A Twisted Complex-Brunn Minkowski Theorem with Applications

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Abstract of the Dissertation

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In his *Annals of Mathematics* paper [Ber09b], Berndtsson proves an important result on the Nakano positivity of holomorphic infinite-rank vector bundles whose fibers are Hilbert spaces consisting of holomorphic L^2-functions with respect to a family of weight functions $\left\{e^{-\varphi(t,\cdot)}\right\}_{t\in U}$, varying in $t \in U \subset \mathbb{C}^m$, over a pseudoconvex domain. Using a variant of Hörmander's theorem due to Donnelly and Fefferman, we show that Berndtsson's Nakano positivity result holds under different (in fact, more general) curvature assumptions. This is of particular interest when the manifold admits a negative non-constant plurisubharmonic function, as these curvature assumptions then allow for some curvature negativity. We describe this setting as a "twisted" setting. Furthermore, we extend our main result, and thus Berndtsson's Nakano positivity result, to trivial families of possibly unbounded Stein manifolds. As immediate applications of this result, we prove log-plurisubharmonic variation theorems for Bergman kernels, as well as families of compactly supported measures over trivial families of Stein manifolds. We then generalize these log-plurisubharmonic variation results to a certain class of non-trivial families of Stein manifolds. Finally, we present two complex Prékopa-Leindler type theorems.

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Chapter 1

Introduction

1.1 Description of the main results

In this thesis, we mainly study the curvature properties of the natural L^2-metrics of holomorphic Hilbert bundles associated to trivial families of Stein manifolds. A concrete formulation is as follows. Let X be an n-dimensional relatively compact complete Kähler submanifold of a Stein Kähler manifold (Y, g). Let U be a domain in $\mathbb{C}^m$ containing the origin, and let $V \to \overline{X}$ be a holomorphic vector bundle over the closure $\overline{X}$ of X. Let $\left\{h^{[t]}\right\}_{t\in U}$ be a family of smooth Hermitian metrics for $V \to \overline{X}$. By the latter, we mean taking a metric h for the pullback bundle $\pi_{\overline{X}}^* V \to U \times \overline{X}$, where $\pi_{\overline{X}} : U \times \overline{X} \to \overline{X}$ is the canonical projection, and letting $h^{[t]} := i_t^* h$ where i_t denotes the inclusion map $\overline{X} \hookrightarrow \{t\} \times \overline{X}$. We can consider the space $L^2\left(X, h^{[t]}\right)$ of sections f of $V \to X$ whose norm $|f|_{h^{[t]}}$, with respect to the metric $h^{[t]}$, is square-integrable on X with respect to the volume form dV_g induced by the Kähler metric g. For each $t \in U$, we can then consider the space $\mathcal{H}^2\left(X, h^{[t]}\right)$ of holomorphic sections of $V \to X$ in $L^2\left(X, h^{[t]}\right)$, which is a closed subspace of $L^2\left(X, h^{[t]}\right)$ by our smoothness and boundedness assumptions, and therefore a Hilbert space. The smoothness and boundedness assumptions imply further that the Hilbert spaces in the collection $\left\{\mathcal{H}^2\left(X, h^{[t]}\right)\right\}_{t\in U}$ have equivalent norms. Indeed, for any section f of $V \to X$, for any point $x \in X$, and for any

$s, t \in U$,

$$|f(x)|^2_{h^{[t]}} = \sup_{\sigma \in V_x^* - \{0\}} \frac{|\langle \sigma, f(x) \rangle|^2}{|\sigma|^2_{h^{[t],*}}} = \left(\sup_{\sigma \in V_x^* - \{0\}} \frac{|\sigma|^2_{h^{[s],*}}}{|\sigma|^2_{h^{[t],*}}} \right) |f(x)|^2_{h^{[s]}},$$

where $h^{[t],*}$ and $h^{[s],*}$ are the dual metrics for the dual bundle $V^* \to X$ induced by $h^{[t]}$ and $h^{[s]}$ respectively. Therefore, for any section f of $V \to X$,

$$C_{t,s}^{-1} \int_X |f(x)|^2_{h^{[s]}} \, dV_g(x) \le \int_X |f(x)|^2_{h^{[t]}} \, dV_g(x) \le C_{s,t} \int_X |f(x)|^2_{h^{[s]}} \, dV_g(x),$$

where

$$C_{s,t} := \sup_{\substack{\sigma \in V_x^* - \{0\} \\ x \in X}} \frac{|\sigma|^2_{h^{[s],*}}}{|\sigma|^2_{h^{[t],*}}},$$

and $C_{t,s}$ is defined similarly. In particular, it follows that for any $s, t \in U$, $f \in \mathcal{H}^2\left(X, h^{[t]}\right)$ if and only if $f \in \mathcal{H}^2\left(X, h^{[s]}\right)$. Thus, the underlying vector spaces of the Bergman spaces $\mathcal{H}^2\left(X, h^{[t]}\right)$ are equal as subspaces of the space $\Gamma_{\mathcal{O}}(X, V)$. By fixing $\mathcal{H}^2_0 := \mathcal{H}^2\left(X, h^{[0]}\right)$, we can define the bundle E_h of infinite rank over U with total space $U \times \mathcal{H}^2_0$, whose fiber over $t \in U$ is $\{t\} \times \mathcal{H}^2_0 \cong \mathcal{H}^2_t =: \mathcal{H}^2\left(X, h^{[t]}\right)$. It is a trivial Hilbert bundle equipped with the non-trivial Hermitian metric $(\cdot, \cdot)_{h^{[t]}}$, varying in t, induced by the L^2-norm on $\mathcal{H}^2_t$.

Before stating our main theorem, we need to define what we call a *twisted* curvature operator. Given a smooth Hermitian metric h for $V \to \overline{X}$, let $\Theta_\delta(h)$ be locally defined as

$$\begin{aligned} &\sum_{1 \le j,k \le m} \frac{\partial}{\partial \bar{t}_k} \left(h^{-1} \frac{\partial h}{\partial t_j} \right) d\bar{t}_k \wedge dt_j + \sum_{\substack{1 \le j \le m \\ 1 \le \mu \le n}} \frac{\partial}{\partial \bar{z}_\mu} \left(h^{-1} \frac{\partial h}{\partial t_j} \right) d\bar{z}_\mu \wedge dt_j \\ &\quad + \sum_{\substack{1 \le k \le m \\ 1 \le \nu \le n}} \frac{\partial}{\partial \bar{t}_k} \left(h^{-1} \frac{\partial h}{\partial z_\nu} \right) d\bar{t}_k \wedge dz_\nu + \frac{\delta}{1+\delta} \sum_{1 \le \nu, \mu \le n} \frac{\partial}{\partial \bar{z}_\mu} \left(h^{-1} \frac{\partial h}{\partial z_\nu} \right) d\bar{z}_\mu \wedge dz_\nu, \end{aligned}$$

where $\delta > 0$. We can also express $\Theta_\delta(h)$ as

$$\Theta_\delta(h) = \Theta(h) - \frac{1}{1+\delta} \pi_X^* \Theta\left(h^{[t]}\right).$$

As a block matrix split with respect to the product structure $U \times X$ for our holomorphic trivial family, $\Theta_\delta(h)$ has the form

$$\Theta_\delta(h) = \begin{pmatrix} \bar{\partial}_U \left(h^{-1} \partial_U h\right) & \bar{\partial}_X \left(h^{-1} \partial_U h\right) \\ \bar{\partial}_U \left(h^{-1} \partial_X h\right) & \frac{\delta}{1+\delta} \bar{\partial}_X \left(h^{-1} \partial_X h\right) \end{pmatrix}.$$

Let η be a smooth function on Y and define the *twisted* curvature operator $\Xi_{\delta,\eta}$ by

$$\Xi_{\delta,\eta}(h) := \Theta_\delta(h) + \frac{\delta}{1+\delta}\pi_X^*\left(\left(\mathbf{Ric}(g) + 2\partial\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes \mathrm{Id}_V\right).$$

The operator $\Xi_{\delta,\eta}$ can also be represented as the block matrix

$$\Xi_{\delta,\eta}(h) = \begin{pmatrix} \bar{\partial}_U\left(h^{-1}\partial_U h\right) & \bar{\partial}_X\left(h^{-1}\partial_U h\right) \\ \bar{\partial}_U\left(h^{-1}\partial_X h\right) & \frac{\delta}{1+\delta}\left(\bar{\partial}_X\left(h^{-1}\partial_X h\right) + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\end{pmatrix}.$$

Noting that

$$2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta = \frac{4e^{\frac{1+\delta}{2}\eta}}{1+\delta}\partial_X\bar{\partial}_X\left(-e^{-\frac{1+\delta}{2}\eta}\right),$$

we may also rewrite $\Xi_{\delta,\eta}(h)$ as

$$\Xi_{\delta,\eta}(h) = \begin{pmatrix} \bar{\partial}_U\left(h^{-1}\partial_U h\right) & \bar{\partial}_X\left(h^{-1}\partial_U h\right) \\ \bar{\partial}_U\left(h^{-1}\partial_X h\right) & \frac{\delta}{1+\delta}\left(\bar{\partial}_X\left(h^{-1}\partial_X h\right) + \mathbf{Ric}(g) + \frac{4e^{\frac{1+\delta}{2}\eta}}{1+\delta}\partial_X\bar{\partial}_X\left(-e^{-\frac{1+\delta}{2}\eta}\right)\right)\end{pmatrix}.$$

Note that $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ if and only if $\Xi_{\delta,\eta}(h) \geq 0$ as a block matrix.

Our initial result is stated as follows.

Theorem A. *Let X be an n-dimensional relatively compact complete Kähler submanifold of an ambient Stein Kähler manifold (Y, g). Let $V \to \overline{X}$ be a holomorphic vector bundle. Let $U \subset \mathbb{C}^m$ be a domain, and let $\left\{h^{[t]}\right\}_{t\in U}$ be a family of smooth Hermitian metrics for $V \to \overline{X}$. Let $\delta > 0$ and let η be a smooth function on Y. If $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and*

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes\mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the holomorphic Hermitian bundle $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano positive. Moreover, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes\mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

then $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano semipositive.

When $V \to \overline{X}$ is a line bundle $L \to \overline{X}$ equipped with a smooth Hermitian metric $h = e^{-\varphi}$, $\Theta_\delta\left(e^{-\varphi}\right)$ can be represented as

$$\Theta_{\delta,\eta}\left(e^{-\varphi}\right) = \begin{pmatrix} \partial_U\bar{\partial}_U\varphi & \partial_U\bar{\partial}_X\varphi \\ \partial_X\bar{\partial}_U\varphi & \dfrac{\delta}{1+\delta}\partial_X\bar{\partial}_X\varphi \end{pmatrix}$$

and the twisted curvature operator $\Xi_{\delta,\eta}\left(e^{-\varphi}\right)$ can be represented as

$$\Xi_{\delta,\eta}(h) = \begin{pmatrix} \bar{\partial}_U\left(h^{-1}\partial_U h\right) & \bar{\partial}_X\left(h^{-1}\partial_U h\right) \\ \bar{\partial}_U\left(h^{-1}\partial_X h\right) & \dfrac{\delta}{1+\delta}\left(\bar{\partial}_X\left(h^{-1}\partial_X h\right) + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right) \end{pmatrix}.$$

In this case, Griffiths (semi)positivity and Nakano (semi)positivity are equivalent. In particular, the positivity of $\Xi_{\delta,\eta}\left(e^{-\varphi}\right)$ implies the positivity of

$$\begin{aligned} &\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta \\ &= \partial_X\bar{\partial}_X\varphi^{[t]} + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta, \end{aligned}$$

for each $t \in U$, by Schur complement theory, in which case the second curvature condition becomes redundant.

Theorem A can be seen as a "twisted" variant of Berndtsson's celebrated Nakano positivity theorem [Ber09b, Theorem 1.1]. If $X = \Omega$ is a bounded pseudoconvex domain in $Y = \mathbb{C}^n$, $V \to \overline{X}$ is a trivial bundle $L \to \overline{X}$ of rank 1, and $h = e^{-\varphi}$ is a metric for the pullback bundle $\pi^*_{\overline{X}}L \to U \times \overline{X}$, then we are in the context of Berndtsson's Nakano positivity theorem. In this situation, the family of metrics is a family weight functions $\left\{e^{-\varphi(t,\cdot)}\right\}_{t\in U}$ which allows us to define the family of L^2-spaces $\left\{L^2\left(\Omega,\varphi(t,\cdot)\right)\right\}_{t\in U}$ of measurable functions that are square-integrable with respect to the measure $e^{-\varphi(t,\cdot)}dV$ with dV denoting the Lebesgue measure on Ω. Let $\mathcal{H}^2\left(\Omega,\varphi(t,\cdot)\right)$ denote the space of functions in $L^2\left(\Omega,\varphi(t,\cdot)\right)$ that are holomorphic. Once again, by fixing $\mathcal{H}^2_0 := \mathcal{H}^2\left(\Omega,\varphi(0,\cdot)\right)$, we can define the trivial vector bundle E of infinite rank over U, with total space $U \times \mathcal{H}^2_0$ whose fiber at $t \in U$ is $\{t\} \times \mathcal{H}^2_0 \cong \mathcal{H}^2_t := \mathcal{H}^2\left(\Omega,\varphi(t,\cdot)\right)$.

Theorem 1.1.1. ([Ber09b, Theorem 1.1]) *If Ω is pseudoconvex and φ is plurisubharmonic (resp. strictly plurisubharmonic) on $U \times \Omega$, then the bundle $\left(E, (\cdot,\cdot)_{\varphi(t,\cdot)}\right)$ is Nakano semi-positive (resp. positive).*

By choosing η to be identically constant and letting $\delta \to +\infty$ in our Theorem A, we recover Theorem 1.1.1.

Berndtsson's Nakano positivity theorem is at the center of a long-standing project of Berndtsson aiming at formulating complex analogues of Brunn-Minkowski theory, which first started with his result on the log-plurisubharmonicity of Bergman kernels over pseudoconvex domains ([Ber06]). In addition to extending the Prékopa-Leindler theorem (and thus the Brunn-Minkowski theorem) to the complex setting, Berndtsson's result has deep applications in complex analysis and geometry. For example, his result leads to alternative proofs of existence and uniqueness theorems for Kähler-Einstein metrics ([Ber09a] and [Ber13]), as well as optimal L^2-extension (or Ohsawa-Takegoshi type) theorems ([BL16]).

Now, unlike Berndtsson's case in which the curvature hypothesis would be that of Nakano positivity of the metric h for the pullback bundle in this geometric setting ([Rau13, Theorem 1.5]), our curvature hypothesis is more general, and allows for some amount of curvature negativity along the manifold X in certain cases.

For simplicity, let us focus temporarily on the case where we have a line bundle $L \to Y$ equipped with a family of smooth Hermitian metrics. The twisted curvature condition $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ can seem rather abstract, and it may be unclear how our results constitute an improvement of Berndtsson's theorem. But as we can see from the matrix representation $\Xi_{\delta,\eta}\left(e^{-\varphi}\right)$, if the function $-e^{-\frac{1+\delta}{2}\eta}$ is plurisubharmonic on X, then our metric $e^{-\varphi}$ does not need to be positively curved along the fiber X for the condition $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ to be satisfied. In principle, we may choose a metric $e^{-\varphi}$ so that the curvature along the fiber X is possibly as negative as

$$\frac{4e^{\frac{1+\delta}{2}\eta}}{1+\delta}\partial_X\bar{\partial}_X\left(-e^{-\frac{1+\delta}{2}\eta}\right).$$

Therefore, provided that the manifold X possesses a negative non-constant plurisubharmonic function, our result improves Berndtsson's result. The existence of negative non-constant plurisubharmonic functions is equivalent to the existence of functions *of self-bounded gradient*, on which we expand in Section 4.3. These are functions that satisfy the condition

$$\partial\bar{\partial}\eta \geq c\cdot\left(\partial\eta\wedge\bar{\partial}\eta\right)$$

for some $c>0$, and the latter is equivalent to

$$\partial\bar{\partial}\left(-e^{-c\eta}\right)\geq 0.$$

We offer a few examples in Section 1.2 of this introductory chapter to motivate our results.

With our main result at our disposal, we can prove more general curvature positivity results in the case where X is a possibly unbounded Stein manifold. In this situation, E_h may no longer have the structure of a vector bundle as the fibers may fail to be isometric, resulting in the absence of local triviality. In other words, E_h is simply a family of Hilbert spaces indexed by t or in other words, a *field of Hilbert spaces* (see [LS14, Definition 2.2.1]).

Using alternative characterizations of Griffiths (semi)positivity and Nakano (semi)positivity, we prove the following result.

Theorem B. *Let (X,g) be any Stein Kähler manifold, let U be a domain in $\mathbb{C}^m$, and let $V\to X$ be a holomorphic vector bundle. Let $\left\{h^{[t]}\right\}_{t\in U}$ be a family of smooth Hermitian metrics for $V\to X$, and let $(E_h,(\cdot,\cdot)_{h^{[t]}})$ be the holomorphic Hermitian field of Hilbert spaces whose fiber at t is $\mathcal{H}^2_t:=\mathcal{H}^2\left(X,h^{[t]}\right)$. Let $\delta>0$ and η be a smooth function on X. If $\Xi_{\delta,\eta}(h)>_{\mathrm{Griff}}0$ and*

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right)+\left(\mathbf{Ric}(g)+2\partial_X\bar{\partial}_X\eta-(1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes\mathrm{Id}_V>_{\mathrm{Nak}}0,$$

for each $t \in U$, then the holomorphic Hermitian bundle $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano positive in the sense of definition 6.1.7. Moreover, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta \right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

then $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano semipositive in the sense of definition 6.1.7.

Prior to proving Theorem B, we also show that the holomorphic Hermitian field of Hilbert spaces $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Griffiths positive in a general sense by using a different approach than the one used in our proof of Theorem B.

Theorem C. *Let (X, g) be any Stein Kähler manifold, let U be a domain in $\mathbb{C}^m$, and let $V \to X$ be a holomorphic vector bundle. Let $\left\{h^{[t]}\right\}_{t\in U}$ be a family of smooth Hermitian metrics for $V \to X$ and let $(E_h, (\cdot,\cdot)_{h^{[t]}})$ be the holomorphic Hermitian field of Hilbert spaces whose fiber at t is $\mathcal{H}^2_t := \mathcal{H}^2\left(X, h^{[t]}\right)$. Let $\delta > 0$ and η be a smooth function on X. If $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and*

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta \right) \otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the holomorphic Hermitian bundle $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Griffiths positive in the sense of definition 6.1.6. Moreover, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta \right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in U$, then $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Griffiths semipositive in the sense of definition 6.1.6.

As Berndtsson's work has been the main inspiration for our work, in the spirit of generalizing some of the results of Berndtsson's *complex Brunn-Minkowski theory* ([Ber18]), we use our general curvature positivity theorems to establish log-plurisubharmonic variation results similar to those of Berndtsson. Namely, we prove log-plurisubharmonic variation results for Bergman kernels and families of compactly supported measures for trivial families of possibly unbounded Stein manifolds. As in the case of Berndtsson's complex Brunn-Minkowski theory, these results really only require the Griffiths positivity of the holomorphic Hermitian field of Hilbert spaces $(E_h, (\cdot,\cdot)_{h^{[t]}})$ and follow almost immediately.

Theorem D. *Let (X, g) be any Stein Kähler manifold, let $U \subset \mathbb{C}^m$ be a domain, and let $V \to X$ be a holomorphic vector bundle. Let $\left\{h^{[t]}\right\}_{t \in U}$ be a family of smooth Hermitian metrics for $V \to X$. Let $\delta > 0$ and η be smooth function on X. Denote by K_t the Bergman kernel for the projection $L^2\left(X, h^{[t]}\right) \to \mathcal{H}^2\left(X, h^{[t]}\right)$.*

If $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta \wedge \bar{\partial}_X\eta\right) \otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the family of possibly singular Hermitian metrics for the pullback bundle $\pi_X^* V \to U \times X$ defined by $\left\{K_t^{-1}\right\}_{t \in U}$ on the fiber $(\pi_X^* V)_{(t,z)} \cong V_z$ is positively curved. Otherwise, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta \wedge \bar{\partial}_X\eta\right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the family of possibly singular Hermitian metrics for $\pi_X^* V \to U \times X$ defined by $\left\{K_t^{-1}\right\}_{t \in U}$ on the fiber $(\pi_X^* V)_{(t,z)} \cong V_z$ is semipositively curved.

Theorem E. *Let (X, g) be any Stein Kähler manifold, let $U \subset \mathbb{C}^m$ be a domain, and let $V \to X$ be a holomorphic vector bundle. Let $\left\{h^{[t]}\right\}_{t \in U}$ be a family of smooth Hermitian metrics for $V \to X$. Let $\delta > 0$ and η be smooth function on X. Let $\{\hat{\mu}_t\}_{t \in U}$ be a family of compactly supported V^*-valued complex measures over X. For each section $f \in \Gamma(E_h)$, define the measure $\mu_t^{(f)} = \left\langle f^{[t]}, \hat{\mu}_t \right\rangle$. Suppose that the section $\xi^{(\mu)}$ of E_h^* defined by*

$$f^{[t]} \mapsto \left\langle \xi_t^{(\mu)}, f^{[t]} \right\rangle := \mu_t^{(f)}(X) = \int_X \left\langle f^{[t]}, \hat{\mu}_t \right\rangle$$

is holomorphic. That is, $U \ni t \mapsto \mu_t^{(f)}(X)$ is holomorphic whenever $f \in \Gamma_{\mathcal{O}}(E_h)$.

If $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta \wedge \bar{\partial}_X\eta\right) \otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the function

$$U \ni t \mapsto \log\left(\left\|\xi^{(\mu)}\right\|^2_{h^{[t]},*}\right)$$

is strictly plurisubharmonic or identically $-\infty$*. Otherwise, if either* $\Xi_{\delta,\eta}(h) \geq_{\text{Griff}} 0$ *or*

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V \geq_{\text{Nak}} 0,$$

for each $t \in U$*, then the function*

$$U \ni t \mapsto \log \left(\left\| \xi^{(\mu)} \right\|^2_{h^{[t]},*} \right)$$

is plurisubharmonic or identically $-\infty$*.*

In [Ber17], Berndtsson shows how his log-plurisubharmonic variation results for product domains can be generalized to domains that are subsets of product domains, but not necessarily product domains themselves. His approach consists of a reduction from the latter situation to the former situation. Although Berndtsson's technique is not quite compatible with our twisted curvature condition, we succeed nonetheless at establishing log-plurisubharmonic variation results for a certain class of non-trivial families of Stein manifolds.

Theorem F. *Let* Y *be an* n*-dimensional Stein manifold. Let* ρ *be a smooth plurisubharmonic function on* $\mathbb{C}^m \times Y$ *and let*

$$X = \{\rho(t,z) < 0\} \subset \mathbb{C}^m \times Y.$$

Suppose that for each t*, the restriction* $\rho^{[t]}$ *of* ρ *to*

$$X_t := \{z \in Y : (t,z) \in X\} \subset Y$$

takes values in $[-1,0)$*. Let* g *be a Kähler metric for* Y *and let us equip* $\mathbb{C}^m \times Y$ *with the product metric induced by the Euclidean metric on* $\mathbb{C}^m$ *and the metric* g *on* Y*. Let* $V \to X$ *be a holomorphic vector bundle and let* h *be a smooth Hermitian metric for* $V \to X$ *such that* $\bar{\partial}_t \left(h^{-1}\partial_Y h\right) = 0$*. Let* $V^{[t]} := V\big|_{X_t}$*. If* $\bar{\partial}_t \left(h^{-1}\partial_t h\right) >_{\text{Griff}} 0$ *and*

$$\Theta\left(h^{[t]}\right) + \left(\mathbf{Ric}(g) + \partial_Y \bar{\partial}_Y \rho^{[t]}\right) \otimes \mathrm{Id}_{V^{[t]}} >_{\text{Nak}} 0,$$

over X_t*, for each* $t \in \mathbb{C}^m$*, then for any* $z \in X_t$*, and* $\sigma \in \left(V_z^{[t]}\right)^*$*, the function*

$$(t,z) \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z,\bar{z}) \rangle$$

is strictly plurisubharmonic or identically $-\infty$*. Otherwise, if either* $\bar{\partial}_t\left(h^{-1}\partial_t h\right) \geq_{\text{Griff}} 0$ *and*

$$\Theta\left(h^{[t]}\right) + \left(\mathbf{Ric}(g) + \partial_Y\bar{\partial}_Y\rho^{[t]}\right) \otimes \mathrm{Id}_{V^{[t]}} \geq_{\text{Nak}} 0,$$

over X_t*, for each* $t \in \mathbb{C}^m$*, then for any* $z \in X_t$*, and* $\sigma \in \left(V_z^{[t]}\right)^*$*, then the function*

$$(t, z) \mapsto \log\langle\sigma \otimes \bar{\sigma}, K_t(z, \bar{z})\rangle$$

is plurisubharmonic or identically $-\infty$*.*

Theorem G. *Let* Y *be an* n*-dimensional Stein manifold. Let* ρ *be a smooth plurisubharmonic function on* $\mathbb{C}^m \times Y$ *and let*

$$X = \{\rho(t, z) < 0\} \subset \mathbb{C}^m \times Y.$$

Suppose that for each t*, the restriction* $\rho^{[t]}$ *of* ρ *to*

$$X_t := \{z \in Y : (t, z) \in X\} \subset Y$$

takes values in $[-1, 0)$*. Let* g *be a Kähler metric for* Y *and let us equip* $\mathbb{C}^m \times Y$ *with the product metric induced by the Euclidean metric on* $\mathbb{C}^m$ *and the metric* g *on* Y*. Let* $V \to X$ *be a holomorphic vector bundle and let* h *be a smooth Hermitian metric for* $V \to X$ *such that* $\bar{\partial}_t\left(h^{-1}\partial_Y h\right) = 0$*. Let* $V^{[t]} := V\big|_{X_t}$*. Moreover, let* $\{\hat{\mu}_t\}_{t\in U}$ *be a family of* $\left(V^{[t]}\right)^*$*-valued complex measures over* X_t *that are all locally supported in a compact subset of* X*. For a section* $f \in \Gamma(E_h)$*, define the measure* $\mu_t^{(f)} = \left\langle f^{[t]}, \hat{\mu}_t\right\rangle$*. Suppose that the section* $\xi^{(\mu)}$ *of* E_h^* *defined by*

$$f^{[t]} \mapsto \left\langle \xi_t^{(\mu)}, f^{[t]}\right\rangle := \mu_t^{(f)}(X_t) = \int_{X_t}\left\langle f^{[t]}, \hat{\mu}_t\right\rangle$$

is holomorphic. That is, $U \ni t \mapsto \mu_t^{(f)}(X_t)$ *is holomorphic whenever* $f \in \Gamma_{\mathcal{O}}(E_h)$*.*

If, for each $t \in \mathbb{C}^m$*,* $\bar{\partial}_t\left(h^{-1}\partial_t h\right) >_{\text{Griff}} 0$ *and*

$$\Theta\left(h^{[t]}\right) + \left(\mathbf{Ric}(g) + \partial_Y\bar{\partial}_Y\rho^{[t]}\right) \otimes \mathrm{Id}_{V^{[t]}}$$

over X_t*, for each* $t \in \mathbb{C}^m$*, over* X_t*, then the function*

$$U \ni t \mapsto \log\left(\left\|\xi^{(\mu)}\right\|^2_{h^{[t]},*}\right)$$

is strictly plurisubharmonic or identically $-\infty$*. Otherwise, if either* $\bar{\partial}_t\left(h^{-1}\partial_t h\right) \geq_{\text{Griff}} 0$ *and*

$$\Theta\left(h^{[t]}\right) + \left(\mathbf{Ric}(g) + \partial_Y\bar{\partial}_Y\rho^{[t]}\right) \otimes \mathrm{Id}_{V^{[t]}} \geq_{\text{Nak}} 0,$$

over X_t*, for each* $t \in \mathbb{C}^m$*, then for any* $z \in X_t$*, and* $\sigma \in \left(V_z^{[t]}\right)^*$*, then the function*

$$U \ni t \mapsto \log\left(\left\|\xi^{(\mu)}\right\|^2_{h^{[t]},*}\right)$$

is plurisubharmonic or identically $-\infty$*.*

Finally, we present a couple of generalizations of the complex Prékopa-Leindler theorems of Berndtsson. Recall that a domain Ω in $\mathbb{C}^n$ is *balanced* if $z \in \Omega$ implies that $\lambda z \in \Omega$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

Theorem H. *Let* Ω *be a balanced pseudoconvex domain in* $\mathbb{C}^n$*, let* $\delta > 0$ *and let* η *be a smooth function on* Ω*. Let* U *be a domain in* $\mathbb{C}^m$ *and let* $\varphi \in \mathcal{C}^\infty(U \times \Omega)$ *be* S^1*-invariant in* z *for any* $t \in U$*. If* $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ (resp. > 0) *in* $U \times \Omega$*, then the function*

$$t \mapsto -\log\left(\int_\Omega e^{-\varphi(t,z)} dV(z)\right)$$

is plurisubharmonic (resp. strictly plurisubharmonic) or identically equal to $-\infty$*.*

Theorem I. *Let* $\Omega := \{\zeta : \mathrm{Re}(\zeta) \in D\}$ *for a convex domain* D *in* $\mathbb{R}^n$*. Let* $\delta > 0$ *and let* η *be a smooth function on* Ω*. Let* U *be a domain in* $\mathbb{C}^m$ *and assume that* $\varphi \in \mathcal{C}^\infty\left(U \times \Omega\right)$ *does not depend on the imaginary part of* ζ*. If* $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ (resp. > 0) *in* $U \times \Omega$*, then the function*

$$t \mapsto -\log\left(\int_D e^{-\varphi(t,x)} dx\right)$$

is plurisubharmonic (resp. strictly plurisubharmonic) or identically $-\infty$*.*

1.2 Motivating examples

1.2.1 The unit ball

Let $\delta = 1$. On the unit ball $\mathbb{B}_n(1)$ in $\mathbb{C}^n$, the function $z \mapsto -\log\left(1 - |z|^2\right)$ is a function of self-bounded gradient (with constant 1) on $\mathbb{B}_n(1)$. Therefore,

$$2\partial_z\bar{\partial}_z\eta - (1+\delta)\left(\partial_z\eta \wedge \bar{\partial}_z\eta\right) = \frac{4e^{\frac{1+1}{2}\eta}}{1+1}\partial_z\bar{\partial}_z\left(-e^{-\frac{1+1}{2}\eta}\right) = \frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z},$$

where

$$dz\dot{\wedge}d\bar{z} := dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n.$$

If we equip $\mathbb{B}_n(1)$ with the Euclidean metric, the twisted curvature condition $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ reduces to the following.

$$\begin{pmatrix} \partial_t\bar{\partial}_t\varphi & \partial_t\bar{\partial}_z\varphi \\ \partial_z\bar{\partial}_t\varphi & \frac{1}{2}\left(\partial_z\bar{\partial}_z\varphi + \frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z}\right) \end{pmatrix} \geq 0. \tag{1.2.1}$$

Therefore, $\partial_z\bar{\partial}_z\varphi$ can be chosen to be as negative as $-\frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z}$.

Example 1.2.1. (A diagonal weight on $\mathbb{B}_n(1)$) If we pick a weight φ of the form

$$\varphi(t,z) = \vartheta(t) + \psi(z),$$

then the twisted curvature condition reduces to

$$\begin{pmatrix} \partial_t\bar{\partial}_t\vartheta & 0 \\ 0 & \frac{1}{2}\left(\partial_z\bar{\partial}_z\varphi + \frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z}\right) \end{pmatrix} \geq 0,$$

which is equivalent to

$$\partial_t\bar{\partial}_t\vartheta \geq 0 \text{ and } \partial_z\bar{\partial}_z\psi + \frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z} \geq 0.$$

For instance, let $\varphi(t,z) = |t|^2 - 2|z|^2$. Then

$$\partial_t\bar{\partial}_t\varphi = dt\dot{\wedge}d\bar{t} \geq 0 \text{ and } \partial_z\bar{\partial}_z\psi + \frac{2}{1-|z|^2}dz\dot{\wedge}d\bar{z} = \frac{|z|^2}{1-|z|^2}dz\dot{\wedge}d\bar{z} \geq 0.$$

So the consequence of Theorem 1.1.1 still holds, even though φ is not plurisubharmonic.

Example 1.2.2. (A non-diagonal weight on $\mathbb{D}$) Consider now the case when $n = 1$; the unit disk. Suppose that U is a disk with radius $\sqrt{2}$ centered at the origin. Once again, let $\delta = 1$. Another weight one might consider is

$$\varphi(t, z) = \left(1 - |z|^2\right) |t|^2 = |t|^2 - |t|^2 |z|^2 .$$

Clearly, φ is not plurisubharmonic. However, it satisfies condition (1.2.1). Indeed, the trace of the form

$$\begin{aligned}
&\begin{pmatrix} \sqrt{-1}\partial_t\bar{\partial}_t\varphi & \sqrt{-1}\partial_t\bar{\partial}_z\varphi \\ \sqrt{-1}\partial_z\bar{\partial}_t\varphi & \frac{1}{2}\left(\sqrt{-1}\partial_z\bar{\partial}_z\varphi + \frac{2}{1-|z|^2}dz \wedge \sqrt{-1}d\bar{z}\right) \end{pmatrix} \\
&= \begin{pmatrix} (1-|z|^2)\sqrt{-1}dt \wedge d\bar{t} & -t\bar{z}\sqrt{-1}dt \wedge d\bar{z} \\ -\bar{z}t\sqrt{-1}dz \wedge d\bar{t} & \frac{1}{2}\left(-|t|^2 + \frac{2}{1-|z|^2}\right)\sqrt{-1}dz \wedge d\bar{z} \end{pmatrix},
\end{aligned}$$

is clearly a positive form. Moreover, its determinant

$$\begin{aligned}
&\left[\frac{\delta}{1+\delta}\left(-|t|^2 + \frac{2}{1-|z|^2}\right)(1-|z|^2) + |t|^2|z|^2\right]\sqrt{-1}dt \wedge d\bar{t} \wedge \sqrt{-1}dz \wedge d\bar{z} \\
&= \left[\frac{\delta}{1+\delta}\left(|t|^2|z|^2 + \left(2 - |t|^2\right)\right) + |t|^2|z|^2\right]\sqrt{-1}dt \wedge d\bar{t} \wedge \sqrt{-1}dz \wedge d\bar{z} > 0
\end{aligned}$$

is also a positive form. Thus $\sqrt{-1}\Xi_{1,\eta}\left(e^{-\varphi}\right) > 0$.

1.2.2 Pseudoconvex domains with smooth boundary in $\mathbb{C}^n$

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Let K_Ω denote the Bergman kernel of Ω. By Fefferman's theorem on the asymptotic expansion of K_Ω, (see [Fef74a, Theorem 2] and [Fef74b] for details) the function

$$z \mapsto \eta(z) := \frac{1}{n+1}\log\left(K_\Omega(z, \bar{z})\right)$$

satisfies

$$\partial_z\bar{\partial}_z\eta - \partial_z\eta \wedge \bar{\partial}_z\eta \geq -C\sqrt{-1}\partial_z\bar{\partial}_z\left|\cdot\right|^2$$

for some constant $C > 0$. The result fails if the constant $\dfrac{1}{n+1}$ in η is replaced by a larger constant. (See [MV15, Theorem 3.7.6] and [Var19, pp. 102-103] for details.)

If we let $\varphi(t,z) = |t|^2 - (1-\delta)\left(\dfrac{1}{n+1}\log\left(K_\Omega(z,\bar{z})\right)\right) + C(1+\delta)\,|z|^2$ for $(t,z) \in \mathbb{C}^m \times \Omega$ and $\delta \in (0,1]$, then φ fails to be plurisubharmonic near the boundary of Ω but clearly satisfies the condition $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$. Indeed, near any boundary point,

$$\sqrt{-1}\partial_z\bar{\partial}_z\left(\frac{1}{n+1}\log\left(K_\Omega(z,\bar{z})\right)\right) = \omega_\Omega^B$$

is arbitrarily close to $\omega^B_{\mathbb{B}_n(r_0)}$, the Bergman metric of a ball of radius r_0 centered at the origin in $\mathbb{C}^n$, in suitably chosen coordinates, and the latter is arbitrarily large near any such point in those coordinates.

1.3 Organization of the thesis

Our mathematical contributions are contained in Chapter 6. Chapters 2 through 5 are mainly expository.

The contents of Chapter 2 are very classical in nature, recalling elementary notions from complex analytic and differential geometry, with Section 2.4.8 being the most relevant to the proof of our main result.

Chapter 3 offers an introduction to Bergman spaces and their kernels, including a collection of important properties that are essential to the proofs of our log-plurisubharmonic variation results for Bergman kernels.

Chapter 4 provides a concise exposition of the Donnelly-Fefferman-Ohsawa L^2-estimates for the $\bar{\partial}$-operator in addition to a detailed discussion of functions of self-bounded gradient with numerous examples.

In the penultimate chapter of our thesis, Chapter 5, we give a survey of Berndtsson's complex Brunn-Minkowski theory as a transition to the ultimate chapter of our thesis.

Finally, in Chapter 6, we present our main results and some applications. We start by defining holomorphic Hilbert bundles associated to trivial families of relatively compact complete Kähler submanifolds of Stein manifolds and their geometry, in Section 6.1.1.1. We then lay out the setup for the proof of Theorem A in Section 6.1.1.2, which we prove in Section 6.1.1.3, assuming that the twisted curvature conditions hold strictly. In Section 6.1.1.4, we show how the strict requirement on these twisted curvature conditions can be relaxed via a limiting process. Subsequently, in Section 6.1.2.1, we define general notions of Griffiths (semi)positivity and Nakano (semi)positivity for holomorphic Hermitian fields of Hilbert spaces, and we prove the more general variants of our initial result for general Stein manifolds in Sections 6.1.2.2 and 6.1.2.3. Namely, assuming that the Stein manifold is a possibly unbounded, we show that the conclusions of our theorem in the relatively compact case (Theorem 6.1.1) hold with Nakano (resp. Griffiths) (semi)positivity as defined in Section 6.1.2.1. Finally, we apply these fundamental results to the log-plurisubharmonic variation of Bergman kernels and families of compactly supported measures for general trivial families of Stein manifolds, as well as a class of non-trivial families of Stein manifolds in Section 6.2. We end this thesis with brief proofs of a couple of generalizations of Berndtsson's complex Prékopa-Leindler theorems in Section 6.3.

Chapter 2

Elements of complex analytic and differential geometry

In this expository chapter, we introduce various basic notions from complex analytic geometry which will be of relevance to our study of holomorphic Hilbert bundles. For a comprehensive treatment of complex analysis and geometry in several complex variables, we refer the reader to [Dem12] which contains far more background material and a much more detailed exposition. This chapter of our thesis has been largely adapted from the notes [Var19], in addition to [Ber17] and [Sho14]. We assume familiarity with the basic theories of Riemannian, Hermitian, and Kähler geometry, and of Stein manifolds.

Although the notions relative to vector bundles exposed here pertain to vector bundles of finite rank, this theory can be extended to vector bundles of infinite rank, also known as *Banach* bundles. (See [Lan85].) In particular, if the fibers are Hilbert spaces, we refer to such bundles as *Hilbert* bundles.

We discuss finite rank vector bundles in order to create intuition for the results, but the infinite rank case is much more subtle, and will not be discussed in a direct fashion.

2.1 Holomorphic vector bundles

2.1.1 Definitions

Definition 2.1.1. A holomorphic vector bundle of rank r is a triple $(V, X, \pi : V \to X)$ such that

1. V and X are complex manifolds,
2. π is a holomorphic map,
3. each fiber $V_x := \pi^{-1}(\{x\})$ is a vector space of dimension r, and
4. each $p \in X$ is contained in an open set U on which there are holomorphic maps $e_1, \cdots, e_r : U \to V$ such that

$$\pi e_i = \mathrm{id}_U \text{ and } \mathrm{span}_{\mathbb{C}} \{e_1(x), \cdots, e_r(x)\} = V_x \text{ for all } x \in U.$$

   We call such a collection of holomorphic maps $\{e_1, \cdots, e_r\}$ a *frame for V over U*.

Note that if $\{e_i\}_{1 \le i \le r}$ and $\{\tilde{e}_i\}_{1 \le i \le r}$ are two frames defined over the same open set U, then there are holomorphic functions $g_i^j \in \mathcal{O}(U)$ such that $g_i^j(p) \in \mathrm{GL}(r, \mathbb{C})$ for all $p \in U$ and $\tilde{e}_i = g_i^J e_j$.

Definition 2.1.2. A map of holomorphic vector bundles is a holomorphic map $V \to W$ such that

1. The follow diagram commutes.
$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d] & W \arrow[d] \\ X \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

2. For each $x \in X$, the map $F_x := F\big|_{V_x} : V_x \to W_x$ is linear.

Two vector bundles are said to be isomorphic if there are holomorphic vector bundle maps $F : V \to W$ and $G : W \to V$ such that $F \circ G = \mathrm{Id}_V$ and $G \circ F = \mathrm{Id}_W$.

Definition 2.1.3. A holomorphic vector bundle of rank 1 is called a *holomorphic line bundle*.

Definition 2.1.4. A section s of a holomophic vector bundle $(V, X, \pi : V \to X)$ – i.e. a right inverse for π – is said to be holomorphic (resp. smooth, measurable, etc.) if it is holomorphic (resp. smooth, measurable, etc.) as a map $X \to V$.

Example 2.1.1. (Trivial bundles) The simplest example of a holomorphic vector bundle is the trivial bundle $\pi : X \times \mathbb{C}^r \to X$, where π denotes the projection to the first factor. If a vector bundle is isomorphic to the trivial bundle, then for any basis $\mathbf{e}_1, \cdots, \mathbf{e}_r$ of $\mathbb{C}^r$, the isomorphism $F : X \times \mathbb{C}^r \to V$ defines a frame

$$e_i(x) := F(x, \mathbf{e}_i), 1 \leq i \leq r$$

over all of X. Conversely, a global frame for a vector bundle $V \to X$ defines an isomorphism F^{-1} to the trivial bundle, where F is given the same formula and then extended fiberwise-linearly. In other words, if we fix a basis $\mathbf{e}_1, \cdots, \mathbf{e}_r$ of $\mathbb{C}^r$, we define the isomorphism $F : X \times \mathbb{C}^r \to V$ by

$$F^{-1}\left(\sum_{1\leq i\leq r} f^i(x)\mathbf{e}_i(x)\right) := \left(x, \sum_{1\leq i\leq r} f^i(x)\mathbf{e}_i\right), 1 \leq i \leq r.$$

Therefore, a vector bundle is isomorphic to the trivial bundle if and only if the vector bundle has a global frame. In particular, every (holomorphic) vector bundle is, by definition, locally trivial.

Example 2.1.2. (Operations on bundles)

1. If $V \to X$ and $W \to X$ are holomorphic vector bundles, then so are $V^* \to X, V \otimes W \to X$ and $V \oplus W \to X$. Therefore, $\mathrm{Sym}^k(V) \to X$ and $\Lambda^k(V) \to X$ are holomorphic vector bundles. More generally, all vector bundles obtained from multi-$\mathbb{C}$-linear operations on holomorphic vector bundles are also holomorphic. However, the complex conjugate bundle $V^\dagger \to X$ of a holomorphic vector bundle is generally *not* a holomorphic vector bundle over X, but it is a holomorphic vector bundle over $X^\dagger$, where $X^\dagger$ is the complex manifold with the complex conjugate structure of X.

