{C^{k,\alpha}_\gamma} = ||u||_{C^k_\gamma} + [\nabla^k u]_{\alpha,\gamma-k} \tag{2.2.4}$$

and

$$[\nabla^k u]_{\alpha;\gamma} = \sup_{x \neq y, d(x,y) < inj(x)} \min(\rho(x)^{-\gamma}, \rho(y)^{-\gamma}) \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x,y)^\alpha} \tag{2.2.5}$$

It is well known that the normed spaces $(C^{k,\alpha}_\gamma, ||\cdot||_{k,\alpha;\gamma})$ are Banach for any γ (see for instance [Pac06]). One can further define the weighted Sobolev spaces $W^{k,p}_\gamma$, as in [Beh11]. Next, we introduce the notion of the *isothermal radius*, which plays the same role as that of the harmonic radius in [And90a; AC92].

Definition 2.2.1. *The Isothermal Radius:* Let (Σ, g, β) be a complete Riemann surface without boundary with g a conformal conical metric on Σ representing the divisor $\beta = \sum_{i=1}^n \beta_i p_i$. Let $x \in \Sigma$. Given a constant $C > 1$, $\alpha \in (0,1), \gamma \in \mathbb{R}^n$, we define the isothermal radius $r_I = r_I(g, x, C, \alpha, \gamma)$ as the largest number such that on the geodesic ball $B(x, r_I(x))$ there exists an isothermal coordinate chart $H : B(x, r_I(x)) \to B_0(R) \subset \mathbb{C}$ with

$$(H^{-1})^* g = e^{2\phi(z)} \prod_{i=1}^m |z - z^j|^{2\alpha_j} |dz|^2 \tag{2.2.6}$$

where z is a holomorphic coordinate on $B_0(R)$, $z^j = H(p_j)$ correspond to cone points, $1 \leq m \leq n$, $-1 < \alpha_j$ and $\phi(z) : B_0(R) \to \mathbb{R}$ is smooth and satisfies

A1. $\frac{1}{C} \leq \phi(z) \leq C$

B1. $\sum_{0 \leq |\mu| \leq 1} r_I^{-\gamma + |\mu|} \sup_x |\partial^\mu \phi(x)| + \sum_{|\mu|=1} r_I^{-\gamma+\alpha+1} \sup_{x \neq y} \frac{|\partial^\mu \phi(x) - \partial^\mu \phi(y)|}{d(x,y)^\alpha} \leq C - 1$

We also define the isothermal radius of (Σ, β, g) by

$$r_I(\Sigma) = \inf_{x \in \Sigma} r_I(x) \tag{2.2.7}$$

Observe that α_j either coincide with the β_j or are zero in neighborhoods with no conical points. In the following lemma we prove properties of the isothermal radius that will be needed for the upcoming blow-up argument in the proof of Theorem 2.2.6. All of these facts are true for the harmonic radius and the proofs presented here are essentially the same as in the works of [And90a; AC92; HH97], only slightly modified to fit our setting.

Lemma 2.2.2. *Let (Σ, β, g) be a Riemann surface with a conformal conical metric g representing the divisor β and let $r_I : \Sigma \to \mathbb{R}$ be the isothermal radius. Then the following hold*

1. *$r_I(x)$ is positive and pointwise continuous on Σ*

2. *If $F : (\Sigma, g) \to (\Sigma', g')$ is an isometry, then*

$$F^* r_I(\Sigma') = r_I(\Sigma)$$

3. *r_I scales as the distance: $r_I(\lambda^2 g, x) = \lambda r_I(g, x)$*

Proof. Proof of (1). For the positivity, observe that for any $x \in \Sigma$ we can find a $0 \geq \mu > -1$, a neighborhood U of x and a holomorphic coordinate $z : U \to \mathbb{C}$ such that $g = e^{2\phi}|z|^{2\mu}|dz|^2$ for some smooth $\phi(z) : U \to \mathbb{R}$. Clearly, if x is any of the cone points, such coordinates exist by definition once we choose $\mu = \frac{\theta}{2\pi}$ where θ is the cone angle. If on the other hand x is a smooth point, then such coordinates are guaranteed by the Uniformization Theorem: we can find a neighborhood U of x and a holomorphic coordinate $z : U \to \mathbb{C}$ such that $g = e^{2\phi}|dz|^2$. Finally observe that conditions $A1, B1$ always hold on a fixed conical surface in a given isothermal chart.

For the continuity, given any x close enough to y, we can find a ball of radius a centered at x which contains all the cone points in $B(y, r_I(y))$ and is contained in $B(y, r_I(y))$. Therefore if z is a holomorphic coordinate in $B(y, r_I(y))$ such that

$$g = e^{2\phi(z)} \prod_{i=1}^{m} |z - z_i|^{2\alpha_i} |dz|^2$$

with $\phi(z)$ satisfying conditions $A1, B1$ of Definition 2.2.1, by restricting the coordinates z to $B(x, a)$, we get a holomorphic coordinate on this ball such that in this coordinate the same function $\phi(z)$ still satisfies conditions $A1, B1$ for the same C, α, γ. Therefore $a \leq r_I(x)$. Using this one can directly show

$$|r_I(y) - r_I(x)| = |a + d(x, y) - r_I(x)| \leq \epsilon$$

as wanted.

Proof of (2). We will actually prove something stronger than the statement of (2): computing the isothermal radius with respect to g' at a point $F(x)$ gives the same result as computing the isothermal radius with respect to g at the point x. To begin, fix $x \in \Sigma$ and let $H' : B(F(x), r_I F(x)) \to B_0(R)$ be an isothermal coordinate chart where

$$(H'^{-1})^* g' = e^{2\phi(z)} \prod_{i=1}^{m} |z - z_i|^{2\alpha_i} |dz|^2 \tag{2.2.8}$$

and $\phi(z)$ satisfies conditions $A1, B1$ of Definition 2.2.1. The set $U = F^{-1}(B(F(x), r_I F(x)))$ is open in Σ and contains x. If $B(x, R)$ is the largest geodesic ball centered at x which is still contained in U, then $H := H' \circ F : B(x, R) \to B_0(R)$ is an isothermal coordinate chart for the metric g on $B(x, R)$. Moreover, the metric g on the ball $B(x, R)$ has the form (2.2.6) above, where the conformal factor can be computed to be $2\phi(F(z)) + ||DF||^2$ and satisfies the bounds $A1, B1$ in these coordinates since $||DF||_{(F^{-1})^* g} = 1$. We have thus found a ball centered at x in which all the conditions of the definition of the isothermal radius are satisfied, so we conclude that its radius $R \leq r_I(x)$. In particular, $r_I(F(x), g') \leq r_I(x, g)$. To obtain the opposite inequality, we just follow the same argument with F^{-1} in place of F.
Proof of (3). We want to show that $r_I(\lambda^2 g, x) = \lambda r_I(g, x)$ for any nonzero λ. Start with a conical metric g and after having chosen a holomorphic coordinate for which we can write the metric g as in (2.2.6) of Definition 2.2.1, with $\phi(z)$ satisfying the bounds $A1, B1$, it is straightforward to check that the metric $\lambda^2 g$ satisfies the same bounds in the coordinates $w = \lambda z$. □

We now turn to the question of convergence. For $k \geq 2, \alpha \in (0,1), \gamma \in \mathbb{R}^n$, we say a conformal conical metric g on (Σ, g_0) representing the divisor β is of class $C^{k,\alpha}_{\gamma}$ if in an isothermal chart the coefficients g_{ij} of g are bounded in $C^{k,\alpha}_{\gamma}$. Moreover, a sequence of conformal conical metrics (Σ, g_i, β) of class $C^{k,\alpha}_{\gamma}$ converges in $C^{k,\alpha}_{\gamma}$ to a surface (Σ', g) provided that there exists a sequence of $C^{k+1,\alpha}$ diffeomorphisms $F_i : \Sigma' \to \Sigma$ such that for all i large enough

$$||(F_i^* g_i) - g||_{k,\alpha;\gamma} \to 0 \tag{2.2.9}$$

in any chart of a C^∞ subatlas of the complete C^∞ atlas of Σ. The following lemma addresses the continuity of the isothermal radius in the $C^{1,\alpha}_\gamma$ topology.

