Family Floer program and non-archimedean SYZ mirror construction

A Dissertation Presented

by

Hang Yuan

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics Ph.D. program

Stony Brook University

May 2021

Stony Brook University

The Graduate School

Hang Yuan

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Kenji Fukaya – Dissertation Advisor
Professor, Mathematics

Mark McLean - Chairperson of Defense
Associate Professor, Mathematics

Simon Donaldson - Inside member of Defense
Professor, Mathematics

Eric Zaslow - Outside member of Defense
Professor, Mathematics
(Northwestern University)

This dissertation is accepted by the Graduate School

Eric Wertheimer

Dean of the Graduate School

Abstract of the Dissertation

Family Floer program and non-archimedean SYZ mirror construction

by

Hang Yuan

Doctor of Philosophy

in

Mathematics Ph.D. program

Stony Brook University

2021

Given a Lagrangian fibration, we provide a natural construction of a mirror Landau-Ginzburg model consisting of a rigid analytic space, a superpotential function, and a dual fibration based on Fukaya's family Floer theory. The mirror in the B-side is constructed by the counts of holomorphic disks in the A-side together with the non-archimedean analysis and the homological algebra of the A infinity structures. It fits well with the SYZ dual fibration picture and explains the quantum/instanton corrections and the wall crossing phenomenon. Instead of a special Lagrangian fibration, we only need to assume a weaker semipositive Lagrangian fibration to carry out the non-archimedean SYZ mirror reconstruction.

Contents

1 Introduction

The Strominger-Yau-Zaslow conjecture [SYZ96] asserts that the mirror symmetry for Calabi-Yau manifolds can be interpreted as a duality between (special) Lagrangian torus fibrations. In the language of Kontsevich's homological mirror symmetry [Kon95], the SYZ picture reflects the expectation that a mirror of a Calabi-Yau manifold can be constructed as a moduli space of Lagrangian tori equipped with rank one local systems. Moreover, the mirror symmetry has been also extended to the non-Calabi-Yau cases by considering the Landau-Ginzburg models.

The classical SYZ mirror reconstruction in terms of dual special Lagrangian fibrations needs to be modified by the so-called '*quantum correction*', also known as the '*instanton correction*' (Figure 1). Namely, the counts of holomorphic disks bounding the Lagrangian fibers are expected to correct the moduli space geometry of special Lagrangian submanifolds in some delicate ways.

In [Aur07], Auroux conjectures that if D is an anticanonical divisor in a Kähler manifold X and Ω is a holomorphic volume form defined over $X \setminus D$, then a mirror space $X^\vee_{\mathbb{C}}$ can be constructed as a moduli space of special Lagrangian tori in $X \setminus D$ equipped with flat $U(1)$ connections, and there is a global superpotential $W^\vee_{\mathbb{C}} : X^\vee_{\mathbb{C}} \to \mathbb{C}$ given by Fukaya-Oh-Ohta-Ono's $\mathfrak{m}_0$ obstruction to Floer homology (counts of holomorphic disks). In particular, given a torus fiber L, its SYZ dual torus fiber is expected to be $H^1(L; U(1))$, i.e. the space of flat $U(1)$-connections up to gauge equivalence. In the present paper, we will give a positive answer to Auroux's conjecture but in a non-archimedean setting. Indeed, due to Kontsevich-Soibelman [KS01, KS06], the non-archimedean analysis should play an important role in the formulation of the homological mirror symmetry.

The *Novikov field* Λ is actually a non-archimedean field that is defined as follows:

$$\Lambda = \mathbb{C}((T^{\mathbb{R}})) = \Big\{ \sum_{i\geq 0} a_i T^{E_i} \mid a_i \in \mathbb{C}, E_i \nearrow +\infty \Big\}$$

The Novikov ring Λ_0 consists of those formal series with all $E_i \geq 0$. The multiplicative subgroup U_Λ of Λ consists of those series $a_0 + \sum_{i\geq 1} a_i T^{E_i}$ with $a_0 \neq 0$, so it is roughly a thickening of $\mathbb{C}^*$. Observe that the U_Λ is precisely the set of all the $x \in \Lambda$ with $|x| = 1$ for the natural non-archimedean norm on Λ. So, the U_Λ^n resembles the real torus $T^n = U(1)^n$ and will be called a *non-archimedean torus*.

The main result Theorem 1.3 will construct from a reasonable Lagrangian torus fibration $\{L\}$ in X a mirror Landau-Ginzburg model $(X^\vee, W^\vee)$ over Λ which is also equipped with a dual fibration $\pi^\vee$. To give a quick impression first, we indicate that the mirror space is just set-theoretically given by

$$X^\vee = \bigsqcup_L H^1(L; U_\Lambda) \tag{1}$$

The dual fiber of L is now $H^1(L; U_\Lambda) \cong U_\Lambda^n$ in place of $H^1(L; U(1))$. It can be essentially viewed as the space of (weak) bounding cochains on L. The

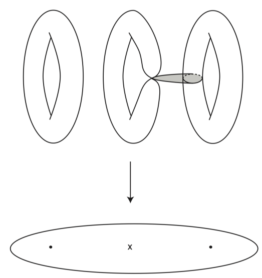

Figure 1: (taken from [ABC+09]) The shadowed disk represents the quantum correction.

main achievement of this paper is that, concerning the quantum correction, we demonstrate how the counts of Maslov index zero disks lead to a rigid analytic space structure on the set (1) and how the counts of Maslov index two disks can give a globally-defined superpotential function $W^\vee$.

The mirror construction of $(X^\vee, W^\vee)$ derives from Fukaya's family Floer theory [Fuk01]. Following Abouzaid [Abo17b], such a Floer-theoretic approach to the SYZ mirror reconstruction is now called the family Floer program. After Fukaya's initiative, Abouzaid [Abo17b, Abo17a, Abo14] and Tu [Tu14] have significant progress on the family Floer program. But, they more or less need to assume that the Lagrangian fibers do not bound holomorphic disks. In this paper, we allow the presence of holomorphic disks. We also discover that the the wall crossing phenomenon [Aur07] can be made precise by the natural A_∞ structures in Lagrangian Floer theory.

1.1 Main theorem

Let (X,ω) be a compact symplectic manifold of dimension $2n$. Sometimes we may also drop the compactness condition. Suppose there is a smooth Lagrangian fibration $\pi : X_0 \to B_0$ in an open domain X_0 of X over an n-dimensional base manifold B_0. By Arnold-Liouville theorem, the fiber $L_q := \pi^{-1}(q)$ over any $q \in B_0$ is a torus, and there exists a natural integral affine structure on B_0.

An example of π can be (the smooth part of) a Gross's fibration [Gro01]. (e.g. we take $X = \mathbb{CP}^2$; let X_0 be the complement of the anticanonical divisor $D = \{[x : y : z] \mid (xy - \epsilon z^2)z = 0\}$ minus the singular Gross's fiber, c.f. [Aur07, §5].) One may also take X to be an elliptic K3 surface, and the π will be the smooth part of an elliptic fibration. To find more examples, we note that, given an integral affine 3-manifold with certain properties, we can construct a symplectic 6-manifold with a Lagrangian 3-torus fibration on a large domain [CBM09]. A Lagrangian fibration in this class usually admits an anti-symplectic involution preserving the fibers [CBMS10]; one may also show the Lagrangian fibers have vanishing weak Maurer-Cartan equations [Sol20].

Definition 1.1. Let $\mathfrak{J}(X,\omega)$ be the space of ω-tame almost complex structures;

let μ be the Maslov index. Given a Lagrangian submanifold L, an almost complex structure J is called *L-semipositive* if $\mu(\beta) \geq 0$ for any $\beta \in \pi_2(X, L)$ that can be represented by a J-holomorphic stable map. Denote by $\mathfrak{J}(X, L, \omega)$ the subspace in $\mathfrak{J}(X, \omega)$ of all the L-semipositive ω-tame almost complex structures.

Assumption 1.2. For every compact subset $K \subset B_0$, the intersection

$$\mathfrak{J}_K := \bigcap_{q \in K} \mathfrak{J}(X, L_q, \omega)$$

has a non-empty interior in $\mathfrak{J}(X, \omega)$. In this case, we call the Lagrangian fibration π is **semipositive**. Without loss of generality, we may assume the base B_0 and $\mathfrak{J}_K$ are connected and open.

We justify the assumption in the following two aspects. On the one hand, we often know the non-emptiness. If $\dim X \leq 8$, a regular J is semipositive for dimension reasons. By [Aur07, Lemma 3.1], a special Lagrangian fibration (or more generally, if the phase functions lift to real-valued functions) is semipositive, at least when X is Calabi-Yau or Fano. On the other hand, the openness condition is also inessential. In fact, we may similarly define $\mathfrak{J}_K^{\leq E}$ for any $E > 0$ by requiring only the stable maps of energy $\leq E$ have non-negative Maslov indices. Then, the Gromov compactness exactly implies that every $\mathfrak{J}_K^{\leq E}$ is open. Clearly, the $\mathfrak{J}_K^{\leq E}$ decreases when K or E increases, and we have $\mathfrak{J}_K = \lim_{E \to \infty} \mathfrak{J}_K^{\leq E}$. So, for simplicity, we would content ourselves with the above assumption in this paper.

Now, we state the main theorem of this paper.

Theorem 1.3. *Given Assumption 1.2, we can naturally associate to the pair (X, π) a triple*

$$\mathbb{X}^\vee := (X^\vee, W^\vee, \pi^\vee)$$

consisting of

(a) a rigid analytic space $X^\vee$ over the Novikov field Λ

(b) a global rigid analytic function $W^\vee$

(c) a dual fibration $\pi^\vee : X^\vee \to B_0$

unique up to isomorphism such that for any $q \in B_0$, the dual fiber $(\pi^\vee)^{-1}(q)$ as a set is the weak Maurer-Cartan solution space of the A_∞ algebra associated to the Lagrangian fiber $L_q := \pi^{-1}(q)$.

Note that in most cases, the weak Maurer-Cartan equations vanish; then, a dual fiber $(\pi^\vee)^{-1}(q)$ is set-theoretically equal to $H^1(L_q; U_\Lambda) \cong U_\Lambda^n$ as mentioned in (1). We will call $X^\vee$, $W^\vee$, and $\pi^\vee$ the mirror space, the mirror (Landau-Ginzburg) superpotential, and the (SYZ) dual fibration map respectively.

The dual fibration $\pi^\vee$ is locally modeled on the following *non-archimedean torus fibration*:

$$\mathfrak{trop} : (\Lambda^*)^n \to \mathbb{R}^n \qquad (y_1, \dots, y_n) \mapsto (\operatorname{val} y_1, \dots, \operatorname{val} y_n) \tag{2}$$

where we set $\Lambda^* = \Lambda \setminus \{0\}$. It resembles the logarithm map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$. The center fiber at $0 \in \mathbb{R}^n$ is a non-archimedean torus $\mathfrak{trop}^{-1}(0) = U_\Lambda^n$, and any other fiber can be identified with it by a shifting map $y_i \mapsto T^{c_i} y_i$. Note that the total space $(\Lambda^*)^n$ is a rigid analytic space. Moreover, the base B_0 can be locally viewed as $\mathbb{R}^n$ up to $GL(n, \mathbb{Z})$-transformations.

The two fibrations $(X^\vee, \pi^\vee)$ and $((\Lambda^*)^n, \mathfrak{trop})$ are locally isomorphic to each other. But globally, a *transition map* ϕ (also called a *gluing map*) for the dual fibration $(X^\vee, \pi^\vee)$ takes the form of $y_i \mapsto T^{c_i} y_i \exp(F_i(y))$. The extra twisting formal power series $F_i(y)$ will be made explicit and precise; roughly, it is obtained by counting Maslov-zero holomorphic disks along a small Lagrangian isotopy among the fibers. By its geometric definition, the $F_i(y)$ vanishes if the Maslov-zero disks are absent, making the gluing map recurs to the usual shifting map $y_i \mapsto T^{c_i} y_i$ for $((\Lambda^*)^n, \mathfrak{trop})$.

The mirror construction actually contains parts of information about singular fibers (see Figure 1): although the torus fibers are always in X_0, the holomorphic disks sweep in X and usually meet $X \setminus X_0$. We expect the ultimate mirror space could be a compactification of $X^\vee$.

On the other hand, the global superpotential $W^\vee$ is exactly given by Fukaya-Oh-Ohta-Ono's $\mathfrak{m}_0$ obstruction to Floer theory as predicted in [Aur07]; the restriction of $W^\vee$ to a fiber is defined by the (one-pointed genus-zero) *open Gromov-Witten invariants*. The $W^\vee$ can have different local expressions yet, and the presence of Maslov-zero disks accounts for the dramatic change of $W^\vee$ across the 'wall', c.f. [CLL12, AAK16, Aur07]. Besides, we will accurately prove that the above gluing maps ϕ must match the various local expressions of $W^\vee$, for which the non-archimedean analysis is crucial.

1.2 Speculations and further directions

Application – Disk counting . By the above discussion, the open GW invariants in one chamber may somehow decide the ones in another chamber. In the recent preprint [Yua21], we achieve some concrete applications of this idea as follows. A Gross's fibration has only two chambers of Clifford and Chekanov tori. First, it is easy to find the superpotential over the Clifford chamber [CO06]; then, we can use the gluing maps (across the wall) to compute the superpotential over the Chekanov chamber. By this means, we can find all the non-trivial open GW invariants for a Chekanov-type Lagrangian torus in $\mathbb{CP}^n$ or $\mathbb{CP}^r \times \mathbb{CP}^{n-r}$. When $n = 2$ (and $r = 1$), we retrieve the previous results of Auroux and Chekanov-Schlenk [Aur07, 5.7 & 5.12] [CS10] without explicitly finding the disks. It also agrees with the recent work of Pascaleff-Tonkonog [PT20, Th 1.4].

Family Floer functor . In a series of papers [Abo14, Abo17b, Abo17a], Abouzaid uses the family Floer program to prove the 'homological mirror symmetry without corrections', assuming the A-side manifold X is equipped with a smooth Lagrangian torus fibration that bounds no holomorphic disks. We have successfully extended it by allowing the holomorphic disks. But, we currently have not worked out Abouzaid's family Floer functor which fully faithfully embeds the

Fukaya category of X into a derived category of twisted sheaves on the B-side mirror space. A work in progress would show that a Lagrangian section $\mathbb{L}$ of π in X gives rise to a 'matrix factorization' of $W^\vee$, and hopefully, we could still obtain a family Floer functor in some ways. Indeed, we can always *locally* obtain a matrix factorization by using A_∞ bimodules associated to $\mathbb{L}$. But, a *global* notion of matrix factorization on a rigid analytic space seems unavailable at present. We expect the correct notion should resemble the *twisted sheaves of perfect complexes* used by Abouzaid [Abo17b].

Semipositive v.s. special Lagrangian . Let X be a Kähler manifold of complex dimension n and D is an anticanonical divisor. Suppose $X \setminus D$ can be equipped with a holomorphic n-form Ω. For a Lagrangian torus $L \subset X \setminus D$, we have a *phase function* $\arg(\Omega|_L) : L \to U(1) \equiv S^1$. If $\arg(\Omega|_L)$ is constant, then L is said to be a *special Lagrangian*. The original SYZ conjecture explains mirror symmetry in terms of dual fibrations with special Lagrangian fibers. But, according to Joyce [Joy03] [Joy07], the strong version of SYZ conjecture that special Lagrangian fibrations exist globally might be false; even when special Lagrangian fibrations exist, they might be ill-behaved. By Theorem 1.3, we would like to propose a weaker condition: *semipositive Lagrangian fibration*. A sufficient condition is to require the phase functions for the fibers can lift to real-valued functions; so, any special Lagrangian fibration is semipositive. (See [Aur07, Lemma 3.1].)

We note that the recent work [Li19] obtains a special Lagrangian fibration on a generic region of a Calabi-Yau Fermat hypersurface near the large complex structure limit. The local structure of such a Calabi-Yau hypersurface is a large annulus region in $(\mathbb{C}^*)^n$, and the special Lagrangian fibration in [Li19] is a small C^∞-perturbation of the logarithm map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$. We expect that adopting the weaker semipositive condition may be helpful. We also note that the $\mathfrak{trop}$ is an analog of the Log.

Non-archimedean approach . Starting from Kontsevich-Soibelman [KS01, KS06], the non-archimedean approach to the mirror symmetry has been studied extensively. For instance, we may count curves from non-archimedean methods [Yu16,Yu21]; the evidence for the relevance of the non-archimedean geometry to the SYZ conjecture is also provided in [Li19,Li20]. Note also that we may define non-archimedean versions of differential (p,q)-forms [CLD12, Gub16], Kähler structures [KT02, Yu18], Monge-Ampère equations [Liu11, BFJ15], etc.

The Novikov field Λ is a non-archimedean field. From the standpoint of Floer theory, there are also some reasons to work over Λ as follows. First, the A_∞ algebras associated to the Lagrangians are in general only defined over the Novikov field. The energies of holomorphic disks may go to infinity. Thus, for the disk-counting, it may be more natural to use formal power series rather than polynomials. Moreover, the complex analytic geometry is insufficient for the convergence issue, but the celebrated Gromov compactness always ensures the convergence in the non-archimedean topology of Λ.

Algebro-geometric viewpoint . Much of what we do is also inspired by the philosophy of the Gross-Siebert program [GS11] (see also [GHK15]). For instance, the Gross-Siebert's work uses the ideas of tropical geometry and formal schemes in some sense. Similarly, our work also uses the non-archimedean tropical geometry [EKL06]; the rigid analytic spaces [Tat71] over Λ is closely related to the formal schemes over Λ_0 due to the celebrated Raynaud's theorem [Bos14, 8.4/3]. Nevertheless, the purposes and methods should be different, as we actually adopt a Floer-theoretic approach following the works of Abouzaid, Auroux, Fukaya, and Tu [Fuk01, Aur07, Abo14, Abo17b, Abo17a, Tu14].

The algebro-geometric approach of Gross-Siebert [GS11] and Gross-Hacking-Keel [GHK15] may be viewed as a tropicalization of the SYZ picture. It is definitely more powerful in dealing with singularities than the Floer-theoretic approach. So, it may offer inspirations and guidelines for further directions. It is also very interesting to explore the relations between the two approaches in the future. Indeed, we expect that the rays in the scattering diagram could correspond to loci of Lagrangian fibers bounding Maslov index zero disks; the slab/wall functions could correspond to the gluing maps, etc. A possible relation for elliptic K3 surfaces has been foreseen in [Fuk09], using the Lagrangian surgery.

1.3 Sketch of proof

The proof of Theorem 1.3 turns out to be technically complicated, but we can still cover most of the essential ideas for the proof here. We also hope the reader can find adequate motivations and intuitions.

1.3.1 Motivating ideas . Pick a Lagrangian fiber L. By counting holomorphic disks with boundary in L and by homological perturbation, we can obtain an A_∞ algebra $\mathfrak{m} = (\mathfrak{m}_{k,\beta})$, where the operator $\mathfrak{m}_{k,\beta} : H^*(L)^{\otimes k} \to H^*(L)$ is labeled by a class $\beta \in \pi_2(X, L)$. These operators satisfy an A_∞ relation with labels (§2.2). Let $E(\beta) = \omega \cap \beta$ be the energy and let $\mu(\beta)$ be the Maslov index. Now, we define the *Maurer-Cartan equation* of $\mathfrak{m}$ to be the following equation

$$\mathfrak{m}_*(b) := \sum_\beta \sum_k T^{E(\beta)} \mathfrak{m}_{k,\beta}(b, \dots, b) = 0 \tag{3}$$

in $b \in H^1(L)\hat{\otimes}\Lambda_0$. By degree reasons and by Assumption 1.2, a class β can contribute to (3) only if $\mu(\beta) = 0$ or 2. Given a basis $\{e_i\}$ of $H^1(L)$, we write $b = x_1 e_1 + \cdots + x_n e_n$ for $x_i \in \Lambda_0$. Following [FOOO10a, §4], the equation (3) is decomposed into a potential function in x_i:

$$\mathscr{W}^0(x_1, \dots, x_n) := \sum_{\mu(\beta)=2} \sum_k T^{E(\beta)} \mathfrak{m}_{k,\beta}(b, \dots, b) = 0 \tag{4}$$

together with the *weak Maurer-Cartan equation*:

$$\mathfrak{m}_{*w}(b) := \sum_{\mu(\beta)=0} \sum_k T^{E(\beta)} \mathfrak{m}_{k,\beta}(b, \dots, b) = 0 \tag{5}$$

A key property of $\mathfrak{m}$ we use is the **divisor axiom**[1] (§2.5) which implies that

$$\mathfrak{m}_{k,\beta}(b,\ldots,b) = \frac{(\partial\beta \cap b)^k}{k!}\ \mathfrak{m}_{0,\beta}\ \in H^{2-\mu(\beta)}(L)\hat{\otimes}\Lambda \tag{6}$$

Therefore, the above series $\mathscr{W}^0$ in x_i can be transformed into a new series in $y_i = e^{x_i}$, denoted by:

$$\mathscr{W}(y_1,\ldots,y_n) := \sum_{\mu(\beta)=2} T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta} = \sum_{\mu(\beta)=2} T^{E(\beta)} y_1^{\partial_1\beta}\cdots y_n^{\partial_n\beta}\mathfrak{m}_{0,\beta} \tag{7}$$

We will only consider $\mathscr{W}$ without recalling $\mathscr{W}^0$. Actually, there is no information loss, and we will even gain more in the end. To see this, we need some *non-archimedean analysis*. Firstly, we note that a formal power series f in y_i is identically zero if and only if $f(y_1,\ldots,y_n)=0$ whenever $y_i \in U_\Lambda$ (Lemma 2.3). Secondly, the Novikov field Λ has the property that every element y in U_Λ can be written in the form $y = e^x$ for some $x \in \Lambda_0$ (Lemma 2.2). Thus, using the divisor axiom of $\mathfrak{m}$ yields $\mathscr{W}^0(x_1,\ldots,x_n) = \mathscr{W}(e^{x_1},\ldots,e^{x_n})$, and we recover $\mathscr{W}^0$ from $\mathscr{W}$. One can also do so for (5).

1.3.2 Non-archimedean torus fibration . In analogy to the field $\mathbb{C}((T))$, the Novikov field $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$ admits a non-archimedean valuation map val : $\Lambda \to \mathbb{R}\cup\{\infty\}$. It is equivalent to a norm defined by $|a| := \exp(-\operatorname{val}(a))$ and defines the *adic-topology* on Λ. Now, we consider

$$\mathfrak{trop} = \operatorname{val}^n : (\Lambda^*)^n \to \mathbb{R}^n \qquad (y_1,\ldots,y_n) \mapsto (\operatorname{val} y_1,\ldots,\operatorname{val} y_n)$$

It resembles the logarithm map $\operatorname{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$. The preimage $\mathfrak{trop}^{-1}(\Delta)$ of a rational polyhedron $\Delta \subset \mathbb{R}^n$ is an affinoid space[2] and is called a *polytopal domain.* Its algebra of functions is the *polyhedral affinoid algebra* $\Lambda\langle\Delta\rangle \subset \Lambda[[Y_1^\pm,\ldots,Y_n^\pm]]$ consisting of all formal Laurent power series $\sum_{\nu\in\mathbb{Z}^n} a_\nu Y^\nu$ so that $\operatorname{val}(a_\nu) + \nu\cdot u \to \infty$ as $|\nu| \to \infty$ for any $u \in \Delta$. Alternatively, $f \in \Lambda\langle\Delta\rangle$ if and only if $f(\mathbf{y})$ converges at any point $\mathbf{y}$ in $\mathfrak{trop}^{-1}(\Delta)$ for the adic-topology in Λ. The set of all maximal ideals, denoted by $\operatorname{Sp}\Lambda\langle\Delta\rangle$, are in bijection with the points in $\mathfrak{trop}^{-1}(\Delta)$. See Appendix A.

By Arnold-Liouville theorem, the base B_0 admits an integral affine structure. So, it locally looks like $\mathbb{R}^n$ up to the action of $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$. Thus, if $\Delta \subset B_0$ is a rational polyhedron in some integral affine chart, it is a one in any other integral affine chart. Suppose Δ is contained in some integral affine chart φ_q : $(U,q) \to (\mathbb{R}^n, 0)$ centered at $q \in B_0$. Then, $\varphi_q(\Delta)$ is a rational polyhedron in $\mathbb{R}^n$. It also induces an isomorphism between

$$\Lambda[[\pi_1(L_q)]] = \left\{ \sum_{i=0}^{\infty} s_i Y^{\alpha_i} \mid s_i \in \Lambda, \alpha_i \in \pi_1(L_q) \right\}$$

[1] An analog in Lagrangian Floer theory of the divisor axiom in Gromov-Witten theory; see [Fuk10, Aur07].

[2] An affinoid space is roughly a rigid-analytic analog of an affine scheme.

and $\Lambda[[Y_1^{\pm},\dots,Y_n^{\pm}]]$; then, we define $\Lambda\langle\Delta,q\rangle$ to be the algebra in $\Lambda[[\pi_1(L_q)]]$ that is identified with $\Lambda\langle\varphi_q(\Delta)\rangle\subset\Lambda[[Y_1^{\pm},\dots,Y_n^{\pm}]]$. The induced map $\mathfrak{trop}_q$: $\mathrm{Sp}\,\Lambda\langle\Delta,q\rangle\to\Delta$ is also identified with the restriction of $\mathfrak{trop}$ over $\varphi_q(\Delta)\subset\mathbb{R}^n$. But for simplicity, if there is no confusion, we often abuse the notations and write $\Delta=\varphi_q(\Delta)$, $\mathfrak{trop}=\mathfrak{trop}_q$, etc.

1.3.3 Mirror local charts . Fix a sufficiently fine rational polyhedron covering $\{\Delta_i\mid i\in\mathfrak{I}\}$ of B_0. Pick points $q_i\in\Delta_i$, and take the Lagrangian fibers $L_i:=L_{q_i}=\pi^{-1}(q)$. Given i, the moduli spaces of holomorphic disks can produce an A_∞ algebra $\check{\mathfrak{m}}:=\check{\mathfrak{m}}^{J,L_i}$ on the de Rham cochain $\Omega^*(L_i)$. To obtain a minimal[3] A_∞ algebra on the de Rham cohomology group $H^*(L_i)$, we need to choose a *contraction* to perform the *homological perturbation* (§4). We always stick to the specific *g-harmonic contraction* $\mathsf{con}(g)=(i(g),\pi(g),G(g))$ (§7) for a metric g. It is basically the data of a Hodge decomposition:

$$\mathsf{con}(g):\qquad \begin{tikzcd} H^*(L) \arrow[r, shift left, "i(g)"] & \Omega^*(L) \arrow[l, shift left, "\pi(g)"] \arrow[loop above, "G(g)"] \end{tikzcd} \tag{8}$$

Denote by $\mathfrak{m}:=\mathfrak{m}^{g,J,L_i}$ the minimal A_∞ algebra on $H^*(L_i)$ obtained by applying the homological perturbation to $\check{\mathfrak{m}}$ (Figure 2). Following [FOOO10b], we call $\mathfrak{m}$ a *canonical model* of $\check{\mathfrak{m}}$ (with respect to g). We will only use $\mathfrak{m}$ regardless of $\check{\mathfrak{m}}$. Now, we consider the *Maurer-Cartan formal power series*:

$$P^i=\sum_\beta T^{E(\beta)}Y^{\partial\beta}\mathfrak{m}_{0,\beta}\qquad\in\Lambda[[\pi_1(L_i)]]\hat{\otimes}H^*(L_i) \tag{9}$$

Since Δ_i is small, the P^i converges on the polytopal domain $\mathfrak{trop}^{-1}(\Delta_i)$ due to the *reverse isoperimetric inequalities* [GS14]. Thus, all the components of P^i live in $\Lambda\langle\Delta_i,q_i\rangle$. As explained before, restricting the P^i to $U^n_\Lambda\equiv\mathfrak{trop}^{-1}(0)\cong\mathfrak{trop}_{q_i}^{-1}(q_i)$ retrieves the MC equation of $\mathfrak{m}$. Besides, it can also somehow recover the MC equation of the A_∞ algebra associated to any adjacent Lagrangian fiber over Δ_i.

Let $\mathbb{1}$ be the constant-one function. Just like how we decompose (3) into (4) and (5), we write

$$P^i=W^i\cdot\mathbb{1}+Q^i$$

where W^i and Q^i consist of all the terms with $\mu(\beta)=2$ and $\mu(\beta)=0$ respectively. Let $\mathfrak{a}_i$ be the ideal generated by the components of Q^i, and we call it *the ideal of weak MC equations.* Define:

$$X_i:=V(\mathfrak{a}_i)=\mathrm{Sp}\left(\Lambda\langle\Delta_i,q_i\rangle/\mathfrak{a}_i\right) \tag{10}$$

We claim that the above X_i will give a **local chart** of the mirror space $X^\vee$. We also remark that in most examples, the ideal $\mathfrak{a}_i$ is just zero; then, the local

[3]Here 'minimal' means the energy-zero part of $\mathfrak{m}_1=\sum T^{E(\beta)}\mathfrak{m}_{1,\beta}$ is zero, i.e. $\mathfrak{m}_{1,0}=0$, rather than $\mathfrak{m}_1=0$.

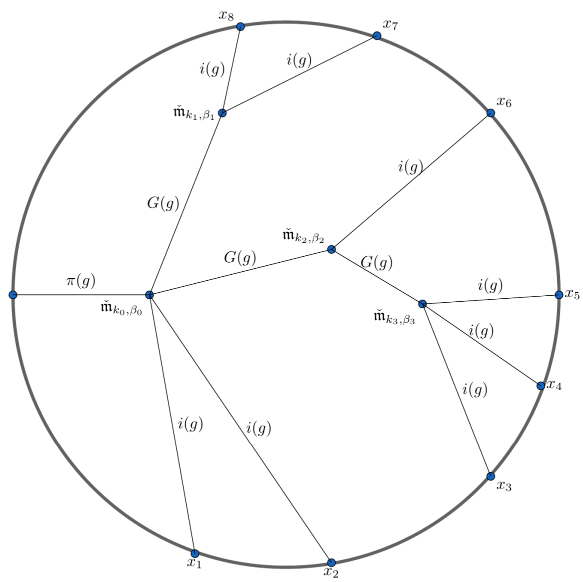

Figure 2: Homological perturbation: from $\check{\mathfrak{m}} = \check{\mathfrak{m}}^{J,L}$ and $\mathsf{con}(g)$, the canonical model $\mathfrak{m} = \mathfrak{m}^{g,J.L}$ is obtained by a Feynman-diagram-like tree summation, e.g. the above contributes to $\mathfrak{m}_{8,\beta_0+\beta_1+\beta_2+\beta_3}$.

chart X_i is just a polytope domain $\mathfrak{trop}^{-1}(\Delta_i) \equiv \operatorname{Sp}\Lambda\langle\Delta_i, q_i\rangle$. On the other hand, the W^i will be a local piece of the mirror superpotential $W^\vee$; the dual fibration $\pi^\vee$ will be locally identified with the $\mathfrak{trop} = \mathfrak{trop}_{q_i}$ on X_i.

Next, we need to construct the *transition maps* (also called the *gluing maps*) among these local charts. To define a transition map, the geometric input is a small Lagrangian isotopy among the fibers; it is equivalent to a small movement of almost complex structure by *Fukaya's trick* (§8). The algebraic output is a *pseudo-isotopy* or an A_∞ *homotopy equivalence* which will define the transition map.

1.3.4 Fukaya's trick . Let L and $\tilde{L}$ be two adjacent Lagrangian fibers over $q, \tilde{q} \in B_0$. Choose a small isotopy F so that $F(L) = \tilde{L}$. Denote by $\mathcal{M}(J, L)$ and $\mathcal{M}(J, \tilde{L})$ the moduli spaces of J-holomorphic disks bounding L and $\tilde{L}$. Recall that they determine the chain-level A_∞ algebras

$$\check{\mathfrak{m}} := \check{\mathfrak{m}}^{J,L} \quad \text{and} \quad \check{\mathfrak{m}}^{\sim} := \check{\mathfrak{m}}^{J,\tilde{L}}$$

on $\Omega^*(L)$ and $\Omega^*(\tilde{L})$ respectively. In the cohomology-level, recall that we can apply the homological perturbation (for a metric g) to obtain two minimal A_∞ algebras

$$\mathfrak{m} := \mathfrak{m}^{g,J,L} \quad \text{and} \quad \mathfrak{m}^{\sim} := \mathfrak{m}^{g,J,\tilde{L}} \tag{11}$$

on $H^*(L)$ and $H^*(\tilde{L})$ respectively. They define two local charts of the mirror space as in (10).

We first have Fukaya's trick in the chain-level. A map $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$ is J-holomorphic if and only if $F \circ u : (\mathbb{D}, \partial\mathbb{D}) \to (X, \tilde{L})$ is F_*J-holomorphic. So, for $F_*J := dF \circ J \circ dF^{-1}$, there is a natural identification $\mathcal{M}(F_*J, \tilde{L}) \cong \mathcal{M}(J, L)$; the left side gives a new A_∞ algebra, denoted by:

$$\check{\mathfrak{m}}^F := \check{\mathfrak{m}}^{F_*J, \tilde{L}}$$

It can be regarded as a pushforward of $\check{\mathfrak{m}}^{J,L}$ due to the following chain-level equation of Fukaya's trick:

$$\check{\mathfrak{m}}^F \equiv (F^{-1})^* \circ \check{\mathfrak{m}} \circ F^* \tag{12}$$

Concretely, this means $\check{\mathfrak{m}}^F_{k,\beta}(x_1, \dots, x_k) = (F^{-1})^* \check{\mathfrak{m}}_{k,\beta}(F^*x_1, \dots, F^*x_k)$ for any $x_1, \dots, x_k \in \Omega^*(\tilde{L})$. Defined by the same counting, the two A_∞ algebras $\check{\mathfrak{m}}^F$ and $\check{\mathfrak{m}}$ are actually almost identical, except the energy is varied by the following formula $E(F_*\beta) = E(\beta) + \langle \partial\beta, \tilde{q} - q \rangle$ for $\beta = [u] \in \pi_2(X, L)$. The purpose of using $\check{\mathfrak{m}}^F$ in place of $\check{\mathfrak{m}}$ is just to unify the underlying Lagrangians.

Next, we take a path $\mathbf{J} = (J_s)$ from F_*J to J. Then, the *parameterized moduli space*

$$\bigsqcup_s \mathcal{M}(J_s, \tilde{L})$$

gives rise to the so-called *pseudo-isotopy* $\check{\mathfrak{M}}$ (see [Fuk10] or §2.3). It is roughly a family of A_∞ algebras, moving from $\check{\mathfrak{m}}^F$ to $\check{\mathfrak{m}}^{\sim}$, but it contains extra information about 'derivatives'. By (12), the pseudo-isotopy $\check{\mathfrak{M}}$ actually relate $\check{\mathfrak{m}}$ to $\check{\mathfrak{m}}^{\sim}$.

Further, we want a relationship between the two cohomology-level A_∞ algebras $\mathfrak{m}$ and $\mathfrak{m}^{\sim}$. Consider the two distinct harmonic contractions $\mathsf{con}(g)$ and $\mathsf{con}(F_*g)$ in the meantime. By homological perturbation, we previously use $\mathsf{con}(g)$ and $\check{\mathfrak{m}}$ (resp. $\check{\mathfrak{m}}^{\sim}$) to obtain the canonical model $\mathfrak{m}$ (resp. $\mathfrak{m}^{\sim}$). Now, we can similarly use $\mathsf{con}(F_*g)$ and $\check{\mathfrak{m}}^F$ to obtain another canonical model, denoted by:

$$\mathfrak{m}^F := \mathfrak{m}^{F_*(g,J), \tilde{L}}$$

Moreover, it also satisfies the following Fukaya's trick equation

$$\mathfrak{m}^F \equiv (F^{-1})^* \circ \mathfrak{m} \circ F^* \tag{13}$$

but in the cohomology-level this time. This holds just because of the various F-relatedness conditions, including both (12) and the fact that $\mathsf{con}(F_*g) = F^{-1*} \circ \mathsf{con}(g) \circ F^*$ (put them together in Figure 2 to get some ideas). Unlike the previous relation (12), only the homotopy class of F matters in (13). This flexibility will become crucial for the well-definedness of transition maps and the cocycle conditions.

Finally, we aim to construct from the previous $\check{\mathfrak{M}}$ a *cohomology-level pseudo-isotopy* $\mathfrak{M}$ between $\mathfrak{m}^F$ and $\mathfrak{m}^{\sim}$. To construct it, we carry out the homological perturbation again, but we need to further find a path of metrics $\mathbf{g} = (g_s)$

between $F_* g$ and g. The path always exists since the metric is also a contractible choice. Then, we can use a family version of harmonic contraction (§7.2) to obtain the pseudo-isotopy $\mathfrak{M}$ (§7.3). Intuitively, the roles of the data $\mathbf{J}$ and $\mathbf{g}$ can be described as follows:

$$\boxed{\mathcal{M}(J,L)\cong\mathcal{M}(F_*J,\tilde{L})\xrightarrow{\mathbf{J}}\mathcal{M}(J,\tilde{L})} \rightsquigarrow \boxed{\check{\mathfrak{m}}\approx\check{\mathfrak{m}}^F\xleftarrow{\check{\mathfrak{M}}}\check{\mathfrak{m}}^{\sim}} \overset{\mathbf{g}}{\rightsquigarrow} \boxed{\mathfrak{m}\approx\mathfrak{m}^F\xleftarrow{\mathfrak{M}}\mathfrak{m}^{\sim}}$$

1.3.5 Mirror transition maps . Recall that the two local charts (10) associated to L and $\tilde{L}$ are defined by the A_∞ algebras $\mathfrak{m}=\mathfrak{m}^{g,J,L}$ and $\mathfrak{m}^{\sim}=\mathfrak{m}^{g,J,\tilde{L}}$ (11). In addition, from the pseudo-isotopy $\mathfrak{M}$, we can naturally construct an A_∞ *homotopy equivalence* (§5):

$$\mathfrak{C}^F:\mathfrak{m}^{\sim}\to\mathfrak{m}^F \tag{14}$$

It consists of a collection of operators $\mathfrak{C}^F_{k,\beta}:H^*(\tilde{L})^{\otimes k}\to H^*(\tilde{L})$ for $k\ge 0$, $\beta\in\pi_2(X,\tilde{L})$ satisfying the obvious A_∞ equations; by definition, we have $\mathfrak{C}^F_{0,0}=0$ and $\mathfrak{C}^F_{1,0}=\mathrm{id}$. As the category of affinoid spaces is equivalent to the opposite category of affinoid algebras (Grothendieck's principle), we first consider an affinoid algebra homomorphism: (c.f. [Tu14] and [Aur07])

$$\phi^F:\Lambda\langle\Delta,q\rangle\to\Lambda\langle\tilde{\Delta},\tilde{q}\rangle \qquad Y^\alpha\mapsto T^{\langle\alpha,\tilde{q}-q\rangle}Y^{\tilde{\alpha}}\exp\Big\langle\tilde{\alpha},\sum_\beta T^{E(\beta)}\mathfrak{C}^F_{0,\beta}Y^{\partial\beta}\Big\rangle \tag{15}$$

where $\alpha\in\pi_1(L)\cong\mathbb{Z}^n$, $\tilde{\alpha}:=F_*\alpha\in\pi_1(\tilde{L})\cong\mathbb{Z}^n$, and $\langle\cdot,\cdot\rangle$ is the natural pairing $\pi_1(L)\otimes H^1(L)\to\mathbb{R}$. It should be more careful to substitute $\Delta\cap\tilde{\Delta}$ for $\Delta,\tilde{\Delta}$ in (15), but we suppress it for simplicity. Heuristically, the coefficient $T^{\langle\alpha,\tilde{q}-q\rangle}$ corresponds to the energy change in Fukaya's trick, and the exponential part of ϕ^F in (15) accounts for the quantum correction. Note that $\mathfrak{C}^F_{0,\beta}\in H^{1-\mu(\beta)}(\tilde{L})$, so by the semipositive assumption, the wall crossing phenomenon is only contributed by the Maslov-zero disks, which agrees with Auroux's geometric observation [Aur07, §3.2-3.3].

On the other side, the reader may worry that the algebra homomorphism ϕ^F seems depend on the various choices, but fortunately this does not matter for the following two points (explained soon later):

- **(A)** What we need is actually a quotient morphism $\varphi:\Lambda\langle\Delta,q\rangle/\mathfrak{a}\to\Lambda\langle\tilde{\Delta},\tilde{q}\rangle/\tilde{\mathfrak{a}}$ with $\varphi(W)=\tilde{W}$.
- **(B)** The φ depends only on the *ud-homotopy class* of $\mathfrak{C}^F$, so finally it does *not* depend on the choices.

Here we define $P,W,Q,\mathfrak{a}$ and $\tilde{P},\tilde{W},\tilde{Q},\tilde{\mathfrak{a}}$ for the A_∞ algebras $\mathfrak{m}$ and $\mathfrak{m}^{\sim}$ as before (§1.3.3). The **ud-homotopy** refers to an amended homotopy theory of A_∞ algebras with extra properties; the prefix 'ud' means 'unitality with divisor axiom'. We will define a category $\mathscr{U}\mathscr{D}$ of these A_∞ algebras to systematically handle them. There are some heavy homological algebras, notably §3.

Suppose $L = L_k$ and $\tilde{L} = L_j$, and we define the local charts as in (10). Then, the **transition map**

$$\psi_{jk} : X_j \to X_k$$

(also called the **gluing map**) is defined to be the induced map φ^* on the maximal ideal spectrums for the quotient morphism $\varphi = [\phi^F]$. In other words, we set $\psi_{jk} = \varphi^* : \mathrm{Sp}(\Lambda\langle\tilde{\Delta}, \tilde{q}\rangle/\tilde{\mathfrak{a}}) \to \mathrm{Sp}(\Lambda\langle\Delta, q\rangle/\mathfrak{a})$. Abusing the terminologies, the homomorphism ϕ^F or its quotient φ is also called a transition map or a gluing map if there is no confusion.

The point (A) will make the transition map well-defined; the point (B) will eliminate the ambiguity caused by various choices and will finally account for the cocycle conditions. The main techniques for the proofs are the wall crossing formula (§1.3.6) and the ud-homotopy theory (§1.3.7).

1.3.6 Wall crossing formula . To obtain (A), it comes to the following significant *wall crossing formula* (Theorem 9.7): for any $\eta \in H_*(L)$, there exists a collection of formal power series R^η such that

$$\phi^F(\langle \eta, P\rangle) = \langle F_*\eta, \mathbb{1}\rangle \cdot \tilde{W} + \sum R^\eta \cdot \tilde{Q} \tag{16}$$

where $\langle\cdot,\cdot\rangle$ is the natural pairing and $\mathbb{1}$ denotes the constant-one function.

(i) If η is dual to $\mathbb{1}$, then $\langle F_*\eta, \mathbb{1}\rangle = 1$ and $\langle \eta, P\rangle = W$. So, $\phi^F(W)$ equals to $\tilde{W}$ modulo $\tilde{\mathfrak{a}}$.

(ii) If η runs over $H_{n-2}(L)$, then we see $\langle F_*\eta, \mathbb{1}\rangle = 0$, and the $\langle \eta, P\rangle$ runs over all components of Q, i.e. the generators of the ideal $\mathfrak{a}$. As $\sum R^\eta \cdot \tilde{Q} \in \tilde{\mathfrak{a}}$, we see $\phi^F(\mathfrak{a}) \subset \tilde{\mathfrak{a}}$.

The above two items (i) (ii) immediately achieve the point (A). Therefore, it suffices to show (16). We give a very rough proof below to convey the essential ideas. First, we fix coordinates and bases for which the wall crossing formula (16) becomes an equation in $\Lambda[[Y_1^\pm, \dots, Y_n^\pm]]$. As mentioned before, we only need to check the equation at any $\mathbf{y} = (y_1, \dots, y_n) \in U_\Lambda^n$ (Lemma 2.3); for every $1 \le i \le n$, we can find $x_i \in \Lambda_0$ so that $y_i = e^{x_i}$ (Lemma 2.2). Next, we write $b := (x_1, \dots, x_n)$ in $H^1(L)\hat{\otimes}\Lambda_0 \cong \Lambda_0^n \cong H^1(\tilde{L})\hat{\otimes}\Lambda_0$. Remark that one may roughly view it as a (weak) bounding cochain. Recall that $\langle \eta, P\rangle = \sum_\beta \langle \eta, \mathfrak{m}_{0,\beta}\rangle T^{E(\beta)} Y^{\partial\beta}$, and using (15) obtains that

$$\phi^F(\langle \eta, P\rangle)|_{Y=\mathbf{y}} = \sum_\beta \langle \eta, \mathfrak{m}_{0,\beta}\rangle T^{E(\beta)} T^{\langle \partial\beta, \tilde{q}-q\rangle} \mathbf{y}^{\partial\tilde{\beta}} \exp\langle \partial\tilde{\beta}, \sum_{\gamma\neq 0} T^{E(\gamma)} \mathfrak{C}^F_{0,\gamma} \mathbf{y}^{\partial\gamma}\rangle$$

where we set $\tilde{\beta} := F_*\beta \in \pi_2(X, \tilde{L})$. First, we have $\langle \eta, \mathfrak{m}_{0,\beta}\rangle = \langle F_*\eta, \mathfrak{m}^F_{0,\tilde{\beta}}\rangle$ and $T^{E(\beta)} T^{\langle \partial\beta, \tilde{q}-q\rangle} = T^{E(\tilde{\beta})}$ due to Fukaya's trick. Moreover, using the divisor axiom of $\mathfrak{C}^F$ yields that (c.f. (6))

$$\mathfrak{C}^F_{0,\gamma} \mathbf{y}^{\partial\gamma} = \mathfrak{C}^F_{0,\gamma} e^{\langle \partial\gamma, b\rangle} = \sum_k \mathfrak{C}^F_{k,\gamma}(b, \dots, b)$$

for any fixed γ (we allow $\gamma = 0$ here, but $\mathfrak{C}^F_{0,0} = 0$). Note also that $\mathfrak{C}^F_{1,0} = \mathrm{id}$ and so $\mathbf{y}^{\partial\tilde\beta} = e^{\partial\tilde\beta\cap b} = \exp\langle\partial\tilde\beta, \mathfrak{C}^F_{1,0}(b)\rangle$. Putting things together, we see that

$$\phi^F(\langle\eta, P\rangle)|_{Y=\mathbf{y}} = \sum\langle F_*\eta, \mathfrak{m}^F_{0,\tilde\beta}\rangle T^{E(\tilde\beta)} \exp\langle\partial\tilde\beta, b'\rangle$$

where

$$b' := \sum_\gamma\sum_k T^{E(\gamma)}\mathfrak{C}^F_{k,\gamma}(b,\dots,b)$$

As $\deg\mathfrak{C}^F_{k,\gamma} = 1 - k - \mu(\gamma)$, the non-negativity of μ (Assumption 1.2) exactly implies that $\deg b' = 1$. Thus, we can apply the divisor axiom of $\mathfrak{m}^F$ to the new b' as well. In consequence, we deduce that

$$\phi^F(\langle\eta, P\rangle)|_{Y=\mathbf{y}} = \big\langle F_*\eta, \sum\mathfrak{m}^F\big(\mathfrak{C}^F(b\dots b),\dots,\mathfrak{C}^F(b,\dots,b)\big)\big\rangle$$

Surprisingly, the sum in the bracket is just one half of the A_∞ equation for $\mathfrak{C}^F : \tilde{\mathfrak{m}} \to \mathfrak{m}^F$. Therefore,

$$\phi^F(\langle\eta, P\rangle)|_{Y=\mathbf{y}} = \langle F_*\eta, \sum\mathfrak{C}^F(b,\dots,b,\tilde{\mathfrak{m}}(b,\dots,b),b,\dots,b)\rangle$$

Eventually, applying the divisor axiom again (but in the converse direction pulling b out) together with the unitality conditions, we can prove the wall crossing formula (16).

1.3.7 Homotopy theory of A_∞ algebras . To see the point (B), we need more efforts. In a word, its proof exactly reflects the philosophy in [FOOO10b, §4.3.2]: an A_∞ homotopy equivalence induces a bijection on the set of *gauge equivalence classes* of bounding cochains. But, this bijection is only set-theoretical, and we need to use this philosophy in a more non-archimedean way as follows.

For a different choice F', we can define $\mathfrak{C}^{F'}$ and $\phi^{F'}$ similarly. Comparing the formulas (15) of the two homomorphisms ϕ^F and $\phi^{F'}$, we see that

$$\phi^{F'}(Y^\alpha) = \phi^F(Y^\alpha)\cdot\exp\Big\langle\alpha, \sum T^{E(\beta)}\ (\mathfrak{C}^{F'}_{0,\beta} - \mathfrak{C}^F_{0,\beta})\ Y^{\partial\beta}\Big\rangle$$

for any $\alpha \in \pi_1(L)$. So, all that matters is the difference in the exponent which we call the **error term**:

$$S := \langle\alpha, \sum T^{E(\beta)}\ (\mathfrak{C}^{F'}_{0,\beta} - \mathfrak{C}^F_{0,\beta})\ Y^{\partial\beta}\rangle$$

In order to prove $\varphi = [\phi^F] = [\phi^{F'}]$, it suffices to show that $S \in \tilde{\mathfrak{a}}$. We explain the proof as follows: By Lemma 2.3 again, one can only focus on the restriction $S|_{U^n_\Lambda}$ as in §1.3.6. With some efforts, we can first show that $\mathfrak{C}^{F'}$ is ud-homotopic to $\mathfrak{C}^F$. Then, we perform the following general argument: If $\mathfrak{f}^0$ and $\mathfrak{f}^1$ are two ud-homotopic A_∞ homomorphisms, say, from $\mathfrak{m}'$ to $\mathfrak{m}$, then the ud-homotopy condition means the existence of operator systems $(\mathfrak{f}^s)_{0\le s\le 1}$ and

$(\mathfrak{h}^s)_{0\le s\le 1}$ satisfying a list of conditions (Corollary 2.40). These conditions will imply that for any degree-one input b, we have

$$\sum T^{E(\beta)}\big(\mathfrak{f}^1_{k,\beta}(b,\dots,b)-\mathfrak{f}^0_{k,\beta}(b,\dots,b)\big)=\sum T^{E(\beta)}\int_0^1 ds\cdot\mathfrak{h}^s(b,\dots,b,\mathfrak{m}'(b,\dots,b),b,\dots,b)$$
$$+\sum\int_0^1 ds\cdot\mathfrak{m}\big(\mathfrak{f}^s(b,\dots,b),\dots,\mathfrak{f}^s(b,\dots,b),(\mathfrak{h}^s)_{\ell,\beta_0}(b,\dots,b),\mathfrak{f}^s(b,\dots,b),\dots,\mathfrak{f}^s(b,\dots,b)\big)$$

On the left side, if $\mathfrak{f}^0=\mathfrak{C}^F$ and $\mathfrak{f}^1=\mathfrak{C}^{F'}$, applying the divisor axiom returns to the error term S. On the right side, by unitality, the first sum vanishes modulo the weak Maurer-Cartan equations (i.e. modulo the ideal $\tilde{\mathfrak{a}}$). Besides, due to Assumption 1.2, $\mu(\beta_0)\ge 0$. By degree reason, if $(\mathfrak{h}^s)_{\ell,\beta_0}(b\cdots b)\ne 0$, it must be degree-zero; by unitality again, we can eliminate the second sum. To sum up, we have $S\in\tilde{\mathfrak{a}}$.

Additionally, the *cocycle conditions* among the transition maps can be proved in almost the same way. Roughly, we can show the ud-homotopy between some $\mathfrak{C}_{ik}$ and some composition $\mathfrak{C}_{jk}\circ\mathfrak{C}_{ij}$, where we can also apply Fukaya's trick to unify the underlying Lagrangian submanifolds. There will be a similar error term, and we can apply the above argument once again with $\mathfrak{f}^0=\mathfrak{C}_{ik}$ and $\mathfrak{f}^1=\mathfrak{C}_{jk}\circ\mathfrak{C}_{ij}$.

Structure of paper. In §2, we mainly define a category $\mathscr{U}\mathscr{D}$ of certain A_∞ algebras in order to handle the geometric inputs in an algebraically consistent way. Further, we slightly generalize some well-known homological algebras in §3 §4 §5 in the setting of $\mathscr{U}\mathscr{D}$. In §6, we geometrically review the A_∞ algebras associated to Lagrangians and explain how the divisor axiom and unitalities naturally arise by using forgetful maps on moduli spaces. In §7 and §8 we study the harmonic contractions and Fukaya's trick respectively based on $\mathscr{U}\mathscr{D}$ as well. In the last §9 we put the various pieces together to complete the non-archimedean SYZ mirror construction of $(X^\vee,W^\vee,\pi^\vee)$.

Acknowledgment. This work is done during my PhD program at Stony Brook University. My foremost thanks are to my advisor, Kenji Fukaya, for his encouragement, guidance and support along the way. I would like to thank my college teacher Jun Sun for useful lessons of differential geometry. I also benefit from conversations with Mohammed Abouzaid, Mohamed El Alami, Catherine Cannizzo, Xujia Chen, Andrew Hanlon, Enrica Mazzon, Mark McLean, Kevin Sackel, Yuhan Sun, Yi Wang, and Aleksey Zinger. I am grateful to Tim Campion, Jérôme Poineau, Xavier Xarles, and some anonymous people for discussions via *mathoverflow.net* from which I learn a lot about non-archimedean geometry; moreover, I am especially grateful to Will Sawin for his answer there which inspires the elegant proof of Lemma 2.3. Finally, I want to thank the organizers of *Western Hemisphere Virtual Symplectic Seminar* for giving me a change to have a video talk in April 2020.

2 Homological algebras

2.1 Novikov field

The Floer theory usually involves the **Novikov field** that is defined as follows:

$$\Lambda = \mathbb{C}((T^{\mathbb{R}})) = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i \to +\infty \right\} \tag{17}$$

The Novikov field has a non-archimedean *valuation map* $\mathrm{val} : \Lambda \to \mathbb{R} \cup \{\infty\}$ defined by sending a nonzero series $\sum_{i=0}^{\infty} a_i T^{\lambda_i}$ to the smallest λ_i with $a_i \neq 0$ and sending the zero series to ∞. The valuation ring $\Lambda_0 := \mathrm{val}^{-1}[0, \infty]$ is called the *Novikov ring*. It has a unique maximal ideal $\Lambda_+ := \mathrm{val}^{-1}(0, \infty]$. Note that the val is equivalent to a (non-archimedean) norm $|y| = \exp(-\mathrm{val}(y))$. In particular, we have the so-called *adic topology* on Λ_0 or Λ, and so it makes sense to talk about convergence. By *energy filtration* we mean the filtration defined by setting $F^a\Lambda = \mathrm{val}^{-1}[a, \infty]$ for various a. The *residual field* Λ_0/Λ_+ coincides with $\mathbb{C}$, and we have $\Lambda_0 \equiv \mathbb{C} \oplus \Lambda_+$.

The *multiplicative group* $U_\Lambda = \{y \in \Lambda \mid \mathrm{val}(y) = 0\} \equiv \{y \in \Lambda \mid |y| = 1\}$ resembles the $S^1 \equiv U(1)$ and satisfies that $U_\Lambda \equiv \mathbb{C}^* \oplus \Lambda_+$. Define $\Lambda^* = \Lambda \setminus \{0\}$, and we call the following map

$$\mathfrak{trop} : (\Lambda^*)^n \to \mathbb{R}^n, \qquad (y_1, \dots, y_n) \mapsto \big(\mathrm{val}(y_1), \dots, \mathrm{val}(y_n)\big)$$

a **non-archimedean torus fibration**. In reality, the center fiber over 0 is given by $\mathfrak{trop}^{-1}(0) \equiv U_\Lambda^n$ which resembles the real n-torus $U(1)^n = T^n$. Besides, any two fibers of $\mathfrak{trop}$ are related by a shifting map in the form $y_i \mapsto T^{c_i} y_i$. Note that the total space $(\Lambda^*)^n$ is a rigid analytic space locally covered by the various polytopal domains $\mathfrak{trop}^{-1}(\Delta)$ for rational polyhedrons $\Delta \in \mathbb{R}^n$ (see Appendix A).

An important property of Λ is that we can similarly define the *exponential* and *logarithm* functions.

Definition 2.1. Given $x \in \Lambda_0$, there is a unique decomposition $x = x_0 + x_+$ with $x_0 \in \mathbb{C}$ and $x_+ \in \Lambda_+$. Given $y \in U_\Lambda$, there is a unique decomposition $y = y_0(1 + y_+)$ with $y_0 \in \mathbb{C}^*$ and $y_+ \in \Lambda_+$. Define

$$\exp(x) := e^{x_0} \textstyle\sum_{k \geq 0} \frac{x_+^k}{k!} \qquad \in U_\Lambda$$

$$\log(y) := (\log(y_0) + 2\pi i\mathbb{Z}) + \textstyle\sum_{k \geq 1} (-1)^{k+1} \frac{y_+^k}{k} \qquad \in \Lambda_0/2\pi i\mathbb{Z}$$

Lemma 2.2. *The standard isomorphism $\mathbb{C}^* \cong \mathbb{C}/2\pi i\mathbb{Z}$ can extend to the following:*

$$U_\Lambda \quad \underset{\exp}{\overset{\log}{\rightleftarrows}} \quad \Lambda_0/2\pi i\mathbb{Z}$$

In particular, for any $y \in U_\Lambda$ (i.e. $\mathrm{val}(y) = 0$*) there exists some $x \in \Lambda_0$ so that $y = \exp(x)$.*

Proof. It is proved by a routine computation. $\square$

Lemma 2.3. *If* $f = \sum_{\nu \in \mathbb{Z}^n} c_\nu \mathbf{z}^\nu \in \Lambda[[z_1^\pm, \dots, z_n^\pm]]$ *vanishes on* U_Λ^n*, then* f *is identically zero.*

Proof. The condition actually tells that f converges on $U_\Lambda^n \equiv \mathfrak{trop}^{-1}(0)$. Thus, by Proposition A.3 we know that $f \in \Lambda\langle\{0\}\rangle \equiv \Lambda\langle z_1^\pm, \dots, z_n^\pm\rangle$. Accordingly, as $|\nu| \to \infty$, we have $\mathrm{val}(c_\nu) \to \infty$ or equivalently $|c_\nu| \to 0$. Arguing by contraction, we suppose f was not identically zero. Without loss of generality we may assume the sequence $|c_\nu|$ attains a maximal value $|c_{\nu_0}| = 1$ for some $\nu_0 \in \mathbb{Z}^n$, and we may further assume $c_{\nu_0} = 1$. Recall that $\Lambda_0 = \{c : \mathrm{val}(c) \geq 0\} = \{c : |c| \leq 1\}$ and $\Lambda_+ = \{c : \mathrm{val}(c) > 0\} = \{c : |c| < 1\}$. So, $\Lambda_0/\Lambda_+ \cong \mathbb{C}$. Since $|c_\nu| \leq 1$ for all ν, we know $f \in \Lambda_0[[z^\pm]]$. Taking the quotient by the ideal of elements with norm < 1, we get a power series $\bar{f} = \sum_\nu \bar{c}_\nu \mathbf{z}^\nu$ over the residue field $\mathbb{C}$. Since $c_\nu \to 0$, we have $|c_\nu| < 1$ and thus $\bar{c}_\nu = 0$ for all large enough ν. Hence, the $\bar{f}$ is in reality a nonzero Laurent polynomial with $\bar{c}_{\nu_0} = 1$. But by assumption $\bar{f}(\bar{\mathbf{y}})$ vanishes for all $\bar{\mathbf{y}} \in (\mathbb{C}^*)^n$; hence, $\bar{f}$ must be identically zero. This is a contradiction. $\square$

The above two lemmas are simple but will turn out to be extremely important: if we want to show some series $f(y_1, \dots, y_n) \equiv 0$, it suffices to show $f(e^{x_1}, \dots, e^{x_n}) \equiv 0$ for all $x_i \in \Lambda_0$.

2.2 Gapped A_∞ algebras and A_∞ homomorphisms

We review the homological algebra of A_∞ structures.

Definition 2.4. We define a **label group** to be a triple $\mathfrak{G} = (\mathfrak{G}, E, \mu)$ that consists of an abelian group $\mathfrak{G}$ and two group homomorphisms $E : \mathfrak{G} \to \mathbb{R}$, $\mu : \mathfrak{G} \to 2\mathbb{Z}$. Denote by 0 the unit of $\mathfrak{G}$.

In practice, the label group we will work on is as follows. Take a symplectic manifold (X, ω) and a compact oriented Lagrangian submanifold $L \subset X$. Consider the image of the Hurewicz homomorphism:

$$\mathfrak{G}(X, L) = \mathrm{im}\ (\pi_2(X, L) \to H_2(X, L)) \tag{18}$$

If there is no ambiguity, we will usually use the notations $\mathfrak{G}(X, L)$ and $\pi_2(X, L)$ interchangeably. Clearly, it comes with the energy map $E : \pi_2(X, L) \to \mathbb{R}$ given by $\beta \mapsto \omega \cap \beta$ and the Maslov index $\mu : \pi_2(X, L) \to 2\mathbb{Z}$. Since L is oriented, the image of μ maps into $2\mathbb{Z}$. We define

$$\mathfrak{G}(L) = \mathrm{im}\ (\pi_1(L) \to H_1(L))$$

to be the image of the abelianization map of the fundamental group. Again, if there is no ambiguity, we will often use the notations $\mathfrak{G}(L)$ and $\pi_1(L)$ interchangeably. Moreover, we have the boundary operator denoted by $\partial : \mathfrak{G}(X, L) \to \mathfrak{G}(L)$ or $\partial : \pi_2(X, L) \to \pi_1(L)$.

Now, we go back to the abstract definitions. We may often think $\mathfrak{G} = (\pi_2(X, L), E, \mu)$ as above.

Definition 2.5. Let C, C' be two graded vector spaces over $\mathbb{R}$; let $\mathfrak{G}$ be a label group. Given $k \in \mathbb{N}$ and $\beta \in \mathfrak{G}$, we define[4] $\mathbf{CC}_{k,\beta}(C', C) := \mathrm{Hom}(C'^{\otimes k}, C)$. Given $\beta \in \mathfrak{G}$, we define $\mathbf{CC}_\beta(C', C) := \prod_{k \geq 0} \mathbf{CC}_{k,\beta}(C', C)$. Finally, we define

$$\mathbf{CC}(C', C) := \mathbf{CC}_{\mathfrak{G}}(C', C)$$

to be the subspace of $\prod_{\beta \in \mathfrak{G}} \mathbf{CC}_\beta(C', C) = \prod_{k,\beta} \mathbf{CC}_{k,\beta}(C', C)$ consisting of $\mathfrak{t} = (\mathfrak{t}_{k,\beta})$ satisfying the following **gappedness condition**:

(a) $\mathfrak{t}_{0,0} = 0$.

(b) If $E(\beta) < 0$ or $E(\beta) = 0$ but $\beta \neq 0$, then we require $\mathfrak{t}_\beta = (\mathfrak{t}_{k,\beta})_{k \in \mathbb{N}}$ vanish identically.

(c) For any $E_0 > 0$, there are only finitely many $\beta \in \mathfrak{G}$ such that $\mathfrak{t}_\beta \neq 0$ and $E(\beta) \leq E_0$.

An element in $\mathbf{CC} = \mathbf{CC}(C', C)$ is called an *operator system*. An operator system $\mathfrak{t}$ satisfying (a) (b) (c) is called *gapped* or $\mathfrak{G}$*-gapped*. For simplicity, if $C = C'$, we often write $\mathbf{CC}(C)$; we also often omit C, C' or $\mathfrak{G}$ and just write $\mathbf{CC}_{k,\beta}$, $\mathbf{CC}_\beta$, $\mathbf{CC}_{\mathfrak{G}}$, $\mathbf{CC}$, and so on, according to the context.

The geometric idea behind is that a non-constant pseudo-holomorphic curve has positive energy; besides, the condition (c) corresponds to the Gromov compactness.

Remark 2.6. Provided the gappedness condition, we can do induction on the pairs (k, β). Indeed, we can introduce an order on the set of all such pairs: $(k', \beta') < (k, \beta)$ if either $E(\beta') < E(\beta)$ or $E(\beta') = E(\beta)$, $k' < k$. The gappedness also tells that there are at most countably many β's involved.

Denote by id the identity operator $x \mapsto x$. To compute signs, we introduce the 'twisted' identity $\mathrm{id}_\# = \mathrm{id}^\#$ defined by $x \mapsto (-1)^{\deg x - 1} x$. We also put

$$x^\# = \mathrm{id}_\#(x) = (-1)^{\deg x - 1} x \tag{19}$$

Given a multi k-linear operator ϕ, we have $(-1)^{\deg \phi + k - 1} \mathrm{id}_\# \circ \phi = \phi \circ (\mathrm{id}_\#)^{\otimes k}$. For the *shifted degree*

$$\deg' x = \deg x - 1 \tag{20}$$

we have $\deg' \phi = \deg \phi - 1 + k$. Given $p \in \mathbb{N}$, we put $\phi^{\# p} := \phi \circ (\mathrm{id}_{\# p})^{\otimes k}$ where $\mathrm{id}_{\# p}$ denotes the p-iteration $\mathrm{id}_\# \circ \cdots \circ \mathrm{id}_\#$ of $\mathrm{id}_\#$. It is straightforward to obtain that

$$\phi^\# = (-1)^{\deg' \phi} \mathrm{id}_\# \circ \phi, \qquad \phi^{\# p} = (-1)^{\deg' \phi} \cdot \mathrm{id}_\# \circ \phi^{\#(p-1)} = \cdots = (-1)^{p \cdot \deg' \phi} \mathrm{id}_{\# p} \circ \phi \tag{21}$$

[4] If $k = 0$, we think $\mathrm{Hom}(\mathbb{R}, C) \cong C$. Taking a different label $\beta \in \mathfrak{G}$ just gives a copy of the same space.

Definition 2.7. Fix operator systems $\mathfrak{f} = (\mathfrak{f}_{k,\beta}) \in \mathbf{CC}(C'', C')$, $\mathfrak{g} = (\mathfrak{g}_{k,\beta}) \in \mathbf{CC}(C', C)$ and $\mathfrak{h} = (\mathfrak{h}_{k,\beta}) \in \mathbf{CC}(C)$. The **composition** $\mathfrak{g} \circ \mathfrak{f} \in \mathbf{CC}(C'', C)$ of $\mathfrak{f}$ and $\mathfrak{g}$ is the operator system defined by

$$(\mathfrak{g} \circ \mathfrak{f})_{k,\beta} = \sum_{\ell \geq 1} \sum_{k_1 + \cdots + k_\ell = k} \sum_{\beta_0 + \beta_1 + \cdots + \beta_\ell = \beta} \mathfrak{g}_{\ell,\beta_0} \circ (\mathfrak{f}_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{f}_{k_\ell,\beta_\ell})$$

The **Gerstenhaber product** $\mathfrak{g} \star \mathfrak{h} \in \mathbf{CC}(C', C)$ is defined by the following operators

$$(\mathfrak{g} \star \mathfrak{h})_{k,\beta} = \sum_{\lambda + \mu + \nu = k} \sum_{\beta' + \beta'' = \beta} \mathfrak{g}_{\lambda+\mu+1,\beta'} \circ (\mathrm{id}_{\#}^{\lambda} \otimes \mathfrak{h}_{\nu,\beta''} \otimes \mathrm{id}^{\mu})$$

The gappedness conditions ensure that the summations in the above definition are all finite. The resulting operator systems $\mathfrak{g} \circ \mathfrak{f}$ and $\mathfrak{g} \star \mathfrak{h}$ are gapped as well. Notice also that $\mathfrak{g}_3 \circ (\mathfrak{g}_2 \circ \mathfrak{g}_1) = (\mathfrak{g}_3 \circ \mathfrak{g}_2) \circ \mathfrak{g}_1$.

Definition 2.8. A $\mathfrak{G}$-gapped A_∞ **algebra** is a graded $\mathbb{R}$-vector space C equipped with an operator system $\mathfrak{m} = (\mathfrak{m}_{k,\beta}) \in \mathbf{CC}_{\mathfrak{G}}(C, C)$ such that $\deg \mathfrak{m}_{k,\beta} = 2 - k - \mu(\beta)$ and the following A_∞ associativity relation holds

$$\sum_{\beta' + \beta'' = \beta} \sum_{\lambda + \mu + \nu = k} \mathfrak{m}_{\lambda+\mu+1,\beta'} \circ (\mathrm{id}_{\#}^{\lambda} \otimes \mathfrak{m}_{\nu,\beta''} \otimes \mathrm{id}^{\mu}) = 0$$

In short, the relation can be written as $\mathfrak{m} \star \mathfrak{m} = 0$. We call $(C, \mathfrak{m})$ **minimal** if $\mathfrak{m}_{1,0} = 0$.

In the literature, we usually work with the monoids instead of groups (c.f. [FOOO10b]); we adopt a slightly different style here. Specifically, for an A_∞ algebra $\mathfrak{m}$ associated to a Lagrangian L, the subset $\{\beta \mid \mathfrak{m}_\beta \neq 0\}$ forms a monoid contained in $\pi_2(X, L)$. However, the monoids may depend on choices. Since the monoids are all contained in the same group, we prefer to use the label group (Definition 2.4) and make all these monoids implicit.

Definition 2.9. Let $(C, \mathfrak{m})$, $(C', \mathfrak{m}')$ be two $\mathfrak{G}$-gapped A_∞ algebra. A $\mathfrak{G}$-gapped A_∞ **homomorphism** from $\mathfrak{m}'$ to $\mathfrak{m}$ is an operator system $\mathfrak{f} = (\mathfrak{f}_{k,\beta}) \in \mathbf{CC}_{\mathfrak{G}}(C', C)$ such that $\deg \mathfrak{f}_{k,\beta} = 1 - k - \mu(\beta)$ and the A_∞ relation $\mathfrak{m} \circ \mathfrak{f} = \mathfrak{f} \star \mathfrak{m}'$ holds, i.e.

$$\sum_{\ell \geq 1} \sum_{\substack{0 = j_0 \leq \cdots \leq j_\ell = k \\ \beta_0 + \beta_1 + \cdots + \beta_\ell = \beta}} \mathfrak{m}_{\ell,\beta_0} \circ (\mathfrak{f}_{j_1 - j_0,\beta_1} \otimes \cdots \otimes \mathfrak{f}_{j_\ell - j_{\ell-1},\beta_\ell}) = \sum_{\substack{\lambda + \mu + \nu = k \\ \beta' + \beta'' = \beta}} \mathfrak{f}_{\lambda+\mu+1,\beta'} \circ (\mathrm{id}_{\#}^{\lambda} \otimes \mathfrak{m}'_{\nu,\beta''} \otimes \mathrm{id}^{\mu})$$

Once again, the summations are all finite thanks to the gappedness conditions. If $\mathfrak{f}$ and $\mathfrak{g}$ are two gapped A_∞ homomorphisms, then it is clear that $\mathfrak{g} \circ \mathfrak{f}$ is also a gapped A_∞ homomorphism. Besides, if $\mathfrak{m}$ is a gapped A_∞ algebra and $\mathfrak{f}$ is a gapped A_∞ homomorphism, then their shifted degrees are $\deg' \mathfrak{m} = 1 - \mu(\cdot) \equiv 1 \pmod 2$ and $\deg' \mathfrak{f} = -\mu(\cdot) \equiv 0 \pmod 2$. So, $\mathfrak{m}^{\#} = -\mathrm{id}_{\#} \circ \mathfrak{m}$ and $\mathfrak{f}^{\#} = \mathrm{id}_{\#} \circ \mathfrak{f}$.

From now on, we often omit to say '$\mathfrak{G}$-gapped' or 'gapped'. Moreover, we will write $(C, \mathfrak{m})$, C or $\mathfrak{m}$ for an A_∞ algebra; we will write $\mathfrak{f} : \mathfrak{m}' \to \mathfrak{m}$ or $\mathfrak{f} : (C', \mathfrak{m}') \to (C, \mathfrak{m})$ for an A_∞ homomorphism.