2. Consider two morphisms $f : X \to Z$ and $g : Y \to Z$. Then one can define the *fiber product*

$$X \times_Z Y := \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

There are projection maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$ given by the restriction of $X \times_Z Y$ to the Cartesian projections $X \times Y \to X$ and $X \times Y \to Y$. If $\pi : V \to Y$ is a holomorphic vector bundle and $f : X \to Y$ is a holomorphic map, then the bundle $f^*V = V \times_Y X \to X$ is a holomorphic vector bundle, called *the pullback of* V *by* f.

3. Given holomorphic vector bundles $V \to X$ and $W \to Y$, one defines the holomorphic vector bundle

$$V \boxtimes W = \pi_X^* V \otimes \pi_Y^* W \to X \times Y,$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the Cartesian projections.

Remark 2.1.3. A vector bundle map $F : V \to W$ can be identified with a holomorphic section of the bundle $W \otimes V^*$.

2.1.2 Holomorphic structure of the tangent bundle

On an n-dimensional complex manifold X, one has the real tangent bundle $T_X \to X$, which is a smooth real vector bundle. One can then define the complex vector bundle $T_X \otimes_{\mathbb{R}} \mathbb{C} \to X$. The points of the total space of $T_X \otimes_{\mathbb{R}} \mathbb{C}$ are called *complex tangent vectors*. Now, for a holomorphic coordinate system $z = (z_1, ..., z_n)$ on an open set $U \subset X$ (i.e., an element of the maximal holomorphic atlas of X) one can define the complex tangent vectors

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), 1 \leq i \leq n,$$

where $z_i = x_i + \sqrt{-1}y_i$. These vectors depend on the local coordinate system, but their span does not. For each $x \in U$, we define

$$T^{1,0}_{X,x} := \operatorname{span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\right\}.$$

The elements of $T^{1,0}_{X,x}$ are called $(1,0)$-vector at x. For each $\xi \in T^{1,0}_{X,x}$, the vector $\bar{\xi} \in T_X \otimes_{\mathbb{R}} \mathbb{C}$ (the complex conjugate of ξ) does not lie in $T^{1,0}_{X,x}$. Defining

$$T^{0,1}_{X,x} := \overline{T^{1,0}_{X,x}},$$

we obtain the decomposition

$$T_X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}.$$

Define the vector bundle $\pi : T^{1,0}_X \to X$ by

$$T^{1,0}_X := \coprod_{x \in X} T^{1,0}_{X,x} \text{ and } \pi^{-1}(\{x\}) := T^{1,0}_{X,x},$$

with the vector bundles structure given by the frames $\{\partial/\partial z_i\}_{1 \leq i \leq n}$. The chain rule shows that $T^{1,0}_X \to X$ is a holomorphic vector bundle.

From basic complex analysis, we see that if U is an open set in X, $x \in U$, $f \in \mathcal{O}(U)$ and $\xi \in T^{1,0}_{X,x}$, then

$$\xi(f) = 2\mathrm{Re}(\xi)(f).$$

Moreover, if $\iota : T_{X,x} \hookrightarrow T_{X,x} \otimes_{\mathbb{R}} \times \mathbb{C}$ denotes the natural inclusion and $\pi^{1,0} : T_{X,x} \otimes_{\mathbb{R}} \mathbb{C} \to T^{1,0}_{X,x}$ denotes the projection to the first factor in the decomposition $T_X \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{X,x} \oplus T^{0,1}_{X,x}$, then the composite map $s^{1,0} := \pi^{1,0} \circ \iota : T_X \hookrightarrow T^{1,0}_X$ is a real isomorphism of vector bundles whose inverse is the map $\xi \mapsto 2\mathrm{Re}(\xi)$.

From these observations, one can give $T_X \to X$ the structure of a holomorphic vector bundle in two ways. The first and most direct way is to map T_X isomorphically to $T^{1,0}_X$. Indeed, since the latter is a holomorphic vector bundle, so is the former.

Definition 2.1.5. The vector bundle $T^{1,0}_X \to X$ is called the holomorphic tangent bundle. The dual vector bundle $T^{*1,0}_X \to X$ is called the holomorphic cotangent bundle of X.

Note that in a local coordinate system, a frame of the holomorphic cotangent bundle is given by the complex 1-forms $dz_1, \cdots, dz_n$.

The *canonical bundle* $K_X \to X$ is the determinant of the holomorphic cotangent bundle:

$$K_X := \det\left(T_X^{*1,0}\right) := T_X^{*1,0} \wedge ... \wedge T_X^{*1,0},$$

where we have n copies of $T_X^{*1,0}$. In a holomorphic local coordinate chart, a frame of K_X is given by the n-form $dz_1 \wedge \cdots \wedge dz_n$. The name "canonical" refers to the fact that $K_X \to X$ is essentially the only natural (typically) nontrivial line bundle defined on every complex manifold.

2.2 Differential forms on complex manifolds

On a complex manifold, it is natural to consider complex-valued differential forms, i.e., sections of the bundle

$$\mathcal{E}_X := \bigoplus_{r=1}^{2n} (\Lambda^r(T_X) \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{r=1}^{2n} \bigwedge^r (T_X \otimes_{\mathbb{R}} \mathbb{C}).$$

The sections of this bundle form an algebra with respect to the complex linear extension of the wedge product. The differential d (extended $\mathbb{C}$-linearly) acts on the sections of $\mathcal{E}_X \to X$, mapping sections of $\Lambda^r(T_X) \to X$ to sections of $\Lambda^{r+1}(T_X) \to X$:

$$d\left(\Gamma\left(X, \Lambda^r(T_X)\right)\right) \subset \Gamma\left(X, \Lambda^{r+1}(T_X)\right).$$

Forms in the kernel of d are called *closed* and forms in the image of d are called *exact*.

2.2.1 Forms of bidegree (p, q)

The splitting $T_X \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus T_X^{0,1}$ induces a splitting

$$\Lambda^r(T_X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=r} \left(\bigwedge^p T_X^{1,0}\right) \wedge \left(\bigwedge^q T_X^{0,1}\right) =: \bigoplus_{p+q=r} \Lambda^{p,q}(T_X),$$

and we have the projections $\pi^{p,q} : \Lambda^r(T_X) \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{p,q}(T_X)$.

The wedge product sends an element of $\Lambda^{p,q}(T_{X,x})$ and an element of $\Lambda^{s,t}(T_{X,x})$ to an element $\Lambda^{p+s,q+t}(T_{X,x})$. The smooth vector bundle $\Lambda^{p,q}(T_X) \to X$ is a holomorphic vector bundle if and only if $q = 0$.

Definition 2.2.1. The sections of $\Lambda^{p,q}(T_X) \to X$ are called forms of bidegree (p,q) or *(p,q)-forms.*

Locally every (p,q)-form is of the form $\alpha = \sum_{|I|=p,|J|=q} f_{I\bar{J}} dz_I \wedge d\bar{z}_J$, for some functions $f_{I\bar{J}}$ where $dz_I \wedge d\bar{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. This local expression is not unique since different choices of coefficient functions $f_{I\bar{J}}$ can result in the same α. However, the coefficient functions are uniquely determined by the form if we impose the assumption of skew-symmetry on the coefficient functions $f_{I\bar{J}}$ of the (p,q)-form α: if $\sigma \in S_p$ and $\tau \in S_q$ are permutations, then we can impose the condition

$$f_{I_\sigma \bar{J}_\tau} = (\mathrm{sgn}(\sigma))(\mathrm{sgn}(\tau)) f_{I\bar{J}}$$

on the coefficients of α, where a permutation $\nu \in S_r$ acts on an r-tuple $K = (k_1, \cdots, k_r)$ by the formula $K_\nu = \left(k_{\nu(1)}, \cdots, k_{\nu(r)}\right)$.

2.2.2 Exterior differential operators and twisted differential forms

The exterior algebra of a complex manifold is equipped with two additional differential operators: the ∂-operator and the $\bar{\partial}$-operator, both defined on sections of $\Lambda^{p,q}(T_X) \to X$ by

$$\partial\alpha := \pi^{p+1,q} d\alpha \text{ and } \bar{\partial}\alpha := \pi^{p,q+1} d\alpha, \alpha \in \Gamma\left(X, \Lambda^{p,q}(T_X)\right).$$

An important property of these operators is that $d = \partial + \bar{\partial}$, and consequently $\bar{\partial}^2 = 0$.

Definition 2.2.2. Let X be a complex manifold and let $V \to X$ be a holomorphic vector bundle. A V-valued (p,q)-form is a section of the vector bundle $\Lambda^{p,q}(T_X) \otimes V \to X$. We also call such sections twisted differential forms.

After tensoring with a holomorphic vector bundle, the exterior differential operator is no longer necessarily well defined. More precisely, if we choose frame $\xi_1, \cdots, \xi_r$ for $V \to X$ and write

$$u = \sum_{\substack{|I|=p,|J|=q,1\\ \leq\nu\leq r}} u^{\nu}_{I\bar{J}} dz_I \wedge d\bar{z}_J \otimes \xi_\nu,$$

then the form

$$\alpha := \sum_{\substack{|I|=p,|J|=q\\ 1\leq\nu\leq r\\ 1\leq k\leq n}} \frac{\partial u^{\nu}_{I\bar{J}}}{\partial z_k} dz_k \wedge dz_I \wedge d\bar{z}_J \otimes \xi_\nu$$

only depends on the frame $\xi_1, \cdots, \xi_r$, and therefore does not define a global section of the bundle $\Lambda^{p+1,q}(T_X) \otimes V \to X$. However, the form

$$\alpha := \sum_{\substack{|I|=p,|J|=q\\ 1\leq\nu\leq r\\ 1\leq k\leq n}} \frac{\partial u^{\nu}_{I\bar{J}}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J \otimes \xi_\nu$$

is globally defined, surprisingly. This implies that the local operator

$$\bar{\partial} : \Gamma\left(X, \Lambda^{p,q}(T_X) \otimes V\right) \to \Gamma\left(X, \Lambda^{p,q+1}(T_X) \otimes V\right)$$

which maps

$$u = \sum_{\substack{|I|=p,|J|=q\\ 1\leq\nu\leq r}} u^{\nu}_{I\bar{J}} dz_I \wedge d\bar{z}_J \otimes \xi_\nu$$

to

$$\bar{\partial} u := \sum_{\substack{|I|=p,|J|=q\\ 1\leq\nu\leq r\\ 1\leq k\leq n}} \frac{\partial u^{\nu}_{I\bar{J}}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J \otimes \xi_\nu$$

is well-defined.

Remark 2.2.1. Note that when considering twisted (p, q)-forms, one can reduce to the case $p = n$ without loss of generality. Indeed, since

$$\Lambda^{p,q}(T_X) \cong \Lambda^{p,0}(T_X) \otimes \Lambda^{0,q}(T_X) \cong \Lambda^{p,0}(T_X) \otimes K^*_X \otimes \Lambda^{n,q}(T_X),$$

one can write $\Lambda^{p,q}(T_X) \otimes V \cong \Lambda^{n,q}(T_X) \otimes W$ where $W = \Lambda^{p,0}(T_X) \otimes K^*_X \otimes V$. Therefore, as W is a holomorphic vector bundle, the $\bar{\partial}$-operator we just defined is computed locally in exactly the same way for V-valued (p, q)-forms as for W-valued (n, q)-forms. As a result, when working with twisted differential forms, especially while focusing on the $\bar{\partial}$-equation, it is often enough to consider only twisted (n, q)-forms.

2.3 Hermitian metrics for complex vector bundles

Let X be a complex manifold and let $V \to X$ be a complex vector bundle.

Definition 2.3.1. A Hermitian metric for $V \to X$ is a section h of the bundle $V^* \otimes (V^*)^\dagger \to X$ such that for all $x \in X$ and $v, w \in V_x$,

1. h is Hermitian-symmetric, i.e. $\langle h, v \otimes \bar{w} \rangle = \overline{\langle h, w \otimes \bar{v} \rangle}$, and

2. h is positive-definite, i.e. $\langle h, v \otimes \bar{v} \rangle > 0$ for all $v \neq 0$.

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing.

In other words, h defines a sesquilinear, positive definite Hermitian form on each fiber V_x of the vector bundle $V \to X$.

If $\alpha_1, \cdots, \alpha_r$ is a frame for V^* over the closure of a relatively compact set U, then one can write

$$h = \sum_{1 \leq i,j \leq r} h_{i\bar{j}} \alpha_i \odot \bar{\alpha}_j := \sum_{1 \leq i,j \leq r} h_{i\bar{j}} \left(\alpha_i \otimes \bar{\alpha}_j + \bar{\alpha}_j \otimes \alpha_i \right)$$

for some functions $\{h_{i\bar{j}}\}_{1 \leq i,j \leq r}$ satisfying

$$\overline{h_{i\bar{j}}} = h_{i\bar{j}} \text{ and } \sum_{1 \leq i,j \leq r} h_{i\bar{j}} a_i a_{\bar{j}} \geq \varepsilon \left\| a \right\|^2,$$

for some positive function $\varepsilon > 0$ on U and all $a = (a_1, \cdots, a_r) \in \mathbb{C}^r$. In other terms, at each $x \in U$, the matrix $\left(h_{i\bar{j}}(x) \right)_{i,j=1}^r$ is Hermitian and positive-definite. The regularity of h is that of the functions $\{h_{i\bar{j}}\}_{1 \leq i,j \leq r}$. Although the Hermitian metric for V is a section of $V^* \otimes (V^*)^\dagger$, we will often treat it as a Hermitian inner product on the fibers of V, and thus we will often write $h(v, w) := \langle h, v \otimes \bar{w} \rangle$.

2.4 Connections and curvature

2.4.1 Connections

Definition 2.4.1. Let X be a complex manifold and let $V \to X$ be a complex vector bundle. A connection for $V \to X$ is a linear map

$$\nabla : \Gamma(X, V) \to \Gamma\left(X, T_X^* \otimes V\right),$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s.$$

Note that if ∇^1 and ∇^2 are two connections for a vector bundle $V \to X$ then their difference is a section of $\Gamma\left(X, \mathcal{C}^\infty\left(T_X^* \otimes \operatorname{End}(V)\right)\right)$. In addition, the definition of a connection implies that it is a local operator and may be restricted to small subsets. If one restricts to a sufficiently small subset $U \subset X$, then the vector bundle $V\big|_U \to U$ is isomorphic to the trivial bundle, and thus admits a frame. With the choice of a frame $e_1, \cdots, e_r$ for $V\big|_U$, one has the trivial connection d defined by

$$d\left(\sum_{1\le i\le r} s_i e_i\right) = \sum_{1\le i\le r} ds_i \otimes e_i.$$

Any other connection ∇ for $V\big|_U \to U$ can then obtained from the trivial connection by adding a section A of $T_X^* \otimes \operatorname{Hom}\left(E\big|_U\right) \to U$, i.e. $\nabla\big|_U = d + A$. The section A is called the *connection form*. Since the trivial connection depends on the frame, so does the connection form A. In terms of the frame $e_1, \cdots, e_r$ for $E\big|_U \to U$, one can write

$$Ae_i = \sum_{1\le j\le r} A_i^j \otimes e_j, 1 \le i \le r$$

and then by the Leibniz rule

$$D\left(\sum_{1\le i\le r} s_i e_i\right) = \sum_{1\le i\le r}\left(ds_i \otimes e_i + s_i \sum_{1\le j\le r} A_i^j \otimes e_j\right).$$

The matrix of 1-forms $\left(A_i^j\right)_{1\le i,j\le r}$ is called *the connection matrix*. Note that the connection matrix is locally given by $A_\beta^\alpha = \sum_{1\le\mu\le r} h^{\alpha\bar{\mu}}\partial h_{\beta\bar{\mu}}$.

2.4.2 Induced connections

A connection for a vector bundle $V \to X$ induces connections on all vector bundles obtained from $V \to X$ via multi-$\mathbb{C}$-linear operations.

2.4.2.1 Dual connection

Given a connection ∇ for a vector bundle $V \to X$, one defines the connection ∇^* for the dual vector bundle $V^* \to X$ as follows. Given local sections s for $V \to X$ and α for $V^* \to X$, one has a pairing $\langle s, \alpha \rangle$, which is a function on X. We require the dual connection ∇^* to satisfy

$$d\langle s, \alpha \rangle = \langle Ds, \alpha \rangle + \langle s, \nabla^* \alpha \rangle. \tag{2.4.1}$$

If we fix a frame $e_1, \cdots, e_r$ for $V \to X$ and denoted by $\alpha_1, \cdots, \alpha_r$ its dual frame, then the connection forms $A(\nabla)$ and $A(\nabla^*)$ satisfy

$$0 = d\delta_i^j = d\langle e_i, \alpha_j \rangle = \left\langle \sum_{1 \le k \le r} A(\nabla)_i^k e_k, \alpha_j \right\rangle + \left\langle e_i, \sum_{1 \le \ell \le r} A(\nabla^*)_\ell^j \alpha_\ell \right\rangle = A(\nabla)_i^j + A(\nabla^*)_i^j.$$

Therefore, the dual connection ∇^* is completely determined by the connection ∇ and the compatibility requirement (2.4.1).

2.4.2.2 Product connections

Let $V_1 \to X$ and $V_2 \to X$ be two vector bundles equipped with connections ∇^1 and ∇^2 respectively. Given any product operation $\times$ (e.g. $\times = \wedge$ or $\times = \otimes$), one defines the product connection ∇ for $V_1 \times V_2 \to X$ by the formula

$$\nabla(s_1 \times s_2) = (\nabla^1 s_1) \times s_2 + s_1 \times (\nabla^2 s_2).$$

One can inductively pass to any finite product of vector bundles.

Example 2.4.1. (Induced connections for determinant bundles) Let $V \to X$ be a vector bundle of rank r and let ∇^V be a connection for $V \to X$. Consider the complex line bundle $\det(V) \to X$ whose transition functions are just the determinants of the corresponding transition functions for $V \to X$. Fix a frame $e_1, \cdots, e_r$ for $V \to X$. Then $e_1 \wedge \cdots \wedge e_r$ is a frame for $\det(V)$, and so

$$e_1 \wedge \cdots \wedge \nabla^V e_j \wedge \cdots \wedge e_r = e_1 \wedge \cdots \wedge A(\nabla^V)^k_j e_k \wedge \cdots \wedge e_r = A(\nabla^V)^k_j \delta^j_k e_1 \wedge \cdots \wedge e_j \cdots \wedge e_r,$$

whence

$$\nabla^{\det(V)}(e_1 \wedge \cdots \wedge e_r) = \left(\sum_{1 \leq j \leq r} A(\nabla^V)^j_j \right) e_1 \wedge \cdots \wedge e_r,$$

i.e. the connection matrix for $\nabla^{\det(V)}$ is the trace of the connection matrix of ∇^V.

2.4.3 Connections with additional symmetry

2.4.3.1 Metric compatibility

Definition 2.4.2. Let $V \to X$ be endowed with a metric g. We say that a connection ∇ for $V \to X$ is compatible with g if

$$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t)$$

for all local sections s, t of $V \to X$.

Viewing the metric g as a section of $V^* \otimes (V^*)^\dagger \to X$, for any connection ∇ – not necessarily g-compatible – for $V \to X$, one has

$$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t) + \nabla g(s,t).$$

Thus, the g-condition compatibility can be expressed as $\nabla g = 0$.

Generally, a given metric g has many compatible connections. If ∇^1 and ∇^2 are two connections for $V \to X$ that are g-compatible, then their difference $\mathfrak{D} := \nabla^1 - \nabla^2$ satisfies

$$g(\mathfrak{D}s, t) + g(s, \mathfrak{D}t) = 0,$$

and so $\mathfrak{D}$ is anti-symmetric (or anti-Hermitian, if g is Hermitian) with respect to g. Equivalently, if we define $\mathfrak{D}^\dagger$ by $g\left(\mathfrak{D}^\dagger s, t\right) = g\left(s, \mathfrak{D}t\right)$, then $\mathfrak{D}^\dagger = -\mathfrak{D}$.

Remark 2.4.2. If X is a complex manifold then the splitting $T^*_X \otimes_{\mathbb{R}} \mathbb{C} = T^{*1,0}_X \oplus T^{*0,1}_X$ induces the decomposition $\mathfrak{D} = \mathfrak{D}^{1,0} + \mathfrak{D}^{0,1} \in \left(\mathrm{End}(V) \otimes T^{1,0}_X\right) \oplus \left(\mathrm{End}(V) \otimes T^{0,1}_X\right)$, and the condition $\mathfrak{D}^\dagger = -\mathfrak{D}$ means that

$$\left(\mathfrak{D}^{1,0}\right)^\dagger = -\mathfrak{D}^{0,1} \text{ and } \left(\mathfrak{D}^{0,1}\right)^\dagger = -\mathfrak{D}^{1,0}.$$

In particular, if $\mathfrak{D}$ is of type $(1,0)$, then it must vanish identically.

2.4.3.2 Symmetric connections

Given a smooth manifold X, we have the splitting

$$T^*_X \otimes T^*_X = \mathrm{Sym}^2(T^*_X) \oplus \Lambda^2(T^*_X).$$

Therefore, every connection ∇ for the cotangent bundle splits as

$$\nabla = \nabla^S + \nabla^\Lambda.$$

On any smooth manifold, there is a natural operator sending 1-forms to 2-forms and satisfying the Leibniz rule with respect to the wedge product: the exterior derivative d. This motivates the following definition.

Definition 2.4.3. A connection ∇ for T^*_X is said to be symmetric if $\nabla^\Lambda = d$.

Since the tangent bundle and the cotangent bundle are dual, we call a connection for T_X *symmetric* if it is the dual of a symmetric connection for T^*_X. If ∇ is a connection for T_X dual to a given connection ∇ for T^*_X, a short computation shows that it is symmetric if and only if its connection matrix $\left(\Gamma^i_{jk}\right)_{1\leq i,j,k\leq n}$ is symmetric, i.e. $\Gamma^i_{jk} = \Gamma^i_{kj}$. One can also observe that ∇ is symmetric if and only if $\nabla_\xi \eta - \nabla_\eta \xi = [\xi, \eta]$ for any vector fields ξ and η.

The fundamental theorem of Riemannian geometry, due to Levi-Civita, states that on a Riemannian manifold, there is exactly one symmetric metric-compatible connection, called the Levi-Civita connection.

2.4.3.3 Complex connections

On a complex manifold X, we have a splitting

$$T_X^* \otimes_{\mathbb{R}} \mathbb{C} = T_X^{*1,0} \oplus T_X^{*0,1}.$$

It follows that for a complex vector bundle $V \to X$, a connection ∇ splits as

$$\nabla = \nabla^{1,0} + \nabla^{0,1}. \tag{2.4.2}$$

If the vector bundle $V \to X$ is, in addition, holomorphic, then there is a canonical choice for the component $\nabla^{0,1} : \Gamma(X, V) \to \Gamma\left(X, \Lambda^{0,1}(T_X) \otimes V\right)$; namely the $\bar{\partial}$-operator.

Definition 2.4.4. A connection ∇ for a holomorphic vector bundle $V \to X$ is said to be complex if $\nabla^{0,1} = \bar{\partial}$ in terms of the splitting (2.4.2).

We then have an analogue of the Levi-Civita theorem for connections for holomorphic Hermitian vector bundles.

Theorem 2.4.3. *On a holomorphic Hermitian vector bundle $(V \to X, h)$, there exists a unique complex connection compatible with the Hermitian metric.*

We call such a connection the *Chern connection* for $(V \to X, h)$.

2.4.4 Induced connections on twisted forms

2.4.4.1 Symmetric connections and exterior derivatives

Consider a differential 1-form $\alpha = \sum_{1\le i\le n} \alpha_i dx_i$ on a manifold X. For a connection ∇ for $T_X^* \to X$, $\nabla\alpha = \sum_{1\le i,j,k\le n} \dfrac{\partial \alpha_i}{\partial x_j} dx_j \otimes dx_i + \alpha_k \theta_{ij}^k dx_j \otimes dx_i$, where θ is the connection matrix of ∇. The skew-symmetric part is then

$$\Lambda^2(\nabla\alpha) = \sum_{1\le i,j,k\le n} \frac{\partial \alpha_i}{\partial x_j} dx_j \wedge dx_i + \alpha_k \theta_{ij}^k dx_j \wedge dx_i.$$

Recall that ∇ is a symmetric connection if and only if

$$d\alpha = \Lambda^2(\nabla\alpha)$$

for any 1-form α. Therefore, ∇ is symmetric if and only if its connection matrix is symmetric, i.e. $\theta^k_{ij} = \theta^k_{ji}$.

Now suppose that β is a differential r-form; that is a section of the product bundle $\Lambda^r(T^*_X)$. For a connection ∇ for $T^*_X \to X$, the product connection ∇^r acts on $\beta = \sum_{1\leq i_1,\cdots,i_r\leq n} \beta_{i_1\cdots i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}$ by

$$\nabla^r\beta = \sum_{1\leq i_0\leq n} \left(\frac{\partial\beta_{i_1\cdots i_r}}{dx_{i_0}} + \sum_{j=1}^{r} \sum_{1\leq \ell\leq n} \beta_{i_1\cdots(\ell)_j\cdots i_r}\theta^\ell_{i_o i_j} \right) dx_{i_0} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

where $(\ell)_j$ denotes ℓ replacing i_j. Taking the $(r+1)^{\text{st}}$ skew-symmetric part of $\nabla^r\beta$ (thought of as an $(r+1)$-tensor), we obtain

$$\nabla^r\beta = \sum_{1\leq i_0\leq n} \left(\frac{\partial\beta_{i_1\cdots i_r}}{dx_{i_0}} + \sum_{j=1}^{r} \sum_{1\leq \ell\leq n} \beta_{i_1\cdots(\ell)_j\cdots i_r} \frac{\theta^\ell_{i_o i_j} - \theta^\ell_{i_j i_0}}{2} \right) dx_{i_0} \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_r},$$

in view of the skew-symmetry of the $\beta_{i_1\cdots i_r}$. Therefore, ∇ is symmetric if and only if $d = \Lambda^{r+1} \circ \nabla^r$ for any integer r with $1 \leq r \leq n$.

2.4.4.2 Twisted exterior derivative

Let $V \to X$ be a vector bundle of rank r, with connection D. We can define a twisted version of the exterior derivative for sections of $T^*_X \otimes V$ – or V-*valued* 1-*forms*. This twisted exterior derivative should produce a V-valued 2-form. As in the previous paragraph, we fix a connection ∇ for T^*_X. For a V-valued 1-form α, we compute that

$$(\nabla \otimes D)\,\alpha = \sum_{\substack{1\leq i,j,k\leq n\\ 1\leq\mu,\nu\leq r}} \left(\frac{\partial\alpha^\nu_i}{dx_j} + \alpha^\mu_i\omega^\nu_{\mu j} + \alpha^\nu_k\theta^k_{ij} \right) dx_j \otimes dx_i \otimes e_\nu$$

and

$$\Lambda^2\left((\nabla \otimes D)\,\alpha\right) = \sum_{\substack{1\leq i,j,k\leq n\\ 1\leq\mu,\nu\leq r}} \left(\frac{\partial\alpha^\nu_i}{dx_j} + \alpha^\mu_i\omega^\nu_{\mu j} + \alpha^\nu_k\theta^k_{ij} \right) dx_j \wedge dx_i \otimes e_\nu,$$

where $\{e_\nu\}_{1\leq\nu\leq r}$ is a frame for $V \to X$ and ω and θ are the connection matrices for D. Once again, if the connection ∇ for T_X^* is symmetric, then the anti-symmetric part $\Lambda^2\left((\nabla\otimes D)\right)$ is independent of the connection ∇. Similarly, if β is a V-valued s-form, then $\Lambda^{s+1}\left((\nabla^r\otimes D)\,\alpha\right)$ is a V-valued $(s+1)$-form, which is again independent of V as soon as ∇ is symmetric.

Definition 2.4.5. Let $V \to X$ be a vector bundle with connection ∇ and let ∇ be a symmetric connection for $T_X^* \to X$. The operator $\nabla^1 : \Gamma\left(M, \mathcal{C}^\infty\left(T_X^*\otimes V\right)\right) \to \Gamma\left(M, \mathcal{C}^\infty\left(\Lambda^2\left(T_X^*\right)\otimes V\right)\right)$ defined by

$$\nabla^1\alpha := \Lambda^2\left((\nabla\otimes D)\,\alpha\right)$$

(which is independent of ∇) is called the twisted exterior derivative associated to ∇. More generally, let ∇^r be the induced product connection for $\Lambda^r\left(T_X^*\right) \to X$. The operator

$$D^r := \Lambda^{r+1}\circ(\nabla^r\otimes D) : \Gamma\left(M, \mathcal{C}^\infty\left(\Lambda^r\left(T_X^*\right)\otimes V\right)\right) \to \Gamma\left(M, \mathcal{C}^\infty\left(\Lambda^{r+1}\left(T_X^*\right)\otimes V\right)\right)$$

is called the twisted r^{th} exterior derivative (for V-valued r-forms) associated to ∇.

Remark 2.4.4. Let $(e_1, \cdots, e_r)$ be a frame for $V \to X$ and let $(x_1, \cdots, x_n)$ be a local coordinate system on X. Then for a section $\sigma \in \Gamma\left(M, V\otimes\Lambda^r\left(T_X^*\right)\right)$ given locally by $\sigma = \sum_{\substack{1\leq i_1,\cdots,i_r\leq n\\ 1\leq\mu\leq r}} \sigma^\mu_{i_1\cdots i_r} dx_{i_1}\wedge\cdots\wedge dx_{i_r}\otimes e_\mu$, one has (with $\nabla = \nabla^r$)

$$\begin{aligned}
\nabla\sigma &= \sum_{\substack{1\leq i_1,\cdots,i_r,j\leq n\\ 1\leq\mu,\nu\leq r}} \frac{\partial\sigma^\mu_{i_1\cdots i_r}}{dx_j} dx_j\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_r}\otimes e_\mu + \omega^\mu_\nu\wedge\sigma^\nu_{i_1\cdots i_r} dx_{i_1}\wedge\cdots\wedge dx_{i_r}\otimes e_\mu \\
&= \sum_{\substack{1\leq i_1,\cdots,i_r,j\leq n\\ 1\leq\mu,\nu\leq r}} \frac{\partial\sigma^\mu_{i_1\cdots i_r}}{dx_j} dx_j\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_r}\otimes e_\mu + (-1)^r\sigma^\nu_{i_1\cdots i_r} dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge\omega^\mu_\nu\otimes e_\mu.
\end{aligned}$$

Informally, one writes $D\sigma = d\sigma + (-1)^r\sigma\wedge\omega$.

2.4.5 Curvature

Definition 2.4.6. Let $V \to X$ be a vector bundle with connection ∇ and, with respect to some frame, connection matrix A. The curvatures of $(V \to X, \nabla)$ are the operators

$$\Theta^k := \nabla^{k+1}\circ\nabla^k : \Gamma\left(M, \mathcal{C}^\infty\left(\Lambda^k\left(T_X^*\right)\otimes V\right)\right) \to \Gamma\left(M, \mathcal{C}^\infty\left(\Lambda^{k+2}\left(T_X^*\right)\otimes V\right)\right)$$

where ∇^j denotes the twisted exterior derivative associated to the connection ∇.

Observe that if s is a V-valued k-form and f is a function, then (informally)

$$\Theta^k(fs) = \nabla\left(f\nabla(s) + df \wedge s\right) = f\left(\nabla \circ \nabla\right)(s) + df \wedge \nabla s - df \wedge \nabla s = f\Theta^k(s),$$

so that $\Theta^k(s)$ is indeed a V-valued $(k+2)$-form. Morever, we have the following proposition.

Proposition 2.4.5. *There exists an* $\mathrm{End}(V)$*-valued* 2*-form* $\Omega(\nabla)$ *such that*

$$\Theta^k(s) = s \wedge \Omega(\nabla)$$

for any $k \in \{0, \cdots, \mathrm{rank}(V)\}$ *and any* V*-valued* k*-form* s*.*

Proof. Let us work in a local trivialization in which the connection matrix is denoted by A. We then have:

$$\begin{aligned}
\Theta^k(s) = \nabla^{k+1}\left(\nabla^k(s)\right) &= \nabla^{k+1}\left(ds + (-1)^k s \wedge A\right) \\
&= d\left(ds + (-1)^k s \wedge A\right) + (-1)^{k+1}\left(ds + (-1)^k s \wedge A\right) \wedge A \\
&= (-1)^k\left(ds \wedge A + (-1)^k s \wedge A\right) + (-1)^k ds \wedge A - (-1)^k s \wedge A \wedge A \\
&= s \wedge \left(dA - A \wedge A\right).
\end{aligned}$$

Therefore, the k-independent local endomorphism $s \mapsto s \wedge (dA - A \wedge A)$ agrees with $\nabla \circ \nabla$ and since $\nabla \circ \nabla$ is globally defined, the proof is complete, with $\Omega(\nabla) := \nabla \circ \nabla$. □

2.4.5.1 Curvature of the Chern connection

Fix a holomorphic Hermitian vector bundle $(V \to X, h)$ of rank r. Observe that since $\nabla = \nabla^{1,0} + \bar{\partial}$ and $\bar{\partial}^2 = 0$, it follows that

$$\nabla^1 \circ \nabla = \nabla^{1,0}_1 \circ \nabla^{1,0} + \nabla^{1,0}_1 \circ \bar{\partial} + \bar{\partial}_1 \circ \nabla^{1,0}.$$

By metric compatibility,

$$\partial h(s,t) = h\left(\nabla^{1,0}s, t\right) + h\left(s, \bar{\partial}t\right) \text{ and } \bar{\partial}h(s,t) = h\left(\bar{\partial}s, t\right) + h\left(s, \nabla^{1,0}t\right),$$

and since $\partial^2 = 0$,

$$0 = \partial^2 h(s,t) = h\left(\nabla^{1,0} \circ \nabla_1^{1,0} s, t\right) + h\left(\nabla^{1,0} s, \bar{\partial} t\right) - h\left(\nabla^{1,0} s, \bar{\partial} t\right) + h(s, \bar{\partial}_1 \bar{\partial} t) = h\left(\nabla_1^{1,0} \circ \nabla^{1,0} s, t\right),$$

and thus $\nabla_1^{1,0} \circ \nabla^{1,0} = 0$. In conclusion, the curvature form Θ of the Chern connection satisfies $\Theta = \nabla_1^{1,0} \circ \bar{\partial} + \bar{\partial}_1 \circ \nabla^{1,0}$. In particular, Θ maps sections to twisted $(1,1)$-forms, and it is thus a twisted $(1,1)$-form.

Proposition 2.4.6. *The curvature of the Chern connection of $(V \to X, h)$ is given by the formula*

$$\Omega_\beta^\alpha = \bar{\partial}\left(\sum_{1 \le \mu \le r} h^{\alpha\bar{\mu}} \partial h_{\beta\bar{\mu}}\right).$$

Proof. Let $h_{\alpha\bar{\beta}} = h(e_\alpha, e_\beta)$ for a holomorphic frame $\{e_\alpha\}_{1 \le \alpha \le r}$ for $V \to X$. Recall that the connection matrix A is given by

$$A_\beta^\alpha = \sum_{1 \le \mu \le r} h^{\alpha\bar{\mu}} \partial h_{\beta\bar{\mu}}.$$

In matrix notation, we have $A = (\partial H)H^{-1}$ with H representing the Hermitian metric. Therefore, using the formulas $\partial\left(H^{-1}\right) = -H^{-1}(\partial H)H^{-1}$ and $\bar{\partial}\left(H^{-1}\right) = -H^{-1}(\bar{\partial} H)H^{-1}$, we calculate that

$$\begin{aligned} dA - A \wedge A &= \left(\partial + \bar{\partial}\right)\left((\partial H)H^{-1}\right) - (\partial H)H^{-1} \wedge (\partial H)H^{-1} \\ &= \bar{\partial}\left((\partial H)H^{-1}\right) + (\partial H)H^{-1}(\partial H)H^{-1} - (\partial H)H^{-1}(\partial H)H^{-1} = \bar{\partial}\left((\partial H)H^{-1}\right). \end{aligned}$$

□

2.4.5.2 Curvature of a line bundle

Let $L \to X$ be a complex line bundle. If ∇ is any connection for $L \to X$ then its curvature is a section of $\mathrm{End}(L) \otimes \Lambda^2(T_X^*) \to X$. Since the line bundle $\mathrm{End}(L) \to X$ is canonically trivial, the curvature of a line bundle is a well-defined 2-form on X. Since the fibers are 1-dimensional, $A(\nabla) \wedge A(\nabla) = 0$ for any local connection form $A(\nabla)$ and so the curvature of ∇ is $d(A(\nabla))$. In particular, the forms yields a globally defined 2-form on X via the isomorphism between $\mathrm{End}(L)$ and the trivial bundle.

Suppose now that X is a complex manifold and the line bundle $L \to X$ is holomorphic. Let h be a Hermitian metric for $L \to X$. If ξ is a holomorphic frame for $L \to X$ over an open set $U \subset M$, then one can define the function

$$\varphi^{(\xi)} := -\log\left(h(\xi, \xi)\right).$$

The curvature of the Chern connection of h is

$$\Theta(h) := \left(\partial\bar{\partial}\varphi^{(\xi)}\right) \otimes \xi \otimes \xi^*,$$

where ξ^* is the frame for $L^* \to X$ dual to ξ. Since $\xi \otimes \xi^*$ is nowhere-zero, one can define the curvature of the holomorphic line bundle to be $\partial\bar{\partial}\varphi^{(\xi)}$. The right hand side of the latter equality is independent of the choice of holomorphic frame. Indeed, given another holomorphic frame $\tilde{\xi}$, it follows that $\tilde{\xi} = f\xi$ for some nowhere zero holomorphic function f and so

$$\varphi^{(\tilde{\xi})} = -\log\left(|f|^2 h(\xi, \xi)\right) = \varphi^{(\xi)} - \log\left(|f|^2\right),$$

whence $\partial\bar{\partial}\varphi^{(\xi)} = \partial\bar{\partial}\varphi^{(\tilde{\xi})}$ since $\log(|f|^2)$ is pluriharmonic due to the holomorphicity and non-vanishing of f. Clearly, $\xi \otimes \xi^* = \tilde{\xi} \otimes \tilde{\xi}^*$.

Consequently, we can use the following global notation for Hermitian metrics of holomorphic line bundles: a metric for a holomorphic line bundle will typically be denoted $e^{-\varphi}$, and its curvature will be denoted by $\partial\bar{\partial}\varphi$.

2.4.6 Curvature of determinant bundles

Proposition 2.4.7. *Let $V \to X$ be a vector bundle of rank r with connection ∇^V and let $\det(V) \to X$ be its determinant line bundle with connection $\nabla^{\det(V)}$. Then*

$$\Omega\left(\nabla^{\det V}\right) = \mathrm{tr}\left(\Omega\left(\nabla^V\right)\right).$$

Proof. Let $e_1, \cdots, e_r$ be a frame for $V \to X$. Then:

$$\begin{aligned}
\left(\nabla^{\det(V)}\right)^2 (e_1 \wedge \cdots \wedge e_r) &= \nabla^{\det(V)} \left(\sum_{j=1}^{r} e_1 \wedge \cdots \wedge \nabla^V(e_j) \wedge \cdots \wedge e_r \right) \\
&= \sum_{j=2}^{r} \sum_{k=1}^{j-1} e_1 \wedge \cdots \wedge \nabla^V(e_k) \wedge \cdots \wedge \nabla^V(e_j) \wedge \cdots \wedge e_r \\
&\quad - \sum_{j=1}^{r-1} \sum_{k=j+1}^{j} e_1 \wedge \cdots \wedge \nabla^V(e_k) \wedge \cdots \wedge \nabla^V(e_k) \wedge \cdots \wedge e_r \\
&\quad + \sum_{j=1}^{r} e_1 \wedge \cdots \wedge \left(\nabla^V\right)^2 (e_j) \wedge \cdots \wedge e_r \\
&= \sum_{j=1}^{r} e_1 \wedge \cdots \wedge \left(\nabla^V\right)^2 (e_j) \wedge \cdots \wedge e_r,
\end{aligned}$$

and so $\Omega\left(\nabla^{\det(V)}\right) = \operatorname{tr}\left(\Omega\left(\nabla^V\right)\right)$ as claimed. □

2.4.6.1 The canonical bundle

Recall that the *canonical bundle* K_X of a complex n-dimensional manifold X is the line bundle $\det\left(T_X^{*1,0}\right)$ whose local sections are $(n,0)$-forms.

If g is a (Riemannian) Hermitian metric on X, Proposition 2.4.7 tells us that the curvature of the Chern connection for $(K_X \to X, \det(g))$ is just the trace of the curvature of $\left(T_X^{*1,0} \to X, g\right)$.

For a general (Riemannian) Hermitian metric g, the latter Chern curvature is unrelated to the curvature of the Levi-Civita connection for g. However, if the metric g is Kähler, the curvature of the Chern connection for $(K_X, \det(g))$ is the negative of the so-called *Ricci curvature* of g:

$$\mathbf{Ric}(g) = -\operatorname{tr}\left(\Omega(g)\right). \tag{2.4.3}$$

In components,

$$\mathbf{Ric}(g)_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \left(\log\left(\det(g_{\mu\bar{\nu}})\right)\right). \tag{2.4.4}$$

2.4.6.2 Curvature of direct sums of vector bundles

As previously seen, given connections ∇^V and ∇^W on vector bundles $V \to X$ and $W \to X$ over a complex manifold X, there is a natural direct sum connection $\nabla^V \oplus \nabla^W$ on $V \oplus W \to X$. The curvature $\Theta^{V\oplus W}$ of $V \oplus W \to X$ satisfies

$$\Theta^{V\oplus W} = \Theta^V \oplus \mathrm{Id}_W + \mathrm{Id}_V \oplus \Theta^W.$$

Additionally, if Ω^V and Ω^W are the curvature forms of $V \to X$ and $W \to X$, then the curvature $\Theta^{V\oplus W}$ form $\Omega^{V\oplus W}$ of $\nabla^V \oplus \nabla^W$ is the direct sum matrix of Ω^V and Ω^W:

$$\Omega^{V\oplus W} = \begin{pmatrix} \Omega^V & 0 \\ 0 & \Omega^W \end{pmatrix}.$$

For further details, we refer the reader to [Dem12, Chapter V, §4].

2.4.7 Curvature positivity of vector bundles

In complex geometry, there are various notions of "positivity" for the curvature of the Chern connection for a Hermitian metric on a holomorphic vector bundle. Indeed, because the curvature $\Theta(h)$ of the Chern connection of a metric h for a holomorphic vector bundle $V \to X$ is a $(1,1)$-form with values in $\mathrm{Hom}(V,V) \to X$, there are many ways to measure its positivity. The strongest notion of positivity is called *Nakano positivity*, and the weakest notion is called *Griffiths positivity.*

Using the metric h, one defines Hermitian forms $\{\cdot,\cdot\}_{h,\Theta(h)}$ on the fibers of $V \otimes T^{1,0}_X$ by letting

$$\{v \otimes \xi, w \otimes \eta\}_{h,\Theta(h)} := h\left(\Theta(h)_{\xi,\bar{\eta}}(v), w\right) \tag{2.4.5}$$

for *indecomposable tensors* – i.e. tensors of the form $v \otimes \xi$ – on a given fiber $V_x \otimes T^{1,0}_{X,x}$ and extending bilinearly to the entire fiber.