Lemma 2.2.3. *For $\alpha \in (0,1)$, $\gamma \in \mathbb{R}^n$, the isothermal radius is continuous under $C^{1,\alpha}_\gamma$ convergence of a sequence of conical metrics (Σ, g_i, β) representing the divisor β.*

Proof. Let (Σ, g_i, β) be a sequence of conformal conical metrics on Σ representing the divisor β and $x \in \Sigma$. As before, $\Sigma^* = \Sigma - \{p_1, \dots p_n\}$, where $p_1, \dots p_n$ are the cone points. Fix $\alpha \in (0,1), \gamma \in \mathbb{R}^n$. Assume that the sequence (g_i) converges in $C^{1,\alpha}_\gamma$ to a $C^{1,\alpha}_\gamma$ metric g on Σ. By the continuity of the isothermal radius we mean explicitly that the following two inequalities hold: given $C > 1$

$$r_I(g, x, C) \geq \limsup_{i\to\infty} r_I(g_i, x, C) \tag{2.2.10}$$

and for any $0 < \epsilon < C - 1$

$$r_I(g, x, C - \epsilon) \leq \liminf_{i\to\infty} r_I(g_i, x, C) \tag{2.2.11}$$

For simplicity, set $r_i := r_I(g_i, x, C)$ and let $H_i : B(x, r_i) \to \mathbb{C}$, $H_i(y) = (H_i^1(y), H_i^2(y)) = z_i \in \mathbb{C}$ be isothermal coordinate charts in which the metrics g_i satisfy $A1, B1$ of Definition 2.2.1.

We begin with some preliminary claims, the first being that for any $r \leq \limsup r_i$, a subsequence of the isothermal charts H_i converges in $C^{2,\alpha}$ to an isothermal chart $H : B(x,r) \to \mathbb{C}$, where $B(x,r)$ is a geodesic ball for the metric g. To this end, suppose $(x_1(y), x_2(y)) \in \mathbb{C}$ is any given local coordinate chart on $B(x,r)$. Observe that it is implicit in the fact that the charts H_i are isothermal that the coordinate functions H_i^k, for $k = 1,2$ are harmonic. In other words,

$$(g_i)^{st} \frac{\partial^2 H_i^k}{\partial x_s \partial x_t} = (g_i)^{st} (\Gamma_i)^l_{st} \frac{\partial H_i^k}{\partial x_l} \tag{2.2.12}$$

where $(g_i)^{st}$ are the components of g_i in the coordinates (x_1, x_2) and $(\Gamma_i)^l_{st}$ are the Christoffel symbols for g_i in these coordinates. Now, condition $A1$ of Definition (2.2.1) implies that in the charts H_i, the components of the metrics g_i satisfy

$$\frac{1}{C}\prod_{j=1}^{m_i}|z_i - z_i^j|^{\alpha_j}\delta_{kl} \leq (g_i)_{kl} \leq C\prod_{j=1}^{m_i}|z_i - z_i^j|^{\alpha_j}\delta_{kl} \tag{2.2.13}$$

where we write $z_i = H_i^1 + iH_i^2$ and the inequality holds as bilinear forms. It then follows from (2.2.13) and the fact that the metrics converge in $C^{1,\alpha}_{\gamma}$ that the charts H_i are bounded in C^1 on Σ^*. Using standard elliptic estimates for (2.2.12) (see [GT83], for instance), we obtain that for each $k = 1, 2$, the sequence (H_i^k) is bounded in $C^{2,\alpha}(\Sigma^*)$. Therefore, by the Arzela-Ascoli theorem, we have that for each $k = 1, 2$, the sequences (H_i^k) converge weakly on a subsequence in $C^{2,\alpha'}$ on Σ^* for $\alpha' \leq \alpha$. In fact, repeating this argument for $H_q^k - H_n^k$ in place of H_i^k one can see that for each $k = 1, 2$, the sequences (H_i^k) are in fact Cauchy. Therefore, they converge strongly on a subsequence in $C^{2,\alpha}_{loc}$ to a limiting map $H : B(x, r) - \{p_1, \ldots p_m\} \to \mathbb{C}$, where $H_i(p_j) = z^j$ for all i. Since the property of being an isothermal chart is preserved under $C^{2,\alpha}$ convergence, H is an isothermal chart for the metric g. Moreover, since there are finitely many cone points, by passing to a subsequence, we may assume m_i in (2.2.13) is independent of i and $z_i^j = z^j$ for all i. Therefore, g can be written in the chart given by H as

$$(H^{-1})^*g = e^{2\phi(z)}\prod_{j=1}^{m}|z - z^j|^{\alpha_j}|dz|^2 \tag{2.2.14}$$

where $\phi(z)$ is a smooth function that satisfies conditions $A1, B1$ of Definition 2.2.1. Hence for any $r \leq \limsup r_i$, a subsequence of the isothermal charts H_i converges in $C^{2,\alpha}$ to an isothermal chart $H : B(x, r) \to \mathbb{C}$ for the metric g. Observe that this argument also shows that if a sequence of conformal conical metrics representing a fixed divisor β converges in $C^{1,\alpha}_{\gamma}$ to a metric g, then g has at most as many cone points as the sequence g_i and no other types of singularities.

Our second preliminary claim is that if (Σ, g, β) is any complete Riemann surface without boundary with a conformal conical metric g representing the divisor β, $x \in \Sigma$, $\gamma \in \mathbb{R}^n$, then for any $1 \leq C' \leq C < \infty$,

$$r_I(C')(x) \leq r_I(C)(x) \tag{2.2.15}$$

and for any $C > 1$

$$\lim_{\epsilon \to 0^+} r_I(C+\epsilon)(x) = r_I(C)(x) \tag{2.2.16}$$

The first inequality follows from the definition, hence to prove the claim, it's enough to show that for any $C > 1$

$$\limsup_{\epsilon \to 0^+} r_I(C+\epsilon, x) \leq r_I(C, x) \tag{2.2.17}$$

Fix $r < \limsup r_I(C+\epsilon, x)$. For a decreasing sequence of $\epsilon > 0$ converging to 0, there are isothermal coordinate charts H_ϵ on $B_x(r)$ satisfying conditions $A1, B1$ of Definition 2.2.1 with $C+\epsilon$ in place of C and r in place of r_I. By the same arguments as above, we get that a subsequence of H_ϵ converges in $C^{2,\alpha}_{loc}$ to a limiting chart H. As before, the bounds $A1, B1$ are preserved under $C^{2,\alpha}$ convergence, hence $r_I(C, x) \geq r$. Since $r < \limsup r_I(C+\epsilon, x)$ was arbitrary, this proves the claim.

We're now ready to prove the first inequality (2.2.10). As before, let $r_i = r_I(g_i)$. We may suppose $\limsup r_i > 0$. The arguments above show that convergence of the metrics in $C^{1,\alpha}_\gamma$ implies convergence of the isothermal charts H_i in $C^{2,\alpha}_{loc}$. Once again, the bounds $A1, B1$ are preserved, so that $r_I(g, C) \geq r$ for any $r \leq \limsup r_i$. Therefore we get the first inequality $r_I(g, C) \geq \limsup r_I(g_i, C)$.

Now fix $r < r_I(g, C)$. To obtain the second inequality (2.2.11), let $H : B(x, r) \to B_0$ be an isothermal coordinate chart for g, so that $(H^{-1})^* g = e^{2\phi(z)} \prod |z - z^j|^{2\alpha_j} |dz|^2$, with z a holomorphic coordinate on $B_0 \subset \mathbb{C}$. Let Δ_i be the Laplace operator for the metric g_i. In the coordinate z, the Laplacian for the metrics g_i has the form

$$\Delta_i = e^{-2(\phi_i(z) + \sum_{j=1}^m \alpha_j \log |z - z^j|)} \Delta \tag{2.2.18}$$

where Δ is the Euclidean laplacian. As observed before, we may assume the cone points z^j are the same for each i after passing to a subsequence. Now, if w_i are solutions to

$$\Delta_i w_i = 0 \text{ on } B \tag{2.2.19}$$

$$w_i = z \text{ on } \partial B \tag{2.2.20}$$

then the functions $u_i := z - w_i$ are harmonic and vanish on the boundary of B. Since, for each i, the function z also solves the boundary value problem (2.22-2.23) it follows from uniqueness that in fact $u_i = 0$. Therefore, $\lim_{i\to\infty} ||u_i||_{2,\alpha} = 0$ and we have that for any compact subset $B' \subset B$ and for any i, H is an isothermal coordinate chart for g_i. Now, since the metrics g_i converge to g in $C^{1,\alpha}_\gamma$, H is an *isothermal* coordinate chart for g_i in which the bounds of Definition 2.2.1 are satisfied with constants $C_i \to C$ as $i \to \infty$. Using (2.2.15), (2.2.16), we have that for any $\epsilon > 0$

$$r \leq \liminf r_I(g_i, C_i) \leq \liminf r_I(g_i, C + \epsilon)$$

Since $r \leq r_I(g, C)$ was arbitrary, this ends the proof of the second inequality. □

An important property of the harmonic radius in the smooth case is that Euclidean space has infinite harmonic radius. The isothermal radius satisfies an analogous condition, but in this case the model space is a flat Riemann surface M with finitely many cone points and angles less than 2π, which is noncompact, complete and of *quadratic area growth* , in the sense that for any $x \in M$ and any $r > 0$

$$\frac{1}{V} r^2 \leq vol(B(r,x)) \leq V r^2 \tag{2.2.21}$$

Theorem 2.2.4. *Any noncompact conical surface with a finite number of conical singularities and angles less than 2π which is flat, complete and of quadratic area growth has infinite isothermal radius.*

Proof. Suppose M is a complete flat conical surface of quadratic area growth, so that there exist $p_1, \ldots, p_n$ in M such that near each p_i we can find a coordinate z and a harmonic function u with $g = e^{2u}|z - z_i|^{2\beta_i}|dz|^2$ where $z_i = z(p_i)$ and (2.2.21) is satisfied. If we smooth out the singularities p_i, the resulting surface M' is still noncompact and complete. Moreover, since the cone angles are less than 2π, the curvature can only increase upon smoothing the singularities, thus M' has Gaussian curvature $K \geq 0$. It further follows from the volume comparison theorem of Bishop-Gromov [Pet16] that if the original (singular) surface has quadratic area growth, then any smoothing M' has at most quadratic area growth.