Definition 2.10. An A_∞ algebra $(C, \mathfrak{m})$ gives a cochain complex $(C, \mathfrak{m}_{1,0})$. By forgetting all $\mathfrak{m}_{k,\beta}$ with $\beta \neq 0$. it gives an A_∞ algebra $(C, \bar{\mathfrak{m}})$, called the **reduction** of $(C, \mathfrak{m})$. An A_∞ homomorphism $\mathfrak{f} : (C, \mathfrak{m}) \to (C', \mathfrak{m}')$ also gives a cochain map $\mathfrak{f}_{1,0} : (C, \mathfrak{m}_{1,0}) \to (C', \mathfrak{m}'_{1,0})$. We call $\mathfrak{f}$ a **$\mathfrak{G}$-gapped A_∞ homotopy equivalence** if $\mathfrak{f}_{1,0}$ is a quasi-isomorphism. Similarly, by forgetting all $\mathfrak{f}_{k,\beta}$ with $\beta \neq 0$, it gives an A_∞ homomorphism $\bar{\mathfrak{f}}$ from $\bar{\mathfrak{m}}$ to $\bar{\mathfrak{m}}'$; we call it the **reduction** of $\mathfrak{f}$.

Definition 2.11. We define the following *incomplete and partial* A_∞ conditions for later use.

(a1) An operator system $\bar{\mathfrak{m}} = (\mathfrak{m}_{j,0})_{1\leq j\leq k} \in \mathbf{CC}_0(C, C)$ is called an A_k **algebra (modulo $T^{E=0}$)**[5] if $\bar{\mathfrak{m}} \star \bar{\mathfrak{m}}|_{\mathbf{CC}_{j,0}} = 0$ and $\deg \mathfrak{m}_{j,0} = 2 - j$ for $1 \leq j \leq k$.

(a2) An operator system $\mathfrak{m} = (\mathfrak{m}_{k,\beta})_{k\geq 0, E(\beta)\leq E} \in \mathbf{CC}(C, C)$ is called an A_∞ **algebra modulo T^E** if $\mathfrak{m} \star \mathfrak{m}|_{\mathbf{CC}_{k,\beta}} = 0$ and $\deg \mathfrak{m}_{k,\beta} = 2 - k - \mu(\beta)$ for $k \in \mathbb{N}$ and β with $E(\beta) < E$.

(b1) Let $(C, \mathfrak{m})$ and $(C', \mathfrak{m}')$ be two A_∞ algebras and let $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}'$ be their reductions. An operator system $\bar{\mathfrak{f}} = (\mathfrak{f}_{j,0})_{1\leq j\leq k} \in \mathbf{CC}_0(C', C)$ is called an A_k **homomorphism (modulo $T^{E=0}$)** if $(\bar{\mathfrak{m}} \circ \bar{\mathfrak{f}} - \bar{\mathfrak{f}} \star \bar{\mathfrak{m}}')|_{\mathbf{CC}_{j,0}} = 0$ and $\deg \mathfrak{f}_{j,0} = 1 - j$ for $1 \leq j \leq k$.

(b2) An operator system $\mathfrak{f} \in \mathbf{CC}(C', C)$ is called an A_∞ **homomorphism modulo T^E** if $(\mathfrak{m} \circ \mathfrak{f} - \mathfrak{f} \star \mathfrak{m}')|_{\mathbf{CC}_{k,\beta}} = 0$ and $\deg \mathfrak{f}_{k,\beta} = 2 - k - \mu(\beta)$ for all $k \in \mathbb{N}$ and β with $E(\beta) < E$.

(b3) Given $\mathsf{B} \in \mathfrak{G}$, an operator system $\mathfrak{f} \in \mathbf{CC}(C', C)$ is called an $A_{\infty,\mathsf{B}}$ **homomorphism** if it is an A_∞ homomorphism modulo $T^{E(\mathsf{B})}$ and moreover $(\mathfrak{m} \circ \mathfrak{f} - \mathfrak{f} \star \mathfrak{m}')|_{\mathbf{CC}_{\mathsf{B}}} = 0$.

2.3 P-pseudo-isotopies

Since now, we always assume that C is a locally convex topological vector space or even assume that it is a direct sum of differential form spaces on some smooth manifolds, which is enough in this paper. Let P be a contractible compact smooth manifold with corners. The definition of smooth functions on a manifold with corner P is delicate; for our purpose, we require such a function to be collared[6] near the boundaries and corners. But for simplicity, we prefer to make this point implicit and still use notations like $C^\infty(P)$, $\Omega^*(P)$, etc. After all, we just use the simple cases when P is a 1-simplex or 2-simplex.

Denote by $C^\infty(P, C)$ the set of all smooth maps from P to C, and we define:

$$C_P := \Omega^*(P) \otimes_{C^\infty(P)} C^\infty(P, C)$$

[5] Probably it is more precise to phrase 'modulo T^ϵ for every small ϵ'.

[6] This means the function looks linear on some neighborhoods of corners and it remains constant near the boundaries away from corners; see e.g. [FOOO17b, p71, Figure 8] for some idea.

Alternatively, one can think of C_P as the set of smooth sections on the bundle $\Lambda^*(P)\otimes C\to P$, i.e. the set of C-valued differential forms on P. Any element in C_P is a linear combination of elements in the form $\eta\otimes x(\cdot)$ with $\eta\in\Omega^*(P)$ and $x(\cdot)\in C^\infty(P,C)$. A grading on C naturally offers a one on $C^\infty(P,C)=\bigoplus_{d\geq 0}C^\infty(P,C^d)$; further, there is a natural **bi-grading** on C_P: an element $\eta\otimes x(\cdot)$ is of degree (p,q) if $\eta\in\Omega^p(P)$ and $x(\cdot)\in C^\infty(P,C^q)$.

The following two kinds of maps interest us. First, we have a natural inclusion map

$$\mathrm{Incl}:=\mathrm{Incl}_P:C\to C_P$$

sending x_0 to $1\otimes x_0$, where we still denote by x_0 the constant map in $C^\infty(P,C)$ at $x_0\in C$. Second, there is a family of evaluation maps

$$\mathrm{Eval}^s:C_P\to C$$

parameterized by $s\in P$, sending $1\otimes x(\cdot)$ to $x(s)$ and $\eta\otimes x(\cdot)$ to zero for $\eta\in\Omega^{>0}(P)$.

Geometrically, if we set $C=\Omega^*(L)$, the identification

$$\eta\otimes x\leftrightarrow\eta\wedge x \tag{22}$$

exactly gives rise to an isomorphism of vector spaces $\Omega^*(L)_P\cong\Omega^*(P\times L)$. In this case, the Incl and Eval^s can be recognized as the pullbacks pr^* and ι_s^* respectively for the projection $\mathrm{pr}:P\times L\to L$ and the inclusions $\iota_s:L\to\{s\}\times L\subset P\times L$. Just like $\iota_s^*\circ\mathrm{pr}^*=\mathrm{id}$, one can easily see that

$$\mathrm{Eval}^s\circ\mathrm{Incl}=\mathrm{id}$$

holds in general. As in [FOOO17b, Remark 21.28] or [ST16, §4.2.1], we make the following definition. It roughly says the $\Omega^*(P)$-linearity up to sign. For $I=(i_1,\dots,i_r)$ we denote $ds_I=ds_{i_1}\wedge\cdots\wedge ds_{i_r}$.

Definition 2.12. A multi-linear operator $\mathfrak{M}:C_P^{\otimes k}\to C'_P$ is said to be P**-pointwise** (or simply **pointwise**) if we have the following *sign compatibility condition*: for any $\sigma\in\Omega^*(P)$ we have

$$\mathfrak{M}(\eta_1\otimes x_1,\dots,\sigma\wedge\eta_i\otimes x_i,\dots,\eta_k\otimes x_k)=(-1)^{\deg\sigma\cdot\left((\deg\mathfrak{M}-1+k)+\sum_{a=1}^{i-1}(\deg\eta_a+\deg x_a-1)\right)}\sigma\wedge\mathfrak{M}(\eta_1\otimes x_1,\dots,\eta_k\otimes x_k)$$

In brief, this means

$$\mathfrak{M}(y_1,\dots,y_{i-1},\sigma\wedge y_i,y_{i+1},\dots,y_k)=(-1)^{\deg\sigma\cdot\deg'\mathfrak{M}}\sigma\wedge\mathfrak{M}(y_1^{\#\deg\sigma},\dots,y_{i-1}^{\#\deg\sigma},y_i,y_{i+1},\dots,y_k)$$

Remark 2.13. If an operator $\mathfrak{M}$ is pointwise, it suffices to consider the inputs in the form of $1\otimes x$ in order to determine it. Namely, we can find a family of operators $\mathfrak{M}_s^I:C^{\otimes k}\to C'$ for a point $s\in P$ and an ordered set $I=\{1\leq i_1<\cdots<i_d\leq k\}$ such that

$$\mathfrak{M}=\sum_I ds_I\otimes\mathfrak{M}_s^I=1\otimes\mathfrak{M}_s^{\varnothing}+\sum_{d\geq 1;i_1<\cdots<i_d}ds_{i_1}\wedge\cdots\wedge ds_{i_d}\otimes\mathfrak{M}_s^{i_1\cdots i_d}$$

and every $\mathfrak{M}_s^I$ is smooth in s for a fixed I. Namely, the above equation says $\mathfrak{M}(1\otimes x_1,\dots,1\otimes x_k)(s)=\sum_I ds_I\otimes \mathfrak{M}_s^I\big(x_1(s),\dots,x_k(s)\big)$. Finally, we note that any operator $\mathfrak{m}: C^{\otimes k}\to C'$ can be trivially extended to a P-pointwise operator $\mathfrak{M}: C_P^{\otimes k}\to C'_P$ simply by setting $\mathfrak{M}=1\otimes \mathfrak{m}$.

So far, the pointwiseness only refers to general multi-linear operators. Further, if we have a cochain complex structure $(C,\mathfrak{m}_{1,0})$ on C, then it naturally induces a cochain complex $(C_P,\mathfrak{M}_{1,0}^P)$ where

$$\mathfrak{M}_{1,0}^P(ds_I\otimes x)=(-1)^{|I|}\big(ds_I\otimes \mathfrak{m}_{1,0}(x)+\textstyle\sum_{i=1}^{\dim P} ds_I\wedge ds_i\otimes \partial_{s_i}x\big) \qquad (23)$$

or equivalently,

$$\mathfrak{M}_{1,0}^P=1\otimes\mathfrak{m}_{1,0}+\textstyle\sum_i ds_i\otimes\partial_{s_i}$$

for the convention in Remark 2.13. One can check the sign here agrees exactly with Definition 2.12. Both Incl and Eval^s become the cochain maps with respect to these differential maps.

In practice, we usually consider the situation $C=\Omega^*(L)$ which is equipped with the standard exterior differential $\mathfrak{m}_{1,0}=d$ on L, then, we can show the following lemma by routine computations:

Lemma 2.14. *The identification (22) gives rise to an isomorphism $\Omega^*(L)_P\equiv \Omega^*(P\times L)$ of cochain complexes so that the differential in (23) corresponds to the exterior derivative d on $P\times L$.*

Following [Sol20, §4.3], the sign convention we choose makes $\mathfrak{m}_{1,0}=d$. It is slightly different from [FOOO17b, Definition 21.29] which makes $\mathfrak{m}_{1,0}(x)=(-1)^{n+\deg x}d(x)$ instead. This is inessential, as we can relate them to each other by the sign transformation $\tilde{\mathfrak{m}}_{k,\beta}(x_1,\dots,x_k)=(-1)^{\epsilon}\mathfrak{m}_{k,\beta}(x_1,\dots,x_k)$ for $\epsilon=\sum_{1\le i\le k}(n+\deg x_i)$.

We also consider the cochain complex $C=H^*(L)$ where the differential is zero: $\mathfrak{m}_{1,0}=0$.

Corollary 2.15. *We have the following natural identifications of cochain complexes $(\Omega^*(L)_{P_1})_{P_2}\equiv\Omega^*(L)_{P_2\times P_1}$ and $(H^*(L)_{P_1})_{P_2}\equiv H^*(L)_{P_2\times P_1}$.*

Proof. By Lemma 2.14, the first one is direct: $(\Omega^*(L)_{P_1})_{P_2}\cong(\Omega^*(P_1\times L))_{P_2}\cong \Omega^*(P_2\times P_1\times L)$. For the second one, just observe that $H^*(L)\cong\mathbb{R}^m$ by selecting a basis, and then $H^*(L)_{P_1}\cong\Omega^*(P)^{\oplus m}$. Hence by the lemma again, $(H^*(L)_{P_1})_{P_2}\cong(\Omega^*(P_1)_{P_2})^{\oplus m}\cong\Omega^*(P_2\times P_1)^{\oplus m}\cong H^*(L)_{P_2\times P_1}$ □

Lemma 2.16. *Given a manifold L, the evaluation maps* $\mathrm{Eval}^s:\Omega^*(L)_P\to\Omega^*(L)$ *or* $\mathrm{Eval}^s: H^*(L)_P\to H^*(L)$ *are quasi-isomorphisms of cochain complexes for all $s\in P$. Therefore,* $\mathrm{Eval}^s:(\Omega^*(L)_{P_1})_{P_2}\to\Omega^*(L)_{P_1}$ *and* $\mathrm{Eval}^s:(H^*(L)_{P_1})_{P_2}\to H^*(L)_{P_1}$ *are also quasi-isomorphisms.*

Proof. For the first map, just observe that the evaluation map Eval^s is identified with the pull-back of inclusion $\iota_s: L\to\{s\}\times L\subset P\times L$. And, since P is contractible, $\mathrm{Eval}^s\equiv\iota_s^*$ is a quasi-isomorphism. For the second map, picking up a basis we may identify $H^*(L) \cong \mathbb{R}^m$ and $H^*(L)_P \cong \Omega^*(P)^{\oplus m}$. Since we use the zero differential for $H^*(L)$, the induced differential on $H^*(L)_P$ obtained by (23) coincides with (up to sign) $(d^P)^{\oplus m}$ where d^P denotes the exterior derivative on P; this completes the proof. $\square$

Definition 2.17. A ($\mathfrak{G}$-gapped) P**-pseudo-isotopy** (of A_∞ algebras) on C is defined to be a P-pointwise $\mathfrak{G}$-gapped A_∞ algebra structure $\mathfrak{M}^P = (\mathfrak{M}^P_{k,\beta}) \in \mathbf{CC}_{\mathfrak{G}}(C_P, C_P)$ so that $\mathfrak{M}^P_{1,0}$ is given by (23).

By definition, a P-pseudo-isotopy is just a special kind of A_∞ algebra defined on some C_P. When $P = [0,1]$, we often just call it a pseudo-isotopy; in this case, we will soon show that the above definition of pseudo-isotopy is equivalent to the more specific definition as follows: [Fuk10, Definition 8.5] or [FOOO17b, Definition 21.25].

Definition 2.18. A (classical) **pseudo-isotopy** of C is defined to be a family of operators $\mathfrak{m}^s_{k,\beta}$ and $\mathfrak{c}^s_{k,\beta}$, $s \in [0,1]$ on C satisfying the following conditions:

(a) The operators $\mathfrak{m}^s_{k,\beta}$ and $\mathfrak{c}^s_{k,\beta}$ are all smooth in $s \in [0,1]$. Their degrees are $2-k-\mu(\beta)$ and $1-k-\mu(\beta)$ respectively.

(b) The $(C, \mathfrak{m}^s)$ is a gapped A_∞ algebra for any s.

(c) If $E(\beta) < 0$, then $\mathfrak{c}^s_{k,\beta} = 0$. For any $E_0 > 0$, we can only find at most finitely many $\beta \in \mathfrak{G}$ with $E(\beta) \le E_0$ so that $\mathfrak{c}^s_{k,\beta} \ne 0$ for some $s \in [0,1]$ and $k \in \mathbb{N}$.

(d) $\mathfrak{m}_{1,0} := \mathfrak{m}^s_{1,0}$ is independent of s and $\mathfrak{c}^s_{1,0} = \frac{d}{ds}$.

(e) For $x_1, \dots, x_k \in C$ we have

$$\frac{d}{ds}\mathfrak{m}^s_{k,\beta} + \sum_{i+j+\ell=k} \sum_{\substack{\beta_1+\beta_2=\beta \\ (i+j+1,\beta_1)\ne(1,0)}} \mathfrak{c}^s_{i+j+1,\beta_1} \circ (\mathrm{id}^i_{\#} \otimes \mathfrak{m}^s_{\ell,\beta_2} \otimes \mathrm{id}^j) - \sum_{i+j+\ell=k} \sum_{\substack{\beta_1+\beta_2=\beta \\ (\ell,\beta_2)\ne(1,0)}} \mathfrak{m}^s_{i+j+1,\beta_1} \circ (\mathrm{id}^i \otimes \mathfrak{c}^s_{\ell,\beta_2} \otimes \mathrm{id}^j) = 0$$

If P is higher dimensional, we have similar but much more complicated conditions. Note that the above definition adopts a slightly different convention than the one in [Fuk10].

Proposition 2.19. *The two definitions of pseudo-isotopy are equivalent.*

Proof. This is proved in [Fuk10, Lemma 8.1] and [FOOO17b, Lemma 21.31]. Assume $\mathfrak{M} = (\mathfrak{M}_{k,\beta})$ is given. By Remark 2.13, we write $\mathfrak{M} = 1 \otimes \mathfrak{m}^s + ds \otimes \mathfrak{c}^s$, which can be used to unfold the A_∞ condition $\mathfrak{M} \star \mathfrak{M} = 0$. Then, we exactly arrive at all the conditions in Definition 2.18, and vice versa. $\square$

Example 2.20. For any gapped A_∞ algebra $(C, \mathfrak{m})$, one can construct the so-called **trivial pseudo-isotopy** $\mathfrak{M}^{\mathrm{tri}}$ about $\mathfrak{m}$. Specifically, if we write $\mathfrak{M}^{\mathrm{tri}}_{k,\beta} = 1 \otimes \mathfrak{m}^s_{k,\beta} \otimes + ds \otimes \mathfrak{c}^s_{k,\beta}$, then $\mathfrak{m}^s_{k,\beta} = \mathfrak{m}_{k,\beta}$ and $\mathfrak{c}^s_{k,\beta} = 0$, except $\mathfrak{c}^s_{1,0} = d/ds$. In other words, $\mathfrak{M}^{\mathrm{tri}} = 1 \otimes \mathfrak{m} + ds \otimes \frac{d}{ds}$. In higher dimensions, the *trivial* P*-pseudo-isotopy* is similarly defined by $\mathfrak{M}^{\mathrm{tri},P} = 1 \otimes \mathfrak{m} + \sum_i ds_i \otimes \frac{d}{ds_i}$.

There is an alternative name for the P-pseudo-isotopy used in [FOOO17b, Definition 21.29]: *P-parametrized family of $\mathfrak{G}$-gapped A_∞ algebra strucres on C*. Indeed, a P-pseudo-isotopy is roughly a P-family of A_∞ algebras together with certain data of derivatives. Heuristically, if $P = [0,1]$, one may think of $\mathfrak{c}^s$ as 'derivatives' of $\mathfrak{m}^s$. In general, let $(C_P, \mathfrak{M}^P)$ be a P-pseudo-isotopy, and we write

$$\mathfrak{M}^P(s) = 1 \otimes \mathfrak{m}^s + \textstyle\sum_{I \neq \varnothing} ds_I \otimes \mathfrak{c}^{I,s}$$

Then, by the proof of Proposition 2.19, we immediately have:

Definition-Proposition 2.21. The $(C, \mathfrak{m}^s)$ is a gapped A_∞ algebra for each $s \in P$. It is called the **restriction** of $(C_P, \mathfrak{M}^P)$ at $s \in P$. Alternatively, we say that the $\mathfrak{M}^P$ **restricts** to the $\mathfrak{m}^s$ at $s \in P$.

Remark 2.22. Note that the Eval^s can be naturally viewed as a gapped A_∞ homomorphism from $(C_P, \mathfrak{M}^P)$ to $(C, \mathfrak{m}^s)$ by setting $\mathrm{Eval}^s_{1,0} = \mathrm{Eval}^s$ and $\mathrm{Eval}^s_{k,\beta} = 0$ for any $(k,\beta) \neq (1,0)$. Namely, we have $\mathfrak{m}^s \circ \mathrm{Eval}^s = \mathrm{Eval}^s \star \mathfrak{M}^P$. In contrast, the Incl *cannot* be viewed as an A_∞ homomorphism in general, unless the $\mathfrak{M}^P$ happens to be a trivial pseudo-isotopy.

2.4 Unitalities and q.c.dR

Definition 2.23. A gapped A_∞ algebra $(C, \mathfrak{m})$ is called (strictly) **unital** if there exists some degree-zero element $\mathbb{1} \in C$ which is called a (strict) **unit** such that

(a0) $\mathfrak{m}_{1,0}(\mathbb{1}) = 0$

(a1) $\mathfrak{m}_{2,0}(\mathbb{1}, x) = (-1)^{\deg x} \mathfrak{m}_{2,0}(x, \mathbb{1}) = x$;

(a2) $\mathfrak{m}_{k,\beta}(\dots, \mathbb{1}, \dots) = 0$ when $(k,\beta) \neq (1,0), (2,0)$

Let $\mathbb{1}_1$ and $\mathbb{1}_2$ be units of $(C_1, \mathfrak{m}_1)$ and $(C_2, \mathfrak{m}_2)$. A gapped A_∞ homomorphism $\mathfrak{f} : \mathfrak{m}_1 \to \mathfrak{m}_2$ is called **unital** (with respect to $\mathbb{1}_1$ and $\mathbb{1}_2$) if

(b1) $\mathfrak{f}_{1,0}(\mathbb{1}_1) = \mathbb{1}_2$

(b2) $\mathfrak{f}_{k,\beta}(\dots, \mathbb{1}_1, \dots) = 0$ when $(k,\beta) \neq (1,0)$.

Definition 2.24. A gapped A_∞ algebra $(C, \mathfrak{m})$ is called **fully unital** if, for any degree-zero $\mathbf{e} \in C$,

(a2') $\mathfrak{m}_{k,\beta}(\dots, \mathbf{e}, \dots) = 0$ when $(k,\beta) \neq (1,0), (2,0)$.

A gapped A_∞ homomorphism $\mathfrak{f} : \mathfrak{m}_1 \to \mathfrak{m}_2$ is called **fully unital** if, for any degree-zero $\mathbf{e} \in C_1$,

(b2') $\mathfrak{f}_{k,\beta}(\dots, \mathbf{e}, \dots) = 0$ when $(k,\beta) \neq (1,0)$.

Definition 2.25. An operator system $\mathfrak{t} \in \mathbf{CC}_{\mathfrak{G}}$ is called **cyclically unital** if, for any degree-zero element $\mathbf{e}$ and any $(k,\beta) \neq (0,0)$, we have:

$$\mathrm{CU}[\mathfrak{t}]_{k,\beta}(\mathbf{e}; x_1, \dots, x_k) := \sum_{i=1}^{k+1} \mathfrak{t}_{k+1,\beta}(x_1^{\#}, \dots, x_{i-1}^{\#}, \mathbf{e}, x_i, \dots, x_k) = 0$$

Here the 'CU' stands for 'cyclical unitality'. Note that it is necessary to require $(k,\beta) \neq (0,0)$, as $\mathrm{CU}[\mathfrak{t}]_{0,0}(\mathbf{e}) = \mathfrak{t}_{1,0}(\mathbf{e})$ can be non-zero in general. Recall that the notation $x^{\#}$ is introduced in (19). Be cautious that the cyclical unitality applies for an arbitrary operator system $\mathfrak{t}$, while the full unitality is defined just for some A_∞ algebra $\mathfrak{m}$ or some A_∞ homomorphism $\mathfrak{f}$.

The following concept may justify our new unitalities.

Definition 2.26. We say a gapped A_∞ algebra $(C, \mathfrak{m})$ is a **quantum correction to de Rham complex**, or in abbreviation, is a **q.c.dR**, if C is isomorphic to some de Rham complex $\Omega^*(N)$ for some manifold N (recall Lemma 2.14 tells $\Omega^*(L)_P \cong \Omega^*(P \times L)$) and the following properties hold:

(a) $\mathfrak{m}_{k,0} = 0$ for $k \geq 3$;

(b) $\mathfrak{m}_{1,0}(x) = dx$ for the exterior derivative $d = d_N$ on N;

(c) $\mathfrak{m}_{2,0}(x_1, x_2) = (-1)^{\deg x_1} x_1 \wedge x_2$.

Remark 2.27. **(1)** To explain why the full unitality is reasonable, the key observation is that an A_∞ algebra $(\Omega^*(L), \mathfrak{m})$ associated to a Lagrangian submanifold (§6) is also proved to be a q.c.dR in the literature[7]. The constant-one $\mathbb{1} \in \Omega^0(L)$ is known to be the unit of $\mathfrak{m}$, but the mere q.c.dR condition is actually sufficient to obtain the conditions (a0) (a1) in Definition 2.23. If we examine more carefully the argument to prove (a2) (see e.g. [Fuk10, (7.3)]), then we will discover that the same argument can be applied equally to any other degree-zero form $\mathbf{e}$. We will go back to this point in §6.3.

(2) The cyclical unitality naturally comes out by a homological algebra consideration. Indeed, it gives a successful induction hypothesis, when we attempt to prove Whitehead theorem with divisor axiom (Theorem 3.1). Moreover, the cyclical unitality fits perfectly with the congruence relations for a divisor-axiom-preserving homotopy theory of A_∞ homomorphisms (see the proof of Lemma 2.41).

Lemma 2.28. *If two gapped A_∞ homomorphisms $\mathfrak{f}, \mathfrak{g}$ are unital (resp. fully unital or cyclically unital), then $\mathfrak{g} \circ \mathfrak{f}$ is also unital (resp. fully unital or cyclically unital).*

Proof. We only show the cyclical unitality, and the other proofs are similar. For $(k,\beta) \neq (0,0)$,

$$\mathrm{CU}[\mathfrak{g} \circ \mathfrak{f}]_{k,\beta}(\mathbf{e}; \dots) = \sum_{(\ell,\beta') \neq (0,0)} \mathfrak{g}(\mathfrak{f}^{\#} \dots \mathfrak{f}^{\#}, \mathrm{CU}[\mathfrak{f}]_{\ell,\beta'}(\mathbf{e}; \dots), \mathfrak{f} \dots \mathfrak{f}) + \sum \mathrm{CU}[\mathfrak{g}]_{m,\beta''}(\mathfrak{f}_{1,0}(\mathbf{e}); \mathfrak{f}, \dots, \mathfrak{f})$$

[7] See [FOOO17b, Definition 21.21 & Theorem 21.35] for the latest de Rham model, and [FOOO10b, Definition 3.5.6 & Remark 3.5.8] for the earlier singular chain model.

The exceptional terms with $(\ell, \beta') = (0,0)$ in the first sum are all collected in the second sum. By condition, we have $\deg \mathfrak{f}_{1,0} = 0$. Then, the second sum vanishes since $\mathfrak{g}$ is cyclically unital. Meanwhile, the first sum also vanishes because $\mathfrak{f}$ is cyclically unital. □

Recall that the notion of a P-pseudo-isotopy of C is slightly stronger than that of an arbitrary A_∞ algebra on C_P. Accordingly, we want to introduce a slightly stronger notion of the unitality as well.

Definition 2.29. A P-pseudo-isotopy $(C_P, \mathfrak{M}^P)$ is said to be P**-unital** if there exists some $\mathbb{1} \in C^0$ so that $\mathrm{Incl}(\mathbb{1})$ is a unit of $\mathfrak{M}^P$. In this case, we call $\mathbb{1} \in C$ or $\mathrm{Incl}(\mathbb{1}) \in C_P$ a P**-unit** of $(C_P, \mathfrak{M}^P)$.

2.5 Divisor axiom

The divisor axiom for A_∞ algebras is known in the literature [Fuk10, Lemma 13.1] [Aur07, Remark 3.5]. Now, we assume the label group $\mathfrak{G} = (\pi_2(X, L), E, \mu)$ is given by the geometric one in (18). Then, we have the boundary homomorphism $\partial : \pi_2(X, L) \to \pi_1(L)$ which is necessary to define the divisor axiom. For an operator system $\mathfrak{t} \in \mathbf{CC}_{\mathfrak{G}}(C, C')$, we define

$$\mathrm{DA}[\mathfrak{t}]_{k,\beta,m}(b; x_1, \dots, x_k) := \sum_{m_0 + \cdots + m_k = m} \mathfrak{t}_{k+m,\beta}(\overbrace{b, \dots, b}^{m_0}, x_1, \overbrace{b, \dots, b}^{m_1}, \dots, \overbrace{b, \dots, b}^{m_{k-1}}, x_k, \overbrace{b, \dots, b}^{m_k})$$

In practice, it suffices to assume $m = 1$, thus, we make the abbreviation $\mathrm{DA}[\mathfrak{t}]_{k,\beta} := \mathrm{DA}[\mathfrak{t}]_{k,\beta,1}$.

Definition 2.30. Given any graded cochain complex (C, δ), we define

$$\mathrm{DI}(C) := \mathrm{DI}(C, \delta) = \{ b \in C \mid b \in \ker \delta, \ \deg b = 1 \} \tag{24}$$

and define $\mathrm{DI}(C, \Lambda) = \mathrm{DI}(C) \hat{\otimes} \Lambda$. We call both of them the spaces of **divisor inputs**. Note that a degree-zero cochain map $\mathfrak{f}_{1,0} : C \to C'$ preserves the spaces of divisor input $\mathfrak{f}_{1,0} : \mathrm{DI}(C) \to \mathrm{DI}(C')$.

Since now, we will only take $C = H^*(L)_P$ or $C = \Omega^*(L)_P$. Recall that their differentials are given as in (23). We claim that in either case, we have the well-defined **cap product** as follows:

$$\partial\beta \cap : \ \mathrm{DI}(C) \to \mathbb{R}, \quad \mathrm{DI}(C, \Lambda) \to \Lambda \tag{25}$$

for any $\beta \in \mathfrak{G}(X, L)$. To see this, let b be a divisor input, and then write

$$b = 1 \otimes \bar{b}^s + \textstyle\sum_{i=1}^{\dim P} ds_i \otimes b_i^s$$

(1) When $C = H^*(L)_P$, the condition that b is a divisor input can imply $\partial_{s_j} \bar{b}^s = 0$. Thus, $\bar{b}^s \in H^1(L)$ is independent of s; we just define $\partial\beta \cap b$ to be $\partial\beta \cap \bar{b}^s$. **(2)** When $C = \Omega^*(L)_P$, the divisor input condition tells $d\bar{b}^s = 0$ and $\partial_{s_j} \bar{b}^s - db_j^s = 0$. Applying the natural quotient $q : Z^1(L) \to H^1(L)$ to the second equation yields that $\partial_{s_j} q(\bar{b}^s) = q(db_j^s) = 0$. Hence, the de Rham cohomology class $q(\bar{b}^s)$ is also independent of s. So we can also define $\partial\beta \cap b := \partial\beta \cap \bar{b}^s = \partial\beta \cap q(\bar{b}^s)$.

Definition 2.31. An operator system $\mathfrak{t} \in \mathbf{CC}_{\mathfrak{G}}(C, C')$ is said to satisfy the **divisor axiom** if for any divisor input $b \in \mathrm{DI}(C)$ and $(k, \beta, m) \neq (0, 0, 1)$, the following *divisor axiom equation* holds:

$$\mathrm{DA}[\mathfrak{t}]_{k,\beta,m}(b; x_1, \dots, x_k) = \frac{(\partial\beta \cap b)^m}{m!}\ \mathfrak{t}_{k,\beta}(x_1, \dots, x_k) \tag{26}$$

Remark 2.32. By combinatorics, there is an equivalent definition: we only require $m = 1$ in the above divisor axiom equation (26). Namely, for all $(k, \beta) \neq (0, 0)$, it suffices to require

$$\mathrm{DA}[\mathfrak{t}]_{k,\beta}(b; x_1, \dots, x_k) = \mathfrak{t}_{k+1,\beta}(b, x_1, \dots, x_k) + \cdots + \mathfrak{t}_{k+1,\beta}(x_1, \dots, x_k, b) = \partial\beta \cap b \cdot \mathfrak{t}_{k,\beta}(x_1, \dots, x_k)$$

Besides, we indicate that the divisor axiom equations for $k = 0$ will be applied to (weak) Maurer-Cartan equations: $\sum_m \mathfrak{m}_{m,\beta}(b, \dots, b) = \sum_m \frac{(\partial\beta\cap b)^m}{m!}\mathfrak{m}_{0,\beta} = e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta}$.

2.6 Category $\mathscr{U}\mathscr{D}$

Our original purpose is to incorporate the divisor axiom into the standard homotopy theory of A_∞ algebras (e.g. [FOOO10b]), but the cyclical unitality need to accompany it, as explained in Remark 2.27 (2). It turns out that we obtain a subcategory of the usual category of A_∞ algebras.

Abusing the notations, we denote by $\mathbb{1}$ the (equivalence class of) constant-one function in any one of $\Omega^*(L)$, $H^*(L)$, $\Omega^*(L)_P$ and $H^*(L)_P$. In this section, we assume $\mathfrak{G} = \mathfrak{G}(X, L) = \pi_2(X, L)$ in (18).

Our claim is that (we will soon prove it in Lemma 2.34)

$$\tilde{\mathscr{U}\mathscr{D}} := \tilde{\mathscr{U}\mathscr{D}}(L) := \tilde{\mathscr{U}\mathscr{D}}(L, X) \tag{27}$$

defined as follows is a category:

- (I) An object in $\tilde{\mathscr{U}\mathscr{D}}$ is a $\mathfrak{G}$-gapped A_∞ algebra with the following properties:
  - (I-0) it extends the natural cochain complex structure on $H^*(L)_P$ or $\Omega^*(L)_P$ for some P.
  - (I-1) it is P-unital, and the $\mathbb{1}$ is a P-unit;
  - (I-2) it is cyclically unital;
  - (I-3) it satisfies the divisor axiom;
  - (I-4) it is a P-pseudo-isotopy (and particularly it is P-pointwise).
- (II) A morphism $\mathfrak{f}$ in $\tilde{\mathscr{U}\mathscr{D}}$ is a $\mathfrak{G}$-gapped A_∞ homomorphism with the following properties
  - (II-1) it is unital with respect to the various $\mathbb{1}$ (so, $\mathfrak{f}_{1,0}(\mathbb{1}) = \mathbb{1}$)
  - (II-2) it is cyclically unital.
  - (II-3) it satisfies the divisor axiom;

(II-4) it satisfies the following identity for any divisor input b:

$$\partial\beta \cap \mathfrak{f}_{1,0}(b) = \partial\beta \cap b \tag{28}$$

Briefly, we aim to describe the (strict/cyclical) **u**nitalities and the **d**ivisor axiom systematically and categorically. First, the item (I-0) just means the $\mathbf{CC}_{1,0}$-component of an object A_∞ algebra must induce the natural cochain complex structures on $H^*(L)_P$ or $\Omega^*(L)_P$ that we mention before. It is also necessary for the definition of the divisor axiom. Second, we need the condition (II-4) to preserve the divisor axiom for compositions. We mention some examples for which the (II-4) holds: (i) $\mathfrak{f}_{1,0} = i(g) : H^*(L) \to \Omega^*(L)$ is a 'harmonic inclusion' which will be discussed in (94); (ii) $\mathfrak{f}_{1,0} = \pi(g) : \Omega^*(L) \to H^*(L)$ is a 'harmonic projection' in (95); (iii) $\mathfrak{f}_{1,0} = \mathrm{id}$; (iv) $\mathfrak{f}_{1,0}$ is so that $\mathrm{Eval}^s \circ \mathfrak{f}_{1,0}$ agrees with one of the above (i), (ii) or (iii).

Definition 2.33. Define

$$\mathscr{U}\mathscr{D} := \mathscr{U}\mathscr{D}(L) := \mathscr{U}\mathscr{D}(L, X)$$

to be the subcategory of $\tilde{\mathscr{U}\mathscr{D}}$ (27) that consists of objects $\mathfrak{m}$ and morphisms $\mathfrak{f}$ satisfying the extra conditions as follows:

(I-5) every β in the set $\mathsf{G}_\mathfrak{m} := \{\beta \in \mathfrak{G} \mid \mathfrak{m}_\beta \neq 0\}$ satisfies $\mu(\beta) \geq 0$

(II-5) every β in the set $\mathsf{G}_\mathfrak{f} := \{\beta \in \mathfrak{G} \mid \mathfrak{f}_\beta \neq 0\}$ satisfies $\mu(\beta) \geq 0$.

Taking Assumption 1.2 into considerations, we will mainly work with the above subcategory $\mathscr{U}\mathscr{D}$ instead of $\tilde{\mathscr{U}\mathscr{D}}$. Note that the $\mathfrak{m}_\beta = (\mathfrak{m}_{k,\beta})_{k\in\mathbb{N}}$ and $\mathfrak{f}_\beta = (\mathfrak{f}_{k,\beta})_{k\in\mathbb{N}}$ are simply the collections of all components of $\mathfrak{m}$ and $\mathfrak{f}$ with β fixed; c.f. Definition 2.5. From now on, an object in $\mathscr{U}\mathscr{D} = \mathscr{U}\mathscr{D}(L, X)$ (similar for $\tilde{\mathscr{U}\mathscr{D}}$ as well) will be usually written as $(C, \mathfrak{m})$, C or $\mathfrak{m}$ according to the context. The set of morphisms from $(C', \mathfrak{m}')$ to $(C, \mathfrak{m})$ will be written as $\mathrm{Hom}_{\mathscr{U}\mathscr{D}}(C', C)$ or $\mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}', \mathfrak{m})$. Denote by $\mathrm{Obj}\,\mathscr{U}\mathscr{D}$ (resp. $\mathrm{Mor}\,\mathscr{U}\mathscr{D}$) the collections of all objects (resp. all morphisms) in $\mathscr{U}\mathscr{D}$.

Notice that the above definition is illegal, unless we prove $\tilde{\mathscr{U}\mathscr{D}}$ and $\mathscr{U}\mathscr{D}$ are indeed categories in the following two lemmas.

Lemma 2.34. *If $\mathfrak{f}, \mathfrak{g} \in \mathrm{Mor}\,\tilde{\mathscr{U}\mathscr{D}}$, then $\mathfrak{g} \circ \mathfrak{f} \in \mathrm{Mor}\,\tilde{\mathscr{U}\mathscr{D}}$. Therefore, $\tilde{\mathscr{U}\mathscr{D}}$ is a category.*

Proof. Since $(\mathfrak{g} \circ \mathfrak{f})_{1,0} = \mathfrak{g}_{1,0} \circ \mathfrak{f}_{1,0}$, the composition $\mathfrak{g} \circ \mathfrak{f}$ also satisfies (II-4). The unitality and cyclical unitality are already proved in Lemma 2.28. It remains to show the divisor axiom. In reality,

$$\begin{aligned} \mathrm{DA}[\mathfrak{g} \circ \mathfrak{f}]_{k,\beta}(b; \dots) = \textstyle\sum_i \sum_{(k_i,\beta_i)\neq(0,0)} \mathfrak{g}_{\ell,\beta_0}\big(\mathfrak{f}_{k_1,\beta_1} \cdots \mathrm{DA}[\mathfrak{f}]_{k_i,\beta_i}(b; \dots) \dots \mathfrak{f}_{k_\ell,\beta_\ell}\big) \\ + \textstyle\sum \mathrm{DA}[\mathfrak{g}]_{\ell,\beta_0}(\mathfrak{f}_{1,0}(b); \mathfrak{f}_{k_1,\beta_1} \cdots \mathfrak{f}_{k_\ell,\beta_\ell}) \end{aligned}$$

holds for $(k, \beta) \neq (0, 0)$ by routine computations. Since $\mathfrak{f}$ and $\mathfrak{g}$ have the divisor axiom, we deduce

$$\begin{aligned} \mathrm{DA}[\mathfrak{g} \circ \mathfrak{f}]_{k,\beta}(b; \cdots) = \textstyle\sum_i \sum_{(k_i,\beta_i)\neq(0,0)} \partial\beta_i \cap b \cdot \mathfrak{g}_{\ell,\beta_0}(\mathfrak{f}_{k_1,\beta_1} \cdots \mathfrak{f}_{k_i,\beta_i} \cdots \mathfrak{f}_{k_\ell,\beta_\ell}) \\ + \textstyle\sum \partial\beta_0 \cap \mathfrak{f}_{1,0}(b) \cdot \mathfrak{g}_{\ell,\beta_0} \circ (\mathfrak{f}_{\beta_1} \cdots \mathfrak{f}_{\beta_\ell}) \end{aligned}$$

By Definition 2.5 (a), we have $\mathfrak{f}_{0,0} = 0$ and so we can actually drop $(k_i, \beta_i) \neq (0,0)$ above. Then using (28) together with $\beta = \beta_0 + \sum_{i=1}^{\ell} \beta_i$, we conclude the divisor axiom for $\mathfrak{g} \circ \mathfrak{f}$. $\square$

Lemma 2.35. *If $\mathfrak{f}, \mathfrak{g} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$, then $\mathfrak{g} \circ \mathfrak{f} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$. Hence, $\mathscr{U}\mathscr{D}$ is a category.*

Proof. Following Lemma 2.34 above, it suffices to check the condition (II-5) in Definition 2.33 is preserved. In reality, by Definition 2.7, if $(\mathfrak{g} \circ \mathfrak{f})_\beta \neq 0$ then for at least one tuple $(\beta_0, \beta_1, \dots, \beta_\ell)$ with $\beta_0 + \beta_1 + \cdots + \beta_\ell = \beta$, we have $\mathfrak{g}_{\beta_0} \circ (\mathfrak{f}_{\beta_1} \otimes \cdots \otimes \mathfrak{f}_{\beta_\ell}) \neq 0$ and thus particularly $\mathfrak{g}_{\beta_0} \neq 0, \mathfrak{f}_{\beta_1} \neq 0, \dots, \mathfrak{f}_{\beta_\ell} \neq 0$. Since $\mathfrak{f}$ and $\mathfrak{g}$ satisfy (II-5), we conclude $\mu(\beta) = \mu(\beta_0) + \cdots + \mu(\beta_\ell) \geq 0$. $\square$

Next, the following lemma is useful for the homotopy theory of A_∞ algebras in $\mathscr{U}\mathscr{D}$.

Lemma 2.36. *If $(C, \mathfrak{m}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$, then its trivial pseudo-isotopy $(C_{[0,1]}, \mathfrak{M}^{\mathrm{tri}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$.*

Proof. By definition, it is routine to check (I-4). By Corollary 2.15, we know the (I-0) holds for $(C_{[0,1]}, \mathfrak{M}^{\mathrm{tri}})$. Observe that $\mathfrak{M}^{\mathrm{tri}}_{k,\beta} = 1 \otimes \mathfrak{m}_{k,\beta}$ for $(k,\beta) \neq (1,0)$, and it is direct to show (I-1), (I-2) or (I-5). It remains to check the divisor axiom (I-3). Fix a divisor input $b = 1 \otimes b_0 + ds \otimes b_1$; without loss of generality, other inputs x_i can be assumed to be in the form $x_i(s) = 1 \otimes y_i(s)$. Then, we have

$$\mathrm{DA}[\mathfrak{M}^{\mathrm{tri}}]_{k,\beta}(b; x_1, \dots, x_k) = 1 \otimes \mathrm{DA}[\mathfrak{m}]_{k,\beta}(b_0; y_1, \dots, y_k)(s) + ds \otimes \mathrm{CU}[\mathfrak{m}]_{k,\beta}(b_1; y_1, \dots, y_k)(s)$$

Because $\mathfrak{m}$ is cyclically unital, the second term vanishes. By definition (25), we have $\partial\beta \cap b = \partial\beta \cap b_0(s)$. In conclusion, the desired divisor axiom equations of $\mathfrak{M}^{\mathrm{tri}}$ follow from that of $\mathfrak{m}$. $\square$

2.7 Homotopy theory with divisor axiom

Our new homotopy theory is supposed to build on $\mathscr{U}\mathscr{D}$ (one can do the same for $\tilde{\mathscr{U}\mathscr{D}}$). Be cautious that the Definition 2.37 below implicitly relies on Lemma 2.36.

Definition 2.37. We call $\mathfrak{f}_0, \mathfrak{f}_1 \in \operatorname{Hom}_{\mathscr{U}\mathscr{D}}((C', \mathfrak{m}'), (C, \mathfrak{m}))$ are **ud-homotopic** to each other **(via $\mathfrak{F}$)** if there is $\mathfrak{F} \in \operatorname{Hom}_{\mathscr{U}\mathscr{D}}((C', \mathfrak{m}'), (C_{[0,1]}, \mathfrak{M}^{\mathrm{tri}}))$ so that $\mathrm{Eval}^0 \circ \mathfrak{F} = \mathfrak{f}_0$ and $\mathrm{Eval}^1 \circ \mathfrak{F} = \mathfrak{f}_1$. Denote this by

$$\mathfrak{f}_0 \overset{\mathrm{ud}}{\sim} \mathfrak{f}_1$$

We can unfold the condition as follows. An image $\mathfrak{F}_{k,\beta}(x_1, \dots, x_k)$ is just an element in $C_{[0,1]}$. So, by considering the bi-grading on $C_{[0,1]}$, we may write $\mathfrak{F} = 1 \otimes \mathfrak{f}_s + ds \otimes \mathfrak{h}_s$ for some linear operators $\mathfrak{f}_s$ and $\mathfrak{h}_s$. Concretely, this says

$$\mathfrak{F}_{k,\beta}(x_1, \dots, x_k)(s) = 1 \otimes (\mathfrak{f}_s)_{k,\beta}(x_1, \dots, x_k) + ds \otimes (\mathfrak{h}_s)_{k,\beta}(x_1, \dots, x_k) \quad (29)$$

It is clear that $\mathfrak{F}$ is $\mathfrak{G}$-gapped if and only if all $\mathfrak{f}_s$ and $\mathfrak{h}_s$ are $\mathfrak{G}$-gapped. We note that despite similar notations used, there is nothing to do with the notion of pointwiseness (Remark 2.13). Technically we should require that these $\mathfrak{f}_s$ and $\mathfrak{h}_s$ are constant in s near the end points (e.g. in order for a sort of gluing); but for simplicity we would rather make this point implicit.

Lemma 2.38. *$\mathfrak{F} = 1\otimes \mathfrak{f}_s + ds\otimes \mathfrak{h}_s \in \mathbf{CC}_{\mathfrak{G}}(C', C_{[0,1]})$ gives a $\mathfrak{G}$-gapped A_∞ homomorphism from $(C',\mathfrak{m}')$ to $(C_{[0,1]},\mathfrak{M}^{\mathrm{tri}})$ if and only if the following conditions hold together:*

- *Every $\mathfrak{f}_s$ is an A_∞ homomorphism from $(C',\mathfrak{m}')$ to $(C,\mathfrak{m})$;*
- $\deg(\mathfrak{h}_s)_{k,\beta} = -k-\mu(\beta)$*;*
- *We have the identity*

$$\frac{d}{ds}\circ \mathfrak{f}_s = \sum \mathfrak{h}_s\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}'\otimes\mathrm{id}^\bullet)+\sum\mathfrak{m}\circ(\mathfrak{f}_s^\#\otimes\cdots\otimes\mathfrak{f}_s^\#\otimes\mathfrak{h}_s\otimes\mathfrak{f}_s\otimes\cdots\otimes\mathfrak{f}_s) \quad (30)$$

Proof. For the degrees, just observe that $\deg\mathfrak{f}_s=\deg\mathfrak{h}_s+1=\deg\mathfrak{F}$. Using Example 2.20 and (29), we expand the A_∞ formula $\sum\mathfrak{M}^{\mathrm{tri}}\circ(\mathfrak{F}\otimes\cdots\otimes\mathfrak{F}) = \sum\mathfrak{F}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}'\otimes\mathrm{id}^\bullet)$, obtaining that $1\otimes\mathfrak{m}\circ(\mathfrak{f}_s\otimes\cdots\otimes\mathfrak{f}_s)+ds\otimes\big(\tfrac{d}{ds}\circ\mathfrak{f}_s-\mathfrak{m}(\mathfrak{f}_s^\#\otimes\cdots\otimes\mathfrak{f}_s^\#\otimes\mathfrak{h}_s\otimes\mathfrak{f}_s\otimes\cdots\otimes\mathfrak{f}_s)\big) = 1\otimes\mathfrak{f}_s\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}'\otimes\mathrm{id}^\bullet)+ds\otimes\mathfrak{h}_s\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}'\otimes\mathrm{id}^\bullet)$. The proof is now complete by comparing both sides. □

Lemma 2.39. *In the above situation of Lemma 2.38, we have:*

(i) The $\mathfrak{F}$ satisfies divisor axiom if and only if all the $\mathfrak{f}_s$ and $\mathfrak{h}_s$ satisfy divisor axiom.

(ii) The $\mathfrak{F}$ is cyclically unital if and only if all the $\mathfrak{f}_s$ and $\mathfrak{h}_s$ are cyclically unital.

(iii) The $\mathfrak{F}$ is unital with respect to the constant-ones $\mathbb{1}$ if and only if all the $\mathfrak{f}_s$ is unital with respect to $\mathbb{1}$'s and meanwhile $(\mathfrak{h}_s)_{k,\beta}(\cdots\mathbb{1}\cdots)=0$ for all s and (k,β).

(iv) The $\mathfrak{F}$ satisfies (II-4) if and only if all the $\mathfrak{f}_s$ satisfy (II-4) (28).

(v) The $\mathfrak{F}$ satisfies (II-5) if and only if all the $\mathfrak{f}_s$ and $\mathfrak{h}_s$ satisfies (II-5) (Definition 2.33).

Proof. By (29), the divisor axiom equation $\mathrm{DA}[\mathfrak{F}]_{k,\beta}(b;x_1,\dots,x_k)=\partial\beta\cap b\cdot\mathfrak{F}_{k,\beta}(x_1,\dots,x_k)$ means:

$$\begin{matrix}1\otimes\mathrm{DA}[\mathfrak{f}_s]_{k,\beta}(b;x_1,\dots,x_k)\\ +ds\otimes\mathrm{DA}[\mathfrak{h}_s]_{k,\beta}(b;x_1,\dots,x_k)\end{matrix} = \partial\beta\cap b\cdot\begin{pmatrix}1\otimes(\mathfrak{f}_s)_{k,\beta}(x_1,\dots,x_k)\\ +ds\otimes(\mathfrak{h}_s)_{k,\beta}(x_1,\dots,x_k)\end{pmatrix}$$

Then, the item (i) is proved by comparison. The properties for the cyclical unitality (ii) and the unitality (iii) can be proved by similar comparisons. Next, by the definition of the cap product (25), we have $\partial\beta\cap\mathfrak{F}_{1,0}(b)=\partial\beta\cap(\mathfrak{f}_s)_{1,0}(b)$ and the item (iv) holds. The item (v) is obvious. □

Corollary 2.40. *$\mathfrak{f}_0$ and $\mathfrak{f}_1$ are ud-homotopic to each other if and only if there exist $(\mathfrak{f}_s)$ and $(\mathfrak{h}_s)$ so that*

(a) All the $\mathfrak{f}_s \in \operatorname{Mor} \mathscr{U}\mathscr{D}$.

(b) The formula (30) holds for $(\mathfrak{f}_s)$ and $(\mathfrak{h}_s)$.

(c) All the $\mathfrak{h}_s$ satisfy the divisor axiom, the cyclical unitality, and $(\mathfrak{h}_s)_{k,\beta}(\cdots \mathbb{1} \cdots) = 0$ for all (k,β).

(d) $\deg(\mathfrak{h}_s)_{k,\beta} = -k - \mu(\beta)$ and all the $\mathfrak{h}_s$ satisfy (II-5) (Definition 2.33).

In particular, we see that $\overset{\text{ud}}{\sim}$ is an equivalence relation.

Proof. This is just a consequence of Lemma 2.38 and Lemma 2.39. □

Lemma 2.41. *If $\mathfrak{f}_0 \overset{\text{ud}}{\sim} \mathfrak{f}_1$ and $\mathfrak{g}_0 \overset{\text{ud}}{\sim} \mathfrak{g}_1$, then $\mathfrak{g}_0 \circ \mathfrak{f}_0 \overset{\text{ud}}{\sim} \mathfrak{g}_1 \circ \mathfrak{f}_1$. So, $\overset{\text{ud}}{\sim}$ is a congruence relation on $\mathscr{U}\mathscr{D}$.*

Proof. Let $\mathfrak{f}_0, \mathfrak{f}_1 \in \operatorname{Hom}_{\mathscr{U}\mathscr{D}}((C'', \mathfrak{m}''), (C', \mathfrak{m}'))$ and $\mathfrak{g}_0, \mathfrak{g}_1 \in \operatorname{Hom}_{\mathscr{U}\mathscr{D}}((C', \mathfrak{m}'), (C, \mathfrak{m}))$. Denote by $\mathfrak{M}'$ and $\mathfrak{M}$ the trivial pseudo-isotopies about $\mathfrak{m}'$ and $\mathfrak{m}$. Recall that they are defined on $C'_{[0,1]}$ and $C_{[0,1]}$ respectively. By definition, there exists morphisms $\mathfrak{F} : C'' \to C'_{[0,1]}$ and $\mathfrak{G} : C' \to C_{[0,1]}$ in $\mathscr{U}\mathscr{D}$ connecting $\mathfrak{f}_0$ and $\mathfrak{f}_1$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ respectively. By Lemma 2.35, the composition $\mathfrak{G} \circ \mathfrak{f}_1 : C'' \to C_{[0,1]}$ is still in $\mathscr{U}\mathscr{D}$. Since $\operatorname{Eval}^i \circ \mathfrak{G} \circ \mathfrak{f}_1 = \mathfrak{g}_i \circ \mathfrak{f}_1$ for $i = 0, 1$, we conclude that $\mathfrak{g}_0 \circ \mathfrak{f}_1 \overset{\text{ud}}{\sim} \mathfrak{g}_1 \circ \mathfrak{f}_1$.

Now, it remains to show $\mathfrak{g}_0 \circ \mathfrak{f}_0 \overset{\text{ud}}{\sim} \mathfrak{g}_0 \circ \mathfrak{f}_1$. In fact, we can first find families $(\mathfrak{f}_s)$ and $(\mathfrak{h}_s)$ with the conditions (a) (b) (c) (d) in Corollary 2.40. Then, we aim to check the same four conditions for the other two families: $\hat{\mathfrak{f}}_s := \mathfrak{g}_0 \circ \mathfrak{f}_s$ and

$$\hat{\mathfrak{h}}_s = \sum \mathfrak{g}_0 \circ (\mathfrak{f}_s^{\#} \cdots \mathfrak{f}_s^{\#}, \mathfrak{h}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s) \tag{31}$$

(a) By condition, $\mathfrak{f}_s \in \operatorname{Mor} \mathscr{U}\mathscr{D}$. Then, by Lemma 2.35, we have $\hat{\mathfrak{f}}_s \in \operatorname{Mor} \mathscr{U}\mathscr{D}$.

(b) To prove the condition (b), we compute

$$\begin{aligned}
\tfrac{d}{ds} \circ \hat{\mathfrak{f}}_s &= \sum \mathfrak{g}_0(\mathfrak{f}_s \cdots \mathfrak{f}_s, \tfrac{d}{ds} \circ \mathfrak{f}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s) \\
&= \sum \mathfrak{g}_0(\mathfrak{f}_s \cdots \mathfrak{f}_s, \mathfrak{h}_s(\mathrm{id}_{\#}^{\bullet} \otimes \mathfrak{m}'' \otimes \mathrm{id}^{\bullet}), \mathfrak{f}_s \cdots \mathfrak{f}_s) + \mathfrak{g}_0(\mathfrak{f}_s \cdots \mathfrak{f}_s, \mathfrak{m}'(\mathfrak{f}_s^{\#} \cdots \mathfrak{f}_s^{\#}, \mathfrak{h}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s), \mathfrak{f}_s \cdots \mathfrak{f}_s) \\
&= \sum \mathfrak{g}_0(\mathfrak{f}_s^{\#} \cdots \mathfrak{f}_s^{\#}, \mathfrak{h}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s)(\mathrm{id}_{\#}^{\bullet} \otimes \mathfrak{m}'' \otimes \mathrm{id}^{\bullet}) + \mathfrak{g}_0(\mathrm{id}_{\#}^{\bullet} \otimes \mathfrak{m}' \otimes \mathrm{id}^{\bullet})(\mathfrak{f}_s^{\#} \cdots \mathfrak{f}_s^{\#}, \mathfrak{h}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s) \\
&= \sum \hat{\mathfrak{h}}_s(\mathrm{id}_{\#}^{\bullet} \otimes \mathfrak{m}'' \otimes \mathrm{id}^{\bullet}) + \mathfrak{m}(\mathfrak{g}_0 \cdots \mathfrak{g}_0)(\mathfrak{f}_s^{\#} \cdots \mathfrak{f}_s^{\#}, \mathfrak{h}_s, \mathfrak{f}_s \cdots \mathfrak{f}_s) \\
&= \sum \hat{\mathfrak{h}}_s(\mathrm{id}_{\#}^{\bullet} \otimes \mathfrak{m}'' \otimes \mathrm{id}^{\bullet}) + \mathfrak{m}\big((\mathfrak{g}_0 \circ \mathfrak{f}_s)^{\#} \cdots (\mathfrak{g}_0 \circ \mathfrak{f}_s)^{\#}, \hat{\mathfrak{h}}_s, (\mathfrak{g}_0 \circ \mathfrak{f}_s) \cdots (\mathfrak{g}_0 \circ \mathfrak{f}_s)\big)
\end{aligned}$$

where the second identity uses the formula (30) for $\mathfrak{f}_s$ and $\mathfrak{h}_s$; the third and fourth identities hold because the $\mathfrak{f}_s$ and the $\mathfrak{g}_0$ are A_∞ homomorphisms. Thus, we show the $\hat{\mathfrak{f}}_s$ and $\hat{\mathfrak{h}}_s$ satisfy (30).

(c) Concerning the divisor axiom, we compute as follows:

$$
\begin{aligned}
\mathrm{DA}[\hat{\mathfrak{h}}_s]_{k,\beta}(b;\cdots) &= \sum \mathrm{DA}[\mathfrak{g}_0]\big((\mathfrak{f}_s)_{1,0}(b);\mathfrak{f}_s^{\#}\cdots\mathfrak{h}_s\cdots\mathfrak{f}_s\big)\\
&+\sum \mathfrak{g}_0\big(\cdots \mathrm{DA}[\mathfrak{f}_s^{\#}](b;\cdots)\cdots\mathfrak{h}_s\cdots\big)+\sum \mathfrak{g}_0\big(\cdots\mathfrak{h}_s\cdots\mathrm{DA}[\mathfrak{f}_s](b;\cdots)\big)\\
&+\sum \mathfrak{g}_0\big(\mathfrak{f}_s^{\#}\cdots\mathrm{DA}[\mathfrak{h}_s](b;\cdots)\cdots\mathfrak{f}_s\big)\\
&+\sum \mathrm{CU}[\mathfrak{g}_0]\big((\mathfrak{h}_s)_{1,0}(b);\mathfrak{f}_s\cdots\cdots\mathfrak{f}_s\big)
\end{aligned}
$$

Since $\deg(\mathfrak{h}_s)_{1,0}(b)=0$, the fifth sum vanishes due to the cyclical unitality of $\mathfrak{g}_0$. Similarly as before, applying the divisor axiom of $\mathfrak{f}_s,\mathfrak{h}_s$ and $\mathfrak{g}_0$, the first four sums together exactly equate $\partial\beta\cap b\cdot\hat{\mathfrak{h}}_s$. Regarding the cyclical unitality, given a degree-zero $\mathbf{e}$, we first notice that $(\mathfrak{h}_s)_{1,0}(\mathbf{e})=0$, since $\deg(\mathfrak{h}_s)_{1,0}(\mathbf{e})=-1$. Then, there is a similar computation:

$$
\begin{aligned}
\mathrm{CU}[\hat{\mathfrak{h}}_s]_{k,\beta}(\mathbf{e};\cdots) &= \sum \mathrm{CU}[\mathfrak{g}_0]\big((\mathfrak{f}_s)_{1,0}(\mathbf{e});\mathfrak{f}_s^{\#}\cdots\mathfrak{h}_s\cdots\mathfrak{f}_s\big)\\
&+\sum \mathfrak{g}_0\big(\cdots \mathrm{CU}[\mathfrak{f}_s^{\#}](\mathbf{e};\cdots)\cdots\mathfrak{h}_s\cdots\big)+\sum \mathfrak{g}_0\big(\cdots\mathfrak{h}_s\cdots\mathrm{CU}[\mathfrak{f}_s](\mathbf{e};\cdots)\cdots\big)\\
&+\sum \mathfrak{g}_0\big(\mathfrak{f}_s^{\#}\cdots\mathrm{CU}[\mathfrak{h}_s](\mathbf{e};\cdots)\cdots\mathfrak{f}_s\big)
\end{aligned}
$$

The above four summations must all vanish thanks to the cyclical unitalities of $\mathfrak{f}_s$, $\mathfrak{h}_s$ and $\mathfrak{g}_0$. Finally, we compute

$$
\begin{aligned}
\hat{\mathfrak{h}}_s(\cdots\mathbb{1}\cdots) &= \sum \mathfrak{g}_0\big(\mathfrak{f}_s^{\#}(\cdots)\cdots\mathfrak{f}_s^{\#}(\cdots)\ \mathfrak{h}_s(\cdots\mathbb{1}\cdots)\ \mathfrak{f}_s(\cdots)\cdots\mathfrak{f}_s(\cdots)\big)\\
&+\sum \mathfrak{g}_0\big(\cdots(\mathfrak{f}_s^{\#})_{k_1,\beta_1}(\cdots\mathbb{1}\cdots)\cdots\mathfrak{h}_s(\cdots)\cdots\mathfrak{f}_s(\cdots)\cdots\big)\\
&+\sum \mathfrak{g}_0\big(\cdots\mathfrak{f}_s^{\#}(\cdots)\cdots\mathfrak{h}_s(\cdots)\cdots\mathfrak{f}_s(\cdots\mathbb{1}\cdots)_{k_2,\beta_2}\cdots\big)
\end{aligned}
$$

The first sum vanishes by the property of $(\mathfrak{h}_s)$. Since the $\mathfrak{f}_s$ is unital with respect to $\mathbb{1}$, a summand term in the second or third summation must be zero, unless $(k_1,\beta_1)=(1,0)$ or $(k_2,\beta_2)=(1,0)$. But, if so, one input is $(\mathfrak{f}_s)_{1,0}(\mathbb{1})=\mathbb{1}$, and the resulting summand term is like $(\mathfrak{g}_0)_{\ell,\beta}(\cdots\mathbb{1}\cdots)$ for some $\ell\geq 2$. Thus, the unitality of $\mathfrak{g}_0$ also enforces this summand to be zero, and the $\hat{\mathfrak{h}}_s(\cdots\mathbb{1}\cdots)$ vanishes as desired. The condition (c) for $\hat{\mathfrak{h}}_s$ is proved.

(d) The degree of $(\hat{\mathfrak{h}}_s)_{k,\beta}$ is clearly $-k-\mu(\beta)$. If $(\hat{\mathfrak{h}}_s)_\beta\neq 0$, by (31) there exists a decomposition $\beta_0+\beta_1'+\cdots+\beta_{\ell_1}'+\beta_3+\beta_1''+\cdots+\beta_{\ell_2}''=\beta$ so that all of the followings are non-zero: $(\mathfrak{g}_0)_{\beta_0}$, $(\mathfrak{f}_s^{\#})_{\beta_\lambda'}$ $(\lambda=1,\dots,\ell_1)$, $(\mathfrak{h}_s)_{\beta_3}$, $(\mathfrak{f}_s)_{\beta_\lambda''}$ $(\lambda=1,\dots,\ell_2)$. Since all of them satisfy (II-5) (Definition 2.33), the additivity of μ implies $\mu(\beta)\geq 0$. Consequently, we see $\hat{\mathfrak{h}}_s$ also satisfies (II-5).

In summary, the families $\hat{\mathfrak{f}}_s$ and $\hat{\mathfrak{h}}_s$ give a ud-homotopy from $\mathfrak{g}_0\circ\mathfrak{f}_0$ to $\mathfrak{g}_0\circ\mathfrak{f}_1$ by Corollary 2.40. □

2.8 Weak Maurer-Cartan equations

Let $(C,\mathfrak{m})\in\operatorname{Obj}\mathscr{U}\mathscr{D}$ and $\mathfrak{f}\in\operatorname{Hom}_{\mathscr{U}\mathscr{D}}((C'',\mathfrak{m}''),(C',\mathfrak{m}'))$ be an object and a morphism in $\mathscr{U}\mathscr{D}$. Recall that $\mathrm{DI}(C)$ denotes the space of divisor inputs (24).

Let R be a coefficient ring: we may choose $R = \Lambda_+$ (which appears in most literature) or $R = \Lambda_0/2\pi i\mathbb{Z}$, Λ_0 (the convergence issue is resolved by the divisor axiom). Define

$$\begin{aligned} \mathfrak{m}_* : \mathrm{DI}(C)\hat{\otimes}R \to C\hat{\otimes}R, \qquad & \mathfrak{m}_*(b) := \sum_\beta \sum_k T^{E(\beta)}\mathfrak{m}_{k,\beta}(b,\dots,b) = \sum_\beta T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta} \\ \mathfrak{f}_* : \mathrm{DI}(C'')\hat{\otimes}R \to C'\hat{\otimes}R, \qquad & \mathfrak{f}_*(b) := \sum_\beta \sum_k T^{E(\beta)}\ \mathfrak{f}_{k,\beta}(b,\dots,b) = \sum_\beta T^{E(\beta)} e^{\partial\beta\cap b}\mathfrak{f}_{0,\beta} \end{aligned} \tag{32}$$

Note that in the above, we always take sum on k first:

$$\mathfrak{m}_*(b) = \textstyle\sum_\beta T^{E(\beta)}\Big(\sum_k \mathfrak{m}_{k,\beta}(b,\dots,b)\Big) =: \sum_\beta T^{E(\beta)} M_\beta$$

If we write $b = b_0 + b_+$ for the decomposition $\Lambda_0 \equiv \mathbb{C}\oplus\Lambda_+$, then by divisor axiom and by Definition 2.1, we obtain that

$$M_\beta = \textstyle\sum_k \mathfrak{m}_{k,\beta}(b,\dots,b) = \sum_k \frac{1}{k!}(\partial\beta\cap b)^k\mathfrak{m}_{0,\beta} = \big(e^{\partial\beta\cap b_0}\sum_k \frac{(\partial\beta\cap b_+)^k}{k!}\big)\mathfrak{m}_{0,\beta} = e^{\partial\beta\cap b}\mathfrak{m}_{0,\beta}$$

So, the $\mathfrak{m}_*(b)$ has no convergence issues. The same argument applies to $\mathfrak{f}_*(b)$ as well.

Definition 2.42. Recall that we denote all the units by the same notation $\mathbb{1}$. We call b a *weak bounding cochain* if $\mathfrak{m}_*(b)$ is a multiple of the unit $\mathbb{1}$, that is, $\mathfrak{m}_*(b) = a\cdot\mathbb{1}$ for some $a\in\Lambda_0$. We call $\mathfrak{m}$ weakly unobstructed if it admits a weak bounding cochain. Finally, the set of all weak bounding cochains is called the *weak Maurer-Cartan solution space* of $\mathfrak{m}$.

3 Whitehead theorem with divisor axiom

In this section, once and for all, we fix $(C',\mathfrak{m}'), (C,\mathfrak{m}) \in \mathrm{Obj}\,\mathscr{U}\mathscr{D}$ and fix

$$\mathfrak{f}\in \mathrm{Hom}_{\mathscr{U}\mathscr{D}}((C',\mathfrak{m}'),(C,\mathfrak{m})) \tag{33}$$

so that the induced cochain map $\mathfrak{f}_{1,0} : (C',\mathfrak{m}'_{1,0}) \to (C,\mathfrak{m}_{1,0})$ is a quasi-isomorphism. In the literature, such a $\mathfrak{f}$ is called a *weak homotopy equivalence* or a *quasi-isomorphism*. Recall that the constant-one functions in various $\Omega^*(L)_P$ and $H^*(L)_P$ are all denoted by $\mathbb{1}$.

Since the whole section §3 is devoted to prove the following theorem, the reader may skip this section in the first reading. It is a mild generalization of [FOOO10b, Theorem 4.2.45], adding the additional properties: the (cyclical) unitality and the divisor axiom.

Theorem 3.1 (Whitehead Theorem)**.** *There exists $\mathfrak{g}\in\mathrm{Hom}_{\mathscr{U}\mathscr{D}}((C,\mathfrak{m}),(C',\mathfrak{m}'))$, unique up to ud-homotopy, so that $\mathfrak{g}\circ\mathfrak{f}$ and $\mathfrak{f}\circ\mathfrak{g}$ are both ud-homotopic to* id*. We call $\mathfrak{g}$ a **ud-homotopy inverse** of $\mathfrak{f}$; it is often denoted by $\mathfrak{f}^{-1}$.*

We will carry out the proof by inductions first on the length filtration in k and then on the energy filtration in β. Initially, since $\mathfrak{f}_{1,0}$ is a quasi-isomorphism, there exist a degree-zero cochain map

$$\mathfrak{g}_{1,0} : C \to C' \tag{34}$$

and a cochain homotopy map $h : C' \to C'$ of degree -1 such that

$$\mathfrak{g}_{1,0}\mathfrak{f}_{1,0} - \mathrm{id} = \mathfrak{m}'_{1,0} h + h\mathfrak{m}'_{1,0} \tag{35}$$

Consider a cochain map $\mathfrak{h}_{1,0} : C' \to C'_{[0,1]}$ defined by

$$\mathfrak{h}_{1,0}(x) = 1 \otimes \big((1-s)x + s\mathfrak{g}_{1,0}\mathfrak{f}_{1,0}(x)\big) + ds \otimes h(x) \tag{36}$$

Then, it satisfies $\mathrm{Eval}^0 \circ \mathfrak{h}_{1,0} = \mathrm{id}$ and $\mathrm{Eval}^1 \circ \mathfrak{h}_{1,0} = \mathfrak{g}_{1,0} \circ \mathfrak{f}_{1,0}$. By definition, we know $\mathfrak{f}_{1,0}(\mathbb{1}) = \mathbb{1}$ and $\mathfrak{m}'_{1,0}(\mathbb{1}) = 0$; by degree reason, we also known $h(\mathbb{1}) = 0$. By (35) and (36), we obtain

$$\mathfrak{g}_{1,0}(\mathbb{1}) = \mathbb{1}, \quad \text{and} \quad \mathfrak{h}_{1,0}(\mathbb{1}) = 1 \otimes \mathbb{1} = \mathrm{Incl}(\mathbb{1}) = \mathbb{1} \tag{37}$$

Lemma 3.2. *The property (II-4) (28) holds for* $\mathfrak{g}_{1,0}$ *and* $\mathfrak{h}_{1,0}$.

Proof. First, we fix a divisor input $b \in \mathrm{DI}(C, \mathfrak{m}_{1,0})$ (24). Since the cochain map $\mathfrak{f}_{1,0}$ is a quasi-isomorphism, there exists some $b' \in \mathrm{DI}(C', \mathfrak{m}'_{1,0})$ so that $\mathfrak{g}_{1,0}(b) = b' + \mathfrak{m}'_{1,0}(c')$ and $\mathfrak{f}_{1,0}(b') = b + \mathfrak{m}_{1,0}(c)$. By degree reason, we have $\deg c = \deg c' = 0$. Since $\mathfrak{f}_{1,0}$ satisfies (II-4) (28), we see that $\partial\beta \cap \mathfrak{g}_{1,0}(b) = \partial\beta \cap b' = \partial\beta \cap \mathfrak{f}_{1,0}(b') = \partial\beta \cap b$. As for $\mathfrak{h}_{1,0}$, by the definition of the cap product (25), using (36) infers that $\partial\beta \cap \mathfrak{h}_{1,0}(b) = \partial\beta \cap \mathrm{Eval}^0\, \mathfrak{h}_{1,0}(b) = \partial\beta \cap b$. □

Remark 3.3. There is certain flexibility in choosing $\mathfrak{g}_{1,0}$ in (34), since we only need to require $\mathfrak{g}_{1,0}$ satisfies (35). In practice, we are always interested in the following cases:

- When $\mathfrak{f} = \mathrm{Eval}^s : H^*(L)_P \to H^*(L)$ or $\Omega^*(L)_P \to \Omega^*(L)$ for some $s \in P$. Then a ud-homotopy inverse $\mathfrak{g} = (\mathrm{Eval}^s)^{-1}$ of Eval^s can be chosen so that $\mathfrak{g}_{1,0} = \mathrm{Incl}$.
- We will later construct a morphism $\mathfrak{i}^g \in \mathrm{Mor}\,\mathscr{U}\mathscr{D}$ in Corollary 7.3 such that $\mathfrak{i}^g_{1,0} = i(g)$, where the $i(g)$ is the harmonic embedding $H^*(L) \to \Omega^*(L)$ for a metric g; see (94). Then, a ud-homotopy inverse $\mathfrak{g} = (\mathfrak{i}^g)^{-1}$ can be chosen so that $\mathfrak{g}_{1,0} = \pi(g)$, where the $\pi(g)$ is the harmonic projection $\Omega^*(L) \to H^*(L)$ as defined in (95).