Note that given Hermitian metrics for h_V and h_W for the holomorphic vector bundles $V \to X$ and $W \to X$ respectively, $h_{V\oplus W} := h_V \oplus h_W$ is a Hermitian metric for $V \oplus W \to X$. Let Θ^V and Θ^W denote the Chern connections for $(V \to X, h_V)$ and $(W \to X, h_W)$. If we denote by θ_V the Hermitian form $\{\cdot,\cdot\}_{h_V,\Theta^V(h_V)}$ and similarly for W and $V \oplus W$, then

$$\theta_{V\oplus W} = \theta_V \oplus h_W + h_V \oplus \theta_W.$$

2.4.7.1 Notions of positivity

Definition 2.4.7. Let $V \to X$ be a holomorphic vector bundle with smooth Hermitian metric h and fix a smooth (Riemannian) Hermitian metric g on X.

1. We say that h has positive curvature in the sense of Griffiths at a point $x \in X$ if there exists $c > 0$ such that

$$\{v \otimes \xi, v \otimes \xi\}_{h,\Theta(h)} \geq c \cdot h(v,v)g(\xi,\xi)$$

   for all $v \otimes \xi \in V_x \otimes T^{1,0}_{X,x}$.

2. We say that h has positive curvature in the sense of Nakano at a point $x \in X$ if there exists $c > 0$ such that

$$\left\{\sum_{j=1}^n v_j \otimes \xi_j, \sum_{k=1}^n v_k \otimes \xi_k\right\}_{h,\Theta(h)} \geq c \sum_{j=1}^n h(v_j,v_j)g(\xi_j,\xi_j)$$

   for all $v_1 \otimes \xi_1, \cdots, v_n \otimes \xi_n \in V_x \otimes T^{1,0}_{X,x}$ where $n = \min(\dim_{\mathbb{C}}(X), \mathrm{rank}(V))$.

We define non-negative curvature by taking $c = 0$, and we define negative and non-positive curvature by simply changing the sign of c and reversing the inequalities.

More generally, given a Hermitian form θ on $V \otimes T^{1,0}_X$, we will say that θ is Griffiths semipositive (resp. positive) at a point $x \in X$ if $\theta(v \otimes \xi) \geq 0$ for all $v \otimes \xi \in V_x \otimes T^{1,0}_{X,x}$. If θ is Griffiths semipositive (resp. positive) at every $x \in X$, we write $\theta \geq_{\mathrm{Griff}} 0$ (resp. $>_{\mathrm{Griff}} 0$).

Similarly, we will say that θ is Nakano semipositive (resp. positive) at a point $x \in X$ if $\theta\left(\sum_{i=1}^{n} v_i \otimes \xi_i\right) \geq 0$ (resp. > 0) for all $v_i \otimes \xi \in V_x \otimes T^{1,0}_{X,x}$, where $i = 1, \cdots, n$ where $n = \min\left(\dim_{\mathbb{C}}, \operatorname{rank}(V)\right)$. If θ is Nakano positive (resp. positive) at every $x \in X$, we write $\theta \geq_{\mathrm{Nak}} 0$ (resp. $>_{\mathrm{Nak}} 0$).

For two Hermitian forms θ_1 and θ_2 on $V \otimes T^{1,0}_X$, we will write $\theta_1 \geq_{\mathrm{Nak}} \theta_2$ (resp. $\theta_1 >_{\mathrm{Nak}} \theta_2$) if $\theta_1 - \theta_2 \geq_{\mathrm{Nak}} 0$ (resp. $\theta_1 - \theta_2 >_{\mathrm{Nak}} 0$), and similarly for $\theta_1 >_{\mathrm{Griff}} \theta_2$ (resp. $\theta_1 \geq_{\mathrm{Griff}} \theta_2$).

2.4.7.2 Duality

The Hermitian holomorphic vector bundle $(V \to X, h)$ is Griffiths positive if and only if its dual $(V^* \to X, h^*)$ is Griffiths negative, but this relationship between positivity and duality is no longer true in the case of Nakano positivity. We refer the reader to [Dem12, (6.8) Example] for a counter-example showing that the Nakano positivity or negativity of a given bundle and its dual are unrelated.

2.4.7.3 Positivity of line bundles

When $V \to X$ is of rank 1 – i.e. when it is a line bundle – Griffiths positivity and Nakano positivity coincide since any map $V_x \otimes T^{1,0}_{X,x} \to V_x \otimes T^{1,0}_{X,x}$ has rank at most 1 due to the fibers V_x being 1-dimensional. In this case, the term "positivity" has a unique meaning and one speaks of positivity of the curvature. As previously seen, the curvature of a Hermitian metric $e^{-\varphi}$ then is

$$\partial\bar{\partial}\varphi = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j,$$

where $n = \dim_{\mathbb{C}}(X)$. Hence the curvature of $e^{-\varphi}$ is (semi)positive if and only if, in any holomorphic coordinate system, the Hermitian matrix

$$H_{\mathbb{C}}(\varphi) := \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right)_{1 \leq i,j \leq n}$$

is positive (semi)definite. In other words, the potential φ of the local representative of the metric is plurisubharmonic (or strictly so, in the case of strict curvature positivity).

2.4.7.4 Analytic characterizations of curvature positivity and negativity

The following proposition will be very useful in the study of the positivity of holomorphic Hilbert bundles.

Proposition 2.4.8. *The metric h for $V \to X$ is non-positive (resp. negative) in the sense of Griffiths if and only if for any holomorphic section s of $V \to X$, the function $X \ni x \mapsto |s(x)|_h^2 := h(s(x), s(x))$ is a plurisubharmonic (resp. strictly plurisubharmonic) function on X.*

Proof. For any holomorphic section s of $V \to X$, we have $\partial h(s,s) = h\left(\nabla^{1,0}(s), s\right)$ since $\nabla^{0,1}(s) = \bar{\partial} s = 0$. Therefore,

$$\begin{aligned}
\partial\bar{\partial} h(s,s) = -\bar{\partial}\partial h(s,s) &= -h\left(\nabla^{0,1} \circ \nabla^{0,1} s, s\right) + h\left(\nabla^{1,0} s, \nabla^{V,(1,0)} s\right) \\
&= -h\left(\Theta^V(s), s\right) + h\left(\nabla^{1,0} s, \nabla^{1,0} s\right) \\
&= -\left(\Theta(s), s\right)_h + \left|\nabla^{1,0} s\right|_h^2 .
\end{aligned}$$

The second summand is clearly nonnegative, so we see that if h is nonpositive (resp. positive) in the sense of Griffiths, then $\partial\bar{\partial}\,|s|_h^2$ is non-negative (resp. positive).

To see the converse, it is enough to work locally since plurisubharmonicity is a local property. We thus assume that the vector bundle $V \to X$ is trivial, but with non-trivial metric. Under the condition of triviality, given any vector $v \in V_x$, there exists a holomorphic section s_v of $V \to X$ such that $s_v(x) = v$ and $\nabla^{1,0} s_v(x) = 0$. Indeed, if $v \in V_x$, the holomorphic section we seek must have the form $s(w) = v + \sum_{k=1}^{\mathrm{rank}(V)} a_k(w_k - x_k)$ if we think of w as local coordinates near x. Then $\nabla^{1,0} s = Av + \sum_{k=1}^{\mathrm{rank}(V)} a_k dw_k$ at x, where A is the connection matrix of ∇ – the Chern connection of $(V \to X, h)$. Choosing $a_k = -A_k v$, where $A = \sum_{k=1}^{\mathrm{rank}(V)} A_k dw_k$, we obtain the desired holomorphic section. Plugging s_v into the previous formula for $\partial\bar{\partial}h(s,s)$ yields

$$\partial\bar{\partial}h(s,s) = -\left(\Theta(s), s\right)_h,$$

which shows that if $x \mapsto |s(x)|_h^2$ is plurisubharmonic, then h is non-positive in the sense of Griffiths. □

We also have the following proposition.

Proposition 2.4.9. *The metric h for $V \to X$ is non-positive (resp. negative) in the sense of Griffiths if and only if for any holomorphic section s of $V \to X$, the function $X \ni x \mapsto \log\left(h(s(x), s(x))\right) = \log\left(|s(x)|_h^2\right)$ is a plurisubharmonic (resp. strictly plurisubharmonic) function on X or identically $-\infty$.*

This can be proved by explicitly computing $\partial\bar{\partial}\log\left(|s|_h^2\right)$ for a non-zero holomorphic section s such that $\nabla^{0,1}s = 0$ at a given point p, which then shows that

$$\partial\bar{\partial}\log\left(|s(p)|_h^2\right) = -\frac{h\left(\Theta(s(p)), s(p)\right)}{|s(p)|_h^2}.$$

Remark 2.4.10. This result can also be obtained by simply observing that a positive function f is plurisubharmonic if and only if $\log(f)$ is plurisubharmonic. See [DAn01, Proposition 2.2].

More generally, if X is an n-dimensional complex manifold and $V \to X$ is a holomorphic vector bundle of rank r with smooth Hermitian metric h, we can formulate a similar test for pointwise Nakano positivity. Set $k = \min(r, n)$ and write

$$\widehat{dz_i \wedge d\bar{z}_j} = c_n dz_1 \wedge \cdots \wedge dz_{i-1} \wedge dz_{i+1} \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{j-1} \wedge d\bar{z}_{j+1} \wedge d\bar{z}_n$$

for a local coordinate z. Here c_n is a unimodular constant chosen so that $\widehat{dz_i \wedge d\bar{z}_j}$ is a positive form. We then have the following proposition. This criterion is primarily due to Berndtsson [Ber09b].

Proposition 2.4.11. *The metric h for $V \to X$ is positively curved in the sense of Nakano at $p \in X$ if and only if, for every local coordinate system z at p, and every tuple of holomorphic* *sections* $(f_1, \cdots, f_k)$ *such that* $\nabla^{1,0} f_i = 0$ *at* p *for each* $i = 1, \cdots, k$*, the* (n,n)*-form*

$$\partial\bar{\partial}\left(\sum_{i,j=1}^{k} h\left(f_i, f_j\right) \widehat{dz_i \wedge d\bar{z}_j}\right)$$

is a negative multiple of the Lebesgue measure $dV(z)$ *near* p*.*

Another similar criterion for Nakano negativity, primarily due to [Rau15], is the following.

Proposition 2.4.12. *The metric* h *for* $V \to X$ *is negatively curved in the sense of Nakano at* $p \in X$ *if and only if, for every local coordinate system* z *at* p*, and every tuple of holomorphic sections* $(f_1, \cdots, f_k)$*, the* (n,n)*-form*

$$\bar{\partial}\partial\left(\sum_{i,j=1}^{k} h\left(f_i, f_j\right) \widehat{dz_i \wedge d\bar{z}_j}\right)$$

is a negative multiple of the Lebesgue measure $dV(z)$ *near* p*.*

2.4.8 Subbundles and Griffiths' curvature formula

Let $(V \to X, h)$ be a holomorphic Hermitian vector bundle over a complex manifold X, and let $W \to X$ be a holomorphic subbundle of $V \to X$. Then $W \to X$ is also Hermitian with the metric induced from h, and thus admits a Chern connection. Note that the Hermitian metric h gives us a fiberwise orthogonal projection map $\mathcal{P}_z : V_z \to W_z$ for each point z in the base. Although $\mathcal{P}_z$ is not holomorphic as z varies, these maps together generate a smooth bundle map from V to W. Similarly, we let $\mathcal{P}_z^{\perp}$ be the orthogonal projection on $W_z^{\perp}$, the orthogonal complement of W_z in V_z. The sum of $\mathcal{P}_z$ and $\mathcal{P}_z^{\perp}$ is the identity map.

Proposition 2.4.13. *Let* ∇^V *and* ∇^W *be the Chern connections of* $V \to X$ *and* $W \to X$ *respectively. Then,*

1. $\nabla^W = \mathcal{P} \circ \nabla^V$*, and*
2. *The map* $s \mapsto \mathfrak{p}(s) := \left(\mathcal{P}^{\perp} \circ \nabla^V\right)(s)$ *satisfies* $\mathfrak{p}(fs) = f\mathfrak{p}(s)$ *if* f *is a smooth function and* s *is a smooth section of* $W \to X$*. Hence,* $\mathfrak{p}(s) = 0$ *at* z *if* $s = 0$ *at* z*. So* $\mathfrak{p}$ *defines a linear map from* W *to* $W^{\perp} \otimes T_X^*$*.*

Proof. 1. Since $\nabla^V(fs) = df \otimes s + f\nabla^V(s)$, it follows that $\mathcal{P} \circ \nabla^V$ satisfies the same property if s is a section of $W \to X$. Therefore, $\mathcal{P} \circ \nabla^V$ is a connection. Moreover, if s is holomorphic, the fact $\nabla^V(s)$ is of bidegree $(1,0)$ implies that $\nabla^W(s)$ is also of bidegree $(1,0)$. Therefore, the connection $\mathcal{P} \circ \nabla^V$ is complex. Finally, if s_1 and s_2 are two holomorphic sections of $W \to X$, then

$$dh(s_1, s_2) = h\left(\nabla^V(s_1), s_2\right) + \left(s_1, \nabla^V(s_2)\right) = h\left((\mathcal{P} \circ \nabla^V)(s_1), s_2\right) + h\left(s_1, (\mathcal{P} \circ \nabla^V)(s_2)\right).$$

Therefore, $\mathcal{P} \circ \nabla^V$ is a complex connection that is compatible with h, and must thus be ∇^W by the uniqueness of the Chern connection.

2. $\mathfrak{p}(fs) = \mathcal{P}^\perp\left(df \otimes s + f\nabla^V(s)\right) = f\mathfrak{p}(s)$ if s is a section of $W \to X$.

□

Theorem 2.4.14. *If $W \to X$ is a holomorphic subbundle of a Hermitian holomorphic vector bundle $(V \to X, h)$, then*

$$\Theta^W = \Theta^V - \mathfrak{p}^*\mathfrak{p},$$

where Θ^W and Θ^V are the curvatures of $W \to X$ and $V \to X$ respectively, and $\mathfrak{p}^*$ is the dual of $\mathfrak{p}$ with respect to h.

Proof. For any holomorphic section s of $V \to X$, we have $\partial h(s,s) = h\left(\nabla^{V,(1,0)}(s), s\right)$ since $\nabla^{V,(0,1)}(s) = \bar{\partial} s = 0$. Therefore,

$$\begin{aligned}\partial\bar{\partial} h(s,s) = -\bar{\partial}\partial h(s,s) &= -h\left(\nabla^{V,(0,1)} \circ \nabla^{V,(0,1)} s, s\right) + h\left(\nabla^{V,(1,0)} s, \nabla^{V,(1,0)} s\right) \\ &= -h\left(\Theta^V(s), s\right) + h\left(\nabla^{V,(1,0)} s, \nabla^{V,(1,0)} s\right).\end{aligned}$$

Therefore,

$$h\left(\Theta^W(s), s\right) - h\left(\Theta^V(s), s\right) = h\left(\nabla^{W,(1,0)} s, \nabla^{W,(1,0)} s\right) - h\left(\nabla^{V,(1,0)} s, \nabla^{V,(1,0)} s\right),$$

for any holomorphic section s of $W \to X$. Therefore, by Proposition 2.4.13, the fact that $\mathcal{P} + \mathcal{P}^{\perp} = \mathrm{Id}$, and by orthogonality,

$$
\begin{aligned}
&h\left(\nabla^{W,(1,0)} s, \nabla^{W,(1,0)} s\right) - h\left(\nabla^{V,(1,0)} s, \nabla^{V,(1,0)} s\right) \\
&= h\left(\mathcal{P} \circ \nabla^{V}(s), \mathcal{P} \circ \nabla^{V}(s)\right) - h\left(\mathcal{P} \circ \nabla^{V}(s), \mathcal{P} \circ \nabla^{V}(s)\right) - h\left(\mathcal{P}^{\perp} \circ \nabla^{V}(s), \mathcal{P}^{\perp} \circ \nabla^{V}(s)\right) \\
&= -h\left(\mathfrak{p}(s), \mathfrak{p}(s)\right)
\end{aligned}
$$

□

The map $\mathfrak{p}$ is called the *second fundamental form* of $W \to X$ in $V \to X$. In the complex case, the quadratic form $-h\left(\mathfrak{p}(s), \mathfrak{p}(s)\right)$ is always a non-positive $(1,1)$-form. If the curvature of $V \to X$ vanishes identically, then the curvature of $W \to X$ is completely determined by the second fundamental form.

Chapter 3

Bergman spaces and their kernels

Let X be a complex manifold with Borel measure μ and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h. Consider the *Bergman space* which is defined as the subspace

$$\mathcal{H}^2_{0,0}(\mu, h) := L^2_{0,0}(\mu, h) \cap \Gamma_{\mathcal{O}}(X, V)$$

of square-integrable holomorphic sections, as a subspace of the space $L^2_{0,0}(\mu, h)$ of (measurable) square-integrable sections. The subspace $\mathcal{H}^2_{0,0}(\mu, h) \subset L^2_{0,0}(\mu, h)$ is closed for many natural choices of the measure μ and the metric h, and therefore, there is an orthogonal projection – i.e. a bounded linear self-adjoint projection operator $P : L^2_{0,0}(\mu, h) \to \mathcal{H}^2_{0,0}(\mu, h)$; the so-called *Bergman projection*. Its Schwarz kernel, called the *Bergman kernel*, possesses many properties and under certain positivity conditions, carries a great amount of information about the data defining it. One can also build the corresponding theory for the Hilbert spaces $L^2_{p,q}(\mu, h)$. In this case, the space $\mathcal{H}^2_{0,0}(\mu, h)$ is replaced by the subspace $\mathcal{H}^2_{p,q}(\mu, h)$ of $\bar{\partial}$-closed V-valued (p, q)-forms in $L^2_{p,q}(\mu, h)$.

The contents of this chapter are largely adapted from the course notes for the topics course MAT 670 – Topics in Complex Analysis: Variation of Bergman Spaces, as taught by Prof. Dror Varolin at Stony Brook University, Fall 2020. These notes are not publicly available, unfortunately, and will be published as a separate manuscript in the future. That said, similar contents on the complex analysis and geometry of Bergman kernels can be found in [Var19, Part III, Lecture 11], [MM07], [Her18, Chapter 3, §3.3], [Ass16, §2.2], [BDS20, §2], [Ber03], [Ohs18, Chapter 4], [Pas94], [BH98], [Kob13, §3.2], [BG15], and [BG16].

3.1 Orthogonal projections in Hilbert spaces

As the notion of an orthogonal projection is at the foundation of the theory of Bergman spaces, we provide here a systematic treatment of orthogonal projections on Hilbert spaces. For a more general treatment of kernels on topological vector spaces, we refer the reader to [Tre06].

Definition 3.1.1. An orthogonal projection on a Hilbert space H is a bounded linear self-adjoint projection – i.e. a bounded linear map $P : H \to H$ satisfying $P^\dagger = P = P \circ P$.

To each orthogonal projection P, one can assign a closed subspace $P(H)$. So there is a map

$$\Pi_O(H) \ni P \mapsto P(H) \in \mathfrak{C}(H)$$

from the set $\Pi_O(H)$ of orthogonal projections on H to $\mathfrak{C}(H)$ of closed subspaces of H. The Fundamental Theorem of Orthogonal Projections states that this map is a bijection.

Theorem 3.1.1. (Fundamental Theorem of Orthogonal Projections) *The map*

$$\Pi_O(H) \ni P \mapsto P(H) \in \mathfrak{C}(H)$$

is a one-to-one correspondence.

Lemma 3.1.2. *Let $V \in \mathfrak{C}(H)$ be a closed subspace. For each $x \in H$, there exists at most one element $y \in V$ such that $x - y \perp V$.*

Proof. Suppose that y_1 and y_2, in V, both have the property that their difference from x is orthogonal to V. Then by the Pythagorean identity and rewriting $y_1 - x = y_1 - y_2 + y_2 - x$, it follows that

$$\|y_1 - y_2\|^2 = \|y_1 - x\|^2 - \|y_2 - x\|^2 = -\left(\|y_2 - x\|^2 - \|x - y\|^2\right)^2 = -\|y_2 - y_1\|^2,$$

hence $\|y_2 - y_1\| = 0$ so that $y_1 = y_2$. $\square$

Proposition 3.1.3. *Let $V \in \mathfrak{C}(H)$ be a closed subspace. For each $x \in H$, there exists a unique element $P_V(x) \in H$ such that*

$$\forall v \in X : \|P_V(x) - x\| \leq \|v - x\| \tag{3.1.1}$$

and

$$P_V \in \Pi_O(H).$$

Proof. Fix $x \in H$. Consider the squared norm function $N_x : V \to \mathbb{R}_{\geq 0}, v \mapsto \|x - v\|^2$. The infimum of N_x exists since N_x is bounded below. Moreover, since V is closed, N_x has a minimum. Let $x_0 \in V$ be the minimizer of N_x. Then for any $v \in V$, the function

$$[0, \infty) \ni \varepsilon \mapsto f(\varepsilon) := \|x_0 + \varepsilon v - x\|^2 = \|x_0 - x\|^2 + 2\varepsilon \mathrm{Re}(v, x_0 - x) + \varepsilon^2 \|v\|^2$$

has a critical point at $\varepsilon = 0$. So $\mathrm{Re}(v, x_0 - x) = 0$. Replacing v by $\sqrt{-1}v$ show that $\mathrm{Im}(v, x_0 - x) = \mathrm{Re}(\sqrt{-1}v, x_0 - x) = 0$, and so $x_0 - x \perp V$. Hence, by Lemma 3.1, the minimizer N_x is unique – i.e. there exists a unique element $P_V(x)$ such that

$$\forall v \in X : \|P_V(x) - x\| \leq \|v - x\|.$$

It remains to show that $P_V \in \Pi_O(H)$. If $v \in V$, the clearly $P_V(v) = v$, and so $P_V P_V = P_V$ so that P_V is a projection. Furthermore, by the Pythagorean identity,

$$\|P_V(x)\|^2 \leq \|P_V(x)\|^2 + \|x - P_V(x)\|^2 = \|x\|^2$$

and so P_V is a bounded operator. Finally, for $v \in V$,

$$\left(x - P_V^\dagger(x), v\right) = (x, v) - \left(P_V^\dagger(x), v\right) = (x, v) - (x, P_V(v)) = 0,$$

so that $x - P_V^\dagger(x) \perp V$; showing that $P_V^\dagger(x) = P_V(x)$ in view of Lemma 3.1. Therefore, P_V is self-adjoint, which completes the proof. $\square$

Proposition 3.1.4. *If $P \in \Pi_O(H)$, then $P = P_{P(H)}$.*

Proof. If $v \in P(H)$, then

$$(v, x - P(x)) = (v, x) - (v, P(x)) = (v, x) - \left(P^\dagger(v), x\right) = (v, x) - (P(v), x) = 0$$

so that $P(x) - x \perp P(H)$, and hence $P = P_{P(H)}$. □

Propositions 3.1.3 and 3.1.4 show that the map $\Pi_O(H) \ni P \mapsto P(H) \in \mathfrak{C}(H)$ is surjective and injective, respectively, thereby proving Theorem 3.1.1.

The kernels we will consider take values in a certain Hilbert space completion of a tensor product of Hilbert spaces. Let us define this tensor product. Let H_1 and H_2 be separable Hilbert spaces. The space $\widehat{H_1 \otimes H_2}$ denotes the Hilbert space completion of the tensor product $H_1 \otimes H_2$ with respect to the norm defined by the orthonormal Riesz basis $\{v_i \otimes w_j\}_{i,j \in \mathbb{N}}$ where $\{v_i\}_{i \in \mathbb{N}}$ and $\{w_j\}_{j \in \mathbb{N}}$ are orthonormal Riesz bases for H_1 and H_2 respectively.

Theorem 3.1.5. *Let H be a separable Hilbert space and let $P \in \Pi_O(H)$ be an orthogonal projection. For any orthonormal Riesz basis $\{v_\alpha\}_{\alpha \in \mathbb{N}}$ of the closed subspace $P(H)$, the sequence of partial sums*

$$K_P^{(N)} := \sum_{\alpha=1}^{N} v_\alpha \otimes \bar{v}_\alpha$$

is weakly convergent on $H \otimes \bar{H}$, and its weak limit K_P along indecomposable tensors is given by

$$(K_P, x \otimes \bar{y})_{H \otimes \bar{H}} = (Px, y)_H.$$

Proof. Let $\{w_\beta\}_{\beta \in \mathbb{N}}$ be an orthonormal Riesz basis for $P(H)^\perp$. Then $\{v_\alpha, w_\beta\}_{\alpha,\beta \in \mathbb{N}}$ is an orthonormal Riesz basis for H. Given $x, y \in H$, one may write $x = x_0^{(\alpha)} v_\alpha + x_\perp^{(\beta)} w_\beta$ and $y = y_0^{(\alpha)} v_a + y_\perp^{(\beta)} w_\beta$. Since $x - \sum_{\alpha \in \mathbb{N}} x_0^{(\alpha)} v_\alpha \in P(H)^\perp$ for all $x \in H$, $P(x) = \sum_{\alpha \in \mathbb{N}} x_0^{(\alpha)} v_\alpha$, and so

$$\left(K_P^{(N)}, x \otimes \bar{x}\right)_{H \otimes \bar{H}} = \sum_{\alpha=1}^{N} \left| x_0^{(\alpha)} \right|^2 \leq \|P(x)\|_h^2,$$

which shows that $\left(K_P^{(N)}, x \otimes \bar{x}\right)_{H\otimes\bar{H}}$ is a bounded increasing – hence convergent – sequence whose limit is clearly $\|P(x)\|_H^2$. Since

$$(x-y)\otimes(\bar{x}-\bar{y}) = x\otimes\bar{x} + y\otimes\bar{y} - x\otimes\bar{y} - y\otimes\bar{x},$$

we see that

$$\begin{aligned}\operatorname{Re}\left(K_P^{(N)}, x\otimes\bar{y}\right)_{H\otimes\bar{H}} &= \frac{1}{2}\left[\left(K_P^{(N)}, (x-y)\otimes(\bar{x}-\bar{y})\right)_{H\otimes\bar{H}}\right] \\ &\quad - \frac{1}{2}\left[\left(K_P^{(N)}, x\otimes\bar{x}\right)_{H\otimes\bar{H}} - \left(K_P^{(N)}, y\otimes\bar{y}\right)_{H\otimes\bar{H}}\right]\end{aligned}$$

converges, as does

$$\operatorname{Im}\left(K_P^{(N)}, x\otimes\bar{y}\right)_{H\otimes H^\dagger} = \operatorname{Re}\left(K_P^{(N)}, \sqrt{-1}x\otimes\bar{y}\right)_{H\otimes\bar{H}},$$

and hence $\left(K_P^{(N)}, x\otimes\bar{y}\right)_{H\otimes H^\dagger}$ converges as well. Clearly,

$$(K_P, x\otimes\bar{y})_{H\otimes H^\dagger} = \frac{1}{2}\left[\|P(x-y)\|_H^2 + \left\|P\left(x+\sqrt{-1}y\right)\right\|_H^2 - \|P(x)\|_H^2 - \|P(y)\|_H^2\right] = (Px, y)_h,$$

which completes the proof. $\square$

Remark 3.1.6. Note that K_P converges in $\widehat{H\otimes H^\dagger}$ if and only if $P(H)$ is finite-dimensional.

3.2 The Bergman projection

We now adapt the general theory of the previous setting to the complex geometric setting by considering Hilbert spaces of sections of holomorphic line bundles briefly mentioned at the beginning of this chapter; Bergman spaces. If such a section is holomorphic and the Hilbert spaces are defined by sufficiently regular geometric data, then the point evaluation of the section is controlled by the L^2-norm of the section. This boundedness of the point evaluation operator allows for some level of interplay between the properties of the sections when viewed as functions, and when viewed as vectors in a Hilbert space. This interplay possesses many important consequences.

3.2.1 Complex reproducing kernel Hilbert space structures

In many natural situations, the point evaluation operator is a bounded linear functional on the spaces of holomorphic sections. This boundedness is at the foundation of the theory of Bergman projections and Bergman kernels. If a given Hilbert space of functions has a bounded point evaluation operator, then one calls such a space a *complex reproducing kernel Hilbert space* ($\mathbb{C}$-RKHS).

Definition 3.2.1. (Complex reproducing kernel Hilbert space) Let X be a complex manifold with Borel measure μ and let $V \to X$ be a holomorphic vector bundle with Lebesgue measurable metric h. The pair (μ, h) is a (local) *complex reproducing kernel Hilbert space structure* ($\mathbb{C}$-RKHS structure for short) if for any compact $K \subset X$, there exists a constant $C_K > 0$ (depending on K) such that

$$|f_0(x)|_h^2 \leq C_K \, \|f_0\|^2_{L^2_{0,0}(\mu,h)}$$

for all $x \in K$ and all

$$f_0 \in \mathcal{H}^2_{0,0}(\mu, h) := \left\{ f \in \Gamma_{\mathcal{O}}(X, V) : \|f\|^2_{L^2_{0,0}(\mu,h)} := \int_X |f|_h^2 \, d\mu < +\infty \right\}.$$

In this case, we say that $\mathcal{H}^2_{0,0}(\mu, h)$ is a *Bergman space*. An $\mathbb{C}$-RKHS structure is said to be *global* if the constant $C_K := C$ may be taken independently of the compact set K, i.e. there exists a uniform constant C such that

$$\forall x \in X, \forall f \in \mathcal{H}^2_{0,0}(\mu, h) : |f(x)|_h^2 \leq C \, \|f\|^2_{L^2_{0,0}(\mu,h)}.$$

In this case, we also say that the pair (μ, h) is a *global* $\mathbb{C}$*-RKHS structure*. The local uniform boundedness of the point evaluation map which defines the $\mathbb{C}$-RKHS structure is clasically known as the *Bergman inequality*.

Proposition 3.2.1. *Let X be a complex manifold with Borel measure μ and let $V \to X$ be a holomorphic vector bundle with singular Hermitian metric h. Assume that*

1. *the Borel measure μ is absolutely continuous with respect to Lebesgue measure, and its local Radon-Nikodym derivatives are locally bounded below,*

2. *the metric h is locally bounded below, i.e., for every local frame $e_1, \cdots, e_r$ of $V \to X$, there exists a constant $c_0 > 0$ such that $|\sum_{i=1}^n v_i e_i|_h^2 \geq c_0 \sum_{i=1}^n |v|_i^2$.*

Then for every set K with compact closure in X, there exists a constant C_K such that if $s \in \mathcal{H}^2_{0,0}(\mu, h)$, then

$$\sup_K |s|_h^2 \leq C_K \int_X |s|_h^2 \, d\mu.$$

Proof. For every point $p \in K$, choose an opset B_p with compact closure in X that is biholomorphic to the unit ball in $\mathbb{C}^n$, and choose a holomorphic frame $e_1, \cdots, e_r$ for $V \to X$ that is orthonormal at p. Choose $R > 0$ and $C_1 > 0$ so that $C_1 |\sum_{i=1}^n v_i e_i|_h^2 \geq \sum_{i=1}^n |v|_i^2$ for all $v \in \mathbb{C}^r$, every $z \in B_R(p)$ and every $p \in K$, where z denotes the coordinates in B_p. Such $R > 0$ and $C_1 > 0$ exists by the assumption on h and the relative compactness of K. Write $s = \sum_{i=1} f_i e_i$ for holomorphic functions $f_1, \cdots, f_r$ on B_p. By the sub-mean value property,

$$|f_i(p)|^2 \leq \frac{1}{\pi^n R^{2n}} \int_{B_R(p)} |f_i|^2 \, dV(z),$$

where dV denotes Lebesgue measure in B_p. By our assumption on μ and the relative compactness of K, there exists $C_2 > 0$ such that $dV/d\mu \leq C_2$ on K. Hence

$$|s(p)|_h^2 \leq \sum_{i=1}^r |f_i(p)|^2 \leq \frac{C_1}{\pi^n R^{2n}} \int_{B_R(p)} \left| \sum_{i=1}^r f_i e_i \right|_h^2 dV(z) \leq \frac{C_1 C_2}{\pi^n R^{2n}} \int_{B_R(p)} |s|_h^2 \, d\mu,$$

which is the desired inequality. □

Remark 3.2.2. This proposition shows us that we can have an $\mathbb{C}$-RKHS structure under rather weak regularity hypotheses.

Remark 3.2.3. If X is compact, the $\mathbb{C}$-RKHS structure is clearly global by definition. The most immediate examples of global $\mathbb{C}$-RKHS structures are obtained when X is a compact complex manifold and h is bounded from below in sup-norm, in the sense of Proposition 3.2.1.

In the sequel, we will use the measure $d\mu = dV_g$ of some (Riemannian) Hermitian metric g. However, it is worth noting that one can define Hilbert spaces without the use of a metric g on the manifold. Indeed, let $(V \to X, h)$ be a holomorphic Hermitian vector bundle. Then, instead of considering sections of $V \to X$, one may consider V-valued canonical sections – that is, sections of $K_X \otimes V$. For each section f of $V \to X$, one has the measure $|f|_h^2$ defined as follows. If z is a local coordinate and $\xi_1, \cdots, \xi_r$ a local frame, then one may write $f = \sum_{i=1}^r f_i dz_1 \wedge \cdots \wedge dz_n \otimes \xi_i$ and thus define

$$|f|_h^2 := \sum_{1 \le i,j \le r} f_i \bar{f}_j h(\xi_i, \xi_j) \left(\frac{\sqrt{-1}}{2} \right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.$$

As noted in a Remark 2.2.1, one may always consider sections of $V \to X$ in lieu of V-valued $(n, 0)$-forms and vice versa. See [Ohs18] for instance, for more contents on this approach to Bergman kernels.

3.2.2 Definition of the Bergman projection

We now proceed to define the Bergman projection. Such a definition is possible in virtue of the following proposition.

Proposition 3.2.4. *If (μ, h) is an $\mathbb{C}$-RKHS structure, then the Bergman space $\mathcal{H}^2_{0,0}(\mu, h)$ is a closed subspace of $L^2_{0,0}(\mu, h)$.*

This proposition follows from the Bergman inequality in the following manner. Since the L^2-norm dominates the L^∞_{loc}-norm on $\mathcal{H}^2_{0,0}(\mu, h)$ by the Bergman inequality, every sequence in $\mathcal{H}^2_{0,0}(\mu, h)$ that is Cauchy with respect to the L^2-norm, converges locally uniformly, and hence its L^2-limit is holomorphic by Montel's theorem. We thus have the following definition.

Definition 3.2.2. The orthogonal projection $P : L^2_{0,0}(\mu, h) \to \mathcal{H}^2_{0,0}(\mu, h)$ is called the *Bergman projection* of $\mathcal{H}^2_{0,0}(\mu, h)$.

3.3 L^2 estimates for the Bergman projection

If (μ, h) is an $\mathbb{C}$-RKHS structure and $f \in L^2_{0,0}(\mu, h)$, then the section $U_0 = f - P(f) \in L^2_{0,0}(\mu, h)$ is a solution of the $\bar{\partial}$-equation

$$\bar{\partial} u = \bar{\partial} f.$$

Since $\mathcal{H}^2_{0,0}(\mu, h) = \mathrm{Ker}\left(\bar{\partial}\right) \cap L^2_{0,0}(\mu, h)$, Proposition 3.1.3 shows that U_0 is the solution of minimal L^2-norm. This minimality is particularly useful in estimating $\|P(f)\|^2_{L^2_{0,0}(\mu,h)}$ from below. Indeed, given *any* weak solution $u \in L^2_{0,0}(\mu, h)$ of $\bar{\partial} u = \bar{\partial} f$,

$$\|P(f)\|^2_{L^2_{0,0}(\mu,h)} = \|f\|^2_{L^2_{0,0}(\mu,h)} - \|f - P(f)\|^2_{L^2_{0,0}(\mu,h)} \geq \|f\|^2_{L^2_{0,0}(\mu,h)} - \|u\|^2_{L^2_{0,0}(\mu,h)}.$$

When $\mu = dV_g$ for some (Hermitian) Riemannian metric g, one can obtain desirable candidate solutions u by making use of the Hörmander-Skoda-Demailly Theorem.

Theorem 3.3.1. (Hörmander-Skoda-Demailly Theorem) *Let X be a complete Kähler manifold equipped with a (not necessarily complete) Kähler metric g, and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h. Assume there exists a non-negative Hermitian $(1,1)$-form Φ such that*

$$\Theta(h) + \mathbf{Ric}(g) \geq \Phi.$$

Then for every $f \in L^2_{0,0}(g, h)$ such that

$$\int_X \left|\bar{\partial} f\right|^2_{\Phi,h} dV_g < +\infty$$

one has the estimate

$$\int_X |f - P(f)|^2_h \, dV_g \leq \int_X \left|\bar{\partial} f\right|^2_{\Phi,h} dV_g.$$

See [Dem12, Chapter VIII, §6, (6.1) Theorem] for a proof.

3.4 The Bergman Kernel

According to Theorem 3.1.5, if $\{f_j\}_{j\geq 1} \subset \mathcal{H}^2_{0,0}(\mu, h)$ is an orthonormal Riesz basis, then the formal series

$$K = \sum_{j\geq 1} f_j \otimes \bar{f}_j$$

associated to the Bergman projection converges weakly on any indecomposable tensor in the Hilbert space $\widehat{H \otimes H}^\dagger$, where $H := L^2_{0,0}(\mu, h)$. It was also pointed out that if $\mathcal{H}^2_{0,0}(\mu, h)$ is not finite-dimensional, then K does not converge in $\widehat{H \otimes H}^\dagger$. If, however, (μ, h) is a local $\mathbb{C}$-RKHS structure then one can represent the formal series K very concretely.

3.4.1 Existence

On a Bergman space, the point evaluation is bounded by definition, and hence for each $x \in X$, there is a bounded linear function $\ell_x : \mathcal{H}^2_{0,0}(\mu, h) \mapsto V_x$ whose norm is locally uniformly bounded as a function of x. By the Riesz Representation Theorem, there is a vector $\xi_x \in \mathcal{H}^2_{0,0}(\mu, h) \otimes V_x^\dagger$ such that $f(x) = (f, \xi_x)$ and $x \mapsto \|\xi_x\|$ is locally uniformly bounded.

Lemma 3.4.1. *Let $\mathcal{H}^2_{0,0}(\mu, h)$ be a Bergman space and let $\{f_j\}_{j\geq 1} \subset \mathcal{H}^2_{0,0}(\mu, h)$ be a sequence of vectors such that for every $x \in X$*

1. *for each $K \Subset X$, there exists $C_K > 0$ such that $\sup\limits_{\substack{j\geq 1 \\ x\in K}} |(f_j, \xi_x)|_h \leq C_K$, and*
2. *$f(x) := \lim_{j\to\infty} (f_j, \xi_x)$ exists in V_x.*

Then $f_j \to f$ locally uniformly on X. In particular, $f \in \Gamma_{\mathcal{O}}(X, V)$.

Proof. Since $(f_j, \xi_x) = f_j(x)$, the first condition and Montel's theorem imply that a subsequence of $\{f_j\}_{j\geq 1}$ converges to a holomorphic section $\tilde{f}$. On the other hand, by the second property, $\{f_j\}_{j\geq 0}$ converges pointwise to f, and hence $\tilde{f} = f$. $\square$

Remark 3.4.2. The limit itself, f, does not have to be in $\mathcal{H}^2_{0,0}(\mu, h)$.

Theorem 3.4.3. *Let X be a complex manifold with Borel measure μ and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h such that (μ, h) is an $\mathbb{C}$-RKHS structure. Then, there exists a holomorphic section $K \in \Gamma_{\mathcal{O}}\left(X \otimes X^{\dagger}, V \boxtimes V^{\dagger}\right)$ such that*

1. $\overline{K(x,\bar{y})} = K(y,\bar{x})$,

2. $K(\cdot,\bar{y}) \in \mathcal{H}^2_{0,0}(\mu,h)$ *for every* $y \in X$, *and*

3. *the Bergman projection* $P : L^2_{0,0}(\mu,h) \to \mathcal{H}^2_{0,0}(\mu,h)$ *is given by*

$$P(f)(x) = \int_X h\left(f(y), K(x,\bar{y})\right) d\mu(y). \tag{3.4.1}$$

Moreover, K is uniquely determined by these three properties.

Proof. Suppose we have two sections K and $\tilde{K}$ that satisfy these three properties. Since $K(\cdot,\bar{x}), \tilde{K}(\cdot,\bar{z}) \in \mathcal{H}^2_{0,0}(\mu,h)$,

$$\tilde{K}(x,\bar{z}) = \left(\tilde{K}(\cdot,\bar{z}), K(x,\bar{\cdot})\right)_{L^2_{0,0}(\mu,h)} = \left(K(\cdot,\bar{x}), \tilde{K}(z,\bar{\cdot})\right)_{L^2_{0,0}(\mu,h)} = K(x,\bar{z}),$$

which establishes the uniqueness of K.

We now turn to the existence of K. Fix an orthonormal basis $\{s_j\}_{j\geq 0}$ of $\mathcal{H}^2_{0,0}(\mu,h)$ and let

$$f_N(x,\bar{y}) := \sum_{j=1}^{N} s_j(x) \otimes \overline{s_j(y)}.$$

Clearly, $f_N \in \Gamma_{\mathcal{O}}\left(X \otimes \bar{X}, V \boxtimes V^{\dagger}\right)$. Suppose that f_N converges locally uniformly, for the moment, and let K denote its limit. K clearly satisfies the first property, while the third one is a consequence of Theorem 3.1.5. Combining the identity

$$\int_X |f_N(\cdot,\bar{y})|_h^2 \, d\mu = f_N(y,\bar{y}),$$

with Fatou's Lemma, we can see that

$$\int_X |K(\cdot,\bar{y})|_h^2 \, d\mu \leq K(y,\bar{y}),$$

which proves the second property.

It remains to prove the convergence. To do so, we will apply Lemma 3.4.1 to $X \times X^\dagger$ with its product measure $\mu \times \mu$ and holomorphic vetor bundle $V \boxtimes V^\dagger \to X \times X^\dagger$ with the Hermitian metric $h \boxtimes \bar{h} := \pi_X^* h + \pi_{\bar{X}}^* \bar{h}$ where $\bar{h}(\xi, \bar{\sigma}) = h(\sigma, \bar{\xi})$. Fubini's Theorem shows that $\left(\mu \times \mu, h \boxtimes \bar{h}\right)$ is a an $\mathbb{C}$-RKHS structure structure. The Riesz-dual vector to point-evaluation vectors in $X \times X^\dagger$ are $\xi_{(x,\bar{y})} = \xi_x \otimes \bar{\xi}_y$, where $\xi_p \in \mathcal{H}^2_{0,0}(\mu, h) \otimes V_x^\dagger$ is the Riesz-dual of point-evaluation at $p \in X$ – i.e. $(s, \xi_p) = s(p)$. To apply 3.4.1, we need to show that

1. $\left|\left(f_N, \xi_{(x,\bar{y})}\right)\right|_{h \otimes H^\dagger}$ is locally uniformly bounded, and
2. $\left(f_N, \xi_{(x,\bar{y})}\right)$ converges in $V_x \otimes V_y^\dagger$.