Now fix a smoothing M' of M and let $\tilde{M}$ be its universal cover. Since M' is complete and noncompact, the pullback of the metric on M' to $\tilde{M}$ by the covering map makes $\tilde{M}$ into a simply connected, complete, noncompact surface with Gaussian curvature $K \geq 0$. A complete surface is hyperbolic if it admits a positive Green's function. On the other hand, a theorem of Yau [Yau75] (see also [BF42]) asserts that a complete surface of nonnegative Gaussian curvature admits no non-constant positive superharmonic functions. Thus $\tilde{M}$ cannot be hyperbolic and by the uniformization theorem, we have $\tilde{M}$ is parabolic, i.e. it is conformally equivalent to the complex plane.

We actually claim that M' is simply connected, so that M' is parabolic. By Bishop-Gromov again, the universal cover $\tilde{M}$ of M' has at most quadratic area growth. Moreover, since M' has at least quadratic area growth, by Proposition 1.2 in [And90b] we then have that M' is the quotient of $\tilde{M}$ by a finite group of isometries Γ. Since M' is smooth, Γ must be trivial.

Since smoothing out the singularities doesn't change the topology or the conformal structure, M is also simply connected and parabolic. Therefore, there exists a global coordinate z such that the metric on M has the form $e^{2v}|dz|^2$. Since M has conical singularities, $e^{2v} = e^{2u}\prod |z - z_i|^{2\alpha_i}$. Hence, there exist global coordinates on M for which the metric has the form

$$g = e^{2u}\prod |z - z_i|^{2\alpha_i}|dz|^2 \tag{2.2.22}$$

where u is harmonic. At this point, we have shown that a noncompact, complete, flat conical surface with cone angles less than 2π and quadratic area growth is conformally equivalent to the complex plane. Our final claim is that the function u in (2.2.22) has to be constant. To see this, define

$$h = \prod |z - z_i|^{-\alpha_i} g \tag{2.2.23}$$

where g is the metric in (2.2.22). In other words, $h = e^{2u}|dz|^2$. Since g is flat by assumption, the function u is harmonic. Therefore, the Gaussian curvature K_h of h satisfies $K_h = e^{-2u}\Delta u = 0$. On the other hand, since the cone angles are less than 2π, $\alpha_i < 0$ for all i. Therefore, if γ is any C^1 curve, then far away from the cone points,

$$\int h(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt = \int \prod |z(\gamma(t)) - z_i(\gamma(t))|^{\frac{-\alpha_i}{2}} g(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt \geq \int g(\dot{\gamma}(t), \dot{\gamma}(t))^{\frac{1}{2}} dt \tag{2.2.24}$$

where the inequality follows since $\prod |z(\gamma(t)) - z_i(\gamma(t))|^{\frac{-\alpha_i}{2}} >> 1$ whenever $|z(\gamma(t)) - z_i(\gamma(t))| >> 1$. The assumption that g is complete together with (2.2.24) now imply that h is complete.

Now, let $F : T_0\mathbb{C} \to \mathbb{C}$ be the exponential map of the origin for the metric h. By the above arguments, the smooth metric h is flat and complete, so by Cartan-Hadarmard's theorem (see [Pet16], Thm 22) the exponential map F is in fact a diffeomorphism of $\mathbb{C}$ that satisfies

$$F^* h = e^{2u(0)}|dz|^2 \tag{2.2.25}$$

In other words,

$$F^* h = e^{u \circ F}|\partial F + \bar{\partial} F|^2 = e^{u \circ F}(|\partial F|^2 + |\bar{\partial} F|^2 + 2Re\partial F \bar{\partial} F) = e^{2u(0)}|dz|^2$$

The last equality implies that either $\partial F = 0$ or $\bar{\partial} F = 0$, and since the orientation of the tangent space is the same as the base manifold, we must have that $\bar{\partial} F = 0$, so F is holomorphic. A standard result in complex analysis is that holomorphic diffeomorphisms of the plane are affine linear maps. Given that F preserves the origin (it is the exponential map at the origin), we conclude that it must be of the form

$$F(\zeta) = c\zeta$$

for some $c \neq 0$. But then $e^{u \circ F} = e^{u \circ (c\zeta)} = e^{2u(0)}$, so u has to be constant.

The sequence of arguments above now show that M admits global coordinates for which the metric has the form

$$g = C \prod |z - z_i|^{2\alpha_i} |dz|^2$$

from which it follows that the isothermal radius must be infinite. □

The following is a generalization of $C^{1,\alpha}$ convergence to weighted $C^{1,\alpha}_\gamma$ convergence of conical metrics on Riemann surfaces.

Theorem 2.2.5. *Let $M_i = (\Sigma, g_i, \beta)$ be a sequence of complete, conformal conical surfaces with metrics g_i representing the divisor β. Let $\{x_i\} \in M_i$ be a sequence of points. Given $\Lambda > 0, C > 1, \alpha \in (0, 1)$, suppose that*

1. *for any i, $||K(g_i)||_0 \leq \Lambda$ away from the cone points*

2. *there exists $r > 0$ such that for any sequence of points (y_i) in M_i there is an isothermal chart $H_i : \Omega_n \to B_0(r)$ where Ω_i is some open set in M_i and $B_0(r) \subset \mathbb{C}$ such that for any i, there exists ϕ_i smooth, with $\frac{1}{C} \leq \phi_i(z) \leq C$ such that*

$$(H_i^{-1})^* g_i = e^{2\phi_i(z)} \prod_{j=1}^{m} |z - z_j|^{2\alpha_j} |dz|^2 \tag{2.2.26}$$

   where z are holomorphic coordinates on $B_0(r)$ and $z_j = H_n(p_j)$ correspond to the cone points p_j

3. *for $\gamma \in \mathbb{R}^n$, a subsequence of $(H_i^{-1})^* g_i$ converges in $C^{1,\alpha}_\gamma$ on $B_0(r)$*

Then there exists a complete Riemannian manifold M of class $C^{2,\alpha}$, there exists a conformal conical metric g of class $C^{1,\alpha}_\gamma$ and a point $x \in M$ such that the following holds: for any compact domain $D \subset M$ with $x \in D$ there exist, up to passing to a subsequence, compact domains $D_i \subset M_i$ with points $x_i \in D_i$ and $C^{2,\alpha}$ diffeomorphisms $\Phi_n : D \to D_i$ satisfying

$$\lim_{i \to \infty} (\Phi_i^{-1})(x_i) = x \tag{2.2.27}$$

$$\Phi_i^* g_i \text{ converges in } C^{1,\alpha}_\gamma \text{ in any chart of the induced } C^{2,\alpha} \text{ complete atlas of } D. \tag{2.2.28}$$

Proof. The proof is exactly as in the smooth case (see [HH97; Pet87] and [And90a] for a summarized version) since conical surfaces are considered to have singularities only in the metric sense, as can be seen from the local coordinate expression in (2.2.26) above. □

Theorem 2.2.6. *: Let Σ be a compact Riemann surface without boundary and fix a divisor $\beta = \sum_{j=1}^{n} \beta_j p_j$ on Σ such that $-1 < \beta_j < 0$ for all j. If $M_i = (\Sigma, g_i, \beta)$ is a sequence of smooth conformal conical metrics on Σ representing the divisor β such that there exist constants $D_0, \Lambda, v_0 > 0$ for which*

1. *$diam(M_i) \leq D_0$*
2. *there exists $t_0 > 0$ such that $vol_{g_i}(B(r)) \geq v_0$ for every $r \leq t_0$*
3. *$||K_{g_i}(x)||_0 \leq \Lambda$ away from the cone points*

Then for any $\gamma \in \mathbb{R}^n$, there exists a subsequence of (g_i), $C^{2,\alpha}$ diffeomorphisms $F_i : \Sigma \to \Sigma$ and a $C^{1,\alpha}_{\gamma}$ conformal conical metric g representing a divisor β' such that

$$||(F_i^* g_i)_{st} - g_{st}||_{1,\alpha;\gamma} \to 0 \tag{2.2.29}$$

as $i \to \infty$, where $\beta' = \sum_{j=1}^{m} \beta'_j q_j$ with $-1 < \beta'_j \leq 0$, $m \leq n$.