Now, the pair $(\mathfrak{g}_{1,0}, \mathfrak{h}_{1,0})$ gives the initial step of the induction.

3.1 Obstruction for the length filtration

Recall that we often omit C, C' and $\mathfrak{G}$ for the $\mathbf{CC}_{\mathfrak{G}}(C, C')$. In this section, we aim to inductively construct the energy-zero part of the $\mathfrak{g}$ in the component

$\mathbf{CC}_0 := \prod_{k=1}^{\infty} \mathbf{CC}_{k,0}$. By the *length filtration* on $\mathbf{CC}_0$ we mean the filtration defined by the following sequences

$$F^k\,\mathbf{CC}_0 := \prod_{j=1}^{k} \mathbf{CC}_{j,0}$$

To begin with, we define a natural differential on the vector space $\mathbf{CC}_0$:

$$\bar{\delta} := \bar{\delta}_k : \mathbf{CC}_{k,0} \to \mathbf{CC}_{k,0} \qquad \phi \mapsto \mathfrak{m}_{1,0}\circ\phi - (-1)^{\deg\phi - 1 + k} \sum_i \phi\circ(\mathrm{id}_{\#}^{i} \otimes \mathfrak{m}'_{1,0} \otimes \mathrm{id}^{k-i-1}) \tag{38}$$

Obviously, the $\bar{\delta}$ has degree one, i.e. $\deg \bar{\delta}\phi = \deg\phi + 1$. By routine computation, we have:

$$\bar{\delta}\circ\bar{\delta} = 0 \tag{39}$$

To include the unitalities and divisor axiom, we introduce the following definition:

Definition 3.4. We define $\mathbf{CC}_{1,0}^{\mathrm{ud}} = \{\phi \mid \phi(\mathbb{1}) = \mathbb{1}\}$ and define $\mathbf{CC}_{k,0}^{\mathrm{ud}} \subset \mathbf{CC}_{k,0}$ for $k \geq 2$ to be the space consisting of ϕ with the following properties:

(L1) $\mathrm{DA}[\phi]_{k-1,0}(b; x_1, \dots, x_{k-1}) = 0$ for any divisor input b.

(L2) $\mathrm{CU}[\phi]_{k-1,0}(\mathbf{e}; x_1, \dots, x_{k-1}) = 0$ for any $\mathbf{e}$ with $\deg \mathbf{e} = 0$.

(L3) $\phi(\dots, \mathbb{1}, \dots) = 0$ for the constant-one $\mathbb{1}$.

Finally, we define $\mathbf{CC}_0^{\mathrm{ud}} = \prod_{k\geq 1} \mathbf{CC}_{k,0}^{\mathrm{ud}}$; more concretely, this says $(\mathbf{CC}_0^{\mathrm{ud}}, \bar{\delta}) = \prod_{k\geq 1} (\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta}_k)$.

Here the condition (L1) just tells the divisor axiom for the special case $\beta = 0$ (Definition 2.31), and the (L2) describes the cyclical unitality for $\beta = 0$ (Definition 2.25). Note that the length filtration in $\mathbf{CC}_0$ can be inherited in $\mathbf{CC}_0^{\mathrm{ud}}$ by setting $F^k\,\mathbf{CC}_0^{\mathrm{ud}} = \prod_{j=1}^{k} \mathbf{CC}_{j,0}^{\mathrm{ud}}$.

Proposition 3.5 (Properties of $\bar{\delta}$ and $\mathbf{CC}_0^{\mathrm{ud}}$)**.**

(i) The $\bar{\delta}$ restricts to a differential on $\mathbf{CC}_{k,0}^{\mathrm{ud}}$, so there is a cohomology $H(\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta})$ for each k.

(ii) If f is a cochain map of degree zero with $f(\mathbb{1}) = \mathbb{1}$, then for any k, it induces a cochain map

$$f_* : (\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta}) \to (\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta})$$

sending ϕ to $f\circ\phi$. Moreover, if f is a quasi-isomorphism, then so is f_*.

(iii) If f is a cochain map of degree zero with $f(\mathbb{1}) = \mathbb{1}$, then for any k, it induces a cochain map

$$\hat{f} : (\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta}) \to (\mathbf{CC}_{k,0}^{\mathrm{ud}}, \bar{\delta})$$

sending ϕ to $\phi\circ f^{\otimes k}$. Moreover, if f is a qusi-isomorphism, then so is $\hat{f}$.

Proof. **(i).** By virtue of (39), it suffices to show $\bar\delta$ maps $\mathbf{CC}^{\mathrm{ud}}_{k,0}$ into $\mathbf{CC}^{\mathrm{ud}}_{k,0}$. Firstly, it is easy to check the $\bar\delta$ preserves the condition (L1). Secondly, given $\phi\in\mathbf{CC}^{\mathrm{ud}}_{k,0}$, we compute

$$\begin{aligned}\textstyle\sum_i \bar\delta\phi(x_1^\#,\dots,x_{i-1}^\#,\mathbf{e},x_i,\dots,x_{k-1}) &= \pm\, \mathfrak{m}_{1,0}\circ \textstyle\sum_i \phi(x_1,\dots,x_{i-1},\mathbf{e},x_i,\dots,x_{k-1})\\ &\pm \textstyle\sum_i \phi(x_1,\dots,x_{i-1},\mathfrak{m}'_{1,0}\mathbf{e},x_i,\dots,x_{k-1})\\ &\pm \textstyle\sum_{i>j}\phi(x_1,\dots,x_{j-1},\mathfrak{m}'_{1,0}(x_j^\#),x_{j+1}^\#,\dots,x_{i-1}^\#,\mathbf{e},x_i,\dots,x_{k-1})\\ &\pm \textstyle\sum_{i<j}\phi(x_1,\dots,x_{i-1},\mathbf{e}^\#,x_i^\#,\dots,x_{j-1}^\#,\mathfrak{m}'_{1,0}(x_j),x_{j+1},\dots,x_{k-1})\end{aligned}$$

The first sum vanishes as ϕ satisfies (L2), and the second sum vanishes as ϕ satisfies (L1). Also, for any fixed j, we set $y_\ell^{(j)}=x_\ell^\#$ for $1\le\ell\le j-1$, $y_\ell^{(j)}=x_\ell$ for $j+1\le\ell\le k-1$ and $y_j^{(j)}=\mathfrak{m}'_{1,0}(x_j)$; so, the last two sums together give $\pm\sum_j\sum_\ell\phi(y_1^{(j)},\dots,y_{\ell-1}^{(j)},\mathbf{e},y_\ell^{(j)},\dots,y_{k-1}^{(j)})$ which equals zero by the (L2) condition of ϕ. Thirdly, the $\bar\delta$ also preserves the unitality condition (L3) because of $\mathfrak{m}'_{1,0}(\mathbb{1})=0$.

(ii). It is straightforward to show that $\bar\delta f_*=f_*\bar\delta$ and that the f_* restricts to a map from $\mathbf{CC}^{\mathrm{ud}}_{k,0}$ into $\mathbf{CC}^{\mathrm{ud}}_{k,0}$. Note that when $k=1$, we need to use the condition $f(\mathbb{1})=\mathbb{1}$. In addition, if f is a quasi-isomorphism, then it admits an inverse f' in the sense that $f'\circ f=\mathrm{id}+\mathfrak{m}_{1,0}h+h\mathfrak{m}_{1,0}$ and $f\circ f'=\mathrm{id}+\mathfrak{m}'_{1,0}k+k\mathfrak{m}'_{1,0}$ for some operators h and k. Now, given $\phi\in\ker\bar\delta\cap\mathbf{CC}^{\mathrm{ud}}_0$, we have $h\circ\phi\in\mathbf{CC}^{\mathrm{ud}}_0$ and $f'_*f_*\phi=\phi+\mathfrak{m}_{1,0}h\phi+h\mathfrak{m}_{1,0}\phi=\phi+\bar\delta(h\phi)$; the similar holds for $f_*f'_*$. Hence, we get an isomorphism $H(f_*):H(\mathbf{CC}^{\mathrm{ud}},\bar\delta)\to H(\mathbf{CC}^{\mathrm{ud}},\bar\delta)$ with an inverse $H(f'_*)$ on the $\bar\delta$-cohomologies.

(iii). As before, it is routine to check that $\hat f$ is a cochain map, and it maps $\mathbf{CC}^{\mathrm{ud}}_{k,0}$ into $\mathbf{CC}^{\mathrm{ud}}_{k,0}$. Now, suppose f is a quasi-isomorphism, and let h,k be as above in (ii). If $\bar\delta\phi=0$, then $\widehat{f'f}(\phi)=\phi(\mathrm{id}+\bar\delta h,\dots,\mathrm{id}+\bar\delta h)=\phi+\bar\delta\eta$ for some η[8]. By similar argument, we complete the proof. □

Corollary 3.6. *The* Eval^s *and* Incl *induce quasi-isomorphisms* $\mathrm{Eval}^s_*,\widehat{\mathrm{Eval}^s}$ *and* $\mathrm{Incl}_*,\widehat{\mathrm{Incl}}$ *on* $\mathbf{CC}^{\mathrm{ud}}_{k,0}$.

Proof. Recall that the Eval^s and Incl are quasi-isomorphisms. Since we have $\mathrm{Eval}^s\circ\mathrm{Incl}=\mathrm{id}$, the proof is complete by Proposition 3.5. □

To perform an induction, we want to study the obstruction of extending an A_{k-1} homomorphism to an A_k homomorphism. If we do not require the divisor axiom, this is fully discussed in [FOOO10b].

Theorem 3.7. *Given an A_{k-1} homomorphism $\bar{\mathfrak{g}}=(\mathfrak{g}_{j,0})_{1\le j\le k-1}\in F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0(C',C)$ between two objects $(C',\mathfrak{m}'),(C,\mathfrak{m})\in\mathrm{Obj}\,\mathscr{U}\mathscr{D}$, there exists a $\bar\delta$-closed*

$$\mathfrak{o}_k(\bar{\mathfrak{g}})\in\mathbf{CC}^{\mathrm{ud}}_{k,0}(C',C)$$

[8]Notice that $\bar\delta(\phi(h_1,\dots,h_k))=(\bar\delta\phi)(h_1,\dots,h_k)\pm\sum_i\phi(h_1,\dots,\bar\delta h_i,\dots,h_k)$. For example, this implies that $\phi(\bar\delta h,\bar\delta h,\mathrm{id},\dots,\mathrm{id})=\bar\delta\big(\phi(h,\bar\delta h,\mathrm{id},\dots,\mathrm{id})\big)$ as $\bar\delta(\mathrm{id})=0$; one can compute $\phi(\mathrm{id}+\bar\delta h,\dots,\mathrm{id}+\bar\delta h)$ similarly.

such that its cohomology $[\mathfrak{o}_k(\bar{\mathfrak{g}})] \in H(\mathbf{CC}^{\mathrm{ud}}_{k,0}, \bar{\delta})$ vanishes if and only if $\bar{\mathfrak{g}}$ can be extended to an A_k homomorphism $\bar{\mathfrak{g}}^+ = (\bar{\mathfrak{g}}, \mathfrak{g}_{k,0})$ with $\mathfrak{g}_{k,0} \in \mathbf{CC}^{\mathrm{ud}}_{k,0}$. Moreover, if so, we have $\mathfrak{o}_k(\bar{\mathfrak{g}}) + \bar{\delta}(\mathfrak{g}_{k,0}) = 0$

Proof. We first make the definition: (which *depends* on the chosen A_∞ algebras $\bar{\mathfrak{m}}'$ and $\bar{\mathfrak{m}}$)

$$\mathfrak{o}_k(\bar{\mathfrak{g}}) := \sum_{\ell \neq 1} \mathfrak{m}_{\ell,0} \circ (\mathfrak{g}_{j_1 - j_0, 0} \otimes \cdots \otimes \mathfrak{g}_{j_\ell - j_{\ell-1}, 0}) - \sum_{\nu \neq 1} \mathfrak{g}_{\lambda+\mu+1,0}(\mathrm{id}_\#^\lambda \otimes \mathfrak{m}'_{\nu,0} \otimes \mathrm{id}^\mu) \tag{40}$$

where the conditions $\ell, \nu \neq 1$ guarantee that the $\mathfrak{o}_k(\bar{\mathfrak{g}})$ involve only $\mathfrak{g}_{i,0}$ for $i < k$. As $\bar{\mathfrak{m}}, \bar{\mathfrak{m}}', \bar{\mathfrak{g}} \in \mathbf{CC}^{\mathrm{ud}}_0$, it is routine to check $\mathfrak{o}_k(\bar{\mathfrak{g}}) \in \mathbf{CC}^{\mathrm{ud}}_0$ as well. By (38), the $\bar{\mathfrak{g}}^+ = (\bar{\mathfrak{g}}, \mathfrak{g}_{k,0})$ being an A_k homomorphism is exactly equivalent to $\mathfrak{o}_k(\bar{\mathfrak{g}}) + \bar{\delta}(\mathfrak{g}_{k,0}) = 0$. Moreover, every term in the expression (40) of $\mathfrak{o}_k(\bar{\mathfrak{g}})$ has degree $2-k$. From $\deg \bar{\delta} = 1$, it follows that the degree of $\mathfrak{g}_{k,0}$ is $1-k$ as desired.

Now, it remains to prove $\mathfrak{o}_k(\bar{\mathfrak{g}})$ is $\bar{\delta}$-closed. In fact, using (38), we first compute:

$$\begin{aligned}\bar{\delta}\mathfrak{o}_k(\bar{\mathfrak{g}}) = &\textstyle\sum_{\ell \neq 1} \mathfrak{m}_{1,0} \circ \mathfrak{m}_{\ell,0} \circ (\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}) + \sum_{\ell \neq 1} \mathfrak{m}_{\ell,0} \circ (\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}) \circ (\mathrm{id}_\#^\bullet \otimes \mathfrak{m}'_{1,0} \otimes \mathrm{id}^\bullet) \\ &- \big(\textstyle\sum_{\nu \neq 1} \mathfrak{m}_{1,0} \circ \mathfrak{g} \circ (\mathrm{id}_\#^\bullet \otimes \mathfrak{m}'_{\nu,0} \otimes \mathrm{id}^\bullet) + \sum_{\nu \neq 1} \mathfrak{g} \circ (\mathrm{id}_\#^\bullet \otimes \mathfrak{m}'_{\nu,0} \otimes \mathrm{id}^\bullet) \circ (\mathrm{id}_\#^\bullet \otimes \mathfrak{m}'_{1,0} \otimes \mathrm{id}^\bullet) \big)\end{aligned}$$

Using the A_∞ equations for $\mathfrak{m}$ and $\mathfrak{m}'$ together with the condition $\bar{\mathfrak{g}}$ is A_{k-1} homomorphism, a tedious but routine calculation deduces $\bar{\delta}\mathfrak{o}_k(\bar{\mathfrak{g}}) = 0$. ☐

By definition, the reduction A_∞ algebras $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}'$ (Definition 2.10) satisfy $\bar{\mathfrak{m}}', \bar{\mathfrak{m}} \in \operatorname{Obj} \mathscr{U}\mathscr{D}$ as well. According to (40), the assignment $\bar{\mathfrak{g}} \mapsto \mathfrak{o}_k(\bar{\mathfrak{g}})$ is viewed as a map from $F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0$ to $\mathbf{CC}^{\mathrm{ud}}_{k,0}$ and is determined exactly by $\bar{\mathfrak{m}}$ and $\bar{\mathfrak{m}}'$.

Lemma 3.8. *In the above situation, let $\bar{\mathfrak{f}}^0 = (\mathfrak{f}^0_{k,0})_k$ and $\bar{\mathfrak{f}}^1 = (\mathfrak{f}^1_{k,0})_k$ be two A_∞ homomoprhisms[9] in $\mathbf{CC}^{\mathrm{ud}}_0$. Then, $\bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0$ is also an A_{k-1} homomorphism and $[\mathfrak{o}_k(\bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0)] = \big[\mathfrak{f}^1_{1,0} \circ \mathfrak{o}_k(\bar{\mathfrak{g}}) \circ (\mathfrak{f}^0_{1,0})^{\otimes k}\big]$*

Proof. Recall that $\bar{\mathfrak{g}} \in F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0 \subset \mathbf{CC}^{\mathrm{ud}}_0$ is an A_{k-1} homomorphism. Abusing the notations, all the involved A_∞ algebras here are denoted by $\mathfrak{m}$. To begin with, it is routine to check $\bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0$ is an A_{k-1} homomorphism. Besides, $\bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0$ restricts to the subspace $F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0 \subset F^{k-1}\,\mathbf{CC}_0$ by the proof of Lemma 2.35. So, the $\mathfrak{o}_k(\bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0)$ is well-defined by Theorem 3.7. Indeed, if we further regard $\bar{\mathfrak{g}}$ as an element in $\mathbf{CC}^{\mathrm{ud}}_0$ by trivial extensions (i.e. we set $\bar{\mathfrak{g}}_k = \bar{\mathfrak{g}}_{k+1} = \cdots = 0$), then the operator system

$$\bar{\mathfrak{h}} := \bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0 = (\mathfrak{h}_{j,0})_{j \geq 1}$$

satisfies that $\bar{\mathfrak{h}} \in \mathbf{CC}^{\mathrm{ud}}_0$ and $\bar{\mathfrak{h}}|_{F^{k-1}\,\mathbf{CC}}$ is an A_{k-1} homomorphism.

Now we consider

$$\mathbf{CC}_{k,0} \ni \mathfrak{O} := \textstyle\sum_r \mathfrak{m}_{r,0} \circ (\bar{\mathfrak{h}} \otimes \cdots \otimes \bar{\mathfrak{h}}) - \sum_s \bar{\mathfrak{h}} \circ (\mathrm{id}_\#^\bullet \otimes \mathfrak{m}_{s,0} \otimes \mathrm{id}^\bullet)$$

[9] Actually, it is enough to assume that they are A_k homomorphisms (Definition 2.11).

It is easy to compute that $\mathfrak{O}=\mathfrak{o}_k(\bar{\mathfrak{h}})+\bar{\delta}\mathfrak{h}_{k,0}$. Also, we can compute the $\mathfrak{O}$ in another way as follows:

$$\begin{aligned}\mathfrak{O} =&\textstyle\sum \bar{\mathfrak{f}}^1_{p,0}\circ(\mathrm{id}_\#\otimes\mathfrak{m}_{r,0}\otimes\mathrm{id})\circ(\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\circ(\bar{\mathfrak{f}}^0\otimes\cdots\otimes\bar{\mathfrak{f}}^0)-\sum\bar{\mathfrak{f}}^1_{q,0}\circ(\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}_{s,0}\otimes\mathrm{id}^\bullet)\circ(\bar{\mathfrak{f}}^0\otimes\cdots\otimes\bar{\mathfrak{f}}^0)\\ =&\textstyle\sum\mathfrak{f}^1_{1,0}\circ\bar{\mathfrak{m}}\circ(\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\circ(\mathfrak{f}^0_{1,0})^{\otimes k}-\sum\mathfrak{f}^1_{1,0}\circ\bar{\mathfrak{g}}\circ(\mathrm{id}^\bullet_\#\otimes\bar{\mathfrak{m}}\otimes\mathrm{id}^\bullet)\circ(\mathfrak{f}^0_{1,0})^{\otimes k}\\ =&\textstyle\sum_{\ell\neq1}\mathfrak{f}^1_{1,0}\circ\bar{\mathfrak{m}}_{\ell,0}\circ(\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\circ(\mathfrak{f}^0_{1,0})^{\otimes k}-\sum_{\nu\neq1}\mathfrak{f}^1_{1,0}\circ\bar{\mathfrak{g}}\circ(\mathrm{id}^\bullet_\#\otimes\bar{\mathfrak{m}}_{\nu,0}\otimes\mathrm{id}^\bullet)\circ(\mathfrak{f}^0_{1,0})^{\otimes k}\end{aligned}$$

where the first identity holds by the A_∞ homomorphism equations for $\bar{\mathfrak{f}}^0$ and $\bar{\mathfrak{f}}^1$, the second identity holds since $\bar{\mathfrak{g}}$ is an A_{k-1} homomorphism, and the third is because we have trivially extended $\mathfrak{g}$ by $\mathfrak{g}_{k,0}=0$. The outcome exactly says $\mathfrak{O}=\mathfrak{f}^1_{1,0}\circ\mathfrak{o}_k(\mathfrak{g})\circ(\mathfrak{f}^0_{1,0})^{\otimes k}$. So, the proof is now complete. ☐

3.2 Extension for the length filtration

Let $\mathfrak{M}$ and $\mathfrak{M}'$ be the trivial pseudo-isotopies about the $\mathfrak{m}$ and $\mathfrak{m}'$ respectively. Clearly, the reduction $\bar{\mathfrak{M}}$ (resp. $\bar{\mathfrak{M}}'$) coincides with the trivial pseudo-isotopies about $\bar{\mathfrak{m}}$ (resp. $\bar{\mathfrak{m}}'$). Denote by $\bar{\mathfrak{f}}=(\mathfrak{f}_{j,0})_{j\geq1}$ the restriction of the $\mathfrak{f}\in\mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}',\mathfrak{m})$ (33) in $\mathbf{CC}^{\mathrm{ud}}_0(C',C)$; then, $\bar{\mathfrak{f}}\in\mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\bar{\mathfrak{m}}',\bar{\mathfrak{m}})$. Now, we are going to show a weaker version of Theorem 3.1 only in the energy-zero component.

Theorem 3.9. *There exists an A_∞ homomorphism $\bar{\mathfrak{g}}\in\mathbf{CC}^{\mathrm{ud}}_0(C,C')$ so that $\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}$ is ud-homotopic to* id *via some $\bar{\mathfrak{h}}\in\mathbf{CC}^{\mathrm{ud}}_0(C,C'_{[0,1]})$ and $\bar{\mathfrak{f}}\circ\bar{\mathfrak{g}}$ is ud-homotopic to* id *via some $\bar{\mathfrak{h}}'\in\mathbf{CC}^{\mathrm{ud}}_0(C',C_{[0,1]})$*

We emphasize that the $\bar{\mathfrak{h}},\bar{\mathfrak{h}}'$ are assumed to be contained in the energy zero part of $\mathbf{CC}(C,C'_{[0,1]})$, which is stronger than merely saying $\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}$ and $\bar{\mathfrak{f}}\circ\bar{\mathfrak{g}}$ is ud-homotopic to id in the sense of Definition 2.37. We plan to prove it by induction on k. The initial step when $k=1$ has completed before.

Proposition 3.10. *Let $k\geq2$. Given $\bar{\mathfrak{f}}\in\mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\bar{\mathfrak{m}}',\bar{\mathfrak{m}})$ as above, we assume*

$$\bar{\mathfrak{g}}=(\mathfrak{g}_{j,0})_{1\leq j\leq k-1}\in F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0(C,C'),\quad \textit{and}\quad \bar{\mathfrak{h}}=(\mathfrak{h}_{j,0})_{1\leq j\leq k-1}\in F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0(C,C'_{[0,1]})$$

are two A_{k-1} homomorphisms such that

$$\mathrm{Eval}^0\circ\bar{\mathfrak{h}}|_{F^{k-1}\mathbf{CC}_0}=\mathrm{id},\quad\textit{and}\quad \mathrm{Eval}^1\circ\bar{\mathfrak{h}}|_{F^{k-1}\mathbf{CC}_0}=\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}$$

Then there exists two A_k homomorphisms $\bar{\mathfrak{g}}^+=(\bar{\mathfrak{g}},\mathfrak{g}_{k,0})\in F^k\,\mathbf{CC}^{\mathrm{ud}}_0$ and $\bar{\mathfrak{h}}^+=(\bar{\mathfrak{h}},\mathfrak{h}_{k,0})\in F^k\,\mathbf{CC}^{\mathrm{ud}}_0$, extending $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$ respectively. Moreover, they satisfy that

$$\mathrm{Eval}^0\circ\bar{\mathfrak{h}}^+|_{CC_{k,0}}=\mathrm{id},\quad\textit{and}\quad \mathrm{Eval}^1\circ\bar{\mathfrak{h}}^+|_{CC_{k,0}}=\bar{\mathfrak{g}}^+\circ\bar{\mathfrak{f}}\tag{41}$$

Proof. By Lemma 3.8 and by condition, we see that $[\mathrm{Eval}^0\circ\mathfrak{o}_k(\bar{\mathfrak{h}})]=[\mathfrak{o}_k(\mathrm{Eval}^0\circ\bar{\mathfrak{h}})]=[\mathfrak{o}_k(\mathrm{id})]=0$. Then, by Corollary 3.6, $[\mathfrak{o}_k(\bar{\mathfrak{h}})]=0$, which implies that there exists some $\alpha\in\mathbf{CC}^{\mathrm{ud}}_{k,0}$ so that $\mathfrak{o}_k(\bar{\mathfrak{h}})+\bar{\delta}\alpha=0$. Recall that naively setting $\mathfrak{h}_{k,0}=\alpha$ gives an A_k homomorphism extension, but if so, we would miss the property (41). In reality, we need to put

$$\mathfrak{h}_{k,0}=\alpha-\mathrm{Incl}\,\mathrm{Eval}^0(\alpha)\in\mathbf{CC}^{\mathrm{ud}}_{k,0}$$

By Proposition 3.5 and by the definition (40) of $\mathfrak{o}_k$, we compute that $\bar\delta\,\mathrm{Eval}^0\,\alpha = \mathrm{Eval}^0\,\bar\delta\alpha = \mathrm{Eval}^0(-\mathfrak{o}_k(\bar{\mathfrak{h}})) = -\mathfrak{o}_k(\mathrm{Eval}^0\,\bar{\mathfrak{h}}) = -\mathfrak{o}_k(\mathrm{id}) = 0$. Due to (38), the fact $\mathfrak{M}^{\mathrm{tri}}_{1,0} \circ \mathrm{Incl} = \mathrm{Incl} \circ \mathfrak{m}_{1,0}$ implies that the $\bar\delta$ commutes with Incl. Thus, we get $\bar\delta\mathfrak{h}_{k,0} = \bar\delta\alpha - \mathrm{Incl}\,\bar\delta\,\mathrm{Eval}^0(\alpha) = \bar\delta\alpha$ and $\mathfrak{o}_k(\bar{\mathfrak{h}}) + \bar\delta\mathfrak{h}_{k,0} = 0$. Then, the $\bar{\mathfrak{h}}^+ = (\bar{\mathfrak{h}}, \mathfrak{h}_{k,0})$ gives an A_k homomorphism extension such that $\mathrm{Eval}^0 \circ \bar{\mathfrak{h}}^+|_{\mathbf{CC}_{k,0}} = \mathrm{Eval}^0\,\mathfrak{h}_{k,0} = 0$. Since Eval^1 is trivially an A_∞ homomorphism, the composition $\mathrm{Eval}^1 \circ \bar{\mathfrak{h}}^+$ must be an A_k homomorphism, and it extends $\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}} \in F^{k-1}\,\mathbf{CC}^{\mathrm{ud}}_0$. Now, we have

$$\mathfrak{o}_k(\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}) = -\bar\delta(\mathrm{Eval}^1 \circ \bar{\mathfrak{h}}^+)_{k,0} = -\bar\delta(\mathrm{Eval}^1 \circ \mathfrak{h}_{k,0}) \tag{42}$$

and so $[\mathfrak{o}_k(\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}})] = 0$. By the proof of Lemma 3.8, we have actually shown that

$$\mathfrak{o}_k(\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}) + \bar\delta\big(\textstyle\sum_{\ell<k} \mathfrak{g}_{\ell,0} \circ (\mathfrak{f} \otimes \cdots \otimes \mathfrak{f})\big) = \mathfrak{o}_k(\bar{\mathfrak{g}}) \circ \mathfrak{f}_{1,0}^{\otimes k} \tag{43}$$

and $[\mathfrak{o}_k(\bar{\mathfrak{g}}) \circ \mathfrak{f}_{1,0}^{\otimes k}] = [\mathfrak{o}_k(\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}})] = 0$. Then, due to Proposition 3.5 (iii), we have $[\mathfrak{o}_k(\bar{\mathfrak{g}})] = 0$. Applying Theorem 3.7 yields an extension A_k homomorphism $\bar{\mathfrak{g}}^+ = (\bar{\mathfrak{g}}, \mathfrak{g}_{k,0}) \in F^k\,\mathbf{CC}^{\mathrm{ud}}_0$ for some $\mathfrak{g}_{k,0}$ so that $\mathfrak{o}_k(\bar{\mathfrak{g}}) = -\bar\delta\mathfrak{g}_{k,0}$. Consider

$$\Xi := \big(\mathrm{Eval}^1 \circ \bar{\mathfrak{h}}^+ - \bar{\mathfrak{g}}^+ \circ \bar{\mathfrak{f}}\big)_{k,0} = \mathrm{Eval}^1 \circ \mathfrak{h}_{k,0} - \mathfrak{g}_{k,0} \circ \mathfrak{f}_{1,0}^{\otimes k} - \textstyle\sum_{\ell<k} \mathfrak{g}_{\ell,0} \circ (\mathfrak{f} \otimes \cdots \otimes \mathfrak{f}) \tag{44}$$

Generally, this is not necessary zero in contrast to (41). So, we need to slightly modify $\mathfrak{g}_{k,0}$ and $\mathfrak{h}_{k,0}$ as follows. First, using (42) and (43) implies that $\bar\delta\Xi = -(\mathfrak{o}_k(\bar{\mathfrak{g}}) + \bar\delta\mathfrak{g}_{k,0}) \circ \mathfrak{f}_{1,0}^{\otimes k} = 0$. Also, it is easy to check $\Xi \in \mathbf{CC}^{\mathrm{ud}}_0$, since so do $\bar{\mathfrak{f}}$, $\bar{\mathfrak{g}}^+$ and $\bar{\mathfrak{h}}^+$. Thus, the Ξ gives rise to a class $[\Xi]$ in $H(\mathbf{CC}^{\mathrm{ud}}_{k,0}, \bar\delta)$. Since $\bar{\mathfrak{f}}_{1,0}$ is a quasi-isomorphism, we can find a $\bar\delta$-closed $\Delta\mathfrak{g} \in \mathbf{CC}^{\mathrm{ud}}_{k,0}(C, C')$ so that $\Xi + \bar\delta\eta = \Delta\mathfrak{g} \circ \mathfrak{f}_{1,0}^{\otimes k}$ for some $\eta \in \mathbf{CC}^{\mathrm{ud}}_{k,0}(C, C')$. We define $\Delta\mathfrak{h} \in \mathbf{CC}^{\mathrm{ud}}_{k,0}(C', C'_{[0,1]})$ by setting $\Delta\mathfrak{h}(s) = 1 \otimes s\eta$. In special, we have $\mathrm{Eval}^0 \circ \Delta\mathfrak{h} = 0$ and $\mathrm{Eval}^1 \circ \Delta\mathfrak{h} = \eta$.

Ultimately, we claim the modified $\mathfrak{h}'_{k,0} := \mathfrak{h}_{k,0} + \bar\delta\Delta\mathfrak{h}$ and $\mathfrak{g}'_{k,0} := \mathfrak{g}_{k,0} + \Delta\mathfrak{g}$ meet our needs. Firstly, both of them lie in $\mathbf{CC}^{\mathrm{ud}}_{k,0}$ by construction. Secondly, since $\Delta\mathfrak{g}$ and $\bar\delta\Delta\mathfrak{h}$ are $\bar\delta$-closed, $(\bar{\mathfrak{h}}, \mathfrak{h}'_{k,0})$ and $(\bar{\mathfrak{g}}, \mathfrak{g}'_{k,0})$ are still the desired A_k homomorphism extensions by Theorem 3.7. Thirdly, the condition $\mathrm{Eval}^0 \circ \bar{\mathfrak{h}}^+ = \mathrm{id}$ is not destroyed, and one can show the modified (44) is given by $\Xi' := \mathrm{Eval}^1 \circ \mathfrak{h}'_{k,0} - \mathfrak{g}'_{k,0} \circ \mathfrak{f}_{1,0}^{\otimes k} - \sum_{\ell<k} \mathfrak{g}_{\ell,0} \circ (\mathfrak{f} \otimes \cdots \otimes \mathfrak{f}) = \Xi + \mathrm{Eval}^1 \circ (\bar\delta\Delta\mathfrak{h}) - \Delta\mathfrak{g} \circ \mathfrak{f}_{1,0}^{\otimes k} = \Xi + \bar\delta\eta - \Delta\mathfrak{g} \circ \mathfrak{f}_{1,0}^{\otimes k} = 0$. Now, the proof is complete. $\square$

Proof of Theorem 3.9. We give a proof by induction. The base case for $\mathfrak{g}_{1,0}$ and $\mathfrak{h}_{1,0}$ has been established in (34), (36), and Lemma 3.2. Note that $\mathrm{Eval}^0\,\mathfrak{h}_{1,0} = \mathrm{id}$ and $\mathrm{Eval}^1\,\mathfrak{h}_{1,0} = \mathfrak{g}_{1,0}\mathfrak{f}_{1,0}$ and both of them indeed live in $\mathbf{CC}^{\mathrm{ud}}_{1,0}$ by (37). Hence, starting from this we can repeatedly use Proposition 3.10 to obtain $\bar{\mathfrak{g}} = (\mathfrak{g}_{k,0})_{k\geq 1}$ and $\bar{\mathfrak{h}} = (\mathfrak{h}_{k,0})_{k\geq 1}$ in $\mathbf{CC}^{\mathrm{ud}}_0$ such that

$$\bar{\mathfrak{g}} \circ \bar{\mathfrak{f}} \overset{\mathrm{ud}}{\sim} \mathrm{id}$$

via $\bar{\mathfrak{h}}$. It remains to show $\bar{\mathfrak{f}} \circ \bar{\mathfrak{g}} \overset{\mathrm{ud}}{\sim} \mathrm{id}$ via some $\bar{\mathfrak{h}}' \in \mathbf{CC}^{\mathrm{ud}}_0$ as well. In reality, applying the above result to $\bar{\mathfrak{g}}$ in place of $\bar{\mathfrak{f}}$, we get some $\bar{\mathfrak{f}}' \in \mathbf{CC}^{\mathrm{ud}}_0$ so that

$\bar{\mathfrak{f}}' \circ \bar{\mathfrak{g}} \overset{\mathrm{ud}}{\sim} \mathrm{id}$ via some $\bar{\mathfrak{k}} \in \mathbf{CC}_0^{\mathrm{ud}}$. Finally, the argument in the proof of Lemma 2.41 can be performed within the energy-zero part $\mathbf{CC}_0^{\mathrm{ud}}$; thus, we also obtain $\bar{\mathfrak{f}}' \overset{\mathrm{ud}}{\sim} \bar{\mathfrak{f}}' \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}} \overset{\mathrm{ud}}{\sim} \bar{\mathfrak{f}}$ and $\bar{\mathfrak{f}} \circ \bar{\mathfrak{g}} \overset{\mathrm{ud}}{\sim} \bar{\mathfrak{f}}' \circ \bar{\mathfrak{g}} \overset{\mathrm{ud}}{\sim} \mathrm{id}$ via some operator system in $\mathbf{CC}_0^{\mathrm{ud}}$. $\square$

3.3 Obstruction for the energy filtration

In Theorem 3.9, we have constructed an ud-homotopy inverse $\bar{\mathfrak{g}}$ for the reduction $\bar{\mathfrak{f}}$ within the energy-zero components. Our next goal is to carry out an induction with respect to the energy filtration, and we treat the $\bar{\mathfrak{g}}$ as the base case. In contrast to the length filtration, a key difference is that given a fixed $\beta \neq 0$, the divisor axiom equations (26) will involve $\mathfrak{f}_{k,\beta}$ for distinct $k \in \mathbb{N}$. Inspired by this, we define

$$\mathbf{CC}_\beta^{\mathrm{ud}} \subset \mathbf{CC}_\beta$$

consisting of a $\mathbb{N}$-labeled operator system $\varphi = (\varphi_{k,\beta})_{k\geq 0}$ such that:

(E1) $\mathrm{DA}[\varphi]_{k,\beta}(b; x_1, \dots, x_k) = \partial\beta \cap b \cdot \varphi_{k,\beta}(x_1, \dots, x_k)$ for every $k \geq 0$ and divisor input b.

(E2) $\mathrm{CU}[\varphi]_{k,\beta}(\mathbf{e}; x_1, \dots, x_k) = 0$ for every $k \geq 0$ and degree-zero input $\mathbf{e}$.

(E3) $\varphi_{k,\beta}(\dots, \mathbb{1}, \dots) = 0$ for every k and the unit $\mathbb{1}$.

The three items correspond to the divisor axiom, the cyclical unitalities and the unitalities in sequence. We remark that (E1) and (E2) only relate $\varphi_{k+1,\beta}$ to $\varphi_{k,\beta}$.

Define

$$\delta := \delta_{\bar{\mathfrak{g}}} : \mathbf{CC}_\beta \to \mathbf{CC}_\beta \tag{45}$$

by sending an element $\varphi = (\varphi_{k,\beta})_{k\geq 0}$, with a fixed shifted degree $\deg' \varphi_{k,\beta} = p$ for all k, to an element $\delta_{\bar{\mathfrak{g}}}(\varphi)$ so that the component $\big(\delta_{\bar{\mathfrak{g}}}(\varphi)\big)_{k,\beta}$ in $\mathbf{CC}_{k,\beta}$ is given by (see (21) for the notations)

$$\sum_{\ell\geq 1} \sum_{\substack{k_1+\cdots+k_\ell=k \\ 1\leq i\leq \ell}} \mathfrak{m}_{\ell,0} \circ (\mathfrak{g}_{k_1,0}^{\#p} \otimes \cdots \otimes \mathfrak{g}_{k_{i-1},0}^{\#p} \otimes \varphi_{k_i,\beta} \otimes \mathfrak{g}_{k_{i+1},0} \otimes \cdots \otimes \mathfrak{g}_{k_\ell,0}) - (-1)^p \sum_{\lambda+\mu+\nu=k} \varphi_{\lambda+\mu+1,\beta} \circ (\mathrm{id}_\#^\lambda \otimes \mathfrak{m}'_{\nu,0} \otimes \mathrm{id}^\mu)$$

(compare [FOOO10b, Sec. 4.4.5 (4.4.39)]). It is an analog of (38). For simplicity, we may write $\sum \bar{\mathfrak{m}} \circ (\bar{\mathfrak{g}}^{\#p} \otimes \cdots \bar{\mathfrak{g}}^{\#p} \otimes \varphi \otimes \bar{\mathfrak{g}} \otimes \cdots \otimes \bar{\mathfrak{g}}) - (-1)^p \sum \varphi \circ (\mathrm{id}_\#^\bullet \otimes \bar{\mathfrak{m}}' \otimes \mathrm{id}^\bullet)$.

There are some useful observations as follows: Firstly, the map $\delta_{\bar{\mathfrak{g}}}$ has degree one in the sense that $\deg' \delta_{\bar{\mathfrak{g}}}(\varphi) = p+1$ whenever $\deg' \varphi = p$. Secondly, the map $\delta = \delta_{\bar{\mathfrak{g}}}$ depends not only on $\bar{\mathfrak{g}}$ but also on the A_∞ algebras $\mathfrak{m}$ and $\mathfrak{m}'$. Thirdly, because $\bar{\mathfrak{g}}, \bar{\mathfrak{m}}, \bar{\mathfrak{m}}' \in \mathbf{CC}_0^{\mathrm{ud}}$, one can directly check that

$$\mathrm{DA}[\delta\varphi]_{\bullet,\beta}(b; x_1, \dots, x_k) = \delta\Big(\mathrm{DA}[\varphi]_{\bullet,\beta}(b; x_1, \dots, x_k)\Big) \tag{46}$$

as elements in $\mathbf{CC}_\beta$. Fourthly, if we can find some k so that $\varphi_{i,\beta} = 0$ for $1 \leq i \leq k-1$, then

$$(\delta\varphi)_{k,\beta} = \bar{\delta}\big(\varphi_{k,\beta}\big) \tag{47}$$

Lemma 3.11. *The $\delta \equiv \delta_{\mathfrak{g}}$ is a differential, that is,*

$$\delta \circ \delta = 0$$

Proof. Pick up an element $\varphi = (\varphi_{k,\beta})_{k\geq 0}$ with fixed degree components, say $\deg' \varphi_{k,\beta} = p$. Then, $\deg' \delta(\varphi) = \deg' \varphi + 1$ by definition of $\delta := \delta_{\bar{\mathfrak{g}}}$. We compute

$$\begin{aligned}
\delta\delta(\varphi) &= \sum \mathfrak{m}\big(\mathfrak{g}^{\#(p+1)} \cdots \mathfrak{g}^{\#(p+1)}(\delta\phi)\mathfrak{g}\cdots\mathfrak{g}\big) + (-1)^{p+2}\sum(\delta\varphi)(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}) \\
&= \sum \mathfrak{m}\big(\mathfrak{g}^{\#(p+1)} \cdots \mathfrak{g}^{\#(p+1)}\mathfrak{m}(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g})\mathfrak{g}\cdots\mathfrak{g}\big) \\
&+ (-1)^{p+1}\sum \mathfrak{m}\big(\mathfrak{g}^{\#(p+1)}\cdots\mathfrak{g}^{\#(p+1)}\varphi(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})\mathfrak{g}\cdots\mathfrak{g}\big) \\
&+ (-1)^{p+2}\sum \mathfrak{m}(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g})\ (\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}) \\
&+ (-1)^{p+2}(-1)^{p+1}\sum\varphi\ (\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})\ (\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}) \qquad =: I + II + III + IV
\end{aligned}$$

where any $\mathrm{id}_{\#}$ or id actually refers to some $\mathrm{id}_{\#}^{k} = \mathrm{id}_{\#}\otimes\cdots\otimes\mathrm{id}_{\#}$ or $\mathrm{id}^{\ell} = \mathrm{id}\otimes\cdots\otimes\mathrm{id}$. We compute:

$$\begin{aligned}
I &= \sum \mathfrak{m}(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g}) \\
&- \sum \mathfrak{m}(\mathfrak{g}^{\#(p+1)}\cdots\mathfrak{g}^{\#(p+1)}\mathfrak{m}(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p})\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g}) \\
&- \sum \mathfrak{m}(\mathfrak{g}^{\#(p+1)}\cdots\mathfrak{g}^{\#(p+1)},(\mathrm{id}_{\#}\varphi),\mathfrak{g}^{\#}\cdots\mathfrak{g}^{\#}\mathfrak{m}(\mathfrak{g}\cdots\mathfrak{g})\mathfrak{g}\cdots\mathfrak{g}) \qquad =: I_1 - I_2 - I_3
\end{aligned}$$

Since $\mathfrak{g}$ is an A_∞ homomorphism, we have $\mathfrak{m}(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}) = (\mathfrak{m}(\mathfrak{g}\cdots\mathfrak{g}))^{\#p} = (\mathfrak{g}(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}))^{\#p} = (-1)^p\mathrm{id}_{\#p}\ \mathfrak{g}\ (\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}) = (-1)^p\mathfrak{g}^{\#p}(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})$ and

$$I_2 = (-1)^p\sum\mathfrak{m}(\mathfrak{g}^{\#(p+1)}\cdots\mathfrak{g}^{\#(p+1)}\mathfrak{g}^{\#p}\ (\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})\ \mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g})$$

Additionally, notice that $\varphi^{\#} = \varphi(\mathrm{id}_{\#},\dots,\mathrm{id}_{\#}) = (-1)^p\mathrm{id}_{\#}\circ\varphi$ by (21), and we conclude

$$I_3 = (-1)^p\sum\mathfrak{m}(\mathfrak{g}^{\#(p+1)}\cdots\mathfrak{g}^{\#(p+1)},\varphi^{\#},\mathfrak{g}^{\#}\cdots\mathfrak{g}^{\#}\mathfrak{g}(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id})\mathfrak{g}\cdots\mathfrak{g})$$

Because the A_∞ equation tells $I_1 = 0$, we conclude that

$$I + II = -(I_2+I_3) + II = (-1)^{p+1}\mathfrak{m}(\mathfrak{g}^{\#p}\cdots\mathfrak{g}^{\#p}\varphi\mathfrak{g}\cdots\mathfrak{g})(\mathrm{id}_{\#}\otimes\mathfrak{m}\otimes\mathrm{id}) = -III$$

Finally the A_∞ equation says $IV = 0$, and hence we have $\delta\delta = 0$. □

Provided Lemma 3.11, we can show an analog to Proposition 3.5. For simplicity, we will use the same symbol $\mathfrak{m}$ for the various A_∞ algebras involved.

Proposition 3.12. *Let $\beta \neq 0$ and and let $\bar{\mathfrak{g}}$ be an A_∞ homomorphism modulo $T^{E=0}$ in $\mathbf{CC}_0^{\mathrm{ud}}$.*

(i) The $\delta = \delta_{\bar{\mathfrak{g}}}$ *restricts to a differential on* $\mathbf{CC}_\beta^{\mathrm{ud}}$*, hence, there is a cohomology*

$$H(\mathbf{CC}_\beta^{\mathrm{ud}}, \delta_{\bar{\mathfrak{g}}})$$

(ii) If f *is a cochain map of degree zero satisfying (28),* $f(\mathbb{1}) = \mathbb{1}$*, and the property*[10] $\mathfrak{m}_{k,0} \circ f^{\otimes k} = f \circ \mathfrak{m}_{k,0}$*, then it induces a cochain map*[11]

$$f_* := (\mathrm{id}, f)_* : (\mathbf{CC}_\beta^{\mathrm{ud}}, \delta_{\bar{\mathfrak{g}}}) \to (\mathbf{CC}_\beta^{\mathrm{ud}}, \delta_{f\bar{\mathfrak{g}}})$$

sending φ *to* $f \circ \varphi$*. Moreover, if* f *is a quasi-isomorphism which admits a right inverse* h *in the sense that* $f \circ h = \mathrm{id}$*, then* f_* *is also a quasi-isomorphism.*

(iii) In the same condition of (ii), there is an induced cochain map

$$\hat{f} := (f, \mathrm{id})_* : (\mathbf{CC}_\beta^{\mathrm{ud}}, \delta_{\bar{\mathfrak{g}}}) \to (\mathbf{CC}_\beta^{\mathrm{ud}}, \delta_{\bar{\mathfrak{g}}\circ f})$$

sending φ *to* $(\varphi_k \circ f^{\otimes k})_{k \geq 0}$*. Moreover, if* f *is a quasi-isomorphism with a left inverse* k *in the sense that* $k \circ f = \mathrm{id}$*, then* $\hat{f}$ *is also a quasi-isomorphism.*

Example 3.13. The very example we care for the item (ii) is $f = \mathrm{Eval}^s$ whose right inverse is given by Incl. It is a quasi-isomorphism due to Lemma 2.16, and it can be viewed as an A_∞ homomorphism due to Remark 2.22. On the other hand, the Incl occasionally serves as an example for (iii), but the Incl is not always an A_∞ homomorphism. For instance, if it involves a non-trivial pseudo-isotopy $\mathfrak{M}$, then $\mathfrak{M}_{k,0} \circ \mathrm{Incl}^{\otimes k} = \mathrm{Incl} \circ \mathfrak{m}_{k,0}$ does *not* hold in general, and the item (iii) cannot apply in this case.

Proof. **(i).** By Lemma 3.11, it suffices to show $\delta := \delta_{\bar{\mathfrak{g}}}$ maps $\mathbf{CC}_\beta^{\mathrm{ud}}$ into $\mathbf{CC}_\beta^{\mathrm{ud}}$. Since $\bar{\mathfrak{m}}'$ and $\bar{\mathfrak{g}}$ also have the unitalities, cyclical unitalities, and divisor axiom, it is routine to check $\delta(\varphi) \in \mathbf{CC}_\beta^{\mathrm{ud}}$.

(ii). Note that $f \in \mathbf{CC}_{1,0}^{\mathrm{ud}} \subset \mathbf{CC}_0^{\mathrm{ud}}$, and one can then easily see that $f\bar{\mathfrak{g}} \in \mathbf{CC}_0^{\mathrm{ud}}$ is also an A_∞ homomorphism module $T^{E=0}$ by the condition of f. This implies that $\delta_{f\bar{\mathfrak{g}}}$ is well-defined. Also, the degree condition on f ensures that $\delta_{f\bar{\mathfrak{g}}}(f\varphi) = f\delta_{\bar{\mathfrak{g}}}(\varphi)$, so f_* is a cochain map. Next, suppose f is a quasi-isomorphism with a right inverse h. Generally, the h does not commute with δ like f.

Take $\varphi = (\varphi_{k,\beta})_{k\geq 0} \in \mathbf{CC}_\beta^{\mathrm{ud}}$ so that $\delta\varphi = 0$ and $f_*\varphi = \delta\xi$ for some $\xi \in \mathbf{CC}_\beta^{\mathrm{ud}}$, and then we aim to show φ is actually δ-exact in $\mathbf{CC}_\beta^{\mathrm{ud}}$. Since $f \circ h = \mathrm{id}$, replacing φ by $\varphi - \delta h \xi$, we may assume

$$f_*\varphi = f\varphi = 0$$

For simplicity, we will often omit β, for instance, φ_k will represent $\varphi_{k,\beta}$, and $\mathfrak{m}_k$ (resp.$\mathfrak{g}_k$) will stand for $\mathfrak{m}_{k,0}$ (resp.$\mathfrak{g}_{k,0}$). Namely, we will use $(\cdot)_k$ to denote

[10]Equivalently, this means f can be viewed as an A_∞ homomorphism in the trivial way

[11]The notation here will become clear soon in Proposition 3.15

the component in both $\mathbf{CC}_{k,\beta}$ and $\mathbf{CC}_{k,0}$. Initially, $\mathfrak{m}_1\varphi_0 = (\delta\varphi)_0 = 0$ and $f\varphi_0 = (f\varphi)_0 = 0$. So $\varphi_0 = \mathfrak{m}_1\eta_0$ for some η_0 as f is a quasi-isomorphism of cochain complexes. Additionally, there is certain freedom: we may replace η_0 by any element in $\eta_0 + \ker\mathfrak{m}_1$. Using the facts that f is a quasi-isomorphism and that $f\circ h = \mathrm{id}$, we may choose η_0 so that $f\eta_0 = 0$. (Indeed, we can first find $\zeta\in\ker\mathfrak{m}_1$ so that $f\eta_0 = f\zeta + \mathfrak{m}_1\zeta'$ for some ζ', then we simply replace η_0 by $\eta_0 - \zeta - \mathfrak{m}_1 h\zeta'$.) In particular, $(\varphi-\delta\eta_0)_0 = \varphi_0 - \mathfrak{m}_1\eta_0 = 0$. Suppose we have $\eta = (\eta_0,\eta_1,\dots,\eta_k,0,0,\dots)$ for $\eta_i\in\mathbf{CC}_{i,\beta}$ satisfying the following *induction hypothesis*

- $\psi := \varphi - \delta\eta$. Moreover, we require $\psi_i = 0$ $(0\le i\le k)$;
- $(f\eta)_i = 0$ $(0\le i\le k)$;
- (E1) holds for every pair (η_i,η_{i+1}) $(0\le i\le k-1)$ and every divisor input b.
- (E2) and (E3) hold for all η_i.

Since $\delta\varphi = 0$, we have $\bar\delta\psi_{k+1} = (\delta\psi)_{k+1} = (\delta(\varphi-\delta\eta))_{k+1} = 0$. Also, since $f\varphi = 0$, we have $f_*\psi_{k+1} = -f(\delta\eta)_{k+1} = -(\delta(f\eta))_{k+1} = 0$. Because of $\varphi\in\mathbf{CC}^{\mathrm{ud}}_\beta$, the property (46) implies that $\mathrm{DA}[\psi]_\beta(b) = \partial\beta\cap b\cdot\varphi - \mathrm{DA}[\delta\eta]_\beta(b) = \delta\big(\partial\beta\cap b\cdot\eta - \mathrm{DA}[\eta]_\beta(b)\big)$ within $F^k\,\mathbf{CC}_\beta$. By the third item of the induction hypothesis, $\partial\beta\cap b\cdot\eta - \mathrm{DA}[\eta]_\beta(b)$ restricts to zero in $F^{k-1}\,\mathbf{CC}_\beta$. Notice also that $\mathrm{DA}[\eta]_\beta(b)$ restricts to zero in $\mathbf{CC}_{k,\beta}$ as $\eta_{k+1}=0$. In conclusion, it follows from (47) that $\mathrm{DA}[\psi]_{k,\beta}(b) = \bar\delta(\partial\beta\cap b\cdot\eta - \mathrm{DA}[\eta]_\beta(b))_k = \partial\beta\cap b\cdot\bar\delta\eta_k$, i.e.

$$\psi_{k+1}(b,x_1,\dots,x_k)+\cdots+\psi_{k+1}(x_1,\dots,x_k,b) = \partial\beta\cap b\cdot(\bar\delta\eta_k)(x_1,\dots,x_k) \quad (48)$$

Before we continue, we prove a sublemma which will be used.

Sublemma 3.14. Fix $u=(u_0,\dots,u_k,0,0\dots)$ so that (E1) holds for (u_i,u_{i+1}) with $0\le i\le k-1$ and that (E2) and (E3) hold for each u_i with $0\le i\le k$. Then there exists a $(k+1)$-multi-linear map u_{k+1} so that (E1) also holds for (u_k,u_{k+1}) and that (E2) and (E3) also hold for u_{k+1}. Moreover, if $fu_i=0$ for all $0\le i\le k$ then $fu_{k+1}=0$; if $u_i f^{\otimes i}=0$ for all $0\le i\le k$ then $u_{k+1}f^{\otimes(k+1)}=0$

Proof of the sublemma. We define $(\partial\beta,x)=\partial\beta\cap x$ if $\deg x = 1$ and define $(\partial\beta,x)$ to be zero if otherwise. We also introduce the following N-multi-linear map

$$u_N^{(m)}(x_1,\dots,x_N) = \textstyle\sum_{i=0}^{N-1} u_{N-m}(x_{i+1},\dots,x_{i+N-m})\cdot(\partial\beta,x_{i+N-m+1})\cdots(\partial\beta,x_{i+N})$$

where $N,m\in\mathbb{N}$ are so that u_{N-m} is already given, i.e. $0\le N-m\le k$, and where the x_j's are N arbitrary inputs with the subscript is modulo N, i.e. $x_{j+N}\equiv x_j$. Observe that $u_N^{(0)} = u_N$ and that these maps are cyclical in the sense $u_N^{(m)}(x_1,\dots,x_N) = u_N^{(m)}(x_{i+1},\dots,x_{i+N})$ for any i. In particular,

$u_N^{(N)}(x_1,\dots,x_N) = N(\partial\beta, x_1)\cdots(\partial\beta, x_N)\cdot u_0$. Now, we put

$$u_{k+1} := \sum_{m=1}^{k} a_m u_{k+1}^{(m)} + a_{k+1}\ u_{k+1}^{(k+1)}$$

for undetermined coefficients a_m. Note that the u_{k+1} depends only on $u_0, u_1, \dots, u_k$.
We compute

$$u_{k+1}^{(k+1)}(b, x_1,\dots,x_k) + \cdots + u_{k+1}^{(k+1)}(x_1,\dots,x_k,b) = \partial\beta\cap b\cdot(k+1)u_k^{(k)}(x_1,\dots,x_k)$$
$$u_{k+1}^{(m)}(b, x_1,\dots,x_k) + \cdots + u_{k+1}^{(m)}(x_1,\dots,x_k,b) = \partial\beta\cap b\cdot\Big(m\ u_k^{(m-1)}(x_1,\dots,x_k) + u_k^{(m)}(x_1,\dots,x_k)\Big)$$

for every $1\le m\le k$, and then

$$\sum_{i=1}^{k+1} u_{k+1}(x_1,\dots,x_{i-1},b,x_i,\dots,x_k) = \partial\beta\cap b\big(a_1 u_k^{(0)}(x_1,\dots,x_k) + \sum_{m=1}^{k}\big(a_m+(m+1)a_{m+1}\big)u_k^{(m)}(x_1,\dots,x_k)\big)$$

Choose $a_m = \frac{(-1)^{m-1}}{m!}$, then $a_1 = 1$ and $a_m + (m+1)a_{m+1} = 0$. It follows that (u_k, u_{k+1}) satisfies (E1). Moreover, since this construction is cyclic, the cyclical unitality holds as well. In fact, we compute

$$\sum_{j=1}^{k} u_{k+1}^{(m)}(x_1^\#,\dots,x_{j-1}^\#,\mathbf{e},x_{j+1},\dots,x_{k+1})$$
$$=\sum_{i=0}^{k-1}(-1)^{\epsilon_i}\sum_{i\le j\le i+k+1-m} u_{k+1-m}(x_{i+1}^\#,\dots,x_{j-1}^\#,\mathbf{e},x_{j+1},x_{i+k+1-m})\cdot(\partial\beta,x_{i+k+2-m})\cdots(\partial\beta,x_{i+k+1})$$

where $\epsilon_i = \sum_{a=1}^{i}(\deg x_a + 1)$ and we think $i,j\in\mathbb{Z}/(k+1)\mathbb{Z}$. The condition $i\le j\le i+k+1-m$ required above comes from the fact that $(\partial\beta,\mathbf{e}) = 0$. For each fixed i, the summation over j gives zero, since u_{k+1-m} satisfies (E2). Hence $u_{k+1}^{(m)}$ and also u_{k+1} satisfy (E2) as we require. As for (E3), recall that we have assumed $\beta\neq 0$, so $u_j(\dots,\mathbb{1},\dots) = 0$ even if $j=1$; furthermore, $(\partial\beta,\mathbb{1}) = 0$ by definition, hence one can easily check that u_{k+1} obeys (E3) as well. Finally, the last statement is straightforward to check. Indeed, if $fu_i = 0$ for $0\le i\le k$, then $fu_{k+1}^{(m)} = 0$ for $1\le m\le k+1$, thus $fu_{k+1} = 0$. Similarly, if $u_i f^{\otimes i} = 0$ for $0\le i\le k$, then because f satisfies (28), we conclude $u_{k+1}^{(m)} f^{\otimes(k+1)} = (u\circ f)_{k+1}^{(m)}$ which also vanishes for $1\le m\le k+1$. □

Back to the proof of Proposition 3.12 (ii). Applying Sublemma 3.14 to $\eta = (\eta_0,\dots,\eta_k,0,0,\dots)$ provides some χ_{k+1} so that $f\chi_{k+1} = 0$, the (E2) and (E3) hold for χ_{k+1}, and

$$\chi_{k+1}(b,x_1,\dots,x_k) + \cdots + \chi_{k+1}(x_1,\dots,x_k,b) = \partial\beta\cap b\cdot\eta_k(x_1,\dots,x_k) \quad (49)$$

for any divisor input b. By (48) and (49), we see that $\psi_{k+1} - \bar\delta\chi_{k+1}\in\mathbf{CC}_{k+1,0}^{\mathrm{ud}}$. Observe that $\bar\delta(\psi_{k+1}-\bar\delta\chi_{k+1}) = 0$ and $f_*(\psi_{k+1}-\bar\delta\chi_{k+1}) = 0$. By Proposition 3.5

(ii), the f_* is a quasi-isomorphism between the cochain complexes $(\mathbf{CC}^{\mathrm{ud}}_{k+1,0}, \bar\delta)$. It follows that there exists some $\theta \in \mathbf{CC}^{\mathrm{ud}}_{k+1,0}$ so that

$$\psi_{k+1} - \bar\delta\chi_{k+1} = \bar\delta\theta \tag{50}$$

By construction, $f\theta \in \mathbf{CC}^{\mathrm{ud}}_{k+1,0}$ is $\bar\delta$-closed. Hence, there exists some $\bar\delta$-closed $\theta' \in \mathbf{CC}^{\mathrm{ud}}_0$ so that $f\theta' = f\theta + \bar\delta\alpha$ for some $\alpha \in \mathbf{CC}^{\mathrm{ud}}_0$. Since $f \circ h = \mathrm{id}$, we have $f\bar\delta h\alpha = \bar\delta\alpha$. Then, replacing θ' by $\theta' - \bar\delta h\alpha$, one can require $\alpha = 0$. So, $f\theta' = f\theta$. Moreover, one can replace θ by $\theta - \theta'$ (which lives in $\theta + \ker\bar\delta$) without affecting (50). In summary, we may assume

$$f\theta = 0$$

in addition to (50). Now, the inductive step can be completed by putting $\eta_{k+1} = \theta + \chi_{k+1}$, $\eta^+ = (\eta_0, \dots, \eta_k, \eta_{k+1}, 0, 0, \dots) = \eta + \eta_{k+1}$ and $\psi^+ = \varphi - \delta\eta^+$. We check the four conditions in the induction hypothesis for them as follows:

- $\psi^+_{k+1} = \big(\psi - \delta(\eta^+ - \eta)\big)_{k+1} = \psi_{k+1} - \bar\delta\eta_{k+1} = 0$ holds by (50), and $\psi^+_j = \psi_j = 0$ for $j \le k$.
- $f\eta_{k+1} = f\theta + f\chi_{k+1} = 0$.
- It remains to show (E1) holds for the new pair (η_k, η_{k+1}), which is true thanks to (49).
- The η_{k+1} satisfies (E2) and (E3) since so so χ_{k+1} and θ.

By induction, the proof of (ii) is now established. Finally, the item (iii) can be proved almost identically and we omit it. We complete the proof of Proposition 3.12. $\square$

Proposition 3.15. *Let $\bar{\mathfrak f}^0 = (\mathfrak f^0_{k,0})_k$ and $\bar{\mathfrak f}^1 = (\mathfrak f^1_{k,0})_k$ be two A_∞ homomorphisms (modulo $T^{E=0}$) in $\mathscr U\mathscr D$. There exists an induced cochain map*

$$(\bar{\mathfrak f}^0, \bar{\mathfrak f}^1)_* := (\bar{\mathfrak f}^0, \bar{\mathfrak f}^1)^{\bar{\mathfrak g}}_* : (\mathbf{CC}^{\mathrm{ud}}_\beta, \delta_{\bar{\mathfrak g}}) \to (\mathbf{CC}^{\mathrm{ud}}_\beta, \delta_{\bar{\mathfrak f}^1 \circ \bar{\mathfrak g} \circ \bar{\mathfrak f}^0}) \tag{51}$$

Moreover, we have the associativity in the sense that $(\bar{\mathfrak f}^2, \bar{\mathfrak f}^3)_* \circ (\bar{\mathfrak f}^0, \bar{\mathfrak f}^1)_* = (\bar{\mathfrak f}^0 \circ \bar{\mathfrak f}^2, \bar{\mathfrak f}^3 \circ \bar{\mathfrak f}^1)_*$.

Proof. We start with the definition: let $\varphi = (\varphi_{k,\beta})_k$ and $\deg' \varphi_{k,\beta} = p$ for a fixed p, we define $(\bar{\mathfrak f}^0, \bar{\mathfrak f}^1)^{\bar{\mathfrak g}}_* \varphi$ to be the operator system as follows:

$$\sum \bar{\mathfrak f}^1 \circ (\bar{\mathfrak g}^{\# p} \otimes \cdots \otimes \bar{\mathfrak g}^{\# p} \otimes \varphi \otimes \bar{\mathfrak g} \otimes \cdots \otimes \bar{\mathfrak g}) \circ (\bar{\mathfrak f}^0 \otimes \cdots \otimes \bar{\mathfrak f}^0)$$

If $\bar{\mathfrak f}^0 = \mathrm{id}$ or $\bar{\mathfrak f}^1 = \mathrm{id}$, we respectively have that

$$(\bar{\mathfrak f}^0, \mathrm{id})^{\bar{\mathfrak g}}_* \varphi = \varphi \circ (\bar{\mathfrak f}^0 \otimes \cdots \otimes \bar{\mathfrak f}^0)$$
$$(\mathrm{id}, \bar{\mathfrak f}^1)^{\bar{\mathfrak g}}_* \varphi = \bar{\mathfrak f}^1 \circ (\bar{\mathfrak g}^{\# p} \otimes \cdots \otimes \bar{\mathfrak g}^{\# p} \otimes \varphi \otimes \bar{\mathfrak g} \otimes \cdots \otimes \bar{\mathfrak g})$$

Remark that although generally (51) depends on $\bar{\mathfrak{g}}$, the special case $(\bar{\mathfrak{f}}^0, \mathrm{id})^{\bar{\mathfrak{g}}}_*$ does not. Since $\deg' \bar{\mathfrak{f}}^0 = 0$, we know $\deg'(\bar{\mathfrak{f}}^0, \mathrm{id})_*\varphi = \deg' \varphi = p$. A direct computation shows that

$$(\mathrm{id}, \bar{\mathfrak{f}}^1)^{\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}^0}_* \circ (\bar{\mathfrak{f}}^0, \mathrm{id})_* = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)^{\bar{\mathfrak{g}}}_*, \quad \text{and} \quad (\bar{\mathfrak{f}}^0, \mathrm{id})^{\bar{\mathfrak{f}}^1\circ\bar{\mathfrak{g}}^0}_* \circ (\mathrm{id}, \bar{\mathfrak{f}}^1)^{\bar{\mathfrak{g}}}_* = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* \tag{52}$$

Because of this decomposition, it suffices to prove Proposition 3.15 for $(\mathrm{id}, \bar{\mathfrak{f}}^1)_*$ and $(\bar{\mathfrak{f}}^0, \mathrm{id})_*$ separately. Firstly, as $\bar{\mathfrak{f}}^0$, $\bar{\mathfrak{f}}^1$ and $\bar{\mathfrak{g}}$ satisfy the divisor axiom, we have

$$\mathrm{DA}[(\bar{\mathfrak{f}}^0, \mathrm{id})_*\varphi]_{k,\beta}(b; x_1, \dots, x_k) = \sum \mathrm{DA}[\varphi]_{k_1,\beta}\big(\mathfrak{f}^0_{1,0}(b); \bar{\mathfrak{f}}^0(x_1, \dots), \dots, \bar{\mathfrak{f}}^0(\dots, x_k)\big)$$

$$\mathrm{DA}[(\mathrm{id}, \bar{\mathfrak{f}}^1)_*\varphi]_{k,\beta}(b; x_1, \dots, x_k) = \sum \bar{\mathfrak{f}}^1 \circ \big(\bar{\mathfrak{g}}^{\# p}(x_1, \dots), \dots, \mathrm{DA}[\varphi]_{k_1,\beta}(b; \dots), \dots, \bar{\mathfrak{g}}(\dots, x_k)\big)$$

Hence, we have the desired divisor axiom. Similarly, we can show the unitality and the cyclical unitalities. To sum up, both $(\mathrm{id}, \bar{\mathfrak{f}}^1)_*$ and $(\bar{\mathfrak{f}}^0, \mathrm{id})_*$ map $\mathbf{CC}^{\mathrm{ud}}_\beta$ into $\mathbf{CC}^{\mathrm{ud}}_\beta$. In the next, we want to check

$$\delta(\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_*(\varphi) = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_*(\delta\varphi) \quad \text{or more precisely} \quad \delta_{\bar{\mathfrak{f}}^1\circ\bar{\mathfrak{g}}\circ\bar{\mathfrak{f}}^0}(\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_*(\varphi) = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_*(\delta_{\bar{\mathfrak{g}}}\varphi) \tag{53}$$

Without loss of generality, we assume $\deg' \varphi = p$. Consider

$$F_1 := \bar{\mathfrak{f}}^1 \circ (\mathrm{id}^\bullet_\# \otimes \bar{\mathfrak{m}} \otimes \mathrm{id}^\bullet) \circ (\bar{\mathfrak{g}}^{\# p} \otimes \cdots \otimes \bar{\mathfrak{g}}^{\# p} \otimes \varphi \otimes \bar{\mathfrak{g}} \otimes \cdots \bar{\mathfrak{g}}) \circ (\bar{\mathfrak{f}}^0 \otimes \cdots \bar{\mathfrak{f}}^0)$$
$$F_2 := \bar{\mathfrak{f}}^1 \circ (\bar{\mathfrak{g}}^{\# p} \otimes \cdots \otimes \bar{\mathfrak{g}}^{\# p} \otimes \varphi \otimes \bar{\mathfrak{g}} \otimes \cdots \bar{\mathfrak{g}}) \circ (\mathrm{id}^\bullet_\# \otimes \bar{\mathfrak{m}} \otimes \mathrm{id}^\bullet) \circ (\bar{\mathfrak{f}}^0 \otimes \cdots \bar{\mathfrak{f}}^0)$$

and put $F = F_1 - F_2$. On the one hand, because the $\bar{\mathfrak{f}}^1$ and $\bar{\mathfrak{f}}^0$ are A_∞ homomorphisms, one can show that the F exactly agrees with the left side of (53). On the other hand, as $\bar{\mathfrak{g}}$ is an A_∞ homomorphism, one can show the F agrees with the right side of (53). The proof of (51) is complete. Finally, it is clear that $(\bar{\mathfrak{f}}^2, \mathrm{id})_*(\bar{\mathfrak{f}}^0, \mathrm{id})_* = (\bar{\mathfrak{f}}^2 \circ \bar{\mathfrak{f}}^0, \mathrm{id})_*$ and $(\mathrm{id}, \bar{\mathfrak{f}}^3)_*(\mathrm{id}, \bar{\mathfrak{f}}^1)_* = (\mathrm{id}, \bar{\mathfrak{f}}^3 \circ \bar{\mathfrak{f}}^1)_*$, then the associativity in general, i.e. $(\bar{\mathfrak{f}}^2, \bar{\mathfrak{f}}^3)_* \circ (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* = (\bar{\mathfrak{f}}^0 \circ \bar{\mathfrak{f}}^2, \bar{\mathfrak{f}}^3 \circ \bar{\mathfrak{f}}^1)_*$, can be proved by the decomposition in (52). □

By declaring $F^E\,\mathbf{CC} := \prod_{\beta\in\mathfrak{G}; E(\beta)<E} \mathbf{CC}_\beta = \prod_{k\in\mathbb{N}} \prod_{E(\beta)<E} \mathbf{CC}_{k,\beta}$ we obtain the so-called *energy filtration* on $\mathbf{CC}$. We remark that since it may happen that $\partial\beta \cap b \neq \partial\beta' \cap b$ even if $E(\beta) = E(\beta')$, we cannot directly follow [FOOO10b]. This gives one reason why we have to consider a copy $\mathbf{CC}_\beta$ separately, one for each $\beta \in \mathfrak{G}$ (Definition 2.5). For every $\mathsf{B} \in \mathfrak{G}$, we define

$$F^{\mathsf{B}}\,\mathbf{CC} = F^{E(\mathsf{B})}\,\mathbf{CC} \times \mathbf{CC}_{\mathsf{B}} = \prod_{E(\beta)<E(\mathsf{B})} \mathbf{CC}_\beta \times \mathbf{CC}_{\mathsf{B}} \tag{54}$$

Similarly, we can define $F^E\,\mathbf{CC}^{\mathrm{ud}}$ and $F^{\mathsf{B}}\,\mathbf{CC}^{\mathrm{ud}}$ adding the conditions of unitality, cyclical unitality, and divisor axiom. Now, in analogy to Theorem 3.7, we can prove the following:

Theorem 3.16. *Fix $\mathsf{B} \in \mathfrak{G}$ with $E(\mathsf{B}) > 0$, and fix $(C_1, \mathfrak{m}'), (C_2, \mathfrak{m}) \in \mathrm{Obj}\,\mathscr{U}\mathscr{D}$. Suppose $\mathfrak{g} = (\mathfrak{g}_{k,\beta})_{k,\beta} \in F^{E(\mathsf{B})}\,\mathbf{CC}^{\mathrm{ud}}(C_1, C_2)$ is an A_∞ homomorphism modulo $T^{E(\mathsf{B})}$ so that (28) holds for $\mathfrak{g}_{1,0}$. Then, there exists a $\delta_{\bar{\mathfrak{g}}}$-closed*

$$\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}(C_1, C_2)$$

such that its cohomology $[\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})]$ vanishes if and only if $\mathfrak{g}$ can be extended to an $A_{\infty,\mathsf{B}}$ homomorphism $\mathfrak{g}^+ = (\mathfrak{g}, \mathfrak{g}_{\mathsf{B}})$. Moreover, in this case, we have $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) + \delta_{\bar{\mathfrak{g}}}(\mathfrak{g}_{\mathsf{B}}) = 0$.