Fix $K \Subset X$ and $x \in X$, and let $a_j := s_j(x)$. By the Bergman inequality applied to the holomorphic section $\sum_{j=1}^N s_j \otimes \bar{a}_j$ of the vector bundle $V \otimes V_x^\dagger \to X$, there exists a constant $C = C_K$ such that for all $\zeta \in K$

$$
\begin{aligned}
& h_\zeta \otimes \bar{h}_x \left(\sum_{i=1}^N s_j(\zeta) \otimes \bar{a}_i, \sum_{j=1}^N s_j(\zeta) \otimes \bar{a}_j \right) \\
& \leq C_K \int_X h \otimes \bar{h}_x \left(\sum_{i=1}^N s_j(z) \otimes \bar{a}_i, \sum_{j=1}^N s_j(z) \otimes \bar{a}_j \right) d\mu(z) \\
& = C_K \sum_{1 \leq i,j \leq N} \overline{h_x(a_i, a_j)} \int_X h\left(s_i(z), s_j(z)\right) d\mu(z) = C_K \sum_{j=1}^N h\left(s_i(x), s_i(x)\right),
\end{aligned}
$$

and setting $\zeta = x$ yields

$$
\left(\sum_{j=1}^N h\left(s_i(x), s_i(x)\right) \right)^2 \leq C_K \sum_{j=1}^N h\left(s_i(x), s_i(x)\right).
$$

$\square$

Hence,

$$
\sum_{j=1}^N h\left(s_i(x), s_i(x)\right) \leq C_K,
$$

and therefore,

$$
\left|\left(f_N, \xi_{(x,\bar{y})}\right)\right|^2_{h \otimes \bar{h}} = \left| \sum_{j=1}^N h\left(s_i(x), s_i(y)\right) \right|^2 \leq C_K^2, \forall x, y \in K.
$$

Since every compact subset of $X \times X^{\dagger}$ is contained in a set of the form $K \otimes \bar{K}$ for some $K \Subset X$, the first requirement of Lemma 3.4.1 is established. To prove the second one, note that for $N > M$,

$$|f_N(x,\bar{y}) - f_M(x,\bar{y})|_h^2 \leq \left(\sum_{i=1}^{M+1} |s_i(x)|_h^2\right)\left(\sum_{i=1}^{M+1} |s_j(x)|_h^2\right).$$

Hence the sequence $\{f_N(x,\bar{y})\}_{N\geq 1}$ is Cauchy and so it converges.

Definition 3.4.1. The section $K \in \Gamma_{\mathcal{O}}\left(X \times X^{\dagger}, V \boxtimes V^{\dagger}\right)$ is called the *Bergman kernel* of the Bergman space $\mathcal{H}^2_{0,0}(\mu, h)$.

3.4.2 Properties of the Bergman kernel

3.4.2.1 Extremal property

As previously seen, the Bergman kernel of $\mathcal{H}^2_{0,0}(\mu, h)$ is given by the series

$$K = \sum_{j\geq 0} s_j \otimes \bar{s}_j$$

which is locally uniformly convergent, but not necessarily convergent in $\mathcal{H}^2_{0,0}(\mu,h)\otimes\left(\mathcal{H}^2_{0,0}(\mu,h)\right)^{\dagger}$. Here, $\{s_j\}_{j\geq 0}$ denotes any orthonormal Riesz basis of $\mathcal{H}^2_{0,0}(\mu, h)$. This series expansion is useful in proving the following extremal characterization of the Bergman kernel.

Theorem 3.4.4. *Let $\mathcal{H}^2_{0,0}(\mu, h)$ be a Bergman space. The Bergman kernel $K \in \Gamma_{\mathcal{O}}\left(X \times X^{\dagger}, V \boxtimes V^{\dagger}\right)$ of $\mathcal{H}^2_{0,0}(\mu, h)$ is uniquely determined by the extremal property*

$$\forall x \in X, \sigma \in V_x^* : \langle \sigma \otimes \bar{\sigma}, K(x,\bar{x})\rangle = \sup_{u\in\mathcal{H}^2_{0,0}(\mu,h)-\{0\}} \frac{|\langle \sigma, u(x)\rangle|^2}{\|u\|^2_{L^2_{0,0}(\mu,h)}} \tag{3.4.2}$$

Moreover, the supremum is a maximum – i.e., for each $\sigma \in V^*$, there exists a section $u \in \mathcal{H}^2_{0,0}(\mu, h)$ that is unique up to a unimodular constant factor, such that $\|u\|_{L^2_{0,0}(\mu,h)} = 1$ and $\langle \sigma \otimes \bar{\sigma}, K(\pi\sigma, \overline{\pi\sigma})\rangle = |\langle \sigma, u(\pi\sigma)\rangle|^2$, for all $\sigma \in V^*$, where $\pi : V^* \to X$ denotes the bundle projection.

Remark 3.4.5. If $x \in X$ is a point at which the metric h is bounded, then

$$\sup_{\sigma \in V_x^* - \{0\}} \frac{|\langle \sigma, v \rangle|^2}{|\sigma|_{h^*}^2} = |v|_h^2 .$$

Therefore $|K(x, \bar{x})|_h$ is the optimal constant for the Bergman inequality at the point x.

Lemma 3.4.6. *Let $V \to X$ be a holomorphic vector bundle. Then any holomorphic section of $V \boxtimes V^\dagger \to X \times X^\dagger$ is uniquely determined by its value along the sesqui-diagonal set $\Delta_X := \{(x, \bar{x}); x \in X\} \subset X \times X^\dagger$.*

Proof. By the identity principle, it suffices to show that Δ_X is a totally real submanifold of $X \times X^\dagger$. The vector $\xi = \pi_X^* \xi_1 + \pi_{X^\dagger}^* \bar{\xi}_2 \in T^{1,0}_{X \times X^\dagger, (x, \bar{x})}$ is tangent to Δ_X if and only if $\xi_1 = \xi_2$. Since

$$J_{X \times X^\dagger} \xi = \sqrt{-1} \left(\pi_X^* \xi_1 + \pi_{X^\dagger}^* \bar{\xi}_2 \right) = \pi_X^* \left(\sqrt{-1} \xi_1 \right) + \pi_{X^\dagger}^* \left(\overline{-\sqrt{-1} \xi_2} \right),$$

it follows that $J_{X \times X^\dagger} T_{\Delta_X} \cap T_{\Delta_X} = \{0\}$. $\square$

Lemma 3.4.7. *Let $\mathcal{H}^2_{0,0}(\mu, h)$ be a Bergman space. For each $\sigma \in V_x^*$, there exists an orthonormal Riesz basis $\{s_j\}_{j \geq 0}$ for $\mathcal{H}^2_{0,0}(\mu, h)$ such that $\langle \sigma, s_j(x) \rangle = 0$ for all $j \geq 2$.*

Proof. If the point evaluation function $\mathcal{E}_\sigma : \mathcal{H}^2_{0,0}(\mu, h) \ni f \mapsto \langle \sigma, f(x) \rangle \in \mathbb{C}$ is identically zero, i.e. if $\mathcal{H}^2_{0,0}(\mu, h) = \mathrm{Ker}\,(\mathcal{E}_\sigma)$, there is nothing to prove. Assume then that $\mathcal{E}_\sigma \neq 0$. By the Bergman inequality, $\mathcal{E}_\sigma$ is a bounded operator and thus its kernel is a closed subspace. Letting $\xi_\sigma \in \mathcal{H}^2_{0,0}(\mu, h)$ denote the Riesz dual of σ, we find that $f \in \mathrm{Ker}\,(\mathcal{E}_\sigma)$ if and only if $(f, \xi_\sigma) = 0$, and hence we have the orthogonal decomposition

$$\mathcal{H}^2_{0,0}(\mu, h) = \mathbb{C} \xi_\sigma \oplus \mathrm{Ker}\,(\mathcal{E}_\sigma) .$$

Letting $s_1 = \xi_\sigma / \left\| \xi_\sigma \right\|_{L^2_{0,0}(\mu, h)}$ and taking $\{s_j\}_{j \geq 2}$ to be any orthonormal basis for $\mathrm{Ker}\,(\mathcal{E}_\sigma)$, we obtain the desired result. $\square$

We are now in a position to prove Theorem 3.4.4.

Proof. Lemma 3.4.6 and the polarization identity

$$\begin{aligned}\sigma \otimes \bar{\tau} = & \frac{1}{4}\left[(\sigma+\tau)\otimes\overline{(\sigma+\tau)} - (\sigma-\tau)\otimes\overline{(\sigma-\tau)}\right] \\ & + \frac{\sqrt{-1}}{4}\left[\left(\sigma-\sqrt{-1}\tau\right)\otimes\overline{\left(\sigma-\sqrt{-1}\tau\right)} - \left(\sigma+\sqrt{-1}\tau\right)\otimes\overline{\left(\sigma+\sqrt{-1}\tau\right)}\right]\end{aligned}$$

imply that K is completely determined by the quantities $\langle \sigma\otimes\bar{\sigma}, K(x,\bar{x})\rangle$ for $\sigma \in V_x^*$ and $x \in X$. Now let $u \in \mathcal{H}^2_{(0,0)}(\mu,h) - \{0\}$. If $\{\sigma_j\}_{j\geq 0}$ is an orthonormal basis for $\{u\}^\perp$, then

$$\frac{|\langle \sigma, u(x)\rangle|^2}{\|u\|^2_{L^2_{0,0}(\mu,h)}} \leq \frac{|\langle \sigma, u(x)\rangle|^2}{\|u\|^2_{L^2_{0,0}(\mu,h)}} + \sum_{j\geq 0} |\langle \sigma, \sigma_j(x)\rangle|^2 = \langle \sigma\otimes\bar{\sigma}, K(x,\bar{x})\rangle,$$

which shows that

$$\langle \sigma\otimes\bar{\sigma}, K(x,\bar{x})\rangle \geq \sup_{u\in\mathcal{H}^2_{0,0}(\mu,h)-\{0\}} \frac{|\langle \sigma, u(x)\rangle|^2}{\|u\|^2_{L^2_{0,0}(\mu,h)}}.$$

On the other hand, by Lemma 3.1, there exists an orthonormal basis $\{s_1, s_2, \cdots\}$ for $\mathcal{H}^2_{0,0}(\mu,h)$ such that $s_j(x) = 0$ for all $j \geq 2$. Thus

$$\langle \sigma\otimes\bar{\sigma}, K(x,\bar{x})\rangle = |\langle \sigma, s_1(x)\rangle|^2 \leq \sup_{u\in\mathcal{H}^2_{0,0}(\mu,h)-\{0\}} \frac{|\langle \sigma, u(x)\rangle|^2}{\|u\|^2_{L^2_{0,0}(\mu,h)}},$$

which completes the proof. □

3.4.2.2 Invariance

Given a holomorphic diffeomorphism $\Phi : X \to Y$ to a complex manifold Y, it follows that the push-forward measure $\Phi_*\mu$ and the metric $(\Phi^{-1})^* h$ for the pullback bundle $(\Phi^{-1})^* V \to Y$ define an $\mathbb{C}$-RKHS structure on Y that is naturally isomorphic to the $\mathbb{C}$-RKHS structure (μ,h) on X. We then immediately obtain the following proposition.

Proposition 3.4.8. *Let X be a complex manifold with measure μ and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h such that (μ,h) is an $\mathbb{C}$-RKHS structure. If $\Phi : X \to Y$ is a holomorphic diffeomorphism, then*

$$K_2\left(\Phi(x), \overline{\Phi(y)}\right) = K_1(x,\bar{y}),$$

where K_1 is the Bergman kernel of $\mathcal{H}^2_{0,0}(\mu,h)$ and K_2 is the Bergman kernel of $\mathcal{H}^2_{0,0}\left(\Phi_*\mu, (\Phi^{-1})^* h\right)$.

3.4.2.3 Monotonicity

Proposition 3.4.9. *Let X be an n-dimensional complex manifold, let μ be a Borel measure on X and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h such that (μ, h) is an $\mathbb{C}$-RKHS structure. Let K denote the Bergman kernel of $\mathcal{H}^2_{0,0}(\mu, h)$. Let Y be a complex manifold of complex dimension n and let $\iota : Y \hookrightarrow X$ be an injective holomorphic map. Then the Bergman kernel $\iota^* K$ of the $\mathbb{C}$-RKHS structure $(\iota^*\mu, \iota^* h)$ satisfies:*

$$\forall x \in X, \forall \sigma \in (\iota^* V^*)_x : \langle \sigma \otimes \bar{\sigma}, (\iota^* K)(x, \bar{x}) \rangle \geq \left\langle \left(\iota^{-1}\right)^* (\sigma \otimes \bar{\sigma}), \iota K(\iota x, \overline{\iota x}) \right\rangle .$$

Proof. Fix $\sigma \in \iota^* V^*$ and denote by $x \in Y$ the basepoint of σ. By Theorem 3.4.4, there exists a section $s \in \mathcal{H}^2_{0,0}(\mu, h)$ such that

$$\int_X |s|^2_h \, d\mu = 1 \text{ and } \left\langle \left(\iota^{-1}\right)^* (\sigma \otimes \bar{\sigma}), K(\iota x, \overline{\iota x}) \right\rangle = \left| \left\langle \left(\iota^{-1}\right)^* \sigma, s(x) \right\rangle \right|^2 .$$

Then $\iota^* s \in \mathcal{H}^2_{0,0}(\iota^*\mu, \iota^* h)$ and

$$\int_Y |\iota^* s|^2_{\iota^* h} \, d\iota^* \mu = \int_{\iota Y} |s|^2_h \, d\mu \leq 1.$$

The desired equality follows from Theorem 3.4.4. □

The following three propositions generalize [Ber06, Lemmas 3.1, 3.2, and 3.3]. We prove the first one. The next two essentially follow from special cases of the generalization of Ramadanov's theorem in [PW16]. They can be shown using the methods of proof of Propositions 3.4.9 and Proposition 3.4.10, or following the methods of proof found in [PW16].

Proposition 3.4.10. *Let Ω_0 and Ω_1 be bounded domains in a complex manifold X such that $\Omega_0 \Subset \Omega_1$. Let $V \to \Omega_1$ be a holomorphic vector bundle and let μ be a Borel measure for Ω_1. Let $\{h_j\}_{j \geq 0}$ be a sequence of Hermitian metrics for $V \to \Omega_1$ such that (μ, h_j) is an $\mathbb{C}$-RKHS structure for each j. Assume further that $h_j\big|_{\overline{\Omega}_0} = h$ for some metric h for $V\big|_{\overline{\Omega}_0} \to \overline{\Omega}_0$, and that $h_j \searrow 0$ almost everywhere in $\Omega_1 - \Omega_0$. Assume that $\mathcal{H}^2_{0,0}(\Omega_1, h)$ is dense in $\mathcal{H}^2_{0,0}(\Omega_0, h)$. Fix a point $z \in \Omega_0$. Let K_j be the Bergman kernel for $\mathcal{H}^2_{0,0}(\Omega_1, h_j)$, and let K be the Bergman kernel for $\mathcal{H}^2_{0,0}(\Omega_0, h)$. Then, denoting by $\iota : \Omega_0 \hookrightarrow \Omega_1$ the inclusion map,*

$$\forall \sigma \in V^*_{\iota z} : \left\langle \left(\iota^{-1}\right)^* (\sigma \otimes \bar{\sigma}), K_j(\iota z, \overline{\iota z}) \right\rangle \nearrow \langle \sigma \otimes \bar{\sigma}, K(z, \bar{z}) \rangle .$$

Proof. Let $\iota : \Omega_0 \hookrightarrow \Omega_1$ be the inclusion map, and let $\sigma \in V_{\iota z}^*$. By Theorem 3.4.4, $\left\{\left\langle\left(\iota^{-1}\right)^*(\sigma\otimes\bar\sigma), K_j\left(\iota z, \overline{\iota z}\right)\right\rangle\right\}_{j\geq 0}$ is an increasing sequence and

$$\left\langle\left(\iota^{-1}\right)^*(\sigma\otimes\bar\sigma), K_j\left(\iota z, \overline{\iota z}\right)\right\rangle \leq \langle\sigma\otimes\bar\sigma, K(z,\bar z)\rangle$$

for each j. Since

$$K_j(\iota z, \bar{\iota z}) = \int_{\Omega_1} |K_j(\iota z, y)|_{h_j}^2 \, d\mu(y)$$

by Theorem 3.4.3, it follows that $\{K_j\}_{j\geq 0}$ has uniformly bounded norm norm in $\mathcal{H}^2_{0,0}(\Omega_1, h_j)$. Therefore, the sequence $\{K_j\}_{j\geq 0}$ has a weakly convergent subsequence in $\mathcal{H}^2_{0,0}(\Omega_0, h)$. Let $\mathfrak{K}$ be the limit of the weakly convergent subsequence. If f is in $\mathcal{H}^2(\Omega_1, h)$, then by the Cauchy-Schwarz inequality

$$\left|\int_{\Omega_1-\Omega_0} \langle f(y), K_j(x,y)\rangle_{h_j} \, d\mu(y)\right| \leq \|K_j\|^2_{L^2_{0,0}(\Omega_1,h_j)} \int_{\Omega_1-\Omega_0} |f(y)|^2_{h_j} \, d\mu(y),$$

and the latter converges to 0 as $j \to +\infty$. Therefore, any weak limit $\mathfrak{K}$ satisfies

$$f(x) = \int_{\Omega_0} \langle f(y), \mathfrak{K}(x,\bar y)\rangle_h \, d\mu(y)$$

for any $f \in \mathcal{H}^2_{0,0}(\Omega_1, h)$, and since $\mathcal{H}^2_{0,0}(\Omega_1, h)$ is dense in $\mathcal{H}^2_{0,0}(\Omega_0, h)$, the same reproducing property holds for any $f \in \mathcal{H}^2_{0,0}(\Omega_0, h)$. Since $\mathfrak{K}$ is holomorphic, $\mathfrak{K} = K$ by uniqueness and the limit is uniform on compact subsets of Ω_0. In particular, for each $z \in \Omega_0$, $K_j(\iota z, \bar{\iota z})$ converges to $K(z,\bar z)$ as $j \to +\infty$, which implies the desired result. $\square$

Proposition 3.4.11. *Let Ω be a bounded domain, with Borel measure μ, in a complex manifold X, and let $V \to \Omega$ be a holomorphic vector bundle with Hermitian metric h such that h is locally bounded below and (μ, h) is an $\mathbb{C}$-RKHS structure. Let $\{\Omega_j\}_{j\geq 1}$ be an increasing family of subdomains with union equal to Ω. Let z be a fixed point in Ω_0 and let K_j and K be the Bergman kernels for $\mathcal{H}^2_{0,0}\left(\Omega_j, \mu\big|_{\Omega_j}, h\right)$ and $\mathcal{H}^2_{0,0}(\Omega, \mu, h)$ respectively. Then, denoting by $\iota_j : \Omega_0 \hookrightarrow \Omega_j$ and $\iota : \Omega_0 \hookrightarrow \Omega$ the inclusion maps,*

$$\forall\sigma \in V^*_{\iota_0 z} = V^*_{\iota_j z} : \left\langle\left(\iota_j^{-1}\right)^*(\sigma\otimes\bar\sigma), K_j\left(\iota_j z, \overline{\iota_j z}\right)\right\rangle \searrow \left\langle\left(\iota_0^{-1}\right)^*(\sigma\otimes\sigma), K\left(\iota_0 z, \overline{\iota_0 z}\right)\right\rangle.$$

Proposition 3.4.12. *Let Ω be a bounded domain, with Borel measure μ, in a complex manifold X, and let $V \to \Omega$ be a holomorphic vector bundle. Suppose that $\{h_j\}_{j\geq 1}$ is a sequence of Hermitian metrics that are increasing to a metric h. Suppose that for each j, (μ, h_j) is an $\mathbb{C}$-RKHS structure, and that (μ, h) is also an $\mathbb{C}$-RKHS structure. Let z be a fixed point in Ω and let K_j and K be the Bergman kernel for $\mathcal{H}^2_{0,0}(\mu, h_j)$ and $\mathcal{H}^2(\mu, h)$ respectively. Then,*

$$\forall \sigma \in V_z^* : \langle \sigma \otimes \bar{\sigma}, K_j(z, \bar{z}) \rangle \searrow \langle \sigma \otimes \bar{\sigma}, K(z, \bar{z}) \rangle \, .$$

The following proposition is a geometric adaptation of [Ber06, Lemma 3.4].

Proposition 3.4.13. *Let $X = \{(t, z) \in \mathbb{C}^m \times Y : \rho(t, z) < 0\}$ where Y is a Stein manifold and ρ is plurisubharmonic near the closure of X. Let $V \to X$ be a holomorphic vector bundle equipped with a Hermitian metric h that is smooth and locally bounded below near the closure of X. Let $V^{[t]} := V\big|_{X_t}$. Then, for fixed z and $\sigma \in \left(V_z^{[t]}\right)^*$, the function $t \mapsto \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$ is upper semicontinuous.*

Proof. Consider a point t and let s be a nearby point tending to t. We may choose $\varepsilon > 0$ so that all the fibers X_s are contained in the open set $V_\varepsilon := \{(t, z) \in \mathbb{C}^m \times Y : \rho(t, z) < \varepsilon\}$. Note that any compact subset of X_t is contained in all X_s for s sufficiently close to t. Let $K_s(\cdot, \bar{z})$ denote the Bergman kernel of X_s for a fixed point z. Let $\sigma \in \left(V_z^{[s]}\right)^*$. Since the domains X_s all contain a fixed open neighborhood of z, the L^2-norms of $\langle \sigma \otimes \bar{\sigma}, K_s(z, \bar{z}) \rangle$ are bounded. Therefore, any sequence of $\langle \sigma \otimes \bar{\sigma}, K_s(z, \bar{z}) \rangle$ has a subsequence that is weakly convergent on any compact subset of X_t. The L^2-norm of any weak limit $\mathfrak{K}$ cannot exceed the lim inf of the L^2-norms of $\langle \sigma \otimes \bar{\sigma}, K_s(z, \bar{z}) \rangle$ over X_s. By Theorem 3.4.4, it follows that for any $\sigma \in \left(V_z^{[s]}\right)^*$ and $\tau \in \left(V_z^{[t]}\right)^*$

$$\limsup_{s \to t} \langle \sigma \otimes \bar{\sigma}, K_s(z, \bar{z}) \rangle \leq \langle \tau \otimes \bar{\tau}, K_t(z, \bar{z}) \rangle \, ,$$

which completes the proof. $\square$

3.4.3 Examples

Example 3.4.14. (The standard unit ball) Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$. Consider the Hilbert space $L^2_{n,0}(\mathbb{B}_n)$ of L^2-sections of the canonical bundle $K_{\mathbb{B}_n}$. Then

$$K_{\mathbb{B}_n}(z,\bar{w}) = \frac{1}{\sigma_n}\left(1-\langle z,w\rangle\right)^{-(n+1)} dz_1\wedge\cdots\wedge dz_n \otimes d\bar{w}_1\wedge\cdots\wedge d\bar{w}_n$$

where $\sigma_n := \dfrac{\pi^n}{n!} = \mathrm{Vol}(\mathbb{B}_n)$. First, note that the monomial top forms

$$m_\alpha(z) := z^\alpha dz_1\wedge\cdots\wedge dz_n$$

are mutually orthogonal and

$$\int_{\mathbb{B}_n} |m_\alpha|^2 = \int_{\mathbb{B}_n} |z^\alpha|^2\, dV(z) = \int_0^1 r^{2|\alpha|+2(n-1)} r dr \int_{S^{2n-1}} |\theta^\alpha|^2\, d\sigma(\theta) = \frac{1}{2\left(|\alpha|+n\right)}\int_{S^{2n-1}} |\theta^\alpha|^2\, d\sigma(\theta).$$

On the other hand:

$$\int_{S^{2n-1}} |\theta^\alpha|^2\, d\sigma(\theta) = \frac{\displaystyle\int_{\mathbb{C}^n} |z^\alpha|^2 e^{-|z|^2} dV(z)}{\displaystyle\int_0^\infty r^{2(|\alpha|+n-1)} e^{-r^2} r dr} = \frac{\pi^n \alpha!}{2\left(|\alpha|+n-1\right)!}.$$

Therefore,

$$\int_{\mathbb{B}_n} |z^\alpha|^2\, dV(z) = \frac{\pi^n \alpha!}{(|\alpha|+n)!}.$$

Thus, an orthonormal basis is given by the monomials $\left\{\sqrt{\dfrac{(|\alpha|+n)!}{\pi^n\alpha!}}\, m_\alpha(z)\right\}_{\alpha\in\mathbb{N}^n}$, and using the metric $dV(z)^{-1}$, for the canonical bundle of $\mathbb{C}^n$, given by the reciprocal of Lebesgue measure,

$$\frac{|K(z,\bar{z})|}{dV(z)} = \sum_{|\alpha|\geq 0} \frac{(|\alpha|+n)!}{\pi^n\alpha!}|z^\alpha|^2 = \frac{n!}{\pi^n}\sum_{j=0}^\infty \frac{(n+j)!}{j!n!}\sum_{|\alpha|=j}\frac{|\alpha|!}{\alpha!}|z^\alpha|^2 = \frac{n!}{\pi^n}\sum_{j=0}^\infty \binom{n+j}{j}|z|^{2j},$$

whence

$$\frac{|K(z,\bar{z})|}{dV(z)} = \frac{1}{\sigma_n}\left(1-|z|^2\right)^{-(n+1)}.$$

As we already noted, the polarization then determines K:

$$K_{\mathbb{B}_n}(z,\bar{w}) = \frac{1}{\sigma_n}\left(1-\langle z,w\rangle\right)^{-(n+1)} dz_1\wedge\cdots\wedge dz_n \otimes d\bar{w}_1\wedge\cdots\wedge d\bar{w}_n.$$

Example 3.4.15. (Ellipsoids) For a positive Hermitian $(n \times n)$-matrix A, we denote by

$$\Omega_A := \{z \in \mathbb{C}^n : (Az) \cdot \bar{z} < 1\}$$

the *ellipsoid of inertia* A. Clearly, Ω_I is the unit ball. Letting $\sqrt{A}$ denote the unique positive Hermitian matrix such that $\sqrt{A} \cdot \sqrt{A} = A$, the map $\Phi_A(z) := \sqrt{A}z$ is a holomorphic diffeomorphism of $\mathbb{B}_n$ onto Ω_A. Denoting by dV the standard volume form in $\mathbb{C}^n$, one has $\Phi_A^*(dV) = (\det(A))\, dV$. Therefore, by the invariance of Bergman kernels (Proposition 3.4.8), the Bergman kernel K_A for $\mathcal{H}^2_{0,0}\left(dV\big|_{\Omega_A}\right)$ is given by

$$K_A(z, \bar{w}) = (\det A)\, K_I\left(\Phi_A(z), \overline{\Phi_A(w)}\right) = \frac{\det(A)}{\sigma_n \left(1 - \langle Az, w\rangle\right)^{n+1}}.$$

Example 3.4.16. (The Bargmann-Fock Space) Assume $X = \mathbb{C}^n$ with its Lebesgue measure dV, let L be trivial, and set $\varphi(z) = |z|^2$. The Hilbert space $\mathcal{H}^2_{0,0}\left(dV, e^{-\varphi}\right)$ is called the *Bargmann-Fock Space*. Since the monomials $\{z^\alpha\}_{\alpha \in \mathbb{N}^n}$ are mutually orthogonal in $\mathcal{H}^2_{0,0}\left(dV, e^{-\varphi}\right)$ and

$$\int_{\mathbb{C}^n} |z^\alpha|^2\, e^{-|z|^2} dV(z) = \pi^n \alpha!,$$

we see from the multinomial theorem that

$$|K(z, \bar{z})| = \frac{1}{\pi^n} \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{|\alpha|=j} \frac{|\alpha|!}{\alpha!} |z^\alpha|^2 = \frac{1}{\pi^n} \sum_{j=0}^{\infty} \frac{|z|^{2j}}{j!} = \pi^{-n} e^{|z|^2},$$

and therefore $K(z, \bar{w}) = \pi^{-n} e^{\langle z, w\rangle}$.

Remark 3.4.17. By rescaling, one can see that the Bergman kernel K_m for $\mathcal{H}^2_{0,0}\left(dV, e^{-m|\cdot|^2}\right)$ is $K_m(z, \bar{z}) = \dfrac{m^n}{\pi^n} e^{m|z|^2}$.

Example 3.4.18. (Finite-rank examples)

1. Consider the Hilbert space $L^2(\mathbb{C})$ of L^2-functions in the entire plane. Then there are no holomorphic function in $L^2(\mathbb{C})$ other than the zero function.

2. The space of functions that are L^2 with respect to the weight $\varphi(z) = (N+2) \log\left(1 + |z|^2\right)$ in the entire complex plane consists exactly of polynomials of degree at most N.

3. If X is a compact complex manifold, and $(L \to X, e^{-\varphi})$ a holomorphic Hermitian line bundle over X, then $\Gamma_{\mathcal{O}}(X, V)$ is finite-dimensional by the Hodge Theorem, and thus so is its subspace $\mathcal{H}^2_{0,0}(\mu, e^{-\varphi})$.

Chapter 4

Twisted L^2-theory for the $\bar{\partial}$-operator

The use of L^2-estimates for the $\bar{\partial}$-operator to obtain estimates for the Bergman kernel goes all the way back to Hörmander's original paper [Hör65]. In this section, we present two theorems on L^2-estimates for the $\bar{\partial}$-operator that are based on the twisted Bochner-Kodaira-Identity.

4.1 The twisted Bochner-Kodaira Identity

Let X be an n-dimensional complete Kähler manifold with complete Kähler metric g and let $V \to X$ be a holomorphic vector bundle of rank r with smooth Hermitian metric h. We start from the (integrated) Bochner-Kodaira Identity. For a detailed exposition in the case of line bundles, we refer the reader to [MV15] and [Var19]. See also [McN06a].

As X possesses a metric, there is a natural way to map sections of the non-holomorphic vector bundle $K_X \otimes \Lambda^q \left(T_X^{*0,1}\right) \otimes V \to X$ to sections of the holomorphic vector bundle $K_X \otimes \Lambda^q \left(T_X^{1,0}\right) \otimes V \to X$. Writing $\varphi = \sum_{\substack{1 \leq \alpha \leq r \\ |J|=q}} \varphi_{\bar{J}}^{\alpha} dz_1 \wedge \cdots \wedge d\bar{z}^J \otimes e_\alpha$ locally, we see that

$$\begin{aligned}
\mathfrak{I}(\varphi) &:= \sum_{\substack{1 \leq \alpha \leq r \\ |I|=|J|=q}} g^{I\bar{J}} \varphi_{\bar{J}}^{\alpha} dz_1 \wedge \cdots \wedge dz_n \otimes \frac{\partial}{\partial z^I} \otimes e_\alpha \\
&:= \sum_{\substack{1 \leq \alpha \leq r \\ |I|=|J|=q}} g^{I\bar{J}} \varphi_{\bar{J}}^{\alpha} dz_1 \wedge \cdots \wedge dz_n \otimes \frac{\partial}{\partial z_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{i_q}} \otimes e_\alpha
\end{aligned}$$

is a section of $K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V \to X$, and clearly, the map $\varphi \mapsto \mathfrak{I}(\varphi)$ is a one-to-one correspondence that depends only on the pointwise values of φ. The map $\mathfrak{I}$ extends naturally to a one-to-one correspondence $\mathfrak{I}_k$ of $\left(K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V\right)$-valued $(0,k)$-forms, i.e., sections of $\Lambda^k\left(T_X^{*0,1}\right) \otimes K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V \to X$, by acting on the factor $K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V$ as follows. For $J = (j_1, \cdots, j_k)$, set

$$\mathfrak{I}\left(d\bar{z}^J \otimes dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}^I \otimes e_\alpha\right) = d\bar{z}^J \otimes \mathfrak{I}_k\left(dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}^I \otimes e_\alpha\right).$$

The vector bundle $K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V \to X$ being a holomorphic vector bundle, is equipped with a $\bar{\partial}$-operator, and thus we have a well-defined $\left(K_X \otimes \Lambda^q\left(T_X^{1,0}\right) \otimes V\right)$-valued $(0,1)$-form $\bar{\partial}\left(\mathfrak{I}(\varphi)\right)$.

Definition 4.1.1. The operator $\overline{\nabla} : \Gamma\left(X, \Lambda^{n,q}\left(T_X^*\right) \otimes V\right) \to \Gamma\left(X, T_X^{*0,1} \otimes \Lambda^{n,q}\left(T_X^*\right) \otimes V\right)$ is defined by $\overline{\nabla}(\varphi) := \mathfrak{I}_1^{-1}\left[\bar{\partial}\left(\mathfrak{I}(\varphi)\right)\right]$.

Proposition 4.1.1. *For every compactly supported $\beta \in \Gamma(X,V)$, one has the formal identity*

$$\int_X \left|\bar{\partial}_h^*\beta\right|_h^2 dV_g + \int_X \left|\bar{\partial}\beta\right|_{h,g}^2 dV g = \int_X \left|\overline{\nabla}\beta\right|_{h,g}^2 dV_g + \int_X \left(\left(\Theta(h) + \mathbf{Ric}(g) \otimes \mathrm{Id}_V\right)_g \beta, \beta\right)_{h,g} dV_g.$$

Here, $\bar{\partial}_h^*$ denotes the formal adjoint of $\bar{\partial}$ with respect to the metric h.

Now let us replace the metric h by the metric $he^{-\eta}$ for some smooth function $\eta : X \to \mathbb{R}$, and let $\mathfrak{D}_\eta^{0,1}$ denote the $(0,1)$-vector field defined by

$$g\left(\xi, \mathfrak{D}_\eta^{(0,1)}\right) = \overline{\bar{\partial}\eta(\bar{\xi})}, \xi \in T_X^{0,1}.$$

Then $\bar{\partial}_{he^{-\eta}}^*\beta = \bar{\partial}_h^*\beta - \mathfrak{D}_\eta^{0,1} \lrcorner \beta$ and $\Theta\left(he^{-\eta}\right) = \Theta(h) + \partial\bar{\partial}\eta \otimes \mathrm{Id}_V$. Therefore,

$$\begin{aligned}\int_X \left|\bar{\partial}_h^*\beta\right|_{he^{-\eta}}^2 dV_g = \int_X \left|\bar{\partial}_h^*\beta\right|_h^2 e^{-\eta} dV_g &+ \int_X \left(\left(\partial\eta \wedge \bar{\partial}\eta \otimes \mathrm{Id}_V\right)_g \beta, \beta\right)_{h,g} e^{-\eta} dV_g \\ &+ 2\mathrm{Re}\left[\int_X \left(\bar{\partial}_h^*\beta, \mathfrak{D}^{0,1} \lrcorner \beta\right)_h e^{-\eta} dV_g\right]\end{aligned}$$

and

$$\begin{aligned}&\int_X \left(\left(\Theta\left(he^{-\eta}\right) + \mathbf{Ric}(g) \otimes \mathrm{Id}_V\right)_g \beta, \beta\right)_{he^{-\eta},g} dV_g \\ &= \int_X \left(\left(\Theta(h) + \left(\partial\bar{\partial}\eta + \mathbf{Ric}(g)\right) \otimes \mathrm{Id}_V\right)_g \beta, \beta\right)_{h,g} e^{-\eta} dV_g\end{aligned}$$

We thus have the *twisted Bochner-Kodaira Identity*:

$$\begin{aligned}\int_X \left|\bar\partial_h^*\beta\right|_h^2 e^{-\eta}dV_g &+ \int_X \left|\bar\partial\beta\right|_{h,g}^2 e^{-\eta}dV_g \\ = \int_X \left|\overline{\nabla}\beta\right|_{h,g}^2 e^{-\eta}dV_g &+ \int_X \Big(\big(\Theta(h) + \big(\mathbf{Ric}(g) + \partial\bar\partial\eta - \partial\eta\wedge\bar\partial\eta\big)\otimes \mathrm{Id}_V\big)_g \beta,\beta\Big)_{h,g} e^{-\eta}dV_g \\ &- 2\mathrm{Re}\left[\int_X \left(\bar\partial_h^*\beta, \mathfrak{D}^{0,1}\lrcorner\beta\right)_h e^{-\eta}dV_g\right]. \qquad (4.1.1)\end{aligned}$$

Estimating the last term of (4.1.1), one obtains à priori estimates that lead to estimates for $\bar\partial$, as we will see in the next section.

4.2 Donnelly-Fefferman-Ohsawa Estimates for $\bar\partial$

Letting $\delta > 0$, Young's inequality for products yields

$$\begin{aligned}2\mathrm{Re}&\left[\int_X \left(\bar\partial_h^*\beta, \mathfrak{D}^{0,1}\lrcorner\beta\right)_h e^{-\eta}dV_g\right] \\ &\le \frac{1}{\delta}\int_X \left|\bar\partial_h^*\beta\right|_h^2 e^{-\eta}dV_g + \int_X \Big(\delta\left(\partial\eta\wedge\bar\partial\eta\otimes\mathrm{Id}_V\right)_g\beta,\beta\Big)_{h,g} e^{-\eta}dV_g\end{aligned}$$

The twisted Bochner-Kodaira identity yields the Donnelly-Fefferman-Ohsawa à priori estimate

$$\begin{aligned}\left(\frac{1+\delta}{\delta}\right)\int_X \left|\bar\partial_h^*\beta\right|_h^2 e^{-\eta}dV_g &+ \int_X \left|\bar\partial\beta\right|_{h,g}^2 e^{-\eta}dV_g \qquad (4.2.1)\\ \ge \int_X \Big(\big(\Theta(h) &+ \big(\mathbf{Ric}(g) + \partial\bar\partial\eta - (1+\delta)\partial\eta\wedge\bar\partial\eta\big)\otimes\mathrm{Id}_V\big)_g\beta,\beta\Big)_{h,g} e^{-\eta}dV_g,\end{aligned}$$

for all compactly supported smooth V-valued $(0,1)$-forms β. The estimate (4.2.1) leads to the following L^2-estimate for $\bar\partial$.

Theorem 4.2.1. *Let X be a complete Kähler manifold equipped with a Kähler metric g that is not necessarily complete, and let $V \to X$ be a holomorphic vector bundle with Hermitian metric h_0. Assume there exist a smooth function η, a positive number δ and a non-negative $(1,1)$-form Φ such that*

$$\Theta(h_0) + \big(\mathbf{Ric}(g) + 2\partial\bar\partial\eta - (1+\delta)\partial\eta\wedge\bar\partial\eta\big)\otimes\mathrm{Id}_V \ge_{\mathrm{Nak}} \Phi\otimes\mathrm{Id}_V.$$

Then for every V-valued $(0,1)$-form α such that

$$\bar{\partial}\alpha = 0 \ \textit{and} \ \int_X |\alpha|^2_{h_0,\Phi}\, dV_g < +\infty$$

there exists a measurable section u of $V \to X$ such that

$$\bar{\partial}u = \alpha \ \textit{and} \ \int_X |u|^2_{h_0}\, dV_g \leq \left(\frac{1+\delta}{\delta}\right) \int_X |\alpha|^2_{h_0,\Phi}\, dV_g.$$

Proof. If the metric g is not complete, we may replace g by $g_\varepsilon = g + \varepsilon g_0$ where $\varepsilon > 0$ and g_0 is a complete metric for X. So we assume that g is complete for the remainder of the proof. By Remark 2.2.1, we can think of sections of $V \to X$ and V-valued $(0,1)$-forms as V-valued $(n,0)$-forms and V-valued $(n,1)$-forms respectively. We can then apply the L^2-estimates to g_ε and let ε tend to 0 at the end. (For further details, we refer the reader to [Dem12, Chapter VIII, §6, (6.1) Theorem and (6.3) Lemma].) We are going to apply the estimate (4.2.1) to the metric $h = h_0 e^{-\eta}$. Since g is complete, (4.2.1) holds for all β in the domains of the operators T^* and S, where

$$T(f) := \bar{\partial}\left(\sqrt{\left(\frac{1+\delta}{\delta}\right) e^{-\eta}} \cdot f\right) \ \text{and} \ S(\beta) := \sqrt{e^{-\eta}}\bar{\partial}\beta.$$

Together with the curvature assumption, (4.2.1) becomes

$$\int_X |T^*\beta|^2_h\, dV_g + \int_X |S\beta|^2_{h,g}\, dV_g \geq \int_X \left(\left(e^{-\eta}\Phi \otimes \mathrm{Id}_V\right)_g \beta, \beta\right) dV_g.$$

Since $S \circ T = 0$, the same functional analysis argument used in the proof of Hörmander's Theorem, together with the Young's inequality for products used to establish the Demailly-Hörmander-Skoda Theorem (see [Dem12, Chapter VIII]), shows that there exists a measurable section U satisfying

$$TU = \alpha \ \text{and} \ \int_X |U|^2_h\, dV_g \leq \int_X |\alpha|^2_{h,e^{-\eta}\Phi}\, dV_g = \int_X |\alpha|^2_{h_0,\Phi}\, dV_g.$$

Setting $u := U\sqrt{\left(\frac{1+\delta}{\delta}\right) e^{-\eta}}$ shows that

$$\bar{\partial}u = TU \ \text{and} \ \int_X |u|^2_{h_0}\, dV_g = \left(\frac{1+\delta}{\delta}\right) \int_X |U|^2_h\, dV_g,$$

which completes the proof. □

4.3 Functions of self-bounded gradient

Note that if $\eta \equiv 0$, then the curvature hypothesis of Theorem 4.2.1 is exactly that of the Hörmander-Skoda-Demailly Theorem, which we recover by letting $\delta \to +\infty$. On the other hand, if $-e^{-\eta}$ is plurisubharmonic, then so is η. Indeed,

$$\partial\bar{\partial}\left(-e^{-\eta}\right) = e^{-\eta}\left(\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta\right),$$

so that

$$\partial\bar{\partial}\eta \geq \partial\eta \wedge \bar{\partial}\eta \geq 0.$$

So if $-e^{-\eta}$ is plurisubharmonic, then the condition

$$\Theta(h_0) + \left(\mathbf{Ric}(g) + (1-\delta)\partial\bar{\partial}\eta\right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} \Phi \otimes \mathrm{Id}_V$$

implies the curvature hypothesis of Theorem 4.2.1. Therefore, if such a nonconstant function η exists, then one obtains an imporvement of the Hörmander-Skoda-Demailly theorem in the sense that the curvature condition of the Hörmander-Skoda-Demailly has been weakened by allowing negativity up to $(1-\delta)\partial\bar{\partial}\eta$, provided that $\delta \in (0,1)$. For a more detailed discussion in the case where $V \to X$ is a line bundle, we refer the reader to [MV15].

4.3.1 Definition and examples

Definition 4.3.1. Let X be a complex manifold. A function $\eta \in W^{2,1}_{\mathrm{loc}}(X)$ has *self-bounded gradient with constant* $c > 0$ if

$$\partial\bar{\partial}\eta \geq c \cdot \left(\partial\eta \wedge \bar{\partial}\eta\right).$$

We denote the set of such functions by $\mathcal{SBG}_c(X)$. McNeal introduced this notion in [McN02]. The nomenclature is motivated by the fact that if η also happens to be strictly plurisubharmonic, then η is of self-bounded gradient with constant c if and only if

$$|\partial\eta|^2_{\sqrt{-1}\partial\bar{\partial}\eta} \leq c.$$

Remark 4.3.1. For any $c > 0$, $\eta \in \mathcal{SBG}_c(X)$ if and only if $c\eta \in \mathcal{SBG}_1(X)$, and so we only need to consider functions belonging to $\mathcal{SBG}_1(X)$. From now on, we refer to any function in $\mathcal{SBG}_1(X)$ as a *function of self-bounded gradient.*

The typical examples of functions of self-bounded gradient are the potentials for the Poincaré metric on the unit ball and the punctured disk, respectively.