Proof. The first part of the proof is a blow-up argument to show that, under the hypotheses of the theorem, there is a uniform lower bound on the isothermal radius. So to begin, let D_0, Λ, v_0 be positive constants as in the statement of the theorem and $\alpha \in (0, 1)$. Given a conical Riemann surface (Σ, g, β) and $C, t_0 > 0$ satisfying

$$||K||_0 \leq \Lambda \tag{2.2.30}$$

$$vol_g(B(r)) \geq v_0 > 0 \ \forall r \leq t_0 \tag{2.2.31}$$

we show that there exists $r_0 = r_0(\Lambda, v_0) > 0$ such that for every $x \in B(r)$ and any $\gamma \in \mathbb{R}^n$

$$\frac{r_I(x, C, \alpha)}{d_g(x, \partial B(r))} \geq r_0 > 0 \tag{2.2.32}$$

where $r_I(x, C, p)$ is as usual the isothermal radius. Indeed, if (2.2.32) does not hold, there exists a sequence of conical metrics $M_i = (\Sigma_i, g_i, \beta)$ representing the same divisor β with Gaussian curvature $|K_i|_0 \leq \Lambda$, there exists a sequence of balls $B_i = B_i(r) \subset M_i$ of radius $r \leq t_0$, there exists $\gamma \in \mathbb{R}^n$ and there exists a sequence of points $x_i \in B_i$ such that

$$\frac{r_i(x_i)}{d_i(x_i, \partial B_i)} \to 0 \tag{2.2.33}$$

where $d_i = d_{g_i}$ is the induced distance on M_i from g_i and $r_i(x) = r_I(g_i, x)$ is the isothermal radius of the metric g_i at x.

Set $R_i(x) := \frac{r_i(x)}{d_i(x, \partial B_i)}$. By the same arguments as in [HH97; And90a], we may as well assume the points x_i minimize $R_i(x)$. Rescaling the metrics g_i as

$$h_i = \frac{1}{r_i(x_i)^2} g_i \tag{2.2.34}$$

we get

$$\lim_{i\to\infty} ||K_{(B_i,h_i)}||_0 = r_i(x_i)^2 ||K_i||_0 \leq r_i(x_i)^2 \Lambda \to 0 \tag{2.2.35}$$

$$\lim_{i\to\infty} vol(B_i, h_i) = \infty \tag{2.2.36}$$

$$\lim_{i\to\infty} d(x_i, \partial B_i) = \infty \tag{2.2.37}$$

where (2.37) holds away from the cone points. By Lemma 2.2.2 (3), the isothermal radius scales as the distance under rescalings of the metric, thus the isothermal radius of the new metrics h_i satisfies

$$r_i'(x_i) := r_I(h_i, x_i) = 1 \tag{2.2.38}$$

Moreover, for every $y \in B_i$ and for every i

$$r_i'(y) = \frac{r_i(y)}{r_i(x_i)} \geq \frac{d_i(y, \partial B_i)}{d_i(x_i, \partial B_i)} = \frac{d_i'(y, \partial B_i)}{d_i'(x_i, \partial B_i)}$$

(where the first inequality follows since $R_i(y) \geq R_i(x_i)$ and d_i' is the induced distance from h_i). Set

$$\delta_i := \frac{1}{d_i'(x_i, \partial B_i)} \tag{2.2.39}$$

Then $\lim_{i\to\infty}\delta_i = 0$ and for all $y \in B(x_i, \frac{1}{2\delta_i})$ (the geodesic ball for the metric h_i with center x_i) we have

$$r_i'(y) \geq \frac{1}{2} \tag{2.2.40}$$

Hence, the isothermal radius of the rescaled metrics is bounded from below and is at most 1. Now we claim (B_i, x_i, h_i) converges in $C^{1,\alpha}_\gamma$ uniformly on compact sets to a complete (noncompact) manifold (M, y, h). First, the argument above implies that given $R < \infty$, $r_i'(y) \geq \frac{1}{2}$ on $B(x_i, R)$ for i large enough. Thus given $R < \infty$ and a sequence (q_i) in $B(x_i, R)$ we can find isothermal charts $H_i : \Omega_i \to B_0(\frac{1}{2\sqrt{C}})$ centered at q_i such that

$$(H_i^{-1})^* h_i = e^{2\phi_i(z_i)} \prod_{j=1}^{m_i} |z_i - z_i^j|^{2\alpha_j} |dz|^2 \tag{2.2.41}$$

where $\phi_i(z_i) : B_0(\frac{1}{2\sqrt{C}}) \to \mathbb{R}$ are smooth and bounded, the integers $m_i = m_i(q_i)$ and the real numbers α_j satisfy $1 \leq m_i \leq n$, $-1 < \alpha_j \leq 0$ for all $i = 1, \dots$ and $1 \leq j \leq m_i$. As before, z_i is a holomorphic coordinate on $B_0 = B_0(\frac{1}{2\sqrt{C}})$ and $z_i^j = H_i(p_j)$ (p_j are cone points). Moreover,

$$\frac{1}{C} \leq \phi_i(z_i) \leq C \tag{2.2.42}$$

$$||\phi_i(z_i)||_{1,\alpha;\gamma} \leq F(C) \tag{2.2.43}$$

with F depending only on C. By (2.2.43) we have that $(\phi_i(z_i))$ are bounded in $C^{1,\alpha}_\gamma$ on the ball B_0, so after passing to a subsequence we can assume they converge weakly in $C^{1,\alpha}_\gamma$ to some ϕ by the Arzela-Ascoli Theorem. In particular, the metrics $(H_i^{-1})^* h_i =: h_i'$ converge weakly in $C^{1,\alpha}_\gamma$ on B_0. We claim that in fact the metrics h_i' converge strongly in $C^{1,\alpha}_\gamma$. Indeed, from Lemma 2.1.3, we have that the Gaussian curvature K_i of the metrics h_i in the coordinates z_i is given by

$$K_i(z_i) = \frac{e^{-2\phi_i(z_i)} \Delta\phi_i(z_i)}{\prod_{j=1}^{m_i} |z_i - z_i^j|^{2\alpha_j}} \tag{2.2.44}$$

As we observed in the proof of Lemma 2.2.3, we can assume $z_i^j = z^j$ and $m_i = m$ are independent of i by passing to a subsequence. By (2.2.35), $||K_{(B_i,h_i)}||_0 \to 0$ away from the cone points, so that $||(H_i^{-1})^* K_i||_0 = ||K_i(z_i)||_0 \to 0$ as $i \to \infty$ for all $z_i \neq z^j \in B_i$. Now, the limit $\phi(z)$ of the sequence $(\phi_i(z_i))$ solves

$$0 = \frac{e^{-2\phi(z)} \Delta \phi(z)}{\prod_{j=1}^m |z - z^j|^{2\alpha_j}} \tag{2.2.45}$$

weakly. Applying standard elliptic estimates to (2.2.45), we get $\phi(z)$ is actually smooth. In fact, using the same arguments as in the proof of Lemma 2.2.3 (second paragraph following (2.2.13)), we get that the convergence is actually in the *strong* $C^{1,\alpha}_\gamma$ topology, hence the metrics h_i' converge strongly in $C^{1,\alpha}_\gamma$.

It now follows from the arguments used to prove Theorem 2.2.5 (see for instance Proposition 12 in [HH97]) that there exists a $C^{2,\alpha}$ manifold M, $y \in M$ and a $C^{1,\alpha}_\gamma$ conformal conical metric h on M such that for any compact domain $D \subset M$ with $y \in D$ and after passing to a subsequence, there exist compact domains $D_i \subset B_i$ and $y_i \in D_i$ and $C^{2,\alpha}$ diffeomorphisms $\Phi_i : D \to D_i$ such that

$$\lim_{i\to\infty} (\Phi_i^{-1})(y_i) = y \tag{2.2.46}$$

$$||(\Phi_i^{-1})^* h_i - h||_{1,\alpha;\gamma} \to 0 \text{ in } D \tag{2.2.47}$$

where (2.2.47) is in the sense that in any chart of the complete induced atlas on D the components of $(\Phi_i^{-1})^* h_i$ converge in $C^{1,\alpha}_\gamma$ to the components of h.