Proof. Define the k-th component (or more precisely, the $\mathbf{CC}_{k,\mathsf{B}}$-component) of $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})$ to be

$$\sum_{\substack{0=j_0\le\cdots\le j_\ell=k\\ \beta_0+\beta_1+\cdots+\beta_\ell=\mathsf{B}\\ \forall i\neq 0:\beta_i\neq\mathsf{B}}} \mathfrak{m}_{\ell,\beta_0}\circ(\mathfrak{g}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{g}_{j_\ell-j_{\ell-1},\beta_\ell}) - \sum_{\substack{\lambda+\mu+\nu=k\\ \beta'+\beta''=\mathsf{B}\\ \beta'\neq\mathsf{B}}} \mathfrak{g}_{\lambda+\mu+1,\beta'}\circ(\mathrm{id}_{\#}^{\lambda}\otimes\mathfrak{m}'_{\nu,\beta''}\otimes\mathrm{id}^{\mu}) \tag{55}$$

Firstly, parallel to the proof of Theorem 3.7, we can show $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) \in \mathbf{CC}_{\mathsf{B}}^{\mathrm{ud}}$ using the conditions that $\mathfrak{m}'$, $\mathfrak{m}$ and $\mathfrak{g}$ are all contained in $\mathbf{CC}^{\mathrm{ud}}$. Next, being an $A_{\infty,\mathsf{B}}$ homomorphism exactly means that

$$\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) + \delta_{\bar{\mathfrak{g}}}(\mathfrak{g}_{\mathsf{B}}) = 0 \tag{56}$$

Suppose we could find $\mathfrak{g}_{\mathsf{B}}$ to solve this, and we would like to study the degrees beforehand. In reality, due to (55) above, the degree of an arbitrary term in $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})$ can be computed as follows:

$$\deg\mathfrak{m}_{\ell,\beta_0}+\sum\deg\mathfrak{g}_{j_i-j_{i-1},\beta_i} = 2-\ell-\mu(\beta_0)+\sum_{i=1}^{\ell}\big(1-(j_i-j_{i-1})-\mu(\beta_i)\big) = 2-k-\mu(\mathsf{B})$$

$$\deg\mathfrak{g}_{\lambda+\mu+1,\beta'} + \deg m'_{\nu,\beta''} = 1-\lambda-\mu-\mu(\beta')+2-\nu-\mu(\beta'') = 2-k-\mu(\mathsf{B})$$

This is exactly what we desire. Besides, since the $\delta_{\bar{\mathfrak{g}}}$ in (45) has degree one, the degree of $\mathfrak{g}_{k,B}$ must equal to $1-k-\mu(\mathsf{B})$ as we desire. Beware that we use the 'un-shifted' degree here. In terms of shifted degrees (20), we actually have $\deg'\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) = 1-\mu(\mathsf{B}) = 1(\mathrm{mod}\ 2)$ and $\deg'\mathfrak{g}_{\mathsf{B}} = -\mu(\mathsf{B}) = 0(\mathrm{mod}\ 2)$. Now, it remains to check $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})$ is δ-closed. By (45), we compute

$$\begin{aligned}\delta(\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})) &= \sum\mathfrak{m}_{r,0}\circ(\bar{\mathfrak{g}}^{\#}\otimes\cdots\otimes\bar{\mathfrak{g}}^{\#}\otimes\mathfrak{m}_{\ell,\beta_0}(\mathfrak{g}_{\beta_1}\otimes\cdots\otimes\mathfrak{g}_{\beta_\ell})\otimes\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\\ &+\sum\mathfrak{m}_{\ell,\beta_0}\circ(\mathfrak{g}_{\beta_1}\otimes\cdots\otimes\mathfrak{g}_{\beta_\ell})\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}'_{s,0}\otimes\mathrm{id}^{\bullet})\\ &-\sum\mathfrak{m}_{r,0}\circ(\bar{\mathfrak{g}}^{\#}\otimes\cdots\otimes\bar{\mathfrak{g}}^{\#}\otimes\mathfrak{g}_{\beta'}\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}'_{\nu,\beta''}\otimes\mathrm{id}^{\bullet})\otimes\bar{\mathfrak{g}}\otimes\cdots\otimes\bar{\mathfrak{g}})\\ &-\sum\mathfrak{g}_{\beta'}\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}'_{\nu,\beta''}\otimes\mathrm{id}^{\bullet})\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}'_{s,0}\otimes\mathrm{id}^{\bullet}) \qquad =: \Delta_1+\Delta_2-\Delta_3-\Delta_4\end{aligned}$$

As in (55), we need the conditions like $\beta_i\neq\mathsf{B}$ ($i\neq 0$) and $\beta'\neq\mathsf{B}$ (or equivalently $\beta''\neq 0$) in the summations, but for simplicity it may be helpful to add a ghost term $\mathfrak{g}_{\mathsf{B}}=0$. Now, we consider

$$\Delta := \sum_{\gamma+\gamma'+\beta_1+\cdots+\beta_r=\mathsf{B}} \mathfrak{m}_{r,\gamma}\circ(\mathfrak{g}_{\beta_1}\otimes\cdots\otimes\mathfrak{g}_{\beta_r})\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}'_{s,\gamma'}\otimes\mathrm{id}^{\bullet})$$

First, separating the terms with $\gamma\neq 0$ and $\gamma=0$ and using that $\mathfrak{g}$ is an A_∞ homomorphism modulo $T^{E(\mathsf{B})}$, we obtain $\Delta=\Gamma_1+\Gamma_2$ where

$$\begin{aligned}\Gamma_1 &:= \sum_{\gamma\neq 0}\mathfrak{m}_{r,\gamma}\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}\otimes\mathrm{id}^{\bullet})\circ(\mathfrak{g}\otimes\cdots\otimes\mathfrak{g})\\ \Gamma_2 &:= \sum\mathfrak{m}_{r,0}\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}\otimes\mathrm{id}^{\bullet})\circ(\mathfrak{g}\otimes\cdots\otimes\mathfrak{g})-\Delta_1+\Delta_3\end{aligned}$$

Next, separating the terms with $\gamma' \neq 0$ and those with $\gamma' = 0$, we also see that $\Delta = \Gamma'_1 + \Gamma'_2$, where

$$\Gamma'_1 := \sum \mathfrak{g} \circ (\mathrm{id}^\bullet_\# \otimes \mathfrak{m}' \otimes \mathrm{id}^\bullet) \circ (\mathrm{id}^\bullet_\# \otimes \mathfrak{m}' \otimes \mathrm{id}^\bullet) - \Delta_4; \qquad \Gamma'_2 := \Delta_2$$

In the end, the A_∞ equations for $\mathfrak{m}$ and $\mathfrak{m}'$ tells respectively that $\Gamma_1 + \Gamma_2 = -\Delta_1 + \Delta_3$ and $\Gamma'_1 + \Gamma'_2 = \Delta_2 - \Delta_4$. So $\delta(\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})) = (\Delta_1 - \Delta_3) + (\Delta_2 - \Delta_4) = -\Delta + \Delta = 0$. □

Lemma 3.17. *Let $\mathfrak{f}^0$ and $\mathfrak{f}^1$ be two $A_{\infty,\mathsf{B}}$ homomorphisms in $\mathscr{U}\mathscr{D}$. If $\mathfrak{g} \in F^{E(\mathsf{B})}\,\mathbf{CC}^{\mathrm{ud}}$ is any A_∞ homomorphism modulo $T^{E(\mathsf{B})}$, then so is $\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0$, and we have*

$$[\mathfrak{o}_{\mathsf{B}}(\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0)] = \big[(\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* \mathfrak{o}_{\mathsf{B}}(\mathfrak{g})\big] = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* [\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})]$$

where $\bar{\mathfrak{f}}^0$ and $\bar{\mathfrak{f}}^1$ are the reductions. Moreover, $\mathfrak{o}_{\mathsf{B}}(\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0) - (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* \mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) = \delta((\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0)_{\mathsf{B}})$.

Proof. The statement relies on Proposition 3.15; it is analogous to Lemma 3.8. Clearly, $\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0$ is also an A_∞ homomorphism modulo $T^{E(\mathsf{B})}$. As in the proof of Lemma 2.35, one can see $\mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0$ restricts to an element in $F^{E(\mathsf{B})}\,\mathbf{CC}^{\mathrm{ud}}$ as well. Hence, due to Theorem 3.16, the left side can be defined. Let's extend $\mathfrak{g}$ to the whole $\mathbf{CC}$ by adding zero ghost terms. Define $\mathfrak{h} := \mathfrak{f}^1 \circ \mathfrak{g} \circ \mathfrak{f}^0$, and then $\bar{\mathfrak{h}} = \bar{\mathfrak{f}}^1 \circ \bar{\mathfrak{g}} \circ \bar{\mathfrak{f}}^0$. We consider

$$\mathfrak{P} := \sum \bar{\mathfrak{m}} \circ (\mathfrak{h} \otimes \cdots \otimes \mathfrak{h}) - \sum \mathfrak{h} \circ (\mathrm{id}^\bullet_\# \otimes \bar{\mathfrak{m}} \otimes \mathrm{id}^\bullet) \in \mathbf{CC}_{\mathsf{B}}$$

First, it is easy to see that $\mathfrak{P} = \mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) + \delta_{\bar{\mathfrak{h}}}(\mathfrak{h}_{\mathsf{B}})$, where $\mathfrak{h}_{\mathsf{B}} \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}$. Next, we expand $\mathfrak{P}$:

$$\begin{aligned}\mathfrak{P} =& \sum \mathfrak{f}^1 \circ (\mathrm{id}^\bullet_\# \otimes \mathfrak{m} \otimes \mathrm{id}^\bullet) \circ (\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}) \circ (\mathfrak{f}^0 \otimes \cdots \otimes \mathfrak{f}^0) - \sum \mathfrak{f}^1 \circ (\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}) \circ (\mathrm{id}^\bullet_\# \otimes \mathfrak{m} \otimes \mathrm{id}^\bullet) \circ (\mathfrak{f}^0 \otimes \cdots \otimes \mathfrak{f}^0) \\ =& \sum \bar{\mathfrak{f}}^1 \circ \Big(\bar{\mathfrak{g}}^\# \otimes \cdots \otimes \bar{\mathfrak{g}}^\# \otimes \big[\mathfrak{m}(\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}) - \sum \mathfrak{g}(\mathrm{id}^\bullet_\# \otimes \mathfrak{m} \otimes \mathrm{id}^\bullet)\big] \otimes \bar{\mathfrak{g}} \otimes \cdots \otimes \bar{\mathfrak{g}}\Big) \circ (\bar{\mathfrak{f}}^0 \otimes \cdots \otimes \bar{\mathfrak{f}}^0)\end{aligned}$$

Here the terms in the square bracket are elements in $\mathbf{CC}_{\mathsf{B}}$; they agree with $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})$. So, we also obtain $\mathfrak{P} = (\bar{\mathfrak{f}}^0, \bar{\mathfrak{f}}^1)_* \mathfrak{o}_{\mathsf{B}}(\mathfrak{g})$. By comparison, the proof is now complete. □

3.4 Extension for the energy filtration

We are now at the stage to finally prove the most general Theorem 3.1. Recall that using the reduction $\bar{\mathfrak{f}}$ of the fixed $\mathfrak{f} \in \mathrm{Hom}_{\mathscr{U}\mathscr{D}}((C', \mathfrak{m}'), (C, \mathfrak{m}))$ in (33), we have constructed $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$ in Theorem 3.9. They serve as the base case in the energy-zero level. Next, we perform the inductive steps for the energy filtration by the following analogy of Proposition 3.10.

Proposition 3.18. *Let $\mathsf{B} \in \mathfrak{G}$ is so that $E(\mathsf{B}) > 0$. Suppose*

$$\mathfrak{g} = (\mathfrak{g}_{k,\beta}) \in F^{E(\mathsf{B})}\,\mathbf{CC}^{\mathrm{ud}}(C, C'), \quad \textit{and} \quad \mathfrak{h} = (\mathfrak{h}_{k,\beta}) \in F^{E(\mathsf{B})}\,\mathbf{CC}^{\mathrm{ud}}(C, C'_{[0,1]})$$

are two A_∞ homomorphisms modulo $T^{E(\mathsf{B})}$ such that

$$\mathrm{Eval}^0 \circ \mathfrak{h}|_{F^{E(\mathsf{B})}\mathbf{CC}} = \mathrm{id}, \quad \textit{and} \quad \mathrm{Eval}^1 \circ \mathfrak{h}|_{F^{E(\mathsf{B})}\mathbf{CC}} = \mathfrak{g}\circ\mathfrak{f}$$

Then, there exists two $A_{\infty,\mathsf{B}}$ homomorphisms $\mathfrak{g}^+ = (\mathfrak{g}, \mathfrak{g}_{\mathsf{B}}) \in F^{\mathsf{B}}\mathbf{CC}^{\mathrm{ud}}$ and $\mathfrak{h}^+ = (\mathfrak{h}, \mathfrak{h}_{\mathsf{B}}) \in F^{\mathsf{B}}\mathbf{CC}^{\mathrm{ud}}$ extending the $\mathfrak{g}$ and $\mathfrak{h}$ and satisfying the following property:

$$\mathrm{Eval}^0 \circ \mathfrak{h}^+|_{F^{\mathsf{B}}\mathbf{CC}} = \mathrm{id}, \quad \textit{and} \quad \mathrm{Eval}^1 \circ \mathfrak{h}^+|_{F^{\mathsf{B}}\mathbf{CC}} = \mathfrak{g}^+\circ\mathfrak{f} \tag{57}$$

Proof. We think of $\mathfrak{g}$ and $\mathfrak{h}$ as elements in $F^{\mathsf{B}}\mathbf{CC}$ (54) by setting zeros in the component $\mathbf{CC}_{\mathsf{B}}$. By Lemma 3.17, we have $(\mathrm{id}, \mathrm{Eval}^0)_*[\mathfrak{o}_{\mathsf{B}}(\mathfrak{h})] = [\mathfrak{o}_{\mathsf{B}}(\mathrm{Eval}^0\circ\mathfrak{h})] = [\mathfrak{o}_{\mathsf{B}}(\mathrm{id})] = 0$; by Proposition 3.12 (ii), we have $[\mathfrak{o}_{\mathsf{B}}(\mathfrak{h})] = 0$. By Theorem 3.16, we have some $\alpha \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}$ so that $\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) + \delta\alpha = 0$.

To ensure (57), we cannot set $\mathfrak{h}_{\mathsf{B}} = \alpha$ directly, and we need a slight modification of α as before. Firstly, the condition $\mathrm{Eval}^0\circ\mathfrak{h}|_{F^{E}(\mathsf{B})\mathbf{CC}} = \mathrm{id}$ implies that $\mathfrak{o}_{\mathsf{B}}(\mathrm{Eval}^0\circ\mathfrak{h}) = \mathfrak{o}_{\mathsf{B}}(\mathrm{id}) = 0$ by (55). Then, due to Lemma 3.17, we have $\mathfrak{o}_{\mathsf{B}}(\mathrm{Eval}^0\circ\mathfrak{h}) - (\mathrm{id}, \mathrm{Eval}^0)_*\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) = 0$, and thus $(\mathrm{id}, \mathrm{Eval}^0)_*\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) = 0$. Besides, using Proposition 3.12 yields that $(\mathrm{id}, \mathrm{Eval}^0)_*\delta\alpha = \delta(\mathrm{id}, \mathrm{Eval}^0)_*\alpha = \delta(\mathrm{Eval}^0\circ\alpha)$. So, we have $0 = (\mathrm{id}, \mathrm{Eval}^0)_*(\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) + \delta\alpha) = \delta(\mathrm{Eval}^0\circ\alpha)$. Now, we define:

$$\mathfrak{h}_{\mathsf{B}} = \alpha - \mathrm{Incl}\,\mathrm{Eval}^0\circ\alpha$$

Then, we have $\mathrm{Eval}^0\circ\mathfrak{h}_{\mathsf{B}} = 0$, thus, the $\mathfrak{h}^+ = (\mathfrak{h}, \mathfrak{h}_{\mathsf{B}})$ satisfies the first half of (57). Besides, due to (45), we can show that $\delta\mathfrak{h}_{\mathsf{B}} = \delta\alpha - \mathrm{Incl}\,\delta\,\mathrm{Eval}^0\circ\alpha = \delta\alpha$, and hence $\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) + \delta\mathfrak{h}_{\mathsf{B}} = 0$. In other words, setting $\mathfrak{h}^+ = (\mathfrak{h}, \mathfrak{h}_{\mathsf{B}})$ supplies an $A_{\infty,\mathsf{B}}$ extension of $\mathfrak{h}$.

The composition $\mathrm{Eval}^1\circ\mathfrak{h}^+$ is also an $A_{\infty,\mathsf{B}}$ homomorphism which is exactly an extension of $\mathfrak{g}\circ\mathfrak{f}$. It follows that $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}\circ\mathfrak{f}) = \mathfrak{o}_{\mathsf{B}}(\mathrm{Eval}^1\circ\mathfrak{h}) = \mathrm{Eval}^1\circ\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) = -\mathrm{Eval}^1\delta(\mathfrak{h}_{\mathsf{B}}) = -\delta\,\mathrm{Eval}^1\mathfrak{h}_{\mathsf{B}}$. Meanwhile, by Lemma 3.17 and Proposition 3.12 (iii), we also conclude $0 = [\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}\circ\mathfrak{f})] = (\mathfrak{f}, \mathrm{id})_*[\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})]$ and $[\mathfrak{o}_{\mathsf{B}}(\mathfrak{g})] = 0$. By Theorem 3.16, the $\mathfrak{g}$ can be extended to some $\mathfrak{g}^+ = (\mathfrak{g}, \mathfrak{g}_{\mathsf{B}})$ so that $\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) = -\delta\mathfrak{g}_{\mathsf{B}}$. Unfortunately, we cannot ensure the following vanishes:

$$\Pi := (\mathrm{Eval}^1\circ\mathfrak{h}^+ - \mathfrak{g}^+\circ\mathfrak{f})_{\mathsf{B}} = \mathrm{Eval}^1\circ\mathfrak{h}_{\mathsf{B}} - (\bar{\mathfrak{f}}, \mathrm{id})_*\mathfrak{g}_{\mathsf{B}} - \textstyle\sum_{\beta\neq\mathsf{B}}\mathfrak{g}_\beta\circ(\mathfrak{f}\otimes\cdots\otimes\mathfrak{f})$$

So, we need to further modify $\mathfrak{g}_{\mathsf{B}}$ and $\mathfrak{h}_{\mathsf{B}}$. To begin with, observe that since $\mathfrak{f}, \mathfrak{g}^+, \mathfrak{h}^+ \in \mathbf{CC}^{\mathrm{ud}}$, we see that $\Pi \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}$. We compute:

$$\delta\Pi = -\mathrm{Eval}^1\circ\mathfrak{o}_{\mathsf{B}}(\mathfrak{h}) + (\bar{\mathfrak{f}}, \mathrm{id})_*\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) - \delta((\mathfrak{g}\circ\mathfrak{f})_{\mathsf{B}}) = -\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}\circ\mathfrak{f}) + (\bar{\mathfrak{f}}, \mathrm{id})_*\mathfrak{o}_{\mathsf{B}}(\mathfrak{g}) - \delta((\mathfrak{g}\circ\mathfrak{f})_{\mathsf{B}})$$

which vanishes precisely owe to Lemma 3.17.

Next, we claim that $(\bar{\mathfrak{f}}, \mathrm{id})_*$ is a quasi-isomorphism. In fact, by Proposition 3.12(iii), $(\mathrm{Incl}, \mathrm{id})_*$ is a quasi-isomorphism; so is $(\mathrm{Eval}^i, \mathrm{id})_*$ due to the fact $\mathrm{Eval}^i\circ\mathrm{Incl} = \mathrm{id}$ and Proposition 3.15. Also, observe that $(\mathrm{id}, \mathrm{id})_* = (\bar{\mathfrak{h}}, \mathrm{id})_*(\mathrm{Eval}^0, \mathrm{id})_*$ and that $(\bar{\mathfrak{f}}, \mathrm{id})_*(\bar{\mathfrak{g}}, \mathrm{id})_* = (\bar{\mathfrak{h}}, \mathrm{id})_*(\mathrm{Eval}^1, \mathrm{id})_*$. On the other hand, by Theorem 3.9, there is another ud-homotopy $\bar{\mathfrak{h}}'$ between $\bar{\mathfrak{f}}\circ\bar{\mathfrak{g}}$ and id; we similar conclude that $(\mathrm{id},\mathrm{id})_* = (\bar{\mathfrak{h}}',\mathrm{id})_*(\mathrm{Eval}^0,\mathrm{id})_*$ and $(\bar{\mathfrak{g}},\mathrm{id})_*(\bar{\mathfrak{f}},\mathrm{id})_* = (\bar{\mathfrak{h}}',\mathrm{id})_*(\mathrm{Eval}^1,\mathrm{id})_*$. Putting things together, we can show $(\bar{\mathfrak{f}},\mathrm{id})_*$ is a quasi-isomorphism.

Now, by the above claim, one can find some $\Delta\mathfrak{g} \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}(C,C')$ so that $\delta\Delta\mathfrak{g} = 0$ and $\Pi+\delta\eta = (\bar{\mathfrak{f}},\mathrm{id})_*\Delta\mathfrak{g}$ for some $\eta \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}$. Define $\Delta\mathfrak{h} \in \mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}(C',C'_{[0,1]})$ by $\Delta\mathfrak{h}(s) = 1\otimes s\eta$; we have $\mathrm{Eval}^0\circ\Delta\mathfrak{h} = 0$ and $\mathrm{Eval}^1\circ\Delta\mathfrak{h} = \eta$. Finally, we define $\mathfrak{h}'_{\mathsf{B}} := \mathfrak{h}_{\mathsf{B}} + \delta\Delta\mathfrak{h}$ and $\mathfrak{g}'_{\mathsf{B}} = \mathfrak{g}_{\mathsf{B}} + \Delta\mathfrak{g}$. Both are contained in $\mathbf{CC}^{\mathrm{ud}}_{\mathsf{B}}$ by construction, and these modifications also given the $A_{\infty,\mathsf{B}}$ extensions since $\delta\mathfrak{h}'_{\mathsf{B}} = \delta\mathfrak{h}_{\mathsf{B}}$ and $\delta\mathfrak{g}'_{\mathsf{B}} = \delta\mathfrak{g}_{\mathsf{B}}$. It remains to show the new extensions $\mathfrak{h}'^{+} = (\mathfrak{h},\mathfrak{h}'_{\mathsf{B}})$ and $\mathfrak{g}'^{+} = (\mathfrak{g},\mathfrak{g}'_{\mathsf{B}})$ satisfy (57). In fact, the first relation is clearly preserved, and the second holds, since $\Pi' := (\mathrm{Eval}^1\circ\mathfrak{h}'^{+} - \mathfrak{g}'^{+}\circ\mathfrak{f})_{\mathsf{B}} = \Pi + \mathrm{Eval}^1\circ(\delta\Delta\mathfrak{h}) - (\bar{\mathfrak{f}},\mathrm{id})_*\Delta\mathfrak{g} = \Pi + \delta\eta - (\bar{\mathfrak{f}},\mathrm{id})_*\Delta\mathfrak{g} = 0$. ☐

Proof of Theorem 3.1. Denote by $\mathfrak{f}_\beta$ the component of $\mathfrak{f}$ in $\mathbf{CC}_\beta$, and we define $\mathsf{G}_{\mathfrak{f}} := \{\beta \in \mathfrak{G} \mid \mathfrak{f}_\beta \neq 0\}$. One can similarly define $\mathsf{G}_{\mathfrak{m}}$ and $\mathsf{G}_{\mathfrak{m}'}$ for the A_∞ algebras $\mathfrak{m}$ and $\mathfrak{m}'$. Now, we put

$$\mathsf{G} = \mathbb{N}\cdot\mathsf{G}_{\mathfrak{f}} \cup \mathsf{G}_{\mathfrak{m}} \cup \mathsf{G}_{\mathfrak{m}'} \tag{58}$$

to be the additive monoid generated by $\mathsf{G}_{\mathfrak{f}}$, $\mathsf{G}_{\mathfrak{m}}$ and $\mathsf{G}_{\mathfrak{m}'}$. By the gappedness, the image $E(\mathsf{G})$ under the energy morphism $E(\mathrm{cot})$ is a discrete subset of $[0,\infty)$, say, $E(\mathsf{G}) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \lambda_{i+1} < \cdots\}$. Note that every $E^{-1}(\lambda_i)\cap\mathsf{G}$ is finite and $E^{-1}(0)\cap\mathsf{G} = \{0\}$. We aim to inductively construct $\mathfrak{g}$ and $\mathfrak{h}$ within the subspace

$$\mathbf{CC}^{(\infty)} := \prod_{j=0}^{\infty}\prod_{E(\beta)=\lambda_j}\mathbf{CC}^{\mathrm{ud}}_\beta \qquad \subset \mathbf{CC}_{\mathfrak{G}}$$

Suppose we already have two operator systems $\mathfrak{g}^i$ and $\mathfrak{h}^i$ in $\mathbf{CC}^{(i)} := \prod_{j=0}^{i}\prod_{\beta\in\mathsf{G}:E(\beta)=\lambda_j}\mathbf{CC}^{\mathrm{ud}}_\beta$ so that they are $A_{\infty,\beta}$ homomorphisms for every $\beta\in\mathfrak{G}$ with $E(\beta)=\lambda_i$ and the following identities hold in the space $F^{\lambda_i}\,\mathbf{CC}\times\prod_{\beta\in\mathsf{G}:E(\beta)=\lambda_i}\mathbf{CC}_\beta$:

$$\mathrm{Eval}^0\circ\mathfrak{h}^i = \mathrm{id}, \quad \text{and} \quad \mathrm{Eval}^1\circ\mathfrak{h}^i = \mathfrak{g}^i\circ\mathfrak{f} \tag{59}$$

By Theorem 3.9, we have the initial step for $i=0$. Suppose the $\mathfrak{g}^i$ and $\mathfrak{h}^i$ are given as above.

We first claim the two identities in (59) actually hold in the larger space $F^{\lambda_{i+1}}\,\mathbf{CC}$. In fact, pick an arbitrary β with $\lambda_i < E(\beta) < \lambda_{i+1}$; then $\beta\notin\mathsf{G}$. Then, since the $\mathfrak{h}^i$ lives in $\mathbf{CC}^{(i)}$, we have $(\mathrm{Eval}^{0,1}\circ\mathfrak{h}^i)_\beta = \mathrm{Eval}^{0,1}\circ\mathfrak{h}^i_\beta = 0$. First, it is clear that $(\mathrm{id})_\beta = 0$, and so $\mathrm{Eval}^0\circ\mathfrak{h}^i = \mathrm{id}$ in $F^{\lambda_{i+1}}\,\mathbf{CC}$. Second, $(\mathfrak{g}^i\circ\mathfrak{f})_\beta = \sum\mathfrak{g}^i_{\ell,\beta_0}\circ(\mathfrak{f}_{\beta_1}\otimes\cdots\otimes\mathfrak{f}_{\beta_\ell})$ must vanish, otherwise $\beta = \beta_0+\beta_1+\cdots+\beta_\ell\in\mathfrak{G}$. So, $\mathrm{Eval}^1\circ\mathfrak{h}^i = \mathfrak{g}^i\circ\mathfrak{f}$ in $F^{\lambda_{i+1}}\,\mathbf{CC}$ as well. We next claim that the $\mathfrak{g}^i$ and $\mathfrak{h}^i$ can be viewed as A_∞ homomorphisms modulo $T^{\lambda_{i+1}}$. Indeed, pick β with $\lambda_i < E(\beta) < \lambda_{i+1}$; then $\beta\notin\mathsf{G}$. Suppose one can choose β_i's so that $\beta = \sum\beta_i$ and $\mathfrak{m}_{\ell,\beta_0}\circ(\mathfrak{g}^i_{\beta_1}\otimes\cdots\otimes\mathfrak{g}^i_{\beta_\ell})$ is nonzero. Then, by induction hypothesis, we have $\beta_i\in\mathsf{G}$ for each i. It follows that $\beta = \sum\beta_i\in\mathsf{G}$, a contradiction. Suppose one can choose β_1 and β_2 so that $\beta = \beta_1+\beta_2$ and $\mathfrak{g}^i_{\beta_1}(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}'_{\beta_2}\otimes\mathrm{id}^\bullet)$ is nonzero, then $\beta_1 \in \mathsf{G}$ by induction hypothesis. Since $\beta_2 \in \mathsf{G}_{\mathfrak{m}'}$, we deduce $\beta = \beta_1 + \beta_2 \in \mathsf{G}$, a contradiction. Thus $\mathfrak{m} \circ \mathfrak{g}^i - \mathfrak{g}^i \star \mathfrak{m}'$ is zero not only in $F^{\lambda_i}\,\mathbf{CC}$ but also in $\mathbf{CC}_\beta$ for any β with $\lambda_i < E(\beta) < \lambda_{i+1}$. Namely, the $\mathfrak{g}^i$ is an A_∞ homomorphism modulo $T^{\lambda_{i+1}}$. Similarly, one can show so is $\mathfrak{h}^i$.

Now, it is legitimate to apply Proposition 3.18 for every class in the finite set $\mathsf{G} \cap \{\mathsf{B} \mid E(\mathsf{B}) = \lambda_{i+1}\}$, thereby obtaining the extensions $\mathfrak{g}^{i+1}$ and $\mathfrak{h}^{i+1}$ in $\mathbf{CC}^{(i+1)}$ of $\mathfrak{g}^i$ and $\mathfrak{h}^i$ respectively. By construction, they also satisfy the induction hypothesis. Ultimately, we obtain $\mathfrak{g}$ and $\mathfrak{h}$ in $\mathbf{CC}^{(\infty)}$ so that $\mathfrak{g} \circ \mathfrak{f}$ is ud-homotopic to id via $\mathfrak{h}$. Moreover, this construction exactly tells the set $\{\beta \mid \mathfrak{g}_\beta \neq 0 \text{ or } \mathfrak{h}_\beta \neq 0\}$ must be contained in G; thus, the $\mathfrak{g}$ and $\mathfrak{h}$ satisfy the condition (II-5) (Definition 2.33). The conditions (II-1) (II-2) (II-3) for $\mathfrak{g}$ and $\mathfrak{h}$ hold by the definition of $\mathbf{CC}^{\mathrm{ud}}$, and the (II-4) holds by Lemma 3.2.

In summary, we can find $\mathfrak{g}, \mathfrak{h} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$ so that $\mathfrak{g} \circ \mathfrak{f}$ is ud-homotopic to id via $\mathfrak{h}$. It remains to further prove $\mathfrak{f} \circ \mathfrak{g}$ is also ud-homotopic to id as well. Applying the whole argument again to the $\mathfrak{g}$ (instead of $\mathfrak{f}$), there exist some $\mathfrak{f}', \mathfrak{h}' \in \operatorname{Mor} \mathscr{U}\mathscr{D}$ so that $\mathfrak{f}' \circ \mathfrak{g} \overset{\mathrm{ud}}{\sim} \mathrm{id}$ via $\mathfrak{h}'$. Then, using Lemma 2.41 we conclude $\mathfrak{f}' \overset{\mathrm{ud}}{\sim} \mathfrak{f}' \circ (\mathfrak{g} \circ \mathfrak{f}) = (\mathfrak{f}' \circ \mathfrak{g}) \circ \mathfrak{f} \overset{\mathrm{ud}}{\sim} \mathfrak{f}$, and thus $\mathfrak{f} \circ \mathfrak{g} \overset{\mathrm{ud}}{\sim} \mathfrak{f}' \circ \mathfrak{g} \overset{\mathrm{ud}}{\sim} \mathrm{id}$. □

Remark 3.19. There is a useful observation in the above proof: The ud-homotopy inverse $\mathfrak{g}$ of $\mathfrak{f}$ satisfies that the set $\mathsf{G}_{\mathfrak{g}} := \{\beta \mid \mathfrak{g}_\beta \neq 0\}$ is contained in the set $\mathsf{G} = \mathbb{N} \cdot \mathsf{G}_{\mathfrak{f}} \cup \mathsf{G}_{\mathfrak{m}} \cup \mathsf{G}_{\mathfrak{m}'}$ (58).

4 Homological perturbation

A *ribbon tree* is a tree T with an embedding $\mathsf{T} \hookrightarrow \mathbb{D}^2 \subset \mathbb{C}$ such that a vertex v has only one edge if and only if v lies in the unit circle $\partial\mathbb{D}^2$. Such a vertex is called an exterior vertex, and any other vertex is called an interior vertex. The set of all exterior (resp. interior) vertices is denoted by $C_0^{\mathrm{ext}}(\mathsf{T})$ (resp. $C_0^{\mathrm{int}}(\mathsf{T})$). Then, $C_0(\mathsf{T}) = C_0^{\mathrm{ext}}(\mathsf{T}) \cup C_0^{\mathrm{int}}(\mathsf{T})$ is the set of all vertices. Besides, an edge of T is called exterior if it contains an exterior vertex and is called interior otherwise. The set of all exterior edges is denoted by $C_1^{\mathrm{ext}}(\mathsf{T})$ and that of all interior edges is denoted by $C_1^{\mathrm{int}}(\mathsf{T})$.

A *rooted ribbon tree* is a pair (T, v_0) of a ribbon tree T and an exterior vertex v_0 therein. We call v_0 the *root*; it gives a natural partial order on the set of vertices $C_0(\mathsf{T})$ by setting $v < v'$ if $v \neq v'$ and there is a path in T from v to v_0 which passes through v'. Particularly, the root v_0 is the largest vertex with respect to this partial order.

Definition 4.1. For a label group $\mathfrak{G} = (\mathfrak{G}, E, \mu)$, a $\mathfrak{G}$-*decoration* on (T, v_0) is a map $\mathsf{B} : C_0^{\mathrm{int}}(\mathsf{T}) \to \mathfrak{G}$. It is called *stable* if any $v \in C_0^{\mathrm{int}}(\mathsf{T})$ satisfies $E(\mathsf{B}(v)) \geq 0$ and has at least three edges whenever $\mathsf{B}(v) = 0$. Given $k \in \mathbb{N}$ and $\beta \in \mathfrak{G}$, we denote by $\mathscr{T}(k, \beta) := \mathscr{T}(k, \beta; \mathfrak{G})$ the set of $\mathfrak{G}$-decorated *stable rooted ribbon trees* $(\mathsf{T}, v_0, \mathsf{B})$ so that $\# C_0^{\mathrm{ext}}(\mathsf{T}) = k + 1$ and $\sum_{v \in C_0^{\mathrm{int}}(\mathsf{T})} \mathsf{B}(v) = \beta$. Note that $\mathscr{T}(0,0) = \varnothing$; $\mathscr{T}(1,0)$ contains only one element, i.e. the tree $\mathsf{T}_{1,0}$ that consists of exactly two vertices and one edge. For simplicity, we often omit v_0, B and write $\mathsf{T} = (\mathsf{T}, v_0, \mathsf{B})$.

Definition 4.2. A **time allocation** of $\mathsf{T} = (\mathsf{T}, v_0, \mathsf{B}) \in \mathscr{T}(k.\beta)$ is a map $\tau : C_0^{\mathrm{int}}(\mathsf{T}) \to \mathbb{R}$ such that $\tau(v) \le \tau(v')$ whenever $v < v'$. The set of all time allocations τ for T such that all $\tau(v)$ are contained in $[a,b]$ is denoted by $\mathfrak{A}_a^b(\mathsf{T})$. If $[a,b] = [0,1]$, we often just write $\mathfrak{A}(\mathsf{T}) = \mathfrak{A}_0^1(\mathsf{T})$.

Any time allocation τ can be viewed as a point $(x_v = \tau(v))_{v \in C_0^{\mathrm{int}}(\mathsf{T})}$ in $[a,b]^{\# C_0^{\mathrm{int}}(\mathsf{T})}$. Thus, the set $\mathfrak{A}_a^b(\mathsf{T})$ can be identified with a bounded polyhedron cut out by the inequalities $x_v \le x_{v'}$ for $v < v'$ and $a \le x_v \le b$. Hence, it has a natural measure induced from the Lebesgue measure.

4.1 Canonical models

Definition 4.3. Let $(C, \mathfrak{m}_{1,0})$ and (H, δ) be two graded cochain complexes. A triple (i, π, G), consisting of two maps $i : H \to C$, $\pi : C \to H$ of degree zero and a map $G : C \to C$ of degree -1, is called a **contraction** (for H and C) if the following equations hold

$$\mathfrak{m}_{1,0} \circ i = i \circ \delta \qquad \pi \circ \mathfrak{m}_{1,0} = \delta \circ \pi \tag{60}$$

$$i \circ \pi - \mathrm{id}_C = \mathfrak{m}_{1,0} \circ G + G \circ \mathfrak{m}_{1,0} \tag{61}$$

Further, the (i, π, G) is called a **strong contraction**, or say it is **strong**, if we have the extra conditions:

$$\pi \circ i - \mathrm{id}_H = 0 \tag{62}$$

$$G \circ G = 0 \tag{63}$$

$$G \circ i = 0 \tag{64}$$

$$\pi \circ G = 0 \tag{65}$$

We will give examples in §7. Here (63, 64, 65) are called *side conditions*; c.f. [Mar06]. Although a contraction is enough for the the construction of a canonical model and its divisor axiom, a strong contraction is needed to deal with the unitality.

For our purpose, we need to slightly generalize [FOOO10b, Theorem 5.4.2] (see also [Mar06]). Since the generality we need is not available in the literature, we give the full details for the completeness:

Theorem 4.4. *Fix a $\mathfrak{G}$-gapped A_∞ algebra $(C, \mathfrak{m})$ and a graded cochain complex (H, δ). From a contraction (i, π, G), there is a canonical way to construct a $\mathfrak{G}$-gapped A_∞ algebra $\mathfrak{m}^\diamond$ on H together with a $\mathfrak{G}$-gapped A_∞ homomorphism*

$$\mathfrak{i}^\diamond : (H, \mathfrak{m}^\diamond) \to (C, \mathfrak{m})$$

so that $\mathfrak{i}^\diamond_{1,0} = i$ and $\mathfrak{m}^\diamond_{1,0} = \delta$.

Definition 4.5. In Theorem 4.4, we call the triple $(H, \mathfrak{m}^\diamond, \mathfrak{i}^\diamond)$ or just $(H, \mathfrak{m}^\diamond)$ the **canonical model** of $(C, \mathfrak{m})$ (with respect to the contraction (i, π, G)).

Proof. ***Tree construction:*** By induction on $\#C_0^{\mathrm{int}}(\mathsf{T})$, we will construct two sequences of operators

$$\mathfrak{i}_{\mathsf{T}} : C^{\otimes k} \to C, \qquad \mathfrak{m}_{\mathsf{T}} : C^{\otimes k} \to C \tag{66}$$

for all $\mathsf{T} = (\mathsf{T}, v_0, \mathsf{B}) \in \mathscr{T}(k, \beta)$ and all $k \in \mathbb{N}$, $\beta \in \mathfrak{G}$ as follows.

When $\#C_0^{\mathrm{int}}(\mathsf{T}) = 0$, the only possibility is that $\mathsf{T} = \mathsf{T}_{1,0}$, the unique tree in $\mathscr{T}(1, 0)$; then, we define $\mathfrak{i}_{\mathsf{T}_{1,0}} = i$ and $\mathfrak{m}_{\mathsf{T}_{1,0}} = \delta$. When $\#C_0^{\mathrm{int}}(\mathsf{T}) = 1$, we must have $(k, \beta) \neq (0, 0), (1, 0)$; there is only one such tree T in every $\mathscr{T}(k, \beta)$. Then, we define $\mathfrak{i}_{\mathsf{T}} = G \circ \mathfrak{m}_{k,\beta} \circ i^{\otimes k}$ and $\mathfrak{m}_{\mathsf{T}} = \pi \circ \mathfrak{m}_{k,\beta} \circ i^{\otimes k}$.

Suppose now that $\mathfrak{i}_{\mathsf{T}'}$ and $\mathfrak{m}_{\mathsf{T}'}$ have been constructed for $\#C_0^{\mathrm{int}}(\mathsf{T}') \leq n$ and assume $\mathsf{T} = (\mathsf{T}, v_0, \mathsf{B})$ is a decorated stable rooted ribbon tree (Definition 4.1) such that $\#C_0^{\mathrm{int}}(\mathsf{T}) = n + 1$. Consider the edge e of the root v_0. Let v be the other vertex of e. Then, $v \in C_0^{\mathrm{int}}(\mathsf{T})$. We cut all edges of v except e and add a pair of vertices to all the resulting half line segments. Then we get the tree $\mathsf{T}_{\ell,\mathsf{B}(v)} \in \mathscr{T}(\ell, \mathsf{B}(v))$ with one interior vertex v together with the other ℓ trees

$$\mathsf{T}_i := (\mathsf{T}_i, v_0^i, \mathsf{B}^i) \in \mathscr{T}(k_i, \beta_i) \qquad i = 1, \dots \ell \tag{67}$$

where $\mathsf{B}(v) + \sum \beta_i = \beta$ and $\sum k_i = k$; the roots v_0^i are newly-added vertices which are ordered counterclockwise. Define

$$\begin{aligned} \mathfrak{i}_{\mathsf{T}} &= G \circ \mathfrak{m}_{\ell,\mathsf{B}(v)} \circ (\mathfrak{i}_{\mathsf{T}_1} \otimes \cdots \otimes \mathfrak{i}_{\mathsf{T}_\ell}) \\ \mathfrak{m}_{\mathsf{T}} &= \pi \circ \mathfrak{m}_{\ell,\mathsf{B}(v)} \circ (\mathfrak{i}_{\mathsf{T}_1} \otimes \cdots \otimes \mathfrak{i}_{\mathsf{T}_\ell}) \end{aligned} \tag{68}$$

It is well-defined since $\#C_0^{\mathrm{int}}(\mathsf{T}_i) < \#C_0^{\mathrm{int}}(\mathsf{T})$. Summing over all the trees in $\mathscr{T}(k, \beta)$, we define:

$$\begin{aligned} \mathfrak{i}^\diamond_{k,\beta} &= \textstyle\sum_{\mathsf{T} \in \mathscr{T}(k,\beta)} \mathfrak{i}_{\mathsf{T}} \\ \mathfrak{m}^\diamond_{k,\beta} &= \textstyle\sum_{\mathsf{T} \in \mathscr{T}(k,\beta)} \mathfrak{m}_{\mathsf{T}} \end{aligned} \tag{69}$$

For the exceptional cases $(k, \beta) = (0, 0), (1, 0)$, we define $\mathfrak{i}^\diamond_{0,0} = 0$ and $\mathfrak{m}^\diamond_{0,0} = 0$ and $\mathfrak{i}^\diamond_{1,0} = i, \mathfrak{m}^\diamond_{1,0} = \delta$.

We claim that there are only finitely many nonzero terms in (69). Indeed, even though the set $\mathscr{T}(k, \beta)$ may be infinite, a tree in $\mathscr{T}(k, \beta)$ with nonzero contribution to (69) must satisfies that the image of $\mathsf{B} : C_0^{\mathrm{int}}(\mathsf{T}) \to \mathfrak{G}$ is contained in the set $\mathsf{G}_{\mathfrak{m}} := \{\beta \in \mathfrak{G} \mid \mathfrak{m}_\beta \neq 0\}$. Hence, the gappedness of $\mathfrak{m}$ infers that the trees with nonzero contributions form a finite subset of $\mathscr{T}(k, \beta)$.

Combining (68) and (69) yields the inductive formulas for both $\mathfrak{i}^\diamond$ and $\mathfrak{m}^\diamond$: for all $(k, \beta) \neq (0, 0), (1, 0)$, we have

$$\begin{aligned} \mathfrak{i}^\diamond_{k,\beta} &= \sum_{\ell \geq 0} \sum_{k_1 + \cdots + k_\ell = k} \sum_{\substack{\beta_1 + \cdots + \beta_\ell = \beta \\ (\ell, \beta_0) \neq (1,0)}} G \circ \mathfrak{m}_{\ell,\beta_0} \circ (\mathfrak{i}^\diamond_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{i}^\diamond_{k_\ell,\beta_\ell}) \\ \mathfrak{m}^\diamond_{k,\beta} &= \sum_{\ell \geq 0} \sum_{k_1 + \cdots + k_\ell = k} \sum_{\substack{\beta_1 + \cdots + \beta_\ell = \beta \\ (\ell, \beta_0) \neq (1,0)}} \pi \circ \mathfrak{m}_{\ell,\beta_0} \circ (\mathfrak{i}^\diamond_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{i}^\diamond_{k_\ell,\beta_\ell}) \end{aligned} \tag{70}$$

Remark that the condition $(\ell,\beta_0)\neq(1,0)$ in (70) is due to the tree stable condition. The degrees are as we expect: $\deg \mathfrak{i}^\diamond_{k,\beta}=-1+(2-\ell-\mu(\beta_0))+\sum_i(1-k_i-\mu(\beta_i))=1-k-\mu(\beta)$ and also $\deg\mathfrak{m}^\diamond_{k,\beta}=2-k-\mu(\beta)$. To quote later, the argument of proving the gappedness of $\mathfrak{i}^\diamond$ and $\mathfrak{m}^\diamond$ is separately presented in Remark 4.6 below:

Remark 4.6. Denote by $\mathsf{G}_{\mathfrak{m}}$ the set of β with $\mathfrak{m}_\beta\neq 0$. Then by (70) we see both the set $\{\beta\mid \mathfrak{m}^\diamond_\beta\neq 0\}$ and $\{\beta\mid \mathfrak{i}^\diamond_\beta\neq 0\}$ are contained in $\mathbb{N}\cdot\mathsf{G}_{\mathfrak{m}}$. Particularly, if $\mathfrak{m}$ only involves β with $\mu(\beta)\geq 0$, then so do the $\mathfrak{m}^\diamond$ and $\mathfrak{i}^\diamond$. Hence, the conditions (I-5) (II-5) (Definition 2.33) hold for $\mathfrak{m}^\diamond$ and $\mathfrak{i}^\diamond$.

A_∞ ***relations:*** Now, we aim to prove the A_∞ equations

$$(\mathfrak{m}\circ\mathfrak{i}^\diamond-\mathfrak{i}^\diamond\star\mathfrak{m}^\diamond)|_{\mathbf{CC}_{k,\beta}}=0 \tag{71}$$

by an induction on the set of pairs (k,β) with the partial order '$<$' mentioned in Remark 2.6. The case $(k,\beta)=(0,0)$ is trivial. When $(k,\beta)=(1,0)$, it becomes $\mathfrak{m}_{1,0}(\mathfrak{i}^\diamond_{1,0}(x))=\mathfrak{i}^\diamond_{1,0}(\mathfrak{m}^\diamond_{1,0}(x))$ which is just (60). Fix (k,β). Suppose (71) holds for all $(k_1,\beta_1)<(k,\beta)$. By the inductive formula (70), we gain

$$(\mathfrak{m}\circ\mathfrak{i}^\diamond)|_{\mathbf{CC}_{k,\beta}}=(\mathfrak{m}_{1,0}G+\mathrm{id})\circ\textstyle\sum_{(\ell,\beta_0)\neq(1,0)}\mathfrak{m}_{\ell,\beta_0}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)$$

$$\begin{aligned}(\mathfrak{i}^\diamond\star\mathfrak{m}^\diamond)|_{\mathbf{CC}_{k,\beta}}&=i\circ\mathfrak{m}^\diamond_{k,\beta}+\textstyle\sum_{(r,\beta')\neq(1,0)}\mathfrak{i}^\diamond_{r,\beta'}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}^\diamond_{s,\beta''}\otimes\mathrm{id}^\bullet)\\&=i\circ\mathfrak{m}^\diamond_{k,\beta}+\textstyle\sum_{(\sigma,\beta_1)\neq(1,0)}G\circ\mathfrak{m}_{\sigma,\beta_1}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}^\diamond_{s,\beta''}\otimes\mathrm{id}^\bullet)\end{aligned}$$

The conditions $(r,\beta')\neq(1,0)$ and $(\sigma,\beta_1)\neq(1,0)$ together with the tree stability ensure that the induction hypothesis can be applied to those $\mathfrak{i}^\diamond$ in the right side. Consequently, we obtain

$$\begin{aligned}(\mathfrak{i}^\diamond\star\mathfrak{m}^\diamond)|_{\mathbf{CC}_{k,\beta}}&=i\circ\mathfrak{m}^\diamond_{k,\beta}+\textstyle\sum_{(\sigma,\beta_1)\neq(1,0)}G\circ\mathfrak{m}_{\sigma,\beta_1}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}\otimes\mathrm{id}^\bullet)\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)\\&=i\circ\mathfrak{m}^\diamond_{k,\beta}-\textstyle\sum G\circ\mathfrak{m}_{1,0}\circ\mathfrak{m}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)\end{aligned}$$

where we also uses the A_∞ associativity of $\mathfrak{m}$ and the previously-computed degrees of $\mathfrak{i}^\diamond$. To sum up,

$$(\mathfrak{m}\circ\mathfrak{i}^\diamond-\mathfrak{i}^\diamond\star\mathfrak{m}^\diamond)|_{\mathbf{CC}_{k,\beta}}=(\mathfrak{m}_{1,0}\circ G+\mathrm{id}-i\pi+G\circ\mathfrak{m}_{1,0})\circ\textstyle\sum_{(\ell,\beta_0)\neq(1,0)}\mathfrak{m}_{\ell,\beta_0}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)$$

which vanishes exactly due to (61).

Next, we aim to show the A_∞ associativity:

$$\mathfrak{m}^\diamond\star\mathfrak{m}^\diamond|_{\mathbf{CC}_{k,\beta}}=0$$

When $(k,\beta)=(0,0)$, it is trivial; when $(k,\beta)=(1,0)$, it just becomes $\delta^2=0$. For any other (k,β), one can similarly use (70), (60), and (71) to show:

$$\begin{aligned}\mathfrak{m}^\diamond\star\mathfrak{m}^\diamond|_{\mathbf{CC}_{k,\beta}}&=\delta\circ\mathfrak{m}^\diamond_{k,\beta}+\textstyle\sum_{(\sigma,\beta_1)\neq(1,0)}\pi\circ\mathfrak{m}_{\sigma,\beta_1}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}^\diamond\otimes\mathrm{id}^\bullet)\\&=\delta\circ\mathfrak{m}^\diamond_{k,\beta}+\textstyle\sum_{(\sigma,\beta_1)\neq(1,0)}\pi\circ\mathfrak{m}_{\sigma,\beta_1}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}\otimes\mathrm{id}^\bullet)\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)\\&=\textstyle\sum(\delta\circ\pi-\pi\circ\mathfrak{m}_{1,0})\circ\mathfrak{m}\circ(\mathfrak{i}^\diamond\otimes\cdots\otimes\mathfrak{i}^\diamond)=0\end{aligned}$$

The proof is now complete. □

4.2 Properties

As the construction in Theorem 4.4 is quite canonical, it preserves the various conditions like the strict/cyclical unitalities and the divisor axiom as expected:

Proposition 4.7. *The tree construction in Theorem 4.4 has the following properties:*

(i) Assume $\partial\beta \cap i(b) = \partial\beta \cap b$. If $\mathfrak{m}$ satisfies the divisor axiom, then so do both $\mathfrak{m}^\diamond$ and $\mathfrak{i}^\diamond$.

(ii) If $\mathfrak{m}$ is cyclically unital, then so are $\mathfrak{m}^\diamond$ and $\mathfrak{i}^\diamond$.

(iii) Assume (i,π,G) is a strong contraction. If $(C,\mathfrak{m})$ has a unit $\mathbb{1}$ such that $i(\pi(\mathbb{1})) = \mathbb{1}$, then the $\mathfrak{m}^\diamond$ has a unit $\pi(\mathbb{1})$ and the $\mathfrak{i}^\diamond$ is unital with respect to $\pi(\mathbb{1})$ and $\mathbb{1}$.

Proof. ***(i) Divisor axiom.*** We only show the divisor axiom for $\mathfrak{i}^\diamond$ and a similar argument applies for $\mathfrak{m}^\diamond$. By Remark 2.32, it suffices to show

$$\mathrm{DA}[\mathfrak{i}^\diamond]_{k,\beta}(b;x_1,\dots,x_k) = \partial\beta\cap b\cdot \mathfrak{i}^\diamond_{k,\beta}(x_1,\dots,x_k) \tag{72}$$

for all $(k,\beta)\neq(0,0)$. We can easily check the initial case when $(k,\beta)=(1,0)$. Suppose (72) holds for $(k',\beta')<(k,\beta)$, and we aim to show it also holds for (k,β). By the inductive formula (70), we get

$$\begin{aligned}\mathrm{DA}[\mathfrak{i}^\diamond]_{k,\beta}(b;x_1,\dots,x_k) = &\textstyle\sum_{\substack{(k_i,\beta_i)\neq(0,0)\\(\ell,\beta_0)\neq(1,0)}} G\circ\mathfrak{m}_{\ell,\beta_0}\circ\big(\mathfrak{i}^\diamond_{k_1,\beta_1}\cdots \mathrm{DA}[\mathfrak{i}^\diamond]_{k_i,\beta_i}(b;\cdots)\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}\big)\\ &+\textstyle\sum_{\beta_0\neq0} G\circ \mathrm{DA}[\mathfrak{m}]_{\ell,\beta_0}\big(i(b);\mathfrak{i}^\diamond_{k_1,\beta_1}\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}\big)\end{aligned}$$

We may require $\beta_0\neq0$ in the second sum by using the divisor axiom of $\mathfrak{m}$ with $\beta=0\in\mathfrak{G}$. Note that $\mathrm{DA}[\mathfrak{i}^\diamond]_{0,0}(b)=\mathfrak{i}^\diamond_{1,0}(b)=i(b)$; the second sum just consists of the excluded terms with $(k_i,\beta_i)=(0,0)$ in the first sum. We can directly apply the induction hypothesis to the first sum, and we can apply the divisor axiom of $\mathfrak{m}$ to the second sum. In consequence, we obtain

$$\begin{aligned}\mathrm{DA}[\mathfrak{i}^\diamond]_{k,\beta}(b;x_1,\dots,x_k) = &\textstyle\sum_{\substack{(k_i,\beta_i)\neq(0,0)\\(\ell,\beta_0)\neq(1,0)}} \partial\beta_i\cap b\cdot G\circ\mathfrak{m}_{\ell,\beta_0}\circ\big(\mathfrak{i}^\diamond_{k_1,\beta_1}\cdots\mathfrak{i}^\diamond_{k_i,\beta_i}\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}\big)\\ &+\textstyle\sum_{\beta_0\neq0}\partial\beta_0\cap b\cdot G\circ\mathfrak{m}_{\ell,\beta_0}\big(\mathfrak{i}^\diamond_{k_1,\beta_1}\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}\big)\end{aligned}$$

Now, the above conditions $(k_i,\beta_i)\neq(0,0),(\ell,\beta_0)\neq(1,0)$ and $\beta_0\neq0$ can be removed. Finally, as $\beta_0+\sum_i\beta_i=\beta$, we have $\mathrm{DA}[\mathfrak{i}^\diamond]_{k,\beta}(b;x_1,\dots,x_k) = \partial\beta\cap b\cdot\mathfrak{i}^\diamond_{k,\beta}(x_1,\dots,x_k)$.

(ii) Cyclical unitality. Let $\mathbf{e}\in H$ be an degree-zero element. When $(k,\beta)=(0,0)$, we recall that $\mathfrak{i}^\diamond_{1,0}(\mathbf{e})=i(\mathbf{e})$ is exceptional in Definition 2.25. We perform an induction for the pairs (k,β) again. The initial case $(k,\beta)=(1,0)$ is easy to check. For $(k,\beta)\neq(0,0),(1,0)$, suppose it is true for $(k',\beta')<(k,\beta)$. By (70), we get

$$\begin{aligned}\mathrm{CU}[\mathfrak{i}^\diamond]_{k,\beta}(\mathbf{e};x_1,\dots,x_k) = &\textstyle\sum_{\substack{(k_i,\beta_i)\neq(0,0)\\(\ell,\beta_0)\neq(1,0)}} G\circ\mathfrak{m}_{\ell,\beta_0}\circ\big(\mathfrak{i}^{\diamond\#}_{k_1,\beta_1}\cdots \mathrm{CU}[\mathfrak{i}^\diamond]_{k_i,\beta_i}(\mathbf{e};\cdots)\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}\big)\\ &+\textstyle\sum_{(\ell,\beta_0)\neq(0,0)} G\circ\mathrm{CU}[\mathfrak{m}]_{\ell,\beta_0}\big(i(\mathbf{e});\mathfrak{i}^\diamond_{k_1,\beta_1}\cdots\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}(\cdots x_k)\big)\end{aligned}$$

Since $\mathfrak{m}$ is cyclical unital, the second sum is zero. As above, applying the induction hypothesis and the cyclical unitality of $\mathfrak{m}$ completes the induction step. The same argument also works for $\mathfrak{m}^\diamond$.

(iii) Unitality. We first prove the unitality of $\mathfrak{i}^\diamond$. By condition, we have $\mathfrak{i}^\diamond_{1,0}(\pi(\mathbb{1})) = i(\pi(\mathbb{1}) = \mathbb{1}$. So, it remains to show $\mathfrak{i}^\diamond_{k,\beta}(\dots,\pi(\mathbb{1}),\dots) = 0$ for $(k,\beta)\neq(1,0)$. Arguing by contradiction, suppose $(k,\beta)\neq(1,0)$ is the smallest pair so that $\mathfrak{A} := \mathfrak{i}^\diamond_{k,\beta}(\dots,\pi(\mathbb{1}),\dots)\neq 0$ happens for some fixed inputs. Then, we can find some non-zero term in the expansion (70) of $\mathfrak{A}$, say:

$$\mathfrak{a} := G\circ\mathfrak{m}_{\ell,\beta_0}(\mathfrak{i}^\diamond_{k_1,\beta_1}(\dots),\dots,\mathfrak{i}^\diamond_{k_i,\beta_i}(\dots,\pi(\mathbb{1})\dots),\dots,\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}(\dots))\neq 0$$

Since $(\ell,\beta_0)\neq(1,0)$, we must have $(k_i,\beta_i)\neq(k,\beta)$. By the smallest choice of (k,β), we have $(k_i,\beta_i)=(1,0)$. Besides, when $(\ell,\beta_0)\neq(2,0)$, we get $\mathfrak{i}^\diamond_{k_i,\beta_i}(\dots\pi(\mathbb{1})\dots) = i(\pi(\mathbb{1})) = \mathbb{1}$ which is the unit of $\mathfrak{m}$, hence the term $\mathfrak{a}$ vanishes. When $(\ell,\beta_0)=(2,0)$, the above term $\mathfrak{a}$ becomes either $G\circ\mathfrak{m}_{2,0}(\mathfrak{i}^\diamond_{k-1,\beta}(\cdots),\mathbb{1})$ or $G\circ\mathfrak{m}_{2,0}(\mathbb{1},\mathfrak{i}^\diamond_{k-1,\beta}(\cdots))$. By Definition 2.23 (a1), the term turns out to be $\pm G\circ\mathfrak{i}^\diamond_{k-1,\beta}(\cdots)$ in either case. Now, if $(k-1,\beta)\neq(1,0)$, then by the side condition $G\circ G=0$ (63), the term vanishes; if $(k-1,\beta)=(1,0)$, then it is still zero due to the other side condition $G\circ i = 0$ (64). This is a contradiction to $\mathfrak{a}\neq 0$.

As for $\mathfrak{m}^\diamond$, observe that $\mathfrak{m}^\diamond_{1,0}(\pi(\mathbb{1})) = \delta\pi(\mathbb{1}) = \pi\mathfrak{m}_{1,0}(\mathbb{1}) = 0$ by (60). Notice also that $\mathfrak{m}^\diamond_{2,0}(\pi(\mathbb{1}),x) = \pi\circ\mathfrak{m}_{2,0}\circ(i\pi(\mathbb{1}),i(x)) = \pi i(x) = x$ and similarly $(-1)^{\deg x}\mathfrak{m}^\diamond_{2,0}(x,\pi(\mathbb{1})) = x$. Suppose $(k,\beta)\neq(1,0),(2,0)$ is the smallest pair so that $\mathfrak{m}^\diamond_{k,\beta}(\dots,\pi(\mathbb{1}),\dots)\neq 0$ happens. Then as before we may find a non-zero term in the expansion: $\pi\circ\mathfrak{m}_{\ell,\beta_0}(\mathfrak{i}^\diamond_{k_1,\beta_1}(\dots),\dots,\mathfrak{i}^\diamond_{k_i,\beta_i}(\dots,\pi(\mathbb{1})\dots),\dots,\mathfrak{i}^\diamond_{k_\ell,\beta_\ell}(\dots))$. By what we proved above, we may require $(k_i,\beta_i)=(1,0)$. Since $\mathbb{1}\equiv i(\pi(\mathbb{1}))$ is a unit of $\mathfrak{m}$, we can further require $(\ell,\beta_0)=(2,0)$. Then we also arrive at $\pm\pi\circ\mathfrak{i}^\diamond_{k-1,\beta}(\dots)$. Recall that $(k-1,\beta)\neq(1,0)$; so, it is zero again due to the last side-condition $\pi\circ G=0$ (65). □

5 From pseudo-isotopies to A_∞ homomorphisms

5.1 Construction

The following result is basically due to [Fuk10]. It is also obtained by a tree construction.

Theorem 5.1. *There is a canonical way to associate to a $\mathfrak{G}$-gapped pseudo-isotopy $(C_{[0,1]},\mathfrak{M})$ a $\mathfrak{G}$-gapped A_∞ homomorphism*

$$\mathfrak{C} : (C,\mathfrak{m}^0)\to(C,\mathfrak{m}^1)$$

so that $\mathfrak{C}_{1,0}=\mathrm{id}$, where the $\mathfrak{m}^0$ and $\mathfrak{m}^1$ are the restrictions of $\mathfrak{M}$ at $s=0,1$.

Proof. Recall that we may write $\mathfrak{M} = 1\otimes\mathfrak{m}^s + ds\otimes\mathfrak{c}^s$. Like (66), we construct a sequence of operators $\mathfrak{C}_{\mathsf{T},\tau} : C^{\otimes k}\to C$ for a tree $\mathsf{T} := (\mathsf{T},v_0,\mathsf{B})\in\mathscr{T}(k,\beta)$ equipped with a time allocation $\tau\in\mathfrak{A}(\mathsf{T})\equiv\mathfrak{A}^1_0(\mathsf{T})$ (Definition 4.2). Recall that the set $\mathfrak{A}(\mathsf{T})$ of all time allocations embeds into $[0,1]^{\#C_0^{\mathrm{int}}(\mathsf{T})}$ and naturally inherits a measure. We construct these $\mathfrak{C}_{\mathsf{T},\tau}$ inductively on $\#C_0^{\mathrm{int}}(\mathsf{T})$ as well. When $\#C_0^{\mathrm{int}}(\mathsf{T})=0$, $\mathfrak{A}(\mathsf{T})=\varnothing$ and the only possibility is $\mathsf{T}=\mathsf{T}_{1,0}$. We define $\mathfrak{c}_{\mathsf{T}_{1,0},\varnothing}=\mathrm{id}$. When $\#C_0^{\mathrm{int}}(\mathsf{T})=1$, we must have $(k,\beta)\neq(0,0),(1,0)$ and $\mathsf{T}=\mathsf{T}_{k,\beta}$, the tree with one interior vertex v in the set $\mathscr{T}(k,\beta)$. Then, we define $\mathfrak{C}_{\mathsf{T},\tau}=-\mathfrak{c}_{k,\beta}^{\tau(v)}$.

Inductively, suppose the constructions have been established for a tree with $\#C_0^{\mathrm{int}}(\mathsf{T}')\le n$. Let T be a tree so that $\#C_0^{\mathrm{int}}(\mathsf{T})=n+1$. Denote by v the closest vertex to the root v_0. We do a similar surgery as in (67) cutting all the incoming edges of v; then, we get $\mathsf{T}_{\ell,\mathsf{B}(v)}$ and T_i $(1\le i\le \ell)$ like there. Restricting τ on interior vertices of T_i produces a time allocation τ_i in $\mathfrak{A}_0^{\tau(v)}(\mathsf{T}_i)$. Now, we define

$$\mathfrak{C}_{\mathsf{T},\tau}=-\mathfrak{c}_{\ell,\mathsf{B}(v)}^{\tau(v)}\circ(\mathfrak{C}_{\mathsf{T}_1,\tau_1}\otimes\cdots\otimes\mathfrak{C}_{\mathsf{T}_\ell,\tau_\ell}) \tag{73}$$

By the stability, we have $(\ell,\mathsf{B}(v))\neq(1,0)$. Averaging all possible $\tau\in\mathfrak{A}(\mathsf{T})$, we define

$$\mathfrak{C}_{\mathsf{T}}=\int_{\mathfrak{A}(\mathsf{T})}\mathfrak{C}_{\mathsf{T},\tau}d\tau \tag{74}$$

Eventually, we define $\mathfrak{C}_{0,0}=0$ and $\mathfrak{C}_{1,0}=\mathrm{id}$; if $(k,\beta)\neq(0,0),(1,0)$, we define

$$\mathfrak{C}_{k,\beta}:=\textstyle\sum_{\mathsf{T}\in\mathscr{T}(k,\beta)}\mathfrak{C}_{\mathsf{T}} \tag{75}$$

For later use, one can start with a pseudo-isotopy on $C_{[a,b]}$ for an arbitrary interval $[a,b]$. Then, replacing $\mathfrak{A}(\mathsf{T})\equiv\mathfrak{A}_0^1(\mathsf{T})$ by $\mathfrak{A}_a^b(\mathsf{T})$, one can also define $\mathfrak{C}_{\mathsf{T}}^{[a,b]}$ and $\mathfrak{C}_{k,\beta}^{[a,b]}$ in the same way; we still define $\mathfrak{C}_{0,0}^{[a,b]}=0$ and $\mathfrak{C}_{1,0}^{[a,b]}=\mathrm{id}$.

In the special case $a=b$, the set $\mathfrak{A}_a^a(\mathsf{T})$ consists of only one element and has zero measure, and the above integrals as in (74) will become zero. In other words, for $(k,\beta)\neq(1,0)$ we have $\mathfrak{C}_{k,\beta}^{[a,a]}=0$, and so $\mathfrak{C}^{[a,a]}=\mathrm{id}$ is the trivial A_∞ homomorphisms.

Now, using the Fubini's theorem with (73), (74) and (75) implies that

$$\begin{aligned}\mathfrak{C}_{k,\beta}&=\sum_{\mathsf{T}\in\mathscr{T}(k,\beta)}\int_{\mathfrak{A}(\mathsf{T})}-\mathfrak{c}_{\ell,\mathsf{B}(v)}^{\tau(v)}\circ(\mathfrak{C}_{\mathsf{T}_1,\tau_1}\otimes\cdots\otimes\mathfrak{C}_{\mathsf{T}_\ell,\tau_\ell})d\tau\\&=\sum_{\substack{\ell\ge1\\ \beta_0+\beta_1+\cdots+\beta_\ell=\beta\\ k_1+\cdots+k_\ell=k\\ (\ell,\beta_0)\neq(1,0)}}\ \sum_{\substack{\mathsf{T}_1\in\mathscr{T}(k_1,\beta_1)\\ \cdots\cdots\\ \mathsf{T}_\ell\in\mathscr{T}(k_\ell,\beta_\ell)}}\int_0^1du\int_{\mathfrak{A}_0^u(\mathsf{T}_1)\times\cdots\times\mathfrak{A}_0^u(\mathsf{T}_\ell)}-\mathfrak{c}_{\ell,\beta_0}^u\circ\big(\mathfrak{C}_{\mathsf{T}_1,\tau_1}\otimes\cdots\otimes\mathfrak{C}_{\mathsf{T}_\ell,\tau_\ell}\big)\cdot d\tau_1\cdots d\tau_\ell\end{aligned}$$

For a general interval $[a,b]$, it follows from (74) and (75) that we have an *inductive formula*:

$$\mathfrak{C}_{k,\beta}^{[a,b]}=\sum_{\ell\ge1}\sum_{\substack{\beta_0+\beta_1+\cdots+\beta_\ell=\beta\\ k_1+\cdots+k_\ell=k\\ (\ell,\beta_0)\neq(1,0)}}-\int_a^b du\cdot\mathfrak{c}_{\ell,\beta_0}^u\circ\big(\mathfrak{C}_{k_1,\beta_1}^{[a,u]}\otimes\cdots\otimes\mathfrak{C}_{k_\ell,\beta_\ell}^{[a,u]}\big) \tag{76}$$

where $(k,\beta)\neq(0,0),(1,0)$. If we replace $[a,b]$ by $[0,u]$ in (76), then taking the derivative $\frac{d}{du}$ yields:

$$\frac{d}{du}\mathfrak{C}^{[0,u]}_{k,\beta}=\sum_{\ell\geq 1}\sum_{\substack{\beta_0+\beta_1+\cdots+\beta_\ell=\beta\\ k_1+\cdots+k_\ell=k\\ (\ell,\beta_0)\neq(1,0)}}-\mathfrak{c}^u_{\ell,\beta_0}\circ\left(\mathfrak{C}^{[0,u]}_{k_1,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{k_\ell,\beta_\ell}\right) \tag{77}$$

Beware that we no longer need to assume $(k,\beta)\neq(0,0),(1,0)$ in (77), since the two excluded cases can be checked by hand. For instance, when $(k,\beta)=(1,0)$, we have $\frac{d}{du}\mathfrak{C}^{[0,u]}_{1,0}=\frac{d}{du}\mathrm{id}=0$.