Example 4.3.2. The function $\eta : z \mapsto -\log\left(1 - |z|^2\right)$ is of self-bounded gradient on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ since $-e^{-\eta} : z \mapsto |z|^2 - 1$ is plurisubharmonic.

Example 4.3.3. Let X be a complex manifold and let $f : X \to \mathbb{B}_n$ be a holomorphic map with values in $\mathbb{B}_n$. Then the function $\eta := -\log\left(1 - |f|^2\right)$ is of self-bounded gradient because it is the pullback of the function $z \mapsto -\log\left(1 - |z|^2\right)$ from the previous example.

Example 4.3.4. (Relatively compact strongly pseudoconvex domains with smooth boundary)

Let X be a Stein manifold and let $\Omega \Subset X$ be a relatively compact strongly pseudoconvex subdomain.

1. Assuming that the boundary of Ω is smooth, [DF75, Theorem 1] states the existence of a smooth strictly plurisubharmonic function ρ on Ω with negative values and that converges to zero at the boundary. In this situation, we may choose $\eta := -\log(-\rho)$ to be our function in $\mathcal{SBG}_1(\Omega)$. In particular, Example 4.3.2 is the special case when Ω is the unit ball, $X = \mathbb{C}^n$ and $\rho(z) = |z|^2 - 1$.

2. If the boundary of Ω is $\mathcal{C}^r$-smooth; $2 \leq r \leq \infty$, another theorem of Diederich-Fornæss ([DF77, Theorem 1]) states the existence of a defining function ρ that is $\mathcal{C}^r$-smooth in a neighborhood of $\overline{\Omega}$, and such that $\hat{\rho} := -(-\rho)^\gamma$ is a strictly plurisubharmonic bounded exhaustion function on Ω for any small enough number $\gamma \in (0, 1)$. In this situation, we may then choose $\eta = -\log(-\hat{\rho}) = -\gamma \log(-\rho)$ as our function in $\mathcal{SBG}_1(\Omega)$.

Remark 4.3.5. The results of Diederich and Fornæss have first been extended to relatively compact strongly pseudoconvex domains with $\mathcal{C}^1$ boundary by Kerzman-Rosay [KR81], and then further extended to pseudoconvex domains with Lipschitz boundary by Demailly [Dem87]. More recently, Avelin, Hed and Persson extended these results to pseudoconvex domains with Log-Lipschitz boundary (see [AHP12, Theorem 3, Corollary 4]).

Example 4.3.6. Consider the punctured unit disk $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Then the function $\eta : z \mapsto -\log\left(\log\left(|z|^{-2}\right)\right)$ is of self-bounded gradient. Indeed, $-e^{-\eta(z)} = \log\left(|z|^2\right)$ and so

$$\partial\bar{\partial}\left(-e^{-\eta}\right) = \partial\bar{\partial}\left(\log(|z|^2)\right) = 0$$

since $z \mapsto \log\left(|z|^2\right)$ is pluriharmonic for $z \neq 0$.

Note also that

$$\partial\bar{\partial}\eta = \partial\left(-\frac{d\bar{z}}{\bar{z}\log\left(|z|^2\right)}\right) = \frac{dz \wedge d\bar{z}}{|z|^2\left(\log\left(|z|^2\right)\right)^2}.$$

And so we see that

$$\partial\bar{\partial}\eta = \left(-\frac{d\bar{z}}{\bar{z}\log\left(|z|^2\right)}\right) \wedge \left(-\frac{dz}{z\log\left(|z|^2\right)}\right) = \partial\eta \wedge \bar{\partial}\eta,$$

which implies that $\partial\bar{\partial}\left(-e^{-\eta}\right) = 0$.

Example 4.3.7. Let X be a complex manifold. If $T \in \mathcal{O}(X) \cap L^\infty(X)$, then the function $\eta := -\log\left(\log\left(\|T\|_\infty^2 \cdot |T|^{-2}\right)\right)$ is of self-bounded gradient on the manifold $X - \{T = 0\}$. Indeed, by rescaling, we may assume that $\|T\|_\infty = 1$, and then $-e^{-\eta} = \log\left(|T|^2\right)$ is plurisubharmonic, and in fact pluriharmonic in $X - \{T = 0\}$ (by the Poincaré-Lelong formula). Alternatively, we can also see that η is simply the pullback of the function $z \mapsto -\log\left(\log\left(|z|^{-2}\right)\right)$ on $\mathbb{D}^*$ from the previous example.

Example 4.3.8. (Hyperconvex manifolds) A *hyperconvex* manifold X is a manifold that admits a bounded strictly plurisubharmonic exhaustion function $\psi : X \to [-\infty, b)$ (see [Ste74]). Define $\eta := -\log(b - \psi)$. Since $-e^{-\eta} = \psi - b$, it follows that $\eta \in \mathcal{SBG}_1(X)$.

Clearly, we may assume that the exhaustion is negative.

Remark 4.3.9. Note that not every pseudoconvex domain is hyperconvex. A counterexample is the Hartogs triangle $T = \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}$ as observed by Diederich and Fornæss in [DF75] and [DF77].

Remark 4.3.10. Let X be a complete hyperconvex manifold and let $\psi : X \to [-\infty, 0)$ be its plurisubharmonic exhaustion function. By regularizing ψ, one sees that X is hyperconvex if and only if it admits a smooth plurisubharmonic exhaustion function $\tilde{\psi} : X \to [-1, 0)$.

Remark 4.3.11. By Example 4.3.4 and Remark 4.3.5, any relatively compact strongly pseudoconvex with Log-Lipschitz boundary in a Stein manifold is hyperconvex.

Hyperconvex domains and functions of self-bounded gradient are related as follows.

Proposition 4.3.12. [Ohs18, Proposition 4.6] *A manifold X is hyperconvex if and only if there exists a strictly plurisubharmonic exhaustion function ψ on X that is in $\mathcal{SBG}_c(X)$ for some positive constant c.*

Although hyperconvex domains are of interest in other areas of several complex variables (such as pluripotential theory), we will not be directly concerned with them in this thesis.

It is worth noting that the existence of a non-trivial function of self-bounded gradient is a complex geometric hypothesis on a complex manifold. Indeed, on a given complex manifold X, such a non-trivial function exists if and only if X admits a bounded non-constant plurisubharmonic function. For instance, on $\mathbb{C}^n$, there is no such nontrivial function.

4.4 Runge approximation theorem

We state here a Runge approximation theorem for holomorphic sections of a vector bundle over a complex manifold which can be proved using L^2-estimates for the $\bar{\partial}$-operator.

Proposition 4.4.1. *Let Y be a Stein manifold, and let Ω_0 and Ω_1 be smoothly bounded pseudoconvex domains in Y with Ω_0 relatively compact Ω_1. Assume there is a smooth plurisubharmonic function ρ in $\overline{\Omega}_1$ such that $\Omega_0 = \{z \in \Omega_1 : \rho(z) < 0\}$. Let $V \to \Omega_1$ be a* *holomorphic vector bundle, and let h be a Hermitian metric for $V \to \Omega_1$. Then holomorphic sections in $L^2(\Omega_1, h)$ are dense in the space of holomorphic sections in $L^2\left(\Omega_0, h\big|_{\Omega_0}\right)$.*

See [Dem96], for example, for a proof.

4.5 Optimizing the curvature hypothesis of the Donnelly-Fefferman-Ohsawa theorem

Recall that if η is of self-bounded gradient and $\delta \in (0,1)$, then the curvature hypothesis

$$\Theta(h_0) + \big(\mathbf{Ric}(g) + (1-\delta)\partial\bar{\partial}\eta\big) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} \Phi \otimes \mathrm{Id}_V$$

implies the curvature hypothesis of the Donnelly-Fefferman-Ohsawa theorem. In particular, this allows to assume a certain amount of curvature negativity. Specifically, we can afford as much as $-(1-\delta)\partial\bar{\partial}\eta$ curvature negativity.

This begs the question: can we maximize the quantity $\partial\bar{\partial}\eta$ over the space of functions of self-bounded gradient?

A naive approach would be to rescale η. If $\partial\bar{\partial}\eta > 0$, then one can consider $\alpha\partial\bar{\partial}\eta$ for some large positive constant α. However, it is entirely possible that $\alpha\eta$ might no longer be of self-bounded gradient if $\alpha > 1$ since

$$\partial\bar{\partial}\left(\alpha\eta\right) - \partial\left(\alpha\eta\right) \wedge \bar{\partial}\left(\alpha\eta\right) = \alpha\left(\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta\right) - \alpha(\alpha-1)\left(\partial\eta \wedge \bar{\partial}\eta\right).$$

The potential curvature gain from $\alpha\eta$ in Theorem 4.2.1 is given by

$$\begin{aligned} 2\partial\bar{\partial}(\alpha\eta) - (1+\delta)\left(\partial(\alpha\eta) \wedge \bar{\partial}(\alpha\eta)\right) &= (1-\delta)\alpha\left(\partial\bar{\partial}\eta - \left(\frac{\alpha-1}{1-\delta}\right)\left(\partial\eta \wedge \bar{\partial}\eta\right)\right) \\ &\quad + (1+\delta)\alpha\left(\partial\bar{\partial}\eta - \partial\eta \wedge \bar{\partial}\eta\right). \end{aligned}$$

The difference in potential gains for $\alpha \neq 1$ and $\alpha = 1$ can be written as

$$\left[(\alpha-1)\left(2\partial\bar{\partial}\eta-(1+\delta)\left(\partial\eta\wedge\bar{\partial}\eta\right)\right)\right]-\left[\alpha\left(\alpha-1\right)(1+\delta)\left(\partial\eta\wedge\bar{\partial}\eta\right)\right].$$

Assuming that the estimate $\partial\bar{\partial}\eta \geq \partial\eta\wedge\bar{\partial}\eta$ is sharp, it follows that

$$\begin{aligned}
&\left[(\alpha-1)\left(2\partial\bar{\partial}\eta-(1+\delta)\left(\partial\eta\wedge\bar{\partial}\eta\right)\right)\right]-\left[\alpha\left(\alpha-1\right)(1+\delta)\left(\partial\eta\wedge\bar{\partial}\eta\right)\right]\\
&\geq(\alpha-1)\left(1-\delta-\alpha(1+\delta)\right)\partial\bar{\partial}\eta\\
&=-(\alpha-1)\left(\alpha-1+(\alpha+1)\delta\right)\partial\bar{\partial}\sigma,
\end{aligned}$$

which is negative as soon as $\alpha > 1$. On the other hand, if $\alpha < 1$, one gets a curvature gain if $\frac{1-\alpha}{1+\alpha} < \delta$, i.e. if $\alpha > \frac{1-\delta}{1+\delta}$.

Now if $\delta \geq 1$, then the gain

$$(1-\delta)\alpha\left(\partial\bar{\partial}\eta-\left(\frac{\alpha-1}{1-\delta}\right)\left(\partial\eta\wedge\bar{\partial}\eta\right)\right)+(1+\delta)\alpha\left(\partial\bar{\partial}\eta-\partial\eta\wedge\bar{\partial}\eta\right)$$

for $\alpha\eta$ is non-positive (even when $\alpha \geq 1$). Thus, we may as well assume $\delta \in (0,1)$ when thinking about curvature gain, and so the smaller the δ, the larger the gain. However, taking δ to be small increases the lower bound $\frac{1-\delta}{1+\delta}$ (for $\alpha < 1$) to 1. This shows that rescaling does not help when it comes to optimizing the curvature assumption in Theorem 4.2.1.

For the time being, maximizing $\partial\bar{\partial}\eta$ over the space of functions of self-bounded gradient remains an open question, unfortunately.

Chapter 5

Berndtsson's complex Brunn-Minkowski theory

The contents of this chapter have largely been adapted from [Ber17]. The reader may also refer to [Ber18] for a more general (albeit more concise) exposition.

5.1 Introducing complex Brunn-Minkowski theory

5.1.1 Classical Brunn-Minkowski theory

Let A_0 and A_1 be two *convex bodies* in $\mathbb{R}^n$, i.e. compact convex sets with non-empty interior. Their *Minkowski sum* is then defined as

$$A_0 + A_1 := \{a_0 + a_1; a_0 \in A_0, a_1 \in A_1\}. \tag{5.1.1}$$

A fundamental theorem of convex geometry is the Brunn-Minkowski theorem.

Theorem 5.1.1. (Brunn-Minkowski) *Suppose that A_0 and A_1 are nonempty. Then the following inequality holds.*

$$|A_0 + A_1|^{1/n} \geq |A_0|^{1/n} + |A_1|^{1/n}.$$

Here, $|A|$ denotes the Lebesgue measure of the measurable set $A \subset \mathbb{R}^n$. Theorem 5.1.1 was first proved by Brunn in 1887 in his thesis for $n = 2$ and later generalized to arbitrary dimensions by Minkowski in 1896. It was further shown to hold for arbitrary non-empty compact sets by Lyusternik in 1935.

A classical application of the Brunn-Minkowski theorem is a proof of the isoperimetric inequality. Let $f(\varepsilon) = |A + \varepsilon B|$ for $\varepsilon > 0$ small, where B is the unit ball. Then

$$f(\varepsilon) = |A| + \varepsilon\,|\partial A| + o(\varepsilon),$$

where $|\partial A|$ is the $(n-1)$-dimensional volume of the boundary ∂A of A. The Brunn-Minkowski theorem implies

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (f(\varepsilon))^{1/n} \geq |B|^{1/n},$$

whence

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq n\,|B|^{1/n}.$$

If $A = B$, we have equality here, since $B + \varepsilon B = (1+\varepsilon)B$ when B is convex. Thus

$$n\,|B|^{1/n} = \frac{|\partial B|}{|B|^{1-1/n}}$$

and we get

$$\frac{|\partial A|}{|A|^{1-1/n}} \geq \frac{|\partial B|}{|B|^{1-1/n}},$$

which is the classical isoperimetric inequality. This discussion can be generalized by defining the surface area (as Minkowski did) as

$$S(A) := \lim_{\varepsilon \to 0^+} \frac{|A + \varepsilon B| - |A|}{\varepsilon}$$

for a fixed convex body B, which is not necessarily the unit ball, and for a suitable set A in $\mathbb{R}^n$. Then, using a similar argument will produce a more general isoperimetric-type inequality since we only used the convexity of B.

We now discuss an alternative formulation of the Brunn-Minkowski theorem which is more analytic in nature.

First, notice that the Brunn-Minkowski theorem is equivalent to the following inequality for any two nonempty convex bodies A_0 and A_1 in $\mathbb{R}^n$.

$$\forall t \in [0,1] : |(1-t)A_0 + tA_1|^{1/n} \geq (1-t)\,|A_0|^{1/n} + t\,|A_1|^{1/n}\,. \tag{5.1.2}$$

This follows by replacing A_0 and A_1 in the Brunn-Minkowski theorem by $(1-t)A_0$ and tA_1 respectively, and using the positive homogeneity of degree n of the Lebesgue measure in $\mathbb{R}^n$, i.e. $|\lambda A| = \lambda^n\,|A|$ for $\lambda > 0$. In fact, this homogeneity yields another equivalent form of the Brunn-Minkowski theorem, whereby t and $1-t$ can be replaced by any arbitrary positive scalars r and s respectively.

Now, let $A_t := tA_1 + (1-t)A_0$ for $0 \leq t \leq 1$, where A_0 and A_1 are convex bodies. Then $t \mapsto |A_t|^{1/n}$ is concave by (5.1.2), and since each non-negative concave function is log-concave, we obtain

$$|A_t| \geq |A_0|^t\,|A_1|^{1-t}\,. \tag{5.1.3}$$

Conversely, (5.1.3) implies the concavity of the function $t \mapsto |A_t|^{1/n}$. This is interesting since not every non-negative log-concave function is concave. (Simply consider the Gaussian function $x \mapsto e^{-x^2}$.) To see that the concavity of the function $t \mapsto |A_t|^{1/n}$ follows from (5.1.3), let B_0 and B_1 be non-empty convex bodies, let

$$t = \frac{|B_1|^{1/n}}{|B_0|^{1/n} + |B_1|^{1/n}},$$

and apply the inequality (5.1.3) to $A_0 := |B_0|^{-1/n}\,B_0$ and $A_1 := |B_1|^{-1/n}\,B_1$. This results in the classical Brunn-Minkowski inequality, which is equivalent to the inequality (5.1.2).

Therefore, the Brunn-Minkowski theorem is equivalent to the concavity of the function $t \mapsto \log\left(|A_t|\right)$ where A_t is a convex sum of convex bodies.

More generally, let $\mathcal{A} \subset \mathbb{R}^{n+1}$ be a convex body, and let $\mathcal{A}_t := \{x \in \mathbb{R}^n : (t,x) \in \mathcal{A}\}, t \in \mathbb{R}$ be the corresponding t-slice of $\mathcal{A}$.

Theorem 5.1.2. *The function* $t \mapsto |\mathcal{A}_t|^{1/n}$ *is concave on the interval where it is nonzero.*

Interestingly, this theorem is equivalent to the Brunn-Minkowski theorem. Given A_0 and A_1, we may construct a convex body $\mathcal{A} \subset \mathbb{R}^{n+1}$ whose slices are given by $\mathcal{A}_t := tA_1 + (1-t)A_0$ for $0 \leq t \leq 1$. Therefore, Theorem 5.1.2 implies the Brunn-Minkowski theorem by the previous argument. Conversely, given two slices $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A} \subset \mathbb{R}^{n+1}$, then $t\mathcal{A}_1 + (1-t)\mathcal{A}_0 \subset \mathcal{A}_t$ if $\mathcal{A}$ is convex, whence

$$|\mathcal{A}_t|^{1/n} \geq t\,|\mathcal{A}_0|^{1/n} + (1-t)\,|\mathcal{A}_1|^{1/n}$$

by the Brunn-Minkowski theorem. In particular, the following corollary is again equivalent to the Brunn-Minkowski theorem as it implies (5.1.3).

Corollary 5.1.3. *The function* $t \mapsto \log\left(|\mathcal{A}_t|\right)$ *is concave wherever it is defined.*

This corollary, and more specifically, its generalization known as the *Prékopa-Leindler inequality* is central to building the complex analogue of Brunn-Minkowski theory. In what follows, let $\dot{\varphi}$ denote $\partial\varphi/\partial t$, and let $\ddot{\varphi}$ denote $\partial^2\varphi/\partial t^2$. Likewise, let φ' denote $\partial\varphi/\partial x$, let φ'' denote $\partial^2\varphi/\partial x^2$.

Theorem 5.1.4. (Prékopa-Leindler) *Let* $\varphi : (t,x) \mapsto \varphi(t,x)$ *be a convex function in* $\mathbb{R}^{n+1}$. *Let*

$$\Phi(t) := -\log\left(\int_{\mathbb{R}^n} e^{-\varphi(t,x)}dx\right). \tag{5.1.4}$$

Then $t \mapsto \Phi(t)$ *is convex or identically* $-\infty$.

To see how the Brunn-Minkowski theorem follows from the Prékopa-Leindler theorem, we need to define the notion of characteristic function for a convex set. The latter is motivated by log-concave measures – i.e. measures of the form $e^{-\varphi}d\mu$ for some convex function φ – and is more suitable than the usual characteristic function for convex analysis. Given a convex set A, the characteristic function $\chi_A : A \to \mathbb{R} \cup \{+\infty\}$ of A is defined as

$$\chi_A(x) = \begin{cases} 0, & x \in A; \\ +\infty, & x \notin A. \end{cases} \tag{5.1.5}$$

This function is convex, and applying the Prékopa-Leindler theorem to $\varphi = \chi_{\mathcal{A}}$, we recover Corollary 5.1.3.

One proof of the Prékopa-Leindler theorem is that of Brascamp and Lieb ([BL76]) which establishes a real-variable analogue of Hörmander's L^2-estimates for the $\bar{\partial}$-operator, as we will see. We first assume that φ is finite-valued and smooth. The general case follows from the fact that we can write any convex functions as an increasing limit of smooth finite-valued convex functions. Moreover, adding $\varepsilon\left(|t|^2 + |x|^2\right)$ to φ, for some $\varepsilon > 0$, we may assume that φ is strictly convex. Next, we see that we can reduce ourselves to the case $n = 1$. Indeed, let $(t, x) = (t, x_1, \cdots, x_n)$. If we first integrate $e^{-\varphi(t,x)}$ (in (5.1.4)) with respect to x_n, we obtain a function $e^{-\hat{\varphi}(t,x_1,\cdots,x_{n-1})}$, and if the theorem holds for $n = 1$, then this function $\hat{\varphi}$ is convex. We then simply iterate by integrating with respect to x_{n-1}, and so on. We thus assume that $n = 1$ without loss of generality. The rest of the proof is a matter of computing the second derivative of Φ. Adding a linear function to φ, we may assume that $\Phi(0) = 0 = \dot{\Phi}(0)$. Since $\Phi(0) = 0 = \dot{\Phi}(0)$,

$$\int_{\mathbb{R}} e^{-\varphi(0,x)} dx = 1 \text{ and } \int_{\mathbb{R}} \dot{\varphi}(0,x) e^{-\varphi(0,x)} dx = 0.$$

Hence

$$\ddot{\Phi}(0) = \int_{\mathbb{R}} \left(\ddot{\varphi}(0,x) - (\dot{\varphi}(0,x))^2\right) e^{-\varphi(0,x)} dx.$$

The key element of the proof is the following lemma, known as the *Brascamp-Lieb inequality.*

Lemma 5.1.5. (Brascamp-Lieb inequality) *Let ψ be a smooth strictly convex function on $\mathbb{R}$ with $e^{-\psi} \in L^1(\mathbb{R})$ and let u be a function such that*

$$\int_{\mathbb{R}} |u|^2 e^{-\psi} dx < +\infty \textit{ and } \int_{\mathbb{R}} u e^{-\psi} dx = 0.$$

Then

$$\int_{\mathbb{R}} |u|^2 e^{-\psi} dx \leq \int_{\mathbb{R}} \frac{|u'|^2}{\psi''} e^{-\psi} dx.$$

See [BL76] for a proof.

By adding $\varepsilon f(x)$ to $\varphi(t,x)$, where f is a rapidly growing convex function, we may assume that $\dot{\varphi} \in L^2(\mathbb{R}, e^{-\varphi})$ and then let $\varepsilon \to 0$ at the end. We now apply the Brascamp-Lieb inequality to $u = \dot{\varphi}(0,x)$:

$$\ddot{\Phi}(0) \geq \int_{\mathbb{R}} \left(\ddot{\varphi}(0,x) - \frac{(\dot{\varphi}'(0,x))}{\varphi''(0,x)} \right) e^{-\varphi(0,x)} dx$$

$$= \int_{\mathbb{R}} \left(\frac{\varphi''(0,x)\ddot{\varphi}(0,x) - (\dot{\varphi}'(0,x))^2}{\varphi''(0,x)} \right) e^{-\varphi(0,x)} dx.$$

The numerator $\varphi''(0,x)\ddot{\varphi}(0,x) - (\dot{\varphi}'(0,x))^2$ is exactly the determinant of the Hessian of φ, at $t = 0$, with respect to both t and x. Since φ is convex, this is nonnegative, which completes the proof of the Prékopa-Leindler theorem.

Remark 5.1.6. A similar proof can be carried out for general n without resorting to induction. The bound obtained for $\ddot{\Phi}$, assuming that φ is strictly convex, is then

$$\ddot{\Phi} \geq \int_{\mathbb{R}^n} \left(H(\varphi)/D_x^2(\varphi) \right) e^{-\varphi} dx,$$

where $H(\varphi)/D_x^2(\varphi)$ denotes the Schur complement of the block $D_x^2(\varphi)$ – the Hessian of φ with respect to x – of the full Hessian

$$H(\varphi) = \begin{pmatrix} D_t^2(\varphi) & D_t D_x(\varphi) \\ D_x D_t(\varphi) & D_x^2(\varphi) \end{pmatrix}.$$

Here $D_t^2(\varphi)$ and $D_t D_x(\varphi) = D_x D_t(\varphi)$ denote the Hessian with respect to t and the mixed Hessians with respect to both t and x, respectively. This Schur complement interpretation will be central to our work generalizing Berndtsson's Nakano positivity theorem.

5.1.2 Complex analogues of Brunn-Minkowski theory

As mentioned earlier, the Brascamp-Lieb inequality is the real variable version of Hörmander's L^2-estimate for the $\bar{\partial}$-operator. The simplest case of Hörmander's estimate is when φ is a smooth strictly subharmonic function in $\mathbb{C}$. We then let u be a smooth function on $\mathbb{C}$, satisfying

$$\int_{\mathbb{C}} u\bar{h} e^{-\varphi} dV(z) = 0,$$

for all holomorphic functions h satisfying the appropriate L^2-condition, where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}$. This is a direct counterpart to the condition

$$\int_{\mathbb{R}} ue^{-\psi}dx = 0$$

in the real case since the Brascamp-Lieb condition corresponds to the orthogonality of u to all constant functions with respect to the L^2-inner product. In other words, u is orthogonal to all functions in the kernel of the usual real differentiation operator d. The Hörmander condition corresponds to orthogonality to all holomorphic functions; that is all functions in the kernel of $\bar{\partial}$. Under these conditions, we have the following form of Hörmander's L^2-estimate for the $\bar{\partial}$-operator that is due to Skoda.

$$\int_{\mathbb{C}} |u|^2 e^{-\varphi} dV(z) \leq \int_{\mathbb{C}} \frac{|\bar{\partial}u|^2}{\Delta\varphi} dV(z).$$

Once again, this is clearly similar to the Brascamp-Lieb estimate. The orthogonality condition means that u is the solution of minimal L^2-norm to a $\bar{\partial}$-equation, and this is how Hörmander's theorem is typically thought of – a theorem on the existence of solutions to the $\bar{\partial}$-equation with L^2-estimates. Similarly, the Brascamp-Lieb theorem is a theorem providing an L^2-estimate for the d-equation, and is in fact a special case of Hörmander's theorem; when the functions involved do not depend on the imaginary part of z.

Recall that a function u of several complex variables in $\mathbb{C}^n$ is *plurisubharmonic* if it is upper-semicontinuous, not identically $-\infty$, and is subharmonic along any complex line. By the sub-mean value property for subharmonic fuctions, such a function is always locally integrable. If u is smooth, then u is plurisubharmonic if and only if its complex Hessian

$$H_{\mathbb{C}}(u) = \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)_{1 \leq j,k \leq n}$$

is semipositive definite. In general, a function u is plurisubharmonic in the sense of distributions if and only if

$$\sum_{1 \leq j,k \leq n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} a_j \bar{a}_k$$

is a positive measure for any constant vector $a = (a_1, \cdots, a_n) \in \mathbb{C}^n$. The most naive generalization of the Prékopa-Leindler theorem would then be to postulate that if φ be plurisubharmonic in $\mathbb{C}^m \times \mathbb{C}^n$, then

$$t \mapsto \Phi(t) := -\log\left(\int_{\mathbb{C}^n} e^{-\varphi(t,z)} dV(z)\right)$$

is plurisubharmonic in $\mathbb{C}^m$. However, this claim is simply false. As a counter-example, consider $\varphi(t,z) := |z - \bar{t}|^2 - |t|^2 = |z|^2 - \mathrm{Re}(tz)$. Clearly, φ is plurisubharmonic in $\mathbb{C}^2$ but $t \mapsto \Phi(t) = |t|^2 + C$ is not subharmonic.

As observed by Berndtsson, one should think of the integral of $e^{-\varphi}$ as the squared L^2-norm of the function 1 with respect to the weight $e^{-\varphi}$. It is then natural to consider L^2-norms of holomorphic functions in the complex case – that is

$$\|f\|^2_{\varphi(t,\cdot)} := \int_{\mathbb{C}^n} |f(z)|^2 e^{-\varphi(t,z)} dV(z),$$

or similar expressions where the integration is done over slices of pseudoconvex domains in $\mathbb{C}^n$ instead of the total space. One can then consider the Bergman space $\mathcal{H}^2_t$ of holomorphic functions with finite L^2-norm. This gives us a family of Hilbert spaces indexed by t or in other words, a field of Hilbert spaces, as mentioned in Section 1.1. Assuming the Bergman spaces have equivalent norms, we have the structure of a vector bundle of infinite rank or a *Hilbert* bundle. (See Section 1.1 for details.) The complex Brunn-Minkowski type theorem amounts to saying that the curvature of such bundles is non-negative, under certain assumptions on φ. In the case of Berndtsson, these assumptions are the plurisubharmonicity of φ on the total space. In this thesis, we consider situations in which the positivity assumptions can be weakened.

In general, we take two complex manifolds $\mathcal{X}$ and $\mathcal{U}$ of dimensions $n+m$ and n respectively, and a holomorphic submersion $p : \mathcal{X} \to \mathcal{U}$. Rather than holomorphic functions, we consider holomorphic sections of a holomorphic Hermitian vector bundle $\mathcal{V}$ over $\mathcal{X}$. The weight function $e^{-\varphi}$ is replaced by a Hermitian metric h for the bundle $\mathcal{V} \to \mathcal{X}$. The plurisubharmonicity of φ then corresponds to the curvature non-negativity of h. However, to produce a holomorphic

Hilbert bundle, one needs to place further restrictions.

The simplest thing to do is to let $\mathcal{X} = U \times X$ where X is a relatively compact complete Kähler submanifold of a Stein manifold. Since curvature is local, we may assume that U is a domain in $\mathbb{C}^m$. In this case, we have a trivial family of complete Stein manifolds. The underlying vector spaces of the Bergman spaces $\mathcal{H}^2\left(X, h^{[t]}\right)$ are equal as subspaces of $\Gamma_{\mathcal{O}}(X, V)$, and they have equivalent norms as remarked in Section 1.1. Therefore, these Bergman spaces naturally fit together to form a trivial holomorphic Hilbert bundle. Here $h^{[t]}$ denotes the fiberwise restriction of the Hermitian metric h defined on the pullback bundle $\pi_{\overline{X}}^* V \to U \times \overline{X}$ where $\pi_{\overline{X}}$ denotes the canonical projection $U \times \overline{X} \to \overline{X}$. We thus obtain a trivial holomorphic Hilbert bundle E_h whose fiber at $t \in U$ is $\mathcal{H}_t^2$, equipped with the non-trivial Hermitian metric given by the fiberwise L^2-norms $\|\cdot\|_{h^{[t]}}^2$.

Berndtsson's *Annals of Mathematics* paper [Ber09b] treats the case of a trivial family for bounded pseudoconvex domains in $\mathbb{C}^n$ and the more general case of a holomorphic fibration with a Kähler total space and compact Kähler fibers (see also [Ber09a] and [Ber13]). He also studies this problem more generally in [Ber11]. Further general expositions can be found in [Wan17] and [Var19] (for Stein manifolds as the total spaces). For generalities regarding holomorphic Hilbert (and more broadly Banach) bundles, we refer the reader to [LS14], [Tra14] and [Lem15].

For the purposes of this thesis, we will present a weaker version of Berndtsson's Nakano positivity theorem (Theorem 1.1.1) – asserting only Griffiths positivity as a consequence – in the case of trivial families of bounded pseudoconvex domains in $\mathbb{C}^n$, following [Ber17]. We do so because we will present the stronger version of the theorem, under our weaker curvature positivity assumptions, later in the last chapter of this thesis. Another reason for presenting the weaker version is that many important applications of Berndtsson's theorem (e.g. [BL16] and [Ber15]) only require Griffiths positivity.

5.2 Berndtsson's theorem on the curvature positivity of holomorphic Hilbert bundles

Let U be a domain in $\mathbb{C}$ and Ω be a bounded domain in $\mathbb{C}^n$. Let φ be a function on $U \times \Omega$ that is smooth up to the boundary on $U \times \overline{\Omega}$. For each $t \in \Omega$, define

$$\mathcal{H}_t^2 := \mathcal{H}^2\left(\Omega, e^{-\varphi(t,\cdot)}\right) := \left\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f(z)|^2 e^{-\varphi(t,z)} dV(z) < +\infty \right\}.$$

Since φ is smooth up to the boundary and Ω is bounded, the Bergman inequality holds and so $\mathcal{H}_t^2$ is a (reproducing kernel) Hilbert space. In particular, the underlying vector spaces of the $\mathcal{H}_t^2$ are identical as subspaces of $\mathcal{O}(\Omega)$, so we can form the globally trivial holomorphic Hilbert bundle E with total space $U \times \mathcal{H}_0^2$. Since the L^2-norms vary with t, the Hilbert bundle E has a non-trivial metric given by the varying L^2-norms.

Theorem 5.2.1. (Berndtsson) *If Ω is pseudoconvex and φ is strictly plurisubharmonic, then $\left(E, (\cdot,\cdot)_{\varphi(t,\cdot)}\right)$ is Griffiths positive.*

The proof of this theorem uses Theorems 2.4.14 and 3.3.1. To do so, we view E as a subbundle of the globally trivial Hilbert bundle F whose fiber over $t \in U$ is given by $F_t := L^2\left(\Omega, e^{-\varphi(t,\cdot)}\right)$. Given our hypotheses on φ and Ω, the underlying vector spaces of the fibers F_t are also identical as vector spaces, while their Hilbert norms vary with t. This bundle is holomorphic because an L^2 basis of any fiber spans all the other fibers.

A smooth (resp. measurable, resp. holomorphic) section of E is a map $t \mapsto f_t$ that is smooth (resp. measurable, resp. holomorphic) as a map from t to E_t.

Proposition 5.2.2. *The Chern connection of F is the (densely defined) operator defined by*

$$\nabla^F f = d_t f - (\partial_t \varphi) f,$$

where $d_t = \partial_t + \bar{\partial}_t$ and ∂_t are exterior derivatives with respect to t for z fixed.

Proof. Clearly, ∇^F defines a connection and $\nabla^{F,(0,1)} = \bar{\partial}$. Now given two sections f_1 and f_2,

$$\begin{aligned}
d_t\left(f_1, f_2\right)_{\varphi(t,\cdot)} &= d_t \int_\Omega f_1 \bar{f}_2 e^{-\varphi(t,\cdot)} dV \\
&= \int_\Omega \partial_t \left(f_1 \bar{f}_2 e^{-\varphi(t,\cdot)}\right) dV + \int_\Omega \bar{\partial}_t \left(f_1 \bar{f}_2 e^{-\varphi(t,\cdot)}\right) dV \\
&= \int_\Omega \left(\partial_t f_1 \bar{f}_2 e^{-\varphi(t,\cdot)} + f_1 \overline{\bar{\partial}_t f_2} e^{-\varphi(t,\cdot)} - (\partial_t \varphi) f_1 \bar{f}_2 e^{-\varphi(t,\cdot)}\right) dV \\
&\quad + \int_\Omega \left(\bar{\partial}_t f_1 \bar{f}_2 e^{-\varphi(t,\cdot)} + f_1 \overline{\partial_t f_2} e^{-\varphi(t,\cdot)} - (\bar{\partial}_t \varphi) f_1 \bar{f}_2 e^{-\varphi(t,\cdot)}\right) dV \\
&= \int_\Omega \left((\partial_t + \bar{\partial}_t) f_1 - (\partial_t \varphi) f_1\right) \bar{f}_2 e^{-\varphi(t,\cdot)} dV + \int_\Omega f_1 \overline{\left((\partial_t + \bar{\partial}_t) f_1 - (\partial_t \varphi) f_2\right)} e^{-\varphi(t,\cdot)} dV \\
&= \int_\Omega \left(d_t f_1 - (\partial_t \varphi) f_1\right) \bar{f}_2 e^{-\varphi(t,\cdot)} dV + \int_\Omega f_1 \overline{(d_t f_1 - (\partial_t \varphi) f_2)} e^{-\varphi(t,\cdot)} dV \\
&= \left(\nabla^F f_1, f_2\right)_{\varphi(t,\cdot)} + \left(f_2, \nabla^F f_2\right)_{\varphi(t,\cdot)},
\end{aligned}$$

which shows that ∇^F is compatible with the metric $(\cdot,\cdot)_{\varphi(t,\cdot)}$. □

It is clear that the connection form A of ∇^F is given by multiplication by $-\partial_t \varphi$ and so the curvature Θ^F is given by multiplication by $\bar{\partial}_t(-\partial_t \varphi) = \partial_t \bar{\partial}_t \varphi$. To compute the curvature of E, we can use Theorem 2.4.14 which tells us that:

$$\left(\Theta^E(f), f\right)_{\varphi(t,\cdot)} = \left(\Theta^F(f), f\right)_{\varphi(t,\cdot)} - \|\mathfrak{p}(f)\|^2_{\varphi(t,\cdot)}.$$

We now need to compute the map $\mathfrak{p}$. Recall that the action $\mathfrak{p}$ on a smooth section f of E is defined as follows. (See Proposition 2.4.13.)

$$\mathfrak{p}(f) = \mathcal{P}_t^\perp \left(d_t f - (\partial_t \varphi) f\right) = -\mathcal{P}_t^\perp \left((\partial_t \varphi) f\right),$$

where $\mathcal{P}_t^\perp$ denotes the orthogonal complement of the Bergman projection $L^2_t \to \mathcal{H}^2_t$.

Define $u_t := -\mathcal{P}_t^\perp\left((\partial_t \varphi) f\right)$. Now, for each fixed t, u_t is the solution of minimal norm of the $\bar{\partial}_z$-equation

$$\bar{\partial}_z u_t = \bar{\partial}_z \left(\partial_t \varphi f\right) = f \bar{\partial}_z \partial_t \varphi =: \alpha,$$

since it is orthogonal to the space of holomorphic functions by construction. Therefore, by Theorem 3.3.1

$$\int_\Omega |u_t(z)|^2 e^{-\varphi(t,z)} dV(z) \leq \int_\Omega |\alpha|^2_{\partial_z \bar{\partial}_z \varphi} e^{-\varphi(t,z)} dV(z) = \int_\Omega \sum_{\mu,\nu=1}^n \varphi^{z_\mu \bar{z}_\nu} \alpha_\nu \bar{\alpha}_\mu e^{-\varphi(t,z)} dV(z),$$

where $\alpha = \sum_{\mu=1}^{n} \alpha_\mu d\bar{z}_\mu$ and $\varphi^{z_\mu \bar{z}_\nu}$ is the $(\mu, \bar{\nu})$-component of the inverse of the z-Hessian of φ – i.e. the Hessian of φ with respect to z only. Therefore, putting everything together, we obtain:

$$\left(\Theta^E(f), f\right)_{\varphi(t,\cdot)} = \int_\Omega \left(\varphi_{t\bar{t}} - \sum_{\mu,\nu=1}^{n} \varphi^{z_\mu \bar{z}_\nu} \varphi_{t\bar{z}_\mu} \overline{\varphi_{t\bar{z}_\nu}}\right) |f(z)|^2 e^{-\varphi(t,z)} dV(z).$$

The quantity $\varphi_{t\bar{t}} - \sum_{\mu,\nu=1}^{n} \varphi^{z_\mu \bar{z}_\nu} \varphi_{t\bar{z}_\mu} \overline{\varphi_{t\bar{z}_\nu}}$ is exactly the Schur complement of the z-Hessian block of the full Hessian of φ (with respect to both z and t), and so it is positive-definite since φ is strictly plurisubharmonic. (For a geometric proof of this fact, we refer the reader to [Ber09b].) This completes the proof of Berndtsson's theorem.

5.3 Berndtsson's complex interpretations of the Prékopa-Leindler theorem

As discussed, the conclusion of the Prékopa-Leindler theorem is that the function

$$t \mapsto -\log\left(\int_{\mathbb{R}^n} e^{-\varphi(t,x)} dx\right)$$

is convex whenever φ is convex in both variables. The complex geometric setting is different is that the Nakano positivity of our vector bundle does not imply the log-plurisuperharmonicity of the norms of its holomorphic sections.

A holomorphic section s of a line bundle is locally given by $s = fe$ for some holomorphic function f, given a frame e. Given a metric h for the line bundle, the norm of s is given by $|s|_h^2 = |f|^2 e^{-\varphi}$ where $e^{-\varphi} = |e|_h^2$. Thus,

$$\partial\bar{\partial}\left(-\log\left(|s|_h^2\right)\right) = \partial\bar{\partial}\varphi,$$

where $f \neq 0$. If the curvature is (semi)positive – i.e., $\partial\bar{\partial}\varphi \geq 0$ – then $-\log\left(|s|_h^2\right)$ is plurisubharmonic where $h \neq 0$. If E were of rank 1, and f had no zeros, (which is not the case, of course), then it would follow that

$$t \mapsto -\log\left(\int_\Omega |f|^2 e^{-\varphi(t,z)} dV(z)\right)$$

is plurisubharmonic – a clear parallel to the real case.

For the case of higher rank, consider the case in which E splits as a direct sum of trivial line bundles. Then we would have a global frame of holomorphic sections $e_1, \dots, e_r$ and a section would be expressed as $s = \sum_{j=1}^{r} f_j e_j$ for a collection of holomorphic functions $\{f_j\}_{1 \leq j \leq r}$. In particular, the norm of s would be

$$|s|_h^2 = \sum_{j=1}^{r} |f_j|^2 e^{-\varphi_j},$$

where $e^{-\varphi_j} := |e_j|_h^2$. Unfortunately, there is no simple formula for $\partial\bar{\partial}\left(-\log\left(|s|_h^2\right)\right)$, and it is not always positive, even if all the φ_j are identically zero.

In view of these observations, one way to obtain explicit convexity statements from Berndtsson's theorem is to construct other bundles of rank 1 from the bundle E. Alternatively, one can consider the dual bundle E^*. Indeed, the Griffiths positivity of E is equivalent to the Griffiths negativity of E^*, which is in turn equivalent to the plurisubharmonicity of the function $t \mapsto \log\left(\|\xi\|_{*,\varphi(t,\cdot)}^2\right)$ for any non-zero holomorphic section ξ of E^*, by Theorem 2.4.9.

The first approach leads to statements that are analogous to (and do, in fact recover) the Prékopa-Leindler theorem. The second approach leads to more interesting complex analytic and geometric applications, such as optimal L^2-extension theorems (see [BL16]), the log-plurisubharmonic variation of Bergman kernels (see [Ber05] and [Ber06]), and uniqueness theorems for (generalized) Kähler-Einstein metrics (see [Ber15]).

5.3.1 First type of interpretations of the complex Prékopa-Leindler theorem

First, let us consider balanced domains.

Definition 5.3.1. A domain Ω in $\mathbb{C}^n$ is *balanced* if $z \in \Omega$ implies that $\lambda z \in \Omega$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$.

Definition 5.3.2. A domain Ω in $\mathbb{C}^n$ is S^1*-invariant* if $z \in \Omega$ implies that $\lambda z \in \Omega$ for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Definition 5.3.3. A function ψ is S^1*-invariant* if $\psi(\lambda z) = \psi(z)$ for any $\lambda = e^{\sqrt{-1}\alpha}$ where $\alpha \in \mathbb{R}$.

Theorem 5.3.1. (Berndtsson) *Let Ω be a pseudoconvex balanced domain in $\mathbb{C}^n$ and let $\varphi : (t, z) \mapsto \varphi(t, z)$ be a plurisubharmonic in $U \times \Omega$ and S^1-invariant in z for any $t \in U$. Here U is a domain in $\mathbb{C}^m$. Then*

$$t \mapsto -\log\left(\int_\Omega e^{-\varphi(t,z)} dV(z)\right)$$

plurisubharmonic or identically equal to $-\infty$.