We now claim the pointed limit (M, h) is flat, conical and complete. First, the completeness follows from (2.2.37). To see (M, h) is flat, given a compact domain D in M, set $\hat{h}_i := \Phi_i^* h_i$ and for a given $x \in D$, let $U_i : B_x(r) \to \mathbb{C}$, $r > 0$, be an isothermal coordinate chart for $\hat{h}_i$ satisfying $A1, B1$ of Definition 2.2.1. The convergence of the $\hat{h}_i$ implies convergence in $C^{2,\alpha}$ of the charts U_i to a limiting chart $U : B_x(r) \to \mathbb{C}$, by the same arguments as in Lemma 2.2.3. If we write $(U^{-1})^* h = e^{2\psi(z)} \prod_{j=1}^m |z - z_j|^{2\alpha_j} |dz|^2$, then going back to (2.48) we have

$$0 = \frac{e^{-2\psi(z)} \Delta \psi(z)}{\prod_{j=1}^m |z - z_j|^{2\alpha_j}} \tag{2.2.48}$$

It follows from standard elliptic estimates then that the limit metric h is smooth on Σ^*. Since (2.2.48) is also the equation for the Gaussian curvature of h, we also have that it is flat. Moreover, assumption (2) in the Theorem along with the Bishop-Gromov volume comparison Theorem imply that

$$\frac{vol(B(s))}{V^{\Lambda}(s)} \geq \frac{v}{V^{\Lambda}(t_0)} \tag{2.2.49}$$

for all $s \leq t_0$, where V^{Λ} is the volume of a geodesic ball in constant curvature Λ and $vol(B(s))$ is taken in the g_i metric. Using scaling properties of volume and the convergence in $C^{1,\alpha}_{\gamma}$, we have

$$\frac{vol_M(B(s))}{s^2} \geq v' > 0 \tag{2.2.50}$$

for all $s > 0$. On the other hand, since (M, h) is flat with cone angles less than 2π, the volume of a ball of radius s measured in the metric h has to be less than the volume of a ball of radius s measured with respect to the standard metric on $\mathbb{R}^2$, hence

$$\frac{vol_M(B(s))}{s^2} \leq v' \tag{2.2.51}$$

and we conclude that the volume growth must be exactly quadratic.

To obtain a contradiction, observe that since the h_i converge to h in $C^{1,\alpha}_{\gamma}$, by Lemma 2.2.3, we have that

$$r_I(C', p, y) \leq \lim_{i\to\infty} r_I(C, p, y_i) \tag{2.2.52}$$

for some $C' < C$, but by construction $r_I(C, p, y_i) = 1$, while a flat, noncompact, complete surface with finitely many conical singularities and of quadratic area growth has infinite isothermal radius for any C', as follows from Theorem 2.2.4.

Therefore, there is a uniform lower bound on the isothermal radius, which in turn allows us to apply Theorem 2.2.5 to conclude the existence of a limit with the desired properties. In the notation of Theorem 2.2.5, with $D = B_x(R)$, $R > D_0$, we get that for i large enough, $D_i = M_i$, and up to passing to a subsequence there exist diffeomorphisms $\Phi_i : M \to M_i$ such that $\Phi_i^* g_i$ satisfies conditions (2.30-2.31) of Theorem 2.2.5. In particular, M has a smooth structure coming for instance from one of the diffeomorphisms Φ_i with M_i. $\square$

Chapter 3

The Nirenberg problem for a Conical Sphere

3.1 Conformal Geometry of Conical Metrics on S^2

Continuing with the notations of the previous chapter, suppose g_β is a conformal conical metric on $S^2(1) = (S^2, g_{+1})$ representing the divisor β. A diffeomorphism $\psi : S^2 \to S^2$ is called a *conformal transformation* of (S^2, g_β) if $\psi^*(g_\beta) \in [g_\beta]$, i.e. if there exists a function $u : S^2 \to \mathbb{R}$ which is smooth and positive and such that

$$\psi^* g_\beta = e^{2u} g_\beta$$

The set of all conformal transformations forms a group under composition, which we denote by $Conf(S^2, g_\beta)$. It acts on functions $K \in C^{m-2,\alpha}_{\gamma-2}$ by precomposition, i.e.

$$(\phi, K) \to K \circ \phi \tag{3.1.1}$$

There is also an action of $Conf(S^2, g_\beta)$ on the conformal factors u which one can derive as follows. Suppose $\phi \in Conf(S^2, g_\beta)$. By definition,

$$\phi^* g_\beta = \eta_\phi^2 g_\beta$$

where $\eta_\phi = |D\phi| > 0$. Now suppose $g = e^{2u} g_\beta$ then

$$\phi^* g = e^{2(\phi^* u)} \phi^* g_\beta = e^{2(u\circ\phi + \log \eta_\phi)} g_\beta$$

Hence the conformal group $Conf(S^2, g_\beta)$ acts on the conformal factors $u \in C^{m,\alpha}_\gamma$ by

$$(\phi, u) \to u \circ \phi + \log \eta_\phi \tag{3.1.2}$$

Observe that if $\psi \in Diff(S^2)$, then $\psi^* g_\beta$ is always a conical metric with the same number of conical singularities as g_β. Moreover, we have

Lemma 3.1.1. *Every conformal transformation $\psi \in Conf(S^2 - \{p_1, p_2, \ldots p_n\}, g_{+1})$ has an extension $\tilde{\psi} \in Conf(S^2, g_{+1})$.*

Proof. Observe that any conformal map $\psi \in Conf(S^2 - \{p_1, p_2, \ldots p_n\}, g_{+1})$ can be viewed as a biholomorphism $\mathbb{C} - \{q_1, \ldots, q_{n-1}\} \to \mathbb{C} - \{q_1, \ldots, q_{n-1}\}$ after conjugation with the stereographic projection from, say, p_n.

To be precise, let σ_{p_n} be the sterographic projection $S^2 - \{p_n\} \to \mathbb{C}$ and suppose $q_i = \sigma_{p_n}(p_i)$ for $i = 1, \ldots n-1$. The restriction of σ_{p_n} to the punctured sphere $S^2 - \{p_1, \ldots p_n\}$ gives a diffeomorphism with $\mathbb{C} - \{q_1, \ldots q_{n-1}\}$. In particular, if $\psi \in Conf(S^2 - \{p_1, \ldots, p_n\}, g_{+1})$, then

$$\bar{\psi} := \sigma_{p_n}^{-1} \circ \psi \circ \sigma_{p_n} : \mathbb{C} - \{q_1, \ldots q_{n-1}\} \to \mathbb{C} - \{q_1, \ldots q_{n-1}\}$$

is a biholomorphism. We claim now that the points q_i are removable singularities for $\bar{\psi}$. Indeed, if $\bar{\psi}$ had an essentially singularity at any of the points q_i, then by Picard's Theorem, $\bar{\psi}$ would take on all possible values with at most one exception on any neighborhood of q_i, infinitely often, but this would contradict injectivity. The second observation is that ψ has at worst one pole of order one. Again, any higher order pole is excluded because of injectivity. If there were two simple poles at say q_i, q_j, then $\bar{\psi}(q_i) = \infty$, $\bar{\psi}(q_j) = \infty$, but since ψ is an open map, it would map a punctured neighborhood of q_i to a neighborhood of ∞ and a punctured neighborhood of q_j to a neighborhood of ∞. The fact that these two neighborhoods must intersect contradicts injectivity once again.

It then follows that $\bar{\psi}$ extends to a biholomorphism of the Riemann sphere, hence it corresponds to a conformal map $\tilde{\psi}$ of $S^2(1)$. □

Claim 3.1.2. If $p_1, p_2 \dots, p_n$ are cone points for the conformal conical metric g_β, then

$$Conf(S^2 - \{p_1, \dots, p_n\}, g_\beta) = Conf(S^2 - \{p_1, \dots, p_n\}, g_{+1})$$

Proof. We can directly show the two sets are equal since g_β is conformal to g_{+1} by assumption. □

Lemma 3.1.3. *The group $Conf(S^2 - \{p_1, \dots, p_n\}, g_{+1})$ is finite if $n \geq 3$.*

Proof. If $\psi \in Conf(S^2 - \{p_1, \dots p_n\}, g_{+1})$, then its extension $\tilde{\psi}$ to a conformal map of the round sphere fixes the set $\{p_1, \dots p_n\}$. Suppose $\tilde{\psi}(p_1) = p_i, \tilde{\psi}(p_2) = p_j$, $\tilde{\psi}(p_3) = p_k$, where $i, j, k \in (1, \dots n)$ are all distinct. Since $\tilde{\psi}$ is a Mobius transformation, its values are uniquely determined after specifying the image of the points p_1, p_2, p_3. Since there are only finitely many choices for p_i, p_j, p_k, the collection of extensions of $\psi \in Conf(S^2 - \{p_1, \dots p_n\}, g_{+1})$ is finite. In particular, the group $Conf(S^2 - \{p_1, \dots, p_n\}, g_{+1})$ is finite. □