Now, we claim that the above-defined operator system $\mathfrak{C}=(\mathfrak{C}_{k,\beta})$ gives an A_∞ homomorphism. Its proof is essentially given in [Fuk10], but let us repeat it in our notations. Let us check the degrees first: by Definition 2.18 we know $\deg\mathfrak{c}^s_{k,\beta}=1-k-\mu(\beta)$. Inductively, the equations in (76) imply that

$$\deg\mathfrak{C}^{[a,b]}_{k,\beta}=\deg\mathfrak{c}^u_{\ell,\beta_0}+\sum_{i=1}^{\ell}\deg\mathfrak{C}^{[a,u]}_{k_i,\beta_i}=1-\ell-\mu(\beta_0)+\sum_{i=1}^{\ell}(1-k_i-\mu(\beta_i))=1-k-\mu(\beta)$$

To prove the A_∞ relation, we do induction on the pairs (k,β) again (Remark 2.6), but we strengthen the induction statement by allowing any arbitrary intervals $[a,b]\subset[0,1]$. Consider

$$\begin{aligned}\mathfrak{P}^{[a,b]}_{k,\beta}&:=\sum_{\ell\geq1}\sum_{\beta_0+\beta_1+\cdots+\beta_\ell=\beta}\sum_{0=j_0\leq\cdots\leq j_\ell=k}\mathfrak{m}^b_{\ell,\beta_0}\circ(\mathfrak{C}^{[a,b]}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[a,b]}_{j_\ell-j_{\ell-1},\beta_\ell})\\ \mathfrak{Q}^{[a,b]}_{k,\beta}&:=\sum_{\beta'+\beta''=\beta}\sum_{i+j+r=k}\mathfrak{C}^{[a,b]}_{i+j+1,\beta'}\circ(\mathrm{id}^i_\#\otimes\mathfrak{m}^a_{r,\beta''}\otimes\mathrm{id}^j)\end{aligned}$$

and the desired A_∞ relation is $\mathfrak{P}^{[a,b]}_{k,\beta}=\mathfrak{Q}^{[a,b]}_{k,\beta}$. When $(k,\beta)=(0,0)$, it is trivial. When $(k,\beta)=(1,0)$, it reduces to $\mathfrak{m}^b_{1,0}\circ\mathfrak{C}^{[a,b]}_{1,0}=\mathfrak{C}^{[a,b]}_{1,0}\circ\mathfrak{m}^a_{1,0}$, which is obvious. Suppose now $\mathfrak{P}^{[a,b]}_{k',\beta'}=\mathfrak{Q}^{[a,b]}_{k',\beta'}$ holds for all $[a,b]$ and $(k',\beta')<(k,\beta)$. Without loss of generality, we only show $\mathfrak{P}^{[0,1]}_{k,\beta}=\mathfrak{Q}^{[0,1]}_{k,\beta}$. By (77), we get

$$\begin{aligned}\frac{d}{du}\mathfrak{P}^{[0,u]}_{k,\beta}&=\sum\left(\tfrac{d}{du}\mathfrak{m}^u_{\ell,\beta_0}\right)\circ(\mathfrak{C}^{[0,u]}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{j_\ell-j_{\ell-1},\beta_\ell})+\mathfrak{m}^u_{\ell,\beta_0}\circ\tfrac{d}{du}(\mathfrak{C}^{[0,u]}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{j_\ell-j_{\ell-1},\beta_\ell})\\ \frac{d}{du}\mathfrak{Q}^{[0,u]}_{k,\beta}&=\textstyle\sum_{(\ell',\beta_0')\neq(1,0)}-\mathfrak{c}^u_{\ell',\beta_0'}\circ\left(\mathfrak{C}^{[0,u]}_{k_1',\beta_1'}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{k'_{\ell'},\beta'_{\ell'}}\right)\circ(\mathrm{id}^i_\#\otimes\mathfrak{m}^0_{r,\beta''}\otimes\mathrm{id}^j)\end{aligned}$$

According to Definition 2.18 (e), the first sum of $\frac{d}{dt}\mathfrak{P}^{[0,u]}_{k,\beta}$ becomes

$$\begin{aligned}P_1:=P_1'+P_1'':=\quad&-\textstyle\sum_{(r,\beta_0^{(1)})\neq(1,0)}\quad\mathfrak{c}^u_{r,\beta_0^{(1)}}\circ(\mathrm{id}^\lambda_\#\otimes\mathfrak{m}^u_{\nu,\beta_0^{(2)}}\otimes\mathrm{id}^\mu)\circ(\mathfrak{C}^{[0,u]}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{j_\ell-j_{\ell-1},\beta_\ell})\\ &+\textstyle\sum_{(\nu,\beta_0^{(2)})\neq(1,0)}\quad\mathfrak{m}^u_{r,\beta_0^{(1)}}\circ(\mathrm{id}^\lambda\otimes\mathfrak{c}^u_{\nu,\beta_0^{(2)}}\otimes\mathrm{id}^\mu)\circ(\mathfrak{C}^{[0,u]}_{j_1-j_0,\beta_1}\otimes\cdots\otimes\mathfrak{C}^{[0,u]}_{j_\ell-j_{\ell-1},\beta_\ell})\end{aligned}$$

The induction hypothesis implies that $P_1'=\frac{d}{du}\mathfrak{Q}^{[0,u]}_{k,\beta}$. If we denote the second sum of $\frac{d}{du}\mathfrak{P}^{[0,u]}_{k,\beta}$ by P_2, then it follows from (77) that $P_2=-P_1''$. Thus, we get

$\frac{d}{du}\mathfrak{P}^{[0,u]}_{k,\beta} = P_1' + P_1'' + P_2 = P_1' = \frac{d}{du}\mathfrak{Q}^{[0,u]}_{k,\beta}$. Thus, we see that $\mathfrak{P}^{[0,u]}_{k,\beta} = \mathfrak{Q}^{[0,u]}_{k,\beta}$, since their initial values at $u = 0$ also agree with each other. The induction is complete. Finally, the proof of gappedness is given in Remark 5.2 below for later quote. $\square$

Remark 5.2. Similar to Remark 4.6 we have the following observation. Due to (76) or (77), if we write $\mathsf{G} = \{\beta \mid \mathfrak{c}^s_\beta \neq 0\}$ which is a subset of $\{\beta \mid \mathfrak{M}_\beta \neq 0\}$, then the set $\{\beta \mid \mathfrak{C}^{[a,b]}_\beta \neq 0\}$ is contained in $\mathbb{N}\cdot\mathsf{G}$. Particularly, if $\mathfrak{M}$ only involved non-negative index μ, then so does $\mathfrak{C}^{[a,b]}$.

Corollary 5.3. *If $\mathfrak{M}$ is a trivial pseudo-isotopy, then the $\mathfrak{C}$ is the trivial A_∞ homomorphism* id*.*

The proof is straightforward. Heuristically, for a pseudo-isotopy $\mathfrak{M} = 1 \otimes \mathfrak{m}^s + ds \otimes \mathfrak{c}^s$, the systems $\mathfrak{c}^s$ represent the 'derivative' of $\mathfrak{m}^s$; the $\mathfrak{C}^{[a,b]}$ is like an 'integration' on $[a,b]$. Just like the calculus, one can show that $\mathfrak{C}^{[b,c]} \circ \mathfrak{C}^{[a,b]} = \mathfrak{C}^{[a,c]}$. In special, if all the 'derivatives' vanish, only $\mathfrak{C}_{1,0} = \mathrm{id}$ survives.

Finally, we state a technical formula for later use:

Lemma 5.4.

$$\frac{d}{ds}\mathfrak{C}^{[s,1]}_{k,\beta} = \sum_{\lambda+\mu+\nu=k} \sum_{\substack{\beta'+\beta''=\beta \\ (\nu,\beta'')\neq(1,0)}} \mathfrak{C}^{[s,1]}_{\lambda+\mu+1,\beta'} \circ (\mathrm{id}^\lambda \otimes \mathfrak{c}^s_{\nu,\beta''} \otimes \mathrm{id}^\mu)$$

Proof. When $(k,\beta) = (0,0),(1,0)$ it is trivial. We may include $\mathfrak{C}^{[s,b]}$ for an arbitrary upper bound $b \in \mathbb{R}$. Suppose this enhanced statement holds for $(k',\beta') < (k,\beta)$. By virtue of (76), the induction hypothesis deduces that

$$\begin{aligned}
\frac{d}{ds}\mathfrak{C}^{[s,1]}_{k,\beta} &= \sum_{\ell\geq 1} \sum_{(\ell,\beta_0)\neq(1,0)} \sum_{1\leq a\leq \ell} -\int_s^1 du \cdot \mathfrak{c}^u_{\ell,\beta_0} \circ \big(\mathfrak{C}^{[s,u]}_{k_1,\beta_1} \otimes \cdots \otimes \frac{d}{ds}\mathfrak{C}^{[s,u]}_{k_a,\beta_a} \otimes \cdots \otimes \mathfrak{C}^{[s,u]}_{k_\ell,\beta_\ell}\big) \\
&= \sum_{\ell\geq 1} \sum_{\substack{\lambda+\mu+\nu=k \\ \beta'+\beta''=\beta \\ (\nu,\beta'')\neq(1,0)}} \sum_{\substack{\beta_0'+\beta_1'+\cdots+\beta_\ell'=\beta' \\ k_1'+\cdots+k_\ell'=\lambda+\mu+1 \\ (\ell,\beta_0')\neq(1,0)}} \int_s^1 du \cdot \mathfrak{c}^u_{\ell,\beta_0'} \circ \big(\mathfrak{C}^{[s,u]}_{k_1',\beta_1'} \otimes \cdots \otimes \mathfrak{C}^{[s,u]}_{k_\ell',\beta_\ell'}\big) \circ (\mathrm{id}^\lambda \otimes \mathfrak{c}^s_{\nu,\beta''} \otimes \mathrm{id}^\mu)
\end{aligned}$$

Finally, applying (76) again completes the induction. $\square$

5.2 Properties

Since the construction in Theorem 5.1 is also canonical, we have an analog of Proposition 4.7:

Proposition 5.5. *The canonical construction in Theorem 5.1 has the following properties:*

(a) If $\mathfrak{M}$ satisfies the divisor axiom, then so does $\mathfrak{C}$.

(b) If $\mathfrak{M}$ is cyclically unital, then so is $\mathfrak{C}$.

(c) If $\mathfrak{M}$ has a $[0,1]$-unit $\mathbb{1} \in C$, then $\mathfrak{C}$ is unital with respect to $\mathbb{1}$'s.

Proof. ***(a) Divisor axiom.*** Write $\mathfrak{M} = 1 \otimes \mathfrak{m}^s + ds \otimes \mathfrak{c}^s$ as before. Using the divisor input in the form $\mathrm{Incl}(b)$ in the divisor axiom equations of $\mathfrak{M}$, we see that both $\mathfrak{m}^s$ and $\mathfrak{c}^s$ satisfy the divisor axiom. By the inductive formula (76), we get

$$\begin{aligned}\mathrm{DA}[\mathfrak{C}]_{k,\beta}(b;x_1,\dots,x_k) = {} & \textstyle\sum_{\substack{(k_i,\beta_i)\neq(0,0)\\(\ell,\beta_0)\neq(1,0)}} -\int_0^1 du\cdot \mathfrak{c}^u_{\ell,\beta_0}\big(\mathfrak{C}^{[0,u]}_{k_1,\beta_1}\cdots \mathrm{DA}[\mathfrak{C}^{[0,u]}]_{k_i,\beta_i}(b;\cdots)\cdots \mathfrak{C}^{[0,u]}_{k_\ell,\beta_\ell}\big)\\ & + \textstyle\sum_{\beta_0\neq 0} -\int_0^1 du\cdot \mathrm{DA}[\mathfrak{c}^u]_{\ell,\beta_0}\big(b;\mathfrak{C}^{[0,u]}_{k_1,\beta_1}\cdots\mathfrak{C}^{[0,u]}_{k_\ell,\beta_\ell}\big)\end{aligned}$$

Recall that $\mathfrak{C}^{[0,u]}_{1,0} = \mathrm{id}$. In the second sum, we can use the divisor axiom of $\mathfrak{c}^u$. In the first sum, $(k_i,\beta_i) < (k,\beta)$; so we can perform the induction.

(b) Cyclical unitality. Let $\mathbf{e} \in C$ be a degree-zero element. Notice that the cyclical unitality (Definition 2.25) of $\mathfrak{M}$ implies the $\mathfrak{m}^s$ and $\mathfrak{c}^s$ are also cyclically unital. Using the formula (76), we get

$$\begin{aligned}\mathrm{CU}[\mathfrak{C}]_{k,\beta}(\mathbf{e};x_1,\dots,x_k) = {} & \textstyle\sum_{\substack{(k_i,\beta_i)\neq(0,0)\\(\ell,\beta_0)\neq(1,0)}} -\int_0^1 du\cdot \mathfrak{c}^u_{\ell,\beta_0}\big(\mathfrak{C}^{[0,u]\#}_{k_1,\beta_1}\cdots \mathrm{CU}[\mathfrak{C}^{[0,u]}]_{k_i,\beta_i}(\mathbf{e};\cdots)\cdots \mathfrak{C}^{[0,u]}_{k_\ell,\beta_\ell}\big)\\ & + \textstyle\sum_{\beta_0\neq 0} -\int_0^1 du\cdot \mathrm{CU}[\mathfrak{c}^u]_{\ell,\beta_0}\big(\mathbf{e};\mathfrak{C}^{[0,u]}_{k_1,\beta_1}\cdots\mathfrak{C}^{[0,u]}_{k_\ell,\beta_\ell}\big)\end{aligned}$$

By a similar induction, one can show this vanishes.

(c) Unitality. Suppose $\mathbb{1}$ is a $[0,1]$-unit (Definition 2.29) of $\mathfrak{M}$, and then $\mathfrak{c}^s_{k,\beta}(\dots\mathbb{1}\dots) = 0$ for $(k,\beta)\neq(1,0)$. As $\mathfrak{C}_{1,0} = \mathrm{id}$, it suffices to show $\mathfrak{C}_{k,\beta}(\dots\mathbb{1}\dots) = 0$ for $(k,\beta)\neq(1,0)$. Once again, exploiting the formula (76) we obtain that

$$\mathfrak{C}_{k,\beta}(\dots\mathbb{1}\dots) = -\textstyle\sum_{(\ell,\beta_0)\neq(1,0)}\int_0^1 \mathfrak{c}^u_{\ell,\beta_0}(\dots\mathfrak{C}^{[0,u]}_{k_i,\beta_i}(\dots\mathbb{1}\dots)\dots)$$

Since $\mathfrak{C}^{[0,u]}_{1,0}(\mathbb{1}) = \mathbb{1}$, a term with $(k_i,\beta_i) = (1,0)$ vanishes due to the property of $\mathfrak{c}^s$ we just mentioned. So, we may assume all $(k_i,\beta_i)\neq(1,0)$; then, we can use the induction to conclude that $\mathfrak{C}$ is unital. □

In application, we want to work with the category $\mathscr{UD}$ (Definition 2.33). By Whitehead Theorem 3.1 and Remark 2.22, we note that the A_∞ homotopy equivalence $\mathrm{Eval}^1\circ(\mathrm{Eval}^0)^{-1}$ shares the same source and target with $\mathfrak{C}$. Moreover, by Remark 3.3, we can further arrange so that $\mathrm{Eval}^1\circ(\mathrm{Eval}^0)^{-1}|_{\mathbf{CC}_{1,0}} = \mathrm{id} = \mathfrak{C}_{1,0}$. In reality, they are supposed to be ud-homotopic to each other:

Theorem 5.6. *If $(C_{[0,1]},\mathfrak{M}) \in \mathrm{Obj}\,\mathscr{UD}$, then the induced $\mathfrak{C}\in\mathrm{Mor}\,\mathscr{UD}$. Moreover, $\mathfrak{C}\circ\mathrm{Eval}^0$ is ud-homotopic to Eval^1.*

Proof. The first half is basically known already. The conditions (II-3) (II-2) (II-1) for $\mathfrak{C}$ are just consequences of Proposition 5.5 (a) (b) (c) respectively. Besides, the condition (II-4) holds as $\mathfrak{C}_{1,0} = \mathrm{id}$. Finally, the (II-5) holds by Remark 5.2. Hence, $\mathfrak{C}\in\mathrm{Mor}\,\mathscr{UD}$, and actually any $\mathfrak{C}^{[a,b]}\in\mathrm{Mor}\,\mathscr{UD}$.

Now, we aim to show $\mathfrak{C}\circ\mathrm{Eval}^0$ is ud-homotopic to Eval^1. Write $\mathfrak{M}=1\otimes\mathfrak{m}^s+ds\otimes\mathfrak{c}^s$ and denote by $\mathfrak{M}^{\mathrm{tri}}=1\otimes\mathfrak{m}^1+ds\otimes\frac{d}{ds}$ the trivial pseudo-isotopy about $\mathfrak{m}^1$ (Example 2.20). By Definition 2.37, our goal is to find a morphism $\mathfrak{F}$ in $\mathscr{UD}$ from $(C_{[0,1]},\mathfrak{M})$ to $(C_{[0,1]},\mathfrak{M}^{\mathrm{tri}})$ so that $\mathrm{Eval}^{\mathrm{tri},0}\circ\mathfrak{F}=\mathfrak{C}\circ\mathrm{Eval}^0$ and $\mathrm{Eval}^{\mathrm{tri},1}\circ\mathfrak{F}=\mathrm{Eval}^1$. (For clarity, the symbols $\mathrm{Eval}^{\mathrm{tri},i}$ are used for the evaluation maps about the trivial pseudo-isotopy $\mathfrak{M}^{\mathrm{tri}}$). Now, we define

$$\mathfrak{F}=1\otimes\mathfrak{C}^{[s,1]}\quad\in\mathbf{CC}(C_{[0,1]},C_{[0,1]})\tag{78}$$

The notation $1\otimes\mathfrak{C}^{[s,1]}$ means the pointwise extension in Remark 2.13. Explicitly, for $x_i\in C^\infty([0,1],C)$, we have $\mathfrak{F}_{k,\beta}(1\otimes x_1,\dots,1\otimes x_k)(s)=1\otimes\mathfrak{C}^{[s,1]}_{k,\beta}(x_1(s),\dots,x_k(s))$ and $\mathfrak{F}_{k,\beta}(1\otimes x_1,\dots,ds\otimes x_i,\dots,1\otimes x_k)(s)=(-1)^{\sum_{a=1}^{i-1}(\deg x_a+1)}ds\otimes\mathfrak{C}^{[s,1]}_{k,\beta}(x_1(s),\dots,x_k(s))$. By the pointwiseness, we also require $\mathfrak{F}(\cdots ds\otimes y'\cdots ds\otimes y''\cdots)=0$. As the μ maps into $2\mathbb{Z}$, one can check the signs actually agrees with Definition 2.12. Next, we want to show the A_∞ relation:

$$\sum\mathfrak{M}^{\mathrm{tri}}\circ(\mathfrak{F}\otimes\cdots\otimes\mathfrak{F})=\sum\mathfrak{F}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{M}\otimes\mathrm{id}^\bullet)$$

Comparing $1\otimes -$ and $ds\otimes -$ parts separately, it is equivalent to the following two identities:

$$\mathfrak{m}^1\circ(\mathfrak{C}^{[s,1]}\otimes\cdots\otimes\mathfrak{C}^{[s,1]})=\mathfrak{C}^{[s,1]}\circ(\mathrm{id}^\bullet_\#\otimes\mathfrak{m}^s\otimes\mathrm{id}^\bullet)$$
$$\tfrac{d}{ds}\circ\mathfrak{C}^{[s,1]}=\mathfrak{C}^{[s,1]}(\mathrm{id}^\bullet\otimes\mathfrak{c}^s\otimes\mathrm{id}^\bullet)$$

The second one holds exactly because of Lemma 5.4; meanwhile, the first one holds since $\mathfrak{C}^{[s,1]}$ is an A_∞ homomorphism from $\mathfrak{m}^s$ to $\mathfrak{m}^1$. Hence, the $\mathfrak{F}$ constructed above is indeed a gapped A_∞ homomorphism; by construction, it is clear that $\mathrm{Eval}^{\mathrm{tri},0}\circ\mathfrak{F}=\mathfrak{C}\circ\mathrm{Eval}^0$ and $\mathrm{Eval}^{\mathrm{tri},1}\circ\mathfrak{F}=\mathrm{Eval}^1$.

It suffices to check the above $\mathfrak{F}$ is a morphism in $\mathscr{UD}$ (Definition 2.33). It is clear that the condition (II-5) holds for $\mathfrak{F}$. We show (II-1) and (II-4) as follows: Denote by $\mathbb{1}$ the $[0,1]$-unit of the given $\mathfrak{M}$. Then, $\mathbb{1}$ is a unit of any $\mathfrak{m}^s$; particularly, $\mathbb{1}:=\mathrm{Incl}(\mathbb{1})$ is also a $[0,1]$-unit of $\mathfrak{M}^{\mathrm{tri}}$. When $(k,\beta)\neq(1,0)$, we have $\mathfrak{F}_{k,\beta}(\dots,\mathrm{Incl}(\mathbb{1}),\dots)(s)=1\otimes\mathfrak{C}^{[s,1]}_{k,\beta}(\dots,\mathbb{1},\dots)\pm ds\otimes\mathfrak{C}^{[s,1]}_{k,\beta}(\dots,\mathbb{1},\dots)=0$, since $\mathfrak{C}^{[s,1]}$ is unital. Finally, since $\mathfrak{F}_{1,0}=\mathrm{id}_{C_{[0,1]}}$, we have $\mathfrak{F}_{1,0}(\mathbb{1})=\mathbb{1}$ and the condition (28) for $\mathfrak{F}_{1,0}$.

Next, we prove the cyclical unitality (II-2) for $\mathfrak{F}$. A degree-zero element of $C_{[0,1]}$ is like $\mathbf{e}:=1\otimes\mathbf{e}(s)$ for some $\mathbf{e}\in C^\infty([0,1],C^0)$, where C^0 denotes the degree-zero part of C. By the pointwiseness, we may assume the other inputs are in the form of $1\otimes x_i(s)$ for some $x_i\in C^\infty([0,1],C)$. Then, we compute

$$\mathrm{CU}[\mathfrak{F}]_{k,\beta}(\mathbf{e};1\otimes x_1,\dots,1\otimes x_k)(s)=1\otimes\mathrm{CU}[\mathfrak{C}^{[s,1]}]_{k,\beta}(\mathbf{e}(s);x_1(s),\dots,x_k(s))$$

It vanishes due to the cyclical unitality of $\mathfrak{C}^{[s,1]}$.

Finally, we prove the divisor axiom (II-3) for $\mathfrak{F}$. Take a divisor input $b=1\otimes b_0+ds\otimes b_1$ and assume other inputs are in the form of $y_i=1\otimes x_i$ as above. Then, by (78), we have

$$\mathrm{DA}[\mathfrak{F}]_{k,\beta}(b;y_1,\dots,y_k)=1\otimes\mathrm{DA}[\mathfrak{C}^{[s,1]}]_{k,\beta}(b_0;x_1,\dots,x_k)+ds\otimes\mathrm{CU}[\mathfrak{C}^{[s,1]}]_{k,\beta}(b_1;x_1,\dots,x_k)$$

Using both the divisor axiom and cyclical unitality of $\mathfrak{C}^{[s,1]}$, we complete the proof. $\square$

6 A_∞ algebras associated to Lagrangians

Let (X, ω) be a symplectic manifold; let L be a connected compact oriented Lagrangian submanifold equipped with a relatively spin structure. Recall that we denote by $\mathfrak{J}(X, \omega)$ the space of ω-tame almost complex structures and denote by $\mathfrak{J}(X, L, \omega)$ (Definition 1.1) its subset consisting of those almost complex structures J which does not allow negative Maslov index J-holomorphic disks bounding L.

All the A_∞ structures in this section are all in the chain level. Put $\mathfrak{G} = \mathfrak{G}(X, L) = \pi_2(X, L)$ (18).

6.1 Moduli spaces

Let $\beta \in \mathfrak{G}(X, L)$ and fix $J \in \mathfrak{J}(X, L, \omega)$. For $(k, \beta) \neq (0, 0), (1, 0)$, we denote by

$$\mathcal{M}_{k+1,\beta}(J, L) \tag{79}$$

the moduli space of all equivalence classes $[\mathbf{u}, \mathbf{z}]$ of $(k+1)$-*boundary-marked J-holomorphic stable maps* $(\mathbf{u}, \mathbf{z})$ *of genus zero with one boundary component in L in the class β*. We require $(k, \beta) \neq (0, 0), (1, 0)$ for the sake of stability; the $\mathcal{M}_{1,0}(J, L)$ and $\mathcal{M}_{2,0}(J, L)$ are just *not defined*. Here $\mathbf{z} = (z_0, z_1, \dots, z_k)$ represents the boundary marked points ordered counter-clockwisely.

The conception of *stable maps* is standard; see e.g. [FOOO10b], [FOOO17a], [HTK$^+$03], [Sei08], [MS12] and [Fra08]. The equivalence relation roughly refers to a biholomorphism on the domains of two stable maps which identifies the nodal points, the marked points and the boundaries. The moduli space $\mathcal{M}_{k+1,\beta}(J, L)$ admits a natural Hausdorff topology for which it is compact [FOOO10c, Theorem 7.1.43]. To get some intuition, we note that a 'point' in the interior $\mathcal{M}^\circ_{k+1,\beta}(J, L)$ of the moduli space is (the equivalence class of) a J-holomorphic map $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$ with the marked points $\mathbf{z} = (z_1, \dots, z_k)$ in the boundary $\partial\mathbb{D}$.

Let P be a smooth compact contractible oriented manifold with corners, and let $\bigsqcup_i \partial_i P$ be its normalized boundary with the decomposition into connected components. Provided a smooth family $\mathbf{J} = \{J_t \mid t \in P\}$ in $\mathfrak{J}(X, L, \omega)$, we consider the following bifurcation union of moduli spaces:

$$\mathcal{M}_{k+1,\beta}(\mathbf{J}, L) := \bigsqcup_{t \in P} \{t\} \times \mathcal{M}_{k+1,\beta}(J_t, L) \tag{80}$$

It is also called a parameterized moduli space. Denote by $\overline{\mathrm{ev}}_P : \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \to P$ the natural projection onto P. Given $0 \le i \le k$ and $t \in P$, there is a natural evaluation map $\mathcal{M}_{k+1,\beta}(J_t, L) \to L$ sending a stable disk $\mathbf{u}$ to $\mathbf{u}(z_i)$. Taking the union yields a natural map $\overline{\text{ev}}_i : \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \to L$ for each $i = 0, 1, \dots, k$. Now, we call the map

$$\text{ev}_i = (\overline{\text{ev}}_P, \overline{\text{ev}}_i) : \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \to P \times L \tag{81}$$

the i-th *evaluation map* of the parameterized moduli space $\mathcal{M}_{k+1,\beta}(\mathbf{J}, L)$, sending $(t, [\mathbf{u}, \mathbf{z}])$ to $(t, \mathbf{u}(z_i))$. Occasionally, we also call $\overline{\text{ev}}_i$ a evaluation map.

We consider the collection

$$\mathbb{M}(\mathbf{J}) = \Big\{ \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \mid (k, \beta) \in \mathbb{N} \times \mathfrak{G} \Big\} \tag{82}$$

of all the moduli spaces simultaneously, and we call $\mathbb{M}(\mathbf{J})$ a *moduli space system* or a *moduli system*.

The moduli space system $\mathbb{M}(\mathbf{J})$ admits a system of Kuranishi space structures satisfying a list of *axioms* (see [FOOO17b, Condition 21.11] or [FOOO18, Theorem 2.16] when P is a point). Following [FOOO17b, Definition 21.9 & Definition 21.13], the data is called a *P-parametrized tree-like K-system* (or simply a *tree-like K-system*) on $\mathbb{M}(\mathbf{J})$. Note that the letter 'K' stands for 'Kuranishi'. Instead of writing down the whole list of axioms, we just indicate they include the following aspects about the moduli space system:

(A0) Kuranishi structures

(A1) Evaluation maps

(A2) Orientation.

(A3) Compatibilities of boundary decompositions .

(A4) Dimension.

(A5) Energy and Gromov compactness.

A P-parameterized tree-like K-system on $\mathbb{M}(\mathbf{J})$ will produce an A_∞ algebra structure $\check{\mathfrak{M}}$ on the space $\Omega^*(L)_P \equiv \Omega^*(P \times L)$. In this case, we say that 'the $\mathfrak{M}$ is obtained by a tree-like K-system on $\mathbb{M}(\mathbf{J})$' or simply say that 'the $\check{\mathfrak{M}}$ is obtained by $\mathbb{M}(\mathbf{J})$'. We add the checks in the notations (e.g. $\check{\mathfrak{m}}$ or $\check{\mathfrak{M}}$) to emphasize the A_∞ algebras are in the chain-levels (i.e. defined on $\Omega^*(L)$ or $\Omega^*(L)_P$).

Now, we state the following Theorem 6.1 due to Fukaya-Oh-Ohta-Ono.

Theorem 6.1. *Given a smooth family $\mathbf{J} = \{J_t \mid t \in P\}$ in $\mathfrak{J}(X, \omega)$, there exists a P-pseudo-isotopy $(\Omega^*(L)_P, \check{\mathfrak{M}}^P) \in \text{Obj}\,\mathscr{U}\mathscr{D}$, obtained by a P-parameterized tree-like K-system on $\mathbb{M}(\mathbf{J})$, such that*

- *it is a q.c.dR (Definition 2.26);*
- *it is fully unital and strictly unital with the constant-one $\mathbb{1}$ as a unit (Definition 2.23 & 2.24);*

- *it satsifies the divisor axiom (Definition 2.31);*
- *if a $\partial_i P$-pseudo-isotopy $\check{\mathfrak{M}}^{\partial_i P}$ is already obtained like this for all i, then we may further assume $\check{\mathfrak{M}}^P$ restricts to $\check{\mathfrak{M}}^{\partial_i P}$ for each i.*

If the family $\mathbf{J}$ is contained in $\mathfrak{J}(X,L,\omega)$, we can require $(\Omega^*(L)_P, \check{\mathfrak{M}}^P) \in$ $\mathrm{Obj}\,\mathscr{U}\mathscr{D}$. Moreover, if $\mathbf{J}=(J_t)$ is a constant family, we can require $\check{\mathfrak{M}}^P$ to be a trivial P-pseudo-isotopy[12].

A version of Theorem 6.1 is proved in [FOOO17b, Theorem 21.35] and [FOOO18, Theorem 2.16], but we want to use the techniques in [Fuk10] to further show the divisor axiom and the full unitality. Specifically, as explained in Remark 2.27, one can show the full unitality by the method of showing the strict unitality as in [Fuk10, (7.3)]. Moreover, although it is not explicitly stated in the latest [FOOO17b] or [FOOO18], the divisor axiom is proved in [Fuk10, Lemma 13.1].

We will give an explanation of the proof of Theorem 6.1 later in §6.2. In practice, we only focus on the cases when P is a d-simplex for $d=0,1,2$; the three cases respectively correspond to the local charts, the transition maps, and the cocycle conditions for the mirror in Theorem 1.3.

In view of Assumption 1.2, we only pay attention to the category $\mathscr{U}\mathscr{D}$ rather than $\tilde{\mathscr{U}\mathscr{D}}$ (§2.6); the following three theorems we need are all the direct consequences of Theorem 6.1.

Theorem 6.2. *Given $J\in\mathfrak{J}(X,L,\omega)$, there exists an A_∞ algebra $(\Omega^*(L),\check{\mathfrak{m}}^{J,L})$ in $\mathscr{U}\mathscr{D}$ obtained by a tree-like K-system on $\mathbb{M}(J)$.*

Fix $J_0,J_1\in\mathfrak{J}(X,L,\omega)$. Suppose $\check{\mathfrak{m}}^{J_0,L}$ and $\check{\mathfrak{m}}^{J_1,L}$ are obtained as in Theorem 6.2. Let $\mathbf{J}=(J_t)$ be a path in $\mathfrak{J}(X,L,\omega)$ between J_0 and J_1. Then, we have:

Theorem 6.3. *There exists a pseudo-isotopy $(\Omega^*(L)_{[0,1]},\check{\mathfrak{M}}^{\mathbf{J},L})$ in $\mathscr{U}\mathscr{D}$, obtained by a tree-like K-system on $\mathbb{M}(\mathbf{J})$, such that it restricts to $\check{\mathfrak{m}}^{J_0,L}$ and $\check{\mathfrak{m}}^{J_1,L}$ at $s=0,1$. Moreover, if $\mathbf{J}$ is a constant family, then we can require $\check{\mathfrak{M}}^{\mathbf{J},L}$ to be a trivial pseudo-isotopy.*

Convention 6.4. To be more precise, we should write $\check{\mathfrak{m}}^{J,L,\Xi_J}$, further specifying a datum $\Xi=\Xi_J$ of Kuranishi-theory-related choices (Kuranishi structures, CF-perturbations, and so on). We often call Ξ a virtual fundamental chain. Since it is also a contractible choice, we often omit Ξ in the notations. Indeed, applying Theorem 6.3 to the special case $J_0=J_1$ implies that the $\check{\mathfrak{m}}^{J,L,\Xi}$ is independent of the data Ξ up to pseudo-isotopy; see e.g. [Fuk10, Theorem 14.2].

Fix $J_0,J_1,J_2\in\mathfrak{J}(X,L,\omega)$; suppose $\check{\mathfrak{m}}^{J_0,L}$, $\check{\mathfrak{m}}^{J_1,L}$, and $\check{\mathfrak{m}}^{J_2,L}$ are obtained as in Theorem 6.2. Fix $\mathbf{J}_{i,i+1}$ to be a path in $\mathfrak{J}(X,L,\omega)$ from J_i to J_{i+1} for $i=0,1,2 \pmod 3$; suppose $\check{\mathfrak{M}}^{\mathbf{J}_{i,i+1}}$ are obtained as in Theorem 6.3. Let $\mathbb{J}=(J_t)_{t\in\Delta^2}$ be a family in $\mathfrak{J}(X,L,\omega)$ parameterized by the standard 2-simplex $\Delta^2=[v_0,v_1,v_2]$ such that $\mathbb{J}|_{[v_i,v_{i+1}]}=\mathbf{J}_{i,i+1}$ for $i=0,1,2 \pmod 3$. Then, we have:

[12] We cannot say it *must* be a trivial pseudo-isotopy, because it also depends on a choice of 'virtual fundamental chain' on the moduli spaces and one can wildly make the choice.

Theorem 6.5. *There exists a Δ^2-pseudo-isotopy $(\Omega^*(L)_{\Delta^2}, \breve{\mathfrak{M}}^{\mathbb{J}})$ in $\mathscr{U}\mathscr{D}$, obtained by a tree-like K-system on $\mathbb{M}(\mathbb{J})$, such that it restricts to $\breve{\mathfrak{M}}^{\mathbf{J}_{i,i+1}}$ on the edge $[v_i, v_{i+1}] \subset \Delta^2$.*

6.2 A_∞ relations

In this section, we explain how the P-parameterized tree-like K-system on the moduli system gives rise to a P-pseudo-isotopy in Theorem 6.1 based on the foundation works of Fukaya-Oh-Ohta-Ono in [FOOO15, FOOO17b, FOOO17a, FOOO18]. We cannot present the full details, but we will make as precise as possible citations, indicating which statements in the references are used in the process.

6.2.1 Virtual fundamental chains . The aforementioned axiom item (A0) for the tree-like K-system means the existence of a system $\widehat{\mathcal{U}}$ of Kuranishi structures on the moduli system $\mathbb{M}(\mathbf{J})$. There is a general strategy of constructing 'virtual fundamental chains' from a given Kuranishi structure [FOOO15, §6.4]. Roughly speaking, given the above $\widehat{\mathcal{U}}$, we first take a good coordinate system and find a system $\widehat{\mathcal{U}^+}$ of collared Kuranishi structures which can be viewed as a 'thickening' of $\widehat{\mathcal{U}}$ indicating the information of 'gluing'. Next, with regard to $\widehat{\mathcal{U}^+}$, we can find a system $\widehat{\mathfrak{S}}$ of CF-perturbations which basically plays the role of 'virtual fundamental chain' for the moduli system.

To be extremely careful, the $\widehat{\mathcal{U}^+}$ and $\widehat{\mathfrak{S}}$ depend on a parameter $\epsilon > 0$ and are only applied to the system of moduli spaces with an *energy cut* [FOOO17b, Definition 22.8]. In the first place, we only produce A_∞ algebras with energy cuts, but these A_∞ algebras are pseudo-isotopic to each other with an energy cut $E > 0$. We may in a sense pass to a limit as $E \to \infty$ and obtain a true A_∞ algebra over the Novikov field Λ. See the proof of [FOOO17b, Theorem 21.35] and [Fuk17, 3.36-3.39].

Remark that the Kuranishi theory generalizes the manifold theory, and many properties for manifolds still hold. The two most important properties we need in the Kuranishi theory are the *Stokes' formula* [FOOO15, Proposition 9.26] and *composition formula* [FOOO15, Theorem 10.20].

6.2.2 Composition formulas . The axiom item (A1) includes the strong smoothness of ev_ℓ for $\ell \geq 0$ and weakly submersiveness of ev_0 in the sense of [FOOO15, Definition 3.38]. The latter one ensures a definition of the pushforward $\mathrm{ev}_{0!} = \mathrm{ev}_{0!}(-; \widehat{\mathfrak{S}})$ regarding the above-mentioned 'virtual fundamental chain' i.e. the CF-perturbation $\widehat{\mathfrak{S}}$. The map $\mathrm{ev}_{0!}$ is basically like the integration along fibers [BT13] or the Gysin map [LM16] in manifold theory. By [FOOO15, Situation 7.1], a *smooth correspondence* means a tuple $\mathfrak{X} = (X, M, M_0, f, f_0)$ consisting of a compact metrizable space X equipped with a Kuranishi structure $\widehat{\mathcal{U}}$, two smooth manifolds M_0 and M, a strongly smooth map $f : (X, \widehat{\mathcal{U}}) \to M$ and a weakly submersive strongly smooth map $f_0 : (X, \widehat{\mathcal{U}}) \to M_0$. By [FOOO15, (7.1)], we may define a map on the de Rham complexes

$$\mathrm{Corr}_{\mathfrak{X}} \equiv \mathrm{Corr}(X; f, f_0) : \Omega^*(M_0) \to \Omega^{\ell+*}(M) \tag{83}$$

for $\ell = \dim(M) - \mathrm{vdim}(X)$ and we call it a *correspondence map*. Now, suppose we have two smooth correspondences $\mathfrak{X}_{12} = (X_{12}, f_1, f_2)$ and $\mathfrak{X}_{23} = (X_{23}, g_2, g_3)$, we can take the *fiber product Kuranishi space* $X_{13} := X_{12} \times_{(f_2, g_2)} X_{23}$ illustrated as follows:

$$\begin{tikzcd} & & X_{13} \arrow[dl, "h_{12}"'] \arrow[dr, "h_{23}"] & & \\ & X_{12} \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & X_{23} \arrow[dl, "g_2"'] \arrow[dr, "g_3"] & \\ M_1 & & M_2 & & M_3 \end{tikzcd} \tag{84}$$

This turns out to give a new smooth correspondence $\mathfrak{X}_{13} := (X_{13}, f_1 \circ h_{12}, g_3 \circ h_{23})$, called the *composition of smooth correspondences* [FOOO17b, Definition 10.16]. Now, the composition formula [FOOO15, Theorem 10.20] means that

$$\mathrm{Corr}_{\mathfrak{X}_{13}} = \mathrm{Corr}_{\mathfrak{X}_{23}} \circ \mathrm{Corr}_{\mathfrak{X}_{12}}$$

More specifically, by [Fuk10, Proposition 4.3], this means that

$$\mathrm{Corr}_{\mathfrak{X}_{23}}\big(\mathrm{Corr}_{\mathfrak{X}_{12}}(h_1) \times h_2\big) = \mathrm{Corr}_{\mathfrak{X}_{13}}(h_1 \times h_2) \tag{85}$$

6.2.3 Stokes' formulas . The other important formula inherited from manifold theory is the *Stokes' formula*. By [FOOO15, Corollary 8.13], a smooth correspondence $\mathfrak{X} = (X, M, M_0, f, f_0)$ induces a boundary smooth correspondence $\partial\mathfrak{X} = (\partial X, M, M_0, f|_{\partial X}, f_0|_{\partial X})$ which also gives a correspondence map $\mathrm{Corr}_{\partial\mathfrak{X}}$ as before. Then, the Kuranishi-version of Stokes' formula says that

$$d_{M_0} \circ \mathrm{Corr}_{\mathfrak{X}} - \mathrm{Corr}_{\mathfrak{X}} \circ d_M = \mathrm{Corr}_{\partial\mathfrak{X}} \tag{86}$$

where d_{M_0} and d_M denote the exterior derivatives on the spaces of differential forms.

6.2.4 Definition of the operator system . Consider a smooth correspondence $\mathfrak{X} = (X, M, M_0, f, f_0)$ given by $M_0 = P \times L$, $M = (P \times L)^{\times k}$, $f_0 = \mathrm{ev}_0$, $f = \mathrm{ev}_1 \times \cdots \times \mathrm{ev}_k$, and $X = \mathcal{M}_{k+1,\beta}(\mathbf{J}, L)$. Recall that $\mathbf{J} = (J_t)_{t\in P}$ is a P-family of almost complex structures. As in [Fuk10, (7.1)], we define

$$\check{\mathfrak{M}}_{k,\beta}(x_1, \dots, x_k) = \mathrm{Corr}(\mathcal{M}_{k+1,\beta}(\mathbf{J}, L); (\mathrm{ev}_1, \dots, \mathrm{ev}_k), \mathrm{ev}_0)(x_1 \times \cdots \times x_k) \tag{87}$$

Here $x_1 \times \cdots \times x_k$ denotes $\pi_1^* x_1 \wedge \cdots \wedge \pi_k^* x_k$ where π_i's are the projections. Exceptionally, we *define*

$$\check{\mathfrak{M}}_{0,0} = 0 \ \text{ and } \ \check{\mathfrak{M}}_{1,0} = d$$

Notice that when X happens to be a smooth manifold and f_0 is submersive, the correspondence map (83) just gives $\operatorname{Corr}(h) = \pm f_{0!} f^*(h)$ for a form h on M. Thus, instead of (87), it is more convenient to use the following notation:

$$\check{\mathfrak{M}}_{k,\beta}(x_1, \dots, x_k) = (-1)^\epsilon \mathrm{ev}_{0!}(\mathrm{ev}_1^* x_1 \wedge \cdots \wedge \mathrm{ev}_k^* x_k) \tag{88}$$

where

$$\epsilon = \sum_{j=1}^k j(\deg x_j + 1) + 1 \tag{89}$$

Recall that the sign convention we adopt follows [Sol20, §4.3] rather than [Fuk17, (3.40)]. Alternatively, the coefficient of y in $\check{\mathfrak{M}}_{k,\beta}(x_1, \dots, x_k)$ is given by the following 'virtual count':

$$\pm \int^{\mathrm{vir}}_{\mathcal{M}_{k+1,\beta}(\mathbf{J},L)} \mathrm{ev}_0^* y \wedge \mathrm{ev}_1^* x_1 \wedge \cdots \wedge \mathrm{ev}_k^* x_k \tag{90}$$

Now, we have defined an operator system $\check{\mathfrak{M}} = (\check{\mathfrak{M}}_{k,\beta})$ in $\mathbf{CC}_{\mathfrak{G}}$. We also know the degree $\deg \check{\mathfrak{M}}_{k,\beta} = 2 - k - \mu(\beta)$ is as expected due to the dimension axiom (A4): $\mathrm{vdim} \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) = \mu(\beta) + (k+1) - 3 + \dim(P \times L)$. Besides, the condition (A5) for the energy and Gromov compactness can infer that $\check{\mathfrak{M}}$ is $\mathfrak{G}$-gapped. Finally, it remains to show that the A_∞ formula $\check{\mathfrak{M}} \star \check{\mathfrak{M}} = 0$.

6.2.5 Proving the A_∞ formula . By the axioms (A2) and (A3), we have

$$\partial \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \cong \bigsqcup_{\beta_1+\beta_2=\beta;\ k_1+k_2=k;\ 1\le i\le k_2} (-1)^* \mathcal{M}_{k_1+1,\beta_1}(\mathbf{J}, L) \times_{(\mathrm{ev}_i, \mathrm{ev}_0)} \mathcal{M}_{k_2+1,\beta_2}(\mathbf{J}, L)$$

where the left side uses the boundary smooth correspondence and the right side uses the composition of smooth correspondences (both are briefly described before). Note that here we require (k_1, β_1) and (k_2, β_2) are not equal to $(0,0), (1,0)$. Next, we temporarily abbreviate $\mathcal{M}_{k+1,\beta} := \mathcal{M}_{k+1,\beta}(\mathbf{J}, L)$. Then, the above boundary decomposition gives us a diagram like (84) as follows:

$$\begin{tikzcd}
 & & & \mathcal{M}_{k_1+1,\beta_1} \times_{(\mathrm{ev}_i,\mathrm{ev}_0)} \mathcal{M}_{k_2+1,\beta_2} \arrow[dll] \arrow[drr] & & & \\
 & \mathcal{M}_{k_2+1,\beta_2} \arrow[dl, "(\mathrm{ev}_1,\dots,\mathrm{ev}_{k_2})"'] \arrow[drr, "\iota_i \circ \mathrm{ev}_0"] & & & & \mathcal{M}_{k_1+1,\beta_1} \arrow[dll, "(\mathrm{ev}_1,\dots,\mathrm{ev}_{k_1})"'] \arrow[dr, "\mathrm{ev}_0"] & \\
(P \times L)^{\times k_2} & & & (P \times L)^{\times k_1} & & & P \times L
\end{tikzcd}$$

Using (86) and (87), we obtain

$$d \circ \check{\mathfrak{M}}_{k,\beta}(x_1, \dots, x_k) + \sum_j (-1)^* \check{\mathfrak{M}}_{k,\beta}(x_1, \dots, dx_j, \dots, x_k) = \operatorname{Corr}(\partial \mathcal{M}_{k+1,\beta}; (\mathrm{ev}_1, \dots, \mathrm{ev}_k), \mathrm{ev}_0)(x_1 \times \cdots \times x_k) \tag{91}$$

Hence, applying the composition formula (85), the right side of (91) is equal to

$$\sum \pm \operatorname{Corr}(\partial \mathcal{M}_{k_1+1,\beta_1})(x_1 \times \cdots \times \operatorname{Corr}(\partial \mathcal{M}_{k_2+1,\beta_2})(x_{i+1} \times \cdots) \times \cdots \times x_k) = \sum \pm \check{\mathfrak{M}}_{k_1,\beta_1}(x_1, \dots, \check{\mathfrak{M}}_{k_2,\beta_2}(\dots) \dots, x_k)$$

Recall we assume $(k_1,\beta_1) \neq (1,0) \neq (k_2,\beta_2)$ here; but the missing terms with $(k_i,\beta_i)=(1,0)$ can be replenished precisely by the left side of (91) as $\check{\mathfrak{M}}_{1,0}=d$. So, we obtain the desired A_∞ formula. The signs are decided as in [FOOO17b, Condition 21.11] or [FOOO18, Theorem 2.16]; see also [Sol20].

6.2.6 Pointwiseness . We also need to check the P-pointwiseness (Definition 2.12) for a P-pseudo-isotopy (Definition 2.17). This is roughly because of the bifurcation nature of the parameterized moduli spaces (80) and the evaluation maps $\mathrm{ev}_i=(\overline{\mathrm{ev}}_P,\overline{\mathrm{ev}}_i)$ (81). See [FOOO17b, §22.3] for the details. Briefly, fix $\sigma\in\Omega^*(P)$; our goal is to show $\check{\mathfrak{M}}_{k,\beta}(x_1,\dots,x_i,\sigma\wedge x_{i+1},x_{i+1}\dots,x_k)=(-1)^\dagger d\sigma\wedge\check{\mathfrak{M}}_{k,\beta}(x_1,\dots,x_k)$ for the sign $\dagger=\sum_{j=1}^i(\deg x_j-1)\deg\sigma$. By (90), this amounts to count $(-1)^\epsilon \mathrm{ev}^*y\wedge\mathrm{ev}^*x_1\wedge\cdots\wedge\mathrm{ev}^*x_i\wedge\mathrm{ev}^*(\sigma\wedge x_{i+1})\cdots\wedge\mathrm{ev}^*x_k$. The naive switching $\mathrm{ev}^*x_i\wedge\mathrm{ev}^*\sigma=(-1)^{\epsilon_i'}\mathrm{ev}^*\sigma\wedge\mathrm{ev}^*x_i$ gives the sign $\epsilon_i'=\deg x_i\deg\sigma$; however, since the attaching marked point of σ is changed from the $(i+1)$-th to the i-th marked point, there is an extra sign-change $(-1)^{\deg\sigma}$ due to (89). So, we set $\epsilon_i:=\epsilon_i'+\deg\sigma=(\deg x_i-1)\deg\sigma$, and the total sign change is exactly given by $\sum\epsilon_i=\dagger$.

6.2.7 Trivial pseudo-isotopies . Suppose the family of almost complex structures happens to be constant, say, $\mathbf{J}_0=\{J_t=J\mid t\in P\}$ for a fixed J. Then, the moduli space (80) becomes

$$\mathcal{M}_{k+1,\beta}(\mathbf{J}_0,L)\equiv P\times\mathcal{M}_{k+1,\beta}(J,L)$$

and the evaluation maps (81) will reduce to $\mathrm{ev}_i=(\overline{\mathrm{ev}}_P,\overline{\mathrm{ev}}_i)\equiv\mathrm{id}_P\times\overline{\mathrm{ev}}_i: P\times\mathcal{M}_{k+1,\beta}(J,L)\to P\times L$. Accordingly, if the tree-like K-system and 'virtual fundamental chains' on the moduli space system $\mathbb{M}(\mathbf{J}_0)$ are the trivial extensions from the ones on $\mathbb{M}(J)$, then the A_∞ algebra $\check{\mathfrak{M}}$ defined as in (87) will give rise to a *trivial P-pseudo-isotopy* (Example 2.20) about the A_∞ algebra $\check{\mathfrak{m}}$ for $\mathbb{M}(J)$.

6.2.8 Contractible choices . We explain the last item in Theorem 6.1, namely, we aim to obtain $\mathfrak{M}^P$ with the specified restrictions $\mathfrak{M}^{\partial_i P}$. The reference is [FOOO17b, §17]. First of all, we note that the stable map topology on any moduli space is well-understood. If there exists a Kuranishi structure defined on a neighborhood of a compact space, then we can extend it to a global Kuranishi structure without any change in that neighborhood [FOOO17b, Lemma 17.59]. By construction, the Kuranishi structure on the boundary of a moduli space has a collared extension to its small neighborhood; see [FOOO17b, Remark 17.1]. Accordingly, given ∂_iP-parameterized tree-like K-systems on the moduli system $\mathbb{M}(\mathbf{J}|_{\partial_iP})$ for all i, one can find an extended P-parameterized tree-like K-system on the moduli system $\mathbb{M}(\mathbf{J})$. Moreover, the CF-perturbations are actually *contractible* choices, i.e., there is also no obstruction to find an extension $\widehat{\mathfrak{S}}$ from the given $\widehat{\mathfrak{S}}_{\partial_iP}$; see [FOOO17b, §17.7]. Note that Pardon somehow uses a similar philosophy; see around [Par16, (7.0.1)].

To complete Theorem 6.1, it remains to check the following three properties for the $\check{\mathfrak{M}}$: (1) q.c.dR, (2) the divisor axiom, and (3) unitalties. Each of them will require mild restrictions on these contractible choices but there will be still plenty of them.

6.3 Forgetful maps

6.3.1 Quantum corrections to de Rham theory . We first explain that the $\check{\mathfrak{M}}$ defined as above is indeed a q.c.dR in the sense of Definition 2.26. It is known in the literature: [FOOO17b, Definition 21.21 & Theorem 21.35] and [FOOO10b, Definition 3.5.6 & Remark 3.5.8]. We give a brief review as follows.

If $\beta = 0$, the moduli space has a simple form: $\mathcal{M}_{k+1,0}(\mathbf{J}, L) \cong P \times L \times \mathbb{R}^{k-2}$ for every $k \geq 2$; see [FOOO17b, Condition 21.11 (V)]. Geometrically, this is because the moduli space only contains the constant maps into L; the second command $\mathbb{R}^{k-2}$ corresponds to the ordered marked points in $\partial\mathbb{D}$; c.f. [FO97, Lemma 1.3]. In this case, the evaluation maps (81) reduce to the projection $\mathrm{pr} = (\mathrm{id}, \overline{\mathrm{pr}}) : P \times L \times \mathbb{R}^{k-2} \to P \times L$. Thus, $\check{\mathfrak{M}}_{k,0}(x_1, \dots, x_k) = \pm \mathrm{pr}_! \mathrm{pr}^* x_1 \wedge \cdots \wedge \mathrm{pr}^* x_k$. When $k \geq 3$, there is a non-trivial command $\mathbb{R}^{k-2}$ for the fibers of pr, so $\mathrm{pr}_! \mathrm{pr}^* = 0$ and $\check{\mathfrak{M}}_{k,0} = 0$. When $k = 2$, we have $\mathrm{pr} = \mathrm{id}$ and so $\check{\mathfrak{M}}_{2,0}(x_1, x_2) = (-1)^{\deg x_1} x_1 \wedge x_2$. Here the sign is exactly obtained by using (89); c.f. [Sol20, Lemma 4.2] or [ST16, Proposition 3.7].

Notice that we also obtain the divisor axiom for $\beta = 0$ as a byproduct. Indeed, fix $b \in \Omega^1(P \times L)$ with $db = 0$. When $k \geq 2$, since $\check{\mathfrak{M}}_{k+1,0} = 0$, we have $\check{\mathfrak{M}}_{k+1,0}(b, x_1, \dots, x_k) + \cdots + \check{\mathfrak{M}}_{k+1,0}(x_1, \dots, x_k, b) = 0$. When $k = 1$, we have $\check{\mathfrak{M}}_{2,0}(b, x) + \check{\mathfrak{M}}_{2,0}(x, b) = (-1)^{(\deg b - 1) \deg x} b \wedge x + (-1)^{(\deg x - 1) \deg b} x \wedge b = 0$.

6.3.2 divisor axiom and unitality . Now, we explain the divisor axiom based on the forgetful maps as used in [Fuk10, Lemma 13.1]. Roughly, the system of Kuranishi structures on the moduli spaces and the CF-perturbation can be chosen to be *forgetful-map-compatible* in the sense of [Fuk10, §3, §5] [FOOO16, Lemma 2.6.16]. This basically means the 'local triviality' of these forgetful maps so that they behave like a 'fibration', which ensures one can also define the pushforward $\mathfrak{forget}_!$. The divisor axiom for $\beta = 0$ is proved in §6.3.1. So, we may assume $\beta \neq 0$, and we aim to show

$$\mathrm{DA}[\check{\mathfrak{M}}]_{k,\beta}(b; x_1, \dots, x_k) = \partial\beta \cap b \cdot \check{\mathfrak{M}}_{k,\beta}(x_1, \dots, x_k)$$

where b is a divisor input (24), i.e., a closed 1-form. For $\ell = 1, \dots, k+1$, the ℓ-th *forgetful map*

$$\mathfrak{forget}_\ell : \mathcal{M}_{k+2,\beta}(\mathbf{J}, L) \to \mathcal{M}_{k+1,\beta}(\mathbf{J}, L) \tag{92}$$

sends $[\mathbf{u}, \tilde{\mathbf{z}}]$ to the stablization of $[\mathbf{u}, \mathbf{z}]$ where $\mathbf{z}$ is obtained by forgetting the ℓ-th marked point in $\tilde{\mathbf{z}}$. Concretely, if $\tilde{\mathbf{z}} = (z_0, z_1, \dots, z_{k+1})$, we define $\mathbf{z} = (z_0, \dots, z_{\ell-1}, z_{\ell+1}, \dots, z_{k+1})$. Note that the sign of $\mathfrak{forget}_\ell$ is equal to $(-1)^{\ell-1}$. Denote by ev_i $(0 \leq i \leq k+1)$ and ev'_i $(0 \leq i \leq k)$ the evaluation maps (81) for the left and right sides of (92) respectively. Then

$$\mathrm{ev}'_i \circ \mathfrak{forget}_\ell = \begin{cases} \mathrm{ev}_i & \text{if} \quad 0 \leq i \leq \ell - 1 \\ \mathrm{ev}_{i+1} & \text{if} \quad \ell \leq i \leq k \end{cases} \tag{93}$$

$$
\begin{tikzcd}
 & L & \\
\mathcal{M}_{k+2,\beta}(\mathbf{J},L) \arrow[ur, "\mathrm{ev}_i \text{ or } \mathrm{ev}_{i+1}"] \arrow[rr, "\mathfrak{forget}_\ell"] & & \mathcal{M}_{k+1,\beta}(\mathbf{J},L) \arrow[ul, "\mathrm{ev}'_i"']
\end{tikzcd}
$$

The ev_ℓ is missed in (93) and will correspond to the divisor input b. By (88), we have

$$
\begin{aligned}
\check{\mathfrak{M}}_{k+1,\beta}(x_1,\dots,x_{\ell-1},b,x_\ell,\dots,x_k) =&(-1)^\epsilon \mathrm{ev}_{0!}\big(\mathrm{ev}_1^* x_1\wedge\cdots\wedge \mathrm{ev}_{\ell-1}^* x_{\ell-1}\wedge \mathrm{ev}_\ell^* b\wedge \mathrm{ev}_{\ell+1}^* x_\ell\wedge\cdots\wedge \mathrm{ev}_{k+1}^* x_k\big)\\
=&(-1)^{\epsilon+\delta}\mathrm{ev}'_{0!}\mathfrak{forget}_{\ell!}\Big(\mathfrak{forget}_\ell^*\big(\mathrm{ev}_1'^* x_1\wedge\cdots\wedge \mathrm{ev}_k'^* x_k\big)\wedge \mathrm{ev}_\ell^* b\Big)
\end{aligned}
$$

where the sign ϵ is given by (89):

$$
\epsilon = 1+\sum_{j=1}^{\ell-1} j(\deg x_j+1)+\ell(\deg b+1)+\sum_{j=\ell}^{k}(j+1)(\deg x_j+1)
$$

and $\delta=\sum_{j=\ell}^k \deg x_j \deg b$ is an extra sign due to the graded antisymmetricity of the wedge products. In manifold theory, the integration along fiber $\pi_!$ for some map π satisfies that $\pi_!(\alpha\wedge\pi^*\beta)=\pi_!\alpha\wedge\beta$. One can generalize it to the Kuranishi theory, which implies that

$$
\check{\mathfrak{M}}_{k+1,\beta}(x_1,\dots,x_{\ell-1},b,x_\ell,\dots,x_k) = (-1)^{\epsilon+\delta+\ell-1}\mathrm{ev}'_{0!}\big(\mathrm{ev}'^*(x_1\times\cdots\times x_k)\cdot \mathfrak{forget}_{\ell!}\mathrm{ev}_\ell^* b\big)
$$

Now, we put

$$
\mathfrak{F}_\ell := \mathfrak{forget}_{\ell!}\mathrm{ev}_\ell^* b
$$

Since the fibers of $\mathfrak{forget}_\ell$ are one-dimensional, the degree of $\mathfrak{F}_\ell$ is zero, and it can be thought of as a zero form on the moduli space like a 'charge'. Fix a 'point' $\mathbf{p}=(t,[\mathbf{u},\mathbf{z}])$ in $\mathcal{M}_{k+1,\beta}(\mathbf{J},L)$. Then,

$$
\mathfrak{F}_\ell(\mathbf{p}) = \int_{\mathfrak{forget}_\ell^{-1}(\mathbf{p})} \mathrm{ev}_\ell^* b = \int_{\mathrm{ev}_\ell\left(\mathfrak{forget}_\ell^{-1}(\mathbf{p})\right)} b
$$

We may view $\mathrm{ev}_\ell\big(\mathfrak{forget}_\ell^{-1}(\mathbf{p})\big)$ as a singular chain (or current) in $P\times L$ that is roughly described by the set of $(t,\mathbf{u}(z))$ for some marked point $z\in\partial\mathbb{D}$ lies between the $(\ell-1)$-th and $(\ell+1)$-th marked points. But, the order constrains on the marked point z can be eliminated by considering all possible ℓ. In other words, by taking the summation over ℓ, we obtain

$$
\textstyle\sum_\ell \mathrm{ev}_\ell\big(\mathfrak{forget}_\ell^{-1}(\mathbf{p})\big) = \{(t,\mathbf{u}(z))\mid z\in\partial\mathbb{D}\} = (\iota_t\times\partial\mathbf{u})_*\partial\mathbb{D}
$$

where we denote by ι_t the inclusion $\{t\}\to P$. Hence

$$
\textstyle\sum_\ell \mathfrak{F}_\ell(\mathbf{p}) = \int_{(\iota_t\times\partial\mathbf{u})_*\partial\mathbb{D}} b = \int_{\partial\mathbb{D}}(\iota_t\times\partial\mathbf{u})^* b = \partial\beta\cap b
$$

which actually does not depend on the choice of the moduli point $\mathbf{p}$. In summary, we have

$$
\begin{aligned}
\textstyle\sum_\ell \check{\mathfrak{M}}_{k+1,\beta}(x_1,\dots,x_{\ell-1},b,x_\ell,\dots,x_k) &= (-1)^{\epsilon+\delta+\ell-1}\mathrm{ev}'_{0!}\big(\mathrm{ev}'^*(x_1\times\cdots\times x_k)\cdot \textstyle\sum_\ell \mathfrak{F}_\ell\big)\\
&= \partial\beta\cap b\cdot(-1)^{\epsilon+\delta+\ell-1}\cdot \mathrm{ev}'_{0!}\mathrm{ev}'^*(x_1\times\cdots\times x_k) = \partial\beta\cap b\cdot \check{\mathfrak{M}}_{k,\beta}(x_1,\dots,x_k)
\end{aligned}
$$

and thus the $\check{\mathfrak{M}}$ satisfies the divisor axiom; see [Fuk10, Lemma 13.1] for more details.

Next, we aim to show the strict unitality (Definition 2.23) and the full unitality (Definition 2.24); the later one further implies the cyclical unitality (Definition 2.25). As explained in Remark 2.27, because of the q.c.dR property (§6.3.1), we may assume $\beta \neq 0$, and it remains to show $\check{\mathfrak{M}}_{k+1,\beta}(\dots, \mathbf{e}, \dots) = 0$ for any $\mathbf{e} \in \Omega^0(P \times L)$. To see this, we consider the forgetful map in (92) and we also have (93). Performing a similar argument as above yields

$$\check{\mathfrak{M}}_{k+1,\beta}(x_1, \dots, x_{\ell-1}, \mathbf{e}, x_\ell, \dots, x_k) = \mathrm{ev}'_{0!}\big(\mathfrak{forget}_{\ell!}\mathrm{ev}^*_\ell \mathbf{e} \cdot \mathrm{ev}'^*(x_1 \times \cdots \times x_k)\big)$$

But this time $\mathfrak{forget}_{\ell!}\mathrm{ev}^*_\ell \mathbf{e}$ has degree -1 and need to vanish. To be specific, one can focus on its local expression in a single Kuranishi chart by a partition of unity. See [Fuk10, (7.3)-(7.4)] for more details.

7 Harmonic contractions

In §6 above, the A_∞ structures are defined only on the de Rham cochain complexes. To avoid infinite dimensions, we use the homological perturbation (Theorem 4.4) to construct minimal A_∞ algebras on the de Rham cohomologies. But, notice that this requires a choice of contraction (Definition 4.3), and it is not unique in general. In practice, we decide to use the so-called *harmonic contraction* with respect to a metric. The advantage is that it is concrete and can keep track of Fukaya's trick (studied later in §8) from the chain-level to the cohomology-level.

7.1 g-harmonic contractions

Fix a closed manifold L and a Riemannian metric g. Put $C = \Omega^*(L)$ and $H = H^*(L)$. Let $\mathcal{H}^*_g(L)$ be the space of g-harmonic forms. Firstly, we define a cochain map from H to C:

$$i(g) : H^*(L) \xrightarrow{\cong} \mathcal{H}^*_g(L) \subset \Omega^*(L) \tag{94}$$

by the inverse of Hodge isomorphism composed with the inclusion. Secondly, we also define a cochain map from C to H by the g-orthogonal projection $\mathcal{H}_g : \Omega^*(L) \to \mathcal{H}^*_g(L)$ composited with $i(g)^{-1}$:

$$\pi(g) : \Omega^*(L) \xrightarrow{\mathcal{H}_g} \mathcal{H}^*_g(L) \xrightarrow{i(g)^{-1}} H^*(L) \tag{95}$$

Recall that the Hodge decomposition theorem (see e.g. [War13]) tells that

$$\Omega^*(L) = \mathcal{H}^*_g(L) \oplus d\Omega^*(L) \oplus \delta_g \Omega^*(L)$$

where the direct sum is g-orthogonal and δ_g is the dual of d with respect to the metric g. Moreover there is the so-called *Green operator* $\mathrm{Gr}_g : \Omega^*(L) \to$

$\mathcal{H}_g^*(L)^{\perp_g} := d\Omega^*(L) \oplus \delta_g\Omega^*(L)$. It is obtained by declaring $\mathrm{Gr}_g(\alpha)$ to be the unique solution η in $\mathcal{H}_g^*(L)^{\perp_g}$ of the differential equation $\Delta_g\eta = \alpha - \mathcal{H}_g(\alpha)$, where $\Delta_g = d\delta_g + \delta_g d$. So, we have

$$\Delta_g \circ \mathrm{Gr}_g = \mathrm{id} - \mathcal{H}_g \tag{96}$$

It is known that the Green operator Gr_g commutes with both d and δ_g [War13]. Next, we define

$$G(g) := -\mathrm{Gr}_g \circ \delta_g = -\delta_g \circ \mathrm{Gr}_g \tag{97}$$

Since $i(g) \circ \pi(g) = \mathcal{H}_g$, it follows from (96) that

$$d \circ G(g) + G(g) \circ d = \mathcal{H}_g - \mathrm{id} = i(g) \circ \pi(g) - \mathrm{id} \tag{98}$$

Also, since the harmonic forms are all d-closed, we know that

$$d \circ i(g) = 0 \tag{99}$$

Since the d-exact forms are g-orthogonal to the harmonic forms, we also know that

$$\pi(g) \circ d = 0 \tag{100}$$

By definition, we note that the degrees of $i(g)$ and $\pi(g)$ are 0 and the degree of $G(g)$ is -1.

Now, we aim to show that $(i(g), \pi(g), G(g))$ is a strong contraction in the sense of Definition 4.3. It remains to prove the four *side conditions*. Firstly, as $\pi(g) = i(g)^{-1} \circ \mathcal{H}_g$ and $\mathcal{H}_g \circ i(g) = i(g)$, we get

$$\pi(g) \circ i(g) = \mathrm{id} \tag{101}$$

Secondly, since Gr_g commutes with δ_g and $\delta_g \circ \delta_g = 0$, we immediately see that

$$G(g) \circ G(g) = 0 \tag{102}$$

Thirdly, by definition, we know $\mathcal{H}_g \circ G(g) = 0$ and so

$$\pi(g) \circ G(g) = 0 \tag{103}$$

Fourthly, the image of $i(g)$ is a harmonic form which must be δ_g-closed, thus

$$G(g) \circ i(g) = 0 \tag{104}$$

Note that (101, 102, 103, 104) correspond to (62, 63, 65, 64) separately. In summary, we have proved:

Lemma 7.1. *The triple*

$$\mathsf{con}(g) := (i(g), \pi(g), G(g)) \tag{105}$$

is a strong contraction, which we call the ***g-harmonic contraction***

Remark 7.2. Obviously, the constant-one $\mathbb{1} \in \Omega^*(L)$ is harmonic with regard to any metric g. Thus, for its cohomology class which we still denote by $\mathbb{1} = [\mathbb{1}]$, we have $i(g)(\mathbb{1}) = \mathbb{1}$, $\pi(g)(\mathbb{1}) = \mathbb{1}$, and so $i(g) \circ \pi(g)(\mathbb{1}) = \mathbb{1}$. Particularly, the condition $i(\pi(\mathbb{1})) = \mathbb{1}$ in Proposition 4.7 always hold for $\mathsf{con}(g)$

Given an A_∞ algebra $(\Omega^*(L), \check{\mathfrak{m}})$, we can construct its canonical model $(H^*(L), \mathfrak{m}^g, \mathfrak{i}^g)$ from $\mathsf{con}(g)$ by Theorem 4.4. Note that it is also implied that

$$\mathfrak{m}^g_{1,0} = 0, \qquad \mathfrak{i}^g_{1,0} = i(g) \tag{106}$$

Theorem 7.3. *If $(\Omega^*(L), \check{\mathfrak{m}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$, then $(H^*(L), \mathfrak{m}^g) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$ and $\mathfrak{i}^g \in \operatorname{Mor} \mathscr{U}\mathscr{D}$.*

Proof. By Remark 4.6, the (I-5) (II-5) in Definition 2.33 hold for $\mathfrak{m}^g$ and $\mathfrak{i}^g$. Because of $\partial\beta \cap i(g)(b) = \partial\beta \cap b$ and $i(g)\pi(g)(\mathbb{1}) = \mathbb{1}$, the theorem then just follows from Lemma 7.1 and Proposition 4.7. □

We need to further exploit the Hodge theorem. Denote by $Z^*(L) := \ker d \subset \Omega^*(L)$ the space of closed forms; then, $Z^*(L) = \mathcal{H}^*_g(L) \oplus d\Omega^*(L)$. Note that we have an isomorphism:

$$d_g := d : \delta_g \Omega^*(L) \xrightarrow{\cong} d\Omega^*(L) \tag{107}$$

Note that $\ker \delta_g = \mathcal{H}^*_g(L) \oplus \delta_g \Omega^*(L)$. The restriction of δ_g also gives an isomorphism:

$$\delta'_g := \delta_g|_{d\Omega^*(L)} : d\Omega^*(L) \xrightarrow{\cong} \delta_{g_t}\Omega^*(L) \tag{108}$$

7.2 Families of harmonic contractions

Next, we aim to generalize Theorem 7.3 to the case of pseudo-isotopies. Take a smooth family of metrics $\mathbf{g} := (g_t)_{t\in[0,1]}$. We first abbreviate the above-mentioned operators: $(i_t, \pi_t, G_t) := (i(g_t), \pi(g_t), G(g_t))$, $d_t := d_{g_t}$, and $\delta_t := \delta'_{g_t}$. Then, we have the following two isomorphisms:

$$d_t \circ \delta_t : d\Omega^*(L) \to \delta_{g_t}\Omega^*(L) \to d\Omega^*(L); \qquad \delta_t \circ d_t : \delta_{g_t}\Omega^*(L) \to d\Omega^*(L) \to \delta_{g_t}\Omega^*(L)$$

Denote by Δ_t the restriction of the Laplacian $\Delta_{g_t} = d\delta_{g_t} + \delta_{g_t} d$ on the space $\mathcal{H}^*_{g_t}(L)^\perp \equiv d\Omega^*(L) \oplus \delta_{g_t}\Omega^*(L)$. Then, $\Delta_t = d_t\delta_t \oplus \delta_t d_t$ and

$$\Delta_t^{-1} = \delta_t^{-1} d_t^{-1} \oplus d_t^{-1}\delta_t^{-1} : d\Omega^*(L) \oplus \delta_{g_t}\Omega^*(L) \to d\Omega^*(L) \oplus \delta_{g_t}\Omega^*(L) \tag{109}$$

Accordingly, the Green operator Gr_{g_t} can be explicitly expressed by

$$\mathrm{Gr}_{g_t} = \Delta_t^{-1} \circ (\mathrm{id} - \mathcal{H}_{g_t}) \tag{110}$$

where $\mathrm{id} - \mathcal{H}_{g_t}$ is just the projection to $\mathcal{H}^*_{g_t}(L)^\perp = d\Omega^*(L) \oplus \delta_{g_t}\Omega^*(L)$. Now, by (97), we get

$$G_t \circ \delta_{g_t} = 0 \tag{111}$$

Further, it follows from (109) and (110) that

$$G_t|_{Z^*(L)} = -\delta_{g_t} \circ \Delta_t^{-1} \circ \mathrm{pr}_{d\Omega^*(L)} = -d_t^{-1} \circ \mathrm{pr}_{d\Omega^*(L)} \tag{112}$$

where the 'pr' stands for the orthogonal projection. In special, this tells $G_t|_{d\Omega^*(L)} = -d_t^{-1}$.

Lemma 7.4. *There exists two smooth families of operators $h_t : H^*(L) \to \Omega^*(L)$ and $k_t : \Omega^*(L) \to H^*(L)$ of degree -1 with the following properties:*

$$\frac{di_t}{dt} = d \circ h_t; \tag{113}$$

$$\frac{d\pi_t}{dt} = k_t \circ d \tag{114}$$

$$\pi_t \circ h_t = 0 \tag{115}$$

$$k_t \circ i_t = 0 \tag{116}$$

$$G_t \circ h_t = 0 \tag{117}$$

$$k_t \circ G_t = 0 \tag{118}$$

Proof. On the one hand, recall that $d \circ i_t = 0$, hence we may think $i_t : H^*(L) \to Z^*(L)$. For the natural projection $q : Z^*(L) \to H^*(L)$, we have $q \circ i_t = \mathrm{id}$ and so $q \circ \frac{di_t}{dt} = 0$. Thus, the image of $\frac{di_t}{dt} : H^*(L) \to \Omega^*(L)$ is contained in the kernel of q, i.e. the space $d\Omega^*(L)$ of d-exact forms. We define

$$h_t := d_t^{-1} \circ \frac{di_t}{dt} : H^*(L) \to d\Omega^*(L) \to \delta_{g_t}\Omega^*(L) \subset \Omega^*(L) \tag{119}$$

In special, we have $\frac{di_t}{dt} = d \circ h_t$. On the other hand, since $\pi_t \circ d = 0$, we have $\pi_t|_{Z^*(L)} = q$. So, $\frac{d\pi_t}{dt}|_{Z^*(L)} = 0$, and one can view $\frac{d\pi_t}{dt}$ as an operator on $\delta_{g_t}\Omega^*(L)$. Then, we define

$$k_t := \begin{cases} \frac{d\pi_t}{dt} \circ d_t^{-1} & \text{on } \ d\Omega^*(L) \\ 0 & \text{on } \ \mathcal{H}_{g_t}(L) \oplus \delta_{g_t}\Omega^*(L) \end{cases} \tag{120}$$

In particular, $k_t \circ \mathcal{H}_{g_t} = 0$ and $k_t \circ \delta_{g_t} = 0$. As an operator on $\delta_{g_t}\Omega^*(L)$, we have $\frac{d\pi_t}{dt} = k_t \circ d$. The complement of $\delta_{g_t}\Omega^*(L)$ is $Z^*(L)$; both d and $\frac{d\pi_t}{dt}$ vanish on $Z^*(L)$; thereby, the relation $\frac{d\pi_t}{dt} = k_t \circ d$ still holds in $Z^*(L)$.