Proof. We may assume that Ω is bounded since any balanced domain can be exhausted by an increasing sequence of bounded balanced domains, and decreasing limits of plurisubharmonic functions are plurisubharmonic. Similarly, we may assume that φ is smooth by approximation. The fibers of E consist of holomorphic functions on Ω. Let $E_k, k \in \mathbb{N}$, denote the subbundle of E of homogeneous polynomials of degree k. If $f \in E_k$ and $g \in E_m$, then

$$\begin{aligned}\int_\Omega f(z)\bar{g}(z)e^{-\varphi(t,z)} dV(z) &= \int_\Omega f\left(e^{\sqrt{-1}\alpha z}\right) \bar{g}\left(e^{\sqrt{-1}\alpha}\right) e^{-\varphi(t,z)} dV(z) \\ &= e^{(k-m)\sqrt{-1}\alpha} \int_\Omega f(z)\bar{g}(z)e^{-\varphi(t,z)} dV(z),\end{aligned}$$

for any $\alpha \in \mathbb{R}$. Therefore, we see that E_k and E_m are orthogonal if $k \neq m$. This means that E is the direct sum of the holomorphic subbundles E_k. Therefore, by Schur complement theory, each E_k must be Griffiths positive since E is Griffiths positive by Berndtsson's theorem. In particular, E_0 is Griffiths positive. But since E_0 is a trivial line bundle and the constant function $\mathbf{1}$ is a global frame, $t \mapsto -\log\left(\|\mathbf{1}\|^2_{\varphi(t,\cdot)}\right)$ is plurisubharmonic, which proves the claim. $\square$

Next, consider $\mathbb{T}^n$-invariant domains.

Definition 5.3.4. A domain Ω in $\mathbb{C}^n$ is $\mathbb{T}^n$-invariant if $z = (z_1, \cdots, z_n) \in \Omega$ implies that $\left(e^{\sqrt{-1}\alpha_1} z_1, \cdots, e^{\sqrt{-1}\alpha_n} z_n\right) \in \Omega$ for all $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$.

Definition 5.3.5. A function ψ is $\mathbb{T}^n$-invariant if $\psi(z_1, \cdots, z_n) = \psi\left(e^{\sqrt{-1}\alpha_1} z_1, \cdots, e^{\sqrt{-1}\alpha_n} z_n\right)$ for all $\alpha \in \mathbb{R}^n$.

Now suppose that Ω is a bounded $\mathbb{T}^n$-invariant domain, and let φ be a plurisubharmonic function in $U \times \Omega$ that is also $\mathbb{T}^n$-invariant with respect to $z \in \Omega$. Any holomorphic function $f \in \Omega$ can be written as a Laurent series

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha.$$

Therefore, the bundle E decomposes as $E = \bigoplus_{\alpha \in \mathbb{Z}^n} E_\alpha$ where each E_α is spanned by z^α. As in the proof of Theorem 5.3.1, we can see that E_α is orthogonal to E_β for $\alpha \neq \beta$, and so each E_α is Griffiths positive by Berndtsson's theorem, since E is Griffiths positive. Moreover, every E_α is of rank 1 with a constant trivializing section $U \ni t \mapsto z^\alpha$ and so

$$t \mapsto \Phi_\alpha(t) := -\log\left(\|z^\alpha\|^2_{\varphi(t,\cdot)}\right) = -\log\left(\int_\Omega |z^\alpha|^2 e^{-\varphi(t,z)} dV(z)\right)$$

is a plurisubharmonic function of t for all α.

Theorem 5.3.2. (Berndtsson) *Let φ be a plurisubharmonic function in $U \times \Omega$ where U is a domain in $\mathbb{C}^m$ and $\Omega := \{\zeta : \mathrm{Re}(\zeta) \in D\}$ for a convex domain D in $\mathbb{R}^n$. Assume that φ does not does not depend on the imaginary part of ζ. Then*

$$t \mapsto -\log\left(\int_D e^{-\varphi(t,x)} dx\right)$$

is plurisubharmonic or identically $-\infty$.

Proof. Consider the map $\exp : \zeta \mapsto \left(e^{\zeta_1}, \cdots, e^{\zeta_n}\right)$ from $\mathbb{C}^n$ to $(\mathbb{C} - \{0\})^n$, and let $\tilde{\Omega}$ be the image of Ω under this map. Since D is convex, $\tilde{\Omega}$ is pseudoconvex. In addition, $\tilde{\Omega}$ is $\mathbb{T}^n$-invariant. Furthermore, since φ does not depend on $\mathrm{Im}(\zeta)$, there is a plurisubharmonic function Φ in $U \times \tilde{\Omega}$ such that $\tilde{\varphi}\,(t, \exp(\zeta)) = \varphi(t, \zeta)$. Clearly, $\tilde{\varphi}$ is $\mathbb{T}^n$-invariant with respect to z for each t. Upon exhausting D by an increasing sequence of strictly convex domains with smooth boundary, we may assume that D is bounded. We may also assume that φ is smooth by approximating it smoothly near the boundary of each exhausting domain. Then it follows that $\tilde{\Omega}$ is also bounded and that $\tilde{\varphi}$ is also smooth. Hence,

$$t \mapsto -\log\left(\int_{\tilde{\Omega}} |z^\alpha|^2\, e^{-\varphi(t,z)} dV(z)\right)$$

is plurisubharmonic. Changing variables $z = \exp(\zeta)$, it follows that the integral over $\tilde{\Omega}$ equals

$$\int_{[0,2\pi]^n \times D} e^{2\alpha\cdot x} e^{-\varphi(t,x)} e^{2\sum_{j=1}^n x_j} dxdy.$$

Hence

$$-\log\left(\int_D e^{2\alpha\cdot x - \varphi(t,x) + 2\sum_{j=1}^n x_j} dx\right)$$

is plurisubharmonic. Since $A(x) = 2\left(\alpha\cdot x + \sum_{j=1}^n x_j\right)$ is an affine function of x, we may replace φ by $\varphi + A$ and the theorem follows. □

Clearly, the Prékopa-Leindler theorem follows from Theorem 5.3.2. An important corollary of this result is Kiselman's minimum principle:

Theorem 5.3.3. (Kiselman's minimum principle) *Let φ satisfy the hypotheses of Theorem 5.3.2. Then $t \mapsto \inf_{\zeta\in\Omega} \varphi(t, \zeta)$ is subharmonic.*

Proof. We have

$$\begin{aligned}
\inf_{\zeta\in\Omega} \varphi(t,\zeta) = -\sup_{\zeta\in\Omega} -\varphi(t,\zeta) = -\sup_{\zeta\in\Omega} \log\left(e^{-\varphi(t,\zeta)}\right) &= -\log\left(\sup_{\zeta\in\Omega} e^{-\varphi(t,\zeta)}\right) \\
&= -\log\left(\lim_{p\to+\infty}\left[\int_D \left|e^{-\varphi(t,x)}\right|^p dx\right]^{1/p}\right) \\
&= \lim_{p\to+\infty} -\frac{1}{p}\log\left(\int_D e^{-p\varphi(t,x) - x^2} dx\right),
\end{aligned}$$

and by Theorem 5.3.2, the functions

$$-\frac{1}{p}\log\left(\int_D e^{-p\varphi(t,x)-x^2} dx\right)$$

are plurisubharmonic for all $p > 0$. Since the decreasing limit as p goes to $+\infty$ equals $\inf_{\zeta\in\Omega} \varphi(t,\zeta)$, the claim follows. $\square$

5.3.2 Second kind of interpretations of the complex Prékopa-Leindler theorem

A more general situation that we can consider is when the domains (in addition to the weight) vary with t. More explicitly, instead of simply looking at product domains of the form $U \times \Omega$, we can let $\mathcal{D}$ be a pseudoconvex domain in $\mathbb{C}^m_t \times \mathbb{C}^n_z$. The subscripts indicate that we take the t variable to be in $\mathbb{C}^m$ and z variable to be in $\mathbb{C}^n$.

Let U be the image of $\mathcal{D}$ under the projection to the t-coordinate, that is also open. For $t \in U$, we let $\mathcal{D}_t := \{z \in \mathbb{C}^n : (t,z) \in \mathcal{D}\}$ be the corresponding slice of $\mathcal{D}$. Furthermore, let φ be a plurisubharmonic function in $\mathcal{D}$. This time, the Bergman spaces of holomorphic functions are defined as

$$\mathcal{H}^2_t := \mathcal{H}^2\left(\mathcal{D}_t, e^{-\varphi(t,\cdot)}\right) := \left\{ f \in \mathcal{O}\left(\mathcal{D}_t\right) : \|f\|^2_{\varphi(t,\cdot)} := \int_{\mathcal{D}_t} |f(z)|^2 e^{-\varphi(t,z)} dV(z) < +\infty \right\}.$$

The situation that was previously considered was $\mathcal{D} = U \times \Omega$, so that all the $\mathcal{D}_t$ are identical, and our weight function was additionally bounded. However, as $\mathcal{D}_t$ varies, our family of domains is no longer locally trivial, and we do not necessarily have the structure of a bundle. That said, we may still define some kind of holomorphic structure by declaring that given a function $f(t,z)$ with $f_t := f(t,\cdot)$ in $\mathcal{H}^2_t$, f_t is a holomorphic section if f is holomorphic as a function of t and z jointly (or equivalently, separately in t and z by Hartogs's theorem on separate holomorphicity). In this setting, Xu Wang [Wan17] gives a formula for a Chern connection and a curvature operator, in addition to generalizing the curvature formula of Berndtsson. That said, Xu Wang has stronger hypotheses on the domain $\mathcal{D}$.

In this setting, Berndtsson proves a theorem that corresponds to Griffiths positivity by studying families of sections of the duals of $\mathcal{H}^2_t$. A holomorphic section of the dual family is defined to be a map $t \mapsto \xi_t \in \left(\mathcal{H}^2_t\right)^*$ such that $t \mapsto \xi_t(f_t)$ is holomoprhic in t for any holomorphic section f_t in the sense defined above. For such sections, the dual norm is defined as the operator norm:

$$\|\xi_t\|^2_{*,\varphi(t,\cdot)} := \sup_{f_t \in \mathcal{H}^2_t - \{0\}} \frac{|\xi_t(f_t)|^2}{\|f_t\|^2_{\varphi(t,\cdot)}}.$$

Theorem 5.3.4. (Berndtsson) *For each $t \in U$, let μ_t be a compactly supported measure in $\mathcal{D}_t$ with the property that*

$$t \mapsto \xi_t(f_t) := \int_{\mathcal{D}_t} f(t,z) d\mu_t(z)$$

is a holomorphic function if f is holomorphic in $\mathcal{D}$. Then the function $t \mapsto \log\left(\|\xi_t\|^2_{*,\varphi(t,\cdot)}\right)$ is plurisubharmonic or identically $-\infty$.

Proof. The proof consists of a reduction of the general case to the product case. Suppose for the moment that $\mathcal{D} = U \times \Omega$. As before, we may assume that the pseudoconvex domain Ω is bounded since we can exhaust it by an increasing sequence of bounded strictly pseudoconvex domains with smooth boundary. Near the closure of each such domain Ω_k, we can approximate φ by a decreasing sequence of smooth plurisubharmonic functions $\{\varphi_{(k),j}\}_{j=1}^{\infty}$. Since ξ_t is given by integraton against compactly supported measures, we can choose Ω_k to be large enough to contain the support of all the ξ_t. We then let j tend to $+\infty$ first, for k fixed. This gives us a decreasing family of plurisubharmonic functions $t \mapsto \log\left(\|\xi_t\|^2_{*,\varphi_{(k),j}(t,\cdot)}\right)$ tending to $t \mapsto \log\left(\|\xi_t\|^2_{*,\varphi(t,\cdot)}\right)$. Hence the theorem, when proved under the smoothness and boundedness assumptions, holds for each Ω_k without the smoothness assumption on φ. Afterwards, we let k tend to $+\infty$, and thus obtain that the theorem holds for not necessarily bounded Ω. But by definition, ξ_t is a holomorphic section of E^*, the dual of the bundle with fiber $E_t = \mathcal{H}^2_t$ at $t \in U$. Therefore, Berndtsson's theorem implies the result in the product domain situation.

Now consider the case when $\mathcal{D}$ is not a product domain. Arguing as above, we may assume that $\mathcal{D}$ is a bounded strictly pseudoconvex with smooth boundary. We can then write $\mathcal{D} = \{(t,z) \in \mathbb{C}^{n+1} : \rho(t,z) < 0\}$ where ρ is strictly plurisubharmonic in a neighborhood W_0 of the closure of $\mathcal{D}$. Since the result is local, we may after restricting t to lie in a small neighborhood V_0 of a given point, assume that $W_0 = V_0 \times \Omega$ where Ω is a pseudoconvex domain. Then we apply the theorem to W_0 with φ replaced by $\varphi_j := \varphi + j \max(0, \rho)$. Since

$$\int_{\{t\} \times \Omega} |f|^2 e^{-\varphi_j} dV(z)$$

tends to

$$\int_{\mathcal{D}_t} |f|^2 e^{-\varphi} dV(z)$$

as j tends to $+\infty$, the theorem follows in general. □

When all the measures μ_t are point-masses, we obtain Berndtsson's theorem on the plurisubharmonic variation of Bergman kernels. This follows from the extremal characterization of Bergman kernels.

Theorem 5.3.5. ([Ber06, Theorem 1.1]) *Let $\mathcal{D}$ is a pseudoconvex domain in $\mathbb{C}_t^m \times \mathbb{C}_z^n$, let φ be a plurisubharmonic function on $\mathcal{D}$, and let K_t denote the bergman kernel of $\mathcal{H}^2\left(\mathcal{D}_t, e^{-\varphi(t,\cdot)}\right)$. Then the function $(t, z) \mapsto \log\left(K_t(z, z)\right)$ is plurisubharmonic or identically $-\infty$.*

In [Ber06], Berndtsson offers a more detailed proof of this theorem. The reduction to the $m = 1$ case follows from [Ber06, Lemma 3.4] on the upper semicontinuity of Bergman kernels and the fact that a function is plurisubharmonic if and only if it is subharmonic along every complex line intersecting its domain of definition.

Finally, note that Theorem 5.3.4 implies the Prékopa-Leindler theorem as follows. Take $\mathcal{D} := (\mathbb{C} - \{0\})^n$, and define μ by taking averages over the n-dimensional real torus:

$$\mu(f) = \int_{\mathbb{T}^n} f\left(e^{\sqrt{-1}\alpha_1} z_1, \cdots, e^{\sqrt{-1}\alpha_n} z_n\right) d\alpha.$$

These averages do not depend on z, and computing the norm of μ as a functional on $\mathcal{H}^2\left(\mathcal{D}_t, e^{-\varphi(t,\cdot)}\right)$, where φ only depends on $|z_j|$, we recover the Prékopa-Leindler theorem.

Chapter 6

A twisted complex Brunn-Minkowski theorem with applications

6.1 Twisted Nakano positivity of Hilbert bundles

6.1.1 For families of relatively compact complete Kähler submanifolds of Stein manifolds

Let X be an n-dimensional relatively compact complete Kähler submanifold of a Stein Kähler manifold (Y, g). Let $V \to \overline{X}$ be a holomorphic vector bundle, and consider a family $\left\{h^{[t]}\right\}_{t\in U}$ of smooth Hermitian metrics for $V \to \overline{X}$ where U is a domain in $\mathbb{C}^m$. Denote by $\mathcal{H}_t^2 := \mathcal{H}^2\left(X, h^{[t]}\right)$ the Hilbert space of holomorphic sections in $L^2\left(X, h^{[t]}\right)$. Consider the holomorphic Hilbert bundle E_h whose fiber at $t \in U$ is $\mathcal{H}_t^2$. Let $\delta > 0$ and η be a smooth function on Y.

For the time being, suppose that

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, and that $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$, where $\Xi_{\delta,\eta}$ is the operator introduced in Section

1.1. We will first prove Theorem A under this assumption. Relaxing our twisted curvature assumptions from positivity to semipositivity is a more delicate process that requires a limiting argument. We show how to do so in Section 6.1.1.4.

Before proceeding to the proof, we recall some of the definitions and notation introduced in Section 1.1 in more detail, and introduce a few more definitions and some useful notation.

6.1.1.1 Definitions

Let $\pi_{\overline{X}}$ denote the projection $U \times \overline{X} \to \overline{X}$ where U is a domain in $\mathbb{C}^m$. Assume that $0 \in U$ without loss of generality. We define a *family* $\{h^{[t]}\}_{t\in U}$ *of smooth Hermitian metrics* for $V \to \overline{X}$ to be a smooth Hermitian metric h for the pullback bundle $\pi_{\overline{X}}^* V \to U \times \overline{X}$. It follows that for each $t \in U$, $h^{[t]} := i_t^* h$ is a smooth Hermitian metric, where

$$i_t : V \to \pi_{\overline{X}}^* V\big|_{\{t\}\times\overline{X}}$$

is the natural isomorphism of vector bundles induced by the inclusion of $\overline{X}$ into the fiber $\{t\} \times \overline{X}$ of $\pi_{\overline{X}}^* V \to U \times \overline{X}$.

In this setting, we can define for each $t \in U$ a Hilbert space

$$\begin{aligned} \mathcal{H}^2\left(X, h^{[t]}\right) &:= \left\{ f \in \Gamma_{\mathcal{O}}(X,V) : \|f\|^2_{h^{[t]}} := \int_X |f|^2_{h^{[t]}}\, dV_g := \int_X h^{[t]}\left(f,f\right) dV_g < +\infty \right\} \\ &= \Gamma_{\mathcal{O}}(X,V) \cap L^2\left(X, h^{[t]}\right), \end{aligned}$$

where:

$$L^2\left(X, h^{[t]}\right) := \left\{ f \in \Gamma(X,V) : \|f\|^2_{h^{[t]}} := \int_X |f|^2_{h^{[t]}}\, dV_g := \int_X h^{[t]}\left(f,f\right) dV_g < +\infty \right\}.$$

Here, $\Gamma(X,V)$ denotes the space of measurable sections of $V \to X$, $\Gamma_{\mathcal{O}}(X,V)$ denotes the space of holomorphic sections of $V \to X$, and dV_g denotes the volume form induced by the metric g. The norm on L^2_t and its corresponding inner product will be denoted by $\|.\|_{h^{[t]}}$ and $(\cdot,\cdot)_{h^{[t]}}$ respectively. We can then define the holomorphic Hilbert bundle E_h as the infinite-rank vector bundle – or Hilbert bundle –

$$U \times \mathcal{H}_0^2 \to U,$$

and we define a Hermitian metric on it by endowing the Hilbert space fiber

$$\{t\} \times \mathcal{H}_0^2 \cong \mathcal{H}_t^2$$

with the norm $\|\cdot\|_{h^{[t]}}$.

We define the space of sections of E_h as

$$\Gamma(E_h) := \left\{ f \in \Gamma(U \times X, \pi_X^* V) : i_t^* \hat{f} \in \mathcal{H}_t^2, \forall t \in U \right\}.$$

In particular, the space of holomorphic sections E_h is defined as

$$\Gamma_{\mathcal{O}}(E_h) := \left\{ \hat{f} \in \Gamma(E_h) : \hat{f} \in \Gamma_{\mathcal{O}}(U \times X, \pi_X^* V) \right\}.$$

For $\hat{f} \in \Gamma(E_h)$, we denote $i_t^* \hat{f}$ by $\hat{f}^{[t]}$. So all sections are holomorphic on the fibers, and for a section is holomorphic if it is holomorphic in the base variable as well.

We denote by E_h^* the dual bundle to E_h. This bundle is also trivial. The space of sections of E_h^* is defined as

$$\Gamma(E_h^*) := \left\{ \xi : E_h \to \mathbb{C}; \xi_t := \xi\big|_{\mathcal{H}_t^2} \in \left(\mathcal{H}_t^2\right)^* \right\}.$$

The bundle E_h^* is equipped fiberwise with the non-trivial Hermitian dual norm

$$\|\xi\|_{*,h^{[t]}} = \sup_{f \in \mathcal{H}_t^2 - \{0\}} \frac{|\langle \xi_t, f \rangle|}{\|f\|_{h^{[t]}}},$$

for each $t \in U$, where ξ is a section of E_h^* and $\xi_t := \xi\big|_{\mathcal{H}_t^2}$.

A section ξ is smooth (resp. holomorphic) if for each $\hat{f} \in \Gamma(E_h)$ that is smooth (resp. holomorphic), the function

$$U \ni t \mapsto \left\langle \xi_t, i_t^* \hat{f} \right\rangle \in \mathbb{C}$$

is a smooth (resp. holomorphic) function of U.

6.1.1.2 Preliminaries

Let F_h be the Hilbert bundle whose fiber over $t \in U$ is L^2_t and let E_h be its subbundle whose fiber over $t \in U$ is $\mathcal{H}^2_t$. Let $\mathcal{P}_t : L^2_t \to \mathcal{H}^2_t$ denote the fiberwise Bergman projection, and let $\mathcal{P}^{\perp}_t$ the fiberwise orthogonal projection of L^2_t onto the orthogonal complement of $\mathcal{H}^2_t$. Additionally, denote by ∇^{F_h} and ∇^{E_h} the Chern connections of each of F_h and E_h; and let Θ^{F_h} and Θ^{E_h} denote their respective curvature forms.

Choose local coordinates $(z_1, \cdots, z_n)$ for an arbitrary point $z \in X$, and denote by $(t_1, \cdots, t_m)$ the global coordinates $t \in U \subset \mathbb{C}^m$. Let $e_1, \ldots, e_r$ be a holomorphic frame for $V \to X$ and let H denote the local matrix representation of h in this frame, i.e.

$$H = (h(e_j, e_k))_{j,k=1}^r = \begin{pmatrix} h(e_1,e_1) & h(e_1,e_2) & \cdots & h(e_1,e_{r-1}) & h(e_1,e_r) \\ h(e_2,e_1) & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & h(e_{r-1},e_r) \\ h(e_r,e_1) & h(e_r,e_2) & \cdots & h(e_{r-1},e_r) & h(e_r,e_r) \end{pmatrix}.$$

In addition, for any operator $\mathfrak{d} \in \{\partial, \bar{\partial}, d\}$ and for any variable $w_i \in \{t_j, \bar{t}_k, z_\mu, \bar{z}_\nu\}$ with $1 \leq j,k \leq m$ and $1 \leq \nu, \mu \leq n$, we let $\mathfrak{d}_{w_i} H$ denote the following matrix in the same holomorphic frame,

$$\mathfrak{d}_{w_i} H := (\mathfrak{d}_{w_i} h(e_j, e_k))_{j,k=1}^r = \begin{pmatrix} \mathfrak{d}_{w_i}h(e_1,e_1) & \mathfrak{d}_{w_i}h(e_1,e_2) & \cdots & \mathfrak{d}_{w_i}h(e_1,e_{r-1}) & \mathfrak{d}_{w_i}h(e_1,e_r) \\ \mathfrak{d}_{w_i}h(e_2,e_1) & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathfrak{d}_{w_i}h(e_{r-1},e_r) \\ \mathfrak{d}_{w_i}h(e_r,e_1) & \mathfrak{d}_{w_i}h(e_r,e_2) & \cdots & \mathfrak{d}_{w_i}h(e_{r-1},e_r) & \mathfrak{d}_{w_i}h(e_r,e_r) \end{pmatrix}.$$

We adopt the same notation for $h^{[t]}$.

Finally, given a section s expressed as $s = \sum_{i=1}^r s_i e_i$ in this holomorphic frame, for a collection of holomorphic functions $s_1, \ldots, s_r$ such that $(s_1, \cdots, s_r) \neq (0, \cdots, 0)$, we represent s by the column vector $S = [s_1 \cdots s_r]^T$.

From the definition of the Chern connection,

$$d_{t_j}\left(u^{[t]}, v^{[t]}\right)_{h^{[t]}} = \left(\nabla^{F_h}_{t_j} u^{[t]}, v^{[t]}\right)_{h^{[t]}} + \left(u^{[t]}, \nabla^{F_h}_{t_j} v^{[t]}\right)_{h^{[t]}},$$

for any two smooth sections u and v of F_h. Therefore, letting † denote the complex conjugate transpose, we have the following for any two sections u and v of F_h.

$$\begin{aligned}
d_{t_j}\left(u^{[t]}, v^{[t]}\right)_{h^{[t]}} &= d_{t_j} \int_X h^{[t]}\left(u^{[t]}, v^{[t]}\right) dV_g \\
&= \int_X \partial_{t_j}\left[h^{[t]}\left(u^{[t]}, v^{[t]}\right) dV_g\right] + \int_X \bar{\partial}_{t_j}\left[h^{[t]}\left(u^{[t]}, v^{[t]}\right) dV_g\right] \\
&= \int_X \partial_{t_j}\left(V^{[t],\dagger} H^{[t]} U^{[t]}\right) dV_g + \int_X \bar{\partial}_{t_j}\left(V^{[t],\dagger} H^{[t]} U^{[t]}\right) dV_g \\
&= \int_X \left(\bar{\partial}_{t_j} V^{[t]}\right)^\dagger H^{[t]} U^{[t]} dV_g + \int_X \left(V^{[t]}\right)^\dagger \bar{\partial}_{t_j} H^{[t]} U^{[t]} dV_g + \int_X \left(V^{[t]}\right)^\dagger H^{[t]} \bar{\partial}_{t_j} U^{[t]} dV_g \\
&\quad + \int_X \left(\partial_{t_j} V^{[t]}\right)^\dagger H^{[t]} U^{[t]} dV_g + \int_X V^{[t],\dagger} \partial_{t_j} H^{[t]} U^{[t]} dV_g + \int_X \left(V^{[t]}\right)^\dagger H^{[t]} \partial_{t_j} U^{[t]} dV_g.
\end{aligned}$$

Rearranging the terms,

$$\begin{aligned}
d_{t_j}\left(u^{[t]}, v^{[t]}\right)_{h^{[t]}} &= \int_X \left[\left(\partial_{t_j} + \bar{\partial}_{t_j} + \left(H^{[t]}\right)^{-1} \partial_{t_j} H^{[t]}\right) V^{[t]}\right]^\dagger H^{[t]} U^{[t]} dV_g \\
&\quad + \int_X \left(V^{[t]}\right)^\dagger H^{[t]} \left[\left(\partial_{t_j} + \bar{\partial}_{t_j} + \left(H^{[t]}\right)^{-1} \partial_{t_j} H^{[t]}\right) U^{[t]}\right] dV_g \\
&= \int_X h^{[t]}\left(\left[d_{t_j} + \left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] u^{[t]}, v^{[t]}\right) dV_g \\
&\quad + \int_X h^{[t]}\left(u^{[t]}, \left[d_{t_j} + \left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] v^{[t]}\right) dV_g \\
&= \left(\left[d_{t_j} + \left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] u^{[t]}, v^{[t]}\right)_{h^{[t]}} + \left(u^{[t]}, \left[d_{t_j} + \left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] u^{[t]}\right)_{h^{[t]}}.
\end{aligned}$$

Clearly, $u^{[t]} \mapsto d_{t_j} u^{[t]} - \left[\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] u^{[t]}$ defines a connection. Moreover, if $u^{[t]}$ is holomorphic in t, then $d_{t_j} u^{[t]} - \left[\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right] u^{[t]}$ is of bidegree $(1,0)$ and so this operator defines a holomorphic connection. Our previous computation shows that the connection ∇^{F_h} is metric-compatible, and so it must be the Chern connection of F_h. In particular, its $(1,0)$-part is given by $\nabla^{F_h,(1,0)}_{t_j} = \partial_{t_j} - \left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}$.

Thus, the connection form of the Chern connection is given by (wedging with) $\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}$, and so the coefficients of the curvature of F_h are given by $\Theta^{F_h}_{t_j \bar{t}_k} = \bar{\partial}_{t_k}\left[\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right]$.

Therefore, by Theorem 2.4.14

$$\left(\Theta^{F_h}_{t_j\bar{t}_k} u^{[t]}, v^{[t]}\right)_{h^{[t]}} = \left(\mathcal{P}^{\perp}_t\left(\nabla^{F_h}_{t_j} u^{[t]}\right), \mathcal{P}^{\perp}_t\left(\nabla^{F_h}_{t_k} v^{[t]}\right)\right)_{h^{[t]}} + \left(\Theta^{E_h}_{t_j\bar{t}_k} u^{[t]}, v^{[t]}\right)_{h^{[t]}}$$

for any two smooth sections u and v of E_h. Whence if we let $u_1, \ldots, u_m$ be any m smooth sections of the bundle E_h, then

$$\sum_{1\le j,k\le m} \left(\Theta^{F_h}_{t_j\bar{t}_k} u^{[t]}_j, u^{[t]}_k\right)_{h^{[t]}} = \left\|\mathcal{P}^{\perp}_t\left(\sum_{1\le j\le m} \nabla^{F_h}_{t_j} u^{[t]}_j\right)\right\|^2_{h^{[t]}} + \sum_{1\le j,k\le m} \left(\Theta^{E_h}_{t_j\bar{t}_k} u^{[t]}_j, u^{[t]}_k\right)_{h^{[t]}}. \quad (6.1.1)$$

The Nakano positivity of E_h will be established by estimating $\sum_{1\le j,k\le m}\left(\Theta^{E_h}_{t_j\bar{t}_k} u^{[t]}_j, u^{[t]}_k\right)_{h^{[t]}}$ using the curvature formula (6.1.1). Doing so amounts to estimating the following norm.

$$\begin{aligned}\left\|\mathcal{P}^{\perp}_t\left(\sum_{1\le j\le m} \nabla^{F_h}_{t_j} u^{[t]}_j\right)\right\|^2_{h^{[t]}} &= \left\|\mathcal{P}^{\perp}_t\left(\sum_{1\le j\le m} \partial_{t_j} u^{[t]}_j - \left(h^{[t]}\right)^{-1}\partial_{t_j} h^{[t]} u^{[t]}_j\right)\right\|^2_{h^{[t]}} \\ &= \left\|\mathcal{P}^{\perp}_t\left(\sum_{1\le j\le m} \left(h^{[t]}\right)^{-1}\partial_{t_j} h^{[t]} u^{[t]}_j\right)\right\|^2_{h^{[t]}}. \end{aligned} \quad (6.1.2)$$

6.1.1.3 Proof of Theorem A under the assumption of strict curvature positivity

By assumption, for each $t \in U$

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0.$$

By letting

$$\Phi = \bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta,$$

we obtain:

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta - \Phi = 0$$

so that the hypotheses of Theorem 4.2.1 are trivially satisfied since $\Theta\left(h^{[t]}\right) = \bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right)$. Now let $(t_1, \ldots, t_m)$ denote the global coordinates of $t \in U$ and let $(z_1, \ldots, z_n)$ denote local coordinates for $z \in X$. Consider the section

$$u^{[t]} := \sum_{1\le j\le m}\left(\left(h^{[t]}\right)^{-1}\frac{\partial h^{[t]}}{\partial t_j}\right) u^{[t]}_j.$$

Then $u^{[t]}$ solves the $\bar{\partial}_z$-equation

$$\bar{\partial}_X u^{[t]} =: \bar{\partial}_z u^{[t]} = \sum_{\substack{1\le j\le m\\ 1\le\mu\le n}} \frac{\partial}{\partial \bar{z}_\mu}\left(\left(h^{[t]}\right)^{-1}\frac{\partial h^{[t]}}{\partial t_j}\right) u_j^{[t]} d\bar{z}_\mu =: \alpha,$$

since every single $u_j^{[t]}$ depends holomorphically on $z = (z_1, \cdots, z_n)$. Note also that

$$\int_X |\alpha|^2_{\Phi,h^{[t]}}\, dV_g < +\infty.$$

Clearly, the $(0,1)$-form α satisfies $\bar{\partial}_X\alpha = 0$. Moreover, since each $u_j^{[t]}$ is in $\mathcal{H}_t^2 \subset L_t^2$, and since the metric h is smooth up the boundary of X, it follows that $u^{[t]}$ is in $\in L_t^2$ as well. By Theorem 4.2.1, and the fact that $u_0^{[t]} = \mathcal{P}_t^{\perp}\left(u^{[t]}\right)$ is the minimal-norm solution of $\bar{\partial}_X v = \alpha$, we have the estimate

$$\int_X \left|u_0^{[t]}\right|^2_{h^{[t]}} dV_g \le \left(\frac{1+\delta}{\delta}\right)\int_X |\alpha|^2_{\Phi,h^{[t]}}\, dV_g. \tag{6.1.3}$$

Set $\Psi := \Xi_{\delta,\eta}(h)$. Let Ψ_{ab} and Ψ^{cd} denote the components of Ψ in the directions a and b, and those of the inverse of Ψ in the directions c and d respectively where $a, b, c, d \in \{t_j, \bar{t}_k, z_\mu, \bar{z}_\nu\}$. By (6.1.1), (6.1.2) and (6.1.3),

$$\sum_{1\le j,k\le m}\left(\Theta^{E_h}_{t_j\bar{t}_k} u_j^{[t]}, u_k^{[t]}\right)_{h^{[t]}} \ge \int_X \sum_{1\le j,k\le m} h^{[t]}\left(\Psi_{t_j\bar{t}_k} u_j^{[t]}, u_k^{[t]}\right) dV_g - \left\|u_0^{[t]}\right\|^2_{h^{[t]}}. \tag{6.1.4}$$

Furthermore, combining (6.1.4) with the estimate (6.1.3), we can see that:

$$\sum_{1\le j,k\le m}\left(\Theta^{E_h}_{t_j\bar{t}_k} u_j^{[t]}, u_k^{[t]}\right)_{h^{[t]}} \ge \int_X \sum_{1\le j,k\le m} h^{[t]}\left(\left[\Psi_{t_j\bar{t}_k} - \sum_{1\le\mu,\nu\le n}\Psi^{z_\mu\bar{z}_\nu}\Psi_{t_j\bar{z}_\mu}\overline{\Psi_{t_k\bar{z}_\nu}}\right] u_j^{[t]}, u_k^{[t]}\right) dV_g. \tag{6.1.5}$$

Now let M_Ψ be the matrix whose (j,k)-entries are

$$\Psi_{t_j\bar{t}_k} - \sum_{1\le\mu,\nu\le n}\Psi^{z_\mu\bar{z}_\nu}\Psi_{t_j\bar{z}_\mu}\overline{\Psi_{t_k\bar{z}_\nu}}.$$

By Schur complement theory, $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ implies $M_\Psi >_{\mathrm{Griff}} 0$. Therefore, we conclude that if $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$, then

$$\exists c_0 > 0 \text{ (resp. } \ge 0) : \sum_{1\le j,k\le m}\left(\Theta^{E_h}_{t_j\bar{t}_k} u_j^{[t]}, u_k^{[t]}\right)_{h^{[t]}} \ge c_0 \sum_{j=1}^m \left\|u_j^{[t]}\right\|^2_{h^{[t]}}. \tag{6.1.6}$$

6.1.1.4 Relaxing the strict curvature positivity requirement

Let $h_\varepsilon := h e^{-\varepsilon\left(P_X^*\psi + |t|^2\right)}$ where $\varepsilon > 0$ and ψ is a smooth strictly plurisubharmonic function for the ambient Stein manifold containing X, that is smooth up the boundary of X. Since ψ is bounded on X, we may assume that $\psi > 0$ after subtracting a constant. Moreover, as the result is local, we may assume that U is bounded. Denote by $\Theta^{E_{h_\varepsilon}}_{t_j \bar{t}_k}$ the coefficients for the curvature of E_{h_ε}, the Hilbert bundle whose fiber at $t \in U$ is $\mathcal{H}^2_{\varepsilon,t} := \mathcal{H}^2\left(X, h_\varepsilon^{[t]}\right)$. For any $\varepsilon > 0$, the underlying vector spaces of $\mathcal{H}^2_{\varepsilon,t}$ and $\mathcal{H}^2_t$ are equal as subspaces of $\Gamma_{\mathcal{O}}(X, V)$, and so we may act on the same tuple of sections $u_1, \cdots, u_m$.

By construction,

$$\Xi_{\delta,\eta}\left(h_\varepsilon\right) = \Xi_{\delta,\eta}(h) + \varepsilon \begin{pmatrix} |t|^2 & 0 \\ 0 & \partial_X \bar{\partial}_X \psi \end{pmatrix},$$

and

$$\begin{aligned}
&\bar{\partial}_X \left(\left(h_\varepsilon^{[t]}\right)^{-1} \partial_X h_\varepsilon^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V \\
&= \bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V + \varepsilon \left(\partial_X \bar{\partial}_X \psi \otimes \mathrm{Id}_V\right).
\end{aligned}$$

Since

$$\begin{pmatrix} |t|^2 & 0 \\ 0 & \partial_X \bar{\partial}_X \psi \end{pmatrix} >_{\text{Griff}} 0 \text{ and } \partial_X \bar{\partial}_X \psi \otimes \mathrm{Id}_V >_{\text{Nak}} 0,$$

it follows that

$$\Xi_{\delta,\eta}\left(h_\varepsilon\right) >_{\text{Griff}} \Xi_{\delta,\eta}(h)$$

and

$$\begin{aligned}
&\bar{\partial}_X \left(\left(h_\varepsilon^{[t]}\right)^{-1} \partial_X h_\varepsilon^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V \\
&>_{\text{Nak}} \bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V.
\end{aligned}$$

Therefore, if either $\Xi_{\delta,\eta}(h) \geq_{\text{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V \geq_{\text{Nak}} 0,$$

for each $t \in U$, then

$$\forall \varepsilon > 0, \exists c_0^{(\varepsilon)} > 0 : \sum_{1 \le j,k \le m} \left(\Theta^{E_{h_\varepsilon}}_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} \ge c_0^{(\varepsilon)} \sum_{j=1}^{m} \left\| u_j^{[t]} \right\|^2_{h^{[t]}} \ge 0. \tag{6.1.7}$$

Let $\mathcal{P}_t^{\varepsilon,\perp}$ denote the orthogonal projection of $L^2_{\varepsilon,t}$ onto $\left(\mathcal{H}^2_{\varepsilon,t}\right)^\perp$. As before, let $\Psi := \Xi_{\delta,\eta}(h)$, and let Ψ_{ab} and Ψ^{cd} denote the components of Ψ in the directions a and b, and those of the inverse of Ψ in the directions c and d respectively where $a, b, c, d \in \{t_j, \bar{t}_k, z_\mu, \bar{z}_\nu\}$. Let us also adopt the same notation for h_ε by letting Ψ^ε represent the corresponding matrix for h_ε. Furthermore, let Υ and Υ^ε denote the matrices corresponding to $h^{-1}\partial h$ and $h_\varepsilon^{-1}\partial h_\varepsilon$, respectively.

By Theorem 2.4.14, and noting that ψ is independent of t,

$$\begin{aligned}
\sum_{1 \le j,k \le m} \left(\Theta^{E_{h_\varepsilon}}_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} &= \sum_{1 \le j,k \le m} \left(\Psi^\varepsilon_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} - \left\| \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \Upsilon^\varepsilon_{t_j} u_j^{[t]} \right) \right\|^2_{h_\varepsilon^{[t]}} \\
&= \sum_{1 \le j,k \le m} \left[\left(\Psi_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} + \varepsilon \left((\pi_X^* \psi + |t|^2)_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} \right] \\
&\quad - \left\| \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]} \right) - \varepsilon \cdot \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} (\pi_X^* \psi + |t|^2)_{t_j} u_j^{[t]} \right) \right\|^2_{h_\varepsilon^{[t]}} \\
&= \sum_{1 \le j,k \le m} \left(e^{-\varepsilon\left(\psi + |t|^2\right)} \Psi_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h^{[t]}} + \varepsilon \sum_{1 \le j,k \le m} \delta_{j\bar{k}} \left(e^{-\varepsilon\left(\psi + |t|^2\right)} u_j^{[t]}, u_k^{[t]} \right)_{h^{[t]}} \\
&\quad - \left\| e^{-\varepsilon\left(\psi + |t|^2\right)/2} \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \left(\Upsilon_{t_j} - \varepsilon \bar{t}_j \right) u_j^{[t]} \right) \right\|^2_{h^{[t]}}.
\end{aligned}$$

By adding and subtracting $\sum_{1 \le j,k \le m} \left(\Theta^{E_h}_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h^{[t]}}$, we then have:

$$\sum_{1 \le j,k \le m} \left(\Theta^{E_{h_\varepsilon}}_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h_\varepsilon^{[t]}} = \sum_{1 \le j,k \le m} \left(\Theta^{E_h}_{t_j \bar{t}_k} u_j^{[t]}, u_k^{[t]} \right)_{h^{[t]}} + \mathfrak{R}(\varepsilon), \tag{6.1.8}$$

where

$$\mathfrak{R}(\varepsilon) := \sum_{1\le j,k\le m}\left(\left(e^{-\varepsilon\left(\psi+|t|^2\right)}-1\right)\Psi_{t_j\bar{t}_k}u_j^{[t]},u_k^{[t]}\right)_{h^{[t]}} + \varepsilon\sum_{1\le j,k\le m}\delta_{j\bar{k}}\left(e^{-\varepsilon\left(\psi+|t|^2\right)}u_j^{[t]},u_k^{[t]}\right)_{h^{[t]}}$$
$$+\left\|\mathcal{P}_t^{\perp}\left(\sum_{1\le j\le m}\Upsilon_{t_j}u_j^{[t]}\right)\right\|^2_{h^{[t]}} - \left\|e^{-\varepsilon\left(\psi+|t|^2\right)/2}\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\Upsilon_{t_j}u_j^{[t]}\right)\right\|^2_{h^{[t]}}$$
$$+2\varepsilon\mathrm{Re}\left[\left(\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\Upsilon_{t_j}u_j^{[t]}\right), e^{-\left(\psi+|t|^2\right)}\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\bar{t}_j u_j^{[t]}\right)\right)_{h^{[t]}}\right]$$
$$-\varepsilon^2\left\|e^{-\left(\psi+|t|^2\right)/2}\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\bar{t}_j u_j^{[t]}\right)\right\|^2_{h^{[t]}}$$

In particular,

$$\sum_{1\le j,k\le m}\left(\Theta^{E_h}_{t_j\bar{t}_k}u_j^{[t]},u_k^{[t]}\right)_{h^{[t]}} \ge -\mathfrak{R}(\varepsilon), \tag{6.1.9}$$

Since $u_k^{[t]}\in\mathcal{H}_t^2$ for each k and for each $t\in U$, the first two summands in $\mathfrak{R}(\varepsilon)$ converge to 0 as $\varepsilon\to 0$ by smoothness, boundedness, countinuity, and the Cauchy-Schwarz inequality. Since $\psi>0$ by assumption and $\mathcal{P}_t^{\varepsilon,\perp}$ is an orthogonal projection,

$$\left\|e^{-\left(\psi+|t|^2\right)}\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\bar{t}_j u_j^{[t]}\right)\right\|^2_{h^{[t]}} \le \left\|\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\bar{t}_j u_j^{[t]}\right)\right\|^2_{h^{[t]}}\left\|\sum_{1\le j\le m}\bar{t}_j u_j^{[t]}\right\|^2_{h^{[t]}}, \tag{6.1.10}$$

and

$$\left\|\mathcal{P}_t^{\varepsilon,\perp}\left(\sum_{1\le j\le m}\Upsilon_{t_j}u_j^{[t]}\right)\right\|^2_{h^{[t]}} \le \left\|\sum_{1\le j\le m}\Upsilon_{t_j}u_j^{[t]}\right\|^2_{h^{[t]}}. \tag{6.1.11}$$

Each of the upper bounds in (6.1.10) and (6.1.11) respectively is finite by smoothness, boundedness, and the fact that $u_k^{[t]}\in\mathcal{H}_t^2$ for each $1\le k\le m$ and for each $t\in U$. Therefore, the last summand in $\mathfrak{R}(\varepsilon)$ converge to 0 as $\varepsilon\to 0$, as does the fifth summand by the Cauchy-Schwarz inequality.