Theorem 3.1.4. *Suppose $\beta = \sum_{i=1}^n \beta_i p_i$ is a divisor on S^2, $n \geq 3$ and g_β a conical conformal metric representing β as before. Then $Conf(S^2, g_\beta)$ is finite. Moreover, if there exists i, j, k distinct, such that $\beta_i, \beta_j, \beta_k$ are all distinct, then the conformal group $Conf(S^2, g_\beta) = 0$.*

Proof. Let $supp(\beta) = \{p_1, \dots p_n\}$. Define $F : Conf(S^2, \beta, g) \to Conf(S^2 - supp(\beta), g)$ by $F(\phi) = \phi\big|_{S^2 - supp(\beta)}$. Since any $\phi \in Conf(S^2, \beta, g)$ fixes the set $supp(\beta)$, we have that F is a surjective group homomorphism. Moreover,

$$ker(F) = \{\phi \in Conf(S^2, \beta, g) : F(\phi) = \phi|_{S^2 - supp(\beta)} = Id|_{S^2 - supp(\beta)}\}$$

The only freedom is in where the points $p_1, \dots p_n$ are sent, and we know ϕ fixes them on (S^2, β, g). Hence we have $ker(F)$ is isomorphic to a subgroup of S_n, the symmetric group on n elements. Finally, by Claim 3.1.2 and Theorem 3.1.4, $Conf(S^2 - supp(\beta), g)$ is finite for $n \geq 3$, and since $ker(F)$ is also finite, we must have $Conf(S^2, \beta, g)$ is finite. If there are three distinct angles, any conformal map has to fix them, but every conformal map of the unit disk fixing three points is the identity. So the conformal group must be trivial in this case. This concludes the proof of the theorem. □

Remark. The condition that $n \geq 3$ is only sufficient. In the examples in the next section we show that there is a metric on $S^2(1)$ with two conical singularities and noncompact conformal group.

3.2 The Curvature Map π

For $\gamma = (\gamma_1, \ldots \gamma_n)$, let $C^{m,\alpha}_{\gamma}$ be the Banach space of $C^{m,\alpha}_{\gamma}$ functions $u : S^2 \to \mathbb{R}$ considered as conformal factors of $g = e^{2u} g_\beta$ and let $C^{m-2,\alpha}_{\gamma-2}$ be the Banach space of $C^{m-2,\alpha}_{\gamma-2}$ functions K. Our main goal is to study the image of the *curvature map* π, defined to be the map $C^{m,\alpha}_{\gamma} \to C^{m-2,\alpha}_{\gamma-2}$ sending

$$u \mapsto K_g$$

As before, if g is a conformal conical metric on (S^2, g_{+1}), then

$$g = e^{2u} g_\beta = e^{2u} \rho^{2\beta} g_{+1}$$

where ρ is a radius function as in Definition 2.1.1. The Gaussian curvature of g is then

$$K_g = K(e^{2u} g_\beta) = e^{-2u}(K_\beta - \Delta_\beta u)$$

where Δ_β is the Laplacian with respect to the conical metric g_β and K_β the Gaussian curvature of g_β. One can compute

$$\Delta_\beta = \rho^{-2\beta} \Delta_{+1} \tag{3.2.1}$$

$$K_\beta = \rho^{-2\beta}(1 - \beta \Delta_{+1} \log \rho) \tag{3.2.2}$$

Observe that the function $\beta \Delta_{+1} \log \rho$ is defined to be $\beta_i \Delta_{+1} \rho$ in a neighborhood of the cone point p_i and vanishes identically away from the cone points (since $\rho \equiv 1$).

Recall that a C^1 map $F : \mathcal{B}_1 \to \mathcal{B}_2$ is a Fredholm map between Banach manifolds $\mathcal{B}_i$ if the differential

$$D_u F(h) := \frac{d}{dt} F(u + th)\bigg|_{t=0} = \lim_{t \to 0} \frac{F(u + th) - F(u)}{t}$$

is a Fredholm operator at each $u \in \mathcal{B}_i$. As it is well-known, Fredholm maps between Banach spaces are bounded linear operators characterized by having finite-dimensional kernel and cokernel. The index of a Fredholm operator is defined as

$$ind(F) = dim(KerF) - dim(coKerF)$$

Moreover, the index of the Fredholm map F is defined to be the index of its differential, which is independent of the choice of u. For more on the theory of Fredholm maps on Banach manifolds, see [ET70; Nir01].

Theorem 3.2.1. *Let (Σ, g, β) be a conical surface with g representing the divisor $\beta = \sum_{j=1}^{n} \beta_j p_j$. If $\gamma = (\gamma_1, \dots \gamma_n) \in \mathbb{R}^n$ satisfies $\gamma_i > 0$ and $\gamma_i \neq \frac{m}{\beta_j}$ for any $i, j \in (1, \dots n)$, where m is an integer, then the curvature map π is Fredholm of index 0.*

Proof. Fix $u \in C^{m,\alpha}_{\gamma}$. If we set $g = e^{2u} g_\beta$, then as observed above

$$\pi(u) = e^{-2u}(K_\beta - \Delta_\beta u) \tag{3.2.3}$$

Let $h \in C^{m,\alpha}_{\gamma}$, thought of as the tangent space to $C^{m,\alpha}_{\gamma}$ at u. Then

$$\begin{aligned} D_u \pi(h) &= \frac{d}{dt}\bigg|_{t=0} \pi(u + th) \\ &= \frac{d}{dt}\left(e^{-2(u+th)}(K_\beta - \Delta_\beta(u + th))\right)\bigg|_{t=0} \\ &= -2hK_g - e^{-2u}\Delta_\beta h \end{aligned}$$

with $K_g = e^{-2u}(K_\beta - \Delta_\beta u)$, which is the Gaussian curvature of g. At $u = 0$, we get $K_g = K_\beta$, hence

$$D_0\pi(h) = -2hK_\beta - \Delta_\beta h \tag{3.2.4}$$

$$= -\rho^{-2\beta}(2h(1 - \beta\Delta_{+1} \log \rho)) + \Delta_{+1} h) \tag{3.2.5}$$

If we let $a := 2(1 - \beta\Delta_{+1} \log \rho)$, then

$$-L_\beta(h) := D_0\pi(h) = \rho^{-2\beta}(ah + \Delta_{+1} h) \tag{3.2.6}$$

It is known that if $\gamma = (\gamma_1, \dots \gamma_n) \in \mathbb{R}^n$ satisfies $\gamma_i \neq \frac{m}{\beta_j}$ for any $i, j \in (1, \dots n)$, where m is an integer, then the linear operator $L_\beta : C^{k,p}_\gamma \to C^{k-2,p}_{\gamma-2}$ is Fredholm [MW15; Beh11]. We further claim L_β is formally self-adjoint. Observe there is a natural inner product on (S^2, β) given by

$$\langle u, v\rangle = \int_{S^2} u \cdot v\rho^{2\beta} dV_{+1} \tag{3.2.7}$$

Thus for all $u, v \in C^{m,\alpha}_\gamma$ we have

$$\begin{aligned}
\langle v, L_\beta u\rangle &= \int_{S^2} v(\rho^{-2\beta}(\Delta_{+1}u + au))\rho^{2\beta} dV_{+1} && (3.2.8)\\
&= \int_{S^2} v(\Delta_{+1}u + au) dV_{+1} && (3.2.9)\\
&= \int_{S^2} v(\Delta_{+1}u + au) dV_{+1} && (3.2.10)\\
&= \int_{S^2} u(\Delta_{+1}v + av) dV_{+1} && (3.2.11)\\
&= \int_{S^2} u\rho^{-2\beta}(\Delta_{+1}v + av)\rho^{2\beta} dV_{+1} && (3.2.12)\\
&= \langle L_\beta v, u\rangle && (3.2.13)
\end{aligned}$$

The integration by parts in (3.12)-(3.13) needs some justification, so fix $R > 0$ small enough and let $B_R(p_i)$ be a geodesic ball of radius R around the cone point p_i. Let $S_R(p_i)$ denote the circle of radius R with center p_i, $\partial_\nu u$ denotes the normal derivative of u, and dS the volume element of $S_R(p_i)$. We then have

$$\begin{aligned}
\int_{S^2 - \{p_1 \dots p_n\}} v\Delta_{+1}u \quad dV_{+1} &= \lim_{R\to 0} \int_{S^2 - \cup_{i=1}^n B_R(p_i)} v\Delta_{+1}u \quad dV_{+1}\\
&= \lim_{R\to 0}\left(\int_{S^2 - \cup_{i=1}^n B_R(p_i)} u\Delta_{+1}v \quad dV_{+1} + \int_{\cup_{i=1}^n S_R(p_i)} u\partial_\nu v - v\partial_\nu u \quad dS\right)\\
&= \int_{S^2 - \{p_1,\dots p_n\}} u\Delta_{+1}v \quad dV_{+1} + \lim_{R\to 0}\sum_{i=1}^n \int_{S_R(p_i)} u\partial_\nu v - v\partial_\nu u \quad dS
\end{aligned}$$