Now, we show the last four properties: First, since the image of h_t is contained in $\delta_{g_t}\Omega^*(L)$, we have $\pi_t \circ h_t = 0$. Second, since the image of i_t is in $\mathcal{H}^*_{g_t}(L)$, we also have $k_t \circ i_t = 0$. Third, since the image of h_t is contained in $\delta_{g_t}\Omega^*(L)$, it follows from (111) that $G_t \circ h_t = 0$. Fourth, using (97) the image of G_t is contained in $\delta_{g_t}\Omega^*(L)$ on which k_t is defined to be zero; hence, $k_t \circ G_t = 0$. □

Next, we want to study the relations to G_t. Consider the following degree-(-1) operator

$$\Gamma_t := \frac{dG_t}{dt} - i_t \circ k_t - h_t \circ \pi_t : \Omega^*(L) \to \Omega^*(L) \tag{121}$$

Using $d \circ i_t = \pi_t \circ d = 0$ and Lemma 7.4, we compute

$$d \circ \Gamma_t + \Gamma_t \circ d = d \circ \tfrac{dG_t}{dt} + \tfrac{dG_t}{dt} \circ d - d \circ h_t \circ \pi_t - i_t \circ k_t \circ d = \tfrac{d}{dt}\big(d \circ G_t + G_t \circ d - i_t \circ \pi_t\big)$$

Then, applying (98) yields that

$$d \circ \Gamma_t + \Gamma_t \circ d = 0 \tag{122}$$

Therefore, up to a sign, the Γ_t is a cochain map from $\Omega^*(L)$ to $\Omega^*(L)$. The induced map $H^*(L) \to H^*(L)$ on the cohomologies will be zero, and actually we can prove the following stronger result:

$$\Gamma_t \circ i_t = 0 \tag{123}$$

In fact, using $\pi_t \circ i_t = \mathrm{id}$, (116), (104), and (112), we obtain $\Gamma_t \circ i_t = \frac{dG_t}{dt} \circ i_t - h_t = -G_t \circ \frac{di_t}{dt} - d_t^{-1} \circ \frac{di_t}{dt} = 0$.

Lemma 7.5. *In the situation of Lemma 7.4, there exists a smooth family of operators $\sigma_t : \Omega^*(L) \to \Omega^*(L)$ of degree -2, satisfying the following properties:*

$$\frac{dG_t}{dt} - i_t \circ k_t - h_t \circ \pi_t = d \circ \sigma_t - \sigma_t \circ d \tag{124}$$

$$\sigma_t \circ G_t = G_t \circ \sigma_t = 0 \tag{125}$$

$$\pi_t \circ \sigma_t = 0 \tag{126}$$

$$\sigma_t \circ i_t = 0 \tag{127}$$

Proof. By (122) and (123), the operator $\Gamma_t := \frac{dG_t}{dt} - i_t \circ k_t - h_t \circ \pi_t$ maps $Z^*(L)$ into $d\Omega^*(L)$. Define:

$$\sigma_t := \begin{cases} d_t^{-1} \circ \Gamma_t & \text{on} \ \ Z^*(L) \\ 0 & \text{on} \ \ \delta_{g_t}\Omega^*(L) \end{cases} \tag{128}$$

It remains to check the four properties. The first one can be checked as follows: $(d \circ \sigma_t - \sigma_t \circ d)|_{Z^*(L)} = d \circ \sigma_t|_{Z^*(L)} = \Gamma_t|_{Z^*(L)}$ and $(d \circ \sigma_t - \sigma_t \circ d)|_{\delta_{g_t}\Omega^*(L)} = -\sigma_t \circ d|_{\delta_{g_t}\Omega^*(L)} = -d_t^{-1} \circ \Gamma_t \circ d|_{\delta^*_{g_t}\Omega^*(L)} = d_t^{-1} \circ d \circ \Gamma_t|_{\delta^*_{g_t}\Omega^*(L)} = \Gamma_t|_{\delta^*_{g_t}\Omega^*(L)}$ by (122). Next, note that the images of both σ_t and G_t are contained in $\delta_{g_t}\Omega^*(L)$ and that $\sigma_t \circ \delta_{g_t} = G_t \circ \delta_{g_t} = \pi_t \circ \delta_{g_t} = 0$; so, we have $\sigma_t \circ G_t = G_t \circ \sigma_t = 0$ and $\pi_t \circ \sigma_t = 0$. Finally $\sigma_t \circ i_t = 0$ holds just because of (123). $\square$

The purpose to find the above operators is to obtain a *strong* contraction for the pair $(H^*(L)_{[0,1]}, d^{[0,1]})$ and $(\Omega^*(L)_{[0,1]}, \mathfrak{M}_{1,0})$. Recall that the two differentials $d^{[0,1]}$ and $\mathfrak{M}_{1,0}$ are defined as in (23):

$$d^{[0,1]} : H^*(L)_{[0,1]} \to H^*(L)_{[0,1]}, \qquad \begin{cases} 1 \otimes \bar{x} & \mapsto ds \otimes \partial_s(\bar{x}) \\ ds \otimes \bar{x} & \mapsto 0 \end{cases}$$

$$\mathfrak{M}_{1,0} : \Omega^*(L)_{[0,1]} \to \Omega^*(L)_{[0,1]}, \qquad \begin{cases} 1 \otimes x & \mapsto 1 \otimes \mathfrak{m}_{1,0}(x) + ds \otimes \partial_s(x) \\ ds \otimes x & \mapsto -ds \otimes \mathfrak{m}_{1,0}(x) \end{cases}$$

where we denote by $\mathfrak{m}_{1,0} = d$ the exterior differential on $\Omega^*(L)$. The notations $\mathfrak{m}_{1,0}$ or $\mathfrak{M}_{1,0}$ hint but are not related to any A_∞ algebra temporarily. In addition, we define:

$$\begin{aligned} &i(\mathbf{g}) : H^*(L)_{[0,1]} \to \Omega^*(L)_{[0,1]}, && \begin{cases} 1 \otimes \bar{x} & \mapsto 1 \otimes i_s(\bar{x}) + ds \otimes h_s(\bar{x}) \\ ds \otimes \bar{x} & \mapsto ds \otimes i_s(\bar{x}) \end{cases} \\ &\pi(\mathbf{g}) : \Omega^*(L)_{[0,1]} \to H^*(L)_{[0,1]}, && \begin{cases} 1 \otimes x & \mapsto 1 \otimes \pi_s(x) + ds \otimes k_s(x) \\ ds \otimes x & \mapsto ds \otimes \pi_s(x) \end{cases} \\ &G(\mathbf{g}) : \Omega^*(L)_{[0,1]} \to \Omega^*(L)_{[0,1]}, && \begin{cases} 1 \otimes x & \mapsto 1 \otimes G_s(x) + ds \otimes \sigma_s(x) \\ ds \otimes x & \mapsto -ds \otimes G_s(x) \end{cases} \end{aligned} \tag{129}$$

The signs above respect the pointwiseness in Definition 2.12. Hence, in view of Remark 2.13, we may more concisely write $d^{[0,1]} = ds \otimes \partial_s$ and $\mathfrak{M}_{1,0} = 1 \otimes d + ds \otimes \partial_s$ for the differentials and write $i(\mathbf{g}) = 1 \otimes i_s + ds \otimes h_s$, $\pi(\mathbf{g}) = 1 \otimes \pi_s + ds \otimes k_s$ and $G(\mathbf{g}) = 1 \otimes G_s + ds \otimes \sigma_s$.

The following statement is basically a consequence of Lemma 7.4 and 7.5:

Lemma 7.6. *The above triple*

$$\mathsf{con}(\mathbf{g}) := (i(\mathbf{g}), \pi(\mathbf{g}), G(\mathbf{g}))$$

is a strong contraction for $H^*(L)_{[0,1]}$ and $\Omega^*(L)_{[0,1]}$.

Proof. Our goal now is to show all the properties listed in Definition 4.3. To begin with, the degrees are clearly as expected: $\deg i(\mathbf{g}) = \deg \pi(\mathbf{g}) = 0$ and $\deg G(\mathbf{g}) = -1$. We note that $\partial_s i_s = \partial_s \circ i_s - i_s \circ \partial_s$, $\partial_s \pi_s = \partial_s \circ \pi_s - \pi_s \circ \partial_s$, and $\partial_s G_s = \partial_s \circ G_s - G_s \circ \partial_s$.

Firstly, we aim to prove $i(\mathbf{g})$, $\pi(\mathbf{g})$ are cochain maps. By (99) and (113), we have

$$\mathfrak{M}_{1,0} \circ i(\mathbf{g}) = 1 \otimes d \circ i_s + ds \otimes (\partial_s \circ i_s - d \circ h_s) = ds \otimes i_s \circ \partial_s = i(\mathbf{g}) \circ d^{[0,1]}$$

By (100) and (114), we have

$$\pi(\mathbf{g}) \circ \mathfrak{M}_{1,0} = 1 \otimes \pi_s \circ d + ds \otimes (k_s \circ d + \pi_s \circ \partial_s) = ds \otimes \partial_s \circ \pi_s = d^{[0,1]} \circ \pi(\mathbf{g})$$

Secondly, we compute

$$\begin{aligned} i(\mathbf{g}) \circ \pi(\mathbf{g}) - \mathrm{id} &= 1 \otimes (i_s \circ \pi_s - \mathrm{id}) + ds \otimes (i_s \circ k_s + h_s \circ \pi_s) \\ \mathfrak{M}_{1,0} \circ G(\mathbf{g}) &= 1 \otimes d \circ G_s + ds \otimes (\partial_s \circ G_s - d \circ \sigma_s) \\ G(\mathbf{g}) \circ \mathfrak{M}_{1,0} &= 1 \otimes G_s \circ d + ds \otimes (\sigma_s \circ d - G_s \circ \partial_s) \end{aligned}$$

Applying Lemma 7.1 to g_s infers that $i_s \circ \pi_s - \mathrm{id} = d \circ G_s + G_s \circ d$. Then, by (124), we get $i(\mathbf{g}) \circ \pi(\mathbf{g}) - \mathrm{id} = \mathfrak{M}_{1,0} \circ G(\mathbf{g}) + G(\mathbf{g}) \circ \mathfrak{M}_{1,0}$. Thirdly, since $\pi_s \circ i_s = \mathrm{id}$, by (116) and (115) we obtain

$$\pi(\mathbf{g}) \circ i(\mathbf{g}) = 1 \otimes \pi_s \circ i_s + ds \otimes (\pi_s \circ h_s + k_s \circ i_s) = 1 \otimes \mathrm{id} = \mathrm{id}$$

Fourthly, note that $G_s \circ G_s = 0$; by using (125) we obtain

$$G(\mathbf{g}) \circ G(\mathbf{g}) = 1 \otimes G_s \circ G_s + ds \otimes (\sigma_s \circ G_s - G_s \circ \sigma_s) = 0$$

Fifthly, note that $G_s \circ i_s = 0$; by using (127) and (117) we obtain

$$G(\mathbf{g}) \circ i(\mathbf{g}) = 1 \otimes G_s \circ i_s + ds \otimes (\sigma_s \circ i_s - G_s \circ h_s) = 0$$

Sixthly, note that $\pi_s \circ G_s = 0$; by using (118) and (126) we obtain

$$\pi(\mathbf{g}) \circ G(\mathbf{g}) = 1 \otimes \pi_s \circ G_s + ds \otimes (k_s \circ G_s + \pi_s \circ \sigma_s) = 0$$

In conclusion, we have checked all the conditions in Definition 4.3, and the lemma is now proved. $\square$

Lemma 7.7. *The operators $i(\mathbf{g})$, $\pi(\mathbf{g})$ and $G(\mathbf{g})$ are compatible with* Eval^s *in the sense that*

$$\begin{aligned} \mathrm{Eval}^s \circ i(\mathbf{g}) &= i_s \circ \mathrm{Eval}^s \\ \mathrm{Eval}^s \circ \pi(\mathbf{g}) &= \pi_s \circ \mathrm{Eval}^s \\ \mathrm{Eval}^s \circ G(\mathbf{g}) &= G_s \circ \mathrm{Eval}^s \end{aligned}$$

Proof. The proof is an easy computation. $\mathrm{Eval}^s \circ i(\mathbf{g})(1 \otimes \bar{x}) = \mathrm{Eval}^s \big(1 \otimes i_s(\bar{x}(s)) + ds \otimes h_s(\bar{x}(s))\big) = i_s(\bar{x}(s)) = i_s \circ \mathrm{Eval}^s(1 \otimes \bar{x})$ and $\mathrm{Eval}^s \circ i(\mathbf{g})(ds \otimes \bar{x}) = \mathrm{Eval}^s \big(ds \otimes i_s \bar{x}(s)\big) = 0 = i_s \circ \mathrm{Eval}^s(ds \otimes \bar{x})$. Similarly, one can show the lemma for $\pi(\mathbf{g})$ and $G(\mathbf{g})$. $\square$

7.3 Pseudo-isotopies of canonical models

Fix a gapped pseudo-isotopy $(\Omega^*(L)_{[0,1]}, \check{\mathfrak{M}})$, and we write

$$\check{\mathfrak{M}} = 1 \otimes \check{\mathfrak{m}}^s + ds \otimes \check{\mathfrak{c}}^s \tag{130}$$

Take a smooth path $\mathbf{g} = (g_s)_{0 \le s \le 1}$ of metrics on L for which we have a strong contraction $\mathsf{con}(\mathbf{g}) = (i(\mathbf{g}), \pi(\mathbf{g}), G(\mathbf{g}))$ by Lemma 7.6. Moreover, for each single g_s, we also have a harmonic contraction $\mathsf{con}(g_s) = (i(g_s), \pi(g_s), G(g_s))$

by Lemma 7.1. Abusing the notations, the various constant-one functions are all denoted by $\mathbb{1}$. By Remark 7.2, we have $i(g_s)(\mathbb{1}) = \mathbb{1}$, $\pi(g_s)(\mathbb{1}) = \mathbb{1}$, and $i(g_s) \circ \pi(g_s)(\mathbb{1}) = \mathbb{1}$.

Every $\check{\mathfrak{m}}^s$ given in (130) is an A_∞ algebra on $\Omega^*(L)$; by Theorem 7.3, we get its canonical model with respect to $\mathsf{con}(g_s)$ (Definition 4.5):

$$(H^*(L), \mathfrak{m}^{g_s}, \mathfrak{i}^{g_s}) \qquad \text{with } \mathfrak{m}^{g_s} \in \operatorname{Obj} \mathscr{U}\mathscr{D}, \quad \mathfrak{i}^{g_s} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$$

Note that Theorem 7.3 is just a special situation of Theorem 4.4; our purpose is to show an analog of Theorem 7.3 for a pseudo-isotopy of canonical models. To be specific, we first apply Theorem 4.4 to the chain-level pseudo-isotopy $\check{\mathfrak{M}}$ and the $\mathsf{con}(\mathbf{g})$, thereby obtaining a canonical model

$$(H^*(L)_{[0,1]}, \mathfrak{M}^{\mathbf{g}}, \mathfrak{I}^{\mathbf{g}}) \tag{131}$$

of $(\Omega^*(L)_{[0,1]}, \check{\mathfrak{M}})$ such that

$$\mathfrak{I}^{\mathbf{g}}_{1,0} = i(\mathbf{g}) \quad \text{and} \quad \mathfrak{M}^{\mathbf{g}}_{1,0} = ds \otimes \tfrac{d}{ds} \tag{132}$$

Lemma 7.8. *Both $\mathfrak{M}^{\mathbf{g}}$ and $\mathfrak{I}^{\mathbf{g}}$ are $[0,1]$-pointwise. Moreover, we have:*

$$\operatorname{Eval}^s \circ \mathfrak{M}^{\mathbf{g}} = \mathfrak{m}^{g_s} \circ \operatorname{Eval}^s, \qquad \textit{and} \quad \operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}} = \mathfrak{i}^{g_s} \circ \operatorname{Eval}^s \tag{133}$$

Proof. By Lemma 7.7, the $i(\mathbf{g})$, $\pi(\mathbf{g})$, and $G(\mathbf{g})$ are $[0,1]$-pointwise. Besides, an iterated composition of pointwise operators is still pointwise. So, by construction, the $\mathfrak{M}^{\mathbf{g}}$ and $\mathfrak{I}^{\mathbf{g}}$ are also $[0,1]$-pointwise. By Remark 2.13, we write $\mathfrak{I}^{\mathbf{g}} = 1 \otimes \mathfrak{i}^s + ds \otimes \mathfrak{j}^s$ and $\mathfrak{M}^{\mathbf{g}} = 1 \otimes \mathfrak{m}^s + ds \otimes \mathfrak{c}^s$. Note that $\mathfrak{i}^s = \operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}} \circ \operatorname{Incl}$.

Now, it remains to show $\mathfrak{i}^s$ and $\mathfrak{m}^s$ coincide with $\mathfrak{i}^{g_s}$ and $\mathfrak{m}^{g_s}$ respectively. To see this, we look at the inductive formula (70). By Lemma 7.7 and Remark 2.22, we have

$$\begin{aligned} \operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}}_{k,\beta} &= \sum \operatorname{Eval}^s \circ G(\mathbf{g}) \circ \check{\mathfrak{M}}_{\ell,\beta_0} \circ (\mathfrak{I}^{\mathbf{g}}_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{I}^{\mathbf{g}}_{k_\ell,\beta_\ell}) \\ &= \sum G(g_s) \circ \operatorname{Eval}^s \circ \check{\mathfrak{M}}_{\ell,\beta_0} \circ (\mathfrak{I}^{\mathbf{g}}_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{I}^{\mathbf{g}}_{k_\ell,\beta_\ell}) \\ &= \sum G(g_s) \circ \check{\mathfrak{m}}^s_{\ell,\beta_0} \circ (\operatorname{Eval}^s \ \mathfrak{I}^{\mathbf{g}}_{k_1,\beta_1} \otimes \cdots \otimes \operatorname{Eval}^s \ \mathfrak{I}^{\mathbf{g}}_{k_\ell,\beta_\ell}) \end{aligned} \tag{134}$$

On the one hand, we can further pre-compose with the Incl map in the above, thereby obtaining that

$$\mathfrak{i}^s_{k,\beta} \equiv \operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}}_{k,\beta} \circ \operatorname{Incl} = \sum G(g_s) \circ \check{\mathfrak{m}}^s_{\ell,\beta_0} \circ (\mathfrak{i}^s_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{i}^s_{k_\ell,\beta_\ell})$$

In other words, the $\mathfrak{i}^s$ and $\mathfrak{i}^{g_s}$ share the same inductive formulas (70) with respect to the same $\check{\mathfrak{m}}^s$. Moreover, by (129) (132) (106), the initial cases agree: $\mathfrak{i}^s_{1,0} = \operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}}_{1,0} \circ \operatorname{Incl} = \operatorname{Eval}^s \circ i(\mathbf{g}) \circ \operatorname{Incl} = i(g_s) = \mathfrak{i}^{g_s}_{1,0}$. Hence, by induction, we conclude that $\mathfrak{i}^s = \mathfrak{i}^{g_s}$. Arguing in the same way but replacing $G(g_s)$ by $\pi(g_s)$ everywhere, we can also inductively prove $\mathfrak{m}^s = \mathfrak{m}^{g_s}$. On the other hand, if we do not pre-compose with Incl's as above, then performing the induction arguments with (134) directly yields that $\operatorname{Eval}^s \circ \mathfrak{I}^{\mathbf{g}} = i^{g_s} \circ \operatorname{Eval}^s$ and $\operatorname{Eval}^s \circ \mathfrak{M}^{\mathbf{g}} = \mathfrak{m}^{g_s} \circ \operatorname{Eval}^s$. $\square$

The following statement is an analog of Theorem 7.3 for pseudo-isotopies.

Theorem 7.9. *If* $(\Omega^*(L)_{[0,1]}, \check{\mathfrak{M}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$*, then* $(H^*(L)_{[0,1]}, \mathfrak{M}^{\mathbf{g}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$ *and* $\mathfrak{I}^{\mathbf{g}} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$*.*

Proof. First, by Lemma 7.8 we know $\mathfrak{M}^{\mathbf{g}}$ is a pseudo-isotopy. By degree reasons, we see $\pi(\mathbf{g})(\mathbb{1}) = 1 \otimes \pi(g_s)(\mathbb{1}) = 1 \otimes \mathbb{1} = \mathbb{1}$ and $i(\mathbf{g})(\mathbb{1}) = 1 \otimes i(g_s)(\mathbb{1}) = 1 \otimes \mathbb{1} = \mathbb{1}$. In particular, $i(\mathbf{g})(\pi(\mathbf{g})(\mathbb{1})) = \mathbb{1}$ holds. For a divisor input $b \in H^*(L)_{[0,1]}$, the cap product (25) by definition satisfies that $\partial\beta \cap i(\mathbf{g})(b) = \partial\beta \cap b$. Finally, one just use Proposition 4.7 (i)(ii)(iii) and Remark 4.6 to complete the proof. $\square$

7.4 An upshot for pseudo-isotopy-induced A_∞ homomorphisms

$$\begin{tikzcd}
\mathfrak{m}^{g_0} \arrow[r, "\mathfrak{C}^{\mathbf{g}}"] \arrow[d, "\mathfrak{i}^{g_0}"] & \mathfrak{m}^{g_1} \arrow[d, "\mathfrak{i}^{g_1}"] \\
\check{\mathfrak{m}}^0 \arrow[r, "\check{\mathfrak{C}}"] & \check{\mathfrak{m}}^1
\end{tikzcd}$$

Suppose we are in the situation of Theorem 7.9. From the two pseudo-isotopies $(\Omega^*(L)_{[0,1]}, \check{\mathfrak{M}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$ and $(H^*(L)_{[0,1]}, \mathfrak{M}^{\mathbf{g}}) \in \operatorname{Obj} \mathscr{U}\mathscr{D}$, the natural construction in Theorem 5.1 produces two A_∞ homomorphisms $\check{\mathfrak{C}}$ and $\mathfrak{C}^{\mathbf{g}}$ respectively. Moreover, by Theorem 5.6, we also know that $\check{\mathfrak{C}} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$ and $\mathfrak{C}^{\mathbf{g}} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$.

Lemma 7.10. *In the above situation, we have* $\mathfrak{i}^{g_1} \circ \mathfrak{C}^{\mathbf{g}} \overset{\text{ud}}{\sim} \check{\mathfrak{C}} \circ \mathfrak{i}^{g_0}$

Proof. According to Theorem 5.6, we know $\check{\mathfrak{C}} \circ \mathrm{Eval}^0 \overset{\text{ud}}{\sim} \mathrm{Eval}^1$ and $\mathfrak{C}^{\mathbf{g}} \circ \mathrm{Eval}^0 \overset{\text{ud}}{\sim} \mathrm{Eval}^1$. Also, by Lemma 7.8, we have $\mathrm{Eval}^s \circ \mathfrak{I}^{\mathbf{g}} = \mathfrak{i}^{g_s} \circ \mathrm{Eval}^s$ for $s = 0, 1$. Consequently,

$$\begin{aligned}
\check{\mathfrak{C}} \circ \mathfrak{i}^{g_0} \circ \mathrm{Eval}^0 &= \check{\mathfrak{C}} \circ \mathrm{Eval}^0 \circ \mathfrak{I}^{\mathbf{g}} \\
&\overset{\text{ud}}{\sim} \mathrm{Eval}^1 \circ \mathfrak{I}^{\mathbf{g}} = \mathfrak{i}^{g_1} \circ \mathrm{Eval}^1 \\
&\overset{\text{ud}}{\sim} \mathfrak{i}^{g_1} \circ \mathfrak{C}^{\mathbf{g}} \circ \mathrm{Eval}^0
\end{aligned}$$

Take a ud-homotopy inverse of Eval^0 which exists by Theorem 3.1, and we obtain $\check{\mathfrak{C}} \circ \mathfrak{i}^{g_0} \overset{\text{ud}}{\sim} \mathfrak{i}^{g_1} \circ \mathfrak{C}^{\mathbf{g}}$. $\square$

The Lemma 7.10 above holds only for the harmonic contractions and may fail in general. Indeed, the pointwiseness of $\mathsf{con}(\mathbf{g})$ ensures Lemma 7.8 which is essential to prove Lemma 7.10.

8 Fukaya's trick

8.1 Chain-level

Let L and $\tilde{L}$ be two Lagrangian submanifolds in X that are sufficiently adjacent; suppose there exists a small diffeomorphism $F \in \mathrm{Diff}_0(X)$ so that $F(L) = \tilde{L}$.

Since the ω-tameness is an open condition, we may assume both J and F_*J are in $\mathfrak{J}(X,\omega)$, where $F_*J := dF \circ J \circ dF^{-1}$. Note that whenever $J \in \mathfrak{J}(X,L,\omega)$ (Definition 1.1), we have $F_*J \in \mathfrak{J}(X,\tilde{L},\omega)$.

First, immediately from the definition in (18), the F induces a natural isomorphism

$$F_* : \mathfrak{G}(X,L) \cong \mathfrak{G}(X,\tilde{L}) \tag{135}$$

which preserves the Maslov indices and only depends on the homotopy class of F. We denote $\tilde{\beta} = F_*\beta$.

However, the energy is not preserved by the above F_*. If we take a Weinstein neighborhood of L and $\tilde{L}$, then one can find a tautological 1-form λ such that it vanishes exactly on $\tilde{L}$ and $\omega = d\lambda$. Then, for an isotopy from F to id, one can use Stokes' formula to show that $E(F_*\beta) - E(\beta) = -\partial\beta \cap \lambda|_L$.

Remark 8.1. In our situation, the L and $\tilde{L}$ are usually two adjacent Lagrangian torus fibers of π. We may take some action-angle coordinates $\alpha_i \in \mathbb{R}/2\pi\mathbb{Z}$, $x_i \in \mathbb{R}$ such that $L = \{x_i = 0\}$ and $\tilde{L} = \{x_i = c_i\}$. Then, we have $\lambda = \sum_i (x_i - c_i) d\alpha_i$, so $\lambda|_{\tilde{L}} = 0$ and $\lambda|_L = -\sum c_i d\alpha_i$. Assume $L = L_q$ and $\tilde{L} = L_{q'}$ for some base points q and q' in B_0; then $q' - q = (c_1, \dots, c_n)$ can be viewed as a vector in $H^1(L_q) \cong T_qB_0 \cong \mathbb{R}^n$ (c.f. §9.1.1). Now, the energy change is given by $E(\tilde{\beta}) - E(\beta) = \partial\beta \cap (q' - q)$.

As explained in [Fuk10], the diffeomorphism F gives an isomorphism of moduli spaces:

$$F_{\mathcal{M}} : \mathcal{M}_{k,\beta}(J,L) \xrightarrow{\cong} \mathcal{M}_{k,\tilde{\beta}}(F_*J,\tilde{L}) \tag{136}$$

In fact, a J-holomorphic disk u on the left side corresponds to $F \circ u$ on the right side, and vice versa. Furthermore, a Kuranishi-theory choice Ξ (see e.g. Convention 6.4) for the left side can induce via F the other choice $F_*\Xi$ for the right side moduli. From these data, using Theorem 6.2 obtains two chain-level A_∞ algebras $\check{\mathfrak{m}}^{J,\Xi,L}$ on $\Omega^*(L)$ and $\check{\mathfrak{m}}^{F_*J,F_*\Xi,\tilde{L}}$ on $\Omega^*(L)$ which are closely related to each other. To emphasize that the data are F-related, we introduce the following notation:

$$\check{\mathfrak{m}}^{F_*(J,\Xi),\tilde{L}} := \check{\mathfrak{m}}^{F_*J,F_*\Xi,\tilde{L}} \tag{137}$$

Heuristically, we may think of it as a 'pushforward' of $\check{\mathfrak{m}}^{J,L}$ where the data J, L, and Ξ are 'pushed' to F_*J, $\tilde{L} = F(L)$ and $F_*\Xi$ respectively. For simplicity, we will omit Ξ and $F_*\Xi$ in the notations since now. Recall that $F^* : \Omega^*(\tilde{L}) \to \Omega^*(L)$ and $F^* : H^*(\tilde{L}) \to H^*(L)$ are naturally the cochain maps. To begin with, we have the following *chain-level* Fukaya trick (see [Fuk10, Fuk11]):

Lemma 8.2 (Chain-level Fukaya's trick)**.**

$$\check{\mathfrak{m}}^{F_*J,\tilde{L}}_{k,\tilde{\beta}}(x_1,\dots,x_k) = (F^{-1})^*\check{\mathfrak{m}}^{J,L}_{k,\beta}(F^*x_1,\dots,F^*x_k) \tag{138}$$

Sketch of proof. Since the virtual foundation for the proof goes far beyond our scope, we only provide an intuitive description as follows. The argument can be made precise using the virtual techniques; see e.g. [Fuk10, Lemma 13.4].

First, geometrically, we observe that the isomorphism (136) of moduli spaces is compatible with the evaluation maps as illustrated in the following diagram

$$\begin{tikzcd} \mathcal{M}_{k,\beta}(J,L) \arrow[r, "F_{\mathcal{M}}"] \arrow[d, "\mathrm{ev}_i"] & \mathcal{M}_{k,\tilde{\beta}}(F_*J,\tilde{L}) \arrow[d, "\tilde{\mathrm{ev}}_i"] \\ L \arrow[r, "F"] & \tilde{L} \end{tikzcd}$$

In fact, the top horizontal map sends $[\mathbf{u},\mathbf{z}]$ to $[F\circ\mathbf{u},\mathbf{z}]$, and it follows $\tilde{\mathrm{ev}}_i([F\circ\mathbf{u},\mathbf{z}]) = F(\mathbf{u}(z_i)) = F\circ \mathrm{ev}_i([\mathbf{u},\mathbf{z}])$. Next, we denote $\check{\mathfrak{m}}^{\sim} = \check{\mathfrak{m}}^{F_*J,\tilde{L}}$ and $\check{\mathfrak{m}} = \check{\mathfrak{m}}^{J,L}$. Then, by (88) or (90), we obtain:

$$\begin{aligned}
\langle \check{\mathfrak{m}}_{k,\beta}(F^*x_1,\dots,F^*x_k),F^*x_0\rangle &= \textstyle\int_{\mathcal{M}_{k,\beta}(J,L)} {\mathrm{ev}_0}^*F^*x_0\wedge {\mathrm{ev}_1}^*F^*x_1\wedge\cdots\wedge {\mathrm{ev}_k}^*F^*x_k \\
&= \textstyle\int_{\mathcal{M}_{k,\beta}(J,L)} F^*_{\mathcal{M}}\left(\tilde{\mathrm{ev}}_0^*x_0\wedge \tilde{\mathrm{ev}}_1^*x_1\wedge\cdots\wedge\tilde{\mathrm{ev}}_k^*x_k\right) \\
&= \textstyle\int_{\mathcal{M}_{k,\tilde{\beta}}(F_*J,\tilde{L})} \tilde{\mathrm{ev}}_0^*x_0\wedge \tilde{\mathrm{ev}}_1^*x_1\wedge\cdots\wedge\tilde{\mathrm{ev}}_k^*x_k \\
&= \langle \check{\mathfrak{m}}^{\sim}_{k,\tilde{\beta}}(x_1,\dots,x_k),x_0\rangle = \langle F^*\check{\mathfrak{m}}^{\sim}_{k,\tilde{\beta}}(x_1,\dots,x_k),F^*x_0\rangle
\end{aligned}$$

Hence, we obtain $\check{\mathfrak{m}}_{k,\beta}(F^*x_1,\dots,F^*x_k) = F^*\check{\mathfrak{m}}^{\sim}_{k,\tilde{\beta}}(x_1,\dots,x_k)$. □

Corollary 8.3 (Properties of the chain-level Fukaya's trick)**.**

(i) If $\mathbb{1}$ is a unit of $\check{\mathfrak{m}}^{J,L}$, then $(F^{-1})^*\mathbb{1}\equiv\mathbb{1}$ is a unit of $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$.

(ii) If $\check{\mathfrak{m}}^{J,L}$ is cyclically unital (resp. fully unital), then so is $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$.

(iii) If $\check{\mathfrak{m}}^{J,L}$ satisfies the divisor axiom, then $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ satisfies the divisor axiom.

(iv) If $\check{\mathfrak{m}}^{J,L}$ is a q.c.dR, then $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ is a q.c.dR.

(v) If $\check{\mathfrak{m}}^{J,L}$ satisfies (I-5), then $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ satisfies (I-5) (Definition 2.33).

In particular, if $\check{\mathfrak{m}}^{J,L}\in\mathrm{Obj}\,\mathscr{U}\mathscr{D}(L,X)$, then $\check{\mathfrak{m}}^{F_*J,\tilde{L}}\in\mathrm{Obj}\,\mathscr{U}\mathscr{D}(\tilde{L},X)$.

Proof. (i) Just observe that if $\mathbb{1}$ is the constant-one function in $\Omega^0(L)$, then $(F^{-1})^*\mathbb{1}$ is also the constant-one function in $\Omega^0(\tilde{L})$. (ii) Given $\mathbf{e}\in\Omega^0(\tilde{L})$, it follows from Lemma 8.2 that

$$\mathrm{CU}[\check{\mathfrak{m}}^{F_*J,\tilde{L}}]_{k,\beta}(\mathbf{e};x_1,\dots,x_k) = F^{-1*}\circ\mathrm{CU}[\check{\mathfrak{m}}^{J,L}]_{k,\beta}(F^*\mathbf{e};F^*x_1,\dots,F^*x_k)$$

This shows the cyclical unitality. Similarly, one can show the full unitality. (iii) For the divisor axiom, we first note that F also induces an isomorphism between the divisor input spaces (24), namely, $F^*: Z^1(\tilde{L})\cong Z^1(L)$. So, the lemma together with the divisor axiom for $\check{\mathfrak{m}}^{J,L}$ implies that

$$\begin{aligned}
\mathrm{DA}[\check{\mathfrak{m}}^{F_*J,\tilde{L}}]_{k,\tilde{\beta}}(b;x_1,\dots,x_k) &= F^{-1*}\circ\mathrm{DA}[\check{\mathfrak{m}}^{J,L}]_{k,\beta}(F^*b;F^*x_1,\dots,F^*x_k) \\
&= (\partial\beta\cap F^*b)\cdot F^{-1*}\circ\check{\mathfrak{m}}^{J,L}_{k,\beta}(F^*x_1,\dots,F^*x_k)
\end{aligned}$$

Since $\partial\beta\cap F^*b = \partial\tilde{\beta}\cap b$ for $b\in Z^1(\tilde{L})$, the divisor axiom for $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ is proved. (iv) Just use Definition 2.26 and the fact that $F^{-1*}(F^*x_1\wedge F^*x_2) = x_1\wedge x_2$. (v) Use (135) and (138). □

8.2 From chain-level to cohomology-level

There is an induced Fukaya's trick in the cohomology-level. The basic idea is that for the induced metric $F_*g = (F^{-1})^*g$, the two distinct harmonic contractions $\mathsf{con}(g)$ and $\mathsf{con}(F_*g)$ (105) are also F-related, which can be incorporated into the homological perturbation.

Lemma 8.4. *The harmonic contraction* $\mathsf{con}(F_*g) = \big(i(F_*g), \pi(F_*g), G(F_*g)\big)$ *satisfies:*

$$\begin{aligned} i(F_*g) =&(F^{-1})^* \circ i(g) \circ F^* \\ \pi(F_*g) =&(F^{-1})^* \circ \pi(g) \circ F^* \\ G(F_*g) =&(F^{-1})^* \circ G(g) \circ F^* \end{aligned} \tag{139}$$

Proof. The three relations are easy to prove from the definitions in (94, 95, 97). ☐

In view of Theorem 7.3, we denote by

$$\big(H^*(L), \mathfrak{m}^{g,J,L}, \mathfrak{i}^{g,J,L}\big) \tag{140}$$

the canonical model of $\check{\mathfrak{m}}^{J,L}$ with regard to $\mathsf{con}(g)$. Meanwhile, in analogous to (137), we denote by

$$(H^*(\tilde{L}), \mathfrak{m}^{F_*(g,J),\tilde{L}}, \mathfrak{i}^{F_*(g,J),\tilde{L}}) \tag{141}$$

the canonical model of $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ with respect to $\mathsf{con}(F_*g)$. Recall that the chain-level Fukaya's trick relates $\check{\mathfrak{m}}^{F_*J,\tilde{L}}$ to $\check{\mathfrak{m}}^{J,L}$ (Lemma 8.2). Further, the use of harmonic contractions ensures an analog of Fukaya's trick for (140, 141).

Lemma 8.5 (Cohomology-level Fukaya's trick)**.**

$$\mathfrak{m}_{k,\tilde{\beta}}^{F_*(g,J),\tilde{L}}(x_1,\dots,x_k) = (F^{-1})^*\mathfrak{m}_{k,\beta}^{g,J,L}(F^*x_1,\dots,F^*x_k) \tag{142}$$

$$\mathfrak{i}_{k,\tilde{\beta}}^{F_*(g,J),\tilde{L}}(x_1,\dots,x_k) = (F^{-1})^*\mathfrak{i}_{k,\beta}^{g,J,L}(F^*x_1,\dots,F^*x_k) \tag{143}$$

Proof. Write $h = F_*g$. Denote $\mathfrak{m} := \mathfrak{m}^{g,J,L}$, $\mathfrak{m}^F := \mathfrak{m}^{F_*(g,J),\tilde{L}}$ and $\check{\mathfrak{m}} := \check{\mathfrak{m}}^{J,L}$, $\check{\mathfrak{m}}^F := \check{\mathfrak{m}}^{F_*J,\tilde{L}}$; also, denote $\mathfrak{i} := \mathfrak{i}^{g,J,L}$, $\mathfrak{i}^F := \mathfrak{i}^{F_*(g,J),\tilde{L}}$. We perform an induction on the pairs (k,β) as before. We prove the lemma for $\mathfrak{i}^F$ first. The initial case for $(k,\beta) = (1,0)$ is true by (139) and (106). Suppose the formula (143) for $\mathfrak{i}^F$ is correct whenever $(k',\beta') < (k,\beta)$. Then, the inductive formula (70) implies that

$$\mathfrak{i}^F_{k,\beta} = \textstyle\sum_{(\ell,\beta_0)\neq(1,0)} G(h) \circ \check{\mathfrak{m}}^F_{\ell,\beta_0} \circ (\mathfrak{i}^F_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{i}^F_{k_\ell,\beta_\ell})$$

Furthermore, applying Lemma 8.2, Lemma 8.4, and the induction hypothesis, we conclude that

$$\mathfrak{i}^F_{k,\beta} = (F^{-1})^* \textstyle\sum_{(\ell,\beta_0)\neq(1,0)} G(g) \circ \check{\mathfrak{m}}_{\ell,\beta_0} \circ (\mathfrak{i}_{k_1,\beta_1} \otimes \cdots \otimes \mathfrak{i}_{k_\ell,\beta_\ell}) \circ (F^*)^{\otimes k} = (F^{-1})^* \mathfrak{i}_{k,\beta} \circ (F^*)^{\otimes k}$$

Replacing $G(g)$, $G(h)$ by $\pi(g)$, $\pi(h)$, the statement for $\mathfrak{m}^F$ can be proved by a similar induction. ☐

Corollary 8.6 (Properties of cohomology-level Fukaya's trick)**.**

(i) *If* $\mathbb{1}$ *is a unit of* $\mathfrak{m}^{g,J,L}$*, then* $(F^{-1})^*\mathbb{1}$ *is a unit of* $\mathfrak{m}^{F_*(g,J),\tilde{L}}$.

(ii) *If* $\mathfrak{m}^{g,J,L}$ *is cyclical unital, then* $\mathfrak{m}^{F_*(g,J),\tilde{L}}$ *is cyclical unital.*

(iii) *If* $\mathfrak{m}^{g,J,L}$ *satisfies the divisor axiom, then* $\mathfrak{m}^{F_*(g,J),\tilde{L}}$ *satisfies the divisor axiom.*

(iv) *If* $\mathfrak{m}^{g,J,L}$ *satisfies (I-5), then* $\mathfrak{m}^{F_*(g,J),\tilde{L}}$ *satisfies (I-5) as well (Definition 2.33).*

In particular, if $\mathfrak{m}^{g,J,L} \in \operatorname{Obj} \mathscr{U}\mathscr{D}$*, then* $\mathfrak{m}^{F_*(g,J),\tilde{L}} \in \operatorname{Obj} \mathscr{U}\mathscr{D}$.

Proof. Just like the proof of Corollary 8.3, it is a direct consequence of Lemma 8.5. □

Corollary 8.7. *If* $F_0, F_1 \in \operatorname{Diff}(X)$ *with* $F_0(L) = F_1(L) = \tilde{L}$ *are isotopic, then* $\mathfrak{m}^{F_{0*}(g,J),\tilde{L}} = \mathfrak{m}^{F_{1*}(g,J),\tilde{L}}$.

Proof. Use Lemma 8.5 and the homotopy invariance of $F^* : H^*(\tilde{L}) \to H^*(L)$. □

The statement of Corollary 8.7 basically means that the application of Fukaya's trick in the cohomology-level A_∞ algebras only depends on the homotopy class of the diffeomorphism F. In contrast, since $F_0^* \neq F_1^* : \Omega^*(\tilde{L}) \to \Omega^*(L)$, it is *not* necessary that $\mathfrak{i}^{F_*(g,J),\tilde{L}}$ agrees with $\mathfrak{i}^{F_{1*}(g,J),\tilde{L}}$.

8.3 Fukaya's trick for pseudo-isotopies

In practice, we need an upgrade of Fukaya's trick that can be applied to certain pseudo-isotopies. Fix a smooth path $\mathbf{J} = (J_s)_{s\in[0,1]}$ in $\mathfrak{J}(X,\omega)$; fix a smooth family $\mathbf{F} = (F_s)_{s\in[0,1]}$ near the identity in $\operatorname{Diff}_0(X)$ such that $F_s(L) = \tilde{L}$ for each s. Define a diffeomorphism on $X \times [0,1]$, still denoted by

$$\mathbf{F} : X \times [0,1] \to X \times [0,1], \quad (x,s) \to (F_s(x), s) \tag{144}$$

We assume L and $\tilde{L}$ are so close to each other that the family $\mathbf{F}_*\mathbf{J} := (F_{s*}J_s)_{s\in[0,1]}$ is still contained in $\mathfrak{J}(X,\omega)$. Every F_s induces an isomorphism $F_{s*} : \mathfrak{G}(X,L) \to \mathfrak{G}(X,\tilde{L})$ by (135), but these isomorphisms are the same by the homotopy invariance. So, we denote any of them by the same notation below:

$$\mathbf{F}_* : \mathfrak{G}(X,L) \cong \mathfrak{G}(X,\tilde{L}) \tag{145}$$

As previously, we will write $\tilde{\beta} = \mathbf{F}_*\beta$. Moreover, we have the following two natural maps

$$\mathbf{F}^* : \Omega^*(\tilde{L})_{[0,1]} \to \Omega^*(L)_{[0,1]}, \quad \text{and} \quad \mathbf{F}^* : H^*(\tilde{L})_{[0,1]} \to H^*(L)_{[0,1]} \tag{146}$$

Explicitly, a differential form $\alpha_1(s) + ds \wedge \alpha_2(s)$ is sent to $F_s^*\alpha_1(s) + ds \wedge (F_s^*\alpha_2(s) + O(\partial_s F_s))$ where $O(\partial_s F_s)$ represents some terms in $\partial_s F_s$; particularly, we have

$$\operatorname{Eval}^s \circ \mathbf{F}^* = F_s^* \circ \operatorname{Eval}^s : \qquad \Omega^*(\tilde{L})_{[0,1]} \to \Omega^*(L) \tag{147}$$

8.3.1 Cochain level . Every F_s induces an identification $\mathcal{M}_{k,\beta}(J_s, L) \cong \mathcal{M}_{k,\tilde{\beta}}(F_{s*}J_s, \tilde{L})$ as in (136), and the union of them gives an identification of the parameterized moduli spaces (80):

$$\mathbf{F}_{\mathcal{M}} := \bigsqcup_{s\in[0,1]} (F_s)_{\mathcal{M}} : \ \mathcal{M}_{k,\beta}(\mathbf{J}, L) \xrightarrow{\cong} \mathcal{M}_{k,\tilde{\beta}}(\mathbf{F}_*\mathbf{J}, \tilde{L}) \tag{148}$$

Taking all the pairs (k, β) simultaneously yields an identification of the moduli space systems (82): $\mathbb{M}(\mathbf{J}) \cong \mathbb{M}(\mathbf{F}_*\mathbf{J})$. Note that for every s, we have an A_∞ algebra $\check{\mathfrak{m}}^{J_s,L}$ in $\mathscr{U}\mathscr{D}$ obtained by the moduli system $\mathbb{M}(J_s)$ (Theorem 6.2). Also, by the moduli space system $\mathbb{M}(\mathbf{J})$, we obtain a pseudo-isotopy

$$(\Omega^*(L)_{[0,1]}, \check{\mathfrak{M}}^{\mathbf{J},L})$$

in $\mathscr{U}\mathscr{D}$ such that it restricts to $\check{\mathfrak{m}}^{J_s,L}$ at $s = 0, 1$ (Theorem 6.3). Just like (136, 137) before, the above identification (148) yields a new pseudo-isotopy

$$(\Omega^*(\tilde{L})_{[0,1]}, \check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}})$$

that comes from the moduli space system $\mathbb{M}(\mathbf{F}_*\mathbf{J})$. Similar to Lemma 8.2, we can prove the following:

Lemma 8.8 (Chain-level Fukaya's trick for pseudo-isotopies)**.** *For $x_1, \dots, x_k \in \Omega^*(L)_{[0,1]}$, we have*

$$\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}_{k,\tilde{\beta}}(x_1, \dots, x_k) = (\mathbf{F}^{-1})^* \check{\mathfrak{M}}^{\mathbf{J},L}_{k,\beta}(\mathbf{F}^* x_1, \dots, \mathbf{F}^* x_k) \tag{149}$$

Corollary 8.9.

(i) If $\mathbb{1}$ is a unit of $\check{\mathfrak{M}}^{\mathbf{J},L}$, then $\mathbb{1} \equiv (\mathbf{F}^{-1})^*\mathbb{1}$ is a unit of $\check{\mathfrak{M}}^{\mathbf{F}_*(\mathbf{J},L)}$.

(ii) If $\check{\mathfrak{M}}^{\mathbf{J},L}$ is cyclically unital (resp. fully unital), then so is $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$.

(iii) If $\check{\mathfrak{M}}^{\mathbf{J},L}$ satisfies the divisor axiom, then $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ satisfies the divisor axiom.

(iv) If $\check{\mathfrak{M}}^{\mathbf{J},L}$ is a q.c.dR, then $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ is a q.c.dR.

(v) If $\check{\mathfrak{M}}^{\mathbf{J},L}$ satisfies (I-5), then $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ satisfies (I-5) as well (Definition 2.33).

In particular, since $\check{\mathfrak{M}}^{\mathbf{J},L} \in \operatorname{Obj}\mathscr{U}\mathscr{D}(L, X)$, we know $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}} \in \operatorname{Obj}\mathscr{U}\mathscr{D}(\tilde{L}, X)$.

Proof. The proof is almost the same as Corollary 8.3. □

The restriction of $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ at s gives rise to an A_∞ algebra that comes from the moduli space system $\mathbb{M}(F_{s*}J_s)$, and we denote it by $\check{\mathfrak{m}}^{F_{s*}J_s,\tilde{L}}$ by our previous convention. Then, the $\check{\mathfrak{M}}^{\mathbf{J},L}$ (resp.$\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$) restricts at $s = 0, 1$ to the $\check{\mathfrak{m}}^{J_s,L}$ (resp. $\check{\mathfrak{m}}^{F_{s*}J_s,\tilde{L}}$), i.e.

$$\operatorname{Eval}^s \circ \check{\mathfrak{M}}^{\mathbf{J},L} = \check{\mathfrak{m}}^{J_s,L} \circ \operatorname{Eval}^s, \qquad \operatorname{Eval}^s \circ \check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}} = \check{\mathfrak{m}}^{F_{s*}J_s,\tilde{L}} \circ \operatorname{Eval}^s \tag{150}$$

Note that the pair $\check{\mathfrak{m}}^{J_s,L}$ and $\check{\mathfrak{m}}^{F_{s*}J_s,\tilde{L}}$ also satisfies Fukaya's trick, i.e. Lemma 8.2.

8.3.2 Cohomology level . Let $\mathbf{g} = (g_s)_{s\in[0,1]}$ be a family of metrics; then $\mathbf{F}_*\mathbf{g} = (F_{s*}g_s)_{s\in[0,1]}$. Due to Lemma 7.6, we have the two strong contractions $\mathsf{con}(\mathbf{g})$ and $\mathsf{con}(\mathbf{F}_*\mathbf{g})$ for the two pairs of cochain complexes: $(H^*(L)_{[0,1]}, \Omega^*(L)_{[0,1]})$ and $(H^*(\tilde{L})_{[0,1]}, \Omega^*(\tilde{L})_{[0,1]})$ respectively. Then, we denote by

$$\big(H^*(L)_{[0,1]}, \mathfrak{M}^{\mathbf{g},\mathbf{J},L}, \mathfrak{I}^{\mathbf{g},\mathbf{J},L}\big) \qquad \text{and} \qquad \big(H^*(\tilde{L})_{[0,1]}, \mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}, \mathfrak{I}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}\big) \tag{151}$$

the two canonical models of $\check{\mathfrak{M}}^{\mathbf{J},L}$ and $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ with regard to the two contractions $\mathsf{con}(\mathbf{g})$ and $\mathsf{con}(\mathbf{F}_*\mathbf{g})$ respectively. Due to Theorem 7.9, they are all in the category $\mathscr{U}\mathscr{D}$. Furthermore, one can use (150) and Lemma 7.8 to show that the restriction of $\mathfrak{M}^{\mathbf{g},\mathbf{J},L}$ (resp. $\mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}$) at $s=0,1$ agrees with $\mathfrak{m}^{g_s,J_s,L}$ (resp. $\mathfrak{m}^{F_{s*}(g_s,J_s),\tilde{L}}$), i.e.

$$\mathrm{Eval}^s \circ \mathfrak{M}^{\mathbf{g},\mathbf{J},L} = \mathfrak{m}^{g_s,J_s,L} \circ \mathrm{Eval}^s, \qquad \mathrm{Eval}^s \circ \mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}} = \mathfrak{m}^{F_{s*}(g_s,J_s),\tilde{L}} \circ \mathrm{Eval}^s \tag{152}$$

Lemma 8.10. *Suppose $\partial_s F_s = 0$. Then we have $\mathsf{con}(\mathbf{F}_*\mathbf{g}) = \mathbf{F}^{-1*} \circ \mathsf{con}(\mathbf{g}) \circ \mathbf{F}^*$. Consequently,*

$$\begin{aligned} \mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}_{k,\tilde{\beta}}(x_1,\dots,x_k) &= \mathbf{F}^{-1*}\mathfrak{M}^{\mathbf{g},\mathbf{J},L}_{k,\beta}(\mathbf{F}^*x_1,\dots,\mathbf{F}^*x_k) \\ \mathfrak{I}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}_{k,\tilde{\beta}}(x_1,\dots,x_k) &= \mathbf{F}^{-1*}\mathfrak{I}^{\mathbf{g},\mathbf{J},L}_{k,\beta}(\mathbf{F}^*x_1,\dots,\mathbf{F}^*x_k) \end{aligned} \tag{153}$$

Proof. By (129), we write $i(\mathbf{g}) = 1\otimes i_s + ds\otimes h_s$, $\pi(\mathbf{g}) = 1\otimes \pi_s + ds\otimes k_s$, and $G(\mathbf{g}) = 1\otimes G_s + ds\otimes \sigma_s$; meanwhile we write $i(\mathbf{F}_*\mathbf{g}) = 1\otimes \tilde{i}_s + ds\otimes \tilde{h}_s$, $\pi(\mathbf{F}_*\mathbf{g}) = 1\otimes\tilde{\pi}_s + ds\otimes \tilde{k}_s$, and $G(\mathbf{F}_*\mathbf{g}) = 1\otimes \tilde{G}_s + ds\otimes\tilde{\sigma}_s$.

As $\partial_s F_s = 0$, there is no harm to set $F = F_s$ and $F = \mathbf{F}$. Then, the $\mathbf{F}^*$ simply sends $\alpha_1(s) + ds\wedge\alpha_2(s)$ to $F^*\alpha_1(s) + ds\wedge F^*\alpha_2(s)$. Let $d_s = d_{g_s}$ and $\tilde{d}_s = d_{F_*g_s}$ be the isomorphisms as in (107); then, $\tilde{d}_s = F^{-1*}\circ d_s\circ F^*$. By (139), we have $F^*\circ\tilde{i}_s = i_s\circ F^*$, $F^*\circ\tilde{\pi}_s = \pi_s\circ F^*$, and $F^*\circ\tilde{G}_s = G_s\circ F^*$. Moreover, the same properties for the pairs $(h_s,\tilde{h}_s)$, $(k_s,\tilde{k}_s)$, and $(\sigma_s,\tilde{\sigma}_s)$ can be proved by direct computations with their definition formulas (119), (120), (128). In summary, $\mathsf{con}(\mathbf{F}_*\mathbf{g}) = \mathbf{F}^{-1*}\circ\mathsf{con}(\mathbf{g})\circ\mathbf{F}^*$.

Ultimately, by the same argument in the proof of Lemma 8.5, one can also show (153). □

In the above, the second half is analogous to Lemma 8.5. The first half resembles Lemma 8.4, but the condition $\partial_s F_s = 0$ is necessary, because $\mathsf{con}(\mathbf{F}_*\mathbf{g}) \neq \mathbf{F}^{-1*}\circ\mathsf{con}(\mathbf{g})\circ\mathbf{F}^*$ in general. The main issue is about the terms involving non-zero $\partial_s F_s$; but fortunately, these terms only live in $ds\otimes -$ parts in $H^*(L)_{[0,1]}$ or $\Omega^*(L)_{[0,1]}$ and are always killed by Eval^s. From this observation, one can imagine Theorem 5.6 will be useful. Actually, even in the case of $\partial_s F_s\neq 0$, we can still somehow compare the two pseudo-isotopies $\mathfrak{M}^{\mathbf{g},\mathbf{J},L}$ and $\mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}$ 'up to ud-homotopy'. See Lemma 8.11 later.

8.4 Fukaya's trick and category $\mathscr{U}\mathscr{D}$

There is an elegant way to reinterpret the Fukaya's trick equations: one can regard the various F^* as trivial A_∞ homomorphisms (in $\mathscr{U}\mathscr{D}$) which concentrate merely on $\mathbf{CC}_{1,0} \subset \mathbf{CC}$.

Denote by Ω^F the operator system in $\mathbf{CC}(\Omega^*(\tilde{L}), \Omega^*(\tilde{L}))$ defined by setting $\Omega^F_{1,0} = F^*$ and all other $\Omega^F_{k,\beta} = 0$. Similarly, we define $H^F \in \mathbf{CC}(H^*(\tilde{L}), H^*(\tilde{L}))$, $\Omega^{\mathbf{F}} \in \mathbf{CC}(\Omega^*(\tilde{L})_{[0,1]}, \Omega^*(\tilde{L})_{[0,1]})$ and $H^{\mathbf{F}} \in \mathbf{CC}(H^*(\tilde{L})_{[0,1]}, H^*(\tilde{L})_{[0,1]})$ by setting $H^F_{1,0}$, $\Omega^{\mathbf{F}}_{1,0}$, and $H^{\mathbf{F}}_{1,0}$ to be F^* or $\mathbf{F}^*$. Remark that there are two different label groups in the story: $\mathfrak{G} = \mathfrak{G}(X, L)$ and $\tilde{\mathfrak{G}} = \mathfrak{G}(X, \tilde{L})$. But this causes no serious troubles, since we actually have a natural identification $\mathfrak{G} \cong \tilde{\mathfrak{G}}$ from (135) and (145). For simplicity, we will keep this point implicit.

On the one hand, due to the Fukaya's trick equations in (138, 142, 149), all of them except $H^{\mathbf{F}}$ can be viewed as A_∞ homomorphisms in $\operatorname{Mor}\mathscr{U}\mathscr{D}$:

$$\Omega^F : \check{\mathfrak{m}}^{F_*J,\tilde{L}} \to \check{\mathfrak{m}}^{J,L}, \qquad H^F : \mathfrak{m}^{F_*(g,J),\tilde{L}} \to \mathfrak{m}^{g,J,L}, \qquad \Omega^{\mathbf{F}} : \check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}} \to \check{\mathfrak{M}}^{\mathbf{J},L}$$

But in the special case $\partial_s F_s = 0$, it follows from Lemma 8.10 that the exceptional $H^{\mathbf{F}}$ also gives a morphism $H^{\mathbf{F}} : \mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}} \to \mathfrak{M}^{\mathbf{g},\mathbf{J},L}$ in $\mathscr{U}\mathscr{D}$.

On the other hand, the equation (143) describes a relation between the two A_∞ homomorphisms $\mathfrak{i}^{g,J,L}$ and $\mathfrak{i}^{F_*(g,J),\tilde{L}}$; from the above perspective, it exactly means that the following two compositions of morphisms in $\mathscr{U}\mathscr{D}$ agree with each other:

$$\Omega^F \circ \mathfrak{i}^{F_*(g,J),\tilde{L}} = \mathfrak{i}^{g,J,L} \circ H^F \tag{154}$$

Moreover, since every Eval^s can be trivially viewed as a morphism in $\mathscr{U}\mathscr{D}$, one can also translate (147) into the following relation of compositions of morphisms in $\mathscr{U}\mathscr{D}$:

$$\Omega^{F_s} \circ \mathrm{Eval}^s = \mathrm{Eval}^s \circ \Omega^{\mathbf{F}} \tag{155}$$

Eventually, according to Theorem 5.1 and Theorem 5.6, from the four above-mentioned pseudo-isotopies $\check{\mathfrak{M}}^{\mathbf{J},L}$, $\check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J},\tilde{L}}$ and $\mathfrak{M}^{\mathbf{g},\mathbf{J},L}$, $\mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J}),\tilde{L}}$, we have naturally constructed four morphisms in $\mathscr{U}\mathscr{D}$ which we denote by $\check{\mathfrak{C}}^{\mathbf{J}}$, $\check{\mathfrak{C}}^{\mathbf{F}_*\mathbf{J}}$, $\mathfrak{C}^{\mathbf{g},\mathbf{J}}$ and $\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})}$ respectively.

The sources and targets of the four morphisms can be determined by (150) and (152). For instance, the pseudo-isotopy $\check{\mathfrak{M}}^{\mathbf{J}}$ restricts to $\check{\mathfrak{m}}^{J_0,L}$ and $\check{\mathfrak{m}}^{J_1,L}$ at $s=0$ and $s=1$ respectively, so $\check{\mathfrak{C}}^{\mathbf{J}} : \check{\mathfrak{m}}^{J_0,L} \to \check{\mathfrak{m}}^{J_1,L}$.

Lemma 8.11. *In the above situation, $H^{F_1} \circ \mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})}$ is ud-homotopic to $\mathfrak{C}^{\mathbf{g},\mathbf{J}} \circ H^{F_0}$.*

$$
\begin{tikzcd}
\check{\mathfrak{m}}^{F_{0*}J_0} \arrow[rr, "\check{\mathfrak{C}}^{\mathbf{F}_*\mathbf{J}}"] \arrow[ddddd, bend right=60, "\Omega^{F_0}"'] & & \check{\mathfrak{m}}^{F_{1*}J_1} \arrow[ddddd, bend left=60, "\Omega^{F_1}"] \\
& \check{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J}} \arrow[ul, "\mathrm{Eval}^{0+}"] \arrow[ur, "\mathrm{Eval}^{1+}"'] \arrow[ddd, bend left, "\Omega^{\mathbf{F}}"] & \\
\mathfrak{m}^{F_{0*}(g_0,J_0)} \arrow[uu, "\mathfrak{i}^{F_{0*}(g_0,J_0)}"] \arrow[rr, "\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})}"] \arrow[d, bend right, "H^{F_0}"'] & & \mathfrak{m}^{F_{1*}(g_1,J_1)} \arrow[uu, "\mathfrak{i}^{F_{1*}(g_1,J_1)}"'] \arrow[d, bend left, "H^{F_1}"] \\
\mathfrak{m}^{g_0,J_0} \arrow[rr, "\mathfrak{C}^{\mathbf{g},\mathbf{J}}"] \arrow[dd, "\mathfrak{i}^{g_0,J_0}"'] & & \mathfrak{m}^{g_1,J_1} \arrow[dd, "\mathfrak{i}^{g_1,J_1}"] \\
& \check{\mathfrak{M}}^{\mathbf{J}} \arrow[dl, "\mathrm{Eval}^0"'] \arrow[dr, "\mathrm{Eval}^1"] & \\
\check{\mathfrak{m}}^{J_0} \arrow[rr, "\check{\mathfrak{C}}^{\mathbf{J}}"] & & \check{\mathfrak{m}}^{J_1}
\end{tikzcd}
$$

Proof. By Lemma 7.10, the top and bottom rectangle diagrams commute up to ud-homotopy, namely,

$$\mathfrak{i}^{F_{1*}(g_1,J_1)} \circ \mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} \overset{\text{ud}}{\sim} \check{\mathfrak{C}}^{\mathbf{F}_*\mathbf{J}} \circ \mathfrak{i}^{F_{0*}(g_0,J_0)} \tag{156}$$

$$\mathfrak{i}^{g_1,J_1} \circ \mathfrak{C}^{\mathbf{g},\mathbf{J}} \overset{\text{ud}}{\sim} \check{\mathfrak{C}}^{\mathbf{J}} \circ \mathfrak{i}^{g_0,J_0} \tag{157}$$

and Theorem 5.6 concludes that (just to distinguish, we use different notations: Eval^{i+} and Eval^{i})

$$\check{\mathfrak{C}}^{\mathbf{F}_*\mathbf{J}} \circ \mathrm{Eval}^{0+} \overset{\text{ud}}{\sim} \mathrm{Eval}^{1+} \tag{158}$$

$$\check{\mathfrak{C}}^{\mathbf{J}} \circ \mathrm{Eval}^{0} \overset{\text{ud}}{\sim} \mathrm{Eval}^{1} \tag{159}$$

Now, we are going to chase the diagram

$$\begin{aligned}
\mathfrak{i}^{g_1,J_1}\circ H^{F_1}\circ\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} &= \Omega^{F_1}\circ\mathfrak{i}^{F_{1*}(g_1,J_1)}\circ\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} && \text{(use (154))}\\
&\overset{\text{ud}}{\sim}\Omega^{F_1}\circ\check{\mathfrak{C}}^{\mathbf{F}_*\mathbf{J}}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)} && \text{(use (156))}\\
&\overset{\text{ud}}{\sim}\Omega^{F_1}\circ\mathrm{Eval}^{1+}\circ(\mathrm{Eval}^{0+})^{-1}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)} && \text{(use (158))}\\
&\overset{\text{ud}}{\sim}\mathrm{Eval}^{1}\circ\Omega^{\mathbf{F}}\circ(\mathrm{Eval}^{0+})^{-1}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)} && \text{(use (155))}\\
&\overset{\text{ud}}{\sim}\check{\mathfrak{C}}^{\mathbf{J}}\circ\mathrm{Eval}^{0}\circ\Omega^{\mathbf{F}}\circ(\mathrm{Eval}^{0+})^{-1}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)} && \text{(use (159))}\\
&=\check{\mathfrak{C}}^{\mathbf{J}}\circ\Omega^{F_0}\circ\mathrm{Eval}^{0+}\circ(\mathrm{Eval}^{0+})^{-1}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)}\\
& && \text{(use (155) again)}\\
&\overset{\text{ud}}{\sim}\check{\mathfrak{C}}^{\mathbf{J}}\circ\Omega^{F_0}\circ\mathfrak{i}^{F_{0*}(g_0,J_0)}\\
&\overset{\text{ud}}{\sim}\check{\mathfrak{C}}^{\mathbf{J}}\circ\mathfrak{i}^{g_0,J_0}\circ H^{F_0} && \text{(use (154) again)}\\
&\overset{\text{ud}}{\sim}\mathfrak{i}^{g_1,J_1}\circ\mathfrak{C}^{\mathbf{g},\mathbf{J}}\circ H^{F_0} && \text{(use (157))}
\end{aligned}$$

Finally, finding a homotopy inverse of $\mathfrak{i}^{g_1,J_1}$ by Theorem 3.1 completes the proof. $\square$

The main reason why we go around in such a cumbersome way is that, as mentioned previously, the $H^{\mathbf{F}}$ is generally *not* an A_∞ homomorphism (unless $\partial_s F_s = 0$ like Lemma 8.10 (153)). Hence, it cannot be included in the above diagram chasing. But, since what we need is not an identity relation but just a ud-homotopy relation, we have some flexibility and can overcome this issue as above.

As a corollary, the following special case for the constant families will be used later.

Corollary 8.12. *Suppose* $\mathbf{g}=\hat{g}$ *and* $\mathbf{J}=\hat{J}$ *are the constant paths at* g *and* J *but* $\mathbf{F}=(F_s)$ *is arbitrary. If* $\check{\mathfrak{M}}^{\hat{J}}$ *is the trivial pseudo-isotopy about* $\check{\mathfrak{m}}^J$, *then we have* $\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})}\overset{\text{ud}}{\sim}\mathrm{id}$.

Proof. Denote by $\check{\mathfrak{M}}^{\hat{g},\hat{J}}$ the canonical model of $\check{\mathfrak{M}}^{\hat{J}}$ with respect to $\mathsf{con}(\hat{g})$, and then $\mathfrak{M}^{\hat{g},\hat{J}}$ is a trivial pseudo-isotopy about $\check{\mathfrak{m}}^{g,J}$. Denote by $\mathfrak{C}^{\hat{g},\hat{J}}$ the A_∞ homomorphism constructed from $\mathfrak{M}^{\hat{g},\hat{J}}$ (using Theorem 5.1); it follows from Corollary 5.3 that $\mathfrak{C}^{\hat{g},\hat{J}}=\mathrm{id}$. Moreover, because $H^{F_0}=H^{F_1}$, Lemma 8.11 infers that $H^{F_1}\circ\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})}\overset{\text{ud}}{\sim}\mathfrak{C}^{\hat{g},\hat{J}}\circ H^{F_0}=\mathrm{id}\circ H^{F_0}=H^{F_1}$ and so $\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})}\overset{\text{ud}}{\sim}\mathrm{id}$. $\square$

We note that by Corollary 8.7 the source and target of $\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})}$ are actually the same $\mathfrak{m}^{F_{0*}(g,J)}=\mathfrak{m}^{F_{1*}(g,J)}$. Hence, Corollary 8.12 just tells that choosing different F in the application of Fukaya's trick does not cause any ambiguities up to ud-homotopy.

On the other hand, if we keep F_s constant but allow g_s and J_s to vary, then it is easy to see that the ud-homotopic relation in Lemma 8.11 can be actually strengthened to an identity relation:

Proposition 8.13. *Suppose $\partial_s F_s = 0$. Then $H^F \circ \mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} = \mathfrak{C}^{\mathbf{g},\mathbf{J}} \circ H^F$. Concretely, we have*

$$\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})}_{k,\tilde{\beta}} = F^{-1*} \mathfrak{C}^{\mathbf{g},\mathbf{J}}_{k,\beta}(F^* x_1, \dots, F^* x_k)$$

for any $x_1, \dots, x_k \in H^*(\tilde{L})$.

Proof. We write $\mathfrak{M}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} = 1 \otimes \tilde{\mathfrak{m}}^s + ds \otimes \tilde{\mathfrak{c}}^s$ and $\mathfrak{M}^{\mathbf{g},\mathbf{J}} = 1 \otimes \mathfrak{m}^s + ds \otimes \mathfrak{c}^s$. So, $\mathfrak{C}^{\mathbf{g},\mathbf{J}} : \mathfrak{m}^0 \to \mathfrak{m}^1$ and $\mathfrak{C}^{\mathbf{F}_*(\mathbf{g},\mathbf{J})} : \tilde{\mathfrak{m}}^0 \to \tilde{\mathfrak{m}}^1$. In this case, $\tilde{\mathfrak{m}}^s = F^{-1*} \circ \mathfrak{m}^s \circ F^*$ and $\tilde{\mathfrak{c}}^s = F^{-1*} \circ \mathfrak{c}^s \circ F^*$ by Lemma 8.10. Just like the proof of Lemma 8.5, the desired equation can be proved by the inductive formula (76). □

9 Mirror construction

9.1 Set-up

We are going to show the main statement Theorem 1.3. Recall that (X, ω) is a symplectic manifold of dimension $2n$ and we denote by $\mathfrak{J}(X, \omega)$ the space of ω-tame almost complex structures. Suppose $\pi : X_0 \to B_0$ is a Lagrangian torus fibration in an open domain $X_0 \subset X$ over a base manifold B_0. By Arnold-Liouville theorem [Arn13], every Lagrangian fiber $L_q := \pi^{-1}(q)$ over some point $q \in B_0$ is diffeomorphic to the n-torus T^n, and there is a canonical integral affine manifold structure on B_0.

9.1.1 Integral affine structure . In a Weinstein neighborhood of L_q, we consider the so-called *action-angle coordinates* $(\alpha_1, \dots, \alpha_n, x_1, \dots, x_n)$ where $\alpha_i \in \mathbb{R}/2\pi\mathbb{Z} \cong S^1$, $x_i \in \mathbb{R}$, and $\omega = \sum_{i=1}^n dx_i \wedge d\alpha_i$ (see e.g. [KS06, §3.1]). The coordinates $x = (x_1, \dots, x_n)$ give rise to a local chart near $q \in B_0$ over which the fibration π has the local expression $(\alpha, x) \mapsto x$. Such a chart is called an *integral affine chart*. The atlas of all the integral affine charts gives rise to an *integral affine structure* on B_0 (see Appendix C). Besides, taking an integral affine transformation $x' = Ax + b$ and $\alpha' = (A^T)^{-1}\alpha + c$ for some $A \in GL(n, \mathbb{Z})$, $b \in \mathbb{R}^n$ and $c \in (\mathbb{R}/2\pi\mathbb{Z})^n$, we obtain another action-angle coordinate system.

Inserting the vector field ∂_{x_k} to ω supplies a closed one-form $\omega(\frac{\partial}{\partial x_k}, \cdot) = d\alpha_k$ (it is not exact). Also, the assignment $\frac{\partial}{\partial x_k} \mapsto d\alpha_k$ is invariant for another action-angle coordinate system (α', x'), hence, we obtain a canonical operator $\varrho^0 : T_q B_0 \to Z^1(L_q)$ defined by $\xi \mapsto \omega(\xi, \cdot)$. Composing it with the natural quotient $Z^1(L_q) \to H^1(L_q)$ yields a linear map $\varrho : T_q B_0 \to H^1(L_q)$ whose source and target vector spaces have the same dimension. Further, it is surjective, since the cohomology classes $[d\alpha_k]$ exactly form a basis of $H^1(L_q)$, therefore, the ϱ is a vector space isomorphism.

In summary, there is a canonical isomorphism $T_q B_0 \cong H^1(L_q)$ for every $q \in B_0$.

Moreover, gluing these isomorphisms for various q yields an isomorphism $TB_0 \cong \bigcup_{q \in B_0} H^1(L_q)$ of vector bundles. Accordingly, we also have an isomorphism of the $\mathbb{Z}$ lattices (local systems): $T^{\mathbb{Z}}B_0 \cong \bigcup_{q\in B_0} H^1(L_q,\mathbb{Z})$. The right side possesses the Gauss-Manin connection; it naturally gives rise to a flat connection on TB_0 so that $T^{\mathbb{Z}}B_0$ is parallel. To sum up, the fibration π determines an integral affine structure ∇ on the smooth locus B_0. The integral affine connection ∇ induces an exponential map $\exp_q : T_qB_0 \cong H^1(L_q) \to B_0$ at each $q \in B_0$, and we call it an *affine exponential map.*

9.1.2 Diffeomorphisms and isotopies among the fibers . We want to introduce a concrete construction of a small diffeomorphism that sends one fiber to an adjacent fiber. For any $q \in B_0$, there is a sufficiently small neighborhood W_q which admits a smooth map

$$\chi_q : W_q \to \mathrm{Diff}_0(X) \tag{160}$$

such that $\chi_q(q) = \mathrm{id}$; the diffeomorphism $\chi_q(q')$ sends L_q to $L_{q'}$ and is supported in $\pi^{-1}(W_q)$ (see e.g. [Pal60, Theorem B]). For $q'_1, q'_2 \in W_q$, we define

$$F_q^{q'_1,q'_2} := \chi_q(q'_2) \circ \chi_q(q'_1)^{-1} \tag{161}$$

It is a diffeomorphism that sends $L_{q'_1}$ to $L_{q'_2}$. One can choose a neighborhood $\mathcal{U}$ of the identity in $\mathrm{Diff}_0(X)$ such that $\mathcal{U}\cdot\mathcal{U} \subset \mathcal{U}$ and $\mathcal{U}^{-1} \subset \mathcal{U}$. Now, shrinking every W_q if necessary, we may further require $\chi_q(W_q) \subset \mathcal{U}$. Particularly, for any $q'_1, q'_2 \in W_q$ we have $F_q^{q'_1,q'_2} \in \mathcal{U}$.