It now remains to estimate the difference term $\mathfrak{R}(\varepsilon)$. Note that for any section $\mathfrak{u}$ of E_h and any point $w\in X$,

$$\mathcal{P}_t^{\varepsilon,\perp}\mathfrak{u}^{[t]}(w) = \mathcal{P}_t^{\perp}\mathfrak{u}^{[t]}(w) + \left(\mathfrak{u}^{[t]}, K_t(\cdot,w) - e^{-\varepsilon\left(\psi(w)+|t|^2\right)}K_t^{\varepsilon}(\cdot,w)\right)_{h^{[t]}},$$

where K_t and K_t^ε denote the Bergman kernels for $\mathcal{H}_t^2$ and $\mathcal{H}_{\varepsilon,t}^2$ respectively. (The emphasis on w is to indicate that the second variable in each Bergman kernel is fixed.) Thus

$$\begin{aligned}
&\mathcal{P}_t^\perp \mathfrak{u}^{[t]}(w) - e^{-\varepsilon\left(\psi(w)+|t|^2\right)/2} \cdot \mathcal{P}_t^{\varepsilon,\perp} \mathfrak{u}^{[t]}(w) \\
&= (1 - e^{-\varepsilon\left(\psi(w)+|t|^2\right)/2}) \cdot \mathcal{P}_t^\perp \mathfrak{u}^{[t]}(w) + e^{-\varepsilon\left(\psi(w)+|t|^2\right)/2} \cdot \left(\mathfrak{u}^{[t]}, e^{-\varepsilon\left(\psi(w)+|t|^2\right)} K_t^\varepsilon(\cdot, w) - K_t(\cdot, w)\right)_{h^{[t]}},
\end{aligned}$$

and so we have the following estimate.

$$\begin{aligned}
&\left\|\mathcal{P}_t^\perp \mathfrak{u}^{[t]} - e^{-\varepsilon\left(\psi+|t|^2\right)/2} \cdot \mathcal{P}_t^{\varepsilon,\perp} \mathfrak{u}^{[t]}\right\|_{L_t^\infty} \\
&\le M(\varepsilon) \left\|\mathcal{P}_t^\perp \mathfrak{u}^{[t]}\right\|_{L_t^\infty} + e^{-\varepsilon m_0/2} \left\|\mathfrak{u}^{[t]}\right\|_{L_t^\infty} m(\varepsilon) \left\|K_t^\varepsilon(\cdot, w)\right\|_{L_t^\infty} \\
&\quad + e^{-\varepsilon m_0/2} \left\|\mathfrak{u}^{[t]}\right\|_{L_t^\infty} \left\|K_t^\varepsilon(\cdot, w) - K_t(\cdot, w)\right\|_{L_t^\infty},
\end{aligned}$$

where $M(\varepsilon) = \max\left(\left|1 - e^{-\varepsilon(M_0+R_0)/2}\right|, \left|1 - e^{-\varepsilon m_0/2}\right|\right)$, $m(\varepsilon) = \max\left(\left|e^{-\varepsilon(M_0+R_0)} - 1\right|, \left|e^{-\varepsilon m_0} - 1\right|\right)$ and m_0 and M_0 are the minimum and maximum of ψ over $\overline{X}$, respectively, and $R_0 := \sup_{\overline{U}} |t|^2$.

Let $\mathfrak{u} := \sum_{1 \le j \le m} \Upsilon_{t_j} u_j$. By smoothness and boundedness,

$$\left\|\mathfrak{u}^{[t]}\right\|_{L_t^\infty}, \left\|\mathcal{P}_t^\perp \mathfrak{u}^{[t]}\right\|_{L_t^\infty}, \left\|K_t^\varepsilon(\cdot, w)\right\|_{L_t^\infty}, \left\|K_t(\cdot, w)\right\|_{L_t^\infty} < +\infty.$$

Moreover, since $\psi > 0$ by assumption, the sequence of metrics $\left\{h_\varepsilon^{[t]}\right\}_{\varepsilon>0}$ increases to $h^{[t]}$. Therefore, by the generalization of Ramadanov's theorem on Bergman kernels in [PW16],

$$\left\|K_t^\varepsilon(\cdot, w) - K_t(\cdot, w)\right\|_{L_t^\infty} \xrightarrow[\varepsilon \to 0]{} 0.$$

Finally, knowing that $M(\varepsilon), m(\varepsilon) \to 0$ as $\varepsilon \to 0$, we see that

$$\left\|\mathcal{P}_t^\perp \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right) - e^{-\varepsilon\left(\psi+|t|^2\right)/2} \cdot \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right)\right\|_{L_t^\infty} \xrightarrow[\varepsilon \to 0]{} 0,$$

and so

$$\left\|\mathcal{P}_t^\perp \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right) - e^{-\varepsilon\left(\psi+|t|^2\right)/2} \cdot \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right)\right\|_{L_t^2} \xrightarrow[\varepsilon \to 0]{} 0.$$

Thus,

$$\left\|\mathcal{P}_t^\perp \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right)\right\|_{h^{[t]}}^2 - \left\|e^{-\varepsilon\left(\psi+|t|^2\right)/2} \cdot \mathcal{P}_t^{\varepsilon,\perp} \left(\sum_{1 \le j \le m} \Upsilon_{t_j} u_j^{[t]}\right)\right\|_{h^{[t]}}^2 \xrightarrow[\varepsilon \to 0]{} 0.$$

Consequently $\mathfrak{R}(\varepsilon) \to 0$ as $\varepsilon \to 0$, which completes the proof by taking the limit as $\varepsilon \to 0$ in (6.1.9).

6.1.2 For trivial families of possibly unbounded Stein manifolds

Suppose now that X is a possibly unbounded Stein manifold. Then E_h is no longer necessarily a Hilbert bundle, but rather a field of Hilbert spaces whose fibers are not necessarily isomorphic. This means that we will need to *define* Griffiths positivity and Nakano positivity for these fields of Hilbert spaces alternatively. These definitions are inspired by the analytic characterizations that are equivalent to these notions of positivity in the vector bundle case that we saw in Section 2.4.7.4. Our definitions of sections of E_h and its dual are essentially the same as the ones for the locally trivial case.

Since any pseudoconvex subdomain of a Stein manifold is itself a Stein manifold, Theorem A implies the following theorem.

Theorem 6.1.1. *Let X be an n-dimensional relatively compact pseudoconvex subdomain of an ambient Stein Kähler manifold (Y, g). Let $V \to \overline{X}$ be a holomorphic vector bundle. Let $U \subset \mathbb{C}^m$ be a domain, and let $\left\{h^{[t]}\right\}_{t\in U}$ be a family of smooth Hermitian metrics for $V \to \overline{X}$. Let $\delta > 0$ and let η be a smooth function on Y. If $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and*

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta \wedge \bar{\partial}_X\eta\right) \otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then the holomorphic Hermitian bundle $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano positive. Moreover, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta \wedge \bar{\partial}_X\eta\right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in U$, then $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is Nakano semipositive.

We will be making use of Theorem 6.1.1 in our proofs of Theorems C and B.

6.1.2.1 Definitions

Definition 6.1.1. A section $\hat{f}$ of the fields of Hilbert spaces E_h is a section of $\pi_X^* V \to U \times X$ such that $\hat{f}\Big|_{\{t\}\times X} \in E_t := \mathcal{H}_t^2$ for each $t \in U$. We will write $\hat{f}^{[t]} := \hat{f}\Big|_{\{t\}\times X}$.

The section $\hat{f}$ of E_h is said to be holomorphic if it is holomorphic in the total space. In particular, all sections are holomorphic on the fibers.

Definition 6.1.2. Let E_h^* denote the holomorphic Hermitian field of Hilbert spaces dual to E_h – that is the fiber E_t^* of E_h^* over $t \in U$ is the Hilbert space dual of $\mathcal{H}_t^2$ with its usual fiberwise Hilbert norm

$$\|\xi\|_{*,h^{[t]}} := \sup_{f \in \mathcal{H}_t^2 - \{0\}} \frac{\left| \left\langle \xi\big|_{\mathcal{H}_t^2}, f \right\rangle \right|}{\|f\|_{h^{[t]}}}.$$

Definition 6.1.3. A section of E_h^* is a map $\xi : E_h \to \mathbb{C}$ such that $\xi_t := \xi\big|_{\mathcal{H}_t^2} \in (\mathcal{H}_t^2)^*$. The section ξ is said to be smooth (resp. holomorphic) if for each smooth (resp. holomorphic) section $\hat{f}$ of E_h, the function $U \ni t \mapsto \left\langle \xi_t, \hat{f}^{[t]} \right\rangle \in \mathbb{C}$ is smooth (resp. holomorphic).

Now, let F_h denote the field of Hilbert spaces with fiber $F_t := L_t^2$ over $t \in U$ and let $\mathcal{P}_t : L_t^2 \to \mathcal{H}_t^2$ denote the fiberwise Bergman projection.

Definition 6.1.4. (Connections)

- The Chern connection ∇^{F_h} of F_h is formally defined as the following collection of operators $\nabla_{t_j}^{F_h}$ for $1 \le j \le m$ given by $\nabla_{t_j}^{F_h} u^{[t]} = d_{t_j} u^{[t]} - \left(\left(h^{[t]} \right)^{-1} \partial_{t_j} h^{[t]} \right) u^{[t]}$, for a section u of F_h. The domain of each $\nabla_{t_j}^{F_h}$ consists of sections u of F_h such that

$$\partial_{t_j} u^{[t]} - \left(\left(h^{[t]} \right)^{-1} \partial_{t_j} h^{[t]} \right) u^{[t]} \in L_t^2$$

  for each $t \in U$.

  - We may also define $\nabla_{t_j}^{F_h}$, as in the vector bundle case, by the relation

$$\left(\nabla_{t_j}^{F_h} u^{[t]}, v_t \right)_{h^{[t]}} := d_{t_j} (u^{[t]}, v_t)_{h^{[t]}} - \left(u^{[t]}, d_{t_j} v_t \right)_{h^{[t]}}$$

    for any two sections u and v of E_h.

- The $(1,0)$-part of the connection is defined by the collection of operators $\nabla_{t_j}^{F_h,(1,0)}$ mapping $u^{[t]} \in L_t^2$ to $\partial_{t_j} u^{[t]} - \left(\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right) u^{[t]}$. The domain of each $\nabla_{t_j}^{F_h,(1,0)}$ consists of sections u of F_h such that $\partial_{t_j} u^{[t]} - \left(\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right) u^{[t]} \in L_t^2$ for each $t \in U$.

- The $(0,1)$-part of the connection is defined as the collection of $\bar{\partial}$-operators $\bar{\partial}_{t_j}^{F_h}$. The domain of each $\bar{\partial}_{t_j}^{F_h}$ consists of sections u of F_h such that $\bar{\partial}_{t_j} u^{[t]} \in L_t^2$ for each $t \in U$.

  The corresponding connections for E_h are defined as the respective Bergman projections of each connection, with the domains similarly defined. So we have $\nabla_{t_j}^{E_h} := \mathcal{P}_t \circ \nabla_{t_j}^{F_h}$, $\nabla_{t_j}^{E_h,(1,0)} := \mathcal{P}_t \circ \nabla_{t_j}^{F_h,(1,0)}$ and $\bar{\partial}_{t_j}^{E_h} := \mathcal{P}_t \circ \bar{\partial}_{t_j}^{F_h}$.

We will abusively denote the $\bar{\partial}$-operators for E_h and F_h interchangeably.

Definition 6.1.5. (Curvatures)

- The curvature Θ^{F_h} of F_h is the $(1,1)$-form of endomorphisms

$$\Theta^{F_h} = \sum_{1 \leq j,k \leq m} \Theta_{t_j \bar{t}_k}^{F_h} dt_j \wedge d\bar{t}_k,$$

  where the multiplier coefficients $\Theta_{t_j \bar{t}_k}^{F_h}$ are endomorphisms of $V \to X$ defined on X by $\Theta_{t_j \bar{t}_k}^{F_h} = \bar{\partial}_{t_k} \left(\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right)$. The domain of each $\Theta_{t_j \bar{t}_k}^{F_h}$ consists of sections u of F_h such that

$$\bar{\partial}_{t_k} \left(\left(h^{[t]}\right)^{-1} \partial_{t_j} h^{[t]}\right) u^{[t]} \in L_t^2$$

  for each $t \in U$.

- The curvature of Θ^{E_h} of E_h is the $(1,1)$-form of endomorphisms $\Theta_{t_j \bar{t}_k}^{E_h}$ of $V \to X$ defined as by the relation

$$\left(\Theta_{t_j \bar{t}_k}^{F_h} u^{[t]}, v^{[t]}\right)_{h^{[t]}} = \left(\mathcal{P}_t^{\perp}\left(\nabla_{t_j}^{F_h} u^{[t]}\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h} v^{[t]}\right)\right)_{h^{[t]}} + \left(\Theta_{t_j \bar{t}_k}^{E_h} u^{[t]}, v^{[t]}\right)_{h^{[t]}}$$

  for any two sections u and v of E_h. The domain of each endomorphism $\Theta_{t_j \bar{t}_k}^{E_h}$ consists of sections u of E_h such that $\Theta_{t_j \bar{t}_k}^{E_h} u^{[t]} \in L_t^2$ for each $t \in U$.

Definition 6.1.6. (Griffiths positivity)

The holomorphic Hermitian field of Hilbert spaces $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is said to be Griffiths semi-positive (resp. positive) if the function

$$U \ni t \mapsto \log\left(\|\xi\|^2_{*,h^{[t]}}\right)$$

is (strictly) plurisubharmonic or identically $-\infty$ for every holomorphic section ξ of E_h^*.

By Proposition 2.4.9, this definition is equivalent to the usual definition of Griffiths positivity when E_h is bona fide holomorphic Hilbert bundle.

Now, consider the $(m-1,m-1)$-form

$$T_u = \sum_{1\le j,k\le m} \left(u_j^{[t]}, u_k^{[t]}\right)_{h^{[t]}} \widehat{dt_j \wedge d\bar{t}_k},$$

for an m-tuple $u = (u_1, \cdots, u_m)$ of holomorphic sections of E_h. Here,

$$\widehat{dt_j \wedge d\bar{t}_k} = c_n dt_1 \wedge \cdots \wedge dt_{j-1} \wedge dt_{j+1} \wedge \cdots \wedge dt_m \wedge d\bar{t}_1 \wedge \cdots \wedge d\bar{t}_{k-1} \wedge d\bar{t}_{k+1} \wedge d\bar{t}_m,$$

where c_n is a unimodular constant chosen so that $\widehat{dt_j \wedge d\bar{t}_k}$ is a positive form.

Definition 6.1.7. (Nakano positivity)

- The holomorphic Hermitian field of Hilbert spaces $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is said to be Nakano positive (resp. semipositive) at $t \in U$ if

  $$\exists c_0 > 0 \text{ (resp. } = 0) : \partial_U \bar{\partial}_U (-T_u) \ge c_0 \sum_{k=1}^m \left\|u_k^{[t]}\right\|^2_{h^{[t]}} dV(t)$$

  for any m-tuple $(u_1, \cdots, u_m)$ of holomorphic sections of E_h belonging to the domains of $\nabla^{E_h,(1,0)}_{t_j}$ and $\Theta^{E_h}_{t_j\bar{t}_k}$ and such that $\nabla^{E_h,(1,0)}_{t_j} u_j^{[t]} = 0$ at t, for all $1 \le j,k \le m$.

- The holomorphic Hermitian field of Hilbert spaces $(E_h, (\cdot,\cdot)_{h^{[t]}})$ is said to be Nakano (semi)positive if it is Nakano (semi)positive at every $t \in U$.

By Proposition 2.4.11, this definition is equivalent to the usual definition of Nakano positivity when E_h is bona fide holomorphic Hilbert bundle.

6.1.2.2 Griffiths positivity for trivial families of possibly unbounded Stein manifolds

We now proceed to the proof of Theorem C.

Proof. Since the result is local, we may assume that U is bounded. Let X be a possibly unbounded Stein manifold. Let us denote

$$\|\xi\|^2_{*,h^{[t]},X} := \sup_{f^{[t]} \in \mathcal{H}^2_t - \{0\}} \frac{\left|\langle \xi_t, f^{[t]} \rangle\right|^2}{\|f^{[t]}\|^2_{h^{[t]},X}},$$

for a holomorphic section ξ of E_h^*.

Let ξ be an arbitrary holomorphic section of E_h. If $\xi \equiv 0$, then $\|\xi\|^2_{*,h^{[t]},X} = 0$ and so the function $U \ni t \mapsto \log\left(\|\xi\|^2_{*,h^{[t]},X}\right)$ is identically $-\infty$. We thus assume for the remainder of the proof that ξ is not identically 0.

Our goal is to shows that $U \ni t \mapsto \|\xi\|^2_{*,h^{[t]},X}$ is strictly plurisubharmonic or plurisubharmonic depending on the twisted curvature assumption.

Since X is a Stein manifold, we may express X as $X = \bigcup_{j\geq 1} X_j$ where $\{X_j\}_{j\geq 1}$ is an increasing sequence of relatively compact such that for each j, X_j has compact closure in X_{j+1}. Let $\mathcal{P}_{U\times X}$ denote the Bergman projection of $L^2(U\times X, h)$ onto $\mathcal{H}^2(U \times X, h)$ and let $\mathcal{P}^{\perp}_{U\times X}$ denote its orthogonal complement. For each j, let $\chi_j : X \to [0,1]$ be a smooth function supported on X_{j+1} that is identically 1 on $\overline{X}_j$. In addition, let $E_{(j+1),h}$ be the bundle whose fiber over $t \in U$ is $\mathcal{H}^2\left(X_{j+1}, h^{[t]}\right) =: \mathcal{H}^2_{t,(j+1)}$ and let $\hat{\chi}_j := \chi_j \circ \pi_X : U \times X \to X \to [0,1]$.

Since $\{t\} \times X \cong X$ and $\{t\} \times X_j \cong X_j$ for each fixed $t \in U$ and for each j, we will abusively denote $\mathcal{H}^2\left(\{t\} \times X, h^{[t]}\right)$ and $\mathcal{H}^2\left(\{t\} \times X_{j+1}, h^{[t]}\right)$ by $\mathcal{H}^2_t$ and $\mathcal{H}^2_{t,(j+1)}$, respectively. Thus, while we take the fibers of E_h and $E_{(j+1),h}$ to be $\mathcal{H}^2_t$ and $\mathcal{H}^2_{t,(j+1)}$, respectively, we are really thinking of $\mathcal{H}^2\left(\{t\} \times X, h^{[t]}\right)$ and $\mathcal{H}^2\left(\{t\} \times X_{j+1}, h^{[t]}\right)$ respectively.

For any section $f \in \Gamma_{\mathcal{O}}\left(E_{(j+1),h}\right)$, $\mathcal{P}_{U\times X}\left(\hat{\chi}_j f\right) \in \Gamma_{\mathcal{O}}\left(U\times X, \pi_X^* V\right) \cap L^2\left(U\times X, h\right)$. Therefore, the formula

$$\forall f \in \Gamma_{\mathcal{O}}(E_{(j+1),h}) : \left\langle \xi_t^{(j)}, f^{[t]} \right\rangle := \left\langle \xi_t, \mathcal{P}_{U\times X}\left(\hat{\chi}_j f\right)\big|_{\{t\}\times X} \right\rangle,$$

defines a holomorphic section $\xi^{(j)} \in E^*_{(j+1),h}$.

Let us denote the dual square-norm of $\xi_t^{(j)}$ over $\left(\mathcal{H}^2_{t,(j+1)}\right)^*$ by $\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}$ and let $f \in \Gamma_{\mathcal{O}}\left(E_h\right)$ so that $f^{[t]} \in \mathcal{H}_t^2 - \{0\}$. Then, we have the following estimates.

$$\begin{aligned}
\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}} &\geq \frac{\left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\hat{\chi}_j f\right)\big|_{\{t\}\times X} \right\rangle\right|^2}{\left\|\mathcal{P}_{U\times X}\left(\hat{\chi}_j f\right)\big|_{\{t\}\times X}\right\|^2_{h^{[t]},X_{j+1}}} \\
&\geq \frac{\left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} + f^{[t]} \right\rangle\right|^2}{\|f^{[t]}\|^2_{h^{[t]},X}} \\
&= \frac{\left|\left\langle \xi_t, f^{[t]} \right\rangle + \left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} \right\rangle\right|^2}{\|f^{[t]}\|^2_{h^{[t]},X}}.
\end{aligned}$$

Now

$$\begin{aligned}
&\left|\left\langle \xi_t, f^{[t]} \right\rangle + \left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} \right\rangle\right|^2 \\
&= \left|\left\langle \xi_t, f^{[t]} \right\rangle\right|^2 + 2\mathrm{Re}\left[\left\langle \xi_t, f^{[t]} \right\rangle \overline{\left\langle \xi_t, (\chi_j - 1) f^{[t]} - \mathcal{P}^{\perp}_{U\times X}\left(\hat{\chi}_j f\right)\big|_{\{t\}\times X} \right\rangle}\right] \\
&\qquad + \left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} \right\rangle\right|^2 \\
&\leq \left|\left\langle \xi_t, f^{[t]} \right\rangle\right|^2 + 2\left|\left\langle \xi_t, f^{[t]} \right\rangle\right| \left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} \right\rangle\right| \\
&\qquad + \left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X} \right\rangle\right|^2.
\end{aligned}$$

Since ξ_t is a continuous linear functional, there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$\left|\left\langle \xi_t, f^{[t]} \right\rangle\right|^2 \leq C_1 \left\|f^{[t]}\right\|^2_{h^{[t]},X},$$

and

$$\left|\left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X}\right\rangle\right|^2 \leq C_2 \left\|\mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X}\right\|^2_{h^{[t]},X}$$
$$\leq C_2 \left\|(\chi_j - 1) f^{[t]}\right\|^2_{h^{[t]},X}$$
$$\leq C_2 \left\|f^{[t]}\right\|^2_{h^{[t]},X-\bar{X}_{j+1}}.$$

Now since $\left\|f^{[t]}\right\|^2_{h^{[t]},X-\bar{X}_j} \xrightarrow[j\to+\infty]{} 0$, it follows that for any $\varepsilon > 0$,

$$\left|\left\langle \xi_t, f^{[t]}\right\rangle + \left\langle \xi_t, (\chi_j - 1) f^{[t]} - \mathcal{P}^{\perp}_{U\times X}\left(\hat{\chi}_j f\right)\big|_{\{t\}\times X}\right\rangle\right|^2 < \frac{\left|\left\langle \xi_t, f^{[t]}\right\rangle\right|^2}{\left\|f^{[t]}\right\|^2_{h^{[t]},X}} + \varepsilon \leq \|\xi\|^2_{*,h^{[t]},X} + \varepsilon,$$

provided that j is sufficiently large. Therefore, $\|\xi\|^2_{*,h^{[t]},X} \leq \left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}$.

The same exact argument with X_{j+2} replacing X shows that the sequence of dual squared norms $\left\{\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}\right\}_{j\geq 1}$ is decreasing. Moreover, this sequence is bounded below by $\|\xi\|^2_{*,h^{[t]},X}$.

We now need to show that $\|\xi\|^2_{*,h^{[t]},X}$ is indeed the limit of $\left\{\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}\right\}_{j\geq 1}$. In particular, all we need to show is that

$$\lim_{j\to+\infty} \left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}} \leq \|\xi\|^2_{*,h^{[t]},X}.$$

By the definition of $\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}$, for each j, there exists $f_j \in \Gamma_{\mathcal{O}}(E_{(j+1),h})$ such that

$$\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}} = \left|\left\langle \xi^{(j)}, f_j^{[t]}\right\rangle\right|^2 \text{ and } \left\|f_j^{[t]}\right\|^2_{h^{[t]},X_{j+1}} = 1.$$

Extend f_j by 0 on $U \times (X - X_{j+1})$ and let $\tilde{f}_j$ be the extension of f_j. Then $\tilde{f}_j \in L^2(U \times X, h)$ and $\tilde{f}_j$ converges to some $\tilde{f}$ in $L^2(U \times X, h)$. In fact, $\tilde{f} \in \mathcal{H}^2(U \times X, h)$. But then,

$$\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}} = \left|\left\langle \xi_t^{(j)}, \tilde{f}_j^{[t]}\right\rangle\right|^2 = \left|\left\langle \xi_t^{(j)}, \tilde{f}^{[t]}\right\rangle + \left\langle \xi_t^{(j)}, \tilde{f}_j^{[t]} - \tilde{f}^{[t]}\right\rangle\right|^2$$
$$\leq \left|\left\langle \xi_t^{(j)}, \tilde{f}^{[t]}\right\rangle\right|^2 + 2\left|\left\langle \xi_t^{(j)}, \tilde{f}^{[t]}\right\rangle\right| \left|\left\langle \xi_t^{(j)}, \tilde{f}_j^{[t]} - \tilde{f}^{[t]}\right\rangle\right|$$
$$+ \left|\left\langle \xi_t^{(j)}, \tilde{f}_j^{[t]} - \tilde{f}^{[t]}\right\rangle\right|^2.$$

Clearly, $\left|\left\langle \xi_t^{(j)}, \tilde{f}_j^{[t]} - \tilde{f}^{[t]}\right\rangle\right|^2$ converges to 0 as $j \to +\infty$ by continuity since $\left\|\tilde{f}_j^{[t]} - \tilde{f}^{[t]}\right\|^2_{h^{[t]},X}$ converges to 0 as $j \to +\infty$ by L^2-convergence. On the other hand,

$$\left\langle \xi^{(j)}, \tilde{f}^{[t]}\right\rangle = \left\langle \xi_t, \tilde{f}^{[t]}\right\rangle + \left\langle \xi_t, \mathcal{P}_{U\times X}\left(\left(\hat{\chi}_j - 1\right) f\right)\big|_{\{t\}\times X}\right\rangle,$$

and so arguing as we previously did, we can see that

$$\lim_{j\to+\infty}\left|\left\langle \xi^{(j)},\tilde{f}^{[t]}\right\rangle\right|^2 \leq \|\xi\|^2_{*,h^{[t]},X}\,.$$

Altogether, we conclude that $\lim_{j\to+\infty}\left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}} \leq \|\xi\|^2_{*,h^{[t]},X}\,.$

By Theorem 6.1.1 if $\Xi_{\delta,\eta}(h) >_{\mathrm{Griff}} 0$ and

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes \mathrm{Id}_V >_{\mathrm{Nak}} 0,$$

for each $t \in U$, then each of the functions $U \ni t \mapsto \left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}$ is strictly plurisubharmonic by Proposition 2.4.8. Therefore, the function $U \ni t \mapsto \|\xi\|^2_{*,h^{[t]},X}$ is strictly plurisubharmonic and the result follows by Remark 2.4.10 in this case. Otherwise, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \left(\mathbf{Ric}(g) + 2\partial_X\bar{\partial}_X\eta - (1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in U$, then each of the functions $U \ni t \mapsto \left\|\xi^{(j)}\right\|^2_{*,h^{[t]},X_{j+1}}$ is plurisubharmonic by Proposition 2.4.8. Therefore, the function $U \ni t \mapsto \|\xi\|^2_{*,h^{[t]},X}$ is plurisubharmonic and the result follows again, in this case, by Remark 2.4.10. □

6.1.2.3 Nakano positivity for trivial families of possibly unbounded Stein manifolds

We now prove Theorem B.

Proof. Let $t \in U$ be arbitrary. Let X be a possibly unbounded Stein manifold. We may exhaust X as $X = \bigcup_{j\geq 1} X_j$ where $\{X_j\}_{j\geq 1}$ is an increasing sequence of relatively compact such that for each j, X_j has compact closure in X_{j+1}. For each j, let $\chi_j : X \to [0,1]$ be a smooth function supported on X_{j+1} and that is identically 1 on $\overline{X}_j$. Moreover, let F_h denote the field of Hilbert spaces with fiber L^2_t at $t \in U$ and let $F_{(j+1),h}$ and $E_{(j+1),h}$ denote the bundles with fibers $L^2_{t,(j+1)} =: L^2\left(X_{j+1}, h^{[t]}\right)$ and $\mathcal{H}^2_{t,(j+1)} =: \mathcal{H}^2\left(X_{j+1}, h^{[t]}\right)$ at $t \in U$, respectively. Let $\mathcal{P}_t$ denote the orthogonal projection $L^2_t \to \mathcal{H}^2_t$ and let $\mathcal{P}_t^{\perp}$ denote its orthogonal complement.

Similarly, let $\mathcal{P}_t^{(j+1)}$ denote the orthogonal projection $L^2_{t,(j+1)} \to \mathcal{H}^2_{t,(j+1)}$ and let $\mathcal{P}_t^{(j+1),\perp}$ denote its orthogonal complement. Then

$$\nabla^{F_h}_{t_k} = \left(h^{[t]}\right)^{-1} \partial_{t_k} h^{[t]} = \nabla^{F_{(j+1),h}}_{t_k} \text{ and } \Theta^{F_h}_{t_k \bar{t}_\ell} = \bar{\partial}_{t_\ell} \left(\left(h^{[t]}\right)^{-1} \partial_{t_k} h^{[t]} \right) = \Theta^{F_{(j+1),h}}_{t_k \bar{t}_\ell}.$$

Let $u = (u_1, \cdots, u_m)$ be an m-tuple of holomorphic sections of E_h belonging to the domains of $\nabla^{E_h,(1,0)}_{t_k}$ and $\Theta^{E_h}_{t_k \bar{t}_\ell}$ for each $1 \leq k, \ell \leq m$ such that $\nabla^{E_h,(1,0)}_{t_k} u_k^{[t]} = 0$ at t, for each k. Note that:

$$\begin{aligned} T_u &= \sum_{1 \leq k, \ell \leq m} \left(u_k^{[t]}, u_\ell^{[t]} \right)_{h^{[t]}} \widehat{dt_k \wedge d\bar{t}_\ell} \\ &= \sum_{1 \leq k, \ell \leq m} \left(\chi_j^2 u_k^{[t]}, u_\ell^{[t]} \right)_{h^{[t]}} \widehat{dt_k \wedge d\bar{t}_\ell} + \sum_{1 \leq k, \ell \leq m} \left((1-\chi_j^2) u_k^{[t]}, u_\ell^{[t]} \right)_{h^{[t]}} \widehat{dt_k \wedge d\bar{t}_\ell}. \end{aligned}$$

Since each u_k is a holomorphic section of E_h, it follows, by definition, that $u_k^{[t]} \in \mathcal{H}^2_t \subseteq L^2_t$ for each $t \in U$ and that each u_k depends holomorphically on t. Let $\hat{\chi}_j := \chi_j \circ \pi_X$. Then for each j, $\hat{\chi}_j u_k \big|_{\{t\} \times X} = \chi_j u_k^{[t]} \in L^2_t$ for each t, and each $\hat{\chi}_j u_k$ still depends holomorphically on t since $\hat{\chi}_j$ is independent of t. Therefore, each $\hat{\chi}_j u_k$ is a holomorphic section of F_h. So,

$$\begin{aligned} \partial_U \bar{\partial}_U \left(-T_u\right) = &\sum_{1 \leq k, \ell \leq m} \left(\Theta^{F_h}_{t_k \bar{t}_\ell} \left(\chi_j^2 u_k^{[t]} \right), u_\ell^{[t]} \right)_{h^{[t]}} dV(t) \\ &- \sum_{1 \leq k, \ell \leq m} \left(\nabla^{F_h,(1,0)}_{t_k} \left(\chi_j^2 u_k^{[t]} \right), \nabla^{F_h,(1,0)}_{t_\ell} u_\ell^{[t]} \right)_{h^{[t]}} dV(t) \\ &+ \sum_{1 \leq k, \ell \leq m} \left(\Theta^{F_h}_{t_k \bar{t}_\ell} \left((1-\chi_j^2) u_k^{[t]} \right), u_\ell^{[t]} \right)_{h^{[t]}} dV(t) \\ &- \sum_{1 \leq k, \ell \leq m} \left(\nabla^{F_h,(1,0)}_{t_k} \left(\left(1-\chi_j^2\right) u_k^{[t]} \right), \nabla^{F_h,(1,0)}_{t_\ell} u_\ell^{[t]} \right)_{h^{[t]}} dV(t). \end{aligned}$$

But since χ_j does not depend on t, we have $\left[\nabla_{t_k}^{F_h,(1,0)}, \chi_j^2\right] = 0 = \left[\nabla_{t_k}^{F_h,(1,0)}, 1-\chi_j^2\right]$ and $\left[\Theta_{t_j\bar{t}_k}^{F_h,(1,0)}, \chi_j^2\right] = 0 = \left[\Theta_{t_j\bar{t}_k}^{F_h,(1,0)}, 1-\chi_j^2\right]$ for each t_k. Therefore,

$$\begin{aligned}\partial_U\bar{\partial}_U(-T_u) = &\sum_{1\le k,\ell\le m}\left(\Theta_{t_k\bar{t}_\ell}^{F_h}\chi_j u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{h^{[t]}} dV(t)\\ &- \sum_{1\le k,\ell\le m}\left(\nabla_{t_k}^{F_h,(1,0)}\chi_j u_k^{[t]}, \nabla_{t_\ell}^{F_h,(1,0)}\chi_j u_\ell^{[t]}\right)_{h^{[t]}} dV(t)\\ &+ \sum_{1\le k,\ell\le m}\left((1-\chi_j^2)\Theta_{t_k\bar{t}_\ell}^{F_h} u_k^{[t]}, u_\ell^{[t]}\right)_{h^{[t]}} dV(t)\\ &- \sum_{1\le k,\ell\le m}\left(\left(1-\chi_j^2\right)\nabla_{t_k}^{F_h,(1,0)} u_k^{[t]}, \nabla_{t_\ell}^{F_h,(1,0)} u_\ell^{[t]}\right)_{h^{[t]}} dV(t).\end{aligned}$$

We have the following for each k and ℓ.

$$\begin{aligned}&\left(\Theta_{t_k\bar{t}_\ell}^{E_h}\left((1-\chi_j^2)u_k^{[t]}\right), u_\ell^{[t]}\right)_{h^{[t]}}\\ &= \left(\Theta_{t_k\bar{t}_\ell}^{F_h}\left((1-\chi_j^2)u_k^{[t]}\right), u_\ell^{[t]}\right)_{h^{[t]}} - \left(\mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h}\left((1-\chi_j^2)u_k^{[t]}\right)\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right)\right)_{h^{[t]}}\\ &= \left((1-\chi_j^2)\Theta_{t_k\bar{t}_\ell}^{F_h}u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{h^{[t]}} - \left(\mathcal{P}_t^{\perp}\left((1-\chi_j^2)\nabla_{t_k}^{F_h}u_k^{[t]}\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right)\right)_{h^{[t]}}.\end{aligned}$$

Since $\mathcal{P}_t^{\perp}$ is an orthogonal projection, the Cauchy-Schwarz inequality yields

$$\begin{aligned}&\left(\mathcal{P}_t^{\perp}\left((1-\chi_j^2)\nabla_{t_k}^{F_h}u_k^{[t]}\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right)\right)_{h^{[t]}}\\ &\le \left\|\mathcal{P}_t^{\perp}\left((1-\chi_j^2)\nabla_{t_k}^{F_h}u_k^{[t]}\right)\right\|_{h^{[t]},X}^2 \left\|\mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right)\right\|_{h^{[t]},X}^2\\ &\le \left\|(1-\chi_j^2)\nabla_{t_k}^{F_h}u_k^{[t]}\right\|_{h^{[t]},X}^2 \left\|\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right\|_{h^{[t]},X}^2\\ &\le \left\|\nabla_{t_k}^{F_h}u_k^{[t]}\right\|_{h^{[t]},X-\overline{X}_j}^2 \left\|\nabla_{t_\ell}^{F_h}u_\ell^{[t]}\right\|_{h^{[t]},X} \xrightarrow[j\to+\infty]{} 0.\end{aligned}$$

Similarly, for each k and ℓ,

$$\left((1-\chi_j^2)\Theta_{t_k\bar{t}_\ell}^{F_h}u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{h^{[t]}} \le \left\|\Theta_{k\ell}^{F_h}u_k^{[t]}\right\|_{h^{[t]},X-\overline{X}_j}^2 \left\|u_\ell^{[t]}\right\|_{h^{[t]},X}^2 \xrightarrow[j\to+\infty]{} 0.$$

Therefore,

$$\sum_{1\le k,\ell\le m}\left((1-\chi_j^2)\Theta_{t_k\bar{t}_\ell}^{F_h}u_k^{[t]}, u_\ell^{[t]}\right)_{h^{[t]}} dV(t) \xrightarrow[j\to+\infty]{} 0.$$

Estimating each of the terms $\left(\left(1-\chi_j^2\right)\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]},\nabla_{t_\ell}^{F_h,(1,0)}u_\ell^{[t]}\right)_{h^{[t]}}$ in the same fashion, we see that

$$\sum_{1\leq k,\ell\leq m}\left(\left(1-\chi_j^2\right)\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]},\nabla_{t_\ell}^{F_h,(1,0)}u_\ell^{[t]}\right)_{h^{[t]}}dV(t)\xrightarrow[j\to+\infty]{}0.$$

We now focus our attention on the first two sums in the expression of $\partial_U\bar{\partial}_U(-T_u)$. In what follows, let

$$(v_1,v_2)_{[j+1],h^{[t]}}:=\int_{X_{j+1}}h^{[t]}\left(v_1,v_2\right)dV_g.$$

For any two holomorphic sections u and v of F_h, we have

$$\begin{aligned}
&\left(\Theta_{t_k\bar{t}_\ell}^{F_h}u^{[t]},v^{[t]}\right)_{h^{[t]}}-\left(\nabla_{t_k}^{F_h,(1,0)}u^{[t]},\nabla_{t_\ell}^{F_h,(1,0)}v^{[t]}\right)_{h^{[t]}}\\
&=\left(\mathcal{P}_t^\perp\left(\nabla_{t_k}^F u^{[t]}\right),\mathcal{P}_t^\perp\left(\nabla_{t_\ell}^F v^{[t]}\right)\right)_{h^{[t]}}+\left(\Theta_{t_k\bar{t}_\ell}^{E_h}u^{[t]},v^{[t]}\right)_{h^{[t]}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad-\left(\nabla_{t_k}^{F_h,(1,0)}u^{[t]},\nabla_{t_\ell}^{F_h,(1,0)}v^{[t]}\right)_{h^{[t]}}\\
&=\left(\Theta_{t_k\bar{t}_\ell}^{E_h}u^{[t]},v^{[t]}\right)_{h^{[t]}}+\left(\mathcal{P}_t^\perp\left(\nabla_{t_k}^{F_h,(1,0)}u^{[t]}\right),\mathcal{P}_t^\perp\left(\nabla_{t_\ell}^{F_h,(1,0)}v^{[t]}\right)\right)_{h^{[t]}}\\
&\qquad\qquad\qquad-\left(\nabla_{t_k}^{F_h,(1,0)}u^{[t]},\nabla_{t_\ell}^{F_h,(1,0)}v^{[t]}\right)_{h^{[t]}}\\
&=\left(\Theta_{t_k\bar{t}_\ell}^{E_h}u^{[t]},v^{[t]}\right)_{h^{[t]}}-\left(u^{[t]},\mathcal{P}_t\left(\nabla_{t_k}^{F_h,(1,0)}v^{[t]}\right)\right)_{h^{[t]}}-\left(\mathcal{P}_t\left(\nabla_{t_k}^{F_h,(1,0)}u^{[t]}\right),v^{[t]}\right)_{h^{[t]}}\\
&\qquad\qquad\qquad+\left(\mathcal{P}_t\left(\nabla_{t_\ell}^{F_h,(1,0)}u^{[t]}\right),\mathcal{P}_t\left(\nabla_{t_\ell}^{F_h,(1,0)}v^{[t]}\right)\right)_{h^{[t}}\\
&=\left(\Theta_{t_k\bar{t}_\ell}^{E_h}u^{[t]},v^{[t]}\right)_{h^{[t]}}-\left(u^{[t]},\nabla_{t_k}^{E_h,(1,0)}v^{[t]}\right)_{h^{[t]}}-\left(\nabla_{t_k}^{E_h,(1,0)}u^{[t]},v^{[t]}\right)_{h^{[t]}}\\
&\qquad\qquad\qquad+\left(\nabla_{t_k}^{E_h,(1,0)}u^{[t]},\nabla_{t_\ell}^{E_h,(1,0)}v^{[t]}\right)_{h^{[t}}.
\end{aligned}$$

Now, since each u_k satisfies $\nabla_{t_k}^{E_h,(1,0)}u_k^{[t]}=0$ at t, we have the following at t.

$$\begin{aligned}
\nabla_{t_k}^{E_h,(1,0)}\chi_j u_k^{[t]}&=\nabla_{t_k}^{E_h,(1,0)}\left(\left(\chi_j-1\right)u_k^{[t]}\right)\\
&=\mathcal{P}_t\left(\nabla_{t_k}^{F_h,(1,0)}\left(\left(\chi_j-1\right)u_k^{[t]}\right)\right)\\
&=\mathcal{P}_t\left(\left(\chi_j-1\right)\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]}\right)
\end{aligned}$$

But since $\mathcal{P}_t$ is an orthogonal projection,

$$\left\|\mathcal{P}_t\left(\left(\chi_j-1\right)\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]}\right)\right\|_{h^{[t]}}^2\leq\left\|\left(\chi_j-1\right)\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]}\right\|_{h^{[t]}}^2\leq\left\|\nabla_{t_k}^{F_h,(1,0)}u_k^{[t]}\right\|_{h^{[t]},X-\overline{X}_j}^2\xrightarrow[j\to+\infty]{}0,$$

and so by the Cauchy-Schwarz inequality,

$$\left(u_k^{[t]}, \nabla_{t_k}^{E_h,(1,0)} u_\ell^{[t]}\right)_{h^{[t]}}, \left(\nabla_{t_k}^{E_h,(1,0)} u_k^{[t]}, u_\ell^{[t]}\right)_{h^{[t]}}, \left(\nabla_{t_k}^{E_h,(1,0)} u_k^{[t]}, \nabla_{t_\ell}^{E_h,(1,0)} u_\ell^{[t]}\right)_{h^{[t}} \xrightarrow[j\to+\infty]{} 0.$$

On the other hand, note that each u_k is a holomorphic section of $E_{(j+1),h}$ and that each $\hat{\chi}_j u_k$ is simultaneously a holomorphic section of $F_{(j+1),h}$ and a smooth section of $E_{(j+1),h}$. Since $\Theta_{t_k\bar{t}_\ell}^{F_h} = \Theta_{t_k\bar{t}_\ell}^{F_{(j+1),h}}$,

$$\begin{aligned}\left(\Theta_{t_k\bar{t}_\ell}^{E_h} \chi_j u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{h^{[t]}} = & \left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} \chi_j u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{[j+1],h^{[t]}} \\ & + \left(\mathcal{P}_t^{(j+1),\perp}\left(\nabla_{t_k}^{F_h} \chi_j u_k^{[t]}\right), \mathcal{P}_t^{(j+1),\perp}\left(\nabla_{t_\ell}^{F_h} \chi_j u_\ell^{[t]}\right)\right)_{[j+1],h^{[t]}} \\ & - \left(\mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h} \chi_j u_k^{[t]}\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h} \chi_j u_\ell^{[t]}\right)\right)_{h^{[t]}}.\end{aligned}$$