Now, since $u \in C^{k,p}_\gamma(S^2 - \cup_{i=1}^n B_R(p_i))$ for all $R > 0$ small enough, it follows that $\partial_\nu u \in C^{k-1,p}_{\gamma-1}(S^2 - \cup_{i=1}^n B_R(p_i))$, i.e. there is a $C > 0$ such that $||u||_{C^{l,\alpha}_\gamma} \leq C$. In particular, $||\partial_\nu u||_{C^{l-1,\alpha}_{\gamma-1}} \leq C$. It follows from the definition of these norms (see 2.14) that

$$\sup_{x\in S^2 - \{p_1,\dots p_n\}} \rho^{-\gamma}|u| \leq C$$

and

$$\sup_{x\in S^2-\{p_1,\dots p_n\}} \rho^{-(\gamma-1)}|\partial_\nu u| \leq C_1$$

and similarly for v. Therefore

$$\left|\int_{S_R(p_i)} u\partial_\nu v - v\partial_\nu u \quad dS\right| \leq \int_{S_R(p_i)} |u||\partial_\nu v|dS + \int_{S_R(p_i)} |v||\partial_\nu u|dS \tag{3.2.14}$$

$$= \int_{S_R(p_i)} \rho^{-\gamma}|u|\rho^{-\gamma+1}|\partial_\nu v|\rho^{2\gamma-1}dS + \int_{S_R(p_i)} \rho^{-\gamma}|v|\rho^{-\gamma+1}|\partial_\nu u|\rho^{2\gamma-1}dS \tag{3.2.15}$$

$$\leq 2C' \int_{S_R(p_i)} \rho^{2\gamma-1}dS \tag{3.2.16}$$

$$\leq C(\delta)R^{2\gamma} \tag{3.2.17}$$

Thus, provided $\gamma > 0$, taking the limit as $R \to 0$, we see that the boundary terms disappear, as wanted. This concludes the proof that the map L_β is formally self-adjoint and Fredholm, from which it follows that the map π is a Fredholm map of index 0. □

3.3 Properness of the map π

Theorem 3.3.1. *Let $\mathcal{C}_+$ be the subspace of $C^{m-2,\alpha}_{\gamma-2}$ consisting of positive curvature functions K. Define $\mathcal{U} = \pi^{-1}(\mathcal{C}_+)$. If $\gamma > 0$, then the map $\pi_0 : \mathcal{U} \to \mathcal{C}_+$ defined as the restriction of π to $\mathcal{U}$ is proper.*

Proof. If $K_i \to K \in \mathcal{C}_+$, then the sequence K_i is bounded in $C^{k-2,\alpha}_{\gamma-2}$. In particular, K_i is bounded in $C^0_{\gamma-2}$. Hence there exist a constant K such that $||K_i||_0 \leq K$ away from the cone points. Moreover, since the Euler characteristic is positive, we have using Gauss-Bonnet (2.1.5)

$$2\pi\chi(S^2,\beta) = \int_{S^2} K_i dvol_{g_i} \leq K_0 \cdot area(S^2, g_i) \tag{3.3.1}$$

so we get

$$area(S^2, g_i) \geq \frac{2\pi\chi(S^2,\beta)}{K_0} > 0$$

On the other hand, Myers' theorem ([Pet16]) implies that the diameter of each conical surface is finite (since curvature is assumed positive). Furthermore, since the sequence K_i converges in $\mathcal{C}_+$ to a positive function K, we can find constants D_0, v_0 such that

1. $vol(g_i) \geq v_0$
2. $diam(g_i) \leq D_0$

for all i. Under these bounds we can now directly apply Theorem 2.2.6 to conclude that there exists a sequence of diffeomorphisms $F_i : S^2 \to S^2$ such that

$$(F_i^* g_i)_{rs} \to (g_\infty)_{rs} \tag{3.3.2}$$

in $C^{1,\alpha}_\gamma$, where g_∞ is a conical metric on S^2 with m conical singularities of angles $0 < \theta \leq 2\pi$. By passing to a subsequence if necessary, we may assume that the F_i are orientation preserving. On the other hand, since g_∞ is the limit of a sequence of conical metrics in the same conformal class, we claim there exists a diffeomorphism $\psi : S^2 \to S^2$ and a smooth, positive function $u : S^2 \to \mathbb{R}$ such that $\psi^* g = e^{2u} g_\beta$. To see this, suppose $\mathfrak{C}$ denotes the space of conformal classes on the punctured sphere $S^2 - \{p_1, \ldots, p_n\}$, i.e., two smooth (incomplete) metrics h_1, h_2 on $S^2 - \{p_1, \ldots, p_n\}$ represent the same point in $\mathfrak{C}$ if there exists a positive smooth function u such that $h_1 = e^{2u} h_2$. The group $Diff_+$ of orientation preserving diffeomorphisms of S^2 acts on $\mathfrak{C}$ by

$$(\psi, [h]) \to \psi^*[h] = [\psi^* h]$$

Since it is not true in general that $[\psi^* h] = [h]$, we consider the moduli space $\mathcal{M} = \mathfrak{C}/Diff_+$. This space corresponds to Teichmuller space [Pet19; MW15], since the mapping class group of the sphere is trivial (so every orientation preserving diffeomorphism is isotopic to the identity). Observe then that every element of the sequence g_i, when considered as smooth metrics on $S^2 - \{p_1, \ldots p_n\}$, corresponds to the same point in $\mathfrak{C}$, namely the conformal class $[g_\beta]$. If ψ_i is a sequence of orientation preserving diffeomorphisms, then $\psi_i^*[g_i] = [g_i]$

as elements of $\mathcal{M}$, hence the classes $\psi_i^* g_i$ and g_β correspond to the same point in $\mathcal{M}$. Since the topology of Teichmuller space is Hausdorff [Pet19], the sequence $\psi_i^* g_i$ is the constant sequence $[g_\beta] \in \mathcal{M}$. Thus the limit of the $\psi_i^* g_i$ must be in the conformal class of g_β modulo diffeomorphisms, i.e. there exists a diffeomorphism of S^2 such that the limit g satisfies

$$\psi^* g = e^{2u} g_\beta$$

for some positive smooth function u on S^2, as claimed. In particular, there is a diffeomorphism Ψ of S^2 such that $\Psi^* g_\infty = e^{2u} g_\beta$. After precomposing F_i with Ψ^{-1}, we may as well assume that we have a sequence F_i of diffeomorphisms of S^2 such that

$$F_i^*(g_i) = F_i^*(e^{2u_i} g_\beta) \to e^{2u} g_\beta \tag{3.3.3}$$

in $C^{1,\alpha}_\gamma$, where u is some positive smooth function on S^2. As in the arguments preceding Proposition 2.5 in [And17], one now has that the F_i converge on a subsequence to the identity modulo the action of the conformal group, i.e. there exist conformal maps $\phi_i \in Conf(S^2, g_\beta)$ such that $\phi_i^{-1} \circ F_i$ converge to the identity. It follows from Proposition 2.5 in [And17] that the functions $\phi_i^* u_i$ are uniformly bounded in $C^{1,\alpha}_\gamma \cap W^{2,p}_\delta$, for $\gamma > 0$ and some $\delta \leq \gamma$ (see [Beh11] for a definition of weighted Sobolev spaces). Observe that nothing really changes in the presence of conical singularities since the diffeomorphisms F_i are still quasiconformal when restricted to the punctured sphere $S^2 - \{p_1, \dots, p_n\}$ (see [LV73]). By the Arzela-Ascoli theorem, the uniform bound on the sequence $\phi_i^* u_i$ implies convergence on a subsequence to a limit in $C^{1,\alpha}_\gamma$. Moreover, since the conformal group $Conf(S^2, g_\beta)$ is finite, we actually have that $\{u_i\}$ themselves (sub)converge to a limit $u \in C^{1,\alpha}_\gamma$ which satisfies

$$\Delta_{g_\beta} u = K_\beta - K e^{2u} \tag{3.3.4}$$

weakly. Given that the Gaussian curvatures K_i of the metrics g_i are assumed to be in $C^{k-2,\alpha}_{\gamma-2}$, a bootstrapping argument using Proposition 2.7 in [Beh11] implies $u \in C^{k,\alpha}_\gamma$. This then completes the proof that the map π_0 is proper. □

3.4 Degree Computations

We conclude this section with sufficient conditions for a function K to arise as the Gaussian curvature of a conformal conical metric on S^2 having at least three conical singularities and angles less than 2π. Our result follows from computing the degree of the curvature map which we have established in the previous section is a proper Fredholm map of index zero, providing not only an existence theorem but also a signed count of the number of solutions when K is a regular value.