9.1.3 Uniform reverse isoperimetric inequalities . The reverse isoperimetric inequality for pseudo-holomorphic curves is discovered by Groman and Solomon [GS14]. Later, DuVal [Duv16] found a simpler proof of it. It states that the length of the boundary of a pseudo-holomorphic disk u bounding a Lagrangian L is controlled by its energy/area $E(u)$; specifically, there is a constant c so that $E(u) \geq c\ell(u)$. It is used extensively by Abouzaid [Abo17a] in the study of the family Floer program.

However, we need a technical improvement. Not just a single fixed Lagrangian being considered, we need to allow a *uniform constant* for a compact family among Lagrangian fibers. To avoid a digression, the proof of Lemma 9.1 below is postponed in the appendix.

Lemma 9.1 (Corollary B.3)**.** *Fix $J \in \mathfrak{J}(X,\omega)$ and fix a Lagrangian submanifold $L \subset X$. There is a C^1-neighborhood $\mathcal{V}_0$ of J, a small Weinstein neighborhood $\nu_X L$ of L, and a constant $c_0 > 0$ such that: If $\tilde{L} \subset \nu_X L$ is an adjacent Lagrangian submanifold given by the graph of a small closed one-form on L, then for any $\tilde{J} \in \mathcal{V}_0$ and any $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, \tilde{L})$, we have*

$$E(u) \geq c_0 \cdot \ell(\partial u)$$

In our situation, we have the following immediate corollary.

Corollary 9.2. *Given $J \in \mathfrak{J}(X,\omega)$ and a compact domain $K \subset B_0$, there is a C^1-neighborhood $\mathcal{V} = \mathcal{V}_{J,K}$ in $\mathfrak{J}(X,\omega)$ of J together with a constant $c = c_{J,K} > 0$ satisfying the following property: For any $\tilde{J} \in \mathcal{V}$ and $q \in K$, a $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L_q)$ satisfies that $E(u) \geq c \cdot \ell(\partial u)$.*

Proof. For any $q \in K$, one can associate to L_q a small neighborhood $\mathcal{V}(q)$ of J together with a neighborhood $U(q)$ of $q \in K$ in the base B_0 (corresponding to the Weinstein neighborhood $\pi^{-1}(U(q))$ of L_q) as in Lemma 9.1. Since K is compact, one can find finitely many points q_a ($a \in A$) in K such that $\bigcup_{a\in A} U(q_a) = K$. Now, we set $\mathcal{V} = \bigcap_{a\in A} \mathcal{V}(q_a)$. Finally, for an arbitrary $q \in K$, we first take a small diffeomorphism F so that $F(L_q) = L_{q_a}$ for some $a \in A$. Then, for any holomorphic disk u bounding L_q we have $E(u) \gtrsim E(F \circ u) \gtrsim \ell(\partial(F \circ u)) \gtrsim \ell(\partial u)$. □

9.2 Local charts

Fix a compact domain $K \subset B_0$ together with a slightly larger compact domain K' such that

$$K \Subset K' \subset B_0 \tag{162}$$

By Assumption 1.2, we fix an almost complex structure $J \in \mathfrak{J}_K$. We also fix a metric g on X.

9.2.1 Decomposition on the base . Now, we want to find a sufficiently fine covering of K by *rational polyhedrons*. There are plenty of such coverings; for instance, we may find a *polyhedral complexes* (Definition C.2) such that each cell has sufficiently small diameter.

On the one hand, by Corollary 9.2, we have $c = c_{J,K'} > 0$ and a neighborhood $\mathcal{V}' := \mathcal{V}_{J,K'} \subset \mathfrak{J}_K$ of J so that: for any $\tilde{J} \in \mathcal{V}'$ and $q \in K'$, a $\tilde{J}$-holomorphic disk u bounding L_q satisfies $E(u) \geq c \cdot \ell(\partial u)$.

On the other hand, recall that we choose $\mathcal{U}$ and $\{W_q \mid q \in B_0\}$ in §9.1.2 such that $\chi_q(W_q) \subset \mathcal{U}$. Shrinking $\mathcal{U}$ and W_q's if necessary, there exists a smaller neighborhood $\mathcal{V} \subset \mathcal{V}'$ such that the map $\mathrm{Diff}_0(X) \times \mathfrak{J}(X,\omega) \to \mathfrak{J}(X,\omega)$ defined by $(F, J) \mapsto F_*J \equiv dF \circ J \circ dF^{-1}$ sends $\mathcal{U} \times \mathcal{V}$ into $\mathcal{V}'$.

Let $\delta > 0$ be the Lebesgue number of the covering $\{W_q \mid q \in K\}$ of the compact domain K. Take

$$0 < \epsilon \leq \min\{\tfrac{c}{2}, \tfrac{\delta}{3}\} \tag{163}$$

By Lemma C.3, we can find a rational polyhedral complex

$$\mathscr{P} := \mathscr{P}_K := \{\Delta_i \mid i \in \mathfrak{I}\} \tag{164}$$

where $\mathfrak{I}$ is a finite index set, such that $K \subset |\mathscr{P}| \subset K'$ and each cell Δ_i has diameter less that ϵ.

Lemma 9.3. *Given $i \in \mathfrak{I}$, we denote by $\mathscr{P}_i$ the subcomplex consisting of all Δ_j in $\mathscr{P}$ with $\Delta_j \cap \Delta_i \neq \varnothing$. Then, there exists some $q \in K$ such that $|\mathscr{P}_i| \subset W_q$.*

Proof. It suffices to show the diameter of $|\mathscr{P}_i|$ is less than the Lebesgue number δ. In fact, given any two points $x, y \in |\mathscr{P}_i|$, we may assume $x \in \Delta_{j_1}$ and $y \in \Delta_{j_2}$ for some $j_1, j_2 \in \mathfrak{I}$. By definition, there exist some points $x' \in \Delta_{j_1} \cap \Delta_i$ and $y' \in \Delta_{j_2} \cap \Delta_i$. Then, $d(x,y) \le d(x,x') + d(x',y') + d(y',y) \le \operatorname{diam}(\Delta_{j_1}) + \operatorname{diam}(\Delta_i) + \operatorname{diam}(\Delta_{j_2}) \le 3\epsilon < \delta$. □

9.2.2 Polytopal domains . Recall that B_0 is an integral affine manifold. Given $q \in B_0$, we define

$$\Lambda[[\pi_1(L_q)]] = \left\{ \sum_{i=0}^{\infty} s_i Y^{\alpha_i} \mid s_i \in \Lambda, \alpha_i \in \pi_1(L_q) \right\}$$

where Y is a formal symbol. Note that $L_q \cong T^n$; given a basis, there is an isomorphism $\Lambda[[\pi_1(L_q)]] \cong \Lambda[[Y_1^{\pm}, \dots, Y_n^{\pm}]]$ via the identification $Y^\alpha \leftrightarrow Y^{\alpha_1} \cdots Y^{\alpha_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \pi_1(L_q) \cong \mathbb{Z}^n$.

Suppose $\Delta \subset B_0$ is a rational polyhedron (Proposition C.1) such that there is an integral affine chart containing both q and Δ. We define

$$\Lambda\langle \Delta; q\rangle = \left\{ \sum s_\alpha Y^\alpha \in \Lambda[[\pi_1(L_q)]] \mid \operatorname{val}(s_\alpha) + \langle \alpha, \gamma\rangle \to \infty \text{ for all } \gamma \in H^1(L_q) \text{ with } \exp_q(\gamma) \in \Delta \right\}$$

where the bracket denotes the natural pairing $\pi_1(L_q) \times H^1(L_q) \to \mathbb{R}$ and the $\exp_q$ denotes the affine exponential map (§9.1.1). Abusing the terminologies, we also call it a polyhedral affinoid Λ-algebra just like Definition A.2. In reality, given a basis, the $\Delta' := \exp_q^{-1}(\Delta)$ can be naturally identified with a rational polyhedral in $H^1(L_q) \cong \mathbb{R}^n$, thereby inducing a ring isomorphism $\eta : \Lambda\langle \Delta, q\rangle \to \Lambda\langle \Delta'\rangle$ defined by $Y^\alpha \mapsto Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$. It also gives an affinoid space isomorphism $\eta^* : \operatorname{Sp} \Lambda\langle \Delta'\rangle \to \operatorname{Sp} \Lambda\langle \Delta; q\rangle$.

By Proposition A.4, the affinoid space $\operatorname{Sp} \Lambda\langle \Delta, q\rangle$ can be identified with the polytopal domain $\operatorname{Sp} \Lambda\langle \Delta'\rangle \equiv \mathfrak{trop}^{-1}(\Delta')$ in the non-archimedean torus fibration $\mathfrak{trop} : (\Lambda^*)^n \to \mathbb{R}^n$. We define

$$\mathfrak{trop}_q := \mathfrak{trop}_q^\Delta := \exp_q \circ \mathfrak{trop} \circ (\eta^*)^{-1} : \operatorname{Sp} \Lambda\langle \Delta; q\rangle \to \Delta \subset B \tag{165}$$

From now on, for simplicity, we will often not distinguish them if there is no confusion.

$$\begin{tikzcd}
(\Lambda^*)^n \arrow[d, "\mathfrak{trop}"] & \supset & \operatorname{Sp} \Lambda\langle \exp_q^{-1}(\Delta)\rangle \arrow[r, "\cong", "\eta^*"'] \arrow[d, "\mathfrak{trop}"] & \operatorname{Sp} \Lambda\langle \Delta; q\rangle \arrow[d, "\mathfrak{trop}_q"] \\
H^1(L_q) \cong \mathbb{R}^n & \supset & \exp_q^{-1}(\Delta) \arrow[r, "\exp_q"] & \Delta \subset B
\end{tikzcd}$$

9.2.3 Maurer-Cartan formal power series . For any i, we fix a point $q_i \in \Delta_i$. We set $L_i := L_{q_i}$. Since $J \in \mathfrak{J}_K$ and $q_i \in K$, we know $J \in \mathfrak{J}(X, L_i, \omega)$ (Definition 1.1). By Theorem 6.2, we have an A_∞ algebra $\check{\mathfrak{m}}^{J,L_i}$ in $\mathscr{UD}$. By Theorem 7.3, it has a canonical model $(\mathfrak{m}^{g,J,L_i}, \mathfrak{i}^{g,J,L_i})$ in $\mathscr{UD}$ for the g-harmonic contraction using the convention in (140). For simplicity, we simplify their notations:

$$\check{\mathfrak{m}}^{J,i} := \check{\mathfrak{m}}^{J,L_i} \tag{166}$$

$$\mathfrak{m}^{g,J,i} := \mathfrak{m}^{g,J,L_i} \tag{167}$$

We call the following formal power series

$$P^{g,J,i} = \sum_{\beta \in \mathfrak{G}(X,L_i)} T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}^{g,J,i} \quad \in \Lambda[[\pi_1(L_i)]] \hat{\otimes} H^*(L_i)$$

the *Maurer-Cartan formal power series* associated to the $\mathfrak{m}^{g,J,i}$. The gappedness tells that it only involves at most countable many $\beta \in \pi_2(X, L_i) \equiv \mathfrak{G}(X, L_i)$. Note that the coefficient ring of the natural pairing $\langle \cdot, \cdot \rangle : H_*(L_i) \otimes H^*(L_i) \to \mathbb{R}$ can be extended to Λ or $\Lambda[[\pi_1(L_i)]]$. Given $\eta \in H_*(L_i)$, we call the formal power series $\langle \eta, P^{g,J,i} \rangle \equiv \sum_\beta T^{E(\beta)} Y^{\partial\beta} \langle \eta, \mathfrak{m}_{0,\beta}^{g,J,i} \rangle$ the *η-component* of $P^{g,J,i}$.

By degree reason, we have $\mathfrak{m}_{0,\beta}^{g,J,i} \in H^{2-\mu(\beta)}(L_i)$; it can be nonzero only if $\mu(\beta) = 0$ or 2. Hence, we decompose the $P^{g,J,i}$ as follows:

$$W^{g,J,i} = \sum_{\mu(\beta)=2} T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}^{g,J,i} / \mathbb{1}_i$$

and

$$Q^{g,J,i} = \sum_{\mu(\beta)=0} T^{E(\beta)} Y^{\partial\beta} \mathfrak{m}_{0,\beta}^{g,J,i}$$

Then, we have $P^{g,J,i} = W^{g,J,i} \cdot \mathbb{1}_i + Q^{g,J,i}$. Here we denote by $\mathbb{1} = \mathbb{1}_i$ the constant-one function on L_i.

Lemma 9.4. *Every component of $P^{g,J,i}$ is contained in $\Lambda\langle \Delta_i; q_i \rangle$. Namely, $W^{g,J,i} \in \Lambda\langle \Delta_i; q_i \rangle$ and each component of $Q^{g,J,i}$ lies in $\Lambda\langle \Delta_i; q_i \rangle$.*

Proof. Suppose $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L_i)$ is a nontrivial J-holomorphic disk. Note that the base point q lives in the compact domain K' (162); thus, for the constant $c > 0$ and the neighborhood $\mathcal{V}$ of J (introduced in §9.2.1), we have $E(u) \geq c\ell(u)$. By construction, we have $\operatorname{diam}(\Delta_i) \leq \epsilon$ for the ϵ in (163). Hence, $\frac{1}{2}E(u) \geq \frac{c}{2}\ell(\partial u) \geq \epsilon\ell(\partial u)$. Denote the boundary of u by $\sigma(t) = u(e^{2\pi i t})$, and then

$$\langle \partial u, \gamma \rangle = \int_{\partial u} \gamma = \int_0^1 \gamma(\sigma'(t)) dt \leq \int_0^1 |\gamma| \cdot |\sigma'(t)| dt \leq \epsilon \int_0^1 |\sigma'(t)| dt \leq \epsilon \cdot \ell(\partial u)$$

holds for any $\gamma \in \Delta_i \subset H^1(L_i)$. It follows that $\frac{1}{2}E(u) \geq \langle \partial u, \gamma \rangle$.

In general, suppose β is represented by a stable map $\mathbf{u}$. First, discarding a sphere bubble only decreases the energy $E(\beta)$ but does not effect $\partial\beta$. So, applying the above argument to each disk component of $\mathbf{u}$ and taking the summation, we obtain $\frac{1}{2}E(\beta) \geq \langle \partial\beta, \gamma \rangle$ for any $\gamma \in \Delta_i$. It follows that $|\operatorname{val}(T^{E(\beta)}) + \langle \partial\beta, \gamma \rangle| \geq E(\beta) - |\langle \partial\beta, \gamma \rangle| \geq \frac{1}{2}E(\beta) \to \infty$. The proof is now complete. $\square$

9.2.4 Definition of local charts . Denote by

$$\mathfrak{a}_i := \mathfrak{a}(g, J, i) \tag{168}$$

the ideal in $\Lambda\langle\Delta; q_i\rangle$ generated by the η-components $\langle \eta, Q^{g,J,i}\rangle$ for all $\eta \in H_*(L_i)$, and we call it *the ideal of weak Maurer-Cartan equations* associated to the $\mathfrak{m}^{g,J,i}$. We remark that in many examples, the Lagrangian fibers have vanishing weak Maurer-Cartan equations, and these ideals are just zero. Note that the $\mathfrak{a}_i$ is a finitely generated ideal. For any $i \in \mathfrak{I}$, we consider the quotient affinoid algebra $A_i := \Lambda\langle\Delta_i, q_i\rangle/\sqrt{\mathfrak{a}_i}$. Now, we define

$$X_i := \operatorname{Sp} A_i \tag{169}$$

It is supposed to be a *local chart* of the mirror space $X^\vee$ in Theorem 1.3; besides, a local piece of the superpotential $W^\vee$ will be given by $W^{g,J,i}$. Next, we aim to define the gluing maps (i.e. transition maps) among these local charts X_i.

9.3 Transition maps

9.3.1 Fukaya's trick . Let Δ_j and Δ_k be two adjacent rational polyhedrons in $\mathscr{P}$. By Lemma 9.3, there is some neighborhood $W_q \supset \Delta_j \cup \Delta_k$ (it is not unique in general). Then, there exists a diffeomorphism $F \in \operatorname{Diff}_0(X)$ such that $F(L_k) = L_j$ and $F \in \mathcal{U}$ (§9.1.2). Recall that we have $F_*J \in \mathcal{V}'$ (§9.2.1), and so the reverse isoperimetric inequality can be applied.

Recall the A_∞ algebras $\check{\mathfrak{m}}^{J,j}$, $\check{\mathfrak{m}}^{J,k}$, $\mathfrak{m}^{g,J,j}$, and $\mathfrak{m}^{g,J,k}$ are defined in (166) and (167). By Fukaya's trick, we can obtain the pushforward A_∞ algebras $\check{\mathfrak{m}}^{F_*J,L_j}$ and $\mathfrak{m}^{F_*(g,J),L_j}$ as in (137) and (141). Again, for clarity, we slightly change the notations as follows:

$$\check{\mathfrak{m}}^{F_*(J,k)} := \check{\mathfrak{m}}^{F_*J,L_j}, \quad \mathfrak{m}^{F_*(g,J,k)} := \mathfrak{m}^{F_*(g,J),L_j} \tag{170}$$

9.3.2 A_∞ homotopy equivalence . Now, we choose a path $\mathbf{J} := \mathbf{J}_F = (J_s)_{s\in[0,1]}$ of almost complex structures in $\mathcal{V}$ between $J_0 = J$ and $J_1 = F_*J$. Note that the reverse isoperimetric inequality applies for every J_s-holomorphic disks by the definition of $\mathcal{V}$ in §9.2.1. We also choose a path $\mathbf{g} := \mathbf{g}_F = (g_s)_{s\in[0,1]}$ of metrics between $g_0 = g$ and $g_1 = F_*g$.

First, applying Theorem 6.3 to the moduli space system $\mathbb{M}(\mathbf{J})$ yields a pseudo-isotopy

$$\check{\mathfrak{M}}^{F,j} := \check{\mathfrak{M}}^{\mathbf{J},L_j} \in \operatorname{Obj} \mathscr{U}\mathscr{D}(L_j, X) \tag{171}$$

on $\Omega^*(L_j)_{[0,1]}$ such that it restricts to $\check{\mathfrak{m}}^{J,j}$ and $\check{\mathfrak{m}}^{F_*(J,k)}$ at $s = 0$ and $s = 1$. Also, by Theorem 4.4, we obtain from the contraction $\mathsf{con}(\mathbf{g})$ a canonical model of $\check{\mathfrak{M}}^{F,j}$ which we denote by

$$(H^*(L_j)_{[0,1]}, \mathfrak{M}^{F,j}, \mathfrak{I}^{F,j}) := (H^*(L_j)_{[0,1]}, \mathfrak{M}^{\mathbf{g},\mathbf{J}}, \mathfrak{I}^{\mathbf{g},\mathbf{J}}) \tag{172}$$

Due to Theorem 7.9, we know $\mathfrak{M}^{F,j} \in \operatorname{Ob} \mathscr{U}\mathscr{D}$ and $\mathfrak{I}^{F,j} \in \operatorname{Mor} \mathscr{U}\mathscr{D}$. Moreover, applying Theorem 5.1 to the two pseudo-isotopies $\check{\mathfrak{M}}^{F,j}$ and $\mathfrak{M}^{F,j}$ produces the following two A_∞ homomorphisms

$$\check{\mathfrak{C}}^F := \check{\mathfrak{C}}^{F,j} := \check{\mathfrak{C}}^{\mathbf{J},L_j} \quad : \check{\mathfrak{m}}^{J,j} \quad \to \check{\mathfrak{m}}^{F_*(J,k)} \tag{173}$$

$$\mathfrak{C}^F := \mathfrak{C}^{F,j} := \mathfrak{C}^{\mathbf{g},\mathbf{J},L_j} : \mathfrak{m}^{g,J,j} \to \mathfrak{m}^{F_*(g,J,k)} \tag{174}$$

Due to Theorem 5.6, both of them are morphisms in $\mathscr{U}\mathscr{D}$.

Define $\Delta_{jk} = \Delta_{kj} = \Delta_j \cap \Delta_k$, and we can similarly define $\Lambda\langle\Delta_{jk}; q_j\rangle$ and $\Lambda\langle\Delta_{jk}; q_k\rangle$ as before in §9.2.2. For the natural inclusions $\Lambda\langle\Delta_j; q_j\rangle \subset \Lambda\langle\Delta_{jk}; q_j\rangle$ and $\Lambda\langle\Delta_k; q_k\rangle \subset \Lambda\langle\Delta_{jk}; q_k\rangle$, the extension of ideal $\mathfrak{a}_j$ in (168) will be still denoted by $\mathfrak{a}_j$. Then, the quotient algebras $A_{jk} := \Lambda\langle\Delta_{jk}; q_j\rangle/\sqrt{\mathfrak{a}_j}$ and $A_{kj} := \Lambda\langle\Delta_{jk}; q_k\rangle/\sqrt{\mathfrak{a}_k}$ have the natural inclusions $A_j \hookrightarrow A_{jk}$; so, there are the natural embedding maps $X_{jk} := \operatorname{Sp} A_{jk} \hookrightarrow X_j$ and $X_{kj} := \operatorname{Sp} A_{kj} \hookrightarrow X_k$.

9.3.3 Ring homomorphism . We aim to use Proposition A.1 to glue the various local charts. Since $\Lambda\langle\Delta_{jk}; q_j\rangle$ and $\Lambda\langle\Delta_{jk}; q_k\rangle$ are contained in $\Lambda[[\pi_1(L_j)]]$ and $\Lambda[[\pi_1(L_k)]]$ respectively, we start with the following ring homomorphism:

$$\begin{aligned}\phi^F_{jk} &: \Lambda[[\pi_1(L_k)]] \to \Lambda[[\pi_1(L_j)]] \\ &sY^\alpha \mapsto sT^{\langle \alpha, q_j - q_k\rangle} \cdot Y^{F_*\alpha} \cdot \exp\Big\langle F_*\alpha, \sum_{\beta\in\mathfrak{G}(X,L_j)} \mathfrak{C}^F_{0,\beta} T^{E(\beta)} Y^{\partial\beta}\Big\rangle\end{aligned} \tag{175}$$

where $s \in \Lambda$, $\alpha \in \pi_1(L_k)$ and $q_j - q_k := \exp^{-1}_{q_k}(q_j)$ represents the preimage of q_j in $H^1(L_k) \cong T_{q_k}B_0 \cong \mathbb{R}^n$ under the affine exponential map $\exp_{q_k}$ (§9.1.1). Recall that $\mathfrak{C}^F_{0,0} = 0$ by definition.

More importantly, we observe that the class $\mathfrak{C}^F_{0,\beta}$ lives in $H^{1-\mu(\beta)}(L_j)$. If it is nonzero, then we must have $\mu(\beta) \geq 0$ (Assumption 1.2), and so the only possibility is $\mu(\beta) = 0$.

The gluing maps among the various local charts are derived from these ring homomorphisms (175) and are therefore only contributed by the counts of Maslov index zero holomorphic disks. This exactly agrees with the observations for the *wall crossing phenomenon* [Aur07]. Remark also that the formulas in [Aur07, Proposition 3.9] or [AAK16, (2.4)] are essentially compatible with (175).

9.3.4 Affinoid algebra homomorphism . We need a technical lemma for the convergence.

Lemma 9.5. *If $\mathfrak{C}^F_{0,\beta} \neq 0$, then $\frac{1}{2}E(\beta) \geq |\langle\partial\beta, q - q_j\rangle|$ for any $q \in \Delta_{jk}$.*

Proof. Without loss of generality, we may assume β can be represented by a J_s-holomorphic disk u for some $s \in [0,1]$.[13] Recall that $J_s \in \mathcal{V}$, thus, one can apply

[13] In fact, the $\mathfrak{C}^F$ is obtained from the pseudo-isotopy $\check{\mathfrak{M}}^{F,j}$. If we denote by G the set of β with $\check{\mathfrak{M}}^{F,j}_\beta \neq 0$, then by Remark 4.6 and Remark 5.2, the set of β with $\mathfrak{C}^F_{0,\beta} \neq 0$ is contained in $\mathbb{N}\cdot\mathsf{G}$. So, for a general β, we have a decomposition $\beta = \sum_{m=1}^N \beta_m$, where every β_m can be represented by some pseudo-holomorphic disk.

the reverse isoperimetric inequality to conclude that $E(u) \geq c \cdot \ell(\partial u)$, where c and $\mathcal{V}$ are chosen as in §9.2.1. By condition, $|q - q_j| \leq \mathrm{diam}(\Delta_j) < \epsilon$. Just like the proof of Lemma 9.4 one can similarly show $|\langle \partial\beta, q - q_j\rangle| \leq \epsilon \cdot \ell(\partial u)$. Then, from (163), it follows that $|\langle \partial\beta, q - q_j\rangle| \leq \frac{1}{2}E(\beta)$ which completes the proof. $\square$

Theorem 9.6. *The ϕ^F_{jk} in (175) restricts to an affinoid algebra homomorphism (using the same notation):*

$$\phi^F_{jk} : \Lambda\langle \Delta_{kj}; q_k\rangle \to \Lambda\langle \Delta_{jk}; q_j\rangle \tag{176}$$

Moreover, $\mathfrak{trop}_{q_k} \circ (\phi^F_{jk})^* = \mathfrak{trop}_{q_j}$.

Proof. Fix $f = \sum_{i=0}^\infty s_i Y^{\alpha_i}$ in $\Lambda\langle \Delta_{kj}, q_k\rangle$. By Proposition A.3, it suffices to show that $\phi^F_{jk}(f)$ converges on $\mathfrak{trop}^{-1}_{q_j}(\Delta_{jk})$. As in §9.2.2, the $\mathfrak{trop}_{q_j}$ is identified with the restriction of $\mathfrak{trop}$ over $\Delta' := \exp^{-1}_{q_j}(\Delta_{jk})$.

Let $\mathbf{y}$ be a point in $\mathfrak{trop}^{-1}_{q_j}(\Delta_{jk}) \cong \mathfrak{trop}^{-1}(\Delta')$; we can view it as a point $\mathbf{y} = (y_1, \dots, y_n)$ in $(\Lambda^*)^n$ such that $q_\mathbf{y} - q_j := \mathfrak{trop}(\mathbf{y}) = (\mathrm{val}(y_1), \dots, \mathrm{val}(y_n)) \in \mathbb{R}^n$ is contained in $\Delta' \subset H^1(L_j) \cong \mathbb{R}^n$. Recall that we may identify both Y^α and $Y^{F_*\alpha}$ with $Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$ for $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \cong \pi_1(L_j) \cong \pi_1(L_k)$. By the substitution at $\mathbf{y}$ we mean that one replaces any such monomial by the value $y_1^{\alpha_1} \cdots y_n^{\alpha_n} \in \Lambda$.

It remains to show that after the substitution at $\mathbf{y}$, the series $\phi^F_{jk}(f)|_{Y=\mathbf{y}} = \sum_{\ell=0}^\infty \phi^F_{jk}(s_\ell Y^{\alpha_\ell})|_{Y=\mathbf{y}}$ converges in the Novikov field Λ. In other words, we aim to show that (see e.g. [Bos14, 2.1/3])

$$\lim_{\ell\to\infty} \mathrm{val}\left(\phi^F_{jk}(s_\ell Y^{\alpha_\ell}) \mid_{Y=\mathbf{y}} \right) = \infty \tag{177}$$

Actually, for arbitrary α, the definition formula (175) first tells that

$$\phi_{jk}(Y^\alpha)|_{Y=\mathbf{y}} = T^{\langle \alpha, q_j - q_k\rangle} \cdot \mathbf{y}^\alpha \cdot \exp\Big(\sum\nolimits_{\mu(\beta)=0, \beta\neq 0} \langle F_*\alpha, \mathfrak{C}^F_{0,\beta}\rangle T^{E(\beta)} \mathbf{y}^{\partial\beta}\Big)$$

The valuation of a nonzero monomial in the exponent is given by $\mathrm{val}(T^{E(\beta)}\mathbf{y}^{\partial\beta}) = E(\beta) + \langle \partial\beta, q_\mathbf{y} - q_j\rangle \geq \frac{1}{2}E(\beta) > 0$ thanks to Lemma 9.5. It follows that the exponential power is contained in $1 + \Lambda_+$. Thus,

$$\mathrm{val}(\phi^F_{jk}(s_\ell Y^{\alpha_\ell})|_{Y=\mathbf{y}}) = \mathrm{val}(s_\ell T^{\langle \alpha_\ell, q_j - q_k\rangle} \mathbf{y}^{\alpha_\ell}) = \mathrm{val}(s_\ell) + \langle \alpha_\ell, q_j - q_k\rangle + \langle \alpha_\ell, q_\mathbf{y} - q_j\rangle = \mathrm{val}(s_\ell) + \langle \alpha_\ell, q_\mathbf{y} - q_k\rangle$$

Moreover, the condition that $f \in \Lambda\langle \Delta_{jk}, q_k\rangle$ exactly means that for any $q \in \Delta_{jk}$ we have $\mathrm{val}(s_\ell) + \langle \alpha_\ell, q - q_k\rangle \to \infty$. Hence, we prove (177). Finally, it is easy to see that the ϕ^F_{jk} preserves the multiplication. This completes the first half of lemma.

We next show the second statement. Note that a point $\mathbf{y} \in \mathfrak{trop}^{-1}_{q_j}(\Delta_{jk})$ corresponds to the maximal ideal $\mathfrak{m}_\mathbf{y} = \{f \mid f(\mathbf{y}) = 0\}$ in $\Lambda\langle \Delta_{jk}; q_j\rangle$. So, the point $\mathbf{z} := (\phi^F_{jk})^*(\mathbf{y})$ is defined by its corresponding maximal ideal $(\phi^F_{jk})^*(\mathfrak{m}_\mathbf{y})$. Write $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{z} = (z_1, \dots, z_n)$. Then, the above argument shows that $z_r = T^{a_r} y_r e_r$, where e_r is some exponential power in $1 + \Lambda_+$ and $(a_1, \dots, a_n) = q_j - q_k$. Thus, $\mathrm{val}(z_r) = a_r + \mathrm{val}(y_r)$ for $1 \leq r \leq n$, and we obtain $\mathfrak{trop}_{q_j}(\mathbf{z}) = \mathfrak{trop}_{q_k}(\mathbf{y})$. $\square$

9.3.5 Wall crossing formula . Recall the local charts are given by the quotient algebras A_i's (§9.2.4). We need to further show that the ϕ^F_{jk} given in Theorem 9.6 induces a quotient homomorphism modulo the ideals of weak Maurer-Cartan equations. To achieve this, we need the following result.

Fix a basis (f_i) of $\pi_1(L_k)$; it induces a dual basis (θ_i) of $H^1(L_k)$. Since $F(L_k) = L_j$, we also choose the bases $(\tilde{f}_i)$ and $(\tilde{\theta}_i)$ of $\pi_1(L_j)$ and $H^1(L_j)$ which are F-related to the previous ones. For any $\beta \in \mathfrak{G}(X, L_k)$, we write $\tilde{\beta} := F_*\beta \in \mathfrak{G}(X, L_j)$ as before. Note that $\theta_{pq} = \theta_p \wedge \theta_q$ $(1 \le p < q \le n)$ form a basis of $H^2(L_k)$. Denote $Q^{g,J,j} = \sum_{1\le p<q\le n} Q^{g,J,j}_{pq} \cdot \theta_{pq}$ (§9.2.3).

Theorem 9.7 (Wall crossing formula)**.** *For $\eta \in H_*(L_k)$, we have*

$$\phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle) = \langle F_*\eta, \mathbb{1}_j\rangle \cdot W^{g,J,j} + \sum_{1\le p<q\le n} R^{F,\eta}_{pq} \cdot Q^{g,J,j}_{pq} \tag{178}$$

where

$$R^{F,\eta}_{pq} = \sum_{\tilde{\beta}\in\mathfrak{G}(X,L_j)} T^{E(\tilde{\beta})} Y^{\partial\tilde{\beta}} \langle F_*\eta, \mathfrak{C}^F_{1,\tilde{\beta}}(\theta_{pq})\rangle \in \Lambda\langle \Delta_j; q_j\rangle$$

Proof. By the same arguments as in Lemma 9.4, every $R^{F,\eta}_{pq}$ above is also contained in $\Lambda\langle\Delta_j, q_j\rangle$. We can think of (178) as an identity in $\Lambda[[\pi_1(L_j)]] \cong \Lambda[[Y_1^{\pm}, \dots, Y_n^{\pm}]]$ for the above bases. By Lemma 2.3 again, it suffices to show the identity holds on U^n_Λ.

Assume $\mathbf{y} = (y_1, \dots, y_n) \in U^n_\Lambda$, i.e. $\operatorname{val}(y_i) = 0$ for all i. By Lemma 2.2, there exists $x_i \in \Lambda_0$ such that $y_i = e^{x_i} = \sum_{k\ge 0} \frac{1}{k!} x_i^k$. We put $b := x_1\theta_1 + \cdots + x_n\theta_n \in H^1(L_k)\hat{\otimes}\Lambda_0$ and $\tilde{b} := F^{-1*}b = x_1\tilde{\theta}_1 + \cdots + x_n\tilde{\theta}_n \in H^1(L_j)\hat{\otimes}\Lambda_0$. Set $\partial_i\beta = \partial\beta \cap \theta_i = \partial\tilde{\beta} \cap \tilde{\theta}_i$, and we note that

$$\partial\beta \cap b = \partial\tilde{\beta} \cap \tilde{b} = \partial_1\beta \cdot x_1 + \cdots + \partial_n\beta \cdot x_n$$

Besides, substituting $y_i = e^{x_i}$ into the monomial $Y^{\partial\beta} \equiv Y^{\partial\tilde{\beta}} \equiv Y_1^{\partial_1\beta} \cdots Y_n^{\partial_n\beta}$ gives the same value in the Novikov field Λ: $e^{\langle\partial\beta, b\rangle} = e^{\langle\partial\tilde{\beta}, \tilde{b}\rangle} = e^{\partial_1\beta\cdot x_1 + \cdots + \partial_n\beta\cdot x_n}$. We put $q_j - q_k := \exp^{-1}_{q_k}(q_j)$ as before; then by Remark 8.1, we have (see also [Fuk10] [Abo17b]):

$$E(\tilde{\beta}) = E(\beta) + \langle \partial\beta, q_j - q_k\rangle \tag{179}$$

Recall that $\mathfrak{C}^F_{1,0} = \mathrm{id}$ and $\mathfrak{m}^{g,J,k}_{0,\beta} = F^*\mathfrak{m}^{F_*(g,J,k)}_{0,\tilde{\beta}}$ (Lemma 8.5). Next, using (175), we compute:

$$\begin{aligned}
\phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle)|_{Y=\mathbf{y}} &= \sum_\beta \langle \eta, \mathfrak{m}^{g,J,k}_{0,\beta}\rangle T^{E(\beta)}\ T^{\langle\partial\beta, q_j - q_k\rangle} Y^{\partial\tilde{\beta}} \exp\Big(\sum_\gamma \langle \partial\tilde{\beta}, \mathfrak{C}^F_{0,\gamma}\rangle T^{E(\gamma)} Y^{\partial\gamma}\Big)\ |_{Y=\mathbf{y}} \\
&= \sum_{\tilde{\beta}} \langle \eta, F^*\mathfrak{m}^{F_*(g,J,k)}_{0,\tilde{\beta}}\rangle\ T^{E(\tilde{\beta})} e^{\langle\partial\tilde{\beta}, \tilde{b}\rangle} \exp\Big(\sum_\gamma \langle\partial\tilde{\beta}, \mathfrak{C}^F_{0,\gamma}\rangle T^{E(\gamma)} e^{\langle\partial\gamma, \tilde{b}\rangle}\Big) \\
&= \sum_{\tilde{\beta}} \langle F_*\eta, \mathfrak{m}^{F_*(g,J,k)}_{0,\tilde{\beta}}\rangle T^{E(\tilde{\beta})} \exp\Big\langle \partial\tilde{\beta}, \tilde{b} + \sum_\gamma \mathfrak{C}^F_{0,\gamma} T^{E(\gamma)} e^{\langle\partial\gamma, \tilde{b}\rangle}\Big\rangle
\end{aligned}$$

Note that $\mathfrak{C}^F \in \operatorname{Mor}\mathscr{U}\mathscr{D}$ (174). We have $\mu(\gamma) \geq 0$ whenever $\mathfrak{C}^F_{0,\gamma} \neq 0$ (Definition 2.33 (II-5)). So, the class $\mathfrak{C}^F_{0,\gamma} \in H^{1-\mu(\gamma)}(L_j)$ is nonzero only if $\mu(\gamma) = 0$; then, $\mathfrak{C}^F_{0,\gamma} \in H^1(L_j)$. By the divisor axiom of $\mathfrak{C}^F$, the exponent is given by bracketing $\partial\tilde\beta$ with the following class:

$$\tilde b + \sum_\gamma \mathfrak{C}^F_{0,\gamma} T^{E(\gamma)} e^{\langle \partial\gamma, \tilde b\rangle} = \sum_{k\geq 0}\sum_\gamma T^{E(\gamma)}\mathfrak{C}^F_{k,\gamma}(\tilde b,\dots,\tilde b) \equiv \mathfrak{C}^F_*(\tilde b) \tag{180}$$

where we recall (32) for the notation. Notice that $\mathfrak{C}^F_*(\tilde b) \in H^1(L_j)\hat\otimes\Lambda_0$ can be also viewed as a divisor input (24). Using the divisor axiom of $\mathfrak{m}^{F_*(g,J,k)}$ (Corollary 8.6) to this new divisor input $\mathfrak{C}^F_*(\tilde b)$ yields:

$$\begin{aligned}
\phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle)|_{Y=\mathbf{y}} &= \Big\langle F_*\eta, \sum_{\tilde\beta} T^{E(\tilde\beta)}\mathfrak{m}^{F_*(g,J,k)}_{0,\tilde\beta}\exp\langle\partial\tilde\beta, \mathfrak{C}^F_*(\tilde b)\rangle\Big\rangle \\
&= \Big\langle F_*\eta, \sum_k\sum_{\tilde\beta} T^{E(\tilde\beta)}\mathfrak{m}^{F_*(g,J,k)}_{k,\tilde\beta}(\mathfrak{C}^F_*(\tilde b),\dots,\mathfrak{C}^F_*(\tilde b))\Big\rangle
\end{aligned}$$

Further, by the A_∞ equation of $\mathfrak{C}^F$ and the divisor axioms of both $\mathfrak{C}^F$ and $\mathfrak{m}^{g,J,j}$, we compute

$$\begin{aligned}
\phi_{jk}(\langle \eta, P^{g,J,k}\rangle)|_{Y=\mathbf{y}} &= \Big\langle F_*\eta, \sum T^{E(\tilde\beta_1)}\mathfrak{C}^F_{\tilde\beta_1}\big(\tilde b,\dots,\tilde b, \sum T^{E(\tilde\beta_2)}\mathfrak{m}^{g,J,i}_{\tilde\beta_2}(\tilde b,\dots,\tilde b),\tilde b,\dots,\tilde b\big)\Big\rangle \\
&= \Big\langle F_*\eta, \sum T^{E(\tilde\beta_1)}e^{\langle\partial\tilde\beta_1,\tilde b\rangle}\mathfrak{C}^F_{1,\tilde\beta_1}\big(\sum T^{E(\tilde\beta_2)}e^{\langle\partial\tilde\beta_2,\tilde b\rangle}\mathfrak{m}^{g,J,j}_{0,\tilde\beta_2}\big)\Big\rangle \\
&= \sum T^{E(\tilde\beta)}\mathbf{y}^{\partial\tilde\beta}\Big\langle F_*\eta, \mathfrak{C}^F_{1,\tilde\beta}(P^{g,J,j}(\mathbf{y}))\Big\rangle
\end{aligned}$$

Since $\mathfrak{C}^F$ is unital, we know $\mathfrak{C}^F_{1,\tilde\beta}(\mathbb{1}_j)$ is zero for $\tilde\beta \neq 0$ and $\mathfrak{C}^F_{1,0}(\mathbb{1}_j) = \mathbb{1}_j$ for $\tilde\beta = 0$. It follows that

$$\begin{aligned}
\phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle)|_{Y=\mathbf{y}} &= \sum T^{E(\tilde\beta)}\mathbf{y}^{\partial\tilde\beta}\Big(\langle F_*\eta, \mathfrak{C}^F_{1,\tilde\beta}(\mathbb{1}_j)\rangle W^{g,J,j}(\mathbf{y}) + \langle F_*\eta, \mathfrak{C}^F_{1,\tilde\beta}(\tilde\theta_{pq})\rangle Q^{g,J,j}_{pq}(\mathbf{y})\Big) \\
&= \langle F_*\eta, \mathbb{1}_j\rangle\cdot W^{g,J,j}(\mathbf{y}) + \sum_{1\leq p<q\leq n} R^{F,\eta}_{pq}(\mathbf{y})\cdot Q^{g,J,j}_{pq}(\mathbf{y})
\end{aligned}$$

In summary, the wall crossing formula (178) holds for any point $\mathbf{y} = (y_1,\dots,y_n)$ in U^n_Λ. Finally, due to Lemma 2.3, this actually holds for all $\mathbf{y}$. The proof is now complete. $\square$

9.3.6 Definition of transition maps . For the quotient algebras A_{jk} and A_{kj} (§9.3.2), the wall crossing formula implies the existence of quotient affinoid algebra homomorphisms:

$$\begin{tikzcd}
\Lambda\langle\Delta_{jk}; q_k\rangle \arrow[r, "\phi^F_{jk}"] \arrow[d, two heads] & \Lambda\langle\Delta_{jk}; q_j\rangle \arrow[d, two heads] \\
A_{kj} \arrow[r, "\varphi_{jk}"] & A_{kj}
\end{tikzcd}$$

Theorem 9.8. *The affinoid algebra homomorphism ϕ^F_{jk} in (176) induces a quotient homomorphism:*

$$\varphi_{jk} := \varphi^F_{jk} : A_{kj} \to A_{jk} \tag{181}$$

Moreover, we have $\varphi_{jk}(W^{g,J,k}) = W^{g,J,j}$.

Proof. This is basically a corollary of Theorem 9.7. In the wall crossing formula (178), if η is dual to θ_{rs} for $1 \le r < s \le n$, then $\langle F_*\eta, \mathbb{1}_j\rangle = 0$ and $\langle \eta, P^{g,J,k}\rangle = Q^{g,J,k}_{rs}$. Hence,

$$\phi^F_{jk}(Q^{g,J,k}_{rs}) = \phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle) = \sum R^{F,\eta}_{pq} Q^{g,J,j}_{pq} \in \mathfrak{a}_j$$

Since the ideal $\mathfrak{a}_k$ is generated by these $Q^{g,J,k}_{rs}$, it follows that $\phi^F_{jk}(\mathfrak{a}_k) \subset \mathfrak{a}_j$. So, the quotient map $\varphi_{jk} : A_{kj} \to A_{jk}$ is well-defined. If η is dual to $\mathbb{1}_k$, then $\langle F_*\eta, \mathbb{1}_j\rangle = 1$ and $\langle \eta, P^{g,J,k}\rangle = W^{g,J,k}$. Hence,

$$\phi^F_{jk}(W^{g,J,k}) = \phi^F_{jk}(\langle \eta, P^{g,J,k}\rangle) = W^{g,J,j} + \sum R^{F,\eta}_{pq} Q^{F,\eta}_{pq} \in W^{g,J,j} + \mathfrak{a}_j$$

It follows that $\varphi_{jk}(W^{g,J,k}) = W^{g,J,j}$. □

Recall that we have defined $X_{jk} = \operatorname{Sp} A_{jk}$ and $X_{kj} = \operatorname{Sp} A_{kj}$ (§9.3.2) which are respectively open domains in the local charts X_j and X_k (§9.2.4). Then, a *gluing map* (or called *transition map*)

$$\psi_{jk} := \varphi^*_{jk} : X_{jk} \to X_{kj} \tag{182}$$

is defined to be the map associated to the quotient homomorphism $\varphi_{jk} : A_{kj} \to A_{jk}$ in (181). Abusing the terminologies, we also call the algebra homomorphism φ_{jk} a gluing map or a transition map. Moreover, using the second statement of Theorem 9.6, we immediately obtain:

Corollary 9.9. *The transition map ψ_{jk} satisfies that $\mathfrak{trop}_{q_k} \circ \psi_{jk} = \mathfrak{trop}_{q_j}$.*

$$\begin{tikzcd}
X_j \arrow[d, "\mathfrak{trop}_{q_j}"] & X_{jk} \arrow[l, hook'] \arrow[r, "\psi_{jk}=\varphi^*_{jk}"] \arrow[d] & X_{kj} \arrow[r, hook] \arrow[d] & X_k \arrow[d, "\mathfrak{trop}_{q_k}"] \\
\Delta_j & \Delta_{jk} \arrow[l, hook'] \arrow[r, equal] & \Delta_{kj} \arrow[r, hook] & \Delta_k
\end{tikzcd}$$

9.4 Choice-independence of transition maps

In this section, we aim to prove the induced homomorphism φ_{jk} in (181) does not depends on the various choices, so the transition map $\psi_{jk} = \varphi^*_{jk}$ in (182) is well-defined. The philosophy behind this statement comes exactly from the gauge equivalence of (weak) bounding cochains as introduced in [FOOO10b, §4.3], but it is used in a more non-archimedean way.

Theorem 9.10. *The transition maps ψ_{jk} or φ_{jk} are independent of the choices of F, $\mathbf{J}$, $\mathbf{g}$.*

Suppose we differently choose $F' \in \mathcal{U} \subset \mathrm{Diff}_0(X)$ (§9.3.1), and we also differently choose $\mathbf{J}' = (J'_s)$, and $\mathbf{g}' = (g'_s)$ (§9.3.2). Similar to (170, 171, 172, 173, 174), we can use these data to obtain

(i) a chain-level A_∞ algebra $\check{\mathfrak{m}}^{F'_*(J,k)}$,

(ii) the canonical model $(H^*(L_j), \mathfrak{m}^{F'_*(g,J,k)}, \mathfrak{i}^{F'_*(g,J,k)})$ of $\check{\mathfrak{m}}^{F'_*(J,k)}$ with respect to $\mathsf{con}(F'_*g)$,

(iii) a chain-level pseudo-isotopy $\check{\mathfrak{M}}^{F',j}$ between $\check{\mathfrak{m}}^{J,j}$ and $\check{\mathfrak{m}}^{F'_*(J,k)}$,

(iv) a cohomology-level pseudo-isotopy $\mathfrak{M}^{F',j}$ between $\mathfrak{m}^{g,J,j}$ and $\mathfrak{m}^{F'_*(g,J,k)}$ (it comes with $\mathfrak{I}^{F',j}$),

(v) the chain-level A_∞ homomorphisms $\check{\mathfrak{C}}^{F'} : \check{\mathfrak{m}}^{J,j} \to \check{\mathfrak{m}}^{F_*(J,k)}$ induced by $\check{\mathfrak{M}}^{F',j}$,

(vi) the cohomology-level A_∞ homomorphism $\mathfrak{C}^{F'} : \mathfrak{m}^{g,J,j} \to \mathfrak{m}^{F'_*(g,J,k)}$ induced by $\mathfrak{M}^{F',j}$.

Notations are used in the same pattern. All of them are contained in the category $\mathscr{U}\mathscr{D}$; the similar Fukaya's trick equations hold. Furthermore, just as (175, 176), we can define the homomorphisms $\phi^{F'}_{jk}$ using the above $\mathfrak{C}^{F'}$ (in place of $\mathfrak{C}^F$). Then, we have two quotient homomorphisms $\varphi^{F'}_{jk}$ and φ^{F}_{jk} which are induced by $\phi^{F'}_{jk}$ and ϕ^{F}_{jk} respectively (181).

Now, by Theorem 9.10 we just mean that we want to show $\varphi^F_{jk} = \varphi^{F'}_{jk}$.

We need some preparations in several aspects. First, we find a path $\mathbf{F} = (F_s)_{s\in[0,1]}$ of diffeomorphisms in the neighborhood $\mathcal{U} \subset \mathrm{Diff}_0(X)$ between $F_0 = F$ and $F_1 = F'$ such that $F_s(L_k) = L_j$ for all $s \in [0,1]$. Let $\hat{J}$ and $\hat{g}$ be the constant $[0,1]$-families at J and g, and we consider

$$\mathbf{F}_*\hat{J} = (F_{s*}J)_{s\in[0,1]} \qquad \mathbf{F}_*\hat{g} = (F_{s*}g)_{s\in[0,1]}$$

9.4.1 Chain-level . On the one hand, we have an 'air-cored triangle' family of almost complex structures whose vertices are given by J, F_*J, F'_*J and whose edges are given by $\mathbf{J}$, $\mathbf{J}'$, $\mathbf{F}_*\hat{J}$. Recall that both $\mathbf{J}$ and $\mathbf{J}'$ are contained in the neighborhood $\mathcal{V}$; also, since $F_s \in \mathcal{U}$ for any s, we have $F_s(\mathcal{V}) \subset \mathcal{V}'$ and $\mathbf{F}_*\hat{J} \subset \mathcal{V}'$ (§9.2.1). Now, we can fill it to obtain a 'solid triangle' family in $\mathcal{V}'$. Namely, there exists a smooth family $\mathbb{J} = (J_x)_{x\in\Delta^2}$ in $\mathcal{V}'$ parameterized by the 2-simplex $\Delta^2 = [v_0, v_1, v_2]$ such that $\mathbb{J}|_{[v_0,v_1]} = \mathbf{J}$, $\mathbb{J}|_{[v_1,v_2]} = \mathbf{F}_*\hat{J}$, and $\mathbb{J}|_{[v_0,v_2]} = \mathbf{J}'$.

On the other hand, by Theorem 6.3, the moduli space system $\mathbb{M}(\hat{J})$ for the constant family $\hat{J}$ can produce a trivial pseudo-isotopy $(\Omega^*(L_k)_{[0,1]}, \check{\mathfrak{M}}^{\hat{J},k})$ about the A_∞ algebra $(\Omega^*(L_k), \check{\mathfrak{m}}^{J,k})$ (166). Due to Fukaya's trick (Lemma 8.8), it gives rise to a 'pushforward' pseudo-isotopy, denoted by:

$$\big(\Omega^*(L_j), \check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}\big) \tag{183}$$

satisfying the equations of Fukaya's trick (149). Moreover, by (150), it is a pseudo-isotopy between $\check{\mathfrak{m}}^{F_*(J,k)}$ and $\check{\mathfrak{m}}^{F'_*(J,k)}$ (in the chain-level); by Corollary 8.9, it lives in $\mathscr{U}\mathscr{D}$. Finally, due to Theorem 5.1 and Theorem 5.6, its induced A_∞ homotopy equivalence

$$\check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} : \check{\mathfrak{m}}^{F_*(J,k)} \to \check{\mathfrak{m}}^{F'_*(J,k)} \tag{184}$$

is a morphism in $\mathscr{U}\mathscr{D}$. It is supposed to satisfy the following property:

$$\begin{tikzcd}
 & F'_*J \arrow[dd, dash, "\mathbf{F}_*\hat{J}"] \\
J \arrow[ur, dash, "\mathbf{J}'"] \arrow[dr, dash, "\mathbf{J}"'] \arrow[r, phantom, "\mathbb{J}"] & \\
 & F_*J
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \check{\mathfrak{m}}^{F'_*(J,k)} \arrow[dd, dash, "\check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}"] \\
\check{\mathfrak{m}}^{J,k} \arrow[ur, dash, "\check{\mathfrak{M}}^{F',j}"] \arrow[dr, dash, "\check{\mathfrak{M}}^{F,j}"'] \arrow[r, phantom, "\check{\mathfrak{M}}^{\mathbb{J}}"] & \\
 & \check{\mathfrak{m}}^{F_*(J,k)}
\end{tikzcd}$$

Lemma 9.11. $\check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} \circ \check{\mathfrak{C}}^{F} \overset{\text{ud}}{\sim} \check{\mathfrak{C}}^{F'}$

Proof. The pseudo-isotopies $\check{\mathfrak{M}}^{F,j}$, $\check{\mathfrak{M}}^{F',j}$ and $\check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}$ coincide at their ends. So, applying Theorem 6.5 to the moduli space system $\mathbb{M}(\mathbb{J})$, we obtain a Δ^2-pseudo-isotopy

$$\check{\mathfrak{M}}^{\mathbb{J}} \in \operatorname{Obj} \mathscr{U}\mathscr{D}(L_j, X) \tag{185}$$

which restricts to $\check{\mathfrak{M}}^{F,j}$, $\check{\mathfrak{M}}^{F',j}$ and $\check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}$ on the edges $[v_0, v_1]$, $[v_0, v_2]$, and $[v_1, v_2]$ respectively.[14]

Let I be an edge of Δ^2. We set

$$\operatorname{Restr}_I^{\Delta^2} : \Omega^*(L_j)_{\Delta^2} \to \Omega^*(L_j)_I \tag{186}$$

to be the natural restriction map. Recall that one can identify $\Omega^*(I \times L_j) \cong \Omega^*(L_j)_I$ and $\Omega^*(\Delta^2 \times L_j) \cong \Omega^*(L_j)_{\Delta^2}$ (Lemma 2.14). Similar to Remark 2.22, one can view $\operatorname{Restr}_I^{\Delta^2}$ as a morphism in $\mathscr{U}\mathscr{D}$, i.e. an A_∞ homotopy equivalence from $\check{\mathfrak{M}}^{\mathbb{J}}$ to one of $\check{\mathfrak{M}}^{F,j}$, $\check{\mathfrak{M}}^{F',j}$, $\check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}$ according to the choice of I. Then, by Theorem 3.1, we get an ud-homotopy inverse $(\operatorname{Restr}_I^{\Delta^2})^{-1}$ which is also a morphism in $\mathscr{U}\mathscr{D}$.

Let v be a vertex of I, and we can similarly define Eval_v^I and $\operatorname{Eval}_v^{\Delta^2}$. Then, $\operatorname{Eval}_v^I \circ \operatorname{Restr}_I^{\Delta^2} = \operatorname{Eval}_v^{\Delta^2}$, thereby obtaining an ud-homotopic relation:

$$\operatorname{Eval}_v^I \overset{\text{ud}}{\sim} \operatorname{Eval}_v^{\Delta^2} \circ (\operatorname{Restr}_I^{\Delta^2})^{-1}$$

[14]Some clarification of the Kuranishi-theory choices may be helpful. These choices are taken in the following order. Firstly, we fix the choices $\Xi_i, i \in \mathfrak{I}$ for the A_∞ algebras $\check{\mathfrak{m}}^{J,i}$ (166). Secondly, the choice that defines $\check{\mathfrak{M}}^{\mathbf{F}_*(\hat{g},\hat{J})}$ in (183) is just induced by Fukaya's trick, thus, there is no extra choice at this stage. Thirdly, we make the choices to define the pseudo-isotopies $\check{\mathfrak{M}}^{F,j}$ and $\check{\mathfrak{M}}^{F',j}$ as in (171). Finally, we make the choices for the definition of $\check{\mathfrak{M}}^{\mathbb{J}}$ by Theorem 6.5.

Further, let v' be the other vertex of I, then one can easily show that

$$\mathrm{Eval}^{I}_{v'} \circ (\mathrm{Eval}^{I}_{v})^{-1} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{\Delta^2}_{v'} \circ (\mathrm{Eval}^{\Delta^2}_{v})^{-1}$$

From Theorem 5.6, it follows that

$$\check{\mathfrak{C}}^{F} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{[v_0,v_1]}_{v_1} \circ (\mathrm{Eval}^{[v_0,v_1]}_{v_0})^{-1}, \quad \check{\mathfrak{C}}^{F'} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{[v_0,v_2]}_{v_2} \circ (\mathrm{Eval}^{[v_0,v_2]}_{v_0})^{-1}, \quad \check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{[v_1,v_2]}_{v_2} \circ (\mathrm{Eval}^{[v_1,v_2]}_{v_1})^{-1}$$

and thus

$$\check{\mathfrak{C}}^{F} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{\Delta^2}_{v_1} \circ (\mathrm{Eval}^{\Delta^2}_{v_0})^{-1}, \quad \check{\mathfrak{C}}^{F'} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{\Delta^2}_{v_2} \circ (\mathrm{Eval}^{\Delta^2}_{v_0})^{-1}, \quad \check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} \overset{\mathrm{ud}}{\sim} \mathrm{Eval}^{\Delta^2}_{v_2} \circ (\mathrm{Eval}^{\Delta^2}_{v_1})^{-1}$$

Now, it is immediate that $\check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} \circ \check{\mathfrak{C}}^{F} \overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{F'}$; the proof is complete. $\square$

9.4.2 Cohomology-level . Let $(H^*(L_j), \mathfrak{M}^{\mathbf{F}_*(\hat{g},\hat{J})}, \mathfrak{I}^{\mathbf{F}_*(\hat{g},\hat{J})})$ be the canonical model (Definition 4.5) of the pseudo-isotopy $\check{\mathfrak{M}}^{\mathbf{F}_*\hat{J}}$ in (183) with respect to $\mathsf{con}(\mathbf{F}_*\hat{g})$. By Theorem 7.9, we know $\mathfrak{M}^{\mathbf{F}_*(\hat{g},\hat{J})} \in \mathrm{Obj}\,\mathscr{U}\mathscr{D}$ and $\mathfrak{I}^{\mathbf{F}_*(\hat{g},\hat{J})} \in \mathrm{Mor}\,\mathscr{U}\mathscr{D}$. By (152), the $\mathfrak{M}^{\mathbf{F}_*(\hat{g},\hat{J})}$ is a pseudo-isotopy between $\mathfrak{m}^{F_*(g,J,k)}$ and $\mathfrak{m}^{F'_*(g,J,k)}$ (in the cohomology-level). Similar to (184), the A_∞ homotopy equivalence induced by $\mathfrak{M}^{\mathbf{F}_*(\hat{g},\hat{J})}$ is also a morphism in $\mathscr{U}\mathscr{D}$, denoted by

$$\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} : \mathfrak{m}^{F_*(g,J,k)} \to \mathfrak{m}^{F'_*(g,J,k)}$$

By Corollary 8.7, the source and target A_∞ algebras of $\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})}$ are identically the same:

$$\mathfrak{m} := \mathfrak{m}^{F_*(g,J,k)} = \mathfrak{m}^{F'_*(g,J,k)} \tag{187}$$

Moreover, it follows from Corollary 8.12 that

$$\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} \overset{\mathrm{ud}}{\sim} \mathrm{id} \tag{188}$$

Ultimately, applying Lemma 7.10 repeatedly to the pairs $(\check{\mathfrak{C}}^{F}, \mathfrak{C}^{F})$, $(\check{\mathfrak{C}}^{F'}, \mathfrak{C}^{F})$, and $(\check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}}, \mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})})$, we obtain the following ud-homotopy relations in sequence:

$$\mathfrak{i}^{F_*(g,J,k)} \circ \mathfrak{C}^{F} \overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{F} \circ \mathfrak{i}^{g,J,j} \tag{189}$$

$$\mathfrak{i}^{F'_*(g,J,k)} \circ \mathfrak{C}^{F'} \overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{F'} \circ \mathfrak{i}^{g,J,j} \tag{190}$$

$$\mathfrak{i}^{F'_*(g,J,k)} \circ \mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} \overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{\mathbf{F}_*\hat{J}} \circ \mathfrak{i}^{F_*(g,J,k)} \tag{191}$$

$$\begin{tikzcd}
 & & & & \mathfrak{m}^{F'_*(g,J,k)} \arrow[dl, "\mathfrak{i}^{F'_*(g,J,k)}"] \arrow[r, equal] & \mathfrak{m} \\
 & & & \check{\mathfrak{m}}^{F'_*(g,J,k)} & & \\
\mathfrak{m}^{g,J,j} \arrow[r, "\mathfrak{i}^{g,J,j}"] \arrow[uurrrr, "\mathfrak{C}^{F'}"] \arrow[ddrrrr, "\mathfrak{C}^{F}"'] & \check{\mathfrak{m}}^{J,j} \arrow[urr, "\check{\mathfrak{C}}^{F'}"] \arrow[drr, "\check{\mathfrak{C}}^{F}"'] \arrow[r, phantom, "\check{\mathfrak{M}}^{\mathbf{J}}"] & {} & & & \\
 & & & \check{\mathfrak{m}}^{F_*(J,k)} \arrow[uu, "\check{\mathfrak{C}}^{\mathbf{F}_*(\hat{h},\hat{J})}"'] & & \\
 & & & & \mathfrak{m}^{F_*(g,J,k)} \arrow[ul, "\mathfrak{i}^{F_*(g,J,k)}"] \arrow[uuuu, "\mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} \overset{\mathrm{ud}}{\sim} \mathrm{id}"'] \arrow[r, equal] & \mathfrak{m}
\end{tikzcd}$$

Lemma 9.12. $\mathfrak{C}^F \overset{\mathrm{ud}}{\sim} \mathfrak{C}^{F'}$.

Proof. The proof is given by chasing the diagram:

$$\begin{aligned}
\mathfrak{i}^{F'_*(g,J,k)} \circ \mathfrak{C}^{F'} &\overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{F'} \circ \mathfrak{i}^{g,J,j} && \text{(use (190))} \\
&\overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{\mathbf{F}_*(\hat{g},\hat{J})I} \circ \check{\mathfrak{C}}^{F} \circ \mathfrak{i}^{g,J,j} && \text{(use Lemma 9.11)} \\
&\overset{\mathrm{ud}}{\sim} \check{\mathfrak{C}}^{\mathbf{F}_*(\hat{g},\hat{J})} \circ \mathfrak{i}^{F_*(g,J,k)} \circ \mathfrak{C}^{F} && \text{(use (189))} \\
&\overset{\mathrm{ud}}{\sim} \mathfrak{i}^{F'_*(g,J,k)} \circ \mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} \circ \mathfrak{C}^{F} && \text{(use (191))}
\end{aligned}$$

Using Theorem 3.1, we conclude $\mathfrak{C}^{F'} \overset{\mathrm{ud}}{\sim} \mathfrak{C}^{\mathbf{F}_*(\hat{g},\hat{J})} \circ \mathfrak{C}^{F}$. By (188), we finally show that $\mathfrak{C}^F \overset{\mathrm{ud}}{\sim} \mathfrak{C}^{F'}$. □

9.4.3 Choice-independence's proof .

Proof of Theorem 9.10. Let ϕ^F_{jk} and $\phi^{F'}_{jk}$ be the homomorphisms defined as in (176) with respect to the different choices. Note that $F_*\alpha = F'_*\alpha$ for all $\alpha \in$

$\pi_1(L_k)$ since F is isotopic to F'. By the definition formulas (175), we only need to study the difference of the series in the exponents

$$\sum_{\beta\neq 0}(\mathfrak{C}^{F'}_{0,\beta}-\mathfrak{C}^{F}_{0,\beta})\ T^{E(\beta)}Y^{\partial\beta}$$

called the *error term*. Due to Corollary 2.40, the ud-homotopy condition $\mathfrak{C}^F \overset{\mathrm{ud}}{\sim} \mathfrak{C}^{F'}$ in Lemma 9.12 can be unpacked to the existence of operators $(\mathfrak{f}_s)_{s\in[0,1]}$ and $(\mathfrak{h}_s)_{s\in[0,1]}$ with the following conditions:

(a) $\mathfrak{f}_s\in \mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}^{g,J,j},\mathfrak{m})$, where $\mathfrak{f}_0=\mathfrak{C}^F$ and $\mathfrak{f}_1=\mathfrak{C}^{F'}$;

(b) $\frac{d}{ds}\circ\mathfrak{f}_s=\sum\mathfrak{h}_s\circ(\mathrm{id}_{\#}^{\bullet}\otimes\mathfrak{m}^{g,J,j}\otimes\mathrm{id}^{\bullet})+\sum\mathfrak{m}\circ(\mathfrak{f}_s^{\#}\cdots\mathfrak{f}_s^{\#},\mathfrak{h}_s,\mathfrak{f}_s\cdots\mathfrak{f}_s)$;

(c) $\mathfrak{h}_s$ satisfies the divisor axiom, the cyclical unitality, and the property $\mathfrak{h}_s(\cdots\mathbb{1}\cdots)=0$;

(d) $\deg(\mathfrak{f}_s)_{k,\beta}=1-k-\mu(\beta)$, $\deg(\mathfrak{h}_s)_{k,\beta}=-k-\mu(\beta)$, and $(\mathfrak{h}_s)_\beta\neq 0$ only if $\mu(\beta)\geq 0$

Recall that the target A_∞ algebra $\mathfrak{m}$ is given in (187). Also, recall that the nonzero terms $\mathfrak{C}^F_{0,\beta},\mathfrak{C}^{F'}_{0,\beta}$ must live in $H^1(L_j)$. So, we fix $\gamma\in\pi_1(L_j)$ and consider the γ-component of the error term:

$$S(Y):=S^\gamma(Y):=\sum_{\beta\neq 0}\langle\gamma,\mathfrak{C}^{F'}_{0,\beta}-\mathfrak{C}^{F}_{0,\beta}\rangle\ T^{E(\beta)}Y^{\partial\beta}$$

Choose a basis $\{\gamma_k\}$ of $\pi_1(L_j)$. Denote by $\{\theta_k\}$ its dual basis of $H^1(L_j)$. Then, we have $\Lambda[[\pi_1(L_j)]]\cong\Lambda[[Y_1^{\pm},\dots,Y_n^{\pm}]]$; we may write $S(Y)=S(Y_1,\dots,Y_n)=\sum_{\beta\neq 0}\langle\gamma,\mathfrak{C}^{F'}_{0,\beta}-\mathfrak{C}^{F}_{0,\beta}\rangle\ T^{E(\beta)}Y_1^{\partial_1\beta}\cdots Y_n^{\partial_n\beta}$ where $\partial_i\beta\in\mathbb{Z}$ and $\partial\beta=\partial_1\beta\cdot\gamma_1+\cdots+\partial_n\beta\cdot\gamma_n$.

We will apply Lemma 2.3 again. Given $\mathbf{y}=(y_1,\dots,y_n)\in U_\Lambda^n$, there exists $x_i\in\Lambda_0$ so that $y_i=e^{x_i}$ for each i due to Lemma 2.2. Put $b=x_1\theta_1+\cdots+x_n\theta_n\in H^1(L_j)\hat{\otimes}\Lambda$, and then $e^{\partial\beta\cap b}=e^{\partial_1\beta\cdot x_1+\cdots+\partial_n\beta\cdot x_n}=\mathbf{y}^{\partial\beta}$. Thus, the divisor axioms of $\mathfrak{C}^F$ and $\mathfrak{C}^{F'}$ infer that (see (32))

$$S(\mathbf{y})=\sum_{k,\beta}T^{E(\beta)}\langle\gamma,\mathfrak{C}^{F'}_{k,\beta}(b,\dots,b)-\mathfrak{C}^{F}_{k,\beta}(b,\dots,b)\rangle=\langle\gamma,\mathfrak{C}^{F'}_*(b)-\mathfrak{C}^F_*(b)\rangle\equiv\langle\gamma,\mathfrak{f}_{1*}(b)-\mathfrak{f}_{0*}(b)\rangle$$

However, from a different point of view, using the condition (b) yields the following computation:

$$\begin{aligned}\mathfrak{f}_{1*}(b)-\mathfrak{f}_{0*}(b)&=\sum T^{E(\beta)}\textstyle\int_0^1 ds\cdot\frac{d}{ds}\circ(\mathfrak{f}_s)_{k,\beta}(b,\dots,b)\\&=\sum T^{E(\beta_1)}\textstyle\int_0^1 ds\cdot(\mathfrak{h}_s)_{\lambda+\mu+1,\beta_1}(b,\dots,b,T^{E(\beta_2)}\mathfrak{m}^{g,J,j}_{\nu,\beta_2}(b,\dots,b),b,\dots,b)\\&\quad+\sum\textstyle\int_0^1 ds\cdot\mathfrak{m}\big((\mathfrak{f}_s)(b,\dots,b),\dots,(\mathfrak{h}_s)_{\ell,\beta_0}(b,\dots,b),\dots,(\mathfrak{f}_s)(b,\dots,b)\big)\end{aligned}$$

The condition (d) implies $\deg(\mathfrak{h}_s)_{\ell,\beta_0}(b,\dots,b)=-\mu(\beta_0)\leq 0$, so we may assume this degree always equals to zero. Then, by the cyclical unitality of $\mathfrak{m}$, the second summation above vanishes. Moreover, the operator $\mathfrak{H} := \int_0^1 ds \cdot \mathfrak{h}_s$ also satisfies the divisor axiom since so does every $\mathfrak{h}_s$. Hence,

$$\begin{aligned}\mathfrak{f}_{1*}(b) - \mathfrak{f}_{0*}(b) &= \sum T^{E(\beta_1)} \mathfrak{H}_{k_1,\beta_1}\big(b,\dots,b, T^{E(\beta_2)}\mathfrak{m}^{g,J,j}_{k_2,\beta_2}(b,\dots,b), b,\dots,b\big) \\ &= \sum T^{E(\beta_1)} e^{\langle \partial\beta_1, b\rangle}\ \mathfrak{H}_{1,\beta_1}\big(T^{E(\beta_2)} e^{\langle\partial\beta_2,b\rangle}\mathfrak{m}^{g,J,j}_{0,\beta_2}\big) = \sum T^{E(\beta)}\mathbf{y}^{\partial\beta}\mathfrak{H}_{1,\beta}(P^{g,J,j}(\mathbf{y}))\end{aligned}$$

Recall that the basis $\{\theta_i\}$ of $H^1(L_j)$ induces a basis of $H^2(L_j)$ given by $\theta_{pq} := \theta_p \wedge \theta_q$ for $1 \le p < q \le n$. Recall also that $P^{g,J,j} = W^{g,J,j}\mathbb{1}_j + \sum_{p<q} Q^{g,J,j}_{pq}\theta_{pq}$. Putting things together, we have

$$S(\mathbf{y}) = \langle \gamma, \mathfrak{f}_{1*}(b) - \mathfrak{f}_{0*}(b)\rangle = S_0(\mathbf{y})\cdot W^{g,J,j}(\mathbf{y}) + \textstyle\sum_{p<q} Q^{g,J,j}_{pq}(\mathbf{y})\cdot S_{pq}(\mathbf{y})$$

where we denote

$$\begin{aligned} S_0 := S_0^\gamma &:= \textstyle\sum_\beta T^{E(\beta)} Y^{\partial\beta}\langle\gamma, \mathfrak{H}_{1,\beta}(\mathbb{1}_j)\rangle \\ S_{pq} := S_{pq}^\gamma &:= \textstyle\sum_\beta T^{E(\beta)} Y^{\partial\beta}\langle\gamma, \mathfrak{H}_{1,\beta}(\theta_{pq})\rangle\end{aligned}$$

But as $\deg \mathfrak{H}_{1,\beta} = -1-\mu(\beta) < 0$, we know $\mathfrak{H}_{1,\beta}(\mathbb{1}_j) = 0$ and so $S_0 = 0$. Consequently,

$$S(\mathbf{y}) = \textstyle\sum_{p<q} S_{pq}(\mathbf{y})\cdot Q^{g,J,j}_{pq}(\mathbf{y}) \tag{192}$$

holds for any arbitrary point $\mathbf{y}$ in U^n_Λ. And, it actually holds everywhere thanks to Lemma 2.3. Thus, the γ-component $S(Y) = S^\gamma(Y) = \sum_{\beta\neq 0}\langle\gamma, \mathfrak{C}^{F'}_{0,\beta} - \mathfrak{C}^F_{0,\beta}\rangle\, T^{E(\beta)}Y^{\partial\beta}$ of the error term is contained in $\mathfrak{a}_j$ for any $\gamma \in \pi_1(L_j)$. Finally, due to (175), we have

$$\begin{aligned}\phi^{F'}_{jk}(sY^\alpha) &= \phi^F_{jk}(sY^\alpha)\cdot \exp\Big(\textstyle\sum\langle F_*\alpha, \mathfrak{C}^{F'}_{0,\beta} - \mathfrak{C}^F_{0,\beta}\rangle T^{E(\beta)}Y^{\partial\beta}\Big) \\ &= \phi^F_{jk}(sY^\alpha)\cdot\exp(S^{F_*\alpha}(Y)) \\ &= \phi^F_{jk}(sY^\alpha)\cdot\big(1 + S^{F_*\alpha}(Y) + \tfrac{S^{F_*\alpha}(Y)^2}{2!} + \cdots\big) = \phi^F_{jk}(sY^\alpha) + (\text{something in } \mathfrak{a}_j)\end{aligned}$$

Hence, the ϕ^F_{jk} and $\phi^{F'}_{jk}$ actually induce the same quotient homomorphism $\varphi_{jk} : A_{kj} \to A_{jk}$ in (181). $\square$

9.5 Cocycle conditions

The last step for the mirror construction is to show the cocycle conditions among the various transition maps ψ_{ij} (182). The idea of the proof is very similar to that of Theorem 9.10. Let Δ_i, Δ_j and Δ_k be three adjacent polyhedrons. We have defined the local charts X_i, X_j, X_k (169) and the transition maps ψ_{ik}, ψ_{jk}, ψ_{ij} (182) which correspond to the quotient algebra homomorphisms φ_{ik}, φ_{jk}, φ_{ij} (181).