Note again that because $\nabla_{t_k}^{E_h,(1,0)} u_k^{[t]} = 0$ at t,

$$\mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h} \chi_j u_k^{[t]}\right) = \mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h,(1,0)} \chi_j u_k^{[t]}\right) = \nabla_{t_k}^{E_h,(1,0)} \chi_j u_k^{[t]} = \nabla_{t_k}^{E_h,(1,0)} \left(\chi_j - 1\right) u_k^{[t]},$$

and so our previous arguments imply that $\left(\mathcal{P}_t^{\perp}\left(\nabla_{t_k}^{F_h} \chi_j u_k^{[t]}\right), \mathcal{P}_t^{\perp}\left(\nabla_{t_\ell}^{F_h} \chi_j u_\ell^{[t]}\right)\right)_{h^{[t]}} \xrightarrow[j\to+\infty]{} 0$. Additionally,

$$\begin{aligned}\left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} \chi_j u_k^{[t]}, \chi_j u_\ell^{[t]}\right)_{[j+1],h^{[t]}} = & \left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} u_k^{[t]}, u_\ell^{[t]}\right)_{[j+1],h^{[t]}} + \left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} \left(\chi_j - 1\right) u_k^{[t]}, u_\ell^{[t]}\right)_{[j+1],h^{[t]}} \\ & + \left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} u_k^{[t]}, \left(\chi_j - 1\right) u_\ell^{[t]}\right)_{[j+1],h^{[t]}} \\ & + \left(\Theta_{t_k\bar{t}_\ell}^{E_{(j+1),h}} \left(\chi_j - 1\right) u_k^{[t]}, \left(\chi_j - 1\right) u_\ell^{[t]}\right)_{[j+1],h^{[t]}}.\end{aligned}$$

The last two summands converge to 0 as $j \to +\infty$ by the Cauchy-Schwarz inequality and the fact that

$$\left\|\left(\chi_j - 1\right) u_k^{[t]}\right\|_{[j+1],h^{[t]}}^2 \leq \left\|u_k^{[t]}\right\|_{h^{[t]},X_{j+1}-\overline{X}_j}^2 \xrightarrow[j\to+\infty]{} 0.$$

As for the second summand,

$$\begin{aligned}
&\left(\Theta^{E_{(j+1),h}}_{t_k\bar{t}_\ell}\left(\chi_j-1\right)u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}\\
&=\left(\Theta^{F_{(j+1),h}}_{t_k\bar{t}_\ell}\left(\chi_j-1\right)u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}-\left(\mathcal{P}^{(j+1),\perp}_t\left(\nabla^{F_h}_{t_k}\left(\chi_j-1\right)u^{[t]}_k\right),\mathcal{P}^{(j+1),\perp}_t\left(\nabla^{F_h}_{t_\ell}u^{[t]}_\ell\right)\right)_{[j+1],h^{[t]}}\\
&=\left(\Theta^{F_h}_{t_k\bar{t}_\ell}\left(\chi_j-1\right)u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}-\left(\mathcal{P}^{(j+1),\perp}_t\left(\nabla^{F_h}_{t_k}\left(\chi_j-1\right)u^{[t]}_k\right),\mathcal{P}^{(j+1),\perp}_t\left(\nabla^{F_h}_{t_\ell}u^{[t]}_\ell\right)\right)_{[j+1],h^{[t]}}\\
&=\left(\left(\chi_j-1\right)\Theta^{F_h}_{t_k\bar{t}_\ell}u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}-\left(\mathcal{P}^{(j+1),\perp}_t\left(\left(\chi_j-1\right)\nabla^{F_h}_{t_k}u^{[t]}_k\right),\mathcal{P}^{(j+1),\perp}_t\left(\nabla^{F_h}_{t_\ell}u^{[t]}_\ell\right)\right)_{[j+1],h^{[t]}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\Theta^{E_{(j+1),h}}_{t_k\bar{t}_\ell}\left(\chi_j-1\right)u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}\\
&\leq\left\|\left(\chi_j-1\right)\Theta^{F_h}_{t_k\bar{t}_\ell}u^{[t]}_k\right\|^2_{[j+1],h^{[t]}}\left\|u^{[t]}_\ell\right\|^2_{[j+1],h^{[t]}}\\
&\qquad+\left\|\mathcal{P}^{(j+1),\perp}_t\left(\left(\chi_j-1\right)\nabla^{F_h}_{t_k}u^{[t]}_k\right)\right\|^2_{[j+1],h^{[t]}}\left\|\mathcal{P}^{(j+1),\perp}_t\nabla^{F_h}_{t_\ell}\left(u^{[t]}_\ell\right)\right\|^2_{[j+1],h^{[t]}}\\
&\leq\left\|\left(\chi_j-1\right)\Theta^{F_h}_{t_k\bar{t}_\ell}u^{[t]}_k\right\|^2_{[j+1],h^{[t]}}\left\|u^{[t]}_\ell\right\|^2_{[j+1],h^{[t]}}+\left\|\left(\chi_j-1\right)\nabla^{F_h}_{t_k}u^{[t]}_k\right\|^2_{[j+1],h^{[t]}}\left\|\nabla^{F_h}_{t_\ell}u^{[t]}_\ell\right\|^2_{[j+1],h^{[t]}}\\
&\leq\left\|\Theta^{F_h}_{t_k\bar{t}_\ell}u^{[t]}_k\right\|^2_{h^{[t]},X_{j+1}-\overline{X}_j}\left\|u^{[t]}_\ell\right\|^2_{[j+1],h^{[t]}}+\left\|\nabla^{F_h}_{t_k}u^{[t]}_k\right\|^2_{h^{[t]},X_{j+1}-\overline{X}_j}\left\|\nabla^{F_h}_{t_\ell}u^{[t]}_\ell\right\|^2_{[j+1],h^{[t]}}\xrightarrow[j\to+\infty]{}0.
\end{aligned}$$

Altogether, we have

$$\partial_U\bar{\partial}_U(-T_u)=\sum_{1\leq k,\ell\leq m}\left(\Theta^{E_{(j+1),h}}_{t_k\bar{t}_\ell}u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}dV(t)+o(j).$$

By Theorem 6.1.1, if $\Xi_{\delta,\eta}(h)>_{\text{Griff}}0$ and

$$\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right)+\left(\mathbf{Ric}(g)+2\partial_X\bar{\partial}_X\eta-(1+\delta)\partial_X\eta\wedge\bar{\partial}_X\eta\right)\otimes\mathrm{Id}_V>_{\text{Nak}}0,$$

for each $t\in U$, then

$$\exists c_0>0:\sum_{1\leq k,\ell\leq m}\left(\Theta^{E_{(j+1),h}}_{t_k\bar{t}_\ell}u^{[t]}_k,u^{[t]}_\ell\right)_{[j+1],h^{[t]}}\geq c_0\sum_{k=1}^m\left\|u^{[t]}_k\right\|^2_{[j+1],h^{[t]}}.$$

For each u_k, $\left\|u^{[t]}_k\right\|^2_{[j+1],h^{[t]}}=\left\|u^{[t]}_k\right\|^2_{h^{[t]}}-\left\|u^{[t]}_k\right\|^2_{h^{[t]},X-\overline{X}_{j+1}}$ and since $u^{[t]}_k\in\mathcal{H}^2_t\subseteq L^2_t$ for each u_k, $\left\|u^{[t]}_k\right\|^2_{h^{[t]},X-\overline{X}_{j+1}}\xrightarrow[j\to+\infty]{}0$. Therefore,

$$\exists c_0>0:\partial_U\bar{\partial}_U(-T_u)\geq c_0\sum_{k=1}^m\left\|u^{[t]}_k\right\|^2_{h^{[t]}}dV(t)+o(j),$$

whence $(E, (\cdot,\cdot)_{h^{[t]}})$ is Nakano positive at t by taking the limit as $j \to +\infty$. The result follows as $t \in U$ was arbitrary.

Otherwise, if either $\Xi_{\delta,\eta}(h) \geq_{\mathrm{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta\right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in U$, then again by Theorem 6.1.1 and our previous observations,

$$\partial_U \bar{\partial}_U (-T_u) \geq o(j),$$

whence $(E, (\cdot,\cdot)_{h^{[t]}})$ is Nakano semipositive at t by taking the limit as $j \to +\infty$. The result follows in this case as well since $t \in U$ was arbitrary. □

6.2 Variations of Bergman kernels and compactly supported measures

6.2.1 Twisted log-plurisubharmonic variation results for trivial families of possibly Stein unbounded manifolds

6.2.1.1 Variations of Bergman kernels

One immediate consequence of Theorem C is Theorem D, which we now prove.

Proof. For $z \in X$ and $t \in U$, define $\hat{\xi}_t^{(z)}$ by $\hat{\xi}_t^{(z)}(\hat{f}) = i_t^* \hat{f}(z)$, for $\hat{f} \in \Gamma_{\mathcal{O}}(E_h)$. By Proposition 3.2.1, $\hat{\xi}_t^{(z)} : \mathcal{H}_t^2 \to V_z$ is a bounded linear map. Now let $\sigma \in V_z^*$ be a non-zero vector. Then

$$\xi_t^{(z,\sigma)} := \sigma \otimes \hat{\xi}_t^{(z)} \in \left(\mathcal{H}_t^2\right)^*.$$

Moreover, if $f \in \Gamma_{\mathcal{O}}(E_h)$, then the function $t \mapsto \left\langle \xi_t^{(z,\sigma)}, f^{[t]} \right\rangle$ is holomorphic. Thus $t \mapsto \xi_t^{(z,\sigma)}$ defines a holomorphic section of E_h^* which we denote by $\xi^{(z,\sigma)}$.

By Theorem 3.4.4 the fiberwise dual squared norm of this section is given by

$$\left\|\xi^{(z,\sigma)}\right\|^2_{*,h^{[t]}} = \sup_{f \in \mathcal{H}_t^2 - \{0\}} \frac{\left|\left\langle \xi^{(z,\sigma)}, f \right\rangle\right|^2}{\|f\|^2_{h^{[t]}}} = \sup_{f \in \mathcal{H}_t^2 - \{0\}} \frac{\left|\langle f(z), \sigma \rangle\right|^2}{\|f\|^2_{h^{[t]}}} = \langle K_t(z,z), \sigma \otimes \bar{\sigma} \rangle .$$

By Theorem 6.1.1, if $\Xi_{\delta,\eta}(h) >_{\text{Griff}} 0$ and

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta \right) \otimes \mathrm{Id}_V >_{\text{Nak}} 0,$$

for each $t \in U$, then

$$\partial_U \bar{\partial}_U \log \langle K_t(z,z), \sigma \otimes \bar{\sigma} \rangle > 0.$$

Otherwise, if either $\Xi_{\delta,\eta}(h) \geq_{\text{Griff}} 0$ or

$$\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \left(\mathbf{Ric}(g) + 2\partial_X \bar{\partial}_X \eta - (1+\delta)\partial_X \eta \wedge \bar{\partial}_X \eta \right) \otimes \mathrm{Id}_V \geq_{\text{Nak}} 0,$$

for each $t \in U$, then

$$\partial_U \bar{\partial}_U \log \langle K_t(z,z), \sigma \otimes \bar{\sigma} \rangle \geq 0,$$

which completes the proof. □

In the Euclidean setting, this result corresponds to Berndtsson's result on the log-plurisubharmonic variation of Bergman kernels for product domains ([Ber06]). In our setting, we have the following theorem.

Theorem 6.2.1. *Let U be a domain in $\mathbb{C}^m$ and Ω a pseudoconvex domain in $\mathbb{C}^n$. Let $\varphi \in \mathcal{C}^\infty(U \times \Omega)$, let $\delta > 0$ and let $\eta \in \mathcal{C}^\infty(\Omega)$. Let K_t denote the Bergman kernel of the projection of $L^2\left(\Omega, e^{-\varphi^{[t]}}\right)$ onto $\mathcal{H}^2\left(\Omega, e^{-\varphi^{[t]}}\right)$. If $\Xi_{\delta,\eta}(e^{-\varphi}) \geq 0$ (resp. > 0) in $U \times \Omega$, then the function $t \mapsto \log(K_t(z,z))$ is plurisubharmonic (resp. strictly plurisubharmonic) or identically $-\infty$.*

6.2.1.2 Variations of families of compactly supported measures

Let $\{\hat{\mu}_t\}_{t\in U}$ be a family of compactly supported V^*-valued complex measures over X. Then for each section f of E_h, the pairing $\mu_t^{(f)} := \left\langle f^{[t]}, \hat{\mu}_t \right\rangle$ defines a compactly supported complex measure on X for each $t \in U$. Now consider the mapping $\xi_t^{(\mu)}$ defined by

$$f^{[t]} \mapsto \left\langle \xi_t^{(\mu)}, f^{[t]} \right\rangle := \mu_t^{(f)}(X) = \int_X \left\langle f^{[t]}, \hat{\mu}_t \right\rangle.$$

Let us represent $\hat{\mu}_t$ locally as $\hat{\mu}_t = \sum_{j=1}^{r} \sigma_j \otimes \mu_t^{(j)}$ where $\sigma_1, \cdots, \sigma_r$ be a local frame for V^* over some open subset $W \subset X$, and let $\mu_t^{(1)}, \cdots, \mu_t^{(r)}$ be complex measures for X over W, all of which are supported on a compact subset K of X. Let $h^{[t],*}$ denote the dual metric to $h^{[t]}$ for the dual bundle $V^* \to X$. Then, locally, we have the following

$$\begin{aligned}
\left| \int_X \left\langle f^{[t]}, \hat{\mu}_t \right\rangle \right| &= \left| \int_X \sum_{j=1}^{r} \left\langle f^{[t]}, \sigma_j \right\rangle d\mu_t^{(j)} \right| \\
&\leq \int_X \sum_{j=1}^{r} \left| \left\langle f^{[t]}, \sigma_j \right\rangle \right| d\mu_t^{(j)} \\
&\leq \int_X |f|_{h^{[t]}} \left(\sum_{j=1}^{r} |\sigma_j|_{h^{[t],*}} \right) d\mu_t^{(j)} \\
&= \int_K |f|_{h^{[t]}} \left(\sum_{j=1}^{r} |\sigma_j|_{h^{[t],*}} \right) d\mu_t^{(j)} \\
&\leq \sup_K \left(\sum_{j=1}^{r} |\sigma_j|_{h^{[t],*}} \right) \sup_K |f|_{h^{[t]}} \int_K d\mu_t^{(j)} \\
&= \sup_K \left(\sum_{j=1}^{r} |\sigma_j|_{h^{[t],*}} \right) \mu_t^{(j)}(K) \sup_K |f|_{h^{[t]}} .
\end{aligned}$$

Now by Proposition 3.2.1, there exists a constant $C_K > 0$ such that

$$\sup_K |f|^2_{h^{[t]}} \leq C_K \int_X |f|^2_{h^{[t]}} \, dV_g.$$

Therefore,

$$\sup_{f^{[t]} \in \mathcal{H}_t^2} \frac{\left| \left\langle \xi_t^{(\mu)}, f^{[t]} \right\rangle \right|^2}{\|f\|^2_{h^{[t]}}}$$

is bounded. Using a partition of unity, we can see that this boundedness does not depend on the choice of frame or local representative measures. So the mapping $\xi_t^{(\mu)}$ defines a smooth section $\xi^{(\mu)}$ of E_h^*. We thus have Theorem E as a consequence of Theorem C.

In the Euclidean setting, Theorem E corresponds to Berndtsson's theorem on the log-plurisubharmonic variation of compactly supported measures for product domains ([Ber18; Ber17]). In our setting, we have the following theorem.

Theorem 6.2.2. *Let U be a domain in $\mathbb{C}^m$ and Ω a pseudoconvex domain in $\mathbb{C}^n$. Let $\varphi \in \mathcal{C}^\infty(U \times \Omega)$, let $\delta > 0$ and let $\eta \in \mathcal{C}^\infty(\Omega)$. Let $\{\mu_t\}_{t \in U}$ be a family of complex measures* *on Ω which are compactly supported inside Ω. Define a section $\xi^{(\mu)}$ of E_φ^* by*

$$\forall t \in U, \forall F \in \mathcal{H}^2_{\varphi(t,\cdot)}(\Omega) : \xi_t^{(\mu)}(F) = \int_\Omega F(z) d\mu_t(z).$$

Suppose that $\xi^{(\mu)}$ is a holomorphic section of E_φ^*. If $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ (resp. > 0) in $U \times \Omega$, then the function

$$U \ni t \mapsto \log\left(\left\|\xi^{(\mu)}\right\|^2_{*,\varphi(t,\cdot)}\right)$$

is plurisubharmonic (resp. strictly plurisubharmonic) or identically $-\infty$.

Furthermore, letting $\hat{\mu}_t := \sigma \otimes \delta_{F(t)}$ for each $t \in U$, where F is a holomorphic map from U to X, $\delta_{F(t)}$ denotes a point-mass measure supported at $F(t)$ and $\sigma \in V^*_{F(t)}$, we obtain a slightly stronger result than Theorem F. Namely, under the hypotheses of Theorem F, the function $U \ni t \mapsto \log\left\langle \sigma \otimes \bar{\sigma}, K_t\left(F(t), F(t)\right)\right\rangle$ is plurisubharmonic (or strictly so) or identically $-\infty$ for every $\sigma \in V^*_{F(t)}$.

6.2.2 Twisted log-plurisubharmonic variation results for a class of non-trivial families of Stein manifolds

Now let Y be an n-dimensional Stein manifold, let ρ be a smooth plurisubharmonic function on $\mathbb{C}^m \times Y$, and suppose that $X := \{(t,z) \in \mathbb{C}^m \times Y : \rho(t,z) < 0\}$ is not necessarily a product manifold in $\mathbb{C}^m \times Y$. We assume further that for each t, the restriction $\rho^{[t]}$ of ρ to

$$X_t := \{z \in Y : (t,z) \in X\} \subset Y$$

takes values in $[-1, 0)$. This assumption is satisfied when X is a bounded strongly pseudoconvex domain, for instance.

Let g be a Kähler metric for Y and choose $\hat{g} := \pi^*_{\mathbb{C}^m} g_0 \oplus \pi_Y^* g$ as a Kähler metric for $\mathbb{C}^m \times Y$, where g_0 is the Euclidean metric on $\mathbb{C}^m$. Let $V \to X$ be a holomorphic vector bundle and let h be Hermitian metric for $V \to X$ such that $\bar{\partial}_t\left(h^{-1}\partial_Y h\right) = 0$, $\bar{\partial}_t\left(h^{-1}\partial_t h\right) \geq 0$, and $h^{[t]} := h\big|_{X_t}$ satisfies

$$\Theta\left(h^{[t]}\right) + \left(\mathbf{Ric}(g) + \partial_Y \bar{\partial}_Y \rho^{[t]}\right) \otimes \mathrm{Id}_{V^{[t]}} \geq_{\mathrm{Nak}} 0 \tag{6.2.1}$$

over X_t, for each $t \in \mathbb{C}^m$. Here, ∂_t and $\bar{\partial}_t$ denote the ∂ and $\bar{\partial}$ operator with respect to the t-variable, on $\mathbb{C}^m$.

In [Ber17], Berndtsson reduces the proof of his log-plurisubharmonic variation results from the case of pseudoconvex subdomains of product domains to product domains. Using a similar approach, we will show two additional log-plurisubharmonic variation results for the class of non-trivial families of Stein manifolds just described.

6.2.2.1 Variations of Bergman kernels

Let K_t denote the Bergman kernel for the Bergman projection $L^2\left(X_t, h^{[t]}, dV_g\right) \to \mathcal{H}^2\left(X_t, h^{[t]}, dV_g\right)$ throughout the following proof of Theorem F.

Proof. Let $z \in X_t$ and $\sigma \in \left(V_z^{[t]}\right)^*$ be fixed. Let us fix $t \in \mathbb{C}^m$ – say $t = 0$. Assume for the moment that $m = 1$. Since the property is local, we may restrict ourselves to a neighborhood of $t = 0$. Let U_0 be a sufficiently small neighborhood of 0 so that all the fibers X_t are contained in a fixed pseudoconvex domain $Y_\varepsilon = \{\zeta \in Y : \rho(0, \zeta) < \varepsilon\}$. As the sublevel set of a smooth plurisubharmonic function inside a Stein manifold, X is a Stein manifold. Upon exhausting X by an increasing sequence of relatively compact strongly pseudoconvex domains, we may assume that X is a bounded strongly pseudoconvex domain with smooth boundary.

For $j \geq 4$, define $\rho_j := \dfrac{1}{j}\log\left(e^{j^2\rho} + 1\right)$ and $h_j := he^{-\rho_j}$, so that $h_j \xrightarrow[j\to+\infty]{} h$ when $\rho \leq 0$ and $h_j \xrightarrow[j\to+\infty]{} 0$ when $\rho > 0$. Then $\Xi_{\delta,\eta}(h_j)$ equals

$$\begin{pmatrix} \bar{\partial}_t\left(h^{-1}\partial_t h\right) + \partial_t\bar{\partial}_t\rho_j & \partial_t\bar{\partial}_Y\rho_j \\ \partial_Y\bar{\partial}_t\rho_j & \dfrac{\delta}{1+\delta}\left[\bar{\partial}_X\left(\left(h^{[t]}\right)^{-1}\partial_X h^{[t]}\right) + \partial_Y\bar{\partial}_Y\rho_j^{[t]} + \mathbf{Ric}(g) + \dfrac{4e^{\frac{1+\delta}{2}\eta}}{1+\delta}\partial_Y\bar{\partial}_Y\left(-e^{-\frac{1+\delta}{2}\eta}\right)\right] \end{pmatrix}.$$

Our goal is to find a function η on Y, depending on $\rho^{[t]}$, so that $\Xi_{\delta,\eta}(h_j) \geq 0$ and

$$\Theta\left(h_j^{[t]}\right) + \left(\mathbf{Ric}(g) + \frac{4e^{\frac{1+\delta}{2}\eta}}{1+\delta}\partial_Y\bar{\partial}_Y\left(-e^{-\frac{1+\delta}{2}\eta}\right)\right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} 0,$$

for each $t \in \mathbb{C}^m$.

Let $\mathfrak{a}, \mathfrak{b} \in \{t, Y\}$ as indices. Then:

$$\partial_{\mathfrak{a}} \bar{\partial}_{\mathfrak{b}} \rho_j = \frac{j e^{j^2 \rho}}{1 + e^{j^2 \rho}} \partial_{\mathfrak{a}} \bar{\partial}_{\mathfrak{b}} \rho + \frac{j^3 e^{j^2 \rho}}{\left(1 + e^{j^2 \rho}\right)^2} \partial_{\mathfrak{a}} \rho \wedge \bar{\partial}_{\mathfrak{b}} \rho.$$

Therefore, $\Xi_{\delta,\eta}(h_j)$ can be decomposed as

$$\begin{pmatrix} \bar{\partial}_t \left(h^{-1} \partial_t h\right) & 0 \\ 0 & \frac{\delta}{1+\delta} \left(\bar{\partial}_X \left(\left(h^{[t]}\right)^{-1} \partial_X h^{[t]} \right) + \mathbf{Ric}(g) + \partial_Y \bar{\partial}_Y \rho^{[t]} \right) \end{pmatrix} + \begin{pmatrix} \partial_t \bar{\partial}_t \rho_j & \partial_t \bar{\partial}_Y \rho_j \\ \partial_Y \bar{\partial}_t \rho_j & \partial_Y \bar{\partial}_Y \rho_j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M_j^{\eta,\rho} \end{pmatrix}$$

where

$$M_j^{\eta,\rho} := -\frac{1}{1+\delta} \left(\frac{j e^{j^2 \rho^{[t]}}}{1 + e^{j^2 \rho^{[t]}}} \partial_Y \bar{\partial}_Y \rho^{[t]} + \frac{j^3 e^{j^2 \rho^{[t]}}}{\left(1 + e^{j^2 \rho^{[t]}}\right)^2} \partial_Y \rho^{[t]} \wedge \bar{\partial}_Y \rho^{[t]} \right) + \frac{\delta}{1+\delta} \left(2 \partial_Y \bar{\partial}_Y \eta - (1+\delta) \partial_Y \eta \wedge \bar{\partial}_Y \eta - \partial_Y \partial_Y \rho^{[t]} \right).$$

Our hypotheses clearly imply that

$$\Xi_{\delta,\eta}(h_j) \geq \begin{pmatrix} 0 & 0 \\ 0 & M_j^{\eta,\rho} \end{pmatrix}$$

and

$$\Theta\left(h_j^{[t]}\right) + \left(\mathbf{Ric}(g) + \frac{4 e^{\frac{1+\delta}{2}\eta}}{1+\delta} \partial_Y \bar{\partial}_Y \left(-e^{-\frac{1+\delta}{2}\eta} \right) \right) \otimes \mathrm{Id}_V \geq_{\mathrm{Nak}} M_j^{\eta,\rho} \otimes \mathrm{Id}_V,$$

for each $t \in \mathbb{C}^m$.

Therefore, all we need to do is find a function η on Y, depending on $\rho^{[t]}$, such that $M_j^{\eta,\rho} \geq 0$.

Now note that

$$\forall j > 0 : \frac{e^{j^2 \rho^{[t]}}}{1 + e^{j^2 \rho^{[t]}}} < 1 \text{ and } \frac{e^{j^2 \rho^{[t]}}}{\left(1 + e^{j^2 \rho^{[t]}}\right)^2} < \frac{1}{4}.$$

So, with $\eta := f\left(\rho^{[t]}\right)$ for some function f to be determined shortly,

$$\begin{aligned}
M_j^{\eta,\rho} &> \frac{\delta}{1+\delta}\left(-\frac{j}{\delta}\partial_Y\bar{\partial}_Y\rho^{[t]} - \frac{j^3}{4\delta}\partial_Y\rho^{[t]}\wedge\bar{\partial}_Y\rho^{[t]} + 2\partial_Y\bar{\partial}_Y\eta - (1+\delta)\partial_Y\eta\wedge\bar{\partial}_Y\eta - \partial_Y\bar{\partial}_Y\rho^{[t]}\right)\\
&= \frac{\delta}{1+\delta}\frac{1}{\delta}\left(2\delta\partial_Y\bar{\partial}_Y\eta - j\partial_Y\bar{\partial}_Y\rho^{[t]} - \delta(1+\delta)\partial_Y\eta\wedge\bar{\partial}_Y\eta - \frac{j^3}{4}\partial_Y\rho^{[t]}\wedge\bar{\partial}_Y\rho^{[t]} - \delta\partial_Y\bar{\partial}_Y\rho^{[t]}\right)\\
&= \frac{1}{1+\delta}\left(2\delta\partial_Y\bar{\partial}_Y\eta - j\partial_Y\bar{\partial}_Y\rho^{[t]} - \delta(1+\delta)\partial_Y\eta\wedge\bar{\partial}_Y\eta - \frac{j^3}{4}\partial_Y\rho^{[t]}\wedge\bar{\partial}_Y\rho^{[t]} - \delta\partial_Y\bar{\partial}_Y\rho^{[t]}\right)\\
&= \frac{1}{1+\delta}\left[\left(2\delta f'\left(\rho^{[t]}\right) - j - \delta\right)\partial_Y\bar{\partial}_Y\rho^{[t]}\right]\\
&\quad + \frac{1}{1+\delta}\left[\left(2\delta f''\left(\rho^{[t]}\right) - \delta(1+\delta)\left(f'\left(\rho^{[t]}\right)\right)^2 - \frac{j^3}{4}\right)\partial_Y\rho^{[t]}\wedge\bar{\partial}_Y\rho^{[t]}\right].
\end{aligned}$$

The function

$$f\left(\rho^{[t]}\right) := C_2 - \frac{2}{1+\delta}\log\left(\cos\left(\frac{\sqrt{1+\delta}\,j^{3/2}}{4\sqrt{\delta}}(\rho^{[t]} + C_1\delta)\right)\right)$$

satisfies

$$2\delta f''\left(\rho^{[t]}\right) - \delta(1+\delta)\left(f'\left(\rho^{[t]}\right)\right)^2 - \frac{j^3}{4} = 0,$$

for any constants C_2 and C_1. The constant C_1 will be determined later and we can just let $C_2 = 0$. Now,

$$f'\left(\rho^{[t]}\right) = \frac{j^{3/2}}{2\sqrt{\delta(1+\delta)}}\tan\left(\frac{j^{3/2}}{4}\sqrt{\frac{1+\delta}{\delta}}(\rho^{[t]} + C_1\delta)\right),$$

and so

$$\begin{aligned}
&-\frac{j}{\delta}\partial_Y\bar{\partial}_Y\rho^{[t]} - \frac{j^3}{4\delta}\partial_Y\rho^{[t]}\wedge\bar{\partial}_Y\rho^{[t]} + 2\partial_Y\bar{\partial}_Y\eta - (1+\delta)\partial_Y\eta\wedge\bar{\partial}_Y\eta - \partial_Y\bar{\partial}_Y\rho^{[t]}\\
&= \frac{1}{\delta}\left(\sqrt{\frac{\delta j^3}{1+\delta}}\tan\left(\frac{j^{3/2}}{4}\sqrt{\frac{1+\delta}{\delta}}(\rho^{[t]} + C_1\delta)\right) - (j+\delta)\right)\partial_Y\bar{\partial}_Y\rho^{[t]}.
\end{aligned}$$

For each $j \geq 4$, we can always find some $\delta := \delta_j > 0$ such that

$$\sqrt{\frac{\delta j^3}{1+\delta}} > j + \delta.$$

Furthermore, if we let $C_1 := C_{\delta,j} := \frac{1}{\delta}\left(\pi\sqrt{\frac{\delta}{j^3(1+\delta)}} + 1\right)$, then $C_1\delta \leq 1 + \pi/8$ for $j \geq 4$, whence $\frac{j^{3/2}}{4}\sqrt{\frac{1+\delta}{\delta}}(\rho^{[t]} + C_1\delta)$ takes values in $[\pi/4, \pi/2)$ since $-1 \leq \rho^{[t]} < 0$. Hence,

$$\sqrt{\frac{\delta j^3}{1+\delta}}\tan\left(\frac{j^{3/2}}{4}\sqrt{\frac{1+\delta}{\delta}}(\rho^{[t]} + C_1\delta)\right) - (j+\delta) > 0.$$

Therefore, Theorem D applies to our situation where the product domain is $U_0 \times Y_\varepsilon$. Hence, by Proposition 3.4.10 (combined with Proposition 4.4.1) $\log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$, over a relatively compact strongly pseudoconvex subdomain of X, can be written as an increasing limit of functions that are subharmonic with respect to t. We then conclude, by Proposition 3.4.13, that the function $t \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$ is also subharmonic by upper semicontinuity. Again, by upper semicontinuity, we get that $t \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$ is plurisubharmonic if $m \geq 1$ since its restriction to any line is subharmonic. Finally, Proposition 3.4.11 implies that the function $t \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$, over X, is the decreasing limit of a sequence of plurisubharmonic functions and is thus plurisubharmonic.

Of course, the plurisubharmonicity is strict if the twisted curvature conditions are strict. However, we will not repeatedly state this in what follows.

So far, we have shown that $t \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$ is plurisubharmonic for z fixed. We will now show that for every $\sigma \in \left(V_z^{[t]}\right)^*$, $(t, z) \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle$ is plurisubharmonic. As before, we can choose a sufficiently small neighborhood U_0 of 0 such that all the fibers X_t are contained in a fixed pseudoconvex domain Y_ε.

In general, we can find a holomorphic tangent vector field $\mathfrak{F}$ on X such that $d\pi_{\mathbb{C}^m}(\mathfrak{F}) = \partial/\partial t$ (see [BP08, Lemma 2.3]). Let $\Phi_{\mathfrak{F}}$ denote the flow of $\mathfrak{F}$ and let $\Phi_{\mathfrak{F}}^t$ denote the flow at time t. Let $\tilde{X}_t := \Phi_{\mathfrak{F}}^t(X_t)$. Since $\Phi_{\mathfrak{F}}^t$ maps X_t to $\tilde{X}_t$ biholomorphically, Proposition 3.4.8 implies that $K_t(z, \bar{z}) = \tilde{K}_t\left(\Phi_{\mathfrak{F}}^t(z), \overline{\Phi_{\mathfrak{F}}^t(z)}\right)$ for any $z \in X_t$. Here K_t is the Bergman kernel for the projection $L^2\left(X_t, h^{[t]}, dV_g\right) \to \mathcal{H}^2\left(X_t, h^{[t]}, dV_g\right)$ and $\tilde{K}_t$ is the one for the projection $L^2\left(\tilde{X}_t, \left(\Phi_{\mathfrak{F}}^{-t}\right)^* h^{[t]}, \left(\Phi_{\mathfrak{F}}^t\right)_* dV_g\right) \to \mathcal{H}^2\left(\tilde{X}_t, \left(\Phi_{\mathfrak{F}}^{-t}\right)^* h^{[t]}, \left(\Phi_{\mathfrak{F}}^t\right)_* dV_g\right)$.

We are now in the previous situation, and so for any fixed $z \in X_t$, the function

$$t \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z}) \rangle = \log \left\langle \sigma \otimes \bar{\sigma}, \tilde{K}_t\left(\Phi_{\mathfrak{F}}^t(z), \overline{\Phi_{\mathfrak{F}}^t(z)}\right) \right\rangle$$

is plurisubharmonic for any $\sigma \in \left(V_z^{[t]}\right)^*$. We may again assume that $m = 1$ without loss of generality by the previous upper semicontinuity arguments.

The flow $\Phi^t_{\mathfrak{F}}$ evaluated at z has the first-order Taylor expansion $\Phi^t_{\mathfrak{F}}(z) = z + t\mathfrak{F}(z) + O(t^2)$ in t around $t = 0$. Therefore, the function $t \mapsto \log \left\langle \sigma \otimes \bar{\sigma}, \tilde{K}_t\left(z, \bar{z}\right)\right\rangle$ is subharmonic in the direction of $\mathfrak{F}$, which is an arbitrary lift of $\partial/\partial t$. We thus have the the subharmonicity of the function $t \mapsto \log \left\langle \sigma \otimes \bar{\sigma}, \tilde{K}_t\left(z, \bar{z}\right)\right\rangle$ in a general non-vertical direction in X.

Now let $z = \Phi^{-t}_{\mathfrak{F}}(w)$. Then the function $t \mapsto \log \left\langle \sigma \otimes \bar{\sigma}, K_t\left(\Phi^{-t}_{\mathfrak{F}}(w), \overline{\Phi^{-t}_{\mathfrak{F}}(w)}\right)\right\rangle$ is subharmonic. Since the vector field $\mathfrak{F}$ is an arbitrary lift of $\partial/\partial t$, this shows that the function $(t, z) \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z})\rangle$ is subharmonic in every non-vertical direction. In the vertical directions $z \mapsto \log \langle \sigma \otimes \bar{\sigma}, K_t(z, \bar{z})\rangle$ is trivially subharmonic, as it is the sum of squares of holomorphic functions. This completes the proof. $\square$

6.2.2.2 Variations of families of compactly supported measures

Now let $\{\hat{\mu}_t\}_{t\in U}$ be a family of $\left(V^{[t]}\right)^*$-valued complex measures over X_t that are all locally supported in a compact subset of X. For each section $f \in \Gamma(E_h)$, define the measure $\mu^{(f)}_t = \left\langle f^{[t]}, \hat{\mu}_t\right\rangle$ and define the mapping $\xi^{(\mu)}_t$ by

$$f^{[t]} \mapsto \left\langle \xi^{(\mu)}_t, f^{[t]}\right\rangle := \mu^{(f)}_t(X_t) = \int_{X_t} \left\langle f^{[t]}, \hat{\mu}_t\right\rangle.$$

Then, similarly to the case of trivial families of Stein manifolds, $\xi^{(\mu)}$ defines a smooth section of E^*_h. We now prove Theorem G.

Proof. By the standard exhaustion argument, we may assume that X is a bounded strictly pseudoconvex domain with smooth boundary. Since the result is local, we may after restricting t to lie in a small neighborhood V_0 of a given point, say 0, assume that $X \subset V_0 \times Y_\varepsilon$ where $Y_\varepsilon := \{z \in Y : \rho(0, z) < \varepsilon\}$ is a pseudoconvex domain in Y. Then we apply Theorem E to $V_0 \times Y_\varepsilon$ with h replaced by h_j, where h_j is the approximating metric used in the proof of Theorem F. Recall that $h_j := he^{-\rho_j}$ where ρ_j is a function of ρ such that $e^{-\rho_j}$ converges to 1 as $j \to +\infty$ when $\rho \le 0$, and $e^{-\rho_j}$ converges to 0 as $j \to +\infty$ when $\rho > 0$. Therefore,

$$\|f\|^2_{L^2\left(\{t\}\times Y_\varepsilon, h^{[t]}_j\right)} := \int_{\{t\}\times Y_\varepsilon} |f|^2_{h^{[t]}_j}\, dV_g \xrightarrow[j\to+\infty]{} \int_{X_t} |f|^2_{h^{[t]}}\, dV_g =: \|f\|^2_{L^2\left(X_t, h^{[t]}\right)},$$

and similarly,

$$\left|\left|\int_{X_t}\left\langle f^{[t]},\hat{\mu}_t\right\rangle\right|-\left|\int_{\{t\}\times Y_\varepsilon}\left\langle f^{[t]},\hat{\mu}_t\right\rangle\right|\right|\leq\left|\int_{X_t}\left\langle f^{[t]},\hat{\mu}_t\right\rangle-\int_{\{t\}\times Y_\varepsilon}\left\langle f^{[t]},\hat{\mu}_t\right\rangle\right|\xrightarrow[j\to+\infty]{}0,$$

so that

$$\left\|\xi^{(\mu)}\right\|^2_{*,h_j^{[t]},\{t\}\times Y_\varepsilon}\xrightarrow[j\to+\infty]{}\left\|\xi^{(\mu)}\right\|^2_{*,h^{[t]},X_t}$$

and so the theorem follows. □

Once again, letting $\hat{\mu}_t := \sigma\otimes\delta_{F(t)}$ for each $t\in U$, where F is a holomorphic map from U to X_t, $\delta_{F(t)}$ denotes a point-mass measure supported at $F(t)$ and $\sigma\in V^*_{F(t)}$, we see that under the hypotheses of Theorem F, the function $U\ni t\mapsto \log\langle\sigma\otimes\bar{\sigma},K_t\left(F(t),F(t)\right)\rangle$ is plurisubharmonic (or strictly so) or identically $-\infty$ for every $\sigma\in\left(V^{[t]}\right)^*_{F(t)}$.

6.3 Twisted Prékopa-Leindler type theorems

We end this thesis with a mention of a couple of theorems that are much more similar in nature to Berndtsson's Prékopa-Leindler type theorems, under weaker assumptions than plurisubharmonicity. These two results follow immediately from Theorem C in the same way that Theorems 5.3.1 and 5.3.2 follow from Berndtsson's Nakano positivity theorem, as shown in Section 5.3.2.

Theorem 6.3.1. *Let Ω be a balanced pseudoconvex domain in $\mathbb{C}^n$, let $\delta>0$ and let η be a smooth function on Ω. Let U be a domain in $\mathbb{C}^m$ and let $\varphi\in\mathcal{C}^\infty(U\times\Omega)$ be S^1-invariant in z for any $t\in U$. If $\Xi_{\delta,\eta}\left(e^{-\varphi}\right)\geq 0$ (resp. >0) in $U\times\Omega$, then the function*

$$t\mapsto -\log\left(\int_\Omega e^{-\varphi(t,z)}dV(z)\right)$$

is plurisubharmonic (resp. strictly plurisubharmonic) or identically equal to $-\infty$.

Proof. As seen in the proof of Theorem 5.3.1, E is the direct sum of the holomorphic subbundles $E_k, k\in\mathbb{N}$, where E_k denotes the subbundle of E of homogeneous polynomials of degree k. Therefore, each E_k must be Griffiths positive in the sense of Definition 6.1.6 since E is Griffiths positive in the sense of Definition 6.1.6 by Theorem C. In particular, E_0 is Griffiths positive in the sense of Definition 6.1.6. But since E_0 is a trivial line bundle and the constant function $\mathbf{1}$ is a global frame, it follows that $t \mapsto -\log\left(\|\mathbf{1}\|^2_{\varphi(t,\cdot)}\right)$ is plurisubharmonic or identically $-\infty$. □

Theorem 6.3.2. *Let $\Omega := \{\zeta : \mathrm{Re}(\zeta) \in D\}$ for a convex domain D in $\mathbb{R}^n$. Let $\delta > 0$ and let η be a smooth function on Ω. Let U be a domain in $\mathbb{C}^m$ and assume that $\varphi \in \mathcal{C}^\infty(U \times \Omega)$ does not depend on the imaginary part of ζ. If $\Xi_{\delta,\eta}\left(e^{-\varphi}\right) \geq 0$ (resp. > 0) in $U \times \Omega$, then the function*

$$t \mapsto -\log\left(\int_D e^{-\varphi(t,x)} dx\right)$$

is plurisubharmonic (resp. strictly plurisubharmonic) or identically $-\infty$.

Proof. As mentioned just before Theorem 5.3.2, if Ω is a $\mathbb{T}^n$-invariant domain, the bundle E can be decomposed as the direct sum of the subbundles $E_\alpha, \alpha \in \mathbb{Z}^n$ where each E_α is spanned by z^α. These subbundles E_α are mutually orthogonal, and so each E_α must be Griffiths positive in the sense of Definition 6.1.6 since E is Griffiths positive in the sense of Definition 6.1.6 by Theorem C. Therefore, as each E_α is of rank 1 with a constant trivializing section $U \ni t \mapsto z^\alpha$, the function

$$t \mapsto -\log\left(\|z^\alpha\|^2_{\varphi(t,\cdot)}\right) := -\log\left(\int_\Omega |z^\alpha|^2 e^{-\varphi(t,z)} dV(z)\right)$$

is a plurisubharmonic function of t for all α.

The image $\tilde{\Omega}$ of Ω under the map $\exp : \mathbb{C}^n \ni \zeta \mapsto \left(e^{\zeta_1}, \cdots, e^{\zeta_n}\right) \in (\mathbb{C} - \{0\})^n$ is pseudoconvex since D is convex. Moreover, as φ is independent of $\mathrm{Im}(\zeta)$, there is a plurisubharmonic function $\tilde{\varphi}$ in $U \times \tilde{\Omega}$ such that $\tilde{\varphi}(t, \exp(\zeta)) = \varphi(t, \zeta)$. Clearly, the domain $\tilde{\Omega}$ is $\mathbb{T}^n$-invariant and the function $\tilde{\varphi}$ is $\mathbb{T}^n$-invariant with respect to z for each t. Therefore, the function

$$t \mapsto -\log\left(\int_\Omega |z^\alpha|^2 e^{-\varphi(t,z)} dV(z)\right)$$

is plurisubharmonic. Changing variables $z = \exp(\zeta)$, it follows that the integral over $\tilde{\Omega}$ equals

$$\int_{[0,2\pi]^n \times D} e^{2\alpha \cdot x} e^{-\varphi(t,x)} e^{2\sum_{j=1}^n x_j} dxdy.$$

Hence the function

$$t \mapsto -\log\left(\int_D e^{2\alpha\cdot x - \varphi(t,x) + 2\sum_{j=1}^n x_j} dx\right)$$

is plurisubharmonic. Since $A(x) = 2\left(\alpha \cdot x + \sum_{j=1}^n x_j\right)$ is an affine function of x, we may replace φ by $\varphi + A$ and the theorem follows. $\square$

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