As mentioned in the introduction, a necessary condition for the existence of a *constant* curvature conical metric on S^2 having at least three conical singularities and angles less than 2π is

$$\sum_{i \neq j} \beta_i < \beta_j, \ \textit{for all } j \tag{3.4.1}$$

In fact, Luo-Tian have shown in [LT92] that if the generalized Euler characteristic is positive, then this condition is sufficient and necessary for uniqueness and existence. Under these assumptions, we can now compute the degree of the curvature map given our previous results. Recall that if F is a proper Fredholm map of index 0 between open subsets of Banach spaces, one can define its *degree* by the formula

$$deg(F) = \sum_{x \in F^{-1}(y)} sign(D_x F)$$

where y is any regular value of F and the sign is $\pm$ according to whether $D_x F$ preserves or reverses orientation. By definition, y is a regular value if $D_x F$ is an isomorphism for all $x \in F^{-1}(y)$. In particular, points with empty preimage are always regular values. We refer the reader to [Nir01] for more on the degree theory of Fredholm maps on Banach manifolds.

Let $\mathcal{C} = \mathcal{C}_+ \cap C^{k-2,\alpha}_{\gamma-2}$, where $\gamma_i > 0, \gamma_i \neq \frac{m}{\beta_j}$ for any $(i,j) \in (1, \ldots n)$. Recall that the restrictions on γ guarantee that the curvature map is proper and Fredholm of index 0. We have,

Theorem 3.4.1. *Suppose $n \geq 3$, and $\beta = \sum_{i=1}^{n} \beta_i p_i$ is a divisor on S^2 satisfying the Troyanov condition (1.1.12) and there exists i, j, k distinct for which $\beta_i, \beta_j, \beta_k$ are all distinct. Assume $\chi(S^2, \beta) > 0$ and let g_β be the unique conical metric on S^2 representing the divisor β of Gaussian curvature $K_\beta = 1$. Then a function K on S^2 is the Gaussian curvature of a metric g conformal to g_β if K is a positive function in $C^{m-2,\alpha}_{\gamma-2}$, $k \geq 2, \alpha \in (0,1)$, where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n, \gamma_i \neq \frac{m}{\beta_j}$, $\gamma_i > 0$ and m is an integer.*

Proof. Suppose $K \in \mathcal{C}$. We want to show there exists a function u such that $e^{2u} g_\beta$ has Gaussian curvature K, where g_β is the unique conformal conical metric with Gaussian curvature 1. The existence of such a metric is equivalent to the existence of a solution u to the equation

$$K = e^{-2u}(1 - \Delta_\beta u) \tag{3.4.2}$$

In the language of this section, it is enough to show that the restriction π_0 of the curvature map to $\pi^{-1}(\mathcal{C})$ has $deg = 1$. The assumption that $K \in \mathcal{C}$ guarantees that the map π_0 is a proper Fredholm map of index 0 (see Theorems 3.3.1, 3.2.1). Observe that for given γ, α, k satisfying the conditions of the theorem, the subset of $C^{m-2,\alpha}_{\gamma-2}$ consisting of positive functions is convex, thus $\mathcal{C}$ is path-connected and there is a well-defined notion of degree. Clearly the function $K = 1 \in \mathcal{C}$. On the other hand, the preimage of $K = 1$ under π_0 is given by all solutions to the equation

$$1 = e^{-2u}(1 - \Delta_\beta u) \tag{3.4.3}$$

By Theorem 2 in [LT92] there exists a unique conical metric g on S^2 representing the divisor β of constant curvature 1. Since $u = 0$ is a solution, it follows that the preimage $\pi_0^{-1}(1) = \{0\}$.

Now, the kernel of the differential of the curvature map π_0 under the assumption that g_β has Gaussian curvature 1 is given by solutions of

$$D_0\pi(h) = -2h - \Delta_\beta h = 0 \tag{3.4.4}$$

We now argue that the first eigenvalue λ of the problem

$$\Delta_\beta h = -\lambda h \tag{3.4.5}$$

satisfies $\lambda \geq 2$. Moreover, if the lowest possible eigenvalue is achieved, namely $\lambda = 2$, then there exists a non-constant solution to the equation $Hess(f) = -fg$. To see this, we follow the same ideas as in the works of Lichnerowicz and Obata [Lic58; Oba62] which have now become standard. Using Bochner's formula away from the cone points, we can write

$$\frac{1}{2}\Delta_\beta|\nabla h|^2 = |Hess(h)|^2 + g_\beta(\nabla\Delta h, \nabla h) + K_\beta|\nabla h|^2 \tag{3.4.6}$$

Using Scharwz inequality and the fact that h is an eigenfunction we get

$$|Hess(h)|^2 \geq \frac{1}{2}(\Delta h)^2 = -\frac{\lambda}{2}h\Delta_\beta h \tag{3.4.7}$$

Combining this with Bochner's formula we get the inequality

$$\Delta_\beta|\nabla h|^2 \geq -\frac{\lambda}{2}h\Delta_\beta h - \lambda|\nabla h|^2 + |\nabla h|^2 \tag{3.4.8}$$

We now claim

$$\int_{S^2-\{p_1,\ldots p_n\}} \Delta_\beta f dvol_\beta = 0 \tag{3.4.9}$$

holds for any function $f \in C^{k-1,\alpha}_{\gamma-1}$. For $R > 0$ small enough, let $B_R(p_k)$ be a geodesic ball of radius R centered at the cone point p_k. Then

$$\begin{aligned}
\int_{S^2-\{p_1,\ldots p_n\}} \Delta_\beta f dvol_\beta = \int_{S^2-\{p_1,\ldots p_n\}} \rho^{-2\beta}\Delta_{+1} f \rho^{2\beta} dvol_{+1} &= \lim_{R\to 0}\int_{S^2-\cup_{k=1}^n B_R(p_k)} \Delta_{+1} f dvol_{+1} \\
&= \lim_{R\to 0}\int_{S^2-\cup_{k=1}^n B_R(p_k)} div(\nabla f) dvol_{+1} \\
&= \lim_{R\to 0}\sum_{k=1}^n \int_{S_R(p_k)} (\nabla f\cdot\nu)dS
\end{aligned}$$

where in the last equality we have used the divergence theorem, with S_R denoting the boundary of $B_R(p_k)$, ν the normal to each boundary circle and dS the area element. As before, the assumption that $f \in C^{k-1,\alpha}_{\gamma-1}$ implies that $\sup_{S^2-\{p_1,\ldots p_n\}} \rho^{-(\gamma-1)+1}|\nabla f| \leq C$, so that

$$\int_{S_R} |\rho^{-\gamma+2}\nabla f\cdot\nu|\rho^{\gamma-2} dS \leq C' R^\gamma$$

Since $\gamma > 0$, taking the limit as $R \to 0$ we get the desired result in (3.4.9).

Now, using (3.4.9) in combination with (3.4.8), we obtain the inequality

$$
\begin{aligned}
0 = \int_{S^2-\{p_1,\dots p_n\}} \Delta_\beta |\nabla h|^2 dvol_\beta &\geq \int_{S^2-\{p_1,\dots p_n\}} -\frac{\lambda}{2} h \Delta_\beta h - \lambda |\nabla h|^2 + |\nabla h|^2 dvol_\beta \\
&= \left(\frac{\lambda}{2} - \lambda + 1\right) \left(\int_{S^2-\{p_1,\dots p_n\}} |\nabla h|^2 dvol_\beta\right)
\end{aligned}
$$

where the integration by parts is justified as in the proof of Theorem 3.2.1. The previous inequality then shows that

$$
-\frac{\lambda}{2} + 1 \leq 0 \tag{3.4.10}
$$

so that $\lambda \geq 2$. Moreover, if $\lambda = 2$, then the inequalities become equality, which forces the trace free part of the Hessian of h to vanish, implying that h solves the equation

$$
Hess(h) = \phi g \tag{3.4.11}
$$

One can further show that $\phi = -h$. Observe that (3.4.11) implies the existence of a nonconstant solution to the equation

$$
\mathcal{L}_{\nabla h} g = -hg \tag{3.4.12}
$$

Now suppose $h \in Ker(D_0 \pi_0)$, that is, h solves $-2h = \Delta_\beta h = \rho^{-2\beta} \Delta_{+1} h$, thus h is an eigenfuction corresponding to the lowest possible eigenvalue. The discussion above shows that h satisfies (3.4.12), in other words, ∇h is a conformal Killing field on (S^2, g, β). Equivalently, this means the locally defined flow of ∇h preserves the conformal structure. Therefore, there exists a nontrivial one-parameter group of conformal transformations. Since the conformal group $Conf(S^2, \beta)$ is trivial, we must have $h = 0$. Thus $K = 1$ is a regular value of the curvature map π_0 . It now follows that $deg \pi_0 = 1$, as wanted. □

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