Theorem 9.13. *$\psi_{ik} = \psi_{jk}\circ\psi_{ij}$ or equivalently $\varphi_{ik} = \varphi_{ij}\circ\varphi_{jk}$.*

We begin with some preparations. Due to Lemma 9.3, we can pick up W_q for some $q \in B_0$ such that $W_q \supset \Delta_i \cup \Delta_j \cup \Delta_k$. Using the construction (161) produces several specific diffeomorphisms $F_{ij} := F_q^{q_j, q_i}$, $F_{jk} := F_q^{q_k, q_j}$, and $F_{ik} := F_q^{q_k, q_i}$ in the neighborhood $\mathcal{U}$. Observe that for $a, b \in \{i, j, k\}$, we have $F_{ab}(L_b) = L_a$ and $F_{ij} \circ F_{jk} = F_{ik}$. By Theorem 9.10, the ambiguities caused by taking different choices have been eliminated, and so we can use these specific choices for the computation.

Just as §9.3.2, we make choices $(\mathbf{J}_{ij}, \mathbf{g}_{ij})$, $(\mathbf{J}_{jk}, \mathbf{g}_{jk})$, and $(\mathbf{J}_{ik}, \mathbf{g}_{ik})$: for $a, b \in \{i, j, k\}$, the $\mathbf{J}_{ab}$ is a path of almost complex structures in $\mathcal{V}$ from J to $F_{ab*}J$, and the $\mathbf{g}_{ab}$ is a path of metrics from g to $F_{ab*}g$. In the same way as (174), these data produce three A_∞ homomorphisms in $\mathscr{U}\mathscr{D}$:

$$\begin{aligned} \mathfrak{C}^{F_{ij}} &: \mathfrak{m}^{g,J,i} \to \mathfrak{m}^{F_{ij*}(g,J,j)} \\ \mathfrak{C}^{F_{jk}} &: \mathfrak{m}^{g,J,j} \to \mathfrak{m}^{F_{jk*}(g,J,k)} \\ \mathfrak{C}^{F_{ik}} &: \mathfrak{m}^{g,J,i} \to \mathfrak{m}^{F_{ik*}(g,J,k)} \end{aligned}$$

Using them, we can similarly construct the algebra homomorphisms $\phi_{ik}^{F_{ik}}$, $\phi_{ij}^{F_{ij}}$, and $\phi_{jk}^{F_{jk}}$ as in (176).

Next, we aim to compare $\phi_{ik}^{F_{ik}}$ with $\phi_{ij}^{F_{ij}} \circ \phi_{jk}^{F_{jk}}$. Heuristically, let us think of the indexes i and k as the 'source' and 'target', while the index j is thought of as the 'bridge'. A subtle point is that the target of $\mathfrak{C}^{F_{ij}}$ does not match the source of $\mathfrak{C}^{F_{jk}}$, but this issue is inessential thanks to Fukaya's trick. In reality, applying Proposition 8.13 to the constant family $\mathbf{F} = (F_s)$ where $F_s = F_{ij}$ for each s, we obtain a push-forward A_∞ homomorphism

$$\tilde{\mathfrak{C}}^{F_{jk}} := \mathfrak{C}^{\mathbf{F}_*(\mathbf{g}_{jk}, \mathbf{J}_{jk})} : \mathfrak{m}^{F_{ij*}(g,J,j)} \to \mathfrak{m}^{F_{ij*}F_{jk*}(g,J,k)} \equiv \mathfrak{m}^{F_{ik*}(g,J,k)}$$

such that $H^{F_{ij}} \circ \tilde{\mathfrak{C}}^{F_{jk}} = \mathfrak{C}^{F_{jk}} \circ H^{F_{ij}}$; see §8.4 for the notation $H^{F_{ij}}$.

$$\begin{tikzcd}
 & \mathfrak{m}^{g,J,j} \arrow[r, dotted, "\mathfrak{C}^{F_{jk}}"] \arrow[d, dotted, "H^{F_{ij}}"'] & \mathfrak{m}^{F_{jk*}(g,J,k)} \arrow[d, dotted, "H^{F_{ij}}"] & & \text{on } H^*(L_j) \\
\mathfrak{m}^{g,J,i} \arrow[r, "\mathfrak{C}^{F_{ij}}"] \arrow[rr, bend right, "\mathfrak{C}^{F_{ik}}"'] & \mathfrak{m}^{F_{ij*}(g,J,j)} \arrow[r, "\tilde{\mathfrak{C}}^{F_{jk}}"] & \mathfrak{m}^{F_{ij*}F_{jk*}(g,J,k)} & = \mathfrak{m}^{F_{ik*}(g,J,k)} & \text{on } H^*(L_i)
\end{tikzcd}$$

Lemma 9.14. $\mathfrak{C}^{F_{ik}}$ *is ud-homotopic to* $\tilde{\mathfrak{C}}^{F_{jk}} \circ \mathfrak{C}^{F_{ij}}$.

Sketch of proof. The proof is almost the same as Lemma 9.12. Write $\mathbf{J}_{ab} = (J_{ab}^s)_{s \in [0,1]}$ and $\mathbf{g}_{ab} = (g_{ab}^s)_{s \in [0,1]}$ for $a, b \in \{i, j, k\}$. Let $\breve{\mathfrak{M}}^{\mathbf{J}_{ij}}$, $\breve{\mathfrak{M}}^{\mathbf{J}_{ik}}$ and $\breve{\mathfrak{M}}^{\mathbf{F}_*\mathbf{J}_{jk}}$ be the underlying chain-level pseudo-isotopies (Theorem 6.3). We can extend them to a Δ^2-pseudo-isotopy as in (185). Just like Lemma 9.11, we can similarly prove $\check{\check{\mathfrak{C}}}^{F_{jk}} \circ \check{\mathfrak{C}}^{F_{ij}} \overset{\text{ud}}{\sim} \check{\mathfrak{C}}^{F_{ik}}$ for the A_∞ homomorphisms associated to the three chain-level pseudo-isotopies. Next, for the cohomology-level, utilizing Lemma 7.10 and chasing diagrams like the proof of Lemma 9.12, we can similarly show $\tilde{\mathfrak{C}}^{F_{jk}} \circ \mathfrak{C}^{F_{ij}} \overset{\text{ud}}{\sim} \mathfrak{C}^{F_{ik}}$. □

Just as how we use Lemma 9.12 to prove Theorem 9.10, we use Lemma 9.14 to show Theorem 9.13.

Proof of Theorem 9.13. There are natural isomorphisms induced by F_{ij}, F_{jk} and F_{ik} among the label groups $\mathfrak{G}(X, L_i)$, $\mathfrak{G}(X, L_j)$, and $\mathfrak{G}(X, L_k)$ (135). Their elements are denoted by β, β' and β'' in sequence. We will always follow this convention in the below. For instance, given $\alpha'' \in \pi_1(L_k)$ we will set $\alpha = F_{ik*}\alpha'' \in \pi_1(L_i)$ and $\alpha' = F_{jk*}\alpha'' \in \pi_1(L_j)$. Note that $F_{ij*}\alpha' = \alpha$.

To start with, we specify the bases as before. Let $\{f''_\ell\}_{1\le \ell\le n}$ be a basis of $\pi_1(L_k)$ which can induce a basis on $\pi_1(L_i)$ (resp. $\pi(L_j)$) given by $f_\ell := F_{ik*}f''_\ell$ (resp. $f'_\ell := F_{jk*}f''_\ell$). Denote the dual bases by $\{\theta_\ell\}$, $\{\theta'_\ell\}$, and $\{\theta''_\ell\}$ respectively. Since the bases are related by the diffeomorphisms, the above α, α', and α'' can be all identified with the same tuple $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ such that $\alpha = \sum \alpha_\ell f_\ell$, $\alpha' = \sum \alpha_\ell f'_\ell$, and $\alpha'' = \sum \alpha_\ell f''_\ell$. Thus, each of Y^α, $Y^{\alpha'}$ or $Y^{\alpha''}$ can be viewed as $Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}$, thereby inducing the natural identifications $\Lambda[[\pi_1(L_a)]] \cong \Lambda[[Y_1^\pm, \dots, Y_n^\pm]]$ for any $a \in \{i, j, k\}$. Particularly, we can regard $\partial\beta$, $\partial\beta'$ or $\partial\beta''$ as the same tuple $(\partial_1\beta, \dots, \partial_n\beta) \in \mathbb{Z}^n$ where $\partial_i\beta = \langle\partial\beta, \theta_i\rangle = \langle\partial\beta', \theta'_i\rangle = \langle\partial\beta'', \theta''_i\rangle$; we can also identify $Y^{\partial\beta}$, $Y^{\partial\beta'}$, or $Y^{\partial\beta''}$ with the same monomial $Y_1^{\partial_1\beta}\cdots Y_n^{\partial_n\beta}$.

Using the energy formula (179) and the definition formulas (175), we first compute as follows:

$$
\begin{aligned}
&\phi_{ij}^{F_{ij}} \circ \phi_{jk}^{F_{jk}}(Y^{\alpha''}) \\
&= \phi_{ij}^{F_{ij}}\Big(T^{\langle\alpha'', q_j - q_k\rangle} Y^{\alpha'} \exp\langle\alpha', \textstyle\sum \mathfrak{C}^{F_{jk}}_{0,\beta'} T^{E(\beta')} Y^{\partial\beta'}\rangle\Big) \\
&= T^{\langle\alpha'', q_j - q_k\rangle} \phi_{ij}^{F_{ij}}(Y^{\alpha'}) \exp\Big(\textstyle\sum \langle\alpha', \mathfrak{C}^{F_{jk}}_{0,\beta'}\rangle T^{E(\beta')} \phi_{ij}^{F_{ij}}(Y^{\partial\beta'})\Big) \\
&= T^{\langle\alpha'', q_j - q_k\rangle} T^{\langle\alpha', q_i - q_j\rangle} Y^\alpha \exp\langle\alpha, \textstyle\sum \mathfrak{C}^{F_{ij}}_{0,\gamma} T^{E(\gamma)} Y^{\partial\gamma}\rangle \exp\Big(\textstyle\sum \langle\alpha', \mathfrak{C}^{F_{jk}}_{0,\beta'}\rangle T^{E(\beta')} T^{\langle\partial\beta', q_i - q_j\rangle} Y^{\partial\beta} \exp\langle\partial\beta, \textstyle\sum \mathfrak{C}^{F_{ij}}_{0,\eta} T^{\cdots} \\
&= T^{\langle\alpha'', q_i - q_k\rangle} Y^\alpha \exp\langle\alpha, \textstyle\sum \mathfrak{C}^{F_{ij}}_{0,\gamma} T^{E(\gamma)} Y^{\partial\gamma}\rangle \exp\Big(\textstyle\sum \langle\alpha, \tilde{\mathfrak{C}}^{F_{jk}}_{0,\beta}\rangle T^{E(\beta)} Y^{\partial\beta} \exp\langle\partial\beta, \textstyle\sum \mathfrak{C}^{F_{ij}}_{0,\eta} T^{E(\eta)} Y^{\partial\eta}\rangle\Big)
\end{aligned}
$$

where in the last step we use Proposition 8.13 to get $\mathfrak{C}^{F_{jk}}_{0,\beta'} = F_{ij}^*\tilde{\mathfrak{C}}^{F_{jk}}_{0,\beta}$ and so $\langle\alpha', \mathfrak{C}^{F_{jk}}_{0,\beta'}\rangle = \langle\alpha, \tilde{\mathfrak{C}}^{F_{jk}}_{0,\beta}\rangle$.

Once again, we will take advantage of Lemma 2.3 for the computations. Assume $\mathbf{y} = (y_1, \dots, y_n)$ is an arbitrary point in U_Λ^n, and we can find $x_\ell \in \Lambda_0$ such that $y_\ell = e^{x_\ell}$ for each $1 \le \ell \le n$ due to Lemma 2.2. We set $b = \sum x_\ell\theta_\ell$, $b' = \sum x_\ell\theta'_\ell$, and $b'' = \sum x_\ell\theta''_\ell$. For a general monomial Y^α, the evaluation at $\mathbf{y}$ gives the value $\mathbf{y}^\alpha = \exp(\alpha \cap b) = \exp(\alpha' \cap b') = \exp(\alpha'' \cap b'') = e^{\alpha_1 x_1 + \cdots + \alpha_n x_n}$ in Λ.

After the substitution $Y = \mathbf{y}$, applying the divisor axiom to the first exponential power, we obtain

$$
\exp\langle\alpha, \textstyle\sum \mathfrak{C}^{F_{ij}}_{0,\gamma} T^{E(\gamma)} Y^{\partial\gamma}\rangle|_{Y=\mathbf{y}} = \sum\langle\alpha, \mathfrak{C}^{F_{ij}}_{0,\gamma}\rangle T^{E(\gamma)} e^{\partial\gamma\cap b} = \langle\alpha, \mathfrak{C}^{F_{ij}}_*(b) - b\rangle
$$

Here we recall (32) for the notation $\mathfrak{C}^{F_{ij}}_*$. The similar holds for the last power replacing α by $\partial\beta$. Thus,

$$\begin{aligned}\phi_{ij}^{F_{ij}}\circ\phi_{jk}^{F_{jk}}(Y^{\alpha''})|_{Y=\mathbf{y}} =&T^{\langle\alpha'',q_i-q_k\rangle}\cdot\mathbf{y}^{\alpha}\cdot\exp\langle\alpha,\mathfrak{C}_*^{F_{ij}}(b)-b\rangle\cdot\exp\Big(\sum\langle\alpha,\tilde{\mathfrak{C}}_{0,\beta}^{F_{jk}}\rangle T^{E(\beta)}\mathbf{y}^{\partial\beta}\exp\langle\partial\beta,\mathfrak{C}_*^{F_{ij}}(b)-b\rangle\Big)\\ =&T^{\langle\alpha'',q_i-q_k\rangle}\cdot\exp\langle\alpha,\mathfrak{C}_*^{F_{ij}}(b)\rangle\cdot\exp\Big(\sum\langle\alpha,\tilde{\mathfrak{C}}_{0,\beta}^{F_{jk}}\rangle T^{E(\beta)}\exp\langle\partial\beta,\mathfrak{C}_*^{F_{ij}}(b)\rangle\Big)\\ =&T^{\langle\alpha'',q_i-q_k\rangle}\cdot\exp\langle\alpha,\widehat{b}\rangle\cdot\exp\Big(\sum\langle\alpha,\tilde{\mathfrak{C}}_{0,\beta}^{F_{jk}}\rangle T^{E(\beta)}\exp\langle\partial\beta,\widehat{b}\rangle\Big)\end{aligned}$$

where we put

$$\widehat{b}:=\mathfrak{C}_*^{F_{ij}}(b)\in H^1(L_i)\hat{\otimes}\Lambda_0$$

Viewing $\widehat{b}$ as a new divisor input (24), we similarly gain

$$\sum\langle\alpha,\tilde{\mathfrak{C}}_{0,\beta}^{F_{jk}}\rangle T^{E(\beta)}\exp\langle\partial\beta,\widehat{b}\rangle=\langle\alpha,\tilde{\mathfrak{C}}_*^{F_{jk}}(\widehat{b})-\widehat{b}\rangle$$

Besides, it is easy to check $\tilde{\mathfrak{C}}_*^{F_{jk}}(\widehat{b})=\tilde{\mathfrak{C}}_*^{F_{jk}}\big(\mathfrak{C}_*^{F_{ij}}(b)\big)=(\tilde{\mathfrak{C}}^{F_{jk}}\circ\mathfrak{C}^{F_{ij}})_*(b)$ from the definition. Therefore,

$$\phi_{ij}^{F_{ij}}\circ\phi_{jk}^{F_{jk}}(Y^{\alpha''})|_{Y=\mathbf{y}}=T^{\langle\alpha'',q_i-q_k\rangle}\exp\langle\alpha,\widehat{b}\rangle\cdot\exp\langle\alpha,\tilde{\mathfrak{C}}_*^{F_{jk}}(\widehat{b})-\widehat{b}\rangle=T^{\langle\alpha'',q_i-q_k\rangle}\exp\langle\alpha,(\tilde{\mathfrak{C}}^{F_{jk}}\circ\mathfrak{C}^{F_{ij}})_*(b)\rangle$$

Just like how we use Lemma 9.12 to show (192) before, we can similarly use Lemma 9.14 to show

$$\langle\alpha,(\tilde{\mathfrak{C}}^{F_{jk}}\circ\mathfrak{C}^{F_{ij}})_*(b)\rangle-\langle\alpha,\mathfrak{C}_*^{F_{ik}}(b)\rangle=\sum_{p<q}S_{pq}^{\alpha}(\mathbf{y})\cdot Q_{pq}^{g,J,i}(\mathbf{y})$$

for some formal series S_{pq}^{α} depending on α. Accordingly, given an arbitrary point $\mathbf{y}$ in U_Λ^n, we have

$$\begin{aligned}\phi_{ij}^{F_{ij}}\circ\phi_{jk}^{F_{jk}}(Y^{\alpha''})|_{Y=\mathbf{y}} =&T^{\langle\alpha'',q_i-q_k\rangle}\exp\langle\alpha,\mathfrak{C}_*^{F_{ik}}(b)\rangle\cdot\exp\Big[\sum_{p<q}S_{pq}^{\alpha}(\mathbf{y})\cdot Q_{pq}^{g,J,i}(\mathbf{y})\Big]\\ =&T^{\langle\alpha'',q_i-q_k\rangle}\mathbf{y}^{\alpha}\exp\langle\alpha,\textstyle\sum_\beta\mathfrak{C}_{0,\beta}^{F_{ik}}T^{E(\beta)}\mathbf{y}^{\partial\beta}\rangle\cdot\exp\Big[\sum_{p<q}S_{pq}^{\alpha}(\mathbf{y})\cdot Q_{pq}^{g,J,i}(\mathbf{y})\Big]\\ =&\phi_{ik}^{F_{ik}}(\mathbf{y}^{\alpha''})\cdot\exp\Big[\sum_{p<q}S_{pq}^{\alpha}(\mathbf{y})\cdot Q_{pq}^{g,J,i}(\mathbf{y})\Big]\end{aligned}$$

where the second equation uses the divisor axiom of $\mathfrak{C}^{F_{ik}}$ in reverse, and the last equation just follows from the definition. By Lemma 2.3 again, we actually know the above equation holds everywhere, and so $\phi_{ij}^{F_{ij}}\circ\phi_{jk}^{F_{jk}}(Y^{\alpha''})=\phi_{ik}^{F_{ik}}(Y^{\alpha''})+$ (something in $\mathfrak{a}_i$). Therefore, $\varphi_{ij}\circ\varphi_{jk}=\varphi_{ik}$. □

Corollary 9.15. $\psi_{ji}\circ\psi_{ij}=\mathrm{id}$ *and* $\psi_{ii}=\mathrm{id}$.

Proof. It is straightforward from Theorem 9.13. In fact, we just need to set $k=i$, $F_{ii}=\mathrm{id}$ and set $\mathbf{J}_{ii},\mathbf{g}_{ii}$ to be the constant families. By construction, we can easily show that $\mathfrak{C}^{F_{ii}}=\mathfrak{C}^{\mathrm{id}}=\mathrm{id}$. From the definition formulas (175, 176), it follows that $\phi_{ii}^{F_{ii}}=\mathrm{id}$ and thus $\psi_{ji}\circ\psi_{ij}=\psi_{ii}=\mathrm{id}$. □

Proof of Main Theorem 1.3. Note that the above construction starts from a chosen compact domain $K\subset B_0$ (162) together with the related choices, such like $J\in\mathfrak{J}_K$. By Theorem 9.13, Corollary 9.15, and Proposition A.1, we can glue X_i (§9.2.4) along X_{ij} (§9.3.2) through ψ_{ij} (182), thereby obtaining a rigid analytic space $X^\vee_{J,K}$ such that the collection (X_i) gives an admissible covering. Moreover, by Theorem 9.8, the various $W^{g,J,i}$ on X_i are compatible with the gluing maps ψ_{ij} and give rise to a global function $W^\vee_{J,K}$ on $X^\vee_{J,K}$. Finally, by Corollary 9.9, one can also glue the various $\mathfrak{trop}_{q_i}$ (165) to obtain a map $\pi^\vee_{J,K} : X^\vee_{J,K} \to K$.

When K is fixed, we claim that the isomorphism class of triple $\mathbb{X}^\vee_{J,K} := (X^\vee_{J,K}, W^\vee_{J,K}, \pi^\vee_{J,K})$ does not depend on J. Indeed, suppose $\tilde{J} \in \mathfrak{J}_K$ is another choice. First, using a path between J and $\tilde{J}$ inside $\mathfrak{J}_K$ and considering the induced pseudo-isotopies, one can construct local isomorphisms between the local pieces just like how we previously construct the transition maps. Then, by the same method of showing the cocycle conditions, they can be glued together to obtain a global isomorphism $\mathbb{X}^\vee_{J,K} \cong \mathbb{X}^\vee_{\tilde{J},K}$. On the other hand, if $\tilde{K}$ is another compact domain, we may assume $\tilde{K} \supset K$; the same construction yields a natural open embedding $\mathbb{X}^\vee_{J,K} \subset \mathbb{X}^\vee_{J,\tilde{K}}$.

In general, we choose a sequence of compact domains $K_1 \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots$ such that $\bigcup_{n \geq 1} K_n = B_0$. Then, we also have a sequence $\mathfrak{J}_{K_1} \supset \cdots \supset \mathfrak{J}_{K_n} \supset \mathfrak{J}_{K_{n+1}} \supset \cdots$. By Assumption 1.2, each $\mathfrak{J}_{K_n}$ is open. Fix some $J_n \in \mathfrak{J}_{K_n}$ and a sufficiently small neighborhood $\mathcal{V}_n$ of J_n in $\mathfrak{J}_{K_n}$, and then we choose some $\tilde{J}_n \in \mathcal{V}_n \cap \mathcal{V}_{n+1}$ to serve as a bridge. By the above arguments, we can obtain an open embedding $\mathbb{X}^\vee_{J_n,K_n} \hookrightarrow \mathbb{X}^\vee_{J_{n+1},K_{n+1}}$. Ultimately, applying Proposition A.1 again to the increasing sequence $(\mathbb{X}^\vee_{J_n,K_n})$, we get the mirror $\mathbb{X}^\vee = (X^\vee, W^\vee, \pi^\vee)$ consisting of an analytic space with a fibration $\pi^\vee : X^\vee \to B_0$ and a global function $W^\vee$. $\square$

Appendices

A Non-archimedean geometry and tropical varieties

In this appendix section, we survey basics of tropical varieties for non-archimedean analytic spaces. For further details, we refer to [Bos14] and [BGR84] for rigid analytic geometry as well as [EKL06] and [Gub07] for non-archimedean tropical geometry.

Let $\mathbb{K}$ be an algebraically closed field. A norm $|\cdot|$ on $\mathbb{K}$ is called **non-archimedean** if $|a+b| \leq \max\{|a|,|b|\}$ for any $a,b \in \mathbb{K}$. It is equivalent to a valuation $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ satisfying (1) $\mathrm{val}(a) = 0$ if and only if $a = 0$; (2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$; (3) $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$. They are related to each other by $\mathrm{val}(a) = -\log|a|$ and $|a| = e^{-\mathrm{val}(a)}$. Let $T_d = \mathbb{K}\langle z_1, \dots, z_d\rangle \subset \mathbb{K}[[z_1,\dots,z_d]]$ be the set of all formal power series $\sum_{\nu \in \mathbb{Z}^d_{\geq 0}} a_\nu \mathbf{z}^\nu$ so that $|a_\nu| \to 0$ as $|\nu| = \sum |\nu_i| \to \infty$. It form a Banach $\mathbb{K}$-algebra, called the d-th **Tate algebra**. An **affinoid algebra** is defined to be a $\mathbb{K}$-Banach algebra A admitting a continuous epimorphism $T_d \to A$ for some d. Let A be an affinoid algebra. Contrast to considering the prime spectrum $\operatorname{Spec} A$ as in algebraic geometry, we look at the maximal spectrum $\operatorname{Sp} A := \operatorname{Max} A$, called the **affinoid $\mathbb{K}$-space** associated to A. In view of Tate's Acyclicity theorem, it is useful to introduce a (strong) Grothendieck topology which consists of the following data: (i) a family of subsets called **admissible open subsets**; (ii) open coverings for any admissible open subset U obeying certain axioms, called **admissible coverings**.

It turns out that $\operatorname{Sp} A$ can be equipped with a strong Grothendieck topology (G-topology) with some completeness conditions [Bos14, 5.1/5]. Then, Tate's Acyclicity Theorem says that the presheaf $\mathcal{O}_{\operatorname{Sp} A}$ of affinoid functions is a sheaf with regard to this G-topology. Note that *points* $x, y, \dots$ in $\operatorname{Sp} A$ can be viewed as maximal ideals $\mathfrak{m}_x, \mathfrak{m}_y, \dots$ in the algebra A given by $\mathfrak{m}_x = \{f \in A \mid f(x) = 0\}$. Also, any $f \in A$ can be regarded as a *function* on the space $\operatorname{Sp} A$ by setting $f(x)$ to be the residue class of f in $A/\mathfrak{m}_x \equiv \mathbb{K}$[15]. Moreover, we define

$$V(F) = \{x \in \operatorname{Sp} A \mid f(x) = 0,\ \forall f \in F\}; \quad \operatorname{id}(E) = \{f \in A \mid f(x) = 0,\ \forall x \in E\} \tag{193}$$

Then, the Hilbert's Nullstellensatz [Bos14, 3.2/4] also holds in the sense that for an ideal $\mathfrak{a} \subset A$, we have $\operatorname{id}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$. Thus, several functions f_i on $\operatorname{Sp} A$ has no common zeros if and only if the unit ideal is generated by f_i.

The affinoid space associated to the n-th Tate algebra is exactly the unit ball in $\mathbb{K}^n$:

$$B^n(\mathbb{K}) \xrightarrow{\cong} \operatorname{Sp} T_n, \quad x \mapsto \mathfrak{m}_x = \{f \in T_n \mid f(x) = 0\} \tag{194}$$

where $B^n(\mathbb{K}) := \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid |x_i| \leq 1\}$. Also, the Tate algebra can be viewed as the space of well-defined functions on the unit ball in the sense that for a formal power series $f \in \mathbb{K}[[z_1, \dots, z_n]]$, we have $f \in T_n$ if and only if f converges on $B^n(\mathbb{K})$. More generally, by [EKL06, Proposition 3.1.8], if $A = T_d/(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in T_d$, then $\operatorname{Sp} A$ agrees with $V(f_1, \dots, f_r)$ of $B^n(\mathbb{K})$.

In general, a **rigid analytic $\mathbb{K}$-space** is a locally ringed space $(X, \mathcal{O}_X)$ which admits a Grothendieck topology with the aforementioned completeness conditions such that it admits an admissible covering by affinoid $\mathbb{K}$-spaces. Moreover, one can also define the morphisms between them. By contrast, a scheme is a locally ringed space admitting a covering by affine schemes. Note that a rigid analytic space can be constructed by gluing local ones as the following proposition tells:

Proposition A.1 ([Bos14, 5.3/5]). *Consider the following data (i) rigid analytic spaces X_i, $i \in I$; (ii) open subspaces $X_{ij} \subset X_i$; (iii) morphisms $\psi_{ij} : X_{ij} \to X_{ji}$. Suppose (a) $X_{ii} = X_i$, $\psi_{ij} \circ \psi_{ji} = \operatorname{id}$ and $\psi_{ii} = \operatorname{id}$; (b) ψ_{ij} induces isomorphisms $\psi_{ij} : X_{ij} \cap X_{ik} \to X_{ji} \cap X_{jk}$ satisfying cocycle conditions $\psi_{ik} = \psi_{jk} \circ \psi_{ij}$. Then these X_i can be glued by identifying X_{ij} with X_{ji} to get a rigid analytic space X admitting $(X_i)_{i \in I}$ as an admissible covering.*

[15] $A/\mathfrak{m}_x$ is first a finite field over $\mathbb{K}$ due to [Bos14, 2.2/12] and must equal to $\mathbb{K}$, since $\mathbb{K}$ is algebraically closed

Every $\mathbb{K}$-scheme X of locally finite type admits a **rigid analytification**, which is a rigid $\mathbb{K}$-space together with a morphism $X^{\mathrm{an}} \to X$, satisfying some universal properties. An instructive fact is that *the map on the underlying sets identifies the points of X^{an} with the closed points of the scheme X*. In reality, we have the so-called GAGA-functor from the category of $\mathbb{K}$ schemes of locally finite type to the category of rigid $\mathbb{K}$-space (see [Bos14, Sec. 5.4] or [Tem15, Sec.5.1]). For example, fix $d > 0$ and $s > 1$. Consider the (scaled) Tate algebra $T_n^{(i)} := \mathbb{K}\langle s^{-i}z_1, \dots, s^{-i}z_n\rangle$ for some $i \in \mathbb{N}$ which consists of formal power series $\sum a^\nu \mathbf{z}^\nu$ so that $|a_\nu| s^{i|\nu|} \to 0$. Just like (194), $\operatorname{Sp} T_n^{(i)}$ agrees with the ball with radius s^i. The rigid analytification of affine n-space $\mathbb{A}^n_{\mathbb{K}}$ is given by $\mathbb{A}^{n,\mathrm{an}}_{\mathbb{K}} = \bigcup_{i=0}^\infty \operatorname{Sp} T_n^{(i)}$.

Furthermore, we need to consider a tropical example as follows. Denote by $\mathbb{G}_m^n = \operatorname{Spec} \mathbb{K}[z_1^\pm, \dots, z_n^\pm]$ the punctured n-affine space. Then, the points of the rigid analytification $\mathbb{G}_m^{n,\mathrm{an}}$ are in bijection with the closed points in $\mathbb{G}_m^n$, i.e. the set $(\mathbb{K}^\times)^n$ where $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$. In fact, the rigid analytification $\mathbb{G}_m^{n,\mathrm{an}}$ admits an admissible covering

$$\mathbb{G}_m^{n,\mathrm{an}} = \bigcup_{r\geq 1} \operatorname{Sp}\left(\mathbb{K}\langle r^{-1}z_i, r^{-1}z_i^{-1} \mid 1 \leq i \leq n\rangle\right)$$

where $\operatorname{Sp}(\mathbb{K}\langle r^{-1}z_i, r^{-1}z_i^{-1}\rangle)$ is the subset of $(\mathbb{K}^\times)^n$ defined by $\frac{1}{r} \leq |x_i| \leq r$. As an analog of $(\mathbb{C}^*)^n \to \mathbb{R}^n$, we have the following map is well-known in out non-archimedean setting:

$$\mathfrak{trop} : (\mathbb{K}^\times)^n \cong \mathbb{G}_m^{n,\mathrm{an}} \to \mathbb{R}^n, \quad \mathfrak{trop}(a_1, \dots, a_n) = (\operatorname{val}(a_1), \dots, \operatorname{val}(a_n))$$

Let $\Gamma = \operatorname{val}(\mathbb{K}^\times)$ be the valuation group. For instance, when $\mathbb{K}$ is the Novikov field, we have $\Gamma = \mathbb{R}$. A convex subset $\Delta \subset \mathbb{R}^n$ is called a **Γ-rational polyhedron** (often omitting 'Γ') if it is defined by finitely many inequalities $\sum_j b_{ij}x_j \geq c_i$ for $c_i \in \Gamma$ and $b_{ij} \in \mathbb{Z}$.

Definition A.2. For a bounded Γ-rational polyhedron $\Delta \subset \mathbb{R}^n$, we define the so-called **polyhedral affinoid algebras** $\mathbb{K}\langle\Delta\rangle \subset \mathbb{K}[[z_1^\pm, \dots, z_n^\pm]]$ to be the set of formal Laurent series $f = \sum_{\nu\in\mathbb{Z}^d} a_\nu \mathbf{z}^\nu$ so that $\operatorname{val}(a_\nu) + \nu \cdot u \to \infty$ for all $u \in \Delta$.

The $\mathbb{K}\langle\Delta\rangle$ is indeed an algebra. We can define a multiplication on $\mathbb{K}\langle\Delta\rangle$ by setting $\left(\sum_\nu a_\nu \mathbf{z}^\nu\right) \cdot \left(\sum_\nu b_\nu \mathbf{z}^\nu\right) = \sum_\nu \left(\sum_{\nu=\nu_1+\nu_2} a_{\nu_1} b_{\nu_2}\right) \cdot \mathbf{z}^\nu$ which converges because $\operatorname{val}(\sum_{\nu=\nu_1+\nu_2} a_{\nu_1}b_{\nu_2}) + \nu\cdot u \geq \min_{\nu=\nu_1+\nu_2}\{\operatorname{val}(a_{\nu_1}) + \operatorname{val}(b_{\nu_2}) + \nu_1 \cdot u + \nu_2 \cdot u\} \to \infty$. Just like the case of Tate algebra mentioned above (194), a useful observation is as follows:

Proposition A.3. *Let $f = \sum_{\nu\in\mathbb{Z}^d} a_\nu \mathbf{z}^\nu \in \mathbb{K}[[z^\pm]]$. Then $f \in \mathbb{K}\langle\Delta\rangle$ if and only if $f(\mathbf{y})$ converges for any $\mathbf{y} \in \mathfrak{trop}^{-1}(\Delta)$. In this case, f can be recognized as a global function defined on $\mathfrak{trop}^{-1}(\Delta)$.*

Proof. Recall [Bos14, 2.1/3] that the convergence means exactly that $a_\nu \mathbf{y}^\nu$ forms a zero sequence, or equivalently $\operatorname{val}(a_\nu) + \nu \cdot \mathfrak{trop}(\mathbf{y}) \to \infty$. This exactly corresponds to the definition of $\mathbb{K}\langle\Delta\rangle$. $\square$

The following is a well-known fact; see [EKL06, 3.1.5/3.18(c)] or [Gub07, Prop. 4.1].

Proposition A.4. *$\mathbb{K}\langle\Delta\rangle$ is an affinoid algebra and $U_\Delta := \mathfrak{trop}^{-1}(\Delta)$ is identified with $\mathrm{Sp}\,\mathbb{K}\langle\Delta\rangle$ via $\mathbf{x} \leftrightarrow \mathfrak{m}_{\mathbf{x}}$. Moreover, $\mathfrak{trop}^{-1}(\Delta)$ is a Weiestrass domain in $(\mathbb{K}^\times)^n$, called a **polytopal domain**.*

B Reverse isoperimetric inequalities

Theorem B.1 (Theorem 1.1 [GS14])**.** *Let (X,ω) be a symplectic manifold and L be a closed Lagrangian submanifold. For any ω-tame almost complex structure J, there exists a constant $c = c(L,J) > 0$ so that $\mathrm{area}(u; g_J) \geq c \cdot \ell(\partial u; g_J)$ for any J-holomorphic curve $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$.*

For our purpose, we need to strengthen it to Theorem B.2 below. As in [Abo17a, Appendix A], we will closely follow DuVal's more flexible argument [Duv16]. First, we note that by [MS12, Lemma 2.2.1], if u is a J-holomorphic curve for some ω-tame almost complex structure J, then the energy $E(u) = \frac{1}{2}\int_\Sigma |du|^2 d\mathrm{vol}_\Sigma$ is actually topological: $E(u) = \mathrm{area}(u) = \int_\Sigma u^*\omega$.

Theorem B.2. *Fix (X,ω), L and J as above in Theorem B.1. There exists a C^1-neighborhood $\mathcal{V}_1$ of J and a constant $c_1 > 0$ such that for any $\tilde{J} \in \mathcal{V}_1$ and $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$, we have*

$$E(u) \geq c_1 \cdot \ell(\partial u)$$

Corollary B.3. *There is a C^1-neighborhood $\mathcal{V}_0$ of J, a Weinstein neighborhood $\nu_X L$ of L and a constant $c_0 > 0$ such that: If $\tilde{L} \subset \nu_X L$ is an adjacent Lagrangian submanifold given by the graph of a closed one-form on L, then for any $\tilde{J} \in \mathcal{V}_0$ and $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, \tilde{L})$, we have*

$$E(u) \geq c_0 \cdot \ell(\partial u)$$

Proof of Theorem B.2 implying Corollary B.3. Suppose a neighborhood $\mathcal{V}_1$ of J and a constant $c_1 > 0$ are obtained by Theorem B.2. Choose the $\mathcal{V}_0 \subset \mathcal{V}_1$ and $\nu_X L$ small enough, and we may require that for any such $\tilde{L} \subset \nu_X L$ and $\tilde{J} \in \mathcal{V}_0$, one can find a small $F \in \mathrm{Diff}_0(X)$ such that $F(\tilde{L}) = L$ and $F_*\tilde{J} \in \mathcal{V}_1$. Then, since $F \circ u$ is a $F_*\tilde{J}$-holomorphic disk bounding L, we have $E(F \circ u) \geq c_1 \ell(\partial(F \circ u))$. Finally, we can compare the energy and the boundary as follows: $E(u) \gtrsim E(F \circ u) \gtrsim \ell(\partial(F \circ u)) \gtrsim \ell(\partial u)$. □

Definition B.4. Define $d^{\tilde{J}} f := -df \circ \tilde{J}$. Given an almost complex structure $\tilde{J}$, a function ρ is called **$\tilde{J}$-plurisubharmonic** (resp. **strict $\tilde{J}$-plurisubharmonic**) if $dd^{\tilde{J}}\rho(v, \tilde{J}v) \geq 0$ (resp. > 0) for any $v \neq 0$.

Now, we aim to show Theorem B.2. We slightly generalize a lemma in [Duv16] by allowing a small neighborhood of the almost complex structure.

Lemma B.5. *There exists a C^1-neighborhood $\mathcal{W}$ of J and a function ρ of class C^2 defined on a tubular neighborhood of L such that the ρ vanishes exactly on L and the following property holds: For any $\tilde{J} \in \mathcal{W}$, we have*

(i) $\sqrt{\rho}$ is $\tilde{J}$-plurisubharmonic outside L

(ii) ρ is strictly $\tilde{J}$-plurisubharmonic

Proof. Let $\nu_X L$ be a small tubular neighborhood of L. Fix a nonnegative C^2 function $\rho \geq 0$ defined on $\nu_X L$ which vanishes exactly on L. And, we are going to modify ρ to meet the requirements.

The question is local. We take a system of local coordinates $z^\alpha = x^\alpha + iy^\alpha$ $(1 \leq \alpha \leq n)$ near L in X so that the coordinates x^α in the subspace $\mathbb{R}^n \cong \{y^\alpha = 0\} \subset \mathbb{C}^n$ gives a local chart of L. Note that since L is compact, one can cover L be finitely many such local charts. Moreover, we may require the restriction of J on $L \cong \mathbb{R}^n$ is the standard complex structure J_0 on $\mathbb{C}^n$. Note that $J = J_0 + O(|y|)$ and $\rho = O(|y|^2)$; besides, we write

$$\rho = \sum a_{\beta\gamma}(x) y^\beta y^\gamma + O(|y|^3) =: q + O(|y|^3)$$

where $(a_{\beta\gamma}(x))$ is a symmetric strictly positive matrix which only depends on $x = (x^\alpha)$. Put $x^{\alpha+n} = y^\alpha$ for $1 \leq \alpha \leq n$; we will use the letters $i, j, \ldots$ to indicate integers from 1 to $2n$, while we will use $\alpha, \beta, \ldots$ for integers from 1 to n. We will also use the Einstein summation convention.

Now, we express an arbitrary almost complex structure $\tilde{J}$ in a small neighborhood $\mathcal{W}$ of J with respect to these local coordinates as follows:

$$\tilde{J} = \sum_{i,j=1}^{2n} \tilde{J}^i_j \frac{\partial}{\partial x^i} \otimes dx^j$$

One may similarly write $J = J^i_j \partial_{x^i} \otimes dx^j$ and $J_0 = (J_0)^i_j \partial_{x^i} \otimes dx^j$. Then, $(J_0)^{\beta+n}_\alpha = \delta^\beta_\alpha$, $(J_0)^\beta_{\alpha+n} = -\delta^\beta_\alpha$ and all other $(J_0)^j_i = 0$; given a function f on $\nu_X L$, we have

$$d^{\tilde{J}} f = -\tilde{J}^j_i \partial_{x^j} f dx^i$$

By Definition B.4, a two-form $\theta = \theta_{ij} dx^i \wedge dx^j$ with $\theta_{ij} = -\theta_{ji}$ is said to be $\tilde{J}$-positive (resp. strictly $\tilde{J}$-positive) if

$$\theta(v, \tilde{J}v) = v^i \theta_{ik} \tilde{J}^k_j v^j \geq 0 \text{ (resp. } > 0)$$

for any $v = v^i \partial_{x^i} \neq 0$, i.e. the (strict) positivity of the following symmetry matrix

$$(\theta_{ik} \tilde{J}^k_j + \theta_{jk} \tilde{J}^k_i)_{1 \leq i,j \leq 2n}$$

Step one. We first study ρ. Compute $dd^{\tilde{J}}\rho = dd^{\tilde{J}} q + O(|y|)$. Beware that this $O(|y|)$ actually depends on $\tilde{J}$ but there is a uniform constant C so that

$O(|y|) \leq C|y|$. Next, we compute

$$\begin{aligned} dd^{\tilde{J}}q &= a_{\beta\gamma}dd^{\tilde{J}}(y^\beta y^\gamma) + O(|y|) = 2a_{\beta\gamma}\tilde{J}_i^{n+\beta}dx^i \wedge dy^\gamma + O(|y|) \\ &= 2a_{\beta\gamma}(J_0)_i^{n+\beta}dx^i \wedge dy^\gamma + O(\tilde{J} - J_0) + O(|y|) \\ &= 2a_{\beta\gamma}dx^\beta \wedge dy^\gamma + O(\tilde{J} - J_0) + O(|y|) \qquad =: 2\theta + O(\tilde{J} - J_0) + O(|y|) \end{aligned}$$

where the $O(\tilde{J} - J_0)$ represents a term bounded by a multiple of the C^1-norm of $\tilde{J} - J_0$. Define a 2-form $\theta = \theta_{ij}dx^i \wedge dx^j = a_{\beta\gamma}dx^\beta \wedge dy^\gamma$, where $\theta_{\beta,\gamma+n} = -\theta_{\gamma+n,\beta} = \frac{1}{2}a_{\beta\gamma}$ and all other $\theta_{ij} = 0$. Then,

$$dd^{\tilde{J}}\rho(\cdot, \tilde{J}\cdot) = 2\theta(\cdot, J_0\cdot) + O(\tilde{J} - J_0) + O(|y|) \tag{195}$$

It can be viewed as a matrix identity, and the $\theta(\cdot, J_0\cdot)$ corresponds to the symmetric strictly positive-definite $(2n) \times (2n)$ matrix $A = (A_{ij})$ defined by setting $A_{\beta\gamma} = A_{\beta+n,\gamma+n} = a_{\beta\gamma}$ and all other $A_{ij} = 0$. Remark that the θ and A only depend on ρ. By shrinking the neighborhood $\nu_X L$, one can make the $O(|y|)$ small; by shrinking $\mathcal{W}$, we can make the $O(\tilde{J} - J_0)$ small. So, by (195), one can ensure that for any $\tilde{J} \in \mathcal{W}$, the following equation holds

$$dd^{\tilde{J}}\rho(\cdot, \tilde{J}\cdot) \geq \theta(\cdot, J_0\cdot) \tag{196}$$

Step two. We next deal with $\sqrt{\rho}$ and aim to prove the following:

$$dd^{\tilde{J}}\sqrt{\rho}(\cdot, \tilde{J}\cdot) \geq O(1) \tag{197}$$

if $\tilde{J}$ is sufficiently close to J. A subtle point is that $dd^{\tilde{J}}\sqrt{\rho}$ may be unbounded near $L \cong \{y^\alpha = 0\}$. The idea is that one can show the unbounded part corresponds to a positive semi-definite matrix. To see this, observe first that the highest unbounded terms are asymptotically $O(\frac{1}{|y|})$. Recall that $q = a_{\beta\gamma}(x)y^\beta y^\gamma$. In the computation of $dd^{\tilde{J}}\sqrt{q}$ modulo $O(1)$, there is no need to differentiate $a_{\beta\gamma}(x)$, since otherwise one cannot produce any $O(\frac{1}{|y|})$ terms. Hence, modulo $O(1)$, we may assume that $a_{\beta\gamma}$ are all constant; up to a linear transformation, we may further assume $q = |y|^2$.

Now, we have $\rho = q + O(|y|^3)$ and $\sqrt{\rho} = \sqrt{q} + O(|y|^2) = |y| + O(|y|^2)$. Then, $dd^{\tilde{J}}\sqrt{\rho} = dd^{\tilde{J}}|y| + O(1)$, and the unbounded part is $dd^{\tilde{J}}|y|$. Hence, to show (197), it suffices to show $dd^{\tilde{J}}|y|(\cdot, \tilde{J}\cdot)$ is positive semi-definite. In reality, we fix $v = v^i\partial_{x^i} \neq 0$ and compute:

$$\begin{aligned} dd^{\tilde{J}}|y|(v, \tilde{J}v) &= \left(\textstyle\sum_{\alpha,\beta} \tilde{J}_\ell^{n+\alpha} \frac{\delta_\beta^\alpha|y|^2 - y^\alpha y^\beta}{|y|^3} dx^\ell \wedge dy^\beta\right)\left(v^i\partial_{x^i}, \tilde{J}_k^j v^k \partial_{x^j}\right) \\ &= \textstyle\sum_{\alpha,\beta} \frac{\delta_\beta^\alpha|y|^2 - y^\alpha y^\beta}{|y|^3} \cdot \left((v^\ell \tilde{J}_\ell^{n+\alpha}) \cdot (v^k \tilde{J}_k^{n+\beta}) + v^{n+\alpha}v^{n+\beta}\right) \end{aligned}$$

where the last equality holds follows from the identity $\tilde{J}_\ell^m \tilde{J}_k^\ell = -\delta_k^m$. By setting $u^\alpha = \sum_\ell v^\ell \tilde{J}_\ell^{n+\alpha}$ or $u^\alpha = v^{n+\alpha}$, it reduces to the obvious positive semi-definiteness of the quadratic form $\sum_{\alpha,\beta}(\delta_\beta^\alpha|y|^2 - y^\alpha y^\beta) \cdot u^\alpha u^\beta = n\sum_\alpha (u^\alpha y^\alpha)^2 - (\sum_\alpha u^\alpha y^\alpha)^2$. Now our claim (197) is now established.

Step three. Finally, we replace ρ by $\rho_1 = (\sqrt{\rho} + B\rho)^2 = \rho + 2B\rho^{\frac{3}{2}} + B^2\rho^2$ for a sufficiently large constant $B > 0$. Since $\rho = O(|y|^2)$ and $\rho^{\frac{3}{2}} = O(|y^3|)$, the ρ_1 is C^2 near L.

(i) Observe that $\sqrt{\rho_1} = \sqrt{\rho} + B\rho$, and so $dd^{\tilde{J}}\sqrt{\rho_1}(\cdot, \tilde{J}\cdot) = dd^{\tilde{J}}\sqrt{\rho}(\cdot, \tilde{J}\cdot) + Bdd^{\tilde{J}}\rho(\cdot, \tilde{J}\cdot)$ is positive-definite by (196) and (197) for a sufficiently large $B > 0$. Thus, the $\sqrt{\rho_1}$ is $\tilde{J}$-plurisubharmonic.

(ii) We compute $dd^{\tilde{J}}\rho_1 = (\frac{3B}{2\sqrt{\rho}} + 2B^2) d\rho \wedge d^{\tilde{J}}\rho + (1 + 3B\sqrt{\rho} + 2B^2\rho) dd^{\tilde{J}}\rho$. For any $v \neq 0$, we have $d\rho \wedge d^{\tilde{J}}\rho(v, \tilde{J}v) = (d\rho(v))^2 + (d\rho(\tilde{J}v))^2 \geq 0$, so $d\rho \wedge d^{\tilde{J}}\rho(\cdot, \tilde{J}\cdot)$ is positive semi-definite. By (196), we also show that $dd^{\tilde{J}}\rho_1(\cdot, \tilde{J}\cdot) > 0$. Thus, the ρ_1 is strictly $\tilde{J}$-plurisubharmonic. □

Proof of Theorem B.2. Let ρ and $\mathcal{W}$ be as in Lemma B.5, and let $\nu_X L$ be the tubular neighborhood of L therein. For any function f we denote by $h^f_{\tilde{J}}$ the symmetric tensor defined as follows:

$$h^f_{\tilde{J}}(v, w) = \tfrac{1}{2}\big(dd^{\tilde{J}} f(v, \tilde{J}w) + dd^{\tilde{J}} f(w, \tilde{J}v)\big)$$

Then, the Lemma B.5 actually tells that $h^{\sqrt{\rho}}_{\tilde{J}}$ is a semi-metric and $h^{\rho}_{\tilde{J}}$ is a metric on their domains for any $\tilde{J} \in \mathcal{W}$. Shrinking $\nu_X L$ if necessary, we may assume $\nu_X L = \{p \in X \mid \operatorname{dist}(p, L) \leq d_0\}$ for some small $d_0 > 0$. Notice that $\rho = O(|y|^2)$ and $|y|$ is comparable to $\operatorname{dist}(p, L)$, thus, there is $c_2 > 0$ so that

$$\tfrac{1}{c_2} \operatorname{dist}(p, L) \leq \sqrt{\rho}(p) \leq c_2 \operatorname{dist}(p, L) \tag{198}$$

for any $p \in \nu_X L$. In particular, there exists a constant $r_0 > 0$ which is independent of $\tilde{J} \in \mathcal{W}$ such that

$$\{p \mid \rho(p) \leq r_0^2\} \subset \nu_X L \tag{199}$$

We choose local coordinates (x^α, y^α) as before. Given $p \in \nu_X L$, the distance $\operatorname{dist}(p, L)$ is comparable to $|y|$. Take a cube Q centered at p so that ∂Q touches $L = \{y^\alpha = 0\}$. Applying the gradient estimate [GT15, Eq.(3.15)] to ρ on Q deduces that $|\nabla\rho| \leq \frac{n}{|y|}\sup_{\partial Q}|\rho| + \frac{|y|}{2}\sup_Q |\Delta\rho| \lesssim (\frac{1}{|y|}|2y|^2 + \frac{|y|}{2}) \lesssim |y|$. Particularly, there is a constant $c_3 > 0$ so that

$$|\nabla\rho| \leq c_3 \operatorname{dist}(p, L) \tag{200}$$

Suppose we have a $\tilde{J}$-holomorphic disk $u : (\mathbb{D}, \partial\mathbb{D}) \to (X, L)$. Denote by j the complex structure on the unit disk $\mathbb{D} \subset \mathbb{C}$. The equation $\tilde{J} \circ du = du \circ j$ implies that for any function f, we have $u^* d^{\tilde{J}} f = d^j u^* f$. Thus, $u^* dd^{\tilde{J}} f = dd^j u^* f$. In our case, ρ is $\tilde{J}$-plurisubharmonic tells that $u^*\rho$ is j-plurisubharmonic; then, $u^* h^\rho_{\tilde{J}}$ is a semi-metric, and we have $u^* dd^{\tilde{J}}\rho = d\operatorname{vol}_{u^* h^\rho_{\tilde{J}}}$. Consider the following function

$$a(r) = \frac{1}{r}\int_{\{u^*\rho \leq r^2\}} u^* dd^{\tilde{J}}\rho = \frac{1}{r}\int_{\{u^*\rho \leq r^2\}} d\operatorname{vol}_{u^* h^\rho_{\tilde{J}}}$$

It is at least well-defined on $[0, r_0]$ thanks to (199). Since $d^{\tilde{J}}\rho = 2\sqrt{\rho} \cdot d^{\tilde{J}}\sqrt{\rho}$, the Stokes formula implies $a(r) = 2\int_{\{u^*\rho = r^2\}} u^* d^{\tilde{J}}\sqrt{\rho}$ and also

$$a(r') - a(r) = 2\int_{\{r^2 \le u^*\rho \le r'^2\}} du^* d^{\tilde{J}}\sqrt{\rho} = 2\int_{\{r^2 \le u^*\rho \le r'^2\}} d\,\mathrm{vol}_{u^* h_{\tilde{J}}^{\sqrt{\rho}}}$$

for $r' \ge r$. By condition, the $h_{\tilde{J}}^{\sqrt{\rho}}$ is a semi-metric away $L \cong \{\rho = 0\}$, and thus the $a(r)$ is increasing. Recall that the constant r_0 is independent of $\tilde{J}$. Now, we have the energy estimate

$$E(u) \gtrsim r_0\ a(r_0) \ge r_0 \lim_{r\to 0} a(r) \gtrsim \lim_{r\to 0} a(r)$$

Due to (198) and (200), we get $|\nabla\rho| \lesssim \mathrm{dist}(\cdot, L) \lesssim \sqrt{\rho} \lesssim r$. So, the coarea formula tells $\int_0^{r^2} \ell(\{u^*\rho = t\})dt = \int_{\{u^*\rho \le r^2\}} |\nabla\rho| \cdot d\,\mathrm{vol}_{u^* h_{\tilde{J}}^{\rho}} \lesssim r \cdot r\ a(r) = r^2 a(r)$ and so

$$E(u) \gtrsim \lim_{r\to 0} a(r) \gtrsim \lim_{r\to 0} \tfrac{1}{r^2} \int_0^{r^2} \ell(\rho = t)dt = \ell(\rho = 0) = \ell(\partial u)$$

□

C Integral affine structures

We state two equivalent definitions of an **integral affine structure** on an n-dimensional manifold Y (see [KS06]): (i) there is an atlas of charts such that the transition functions belong to $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$; (ii) there is a torsion-free flat connection ∇ on TY and a ∇-covariant lattice of maximal rank $TY^{\mathbb{Z}}$.

Given an integral affine structure on Y, a local chart $\phi : U \to \mathbb{R}^n$ on a small open subset $U \subset Y$ is called an **integral affine chart** if the torsion-free flat connection is given by $\nabla = d$ in $TY|_U$. For the coordinates $x_i = x_i \circ \phi$, the lattice $TY_x^{\mathbb{Z}}$ for each $x \in U$ is the free abelian group generated by $\partial/\partial x_i$'s. The transition maps among them belong to $GL(n,\mathbb{Z}) \ltimes \mathbb{R}^n$. If we have two systems of local coordinates (x_i) and (x_i'), then the transition map is of the form

$$x_j' = \textstyle\sum_k a_{jk} x_k + f_j \tag{201}$$

for some matrix $A = (a_{jk}) \in GL(n,\mathbb{Z})$ and $f_j \in \mathbb{R}$.

Proposition C.1. *Let U, U' be two integral affine charts. A subset $\Delta \subset U \cap U'$ is a rational polyhedron in $U \subset \mathbb{R}^n$ if and only if it is a one in $U' \subset \mathbb{R}^n$. Such Δ is also called a* ***rational polyhedron*** *in Y.*

Proof. Assume Δ is defined by $\sum_j b_{ij} x_j' \ge c_i$ in U' for some $b_{ij} \in \mathbb{Z}$ and $c_j \in \mathbb{R}$, and the transition map is given by (201). Then using the coordinated in U, Δ is given by $\sum_k (\sum_j b_{ij} a_{jk}) x_k \ge c_i - \sum_j b_{ij} f_j$. □

Definition C.2. A **rational polyhedral complex** $\mathscr{Q}$ in an integral affine manifold Y is a CW complex so that (i) the underlying space of each cell $\Delta \in \mathscr{Q}$ is a rational polyhedron in Y; (ii) each face of $\Delta \in \mathscr{Q}$ is in $\mathscr{Q}$; (iii) The intersection $\Delta \cap \Delta' \in \mathscr{Q}$ is a face of both $\Delta, \Delta' \in \mathscr{Q}$.

Lemma C.3. *Fix $\epsilon > 0$ and a metric in Y. Suppose $K \Subset K' \subset Y$ are two compact domains in Y. Then, there exists a rational polyhedral complex $\mathscr{P}$ in B_0, and all the cells in $\mathscr{P}$ have diameters less than ϵ, and the underlying topological space satisfies $K \Subset |\mathscr{P}| \Subset K'$.*

Proof. In an integral affine chart at some point $p \in K$, there always exists a rational hypersurface H (codimension-one rational polyhedron) passing through p. By extension, we may require H is maximal with respect to the inclusions among the compact domains in Y. In special, the H must 'escape' K in the sense that $H \cap K$ is closed in K. Now, we cover K' by finitely many open sets $V_1, \dots, V_m$ so that for every $1 \le i \le m$, the closure $\bar{V}_i$ is compact and is contained in some integral affine chart U_i. We may find a sufficiently dense collection of rational hyperplanes in $U_i \cong \phi(U_i) \subset \mathbb{R}^n$ so that every chamber enclosed by them has diameter less that ϵ. Extending all these hyperplanes to be maximal in the above sense, we get a collection $\mathscr{H}$ of closed rational hyperplanes in K'. By Proposition C.1, the $\mathscr{H}$ divides K into rational polyhedrons each of which has diameter less than ϵ. ☐

D ∞-category

Proposition D.1. *$\mathscr{U}\mathscr{D}(L, X)$ is a bicategory.*

Sketch of proof. The objects and 1-morphisms are just as before (Definition 2.33). Given $\mathfrak{f}_0, \mathfrak{f}_1 \in \mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}', \mathfrak{m})$, we define a *2-morphism* to be a morphism $\mathfrak{F}$ as in Definition 2.37 which offers a ud-homotopy between $\mathfrak{f}_i$. Denote it by $\mathfrak{F} : \mathfrak{f}_0 \Longrightarrow \mathfrak{f}_1$. Then, the argument proving above Lemma 2.41 can actually be used to define the so-called *horizontal composition functor*. Namely, for each triple $(\mathfrak{m}'', \mathfrak{m}', \mathfrak{m})$ of objects, we have a bifunctor $c(\mathfrak{m}'', \mathfrak{m}', \mathfrak{m}) : \mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}'', \mathfrak{m}') \times \mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}', \mathfrak{m}) \to \mathrm{Hom}_{\mathscr{U}\mathscr{D}}(\mathfrak{m}'', \mathfrak{m})$ given by sending $\mathfrak{F} : \mathfrak{f}_0 \Longrightarrow \mathfrak{f}_1$ and $\mathfrak{G} : \mathfrak{g}_0 \Longrightarrow \mathfrak{g}_1$ to some 2-morphism $\mathfrak{G} \circ_h \mathfrak{F} : \mathfrak{g}_0 \circ \mathfrak{f}_0 \Longrightarrow \mathfrak{g}_1 \circ \mathfrak{f}_1$. It is tedious but routine to check axioms for the definition of bicategory. ☐

Actually, we do not need the bicategory structure in this paper, but we keep it for independent interest. By definition, every 2-morphism is invertible. The collection of 2-morphisms exactly correspond to the congruence relation of ud-homotopies on the 1-morphism spaces.

Conjecture D.2. *$\mathscr{U}\mathscr{D}(L, X)$ can be realized as an $(\infty, 1)$-category.*

Since higher categories usually have many different definitions, this is however not a complete statement. There seems currently no general form of definitions which could work for all higher categories. An (∞, r)-category was something we want to define so that all k-morphisms for $k > r$ are 'reversible'. For example, when $r = 1$ there are many different ways to make the idea of an $(\infty, 1)$-category precise, like quasi-categories, Segal categories, and so on. The reason why $(\infty, 1)$-category attracts lots of attentions is that it is like a *homotopy theory*: a kind of category with objects, morphisms, homotopies between morphisms, higher homotopies between homotopies and so on [AC16].

Definition D.3 (Definition 1.1.2.4. [Lur09])**.** An $(\infty, 1)$**-category** is a simplicial set K which has the following property: for any $0 < i < n$, any map $f_0 : \Lambda_i^n \to K$ admits an extension $f : \Delta^n \to K$. Here $\Lambda_i^n \subset \Delta^n$ denotes the i-th horn of the standard n-simplex Δ^n, obtained by deleting the interior and the face opposite the i-th vertex.

Proposition D.4. *Fix L and X as before, there is an $(\infty, 1)$-category so that its objects are those A_∞ algebras $(\Omega^*(L), \check{\mathfrak{m}}^{J,L})$ in Theorem 6.2.*

Sketch of proof. We use K to represent the simplicial set we are going to construct, and denote by K_n for $n \in \mathbb{N}$ the set of n-simplices. Then K_0 is already presented in the statement, and K_1 is defined to be the set of pseudo-isotopies in Theorem 6.3 coming from a path of almost complex structures. Inductively, we define K_n to be the set of Δ^n-pseudo-isotopies obtained from a Δ^n-family of ω-tame almost complex structures using Theorem 6.1. The extendability in Definition D.3 is just a result of the fact that ω-tame almost complex structures J (with perturbation data) are contractible choices. $\square$

References

[AAK16] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov. Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces. *Publications mathématiques de l'IHÉS*, 123(1):199–282, 2016.

[ABC+09] Paul S. Aspinwall, Tom Bridgeland, Alastair Craw, Michael R. Douglas, Anton Kapustin, Gregory W. Moore, Mark Gross, Graeme Segal, Balázs Szendrői, and P. M. H. Wilson. *Dirichlet branes and mirror symmetry.* AMS, Providence, RI, 2009.

[Abo14] Mohammed Abouzaid. Family Floer cohomology and mirror symmetry. *arXiv preprint arXiv:1404.2659*, 2014.

[Abo17a] Mohammed Abouzaid. Homological mirror symmetry without corrections. *arXiv preprint arXiv:1703.07898*, 2017.

[Abo17b] Mohammed Abouzaid. The family Floer functor is faithful. *Journal of the European Mathematical Society*, 19(7):2139–2217, 2017.

[AC16] Omar Antolín Camarena. A whirlwind tour of the world of $(\infty, 1)$-categories. In *Mexican mathematicians abroad: recent* *contributions*, volume 657 of *Contemp. Math.*, pages 15–61. Amer. Math. Soc., Providence, RI, 2016.

[Arn13] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*, volume 60. Springer Science & Business Media, 2013.

[Aur07] Denis Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. *Journal of Gökova Geometry Topology*, 1:51–91, 2007.

[BFJ15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Solution to a non-Archimedean Monge-Ampère equation. *Journal of the American Mathematical Society*, 28(3):617–667, 2015.

[BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-Archimedean analysis, volume 261 of Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1984.

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*, volume 2105. Springer, 2014.

[BT13] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology*, volume 82. Springer Science & Business Media, 2013.

[CBM09] Ricardo Castaño-Bernard and Diego Matessi. Lagrangian 3-torus fibrations. *Journal of Differential Geometry*, 81(3):483–573, 2009.

[CBMS10] Ricardo Castaño-Bernard, Diego Matessi, and Jake P Solomon. Symmetries of Lagrangian fibrations. *Advances in mathematics*, 225(3):1341–1386, 2010.

[CLD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. *arXiv preprint arXiv:1204.6277*, 2012.

[CLL12] Kwokwai Chan, Siu-Cheong Lau, and Naichung Conan Leung. SYZ mirror symmetry for toric Calabi-Yau manifolds. *Journal of Differential Geometry*, 90(2):177–250, 2012.

[CO06] Cheol-Hyun Cho and Yong-Geun Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian Journal of Mathematics*, 10(4):773–814, 2006.

[CS10] Yuri Chekanov and Felix Schlenk. Notes on monotone Lagrangian twist tori. *arXiv preprint arXiv:1003.5960*, 2010.

[Duv16] Julien Duval. On a result by Y. Groman and JP Solomon. *Mathematische Annalen*, 364(3-4):1361–1363, 2016.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-archimedean amoebas and tropical varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2006(601):139–157, 2006.

[FO97] Kenji Fukaya and Yong-Geun Oh. Zero-loop open strings in the cotangent bundle and morse homotopy. *Asian Journal of Mathematics*, 1(1):96–180, 1997.

[FOOO10a] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian Floer theory on compact toric manifolds, I. *Duke Mathematical Journal*, 151(1):23–175, 2010.

[FOOO10b] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction, Part I*, volume 1. American Mathematical Soc., 2010.

[FOOO10c] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction, Part II*, volume 2. American Mathematical Soc., 2010.

[FOOO15] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 1. *arXiv preprint arXiv:1503.07631*, 2015.

[FOOO16] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian Floer Theory and Mirror Symmetry on Compact Toric Manifolds*. Société Mathématique de France, 2016.

[FOOO17a] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru On. Construction of Kuranishi structures on the moduli spaces of pseudo holomorphic disks: I. *arXiv preprint arXiv:1710.01459*, 2017.

[FOOO17b] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Kuranishi structure, Pseudo-holomorphic curve, and Virtual fundamental chain: Part 2. *arXiv preprint arXiv:1704.01848*, 2017.

[FOOO18] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru On. Construction of Kuranishi structures on the moduli spaces of pseudo holomorphic disks: II. *arXiv preprint arXiv:1808.06106*, 2018.

[Fra08] Urs Frauenfelder. Gromov convergence of pseudoholomorphic disks. *Journal of Fixed Point Theory and Applications*, 3(2):215–271, 2008.

[Fuk01] Kenji Fukaya. Floer homology for families-a progress report. *CONTEMPORARY MATHEMATICS*, 309:33–68, 2001.

[Fuk09] Kenji Fukaya. Lagrangian surgery and rigid analytic family of Floer homologies. *https://www.math.kyoto-u.ac.jp/ fukaya/Berkeley.pdf*, 2009.

[Fuk10] Kenji Fukaya. Cyclic symmetry and adic convergence in Lagrangian Floer theory. *Kyoto Journal of Mathematics*, 50(3):521–590, 2010.

[Fuk11] Kenji Fukaya. Counting pseudo-holomorphic discs in Calabi-Yau 3-folds. *Tohoku Mathematical Journal, Second Series*, 63(4):697–727, 2011.

[Fuk17] Kenji Fukaya. Unobstructed immersed Lagrangian correspondence and filtered A_∞ functor. *arXiv preprint arXiv:1706.02131*, 2017.

[GHK15] Mark Gross, Paul Hacking, and Sean Keel. Mirror symmetry for log Calabi-Yau surfaces I. *Publications Mathématiques de l'IHES*, 122(1):65–168, 2015.

[Gro01] Mark Gross. Examples of special Lagrangian fibrations. In *Symplectic geometry and mirror symmetry*, pages 81–109. World Scientific, 2001.

[GS11] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Annals of mathematics*, pages 1301–1428, 2011.

[GS14] Yoel Groman and Jake P Solomon. A reverse isoperimetric inequality for J-holomorphic curves. *Geometric and Functional Analysis*, 24(5):1448–1515, 2014.

[GT15] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. springer, 2015.

[Gub07] Walter Gubler. Tropical varieties for non-archimedean analytic spaces. *Inventiones mathematicae*, 169(2):321–376, 2007.

[Gub16] Walter Gubler. Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros). In *Nonarchimedean and tropical geometry*, pages 1–30. Springer, 2016.

[HTK+03] Kentaro Hori, Richard Thomas, Sheldon Katz, Cumrun Vafa, Rahul Pandharipande, Albrecht Klemm, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1. American Mathematical Soc., 2003.

[Joy03] Dominic Joyce. Singularities of special Lagrangian fibrations and the SYZ Conjecture. *Communications in Analysis and Geometry*, 11(5):859–907, 2003.

[Joy07] Dominic D Joyce. *Riemannian holonomy groups and calibrated geometry*, volume 12. Oxford University Press, 2007.

[Kon95] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the international congress of mathematicians*, pages 120–139. Springer, 1995.

[KS01] Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic Geometry And Mirror Symmetry*, pages 203–263. World Scientific, 2001.

[KS06] Maxim Kontsevich and Yan Soibelman. Affine structures and non-archimedean analytic spaces. In *The unity of mathematics*, pages 321–385. Springer, 2006.

[KT02] Maxim Kontsevich and Yuri Tschinkel. Non-archimedean Kähler geometry. 2002.

[Li19] Yang Li. SYZ conjecture for Calabi-Yau hypersurfaces in the Fermat family. *arXiv preprint arXiv:1912.02360*, 2019.

[Li20] Yang Li. Metric SYZ conjecture and non-archimedean geometry. *arXiv preprint arXiv:2007.01384*, 2020.

[Liu11] Yifeng Liu. A non-Archimedean analogue of the Calabi-Yau theorem for totally degenerate abelian varieties. *Journal of Differential Geometry*, 89(1):87–110, 2011.

[LM16] H Blaine Lawson and Marie-Louise Michelsohn. *Spin geometry (pms-38)*, volume 38. Princeton university press, 2016.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mar06] Martin Markl. Transferring A_∞ (strongly homotopy associative) structures. *Proceedings of the 25th Winter School" Geometry and Physics"*, pages 139–151, 2006.

[MS12] Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and symplectic topology*, volume 52. American Mathematical Soc., 2012.

[Pal60] Richard S Palais. Local triviality of the restriction map for embeddings. *Commentarii Mathematici Helvetici*, 34(1):305–312, 1960.

[Par16] John Pardon. An algebraic approach to virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves. *Geometry & Topology*, 20(2):779–1034, 2016.

[PT20] James Pascaleff and Dmitry Tonkonog. The wall-crossing formula and Lagrangian mutations. *Advances in Mathematics*, 361:106850, 2020.

[Sei08] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*, volume 10. European Mathematical Society, 2008.

[Sol20] Jake P Solomon. Involutions, obstructions and mirror symmetry. *Advances in Mathematics*, 367:107107, 2020.

[ST16] Jake P Solomon and Sara B Tukachinsky. Differential forms, Fukaya A_∞ algebras, and Gromov-Witten axioms. *arXiv preprint arXiv:1608.01304*, 2016.

[SYZ96] A Strominger, S-T Yau, and E Zaslow. Mirror symmetry is T-duality. *Nuclear Physics. B*, 479(1-2):243–259, 1996.

[Tat71] John Tate. Rigid analytic spaces. *Inventiones mathematicae*, 12(4):257–289, 1971.

[Tem15] Michael Temkin. Introduction to Berkovich analytic spaces. In *Berkovich spaces and applications*, pages 3–66. Springer, 2015.

[Tu14] Junwu Tu. On the reconstruction problem in mirror symmetry. *Advances in Mathematics*, 256:449–478, 2014.

[War13] Frank W Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94. Springer Science & Business Media, 2013.

[Yu16] Tony Yue Yu. Enumeration of holomorphic cylinders in log Calabi–Yau surfaces. I. *Mathematische Annalen*, 366(3):1649–1675, 2016.

[Yu18] Tony Yue Yu. Gromov compactness in non-archimedean analytic geometry. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018(741):179–210, 2018.

[Yu21] Tony Yue Yu. Enumeration of holomorphic cylinders in log Calabi–Yau surfaces, II: Positivity, integrality and the gluing formula. *Geometry & Topology*, 25(1):1–46, 2021.

[Yua21] Hang Yuan. Disk counting and wall-crossing phenomenon via family Floer theory. *arXiv preprint arXiv:2101.01379*, 2021.